Structured Pseudospectra and Random Eigenvalues Problems in Vibrating Systems

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Abstract

This paper introduces the concept of pseudospectra as a generalized tool for uncertainty quantification and propagation in structural dynamics. Different types of pseudospectra of matrices and matrix polynomials are explained. Particular emphasis is given to structured pseudospectra for matrix polynomials, which offer a deterministic way of dealing with uncertainties for structural dynamic systems. The pseudospectra analysis is compared with the results from Monte Carlo simulations of uncertain discrete systems. Two illustrative example problems, one with probabilistic uncertainty with various types of statistical distributions and the other with interval type of uncertainty, are studied in details. Excellent agreement is found between the pseudospectra results and Monte Carlo simulation results.

1 Introduction

Uncertainties are unavoidable in the description of real-life engineering systems. The propagation of uncertainties through a numerical model plays a crucial role in the analysis and design of an engineering structure. The majority of aerospace structures are subjected to dynamic forces during their period of service. For linear models of such structures, the dynamics are characterized by the eigenvalues and eigenvectors of the system. Therefore, when we take into account the uncertainties of a system, it is necessary to consider *random eigenvalue problems*. Extensive studies have been conducted on this topic since the mid-sixties within the scope of probabilistic modeling. The study of

probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The studies by Boyce[1], and Scheidt and Purkert[2] provide useful points of entry into this area of research. They also provide a systematic account of different approaches to random eigenvalue problems. Furthermore, several review papers, for example, by Ibrahim[3], Benaroya and Rehak[4], Benaroya[5], Manohar and Ibrahim[6], and Manohar and Gupta[7] have appeared in this field. These summarize both current research as well as the earlier works.

Uncertainties can be broadly divided into two categories. The first type is due to the inherent variability in the system parameters; for example, helicopter blades manufactured from a single production line do not have exactly the same properties. This type of uncertainty is often referred to as *aleatoric uncertainty*. If enough samples are present, it is possible to characterize the variability using well established statistical methods and consequently the probably density functions (pdf) of the parameters can be obtained. The second type of uncertainty is due to the lack of knowledge regarding a system, often referred to as *epistemic uncertainty*. This kind of uncertainty generally arises in the modeling of complex systems, for example, cabin noise in helicopters. Due to its very nature, it is comparatively difficult to quantify or model this type of uncertainty.

Broadly speaking, there are two approaches employed to model uncertainties. The first is the *probabilistic approach* and the second is the *possibilistic approach*. In the probabilistic approach the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficients and geometric parameters are quantified using statistical methods and propagated, for example, using the Stochastic Finite Element Method[8,9] (SFEM). This type of approach is suitable to quantify aleatoric uncertainties. On the other hand, epistemic uncertainties do not explicitly depend on the system's parameters. For example, there can be unquantified errors associated with the equations of motion (linear or non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods. For example, discretisation of displacement fields, truncation and roundoff errors, tolerances in the optimization of iterative algorithms, step-sizes in the time-integration methods. A probabilistic approach may not always be suitable to model this type of uncertainties. For this reason possibilistic approaches based on interval algebra [10], convex sets [11], Fuzzy sets [12], info-gap theory [13] or generalized Dempster-Schafer theory have been developed over the past four decades.

The goal of this paper is to quantify the variability in the eigenvalues of linear, uncertain systems where the equations of motion are governed by coupled second-order differential equations. The eigenvalues are complex non-linear

functions of the elements of the system matrices which are uncertain in nature. In spite of extensive research there are still no general and computationally manageable tools to propagate probabilistic and possibilistic uncertainties through such complex non-linear functions. In this paper we investigate the possibility of using pseudospectra as an alternative tool to characterize the variability in the eigenvalues of uncertain systems. The concept of pseudospectra was popularized in the early nineties by Trefethen [14,15] as a tool for analyzing non-normal matrices, that is, matrices with a non-orthogonal set of eigenvectors. They have since found application in the analysis of the transient response of systems and in sensitivity studies of eigenvalues of a linear system. We are interested in the latter problem and will demonstrate how information about the variability of eigenvalues in engineering systems can be obtained from several types of pseudospectra that are found in the literature.

The outline of the paper is as follows. In Section 2 we give an overview on various kinds of pseudospectra that have been discussed in the literature. In Section 3 we introduce a class of structured perturbations and discuss the corresponding pseudospectra. As we will demonstrate by example, in Sections 4 and 5 the use of structured perturbations allows us a direct comparison of pseudospectra with results from Monte-Carlo simulations. Specifically, in Section 4, we demonstrate how pseudospectra can be compared with simulations assuming several types of distributions of the system's parameters for a system with three degrees-of-freedom (DOF). In Section 5 we then demonstrate how pseudospectra with respect to structured perturbations can be used to obtain bounds for the eigenvalues of a system. Finally, in an appendix to the paper we present an introduction to the general theory of structured pseudospectra of matrix polynomials which unifies the several types of pseudospectra introduced before. We also present an algorithm for deriving the shape matrices needed for computing the structured pseudospectra presented in the examples.

2 Background on Pseudospectra

Pseudospectra provide a deterministic way of dealing with uncertainties in a system. They were originally introduced by Trefethen [16] as a tool for the treatment of non-normal matrices, i.e., matrices without a complete set of orthogonal eigenvectors. Two recent books, Refs. [17] and [18], give excellent overviews on pseudospectra, including many examples and historic remarks. Other introductions to pseudospectra can be found in Refs. [14] and [15]. Furthermore, the *Pseudospectra Gateway* provides an extensive list of updated references and links. It can be found at

`http://web.comlab.ox.ac.uk/projects/pseudospectra.`

Various definitions of pseudospectra are discussed in the literature. Below we

will introduce the most relevant ones for applications in structural mechanics. In fact, it turns out that they can all be discussed within the unified framework of *structured pseudospectra of matrix polynomials* [19]. A brief explanation of this is provided in the Appendix of this paper.

2.1 Pseudospectra of matrices

In the simplest setting, given a matrix $A \in \mathbb{C}^{n\times n}$ we investigate the sensitivity of its eigenvalues under additive perturbations by considering

$$\Lambda_\epsilon(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + \delta A) \text{ for some } \delta A \in \mathbb{C}^{n\times n} \text{ with } \|\delta A\| < \epsilon\}, \quad (1)$$

where we denote by $\sigma(\cdot)$ the spectrum of a matrix and by $\|\cdot\|$ the (induced) 2-norm, that is, the largest singular value. The set $\Lambda_\epsilon(A)$ is called the ϵ-*pseudospectrum* (or spectral value set) [18,17] of the matrix A.

The importance of pseudospectra for investigations into the sensitivity of eigenvalues of a system is immediate from the definition. Namely, the ϵ-pseudospectrum bounds all eigenvalues of the perturbed system $A + \delta A$ with a perturbation δA of size less than or equal ϵ.

It is important to note that (1) is based on *complex perturbation matrices*. This may seem inappropriate, since most examples occurring in applications deal with real matrices. However, complex matrices are the natural setting for eigenvalue problems, and consequently the algorithms for computing and approximating pseudospectra are more powerful when complex perturbation matrices are considered. Therefore, throughout this paper we will deal with complex matrices. In applications, this approach will yield upper bounds for the sensitivity of eigenvalues, since non-physical perturbation matrices are included. However, in the examples in Section 4 we will demonstrate how conclusions about the sensitivity of eigenvalues under real perturbations can be drawn from the results.

Formula (1) allows one to approximate pseudospectra by computing the spectrum of $A+\delta A$ with a number of randomly generated matrices δA of norm ϵ. It is, however, not convenient to use (1) for computing pseudospectra. Instead, we consider the resolvent operator of A, defined by

$$R(\lambda, A) = (\lambda I - A)^{-1},$$

and observe that λ is an eigenvalue of A, if $R(\lambda, A)$ does not exist, that is, if $(\lambda I - A)$ is singular. A complex number λ is contained in the ϵ-pseudospectrum of A, if $(\lambda I - A)$ is close to being singular. More precisely, it can be shown [15] that

$$\Lambda_\epsilon(A) = \left\{\lambda \in \mathbb{C} : \left\|(\lambda I - A)^{-1}\right\| > 1/\epsilon\right\}. \quad (2)$$

This formula lies at the heart of pseudospectra computations. A straightforward way of applying (2) is to evaluate the norm of the resolvent on a grid over a region of the complex plane. Plotting the level set contours for different ϵ-values identifies boundaries of the pseudospectra of A. A discussion of more refined methods for the computation of $\Lambda_\epsilon(A)$ can be found in Ref. [15]. What is more, a publicly available package `EigTool` [20] for the computation of pseudospectra exists.

2.1.1 Pseudospectra of a matrix with respect to structured perturbations

In a large number of applications the matrix A has a certain structure, for example, a block-structure or symmetry, which should be respected in the sensitivity analysis. Consequently, pseudospectra with respect to certain structured perturbations have been introduced by a number of authors. For example, pseudospectra with respect to perturbations that are symmetric or circulant are studied by Rump [21], see also the references therein.

Motivated by problems in control theory, pseudospectra with respect to *full-block perturbations* have been introduced by Hinrichsen and co-authors [22,17]. More precisely, they consider affine perturbations of the form

$$A + D\delta AE, \quad (3)$$

with fixed *shape matrices* $D \in \mathbb{C}^{n\times k}$ and $E \in \mathbb{C}^{l\times n}$ describing the perturbation structure, and $\delta A \in \mathbb{C}^{k\times l}$ providing the perturbation matrix. Roughly speaking, such perturbations allow one to perturb blocks in A, with D and E describing the columns and rows to be perturbed, respectively. Pseudospectra with respect to this class of perturbations are defined as

$$\Lambda_\epsilon(A; D, E) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + D\delta AE) \text{ for some } \delta A \text{ with } \|\delta A\| < \epsilon\} \quad (4)$$

It has been shown in Ref. [22] that $\Lambda_\epsilon(A; D, E)$ can be computed using the norm of the transfer function $G(\lambda) := ER(\lambda, A)D$, such that,

$$\Lambda_\epsilon(A; D, E) = \{\lambda \in \mathbb{C} : \|G(\lambda)\| > 1/\epsilon\}. \quad (5)$$

This relationship allows one to compute $\Lambda_\epsilon(A; D, E)$ via the grid approach described above. Note that one can also use the matrices D and E to weight perturbations of blocks in A differently.

This structured approach was recently developed further for perturbations of single elements of the matrix A [23]. More specifically, let $W = (W_{ij}) \in \mathbb{R}^{n\times n}$

denote a matrix of non-negative weights. A *structured ϵ-pseudospectrum* $\Lambda_\epsilon^s(A)$ of A with weights W can be defined as

$$\Lambda_\epsilon^s(A) = \left\{ z \in \mathbb{C} : z \in \sigma \left(A + \sum_{i=1}^{n} \sum_{j=1}^{n} \delta A_{ij} \cdot e_i e_j^T \right) \right.$$
$$\left. \text{for some } \delta A = (\delta A_{ij}) \text{ with } |\delta A_{ij}| < \epsilon W_{ij} \text{ for all } i, j \right\}, \quad (6)$$

where e_i denotes the i-th unit vector, that is, the vector consisting of zeros apart from the i-th position, where it contains a one. Hence, $\Lambda_\epsilon^s(A)$ takes perturbations into account that affect only those elements of A to which a positive weight $W_{ij} > 0$ is associated.

The perturbations in (6) can be rewritten in the form $A + D\Delta E$, where in contrast to (3) the perturbation matrix $\Delta = \text{diag}(A_{ij})$ has to be a diagonal matrix. In this way, the weights W_{ij} can be incorporated into the matrices D and E such that

$$\Lambda_\epsilon^s(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + D\Delta E)$$
$$\text{for some diagonal matrix } \Delta \text{ with } \|\Delta\| < \epsilon\}.$$

A computable formula for $\Lambda_\epsilon^s(A)$ has been derived in Ref. [23]. As we shall see, this formula is a special case of the results presented in Section 3.

2.2 Pseudospectra of matrix polynomials

Vibrating systems have been the motivation for defining another type of pseudospectra. Such systems are described by second-order ordinary differential equations (ODEs) of the form

$$A_2\ddot{x} + A_1\dot{x} + A_0 x = 0, \quad (7)$$

with coefficient matrices $A_i \in \mathbb{C}^{n\times n}$ $(i = 0, 1, 2)$ describing stiffness, damping, and mass effects, respectively. Eigenvalues of the system (7) are eigenvalues of the associated *matrix polynomial*

$$Q(\lambda) = A_2\lambda^2 + A_1\lambda + A_0, \quad (8)$$

that is, solutions of $\det(Q(\lambda)) = 0$. Ref. [24] gives a comprehensive overview on the properties and solution techniques of quadratic eigenvalue problems.

The sensitivity of eigenvalues of (7) in the class of second-order systems can be studied by determining roots of

$$\det(Q+\delta Q)(\lambda) = \det\left((A_2+\delta A_2)\lambda^2 + (A_1+\delta A_1)\lambda + (A_0+\delta A_0)\right), \quad (9)$$

with perturbation matrices $\delta A_i \in \mathbb{C}^{n\times n}$. To introduce corresponding pseudospectra we let $w = (w_0, w_1, w_2)$ be a vector of weights and define

$$\Lambda_\epsilon(Q) := \{\lambda \in \mathbb{C} : (Q+\delta Q)(\lambda) = 0 \\ \text{for some } \delta A_i \text{ with } w_i\|\delta A_i\| < \epsilon,\ i = 0,\ldots,2\}. \quad (10)$$

The w_i allow us to weight perturbations to the coefficient matrices separately. For example, $w_i \equiv 1$ corresponds to an absolute measure of the perturbations, while $w_i = 1/\|A_i\|$ corresponds to a relative measure. We also allow for $w_i = \infty$, in which case A_i must not be perturbed. We call $\Lambda_\epsilon(Q)$ the *weighted pseudospectrum* of Q.

Weighted pseudospectra of matrix polynomials were first introduced by Tisseur and Higham [25], see also Ref. [26]. In those papers it has also been shown that

$$\Lambda_\epsilon(Q) = \left\{\lambda \in \mathbb{C} : \left\|Q(\lambda)^{-1}\right\| > \left(\epsilon(|\lambda|^2/w_2 + |\lambda|/w_1 + 1/w_0)\right)^{-1}\right\}. \quad (11)$$

Observe the similarity of this formula to (2). The polynomial in the inequality in (11) can be interpreted as a weighting factor, induced by the second-order structure. Again, formula (11) can be used for computing weighted pseudospectra of Q, using a grid approach as described above.

Remark 1 *Pseudospectra of quadratic matrix polynomials are a special case of pseudospectra for matrix functions $F(\lambda) = \sum_{i=1}^m A_i p_i(\lambda)$, with p_i as analytic functions, studied in Ref. [27]. In that paper different possibilities for introducing a joint norm for the perturbations of the coefficient matrices A_i are also discussed.*

Full-block perturbations offer an alternative way of preserving the second-order structure of (7). Note that (7) is equivalent to the first-order system $\dot z = \mathcal{A} z$ where $z = (x, \dot x)$ and

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & I \\ -A_2^{-1}A_0 & -A_2^{-1}A_1 \end{pmatrix}. \quad (12)$$

In this setting, one can preserve the second-order structure by considering perturbations that only affect the two lower blocks of $\mathcal{A}$. For example, this can be achieved by setting $D = [\mathbf{0},\ I]^T$ and $E = [I,\ I]$ (the 'T' denotes the transpose of a matrix).

A major drawback in using (12) is that the coefficient matrices A_i are grouped together, and therefore it is nearly impossible to analyze the influence of perturbations to elements of these matrices individually. This difficulty can be overcome by directly considering *s*tructured perturbations of the matrix polynomial. A detailed introduction is presented in Ref. [19]. In the following we recall the most fundamental results for the case of second-order matrix polynomials.

3 Pseudospectra Using Structured Perturbations

In this section, we consider quadratic matrix polynomials of the form (8), that is,

$$Q(\lambda) = A_2\lambda^2 + A_1\lambda + A_0, \tag{13}$$

with $A_i \in \mathbb{C}^{n\times n}$. A complex number λ is called an *eigenvalue* of Q if $\det(Q(\lambda)) = 0$. We are interested in the behaviour of eigenvalues of Q under *structured perturbations* of the individual coefficient matrices A_i. More specifically, we consider perturbations

$$\delta Q(\lambda) = D_2\Delta_2E_2\lambda^2 + D_1\Delta_1E_1\lambda + D_0\Delta_0E_0 \tag{14}$$

with shape matrices $D_i \in \mathbb{C}^{n_i\times k_i}$ and $E_i \in \mathbb{C}^{k_i\times n_i}$ describing the structure of the perturbations and containing possible weights. As in the case of structured pseudospectra of single matrices the perturbation matrices $\Delta_i \in \mathbb{C}^{k_i\times k_i}$ are assumed to be *diagonal matrices*, that is,

$$\Delta_i = \mathrm{diag}(\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,k_i}). \tag{15}$$

Perturbations of the form (14) cover a large number of applications as we will show in the examples of Sections 4 and 5.

The *structured ϵ-pseudospectrum* $\Lambda^s_\epsilon(Q)$ of Q is then defined as

$$\begin{aligned}\Lambda^s_\epsilon(Q) = \{&\lambda \in \mathbb{C} : \det(Q(\lambda) + \delta Q(\lambda)) = 0,\\ &\text{for some } \delta Q \text{ of the form (14) with } \|\mathrm{diag}(\Delta_0, \Delta_1, \Delta_2)\| < \epsilon\}.\end{aligned} \tag{16}$$

Notice that $\|\mathrm{diag}(\Delta_0, \Delta_1, \Delta_2)\| < \epsilon$ is equivalent to $\delta_{i,j} < \epsilon,\ 1 \le j \le k_i,\ \ 0 \le i \le 2$.

Straightforward matrix algebra leads to a computable formula for $\Lambda^s_\epsilon(Q)$. Letting $\Delta = \mathrm{diag}(\Delta_2, \Delta_1, \Delta_0)$, one can show that

$$\det(Q(\lambda) + \delta Q(\lambda)) = 0 \Leftrightarrow \det\left(I + G(\lambda)\Delta\right) = 0,$$

where

$$G(\lambda) := \begin{bmatrix} E_2 \\ E_1 \\ E_0 \end{bmatrix} Q(\lambda)^{-1} \begin{bmatrix} \lambda^2 D_2 & \lambda D_1 & D_0 \end{bmatrix};$$

here it is assumed that λ is not an eigenvalue of Q.

Thus, determining whether $\lambda \in \Lambda^s_\epsilon(Q)$ amounts to determining the minimum norm of a diagonal perturbation Δ such that $\det\left(I + G(\lambda)\Delta\right) = 0$. This is a well-known problem in robust control, where the inverse of this number has been called the *structured singular value* (ssv) or μ-function $\mu_{\mathbf{\Delta}}(G(\lambda))$ [17,28,29]. More precisely, let $\mathbf{\Delta}$ denote the space of complex diagonal matrices. Then, for a matrix G, its ssv with respect to $\mathbf{\Delta}$ is defined as

$$\mu_{\mathbf{\Delta}}(G) = \begin{cases} 0, & \text{if } \det(I - G\Delta) \neq 0 \text{ for all } \Delta \in \mathbf{\Delta}, \\ (\min\{||\Delta|| :\ \Delta \in \mathbf{\Delta},\ \text{and } \det(I - G\Delta) = 0\})^{-1}, & \text{else.} \end{cases}$$

Remark 2 *Structured singular values can be defined for arbitrary closed subsets of the linear space of matrices. The concept can be seen as an extension of the usual singular value, explicitly taking into account a structure of perturbations, that is, their membership to the set of perturbation matrices* $\mathbf{\Delta}$ *(the ssv of a matrix w.r.t. unstructured perturbations is equal to its largest singular value).*

The arguments above show that (16) is equivalent to

$$\Lambda^s_\epsilon(Q) = \left\{\lambda \in \mathbb{C} : \left(\mu_{\mathbf{\Delta}}\left(G(\lambda)\right)\right)^{-1} > 1/\epsilon\right\}. \tag{17}$$

Although we have only rewritten the original problem, we can use (17) for computing structured pseudospectra of Q by the standard grid method.

A number of computational tools are available for computing or approximating the ssv. For example, the Robust Control Toolbox in `Matlab` [30] can be used

to compute the ssv of the matrix $G(\lambda)$ using the routine `mussv`. Therefore, all that is required to compute $\Lambda_\epsilon^s(Q)$ in applications is to formulate $G(\lambda)$.

We also note that `mussv` computes both the upper and lower bounds for the ssv, and thus for the pseudospectra. In the computations in Section 4-4.3, the maximum difference between the values of these bounds was of the order 10^{-4} and thus negligible. Furthermore, the definitions presented above and following the computations assume *complex perturbations* of the matrix polynomial. A restriction to real perturbations is theoretically possible [17]. However, the approximation of the ssv is a lot worse in this case, and therefore the computational results are not helpful.

4 Example of a Three Degree of Freedom System

The connection between pseudospectra and random eigenvalue problems will be illustrated using simple discrete structural dynamic systems.

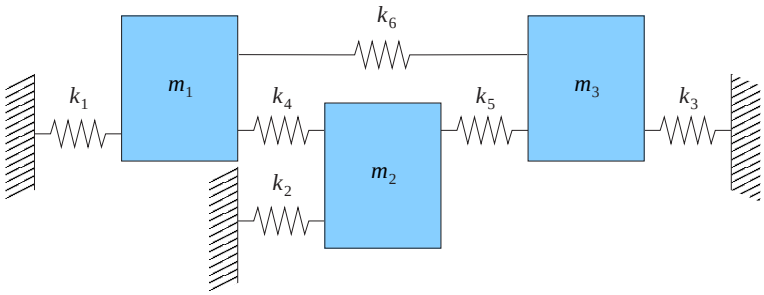

Fig. 1. A three degrees of freedom mass-spring-system.

The first example, taken from Ref. [31], concerns the undamped spring-mass-system shown in Figure 1. The system is described by the second-order equation

$$M\ddot{x} + Kx = 0 \tag{18}$$

where the mass matrix M and the stiffness matrix K are given by

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$K = \begin{pmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_2 + k_4 + k_5 & -k_5 \\ -k_6 & -k_5 & k_3 + k_5 + k_6 \end{pmatrix}.$$

In Ref. [31] the authors deal with the uncertainty in the system by assuming the numbers m_i and k_i to be random variables. Specifically, it is assumed that

$$\begin{aligned} m_i &= \bar{m}_i(1 + \varepsilon_m X_i), \quad i = 1, 2, 3 \\ k_i &= \bar{k}_i(1 + \varepsilon_k X_{i+3}), \quad i = 1, \ldots, 6. \end{aligned} \tag{19}$$

Here the X_i are uncorrelated Gaussian random variables with zero mean and standard deviation. Furthermore, $\bar{m}_i = 1$ for $i = 1, 2, 3$, $\bar{k}_i = 1$ for $i = 1, \ldots, 5$ and $\bar{k}_6 = 3$. The degree of uncertainty is described by

$$\varepsilon_m = 0.15, \ \varepsilon_k = 0.15. \tag{20}$$

A straightforward Monte Carlo simulation indicates the behaviour and sensitivity of the system's eigenvalues. The results from 2000 simulations are illustrated in Figure 2. Note that in order to facilitate later comparison with pseudospectra we assume the variables X_i to be complex with normally distributed real and imaginary part. It can clearly be seen in Figure 2 that the eigenvalues with larger imaginary part are more sensitive than the ones closer to the real axis.

We will perform a rigorous uncertainty analysis using the several pseudospectra definitions given in Sections 2 and 3. The results will be compared with the simulation results shown in Figure 2.

The first step in our analysis is to formulate a corresponding deterministic system. It is natural to choose this system according to the expectations $\bar{m}_i$ and $\bar{k}_i$, such that it is given by

$$M_0\ddot{x} + K_0 x = 0 \tag{21}$$

where

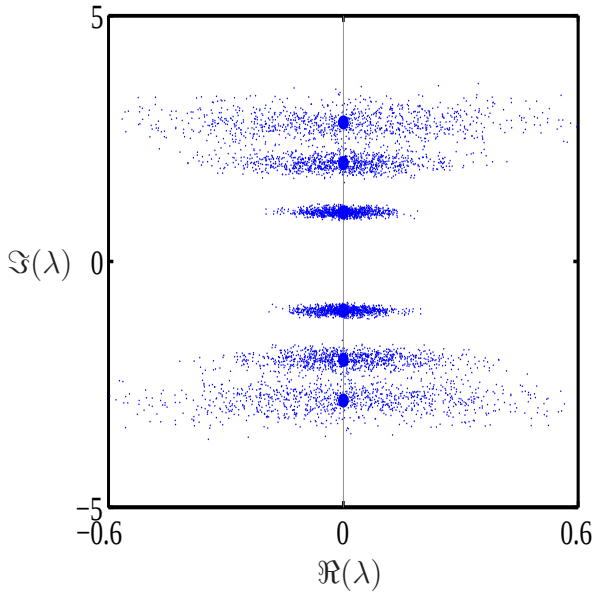

Fig. 2. Eigenvalues of 2000 simulations of (18) in the complex plane. The large dots indicate the expectation for the eigenvalues.

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_0 = \begin{pmatrix} 5 & -1 & -3 \\ -1 & 3 & -1 \\ -3 & -1 & 5 \end{pmatrix}.$$

The eigenvalues λ of (21) are zeros of the matrix polynomial $Q_0(\lambda) = M_0\lambda^2 + K_0$. They are given as

$$\{\lambda_1, \ldots, \lambda_6\} = \{\pm i, \pm 2i, \pm 2\sqrt{2}i\}.$$

These are shown as large dots in Figure 2.

In what follows, we will first consider pseudospectra of (21) rewritten as a first-order system. Pseudospectra with respect to both general and full-block perturbations will be computed. We will then turn to the second-order system and analyse weighted and structured pseudospectra.

4.1 Pseudospectra of the first-order system

Equation (18) can be written as the first-order system $\dot{z} = \mathcal{S}z$, $z = (x, \dot{x})$, where

$$\mathcal{S} = \begin{pmatrix} 0 & I \\ -K_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -5 & 1 & 3 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 \\ 3 & 1 & -5 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

We compute ϵ-pseudospectra of $\mathcal{S}$ via (2). As we consider the 2-norm in the paper, we can exploit the relationship $1/||R(\lambda, \mathcal{S})|| = \sigma_{min}(\lambda I - \mathcal{S})$, where σ_{min} denotes the smallest singular value. The computation of pseudospectra using formula (2) thus amounts to computing $\sigma_{min}(\lambda I - \mathcal{S})$ for values of λ on a grid over the complex plane. A given contour of value ϵ then describes the boundary of ϵ-pseudospectrum of $\mathcal{S}$.

Figure 3(a) shows the results of a pseudospectra computation. We perform the computation, using `Matlab`, on a 200×200 grid. This computation took 1.9 seconds on a 2.8 GHz ZEON processor. The curves in the figure correspond to ϵ values of $\epsilon = 10^{-2.5}$, $10^{-2.0}$, $10^{-1.5}$, $10^{-1.0}$, $10^{-0.5}$, and 1. We clearly see a *qualitative* agreement with the simulation results, in that the eigenvalues with larger imaginary part are more sensitive under perturbations. However, it is not clear how the results can be related quantitatively to the simulations above. In particular, since K_0 and M_0 both enter $\mathcal{S}$ it is not clear which ϵ-value corresponds to the uncertainty measure in the simulations.

Remark 3 *Another reason for the importance of unstructured pseudospectra in the analysis of engineering systems is their use in the description of the transient response of a system [15,18]. Since we are not concerned with this problem in the present paper, we only remark that one can derive estimates for the transient growth in a system from pseudospectra and recommend the references for details.*

We next compute pseudospectra of $\mathcal{S}$ with respect to the full-block perturbation approach (3) with $D = [\mathbf{0},\ I]^T$ and $E = [I,\ \mathbf{0}]$. In this way, only the lower left 3×3 block of $\mathcal{S}$ will be perturbed, and we thus preserve the second-order structure of the problem. Moreover, we do not introduce damping, since no perturbations to the lower-right sub-block of $\mathcal{S}$ are added.

Formula (5) is used for the computations and Figure 3(b) shows the results. Again, we consider a 200×200 grid; computation time was 5.3 seconds. Once more, the boundaries of pseudospectra for $\epsilon = 10^{-2.5}$, 10^{-2}, $10^{-1.5}$, 10^{-1}, $10^{-0.5}$, and 1 are shown. In contrast to the general pseudospectra we now find the eigenvalues closest to the real axis to be more sensitive to perturbation. This is remarkable, since the use of full-block perturbation eliminates non-physical

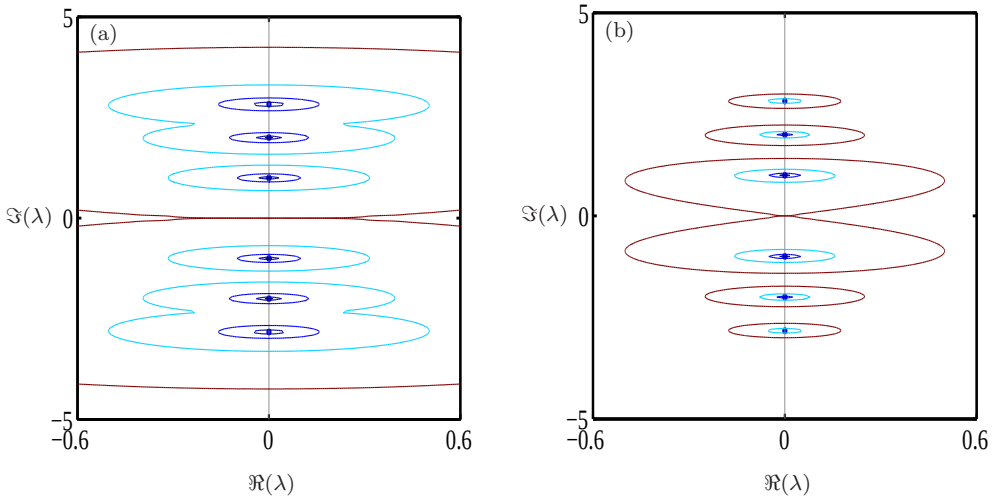

Fig. 3. General pseudospectra of $\mathcal{S}$, (a), and pseudospectra with respect to full-block perturbations, (b), computed using (2) and (5), respectively.

perturbations. They should thus give better quantitative information. It is not understood, however, why they do not reflect the qualitative behaviour in the system.

4.2 Weighted pseudospectra of the second-order system

As discussed in Section 2 the main drawback in an analysis of the first-order system is the difficulty of describing the influence of perturbations to the matrices M_0 and K_0 separately. We therefore turn now to the computation of pseudospectra of the matrix polynomial $Q_0(\lambda) = M_0\lambda^2 + K_0$.

We first compute weighted pseudospectra using formula (11). For this it remains to determine suitable weights. Obviously, $w_1 = \infty$, since no damping should be introduced to the system. The weights w_0 and w_2 are chosen according to the distribution of the m_i and k_i; see (20). Recall from (20) that the m_i and $k_1, \ldots, k_5$ are distributed with standard deviation 0.15, and k_6 with standard deviation 0.45. With these values we have $v_m = 0.15$ and $v_k = 1.2558$ as the 2-norms of the matrices of standard deviations. Consequently, we choose $w_0 = 1/v_k$ and $w_2 = 1/v_m$.

As explained in Section 2 we can use (11) to compute pseudospectra by evaluating $\|Q_0(\lambda)^{-1}\|_2/(|\lambda|^2/w_2 + 1/w_0)$ for complex numbers λ on a grid over a region in the complex plane. Figure 4 contains ϵ-pseudospectrum contours for $\epsilon = 10^{-2.5}$, 10^{-2}, $10^{-1.5}$, 10^{-1}, $10^{-0.5}$, and 1. Again, the computation is per-

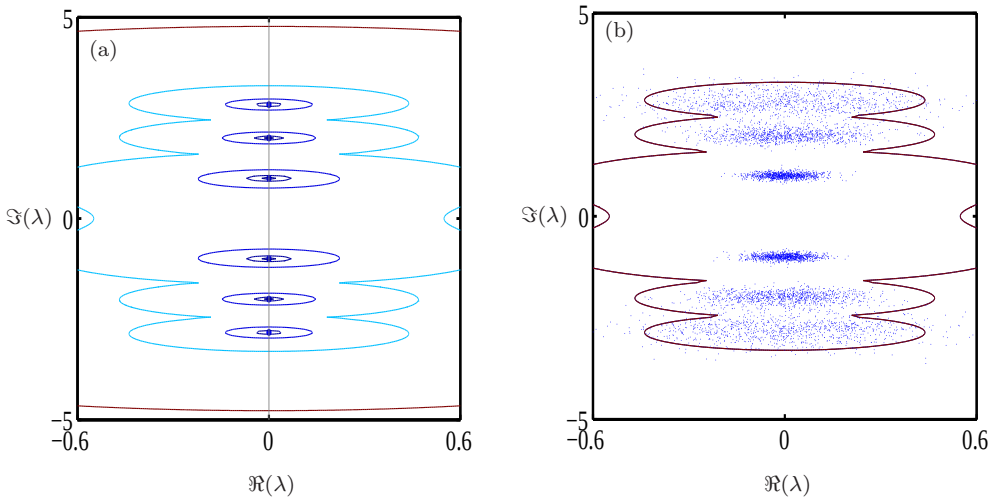

Fig. 4. Weighted pseudospectra of Q_0, computed using (11), (a), and comparison with the simulation results, (b).

formed on a 200×200 grid; computation time was 2.3 seconds. As expected the weighted pseudospectra agree qualitatively with the ones obtained using full-block perturbations of $\mathcal{S}$, in that the eigenvalues closest to the real axis are the most sensitive. Again, it is not clear how to explain the discrepancies with the simulation results.

Note that we have chosen the weights w_0 and w_2, such that the $(\epsilon = 1)$-contour should correspond to the simulation results. Figure 4 (b) shows the $(\epsilon = 1)$-contour again, together with the eigenvalues of 2000 simulations. From this, we conclude that weighted pseudospectra do not provide satisfactory information about the behaviour of eigenvalues in the example. In fact, while estimates about the sensitivity of the eigenvalues closest to the real axis would be far too conservative, the sensitivity of the eigenvalues with largest imaginary part is not reflected accurately, such that a number of eigenvalues lie outside the contour. (We note that the normally distributed variables m_i and k_i can take arbitrarily large values and so we always have to expect some eigenvalues to lie outside the pseudospectra contours.)

4.3 Structured pseudospectra of the second-order system

We now apply the theory presented in Section 3 to the computation of pseudospectra of (21) using structured perturbations. As we will demonstrate, this approach allows us to rigorously investigate the sensitivity of eigenvalues of the system with respect to physical perturbations alone. More precisely, we are interested in the eigenvalues of the perturbed system

$$\left[M_0 + \begin{pmatrix} \delta m_1 & 0 & 0 \\ 0 & \delta m_2 & 0 \\ 0 & 0 & \delta m_3 \end{pmatrix}\right] \lambda^2$$

$$+ \left[K_0 + \begin{pmatrix} \delta k_1 + \delta k_4 + \delta k_6 & -\delta k_4 & -\delta k_6 \\ -\delta k_4 & \delta k_2 + \delta k_4 + \delta k_5 & -\delta k_5 \\ -\delta k_6 & -\delta k_5 & \delta k_3 + \delta k_5 + \delta k_6 \end{pmatrix}\right]. \quad (23)$$

This problem can be treated using the structured pseudospectra approach (16). For this, we rewrite (23) in the form (14)

$$\left[M_0 + D_M \Delta_M E_M\right] \lambda^2 + \left[K_0 + D_K \Delta_K E_K\right]. \quad (24)$$

The main difficulty in this formulation is to find appropriate shape matrices $D_{M,K}$ and $E_{M,K}$. A general method for doing this, based on a singular value decomposition, is described in the Appendix. Application of this method to (23) shows that

$$D_M = 0.15 \cdot I, \quad E_M = I, \quad \Delta_M = \mathrm{diag}(\delta m_1, \delta m_2, \delta m_3), \quad (25)$$

$$D_K = 0.15 \cdot \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -3 \end{pmatrix}, \quad E_K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, \quad (26)$$

$$\Delta_K = \mathrm{diag}(\delta k_1, \delta k_2, \delta k_3, \delta k_4, \delta k_5, \delta \hat{k}_6).$$

Observe that we have introduced $\delta \hat{k}_6 = \delta k_6/3$, in order to give a direct comparison to the random perturbations, where we note that $\bar{k}_6 = 3$ results in a threefold increase in the uncertainty ϵ_6; see (19). The elements in the second column of D_K have been rescaled accordingly, thus reflecting the different weight for perturbations of k_6. This demonstrates how the approach allows us

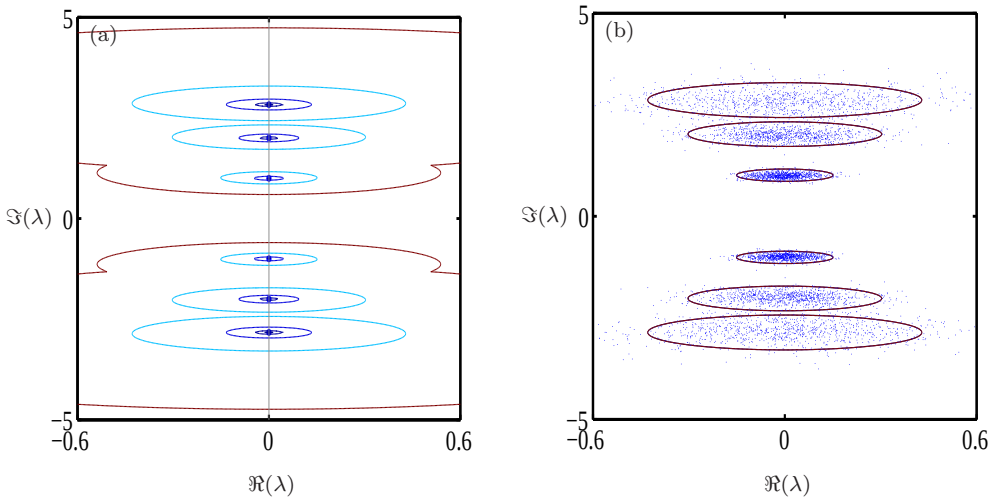

Fig. 5. Structured pseudospectra of Q_0, computed using (17), (a), and comparison with the simulation results, (b).

to weight perturbations of individual entries in the system's matrices differently, similar to the method introduced in Ref. [23]. Furthermore, the degree of uncertainty is reflected in the factor 0.15 in front of the D_M and D_K matrices.

Finally, letting

$$G(\lambda) = \begin{pmatrix} E_M \\ E_K \end{pmatrix} (M_0\lambda^2 + K_0)^{-1} \left(D_M\lambda^2/w_2 \; D_K/w_0 \right), \tag{27}$$

we use the `Matlab` routine `mussv` to compute values of $\mu_{\mathbf{\Delta}}(G(\lambda))$ for λ on a grid over the complex plane. According to (17) a contour plot then yields the boundaries of structured ϵ-pseudospectra. Figure 5 (a) shows the results of this computation. We use a 200×200 grid on the corresponding region in the complex plane. The computation time was 13.50 minutes. As before, ϵ-pseudospectra for $\epsilon = 10^{-2.5}$, 10^{-2}, $10^{-1.5}$, 10^{-1}, $10^{-0.5}$, and 1 are shown. We note that the routine `mussv` computes both upper and lower bounds for the ssv, and thus for the ϵ-values of pseudospectra contours. In this example, the maximal difference between these bounds, on the grid used, is of the order 10^{-4}, such that they are indistinguishable.

The qualitative agreement between the ϵ-pseudospectra and the simulation results is evident in Figure 5. In fact, we can even compare the results quantitatively. Note that the matrices D_M and D_K have been set up such that the $(\epsilon = 1)$-contour again corresponds to the simulation results. In Figure 5 (b), where this contour is shown again, the correspondence is illustrated. Each of the eigenvalues is encircled by a pseudospectrum contour. For the eigenvalues with large imaginary part, however, these contour almost touch each other,

whereas contour around the eigenvalues closest to the real axis is well separated. This is reflected in the simulation results, where it is very difficult to distinguish between perturbations of the eigenvalues at $\pm 2i$ and $\pm 2\sqrt{2}i$.

We conclude that structured pseudospectra of Q_0 provide a rigorous way of analysing the sensitivity of eigenvalues. A disadvantage of the method is the computational cost, which is substantially higher than with all other methods. On the other hand, structured pseudospectra can give information for a whole family of problems, since the different ϵ-contours are related to different units of uncertainty present in the system. Thus, structured pseudospectra may be used for a solution of the inverse problem of establishing bounds of uncertainty that are necessary for a robust stable operation of the engineering system. In other words, they are directly related to the important concept of *stability radii* [17,27].

4.4 Non-Gaussian random perturbations

We have seen that structured pseudospectra of matrix polynomials offer an alternative method for studying the sensitivity of the eigenvalues of (18). In this section we further illustrate the relation between pseudospectra and random elements in the model matrices and consider several different types of probability distributions for the parameters m_i and k_i. As in Section 4 we consider complex random variables, so that the simulation results compare better with the pseudospectra.

Let us first assume that the m_i and k_i are given by

$$\begin{aligned} m_i &= \bar{m}_i(1 + 0.15X_i), \quad i = 1, 2, 3 \\ k_i &= \bar{k}_i(1 + 0.15X_{i+3}), \quad i = 1, \ldots, 6, \end{aligned} \tag{28}$$

where the real and imaginary parts of the random variables X_i are *uniformly distributed* on $[-1, 1]$. The numbers $\bar{m}_i$ and $\bar{k}_i$ have the same values as before, that is, $\bar{k}_6 = 3$ and $\bar{m}_i = \bar{k}_i = 1$ for all other i. This setup is perfectly suited for an analysis using pseudospectra. A major advantage of the uniform distribution is that it provides bounds for the variation of elements, which can be directly translated into weights and ϵ-values for the pseudospectrum computations. The distributions (28) are chosen such that the structured pseudospectrum of Q_0 can be compared to simulation results. Thus, for the pseudospectra computations we employ the unperturbed matrices in (21) according to the expectations of the m_i and k_i, that is, they are given by M_0 and K_0. We also use the same matrices D_K, E_K, D_M and E_M.

In Figure 6 we show the eigenvalues of 2000 simulations of (18) with uniformly

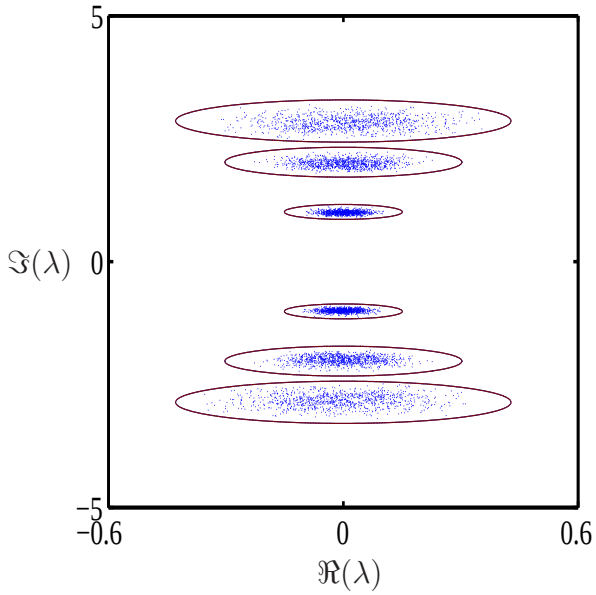

Fig. 6. Pseudospectrum contour of Q_0 and eigenvalues of 2000 simulations of (18) with uniformly distributed m_i and k_i in the complex plane.

distributed m_i and k_i. Also included is the corresponding ($\epsilon = 1$)-contour of the structured pseudospectrum of Q_0. It is clearly shown that the pseudospectrum can be understood as a 'worst case' analysis. In fact, it can be seen that it is very unlikely to generate matrices that push eigenvalues close to the boundary of the pseudospectrum. This effect has been observed before [22].

We finally deal with two further types of probability distribution that are often used in structural mechanics, namely, the *log-normal distribution* and the χ^2-*distribution*. Random variables that are distributed according to these distributions take values on an infinite (positive) interval and are therefore more suitable to model strictly positive quantities like mass and stiffness coefficients. Under these distributions it is more difficult to set up an appropriate pseudospectrum analysis. Similar to the procedure in Section 4.3 we propose the following two steps:

- The deterministic system is constructed according to the expectation of the elements of the system's matrices.
- The matrices D_K and D_M both describe the structure of the perturbation and contain weights chosen according to the standard deviation of the elements.

The first step is conceptually consistent with the non-parametric uncertainty modelling concept proposed by Soize[33,34]. We emphasize that in the second step the relationship between the weights is more important than their absolute value. In fact, for fixed matrices D_K and D_M different pseudospectra contours can be seen as being related to different standard deviations of the

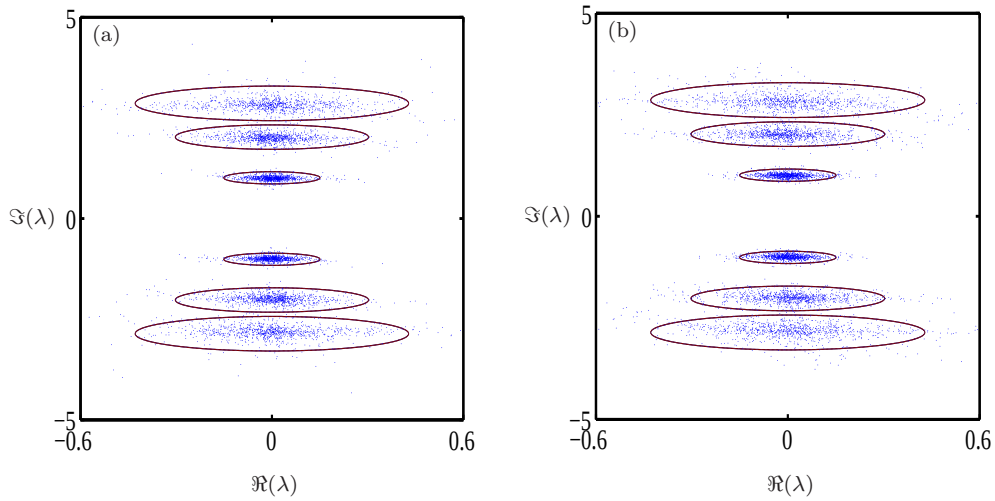

Fig. 7. Comparison between structured pseudospectrum of Q_0 and eigenvalues of random matrices with elements that have a log-normal distribution (a), or a χ^2-distribution (b), respectively.

distributions of elements.

We illustrate this for the log-normal and χ^2-distribution below. Again, we consider the example from Section 4.3 and rescale the variables, such that the results for the structured pseudospectrum of Q_0 can be used.

Firstly, let $X_i, i = 1, \dots 9$ be complex random variables, such that $\log(\Re(X_i)) \sim \mathcal{N}(0,1)$, $\log(\Im(X_i)) \sim \mathcal{N}(0,1)$. Then $\Re(X_i)$ and $\Im(X_i)$ have a log-normal distribution with $\mathbb{E}(\Re(X_i)) = \mathbb{E}(\Im(X_i)) = \sqrt{e}$ and $D^2(\Re(X_i)) = D^2(\Im(X_i)) = \sqrt{e(e-1)}$. Hence, the real and imaginary parts of

$$Y_i = \frac{0.15}{\sqrt{e(e-1)}} X_i - \frac{0.15}{\sqrt{e-1}}$$

have expectation zero and variance $D^2 = 0.15^2$. Consequently, we set

$$\begin{aligned} m_i &= \bar{m}_i(1+Y_i), \quad i = 1,2,3 \\ k_i &= \bar{k}_i(1+Y_{i+3}), \quad i = 1,\dots,6, \end{aligned} \tag{29}$$

with $\bar{m}_i$ and $\bar{k}_i$ as before.

Figure 7 (a) shows the eigenvalues of 2000 simulations of (18) together with the corresponding ($\epsilon = 1$)-contour of the structured pseudospectrum of Q_0. It is shown that the chosen pseudospectrum contour gives a satisfactory bound for the position of eigenvalues of the random system (18) with log-normally distributed m_i and k_i.

Next, we consider the case where the elements of M and K have a χ^2-distribution. For simplicity we consider the χ^2-distribution with one degree of freedom. We let X_i, $i = 1, \ldots, 9$ be random variables, such that $\sqrt{\Re(X_i)} \sim \mathcal{N}(0,1)$, and $\sqrt{\Im(X_i)} \sim \mathcal{N}(0,1)$. Then $\Re(X_i)$ and $\Im(X_i)$ have a χ^2-distribution and, furthermore, $\mathbb{E}(\Re(X_i)) = \mathbb{E}(\Im(X_i)) = 1$, $D^2(\Re(X_i)) = D^2(\Im(X_i)) = 2$. Finally, we rescale $Y_i = 0.15/\sqrt{2}X_i - 0.15/\sqrt{2}$ and set

$$\begin{aligned} m_i &= \bar{m}_i(1 + Y_i), \quad i = 1, 2, 3 \\ k_i &= \bar{k}_i(1 + Y_{i+3}), \quad i = 1, \ldots, 6. \end{aligned} \tag{30}$$

The eigenvalues of 2000 simulations of (18) with these distributions are shown in Figure 7(b). The eigenvalues are slightly more spread out than in the case of the log-normal distribution, but still the pseudospectrum bound is satisfactory.

5 Example of a Five DOF System

Finally, we study a problem from Ref. [32], which deals with the problem of obtaining eigenvalue bounds for systems with interval type parametric uncertainties. This calls for the application of structured pseudospectra.

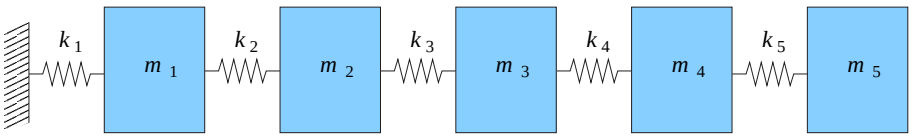

Fig. 8. A five degrees of freedom mass-spring-system.

The example, a five degrees-of-freedom mass-spring system without damping, is shown in Figure 8. Eigenvalues of the system are solutions of $Q(\lambda) = M\lambda^2 + K = 0$, where

$$M = \mathrm{diag}(m_1, \ldots, m_5), \; K = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{pmatrix} \tag{31}$$

It is assumed in Ref. [32] that the parameters k_i and m_i are not known exactly, but only that their values lie in the following intervals

$$k_1 \in [2000, 2020], \quad k_2 \in [1800, 1850], \quad k_3 \in [1600, 1630],$$
$$k_4 \in [1400, 1420], \quad k_5 \in [12001210]$$

and

$$m_1 \in [29, 31],\ m_2 \in [26, 28],\ m_3 \in [26, 28]\ m_4 \in [24, 26],\ m_5 \in [17, 19].$$

We will compute structured pseudospectra of Q to obtain bounds for the eigenvalues of the system. In the unperturbed system $Q_0(\lambda) = M_0\lambda^2 + K_0$ the parameters m_i and k_i are set to the mid-points of their respective intervals. The eigenvalues of the unperturbed system are then given by

$$\lambda_1^{\pm} = \pm 2.4832i,\ \lambda_2^{\pm} = \pm 6.6391i,\ \lambda_3^{\pm} = \pm 10.1768i,\ \lambda_4^{\pm} = \pm 12.8682i,$$
$$\lambda_5^{\pm} = \pm 14.8128i.$$

The differences in the variation of the parameters m_i and k_i are included in the matrices D_i, such that the problem above is equivalent to

$$(M_0 + D_M\Delta_M E_M)\lambda^2 + (K_0 + D_K\Delta_K E_K) = 0, \tag{32}$$

with

$$D_M = E_M = I, \quad \Delta_M = \mathrm{diag}(\delta m_1, \ldots, \delta m_5),$$

and

$$D_K = \begin{pmatrix} 10 & -25 & 0 & 0 & 0 \\ 0 & 25 & -15 & 0 & 0 \\ 0 & 0 & 15 & -10 & 0 \\ 0 & 0 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad E_K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\Delta_K = \mathrm{diag}(\delta k_1, \ldots, \delta k_5),$$

where we need $|\delta m_i|, |\delta k_i| \leq 1$ for all $i = 1, \ldots, 5$.

Hence, the structured ($\epsilon = 1$)-pseudospectrum contour of Q_0 should give us the bounds of m_i and k_i above. Pseudospectra of Q_0 are shown in Figure 9 (a), with the ($\epsilon = 1$)-contour highlighted as the fat contour. Since the pseudospectra are symmetric with respect to the real axis we only plot pseudospectra in the upper half of the complex plane.

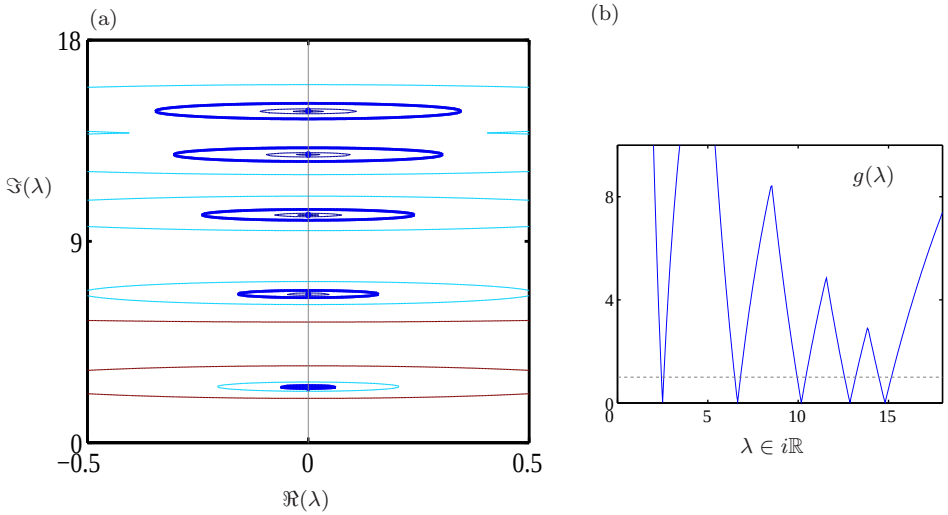

Fig. 9. Structured pseudospectra of the matrix polynomial Q_0 in the upper half of the complex plane (a), and a plot of the function g, introduced in the text (b). Bounds for the eigenvalues of Q are obtained by determining points for which $g(\lambda) = 1$.

To derive the bounds for the eigenvalues, we have to consider where the $(\epsilon = 1)$-pseudospectrum intersects the imaginary axis. Note that for real m_i and k_i satisfying the bounds above the matrices M and K in (31) are symmetric and positive definite. Thus, the system always has purely imaginary eigenvalues [24], and we conclude that the regions of pseudospectra away from the imaginary axis in Figure 9 (a) correspond to physically irrelevant complex perturbations of M_0 and K_0.

A graphical solution for identifying how to compute the bounds is shown in Figure 9(b), where we plot $g(\lambda) := 1/\mu_{\mathbf{\Delta}}(G(\lambda))$ for $\lambda \in i\mathbb{R}$, purely imaginary. By formula (17) the bounds for the variation of eigenvalues are given by points $\lambda \in i\mathbb{R}$, for which $g(\lambda) = 1$; this line is shown as dashed in Figure 9(b). The results are summarized in Table 1, and Table 2 lists the bounds derived in Ref. [32] for comparison. We see that both methods yield very similar results. Our method gives improved lower bounds, whereas the method used in Ref. [32] yields sharper upper bounds for the variation of the eigenvalues of the system.

6 Conclusions

In this paper we have shown that pseudospectra can be used for investigating eigenvalue problems subject to uncertainty. Using both established and recently developed techniques, we have compared results from different types of pseudospectra computations with results from Monte Carlo simulations of ei-

Eigenvalue	Lower Bound	Upper Bound
λ_1	$2.428i$	$2.565i$
λ_2	$6.489i$	$6.811i$
λ_3	$9.947i$	$10.430i$
λ_4	$12.574i$	$13.185i$
λ_5	$14.477i$	$15.170i$

Table 1
Bounds for the eigenvalues of (31), computed using structured pseudospectra of the matrix polynomial.

Eigenvalue	Lower Bound	Upper Bound
λ_1	$2.420i$	$2.550i$
λ_2	$6.483i$	$6.805i$
λ_3	$9.942i$	$10.425i$
λ_4	$12.572i$	$13.182i$
λ_5	$14.474i$	$15.169i$

Table 2
Bounds for the eigenvalues of (31) using an eigenvalue inclusion principle [32].

genvalue problems subject to different types of probabilistic and interval type, parametric uncertainties.

Two spring-mass systems modeled by second-order ODEs were investigated in detail. The first considered probabilistic uncertainties on the entries of the governing system matrices. The computed pseudospectra were shown to agree with the results obtained from Monte Carlo simulation with differing degrees of accuracy. Specifically, unstructured pseudospectra of the problem rewritten as a first-order system were shown to agree qualitatively with the simulation. However, two related concepts of pseudospectra, namely, full-block perturbations of the first-order system and weighted pseudospectra of the matrix polynomial, did not produce satisfactory results. A novel technique to compute structured pseudospectra of the full second-order system was proposed. The computed pseudospectra were shown to agree both qualitatively and quantitatively with the simulation results for normally distributed random variables. Similar agreements were also observed for systems with uniform distribution, log-normal distribution and a χ^2 distribution.

In the second example, a spring-mass system in which the entries of the governing matrices had interval type uncertainties was considered. In this case, we again computed structured pseudospectra of the associated full second-order system using our novel techniques. The pseudospectra computations

were used to provide bounds on the eigenvalues under the interval uncertainties. These bounds were shown to be in good agreement with results obtained via established interval algebra based methods.

The studies reported in this paper show the generality of pseudospectra tools for investigating eigenvalue problems with various types of uncertainties. The techniques proposed here need not be restricted to the second-order systems. The theory of structured pseudospectra is generalized for general matrix functions and can be applied to a wide range of problems such as higher-order matrix polynomials, systems modeled by delay differential equations and differential algebraic equations. Furthermore, these pseudospectra tools may be applied to a system with Fuzzy variables since a Fuzzy variable can be considered as a parameterized interval variable. The computation of pseudospectra is an accessible and computationally efficient tool for analyzing the (robust) stability of all of these systems, where, in some cases, even the computation of the spectra is not a trivial task.

Appendix

Here we give a brief outline of how the different types of pseudospectra introduced in Sections 2 and 3 can be seen as special cases of the general theory of structured pseudospectra developed in Ref. [19]. We also discuss how the matrices D_i and E_i, describing the structure of the perturbation, can be derived in applications.

6.1 A general framework

For all types of pseudospectra defined in this paper, the corresponding computable formulae can be derived by applying the following algorithm:

Step 1 Define $F:\ \mathbb{C} \to \mathbb{C}^{n\times n}$, $F(\lambda) = \sum_{i=1}^{m} A_i p_i(\lambda)$, as the characteristic matrix of the nominal system (typically $F(\lambda) = A - \lambda I$ or $F(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$).

Step 2 Express the structured perturbations on the system as additive uncertainty on the characteristic matrix of the form

$$\delta F(\lambda) = \sum_{i=1}^{f} S_i\ \Pi_i\ T_i\ q_i(\lambda), \tag{33}$$

where $\Pi_i \in \mathbb{C}^{k_i\times k_i}, k_i \geq 1$, denote the underlying *unstructured* perturbations; $S_i \in \mathbb{C}^{n\times k_i}$, $T_i \in \mathbb{C}^{k_i\times n}$ are appropriate constant shape matrices; and

$q_i : \mathbb{C} \to \mathbb{C}$ are entire functions, *in such a way* that the ϵ-pseudospectrum under consideration can be defined as

$$\Lambda_\epsilon^s(F) := \{\lambda \in \mathbb{C} : \det(F(\lambda) + \delta F(\lambda)) = 0, \text{ for some } \delta F \text{ of the form (33) with } ||\Pi_i|| < \epsilon,\ i = 1, \ldots, f\}. \quad (34)$$

Step 3 The general formula is

$$\Lambda_\epsilon^s(F) = \left\{ \lambda \in \mathbb{C} :\ \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} T_1 \\ \vdots \\ T_f \end{bmatrix} \ F(\lambda)^{-1} \ [S_1 \ q_1(\lambda) \cdots S_f \ q_f(\lambda)] \right) > \frac{1}{\epsilon} \right\}, \quad (35)$$

where $\mu_{\mathbf{\Delta}}(\cdot)$ is the structured singular value corresponding to the uncertainty set

$$\mathbf{\Delta} := \left\{ \text{diag}(\Pi_1, \ldots, \Pi_f) :\ \Pi_i \in \mathbb{C}^{k_i \times k_i},\ 1 \le i \le f \right\}.$$

In the special case $S_i = T_i = I,\ i = 1, \ldots f$, expression (35) can be simplified to

$$\Lambda_\epsilon^s(F) = \left\{ \lambda \in \mathbb{C} :\ \|F(\lambda)^{-1}\| \left(\sum_{i=1}^{f} |q_i(\lambda)| \right) > \frac{1}{\epsilon} \right\}.$$

A more general uncertainty structure and corresponding computable formulae can be found in Ref. [19].

Examples

- Pseudospectra (3)-(4). Applying the algorithm with

$$F(\lambda) = A - \lambda I, \quad \delta F(\lambda) = \underbrace{D}_{S_1} \ \underbrace{\delta A}_{\Pi_1} \ \underbrace{E}_{T_1}.$$

  results in (5) as $\mu_{\mathbf{\Delta}}(\cdot) = \|\cdot\|$ if $\mathbf{\Delta}$ is unstructured.
- Pseudospectra (8)-(10). Formula (11) is obtained through the algorithm where $F(\lambda) = G(\lambda)$ and

$$\begin{aligned} \delta F(\lambda) &= \delta A_2 \lambda^2 + \delta A_1 \lambda + \delta A_0 \\ &= \underbrace{w_2 \ \delta A_2}_{\Pi_1} \ \underbrace{\left(\frac{\lambda^2}{w_2}\right)}_{q_1(\lambda)} + \underbrace{w_1 \ \delta A_1}_{\Pi_2} \ \underbrace{\left(\frac{\lambda}{w_1}\right)}_{q_2(\lambda)} + \underbrace{w_0 \ \delta A_0}_{\Pi_3} \ \underbrace{\left(\frac{1}{w_0}\right)}_{q_3(\lambda)}. \end{aligned}$$

- Pseudospectra (13)-(16). With $F(\lambda) = Q(\lambda)$ and

$$\begin{aligned}\delta F(\lambda) &= D_2\Delta_2 E_2\lambda^2 + D_1\Delta_1 E_1\lambda + D_0\Delta_0 E_0 \\ &= \sum_{j=1}^{k_2} \underbrace{D_2^{(j)}}_{S_j}\ \underbrace{\delta_{2,j}}_{\Pi_j}\ \underbrace{E_2^{(j)}}_{T_j}\underbrace{\lambda^2}_{q_j(\lambda)} + \sum_{j=1}^{k_1} D_1^{(j)}\ \delta_{1,j}\ E_1^{(j)}\lambda + \sum_{j=1}^{k_0} D_0^{(j)}\ \delta_{0,j}\ E_0^{(j)},\end{aligned}$$

  where $D_i^{(j)}$ denotes the j-the column of D_i and $E_i^{(j)}$ the j-th row of E_i, the algorithm results in formula (17).

6.2 Computation of the shape matrices

The use of structured pseudospectra in applications requires to set up appropriate shape matrices. In the following we describe a general way how to do this.

Let $F(\lambda) = \sum_{i=1}^{m} A_i p_i(\lambda)$ denote the unperturbed characteristic matrix of the system and let us assume that the coefficient matrices of the perturbed system, $\widetilde{A}_i$, depend on a number of uncertain parameters θ_{ij}, say

$$\widetilde{A}_i = A_i + \sum_j \theta_{ij} P_{ij},$$

with matrices P_{ij} determining where the parameters enter the original matrix. Moreover, assume that the matrices P_{ij} have rank one. Assume that, as in Section 4, we wish to investigate the possible positions of the eigenvalues when $|\theta_{ij}| \leq \epsilon,\ \forall i, j$.

Since P_{ij} has rank one its (reduced) singular value decomposition (svd) [17] yields

$$P_{ij} = u_{ij} v_{ij}^*,$$

where u_{ij} and v_{ij} are vectors. Hence, the perturbation of A_i can be written as

$$A_i + \sum_j \theta_{ij} P_{ij} = A_i + \sum_j u_{ij}\theta_{ij} v_{ij}^*$$

and we have

$$\sum_{i=1}^{m} \widetilde{A}_i p_i(\lambda) = F(\lambda) + \sum_i \sum_j u_{ij}\ \theta_{ij}\ v_{ij}^* p_i(\lambda),$$

where the additive perturbation of F has the form (33), as required above.

Moreover, we alternatively have

$$A_i + \sum_j \theta_{ij} P_{ij} = A_i + [u_{i1} u_{i2} \dots u_{il}] \begin{pmatrix} \theta_{i1} & & & \\ & \theta_{i2} & & \\ & & \ddots & \\ & & & \theta_{il} \end{pmatrix} \begin{bmatrix} v_{i1}^* \\ v_{i2}^* \\ \vdots \\ v_{il}^* \end{bmatrix},$$

which is of the general form used in Section 3.

Remark 4 *The assumption that all matrices P_{ij} have rank one has been motivated by the examples in Section 4 and 5. In general, this will be too restrictive, and the above approach needs to be generalized. The svd then leads to considering diagonal matrices with* repeated scalar blocks *as perturbation matrices. A discussion of this more general perturbation structure can be found in Ref. [19].*

An example

We finally demonstrate the algorithm using the example studied in Section 4. First note that the problem can be written as

$$\begin{bmatrix} M_0 + \delta m_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dots \end{bmatrix} \lambda^2$$

$$+ \begin{bmatrix} K_0 + \delta k_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dots + \delta k_4 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dots \end{bmatrix}.$$

An svd for each of the matrices then shows that this is equivalent to

$$\left[M_0 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta m_1 [1\ 0\ 0] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta m_2 [0\ 1\ 0] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta m_3 [0\ 0\ 1] \right] \lambda^2$$

$$+ \left[K_0 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta k_1 [1\ 0\ 0] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta k_2 [0\ 1\ 0] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta k_3 [0\ 0\ 1] \right.$$

$$\left. + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \delta k_4 [1\ -1\ 0] + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \delta k_5 [0\ 1\ -1] + \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \delta \hat{k}_6 [1\ 0\ -1] \right].$$

Note that we have used the rescaled version $\delta\hat{k}_6$ instead of δk_6 again. From this representation the matrices $D_{K,M}$ and $E_{K,M}$ in (25),(26) can be immediately read off.

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