Champneys, A. R., Vanden-Broek, J-M., & Lord, G. J. (2000). Do true elevation gravity-capillary solitary waves exist? a numerical investigation.

Do true elevation gravity-capillary solitary waves exist? a numerical investigation

A.R. Champneys*, J.-M. Vanden-Broeck† and G.J. Lord‡

December 15, 2000

Abstract

This paper extends the numerical results of Hunter and Vanden-Broek (1983) and Vanden-Broek (1991) which were concerned with studies of solitary waves on the surface of fluids of finite depth under the action of gravity and surface tension. The aim of this paper is to answer the question of whether small-amplitude elevation solitary waves exist. Several analytical results have proved that bifurcating from Froude number $F = 1$, for Bond number τ between 0 and 1/3, there are families 'generalised' solitary waves with periodic tails whose minimum amplitude is an exponentially small function of $F - 1$. An open problem (which, for τ sufficiently close to 1/3, was recently proved by S.-M. Sun to be false) is whether this amplitude can ever be zero, which would give a truly localised solitary wave.

The problem is first addressed in terms of model equations taking the form of generalised 5th-order KdV equations, where it is demonstrated that if such a zero-tail amplitude solution occurs, it does so along codimension-one lines in the parameter plane. Moreover, along solution paths of generalised solitary waves a topological distinction is found between cases where the tail does vanish and those where it does not. This motivates a new set of numerical results for the full problem, formulated using a boundary integral method, namely to probe the size of the tail amplitude as τ varies for fixed $F > 1$. The strong conclusion from the numerical results is that true solitary waves of elevation *do not* exist for the steady gravity-capillary waver wave problem at least for $9/50 < \tau < 1/3$.

1 Introduction

The description of wave propagation under the combined effects of gravity and surface tension on the surface of a liquid above a horizontal bottom is a classical problem in applied mathematics with a long history (Korteweg & de Vries 1895, Wilton 1915). In the last twenty years or so there have been a number of advances to the understanding of this problem, using a variety of numerical, asymptotic and rigorous analytical methods (see the review Dias & Kharif (1999)). We focus exclusively on two–dimensional steady waves. For solitary waves, the problem can be characterised by the Froude number F and the Bond number τ defined by

$$\tau = T/\rho g H^2, \qquad F = c/\sqrt{gH}, \tag{1.1}$$

Here, g is the acceleration due to gravity, T is the coefficient of surface tension, ρ is the density of the fluid, H is its depth and c is the propagation speed of the wave.

When $\tau > 1/3$, given values of $F < 1$ in an appropriate range, it is known that there are isolated solitary waves of depression (with a negative central crest) both via existence theory (Amick & Kirchgassner 1989, Iooss & Kirchgassner 1992, Buffoni, Groves & Toland 1996) and numerical computation

*Department of Engineering Mathematics, University of Bristol, BS8 1TR UK a.r.champneys@bristol.ac.uk
†School of Mathematics, University of East Anglia, Norwich NR4 7TJ J.Vanden-broeck@uea.ac.uk
‡Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS gabriel@ma.hw.ac.uk

(Hunter & Vanden-Broek 1983, Dias, Menasce & J.M. 1996). On the other hand for $\tau < 1/3$, for fixed values of $F > 1$, the solutions become waves of elevation and are no-longer isolated, but form one-parameter families. Moreover, these solutions are not in general true solitary waves, but are characterised by a train of ripples of constant amplitude in the far field (Beale 1991). They are called *generalised solitary waves*, to distinguish them from true solitary waves which are characterised by a flat free surface in the far field. The ripples in the tail are of questionable physical validity because they occur on both sides and therefore do not satisfy radiation conditions without the supply of energy from infinity. Therefore an important question is whether the free parameter can be chosen so that the amplitude of the ripples vanishes. One of the key known features of these ripples, is that the amplitude is an exponentially small function of $F-1$, as has been shown both by exponential asymptotics (Sun & Shen 1993) and by rigorous application of center manifold and normal form theory (Lombardi 2000). Lombardi's analysis (see also the discussion in §3 below) also suggests that for fixed $\tau < 1/3$, it is non-generic that zero-tail-amplitude solutions bifurcate from $F = 1$. What is not known is whether there are isolated τ values less than 1/3 at which such true solitary waves bifurcate. This question has been rigorously answered in the negative for τ *sufficiently close to* 1/3 by Sun (1999). The aim of this paper is to numerically probe the question of whether true solitary waves bifurcate, for general $\tau < 1/3$.

The numerical method we use is to approximate solitary waves by long periodic waves and solving the full nonlinear equations for water waves by a boundary integral method. Hunter & Vanden-Broek (1983) were the first to compute generalised solitary waves in this way. Later, in Vanden-Broek (1991), it was shown for a fixed τ-value, that the amplitude of the ripples computed can be minimised so that they are are invisible on the scale of the waves. Here we present extended calculations to explore whether or not this minimum amplitude is really zero. In order to do this we shall appeal to intuition gained from first studying an extended 5th-order Korteweg–de Vries model (5thKdV) equation.

The results are presented as follows. In §2 we recall briefly a formulation of the classical water wave problem we study and how it has been approximated by various KdV-type model equations. Section 3 then presents a series of numerical experiments on two different forms of an extended 5thKdV equations. The parameter dependence of generalized solitary waves is uncovered both in a case where the tails do vanish and when they do not. Section 4 then introduces the numerical method to be used for the full problem and, taking insight from the results in §3, presents a set numerical experiments specifically designed to probe the minimum size of the tail amplitude of the generalized solitary waves. Finally, §5 draws conclusions.

2 Formulation

We consider a train of periodic waves of wavelength L travelling at a constant velocity C at the surface of a two-dimensional fluid of finite depth H. The fluid is assumed to be inviscid and incompressible, and the flow to be irrotational. In particular we define a velocity potential ϕ and stream function ψ and choose $\phi = 0$ at a crest and $\psi = 0$ on the free surface. Gravity g and surface tension T are both taken into account. The velocity C is defined as the average horizontal fluid velocity at any horizontal level completely within the fluid. The depth H is defined by $H = Q/C$ where Q is the value of $|\psi|$ on the bottom. Solitary waves are defined by taking the limit $L/H \to \infty$. Numerically they will be approximated by periodic waves with large L/H (typically ~ 100). We shall henceforth non-dimensionalise by setting $H = C = 1$ and introducing the dimensionless Froude number F and Bond number τ defined in (1.1). We shall also introduce co-ordinates so that x is horizontal and y is measured vertically upwards from the horizontal bottom.

This problem can be formulated in terms of potential flow with nonlinear dynamic and kinematic boundary conditions on the free surface (e.g. Stoker (1957)). It can also be reformulated as a system of integro–differential equations for the x and y co-ordinates of a material point on the fluid surface as a function of the complex potential function $f = \phi + i\psi$ where $\psi = 0$ defines the free surface. We restrict out attention to waves which are symmetric with respect to $\phi = 0$. Using a Cauchy integral formula we obatain (Vanden-Broek & Schwartz 1979, Hunter & Vanden-Broek 1983, Vanden-Broek 1991)

$$\begin{aligned} x'(\phi) - 1 &= -\frac{1}{L}\int_0^{L/2} y'(s)\left(\cot\frac{\pi(s-\phi)}{L} + \cot\frac{\pi(s+\phi)}{L}\right) ds \\ &+ \frac{2r_0}{L}\int_0^{L/2} \frac{[x'(s)-1]\{r_0^2 - \cos[(2\pi/L)(s-\phi)] - y'(s)\sin[(2\pi/L)(s-\phi)]}{1 + r_0^4 - 2r_0^2\cos[(2\pi/L)(s-\phi)]} ds \\ &+ \frac{2r_0}{L}\int_0^{L/2} \frac{[x'(s)-1]\{r_0^2 - \cos[(2\pi/L)(s+\phi)] - y'(s)\sin[(2\pi/L)(s+\phi)]}{1 + r_0^4 - 2r_0^2\cos[(2\pi/L)(s+\phi)]} ds, \end{aligned} \tag{2.1}$$

$$\frac{F^2}{2}\left(\frac{1}{x'(\phi)^2 + y'(\phi)^2} - 1\right) + y + \tau\frac{x'(\phi)y''(\phi) - x''(\phi)y'(\phi)}{[x'(\phi)^2 + y'(\phi)^2]^{3/2}} = 0, \tag{2.2}$$

where F is the Froude number and

$$r_0 = e^{\frac{-2\pi}{L}}.$$

This latter formulation will be adopted below since it is more convenient for numerics as mesh points are only required to be stationed on the free surface rather than throughout the fluid domain.

In the limit $F \to 1$, the waves are of small amplitude. Therefore solutions may be described by weakly nonlinear theories. To derive them, we move to a steady frame and denote the equation for the free surface as $y = H + A\eta$. Here A is a parameter measuring the amplitude of the wave. For small A, $\eta(x)$ satisfies the KdV equation

$$2(F-1)\eta' - 3\eta\eta' + (\tau - \frac{1}{3})\eta''' = 0, \tag{2.3}$$

(see Korteweg & de Vries (1895) for a derivation). This equation has This equation has periodic travelling solutions (cnoidal waves) which tend in the limit of long wave length to the famous solitary wave solution described by the function sech^2. This describes a depression wave for $\tau > 1/3$, $F < 1$ and an elevation wave for $\tau < 1/3$, $F > 1$.

When $\tau = 1/3$, the dispersive term in (2.3) vanishes and hence there are no periodic or solitary wave solutions. As $\tau \to 1/3$, the appropriate long-wave equation is the 5th-order KdV equation (5thKdV) also known as the Kawahara equation

$$2(F-1)\eta' - 3\eta\eta' + (\tau - \frac{1}{3})\eta''' - \frac{1}{45}\eta^{(5)}, \tag{2.4}$$

which may be derived by a regular asymptotic expansion near $\tau = 1/3$, $F = 1$ (Hunter & Vanden-Broek 1983, Sect.2). There are other derivations of this equation, with different co-efficients, in several other physical contexts, e.g. (Kakutani & Ono 1969, Hasimoto 1970, Kawahara 1972, Zufiria 1987, Hunter & Scheurle 1988, Karpman 1994). The properties of its solutions are discussed in §3 which follows. Also, equations have been derived with extra nonlinear terms, see Kichenassamy & Olver (1996) and references therein. It is precisely such a model that we study in the §3. The values of the parameters we take there are not necessarily ones in which the 5thKdV is a good approximation of the full water-wave problem. Instead, we merely treat the model as a guide of what to expect *qualitatively* when studying the full model numerically.

3 Solitary waves of 5th-order KdV equations

In Champneys & Groves (1997) the following extended 5KdV model for the free surface $u(x)$ of the capillary gravity water-wave problem was considered:

$$u_t + \frac{2}{15}u_{xxxxx} - bu_{xxx} + 3uu_x + \mu\left[u_x u_{xx} + uu_{xxx}\right] = 0 \tag{3.1}$$

It reduces to the fourth-order ODE

$$(2/15)u'''' - bu'' + au + (3/2)u^2 + \mu\left[(uu')' - (1/2)(u')^2\right] = 0 \tag{3.2}$$

upon moving to a steady frame moving at speed a and integrating once choosing the integration constant to be zero in order to describe solitary waves. This model may be derived using Hamiltonian perturbation theory from the full water wave problem (Craig & Groves 1994), and is a special case of the more general form studied by Kichenassamy & Olver (1996). The parameters a and b are related to the Froude and Bond numbers via

$$\frac{a}{2} = \frac{1}{F} - 1, \quad b = \frac{\tau - 1/3}{F^2}.$$

Hence as $\tau \to 1/3$ and $F \to 1$ the parameters a and b play the rôles of the difference of Froude and Bond numbers respectively from the critical codimension-two point. Specifically we have the following scalings

$$a \sim -2(F-1), \quad b \sim \tau - \frac{1}{3} \quad \text{for} \quad F \sim 1, \quad \tau \sim \frac{1}{3}.$$

Finally, μ is an artificial parameter that represents the relative importance of various nonlinear terms in the long-wave expansion. Only its sign is important, since nonzero μ can be rescaled to unity; here we study only $\mu = 0$ or 1. The case $\mu = 0$ gives (a scaling of) the usual 5thKdV equation (2.4) An overview of parameter space may be found in Champneys & Groves (1997). Two parameter regions are of interest: $a > 0$ and $a < 0$ and we discuss these below for both $\mu = 0$ and $\mu = 1$.

Essentially with $a > 0$, solitary wave solutions are all waves of depression, which tend to envelope solitary waves in the limit of small amplitude as b is decreased to $-\sqrt{8a/15}$. In fact there are also infinitely many multi-troughed 'bound state' versions of these solutions for $-\sqrt{8a/15} < b < \sqrt{8a/15}$ some of which are stable as a solution of the evolutionary problem – see (Buffoni, Champneys & Toland 1996, Buffoni & Séré 1996, Yang & Akylas 1997, Buryak & Champneys 1997, Dias & Kuznetsov 1999, Calvo & Akylas 2000, Bridges & Derks 1999) for recent rigorous, asymptotic and numerical results. In another limit, $a \to 0^+$ for $b > 0$ a unique solitary wave solution bifurcates at zero amplitude, in so doing tending to the soliton solution of the usual 3rd-order KdV equation, which can be recovered in this limit after rescaling. Many of these results for $a > 0$ extend to the case $\mu > 0$ and can also be shown to have rigorous implications via spatial centre-manifold reduction, for the existence of qualitatively similar solutions for the full water wave problem

The existence of solitary wave solutions for $a < 0$ is much more subtle and forms the subject of this paper. Essentially, the limit $a \to 0^-$ for $b < 0$ also captures the usual 3rd-order KdV equation after rescaling (as did the limit $a \to 0^+$, $b > 0$). The perturbation that the 5thKdV adds to the soliton solution of the KdV equation (which is a wave of elevation in this case) is a rapid oscillatory term. This then is a beyond-all-orders asymptotics problem and has been studied by many authors, e.g. (Pomeau, Ramani & Grammaticos 1988, Amick & Toland 1992, Grimshaw & Joshi 1995, Sun 1998). Recent rigorous theory by Lombardi (2000) for a class of systems including (3.2) is the most comprehensive. He shows that *generically* in the limit of $a \to 0$ there do not exist true solitary waves of elevation, but there are one-parameter families of generalized solitary waves, which represent homoclinic connections between periodic orbits, the minimum amplitude of which is an exponentially small function of a as $a \to 0$. For (3.2) with $\mu = 0$, it is known that there are no true solitary waves as $a \to 0$ (Amick & McLeod 1991), that is the tail-amplitude never vanishes. Also, the family of generalised solitary waves (for fixed a and b) traces a 'u-shaped' locus in a plot of amplitude-of-tail vs. phase shift between tails (Grimshaw & Joshi 1995). This diagram has a slight asymmetry as was first spotted numerically (Champneys & Lord 1997) and confirmed analytically (Sun 1998), see also Fig. 1(a).

The case $\mu = 1$ is very different, since there is an explicit true solitary wave solution that exists along the line

$$a = \frac{3}{5}(2b+1)(b-2), \; b \geq -1/2, \quad u(x) = 3\left(b + \frac{1}{2}\right)\operatorname{sech}^2\left(\sqrt{\frac{3(2b+1)}{4}}x\right). \tag{3.3}$$

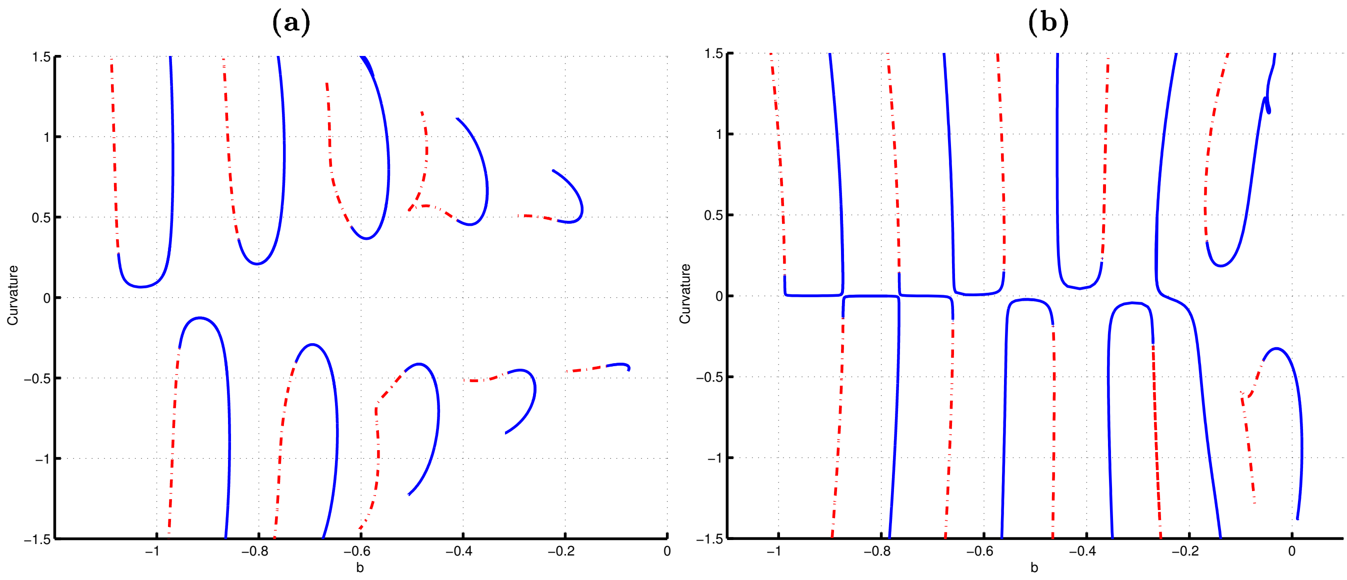

Figure 1: Contrasting the results of curvature versus Bond number for the generalized solitary waves of the 5thKdV model (3.2) with $a = -0.675$ and **(a)** $\mu = 0$ and **(b)** $\mu = 1$. Solid lines correspond to there being a local maximum at the central crest of the wave, and dot-dashed lines to a local minimum. Note that the zero-curvature solutions in (b) correspond to true elevation solitary waves.

Numerically, Champneys & Groves (1997) found this to be the first among a countable number of branches that bifurcate from $a = 0^-$ for smaller values of $b < 0$. In Champneys (2000) many other fourth-order model systems are studied, and it is found that there are many other cases when such 'persistence' of true solitary wave solutions from the singular limit occur. Note that this is not in contradiction with Lombardi's proof of generic *non*-persistence, since the solitary waves occur only along lines in a parameter plane. In fact, a simple dimensionality argument counting dimensions of stable and unstable manifolds of the fourth-order ODE (3.2) shows that, if solitary waves solutions occur, then they should be of codimension one in parameter space, provided that their profiles are even (non-symmetric waves are of codimension two).

The purpose of the new results presented in this section is to see how the persistence or non-persistence of true solitary waves affects the global structure of the generalised solitary wave solutions to the 5thKdV equation. Later on, we then adapt this insight to draw conclusions based on our numerical findings for the full water-wave problem. Some results for fixed $a = -0.675$ are presented in Figs. 1 – 4. Before discussing the results, let us briefly mention the method by which they were obtained, which, in order for a closer analogy to be drawn with results for the full water-wave problem, is rather simpler than that used in our earlier work (Champneys & Lord 1997) (nevertheless, almost identical results have been obtained using the method used in that work). The generalised solitary waves are approximated by a periodic orbit of a fixed long period $-L/2 < x < L/2$ ($L = 43.2$ for the results presented), where periodicity in this time-reversible system is guaranteed by taking boundary conditions $u'(-L/2) = u'''(-L/2) = u'(L/2) = u'''(L/2) = 0$. For each fixed a and b parameter value, this then fixes the phase-shift between the tails. We then perform numerical continuation on this periodic boundary-value problem (using AUTO Doedel *et al.* (1997)) for fixed a, allowing b to vary. This is motivated by the fact that computation into the singular limit $a \to 0$ is undesirable and the observation from the model equations in Champneys (2000) that curves of true solitary wave solutions bifurcate from $a = 0$ at a non-zero angle. Now, this one-parameter sweep is therefore just a slice through a three-parameter surface parametrised by phase shift (effectively L) and the model parameters a and b. Hence it would seem possible that we might miss curves in the (a, b)-plane corresponding to true solitary waves. However, if a true solitary wave is found then by definition the phase shift between the

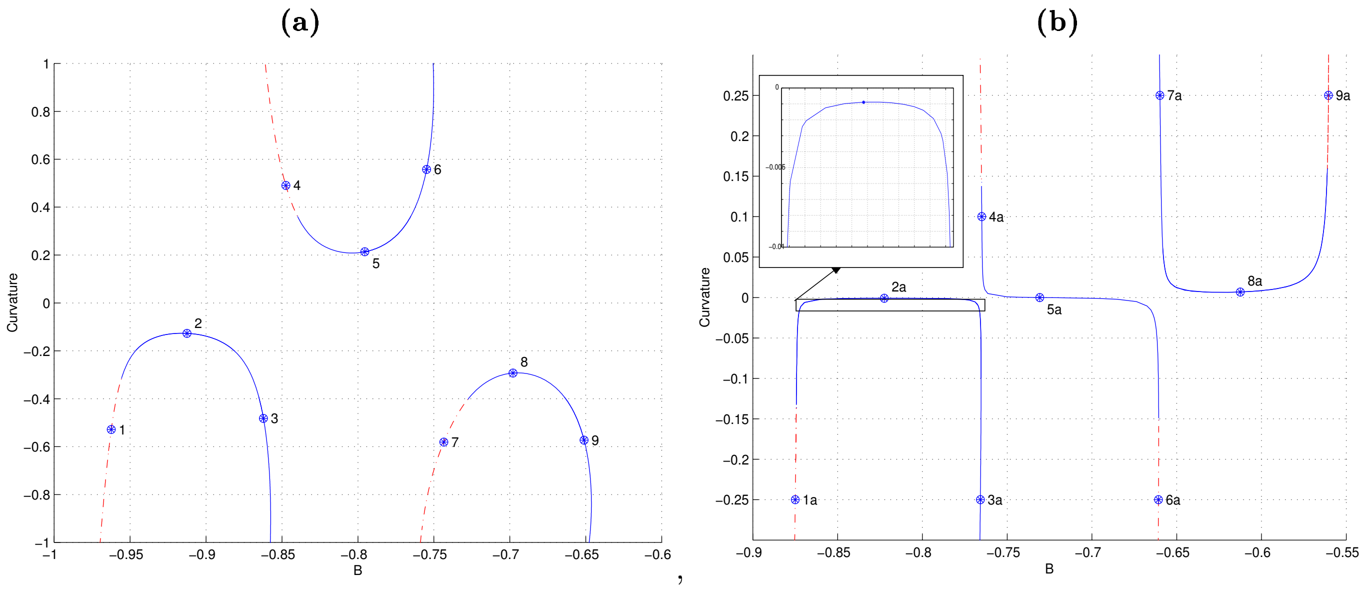

Figure 2: Details of Fig. 1; **(a)** for $\mu = 0$; and **(b)** for $\mu = 1$. The point numbers refer to the solutions shown in Figs. 3 and 4. The inset on **(b)** shows that the left most n-shaped curve is bounded away from zero curvature.

zero solution at $x = -\infty$ and $x = +\infty$ is not defined, and hence computing with *any fixed phase shift* will find (a good numerical approximation to) this homoclinic solution. Indeed, this is precisely what we found by performing calculations with different L-values. Finally, we simplify the problem by looking only for symmetric solutions (with the first two boundary conditions replaced by $u'(0) = u'''(0) = 0$) and as a measure of the amplitude of the tail we take the curvature of the right-hand boundary point $u''(L/2)$. Using curvature in this way has the advantage of that it is a signed quantity, which will become apparent to be helpful for qualitative interpretation of the results.

Let us now compare the results presented for $\mu = 0$ and $\mu = 1$. Consider first $\mu = 0$; Figs. 1(a), 2(a) and 3. Here the generalized solitary waves lie on a succession of disconnected, alternating 'u' and 'n-shaped' curves. At the local minimum or maximum of each curve is the minimum-tail amplitude solution. Note that taking this sweep in b is similar to taking a sweep in L for fixed b and a because, since the period of the tail depends on b, then both b and L effectively control the phase-shift between the two tails. The structure of solutions on each u or n is qualitatively similar, with the large curvature limit of all branches representing a large amplitude, pure periodic wave.

The situation for $\mu = 1$ is qualitatively different, see Figs. 1(b), 2(b) and 4. Here, in addition to u's and n's there are also 's-shaped' curves where the tail amplitude goes through zero. Two such zero tail-amplitude points are detected in Fig. 1(b) — the first of these is at precisely the value $b = -0.25$ given by the formula (3.3); the neighbourhood of the second one is blown up in Fig. 2(b) and occurs at a value of $b \approx -0.7307924$. This then shows how true solitary wave solutions are embedded into loci of generalised solitary waves. That is, there is a topological difference between the cases where true solitary waves occur and those where they do not. Namely, true solitary waves lie on s-shaped curves This is significant, since rather than have to carefully check the size of something that is exponentially small in a, we have produced a numerical criterion for deciding whether true solitary waves exist which relies on computing $O(1)$ quantities.

Note that any attempt to continue the zero curvature solutions on the s–shaped curves in Fig. 1(b) from $\mu = 1$ to $\mu = 0$ fails, since the b-value of the true solitary waves tends to minus infinity as $\mu \to 0$.

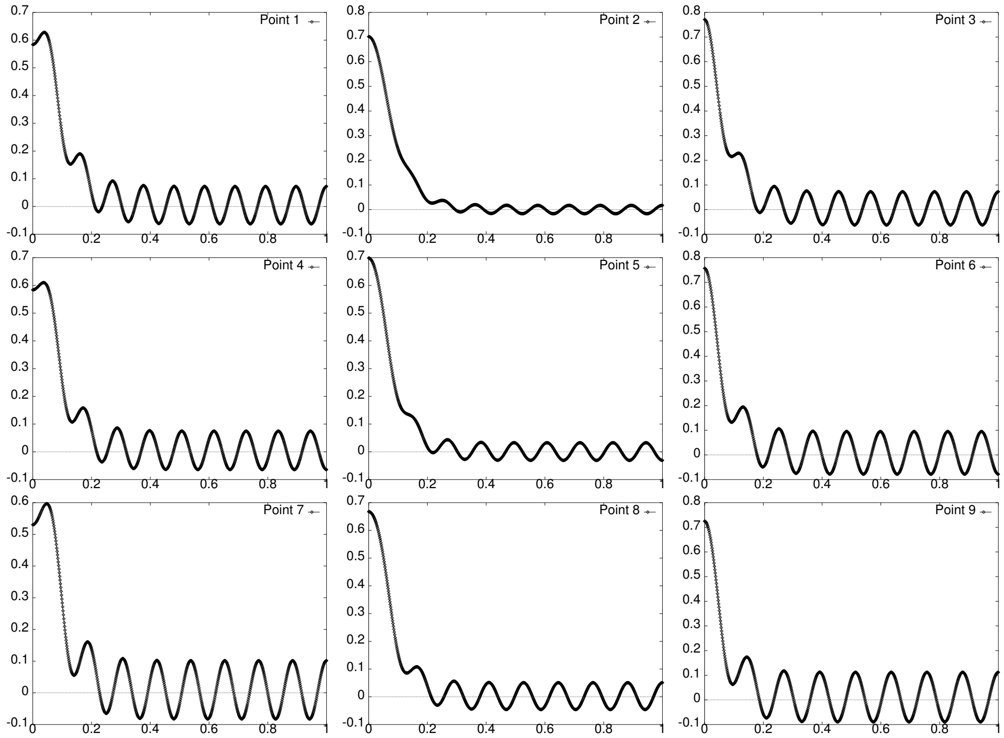

Figure 3: Structure of the solutions on three branches shown in Fig. 2(a) ($\mu = 0$). Only the right-hand portion of the wave from its point of symmetry is depicted, and the horizontal scale has been devided by a factor $L/2 = 21.6$. Note that the central core is at a minimum for solutions at labels 1, 4 and 7 as indicated by the dashed portion of the curves in Fig. 2.

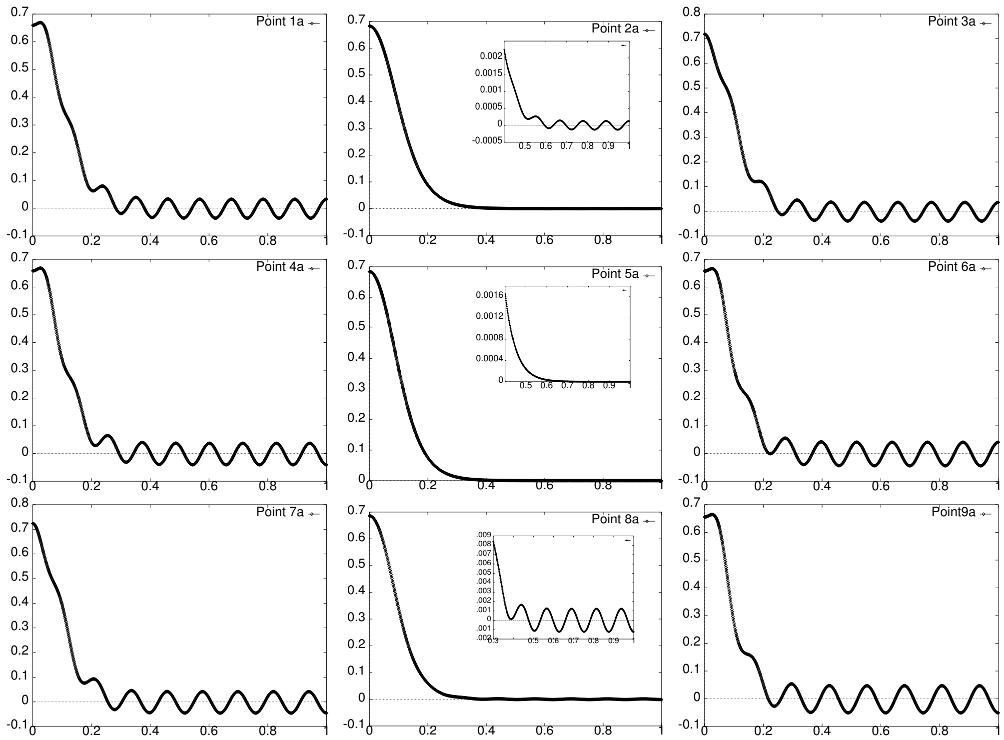

Figure 4: Structure of the solutions on three branches shown in Figure 2(b) ($\mu = 1$), plotted on the same horizongal scale. Insets in the middle three panels show the detail of the tail of the wave at the three points close to zero end curvature. Only point $5a$ is a true solitary wave.

4 Numerical results for the exact water wave problem

Our numerical procedure for solving (2.1), (2.2) follows closely the boundary integral method used by Hunter & Vanden-Broek (1983) and Vanden-Broek (1991), to which the reader is referred for the details. We approximate (generalised) solitary waves by long, even, periodic waves of wavelength L. This approximation enables us to overcome difficulties associated with the appropriate choice of boundary conditions in the far field. As $L \to \infty$, the periodic waves approach generalised solitary waves (for a numerical study of this limit, see Vanden-Broek (1997)). The solutions are characterized by four parameters. The first three parameters are L (or equivalently r_0), τ and F. In Vanden-Broek (1991), the fourth parameter was defined as the velocity of the crest. This is a measure of the wave amplitude, in the sense that when this parameter is close to one, the wave is of small amplitude. In this work we find it more convenient to choose this additional free parameter to be the curvature at the trough of the wave, as this is a measure of the amplitude of the ripples in the tail. Moreover, as illustrated for the 5thKdV models in the previous section, the curvature is a signed quantity which can be used to extract the topological information of whether solutions lie on purely u and n-shaped curves, or whether they are interspersed with s-shaped ones. Practically this curvature is measured using a second-difference formula at the right-hand end point of the free surface.

The system (2.1), (2.2) is discretized by following the procedure described in Hunter & Vanden-Broek (1983) and Vanden-Broek (1991), with the resulting set of nonlinear algebraic equations solved by Newton's method. Error tolerances were typically set at 10^{-7}. The accuracy of solutions was found to be highly dependent on the number of mesh points N used. Solution loci of tail curvature versus τ were computed using simple natural parameter continuation in either τ or the curvature.

Figure 5 presents some results for fixed F and L with $N = 269$. Three successive branches of solutions are plotted as tail curvature vs. τ. The insets show the free surface profiles. Note the striking similarity between the structure of solutions found on the u and n-shaped curves here and those for the 5th-order KdV results in Figures 1–4. Each solution locus connects two end-points corresponding to pure-periodic waves with opposite phases at the central point. In between these large-amplitude extremes is a portion of the locus where the waves have small tail amplitude.

Figures 6 and 7 motivate the choice of F and L taken in these computations. Changing L (keeping τ and F fixed) results in a periodic sequence of identical u and n-shaped curves which correspond to longer and longer approximations to the *same* family of generalised solitary waves. Decreasing the Froude number further towards the critical value $F = 1$ (keeping τ and L fixed) again creates sequences of u's and n's but they become increasingly square, that is the tail amplitude becomes negligible for large portions of the solution locus, separated by almost vertical walls of rapid growth in tail amplitude at almost constant F. For $F < 1.01$, for the τ-value chosen, the size of the tail amplitude was found to be commensurate with error tolerances used in Newton's method and the vertical walls become hard to detect numerically. The rest of the computations presented use the fixed values $L = 98.33$ and $F = 1.02$.

Figure 8 presents results on the convergence of solutions with variation of the number of mesh points N. They compare results for three different values of τ. Panels (a) and (b) illustrate that increasing the number of mesh points from those used for the results in Figure 5 makes little quantitative difference, suggesting that the mesh has effectively converged. Taking a smaller mesh, even as drastically as halving the number of mesh points, makes a difference in the third decimal for τ, but (crucially) only in the fourth or fifth decimal place for the minimum value of the end-curvature. Note also that the agreement becomes better as τ increases (see panels (c) and (d)). So while taking a mesh of $N = 269$ is desirable for accurately reproducing solutions at a given τ-value, it seems that $N = 135$ is sufficient for unfolding the global topology of the solution set. This latter value is also more practical since the typical time taken for Newton's method to converge to one solution point is the order of 500 seconds (on a SUN SPARC 10), compared with about 5000 seconds for $N = 269$.

Finally then, Figure 9 presents the results of a global sweep of $\tau < 1/3$. For the given Froude

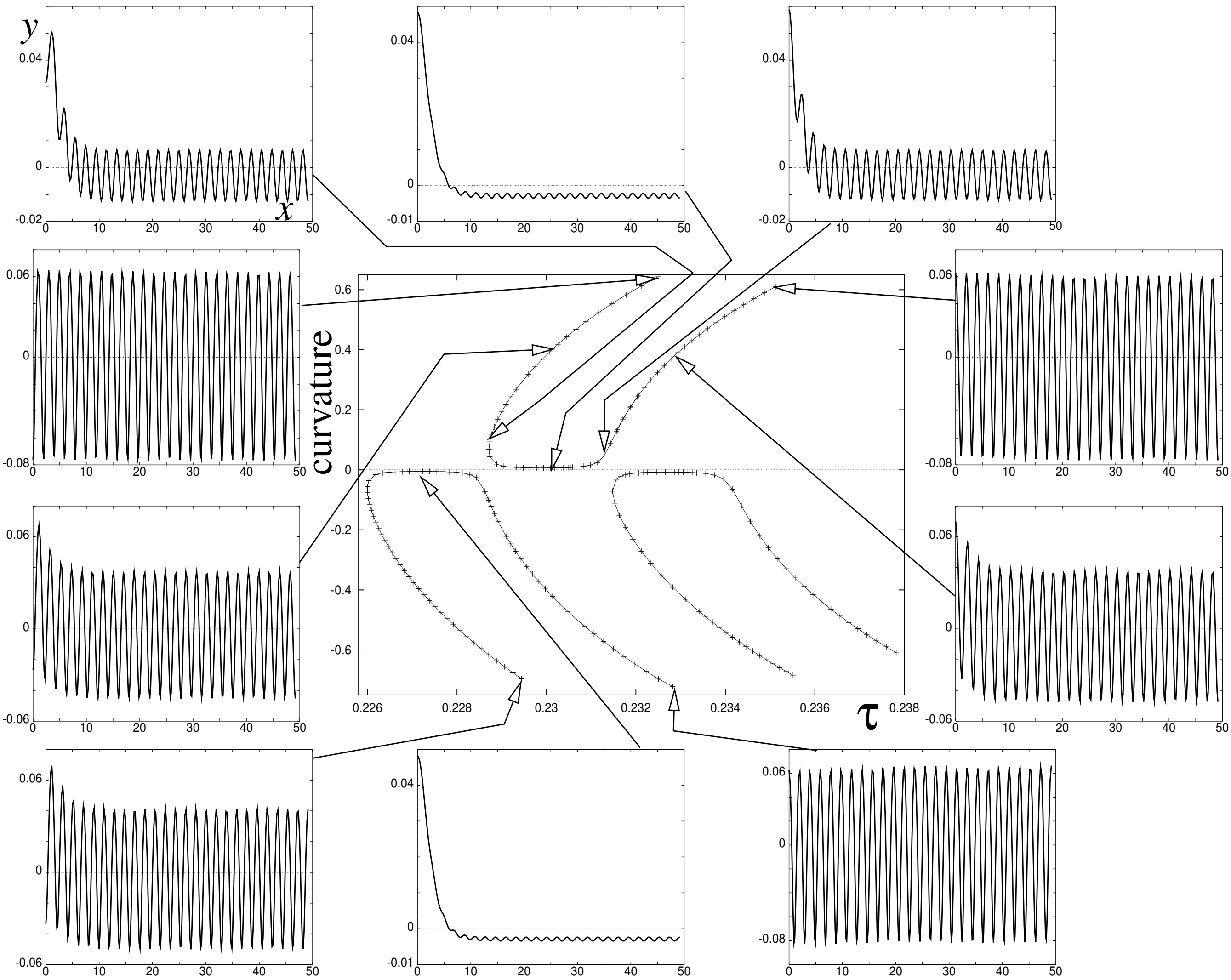

Figure 5: Illustrating the global structure of solution branches for a consecutive sequence of three branches for $F = 1.02$, $L = 98.33$ and $0.226 < \tau < 0.238$, computed with $N = 269$. The insets depict for the upper branch the detailed profiles of the free surface, illustrating the transitions that take place as the u-shaped curve is traversed. Solutions at three points on the left-hand n-shaped curve are also presented.

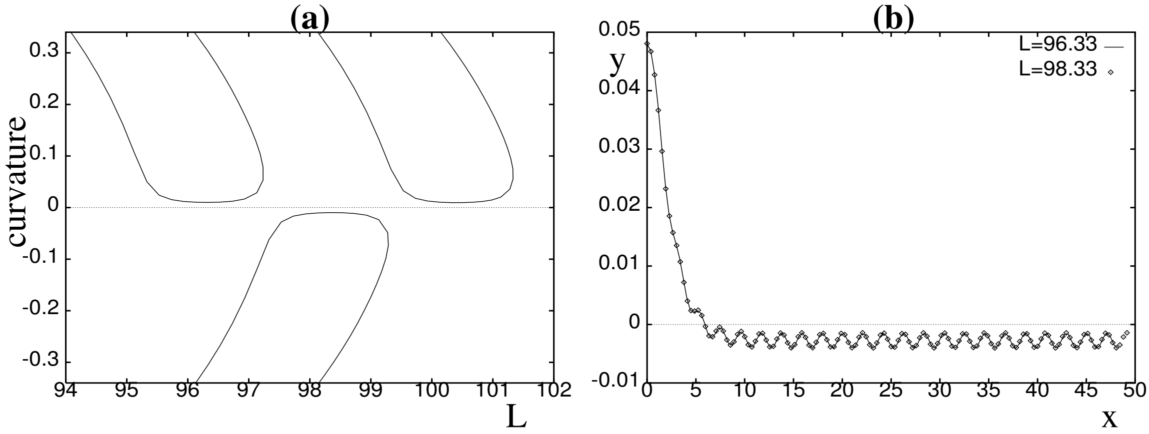

Figure 6: Illustrating the dependence on the length L. **(a)** shows the results of continuation in L for fixed Bond and Froude numbers $\tau = 0.239462$ and $F = 1.02$. **(b)** compares the solutions at approximately the minimum of curvature for the two successive positive branches. Note that the two solutions are overlaid at the scale depicted.

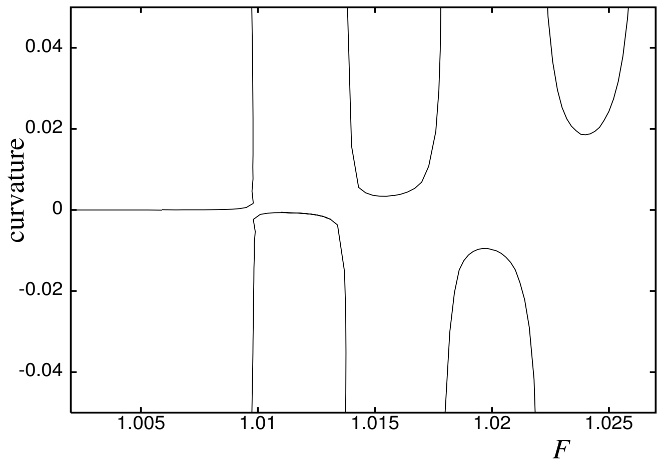

Figure 7: Illustrating the dependence on the Froude number F for fixed $L = 98.33$, $\tau = 0.239462$. Only the small-curvature parts of the branches are depicted. Note that the minimum curvatures on branches for $F < 1.01$ are smaller than the numerical accuracy.

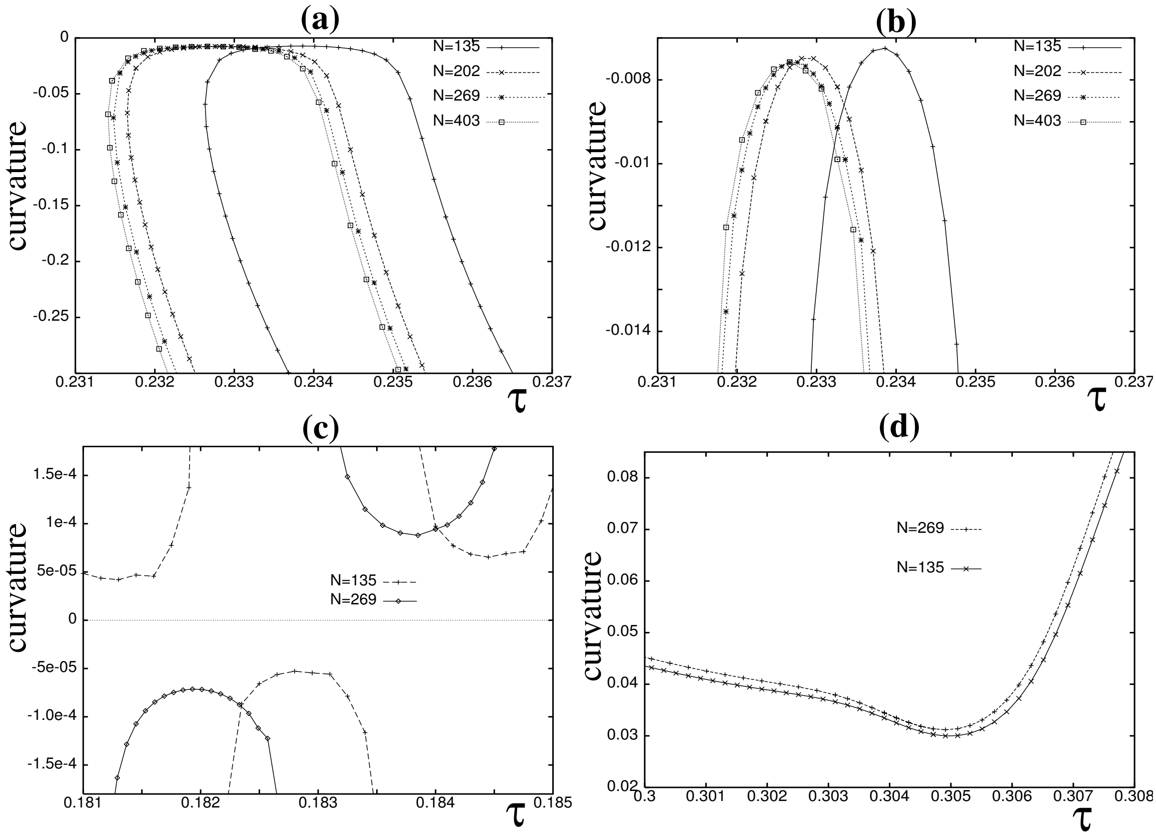

Figure 8: Illustrating the dependence on the number of mesh points N for $L = 98.33$ and $F = 1.02$. In **(a)** results for four different N-values are plotted for one particular branch for $0.231 < \tau < 0.237$, a zoom of the small curvature end of which is depicted in **(b)**. Panels **(c)** and **(d)** depict similar comparisons between two N-values, for the small-curvature tip of branches for two different values of τ.

number, computation for $\tau < 0.18 = 9/50$ was impractical, since the size of the minimum tail amplitude became comparable with the error tolerance in Newton's method. In the figures, we have only depicted the results of continuation of the low-curvature end of the depicted branches. Where branches end 'in mid air' this does not represent the end-point of the branch but where computation was stopped because the magnitude of the curvature was becoming much larger than its minimum value along the branch. Observe from the results that for the entire range of τ-values we have tested, there is no s-shaped curve. We simply get a regular sequence of u's and n's like the case of the 5th-order KdV equation with $\mu = 0$. This then is strong numerical evidence that there is in fact no codimension-one lines in the (F, τ)-plane bifurcating from $F = 1$ for $9/50 < \tau < 1/3$ at which true elevation solitary waves (with zero tail amplitude) occur.

5 Conclusion

There do not exist elevation solitary waves. This is not a proof, but the evidence would appear compelling. At first sight, this conclusion would appear to contradict the results in (Vanden-Broek 1991) which identified generalised solitary waves for which the amplitude of the ripples appear to be zero within graphical accuracy. A closer examination shows that there is no contradiction, since a blow-up of the far field solution in Figure 2(c) of (Vanden-Broek 1991) reveals oscillation on a scale $\sim 10^{-6}$. Note that those results were for fixed $\tau = 0.24$ and, most significantly given the results presented here, a very small F-value of 1.000358. The results presented here suggest that a more reliable numerical test of whether the tail amplitude is ever zero is to vary τ for fixed F and L and to asses the topological structure of the ensuing branches of solutions generalised solitary waves. We have found that the answer is then negative, at least for the τ-values for which the minimum tail amplitude is bigger than numerical precision at the fixed Froude number we chose.

It should be noted that for air-water (for which $T \approx 73$ and $g \approx 9.81$ in cms/gramms/seconds units) $\tau = 0.24$ corresponds to $H \approx$ 6mm. For such depths, viscosity is not irrelevant (Benjamin 1982) and a factor of 10^-6 in an inviscid model appears insignificant. For smaller values of τ (larger H) the effect of viscosity is less significant, but as we have shown, the minimum tail amplitude decreases for fixed F. Hence, realistically, whether the tail amplitude actually vanishes in the inviscid model does not seem to be physically of particular relevance to everyday flows. Nevertheless, this question has proved to be a historically important one in the theory of gravity-capillary water waves, and we believe our results are the first categorical piece of evidence that true solitary waves of elevation do not exist.

Acknowledgements

The authors would like to thank Mark Groves (University of Loughborough) for his key insights at the early stages of this work. The work of ARC is supported by the UK EPSRC which whom he holds an Advanced Fellowship.

References

Amick, C. J. & Kirchgassner, K. (1989), 'A theory of solitary water-waves in the presence of surface tension', *Arch. Rational Mech. Anal.*

Amick, C. J. & McLeod, J. B. (1991), 'A singular perturbation problem in water-waves', *Stability and Applied Analysis of Continuous Media* **1**, 127–148.

Amick, C. J. & Toland, J. F. (1992), 'Global uniqueness of homoclinic orbits for a class of 4th order equations', *Z. Angew. Math. Phys.* **43**, 591–597.

Beale, T. J. (1991), 'Solitary water waves with capillary ripples at infinity', *Commun. Pure Appl. Math.* **64**, 211–257.

Benjamin, T. (1982), 'The solitary wave with surface tension', *Quart. Appl. Math* **40**, 231–234.

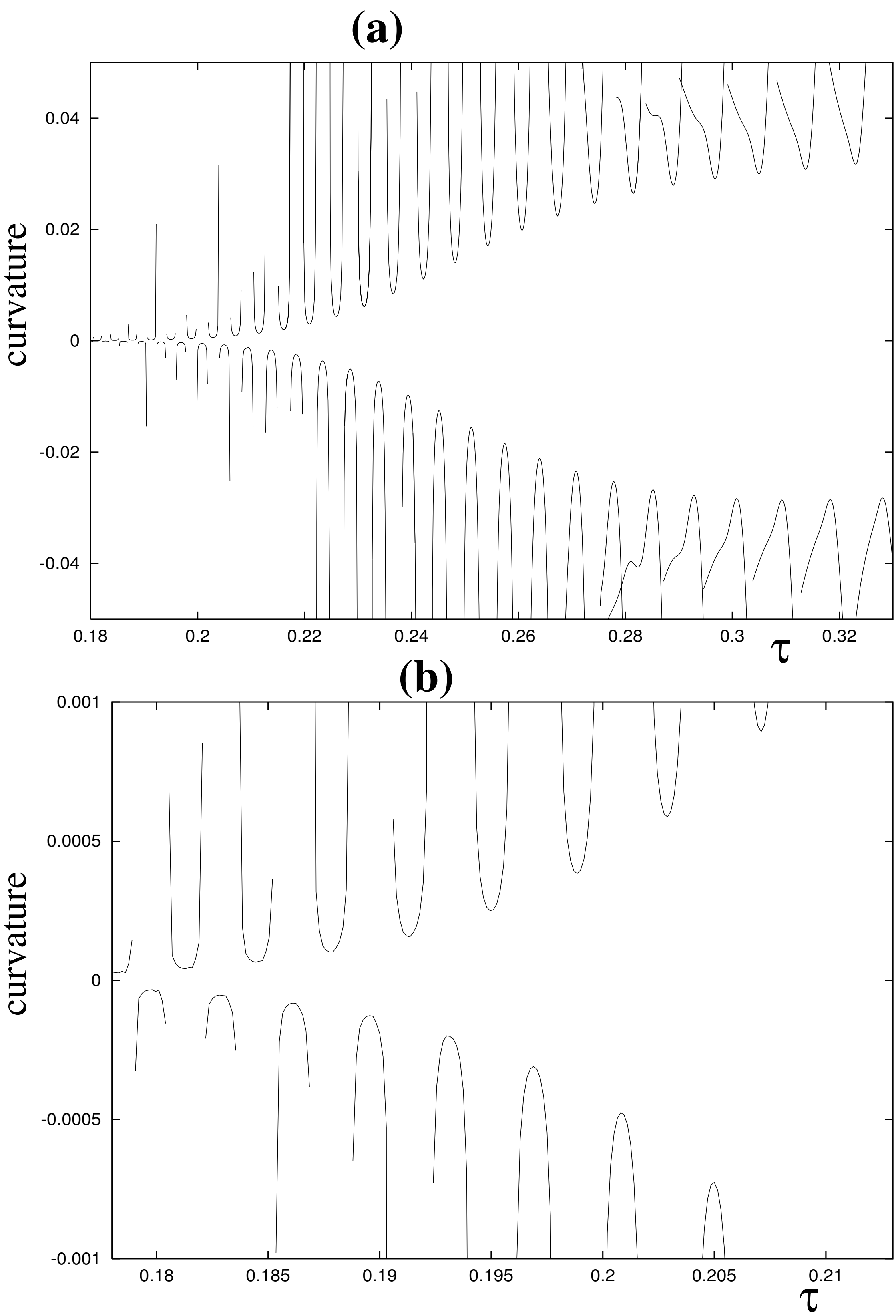

Figure 9: **(a)** A global sweep for $0.18 < \tau < 1/3$ of the small curvature end of the succession of u and n-shaped curves for $F = 1.02$ and $L = 98.33$, computed with $N = 135$. Note that no s-shaped curves are found and consequently, (as illustrated by the zoom in panel **(b)**) the minimum tail amplitude is bounded away from zero.

Bridges, T. & Derks, G. (1999), 'Linear stability of solitary wave solutions of the kawahara equation and its generalizations'. Preprint, Dept. of Maths. and Stats., Univ. of Surrey, Guildford.

Buffoni, B. & Séré, E. (1996), 'A global condition for quasi-random behaviour in a class of conservative systems', *Commun. Pure Appl. Math.* **49**, 285–305.

Buffoni, B., Champneys, A. R. & Toland, J. F. (1996), 'Bifurcation and coalescence of a plethora of homoclinic orbits for a Hamiltonian system', *J. Dyn. Diff. Eqns.* **8**, 221–281.

Buffoni, B., Groves, M. D. & Toland, J. F. (1996), 'A plethora of solitary gravity-capillary water waves with nearly critical bond and froude numbers.', *Phil. Trans. Roy. Soc. Lond. A* **354**, 575–607.

Buryak, A. V. & Champneys, A. R. (1997), 'On the stability of solitary wave solutions of the 5th-order KdV equation', *Phys. Lett. A* **233**, 58–62.

Calvo, D.C. abd Yang, T. & Akylas, T. (2000), 'On the stability of solitary waves with decaying oscillatory tails', *Proc. Roy. Soc. Lond. A* **456**, 469–487.

Champneys, A. (2000), 'Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in revesible systems'.

Champneys, A. R. & Groves, M. D. (1997), 'A global investigation of solitary wave solutions to a two-parameter model for water waves', *J. Fluid Mech.* **342**, 199–229.

Champneys, A. R. & Lord, G. J. (1997), 'Computation of homoclinic solutions to periodic orbits in a reduced water-wave problem', *Physica D* **102**, 101–124.

Craig, W. & Groves, M. D. (1994), 'Hamiltonian long-wave approximations to the water-wave problem', *Wave Motion* **19**, 367–389.

Dias, F. & Kharif, C. (1999), 'Nonlinear gravity and capillary-gravity waves', *Annual review of fluid mechanics* **31**, 301–346.

Dias, F. & Kuznetsov, E. (1999), 'On the nonlinear stability of solitary wave solutions of the fifth-order korteweg-de vries equation', *Phys. Lett. A* **263**, 98–104.

Dias, F., Menasce, D. & J.M., V.-B. (1996), 'Numerical study of gravity capillary solitary waves', *Eur. J. Mech. B Fluids* **15**, 17–36.

Doedel, E., Champneys, A., Fairgrieve, T., Kuznetsov, Y., Sandstede, B. & Wang, X. (1997), 'AUTO97 continuation and bifurcation software for ordinary differential equations'. Available by anonymous ftp from FTP.CS.CONCORDIA.CA, directory PUB/DOEDEL/AUTO.

Grimshaw, R. & Joshi, N. (1995), 'Weakly nonlocal solitary waves in a singularly perturbed Korteweg-de Vries equation', *SIAM J. Appl. Math.* **55**, 124–135.

Hasimoto, H. (1970), '[in Japanese]', *Kagaku* **40**, 401.

Hunter, J. K. & Scheurle, J. (1988), 'Existence of perturbed solitary wave solutions to a model equation for water-waves', *Physica D* **32**, 253–268.

Hunter, J. K. & Vanden-Broek, J.-M. (1983), 'Solitary and periodic gravity-capillary waves of finite amplitude', *J. Fluid Mech.* **134**, 205–219.

Iooss, G. & Kirchgassner, K. (1992), 'Water waves for small surface tension: an approach via normal form', *Proc. Roy. Soc. Edin. A* **112**, 267–200.

Kakutani, T. & Ono, H. (1969), 'Weak non-linear hydromagnetic waves in cold collisionless plasma', *J. Phys. Soc. Japan* **26**, 1305–219.

Karpman, V. (1994), 'Stationary solitary waves of the fifth order kdv-type equations', *Phys. Lett. A* **186**, 300–308.

Kawahara, T. (1972), 'Oscillatroy solitary waves in dispersive media', *J. Phys. Soc. Japan* **33**, 260–264.

Kichenassamy, S. & Olver, P. J. (1996), 'Existence and non-existence of solitary wave solutions to higher-order model evolution equations', *SIAM J. Math. Anal.* **23**, 1141–1166.

Korteweg, D. & de Vries, G. (1895), 'On the change of the form of long waves advancing in a rectangular canal and a new type of long stationary wave', *Philos. Mag.* **39**, 422–443.

Lombardi, E. (2000), *Oscillatory integrals and phenomena beyond all orders; with applications to homolcinic orbits in reversible systems*, Springer-Verlag, Berlin. Lecture Notes in Mathematics 1741.

Pomeau, Y., Ramani, A. & Grammaticos, B. (1988), 'Structrual stability of the korteweg-de vries solitons under a singular perturbation', *Physica D* **31**, 127–134.

Stoker, J. (1957), *Water waves : the mathematical theory with applications*, Interscience, New York.

Sun, S. M. (1998), 'On the oscillatory tails with arbitrary phase shift for solutions of the perturbed KdV equations', *SIAM J. Appl. Math.* **58**, 1163–1177.

Sun, S. M. (1999), 'Non-existence of truly solitary waves in water with small surface tension', *Proc. Roy. Soc. Lond. A* **455**, 2191–2228.

Sun, S. M. & Shen, M. C. (1993), 'Exponentially small estimate for the amplitude of capillary ripples of generalised solitary waves', *J. Math. Anal. Appl.* **172**, 533–566.

Vanden-Broek, J.-M. (1991), 'Elevation solitary waves with surface tension', *Phys. Fluids A* **11**, 2659–2663.

Vanden-Broek, J.-M. (1997), 'Weakly nonlocal gravity-capillary solitary waves – comments', *Phys. Fluids* **9**, 245–346.

Vanden-Broek, J.-M. & Schwartz, L. (1979), 'Numerical computation of steep gravity waves in shallow water', *Phys. Fluids* **22**, 1868–1871.

Wilton, J. (1915), 'On ripples', *Philos. Mag.* **29**, 688–700.

Yang, T. S. & Akylas, T. R. (1997), 'On asymmetric gravity-capillary solitary waves', *J. Fluid Mech* **330**, 215–232.

Zufiria, J. A. (1987), 'Symmetry breaking in periodic and solitary gravity-capillary waves on water of finte depth', *J. Fluid Mech.* **184**, 183–206.