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# Symmetry properties of lasers subject to optical feedback

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## ABSTRACT

We show that a laser with conventional optical feedback is symmetric under any rotation of the electric field, whereas a laser with phase-conjugate feedback is symmetric only under the rotations of the electric field over 0 and  $\pi$ . This has important consequences for dynamics and bifurcations. Most importantly, a PCF laser shows symmetry breaking and restoring bifurcations, something that does not occur in a COF laser.

**Keywords:** (semiconductor) laser, conventional optical feedback, phase-conjugate feedback, symmetry, bifurcations

**PACS numbers:** 42.50.Ne, 42.55.Px, 05.45.+b

## I. INTRODUCTION

The dynamics of a laser subject to external optical feedback is a topic of much experimental and theoretical research; as an entry to the extensive literature see the recent surveys Refs. [1–3]. First, there is the classic problem of how a laser reacts to conventional optical feedback (COF), which is unavoidable in many applications. Second, there is a lot of attention recently for lasers with phase conjugate feedback (PCF) because of possible practical applications [4,5]. Both the COF laser and the PCF laser are examples of delay systems described by quite similar rate equations. They are known to display a wealth of dynamics and bifurcations [6–9], where the main parameter is the feedback strength. The main difference is that the feedback term for COF contains the electric field directly, whereas for PCF it contains the complex conjugate of the electric field, owing to the phase conjugation of the light during reflection. In this paper we point out that this difference results in fundamentally different symmetry properties of the two feedback laser systems, and investigate the consequences for the dynamics and bifurcations.

We begin by considering a (feedback-free) solitary laser. In the semiclassical approach, the dynamical interaction between the single mode of the electric field and the population inversion in the laser medium is adequately described by the classical Maxwell's equations and the quantum mechanical Bloch equations [10]. Thus one obtains three dynamical equations for the electric field, the medium polarization, and the laser inversion. For many laser materials, the dynamics of the medium polarization can be adiabatically eliminated [11,12]. After applying the slowly-varying envelope and rotating-wave approximations, one ends up with first-order ordinary differential equations for the *complex* electric field amplitude  $E$  and for the inversion  $N$ . The complex amplitude  $E$  is related to the true electric field  $\mathcal{E}$  by  $\mathcal{E} = \frac{1}{2}(E \exp(i\omega_0 t) + c.c.)$ , where  $\omega_0$  is the center frequency of the single mode under consideration. Without specifying the details of the laser medium, one obtains [1] the general form:

$$\frac{dE}{dt} = \frac{1}{2}[\mathcal{G}(N, |E|^2) - \Gamma_0(t)]E, \quad (1a)$$

$$\frac{dN}{dt} = J(t) - \frac{N}{T_1(N)} - \text{Re}[\mathcal{G}(N, |E|^2)]|E|^2. \quad (1b)$$

Here  $J$  is the generation rate of inversion through a pump mechanism,  $\mathcal{G}$  is the complex optical gain,  $\Gamma_0$  is the loss rate, and  $T_1$  is the inversion lifetime. We allow these essential parameters to have various dependencies, to keep the analysis as general as possible. Note that the complex optical gain  $\mathcal{G}$  remains largely unspecified, but is only dependent on the inversion  $N$  and the electric field intensity  $|E|^2$  (and higher powers  $|E|^{2k}$  thereof).

Equations (1a,1b) are equivariant [13] under any rotation of the electric field  $E$ , that is, under multiplication of  $E$  with any complex number of modulus one. Only if the gain  $\mathcal{G}$  is real, as is the case for non-chirping lasers like gas lasers, then Eqs. (1a,1b) are also equivariant under complex conjugation. Using group theoretic notation, we conclude that a semiconductor laser (having complex gain due to its nonzero chirp factor  $\alpha$ ) is an  $S^1$ -equivariant system, where  $S^1$  is the group of all rotations of the  $E$ -plane. On the other hand, non-chirping lasers are  $O(2)$ -equivariant, where  $O(2)$  is the symmetry group of all rotations of the  $E$ -plane *and* all reflections in lines through its origin.

## II. SYMMETRY OF THE COF LASER

When the external resonator is much longer than the laser itself, the effect of feedback can be included in the field equation by means of a difference scheme [1]. For a single mode laser subject to COF one needs to add the following feedback term to the equation for the electric field (1a):

$$\frac{(r_1^2 - 1)}{\tau_{in} r_1^2} \sum_{m=1}^M (-r_1 r_3)^m \exp(-im\omega_0\tau) E(t - m\tau). \quad (2)$$

Here  $r_1$  is the amplitude reflection coefficient of the facet facing the external cavity,  $r_3$  the amplitude reflection coefficient of the external resonator. Furthermore,  $\tau_{in}$  is the round-trip

time of the light inside the laser cavity,  $\tau$  is the round-trip time of the light in the external cavity, and  $m$  is an integer denoting multiple round trips, where up to  $M$  round trips are taken into account.

Note that Eqs. (1a,1b) with the feedback terms (2) added to (1a) are of a very general form, allowing for different expressions for the gain and for any number  $M$  of external round-trips. Since we are interested in deterministic effects, spontaneous-emission Langevin-noise terms are intentionally left out. These equations are examples of delay-differential systems [14,15]. We denote a trajectory from time  $t_0$  to  $t_1$  by

$$(E, N)_{t_0}^{t_1} := \{(E(t), N(t)) \mid t \in [t_0, t_1]\}. \quad (3)$$

The trajectory  $(E, N)_{t_0}^{t_1}$  is determined by the values of  $E$  and  $N$  in the time interval  $[t_0 - M\tau, t_0]$ . (Recall that  $M$  is the number of external round trips taken into account.) In other words, after prescribing  $(E, N)_{-M\tau}^0$  as initial condition, the trajectory  $(E, N)_0^{t_1}$  is determined for any  $t_1 \geq 0$ , and can be obtained by numerical integration.

Probably the best known example of a COF model of the very general form discussed here are the Lang-Kobayashi equations [16] for a semiconductor laser subject to weak optical feedback. We used these equations in the scaled and normalized forms given in Ref. [1] to produce the illustrations in Figure 1. Note that only one roundtrip is taken into account in the Lang-Kobayashi equations so that  $M = 1$ . We would like to stress again that the results in this paper are valid more generally for a whole class of COF laser models.

The equations for the COF laser, Eqs. (1a,1b) with the feedback term to (2) added to (1a), are still  $S^1$ -equivariant. However, for non-chirping lasers the addition of the feedback term destroys the invariance under complex conjugation. In conclusion, any COF laser has  $S^1$  symmetry, but not  $O(2)$  symmetry.

The symmetry group  $S^1 = \{c \in \mathbf{C} \mid |c| = 1\}$  acts on a trajectory by

$$c \circ (E, N)_{t_0}^{t_1} = (cE, N)_{t_0}^{t_1} \text{ for all } c \in S^1. \quad (4)$$

Because of the  $S^1$ -symmetry of the COF laser,  $(cE, N)_{t_0}^{t_1}$  is also a trajectory. Physically this means that the phase of a trajectory can be shifted by any prescribed amount. In other words, the trajectories  $(E, N)_{-M\tau}^0$  and  $(cE, N)_{-M\tau}^0$  run entirely parallel, meaning that at any moment in time the only difference between them is the phase-shift  $\arg(c)$  of the electric field. This is the case even when the trajectories themselves are chaotic. Consider now the *group orbit* of a solution

$$S^1 \circ (E, N)_{t_0}^{t_1} = \{(cE, N)_{t_0}^{t_1} \mid c \in S^1\}, \quad (5)$$

the image of the trajectory  $(E, N)_{t_0}^{t_1}$  under all rotations in  $S^1$ . We will call the intersection of the group orbit with the two-dimensional half-plane  $\Sigma = \{(E, N) \mid \text{Im}(E) = 0 \text{ and } \text{Re}(E) \geq 0\}$  the *trace* of the trajectory. If one identifies  $\Sigma$  with the  $(R, N)$ -plane, where  $E(t) = R(t)e^{i\varphi(t)}$ , the trace is simply

$$(R, N)_{t_0}^{t_1} = \{(R(t), N(t)) \mid t \in [t_0, t_1]\}. \quad (6)$$

Note that no extra computation is necessary to obtain the trace; all one needs to do is plot  $N$  versus  $R$ .

We now discuss the symmetry properties of CW states and periodic trajectories. A CW state is a trajectory with constant intensity and inversion, and a phase which depends linearly on time. In other words, it is of the form

$$(R_s \exp(i\omega_s t), N_s)^{2\pi/\omega_s} \quad (7)$$

for fixed values of  $R_s$  and  $N_s$ . A CW state is rotationally symmetric, that is, invariant under the group action of  $S^1$ . The trace of the CW state in Eq. (7) is simply the single point  $(R_s, N_s)$  in the  $(R, N)$ -plane (which we identified with  $\Sigma$ ). Note that the CW states of the Lang–Kobayashi equations can be computed explicitly [7]. It follows immediately from our symmetry considerations that a perturbation in just the  $\varphi$ -direction does not grow or decay and, hence, is the eigenvector for the eigenvalue zero in the stability analysis [1,17–19].

Suppose now that we found a trajectory  $(E, N)_0^T$  such that  $R(0) = R(T)$  and  $N(0) = N(T)$ , where we assume that  $T$  is the smallest number with this property; an example is shown in Fig. 1(a). Note that the phase  $\varphi(t)$  is not periodic; see Fig. 1(b). We call this trajectory  $(R, N)$ -periodic or simply periodic with period  $T$ . Because of periodicity the trace of  $(E, N)_0^T$  is the closed curve  $(R, N)_0^T$  with period  $T$  in Fig. 1(d). The group orbit  $S^1 \circ (E, N)_0^T$  is the torus in  $(E, N)$ -space depicted in Fig. 1(e). There are exactly two possibilities for the dynamics on this torus, depending on the phase difference  $\Phi_T := \varphi(0) - \varphi(T)$ . If  $\Phi_T/(2\pi)$  is rational then the torus consists of an infinite collection of closed orbits in  $(E, N)$ -space. (The period of each of these closed orbits in  $(E, N)$ -space is a multiple of  $T$ .) If  $\Phi_T/(2\pi)$  is irrational then the dynamics is quasiperiodic, meaning that  $(E, N)_0^\infty$  never becomes periodic in  $(E, N)$ -space, but lies dense in the torus. The symmetry allows us to construct the orbit  $(E, N)_0^{kT}$  for any  $k > 0$  without further, expensive integration of the equations. If we let  $c_T := e^{i\Phi_T}$  then the rotated trajectory  $(c_T E, N)_0^T$  can be ‘glued’ to the old trajectory  $(E, N)_0^T$  because  $(R(T), N(T)) = (R(0), N(0))$ . In this fashion we obtain  $(E, N)_0^{2T}$ . Iterating this process gives

$$(E, N)_0^{kT} = \bigcup_{0 \leq j \leq k} (c_T^j E, N)_0^T. \quad (8)$$

This is illustrated in Fig. 1(c), where  $(E, N)_0^{23T}$  was constructed in this way from the boldface  $(E, N)_0^T$ .

An important open problem for COF lasers is to explain the exact nature of the transition from simple dynamics like in Fig. 1 to so-called low frequency fluctuations [6,7]. Of particular interest here are global bifurcations, in which stable and unstable manifolds of CW states change their relative positions. Because the COF laser is an infinite dimensional system, the question arises in which framework one should study the bifurcations in this transition. Stable and unstable manifolds exist in the infinite dimensional space of continuous functions with values in  $(E, N)$ -space, the phase space of the system. However, any of these manifolds is foliated by images of trajectories under the group  $S^1$ . As a consequence, stable and unstable manifolds project down nicely to the  $(R, N)$ -plane, which makes this a good space in which to look for global bifurcations.

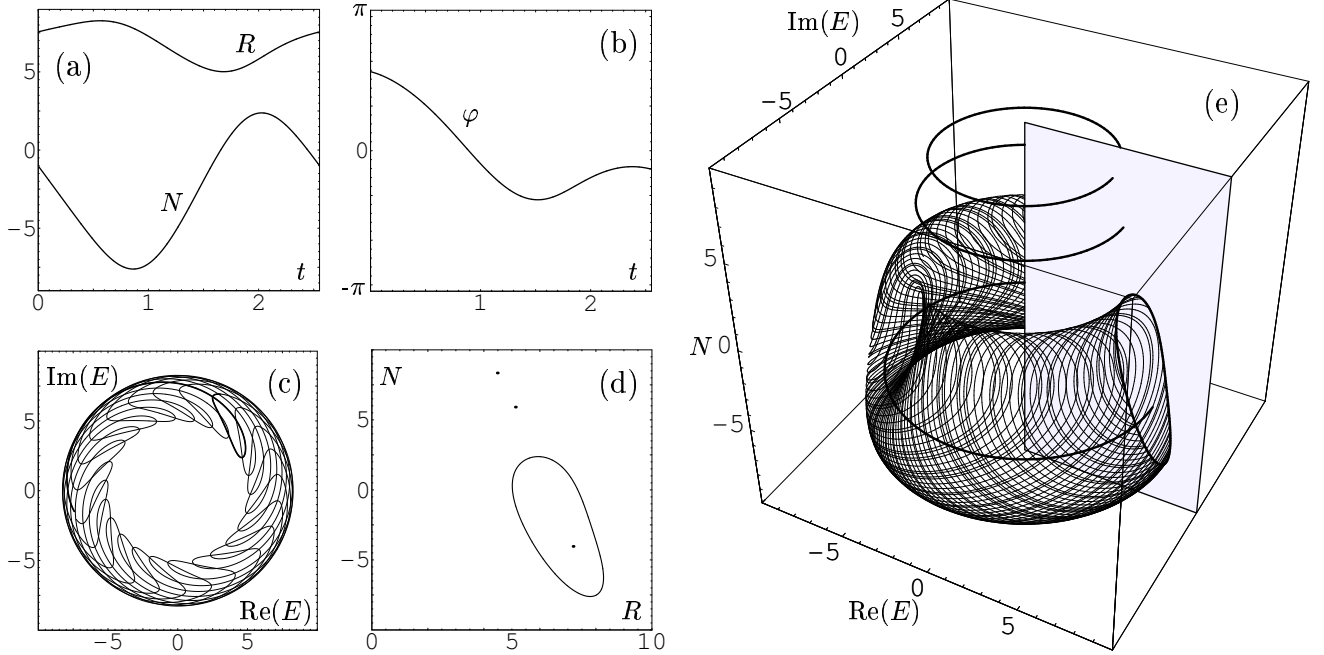


FIG. 1. For a periodic trajectory  $(E, N)_0^T$  the intensity  $R$  and the inversion  $N$  are periodic (a), but the phase  $\varphi$  is not (b). Any orbit  $(E, N)_0^t$  can be constructed by glueing together copies of the original orbit that are phase shifted over consecutive multiples of  $\varphi(0) - \varphi(T)$  (c). The orbit  $(E, N)_0^t$  is a torus (e) that intersects the fixed half-plane  $\{Im(E) = 0; Re(Z) > 0\}$  in the trace  $(R, N)_0^T$  (d).

### III. SYMMETRY OF THE PCF LASER

For a single mode laser subject to PCF from a non-degenerate Four-Wave Mixing (FWM) set-up, the following feedback term needs to be added [1] to (1a):

$$\begin{aligned} \frac{(r_1^2 - 1)}{\tau_{in} r_1} \sum_{m=1}^M \left[ \left( -r_1 \sqrt{R_{PCM}} \right)^{2m-1} \exp[(2m-1)i(\varphi'_{PCM} + \delta\tau) - (m+1)i\delta t] \right. \\ \left. + i(\varphi_{12} - \pi/2) E^*(t - (2m-1)\tau) \right. \\ \left. + \left( -r_1 \sqrt{R_{PCM}} \right)^{2m} \exp[-2mi(\varphi'_{PCM} + \delta\tau)] E(t - 2m\tau) \right] \end{aligned} \quad (9)$$

Here  $R_{PCM}$  is the reflectivity of the phase conjugate mirror,  $\varphi'_{PCM}$  is its phase shift,  $\delta$  is the detuning, and  $\varphi_{12}$  expresses the complex nature of the product of the two pump-fields that drive the phase-conjugation. Due to phase conjugation the complex conjugate  $E^*$  of the electric field enters exactly for odd numbers of roundtrips.

The equations for the PCF laser, Eqs. (1a,1b) with the feedback term (9) added to (1a), are  $\mathbf{Z}_2$ -equivariant, where the symmetry group  $\mathbf{Z}_2 = \{1, -1\}$  is generated by the multiplication of the electric field by  $-1$ . This can be explained by the fact that multiplying

$E$  with a complex number  $c$  of modulus one is a counterclockwise rotation of  $E$ , but a clockwise rotation of  $E^*$ , so that only the rotations over the angles  $0$  and  $\pi$  leave the  $E$ -field invariant. Like for the COF laser, the addition of the feedback term (9) destroys the invariance under complex conjugation. In conclusion, any PCF laser has  $\mathbf{Z}_2$  symmetry.

This discrete symmetry allows for the possibility of symmetry breaking and restoring bifurcations. In symmetry breaking, a symmetric attractor becomes unstable, creating two nonsymmetric attractors. In symmetry restoration, two nonsymmetric attractors merge and give rise to a symmetric attractor. These bifurcations have important physical consequences, as we discuss below. For the illustrations in Figures 2 and 3 we used the PCF equations from Ref. [9], where further details of this model can be found. Like in the Lang–Kobayashi equations, only a single roundtrip is taken into account in this particular model, but again we remark that our results are valid for more general models of PCF lasers.

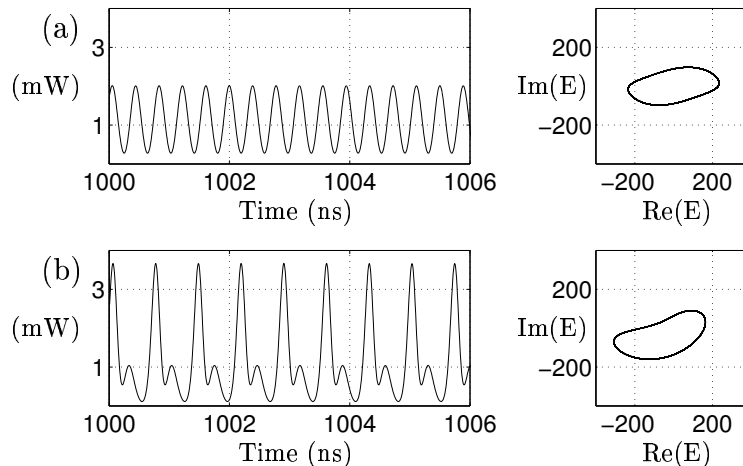


FIG. 2. Symmetry breaking bifurcation of a symmetric limit cycle (a) to a nonsymmetric limit cycle (b). On the left is shown the time series of the power, and on the right the orbit projected onto the complex  $E$ -plane. Notice that the period of the power is doubled in this bifurcation because successive maxima develop different heights.

We first present an example of symmetry breaking of a periodic orbit. When the feedback strength is increased or decreased, a symmetric periodic orbit can lose its stability and create two symmetric stable limit cycles as is depicted in Fig. 2. The symmetric limit cycle is of the form  $(E(t), N(t))_0^T$ , where  $T$  is its period. Because the limit cycle is symmetric, it surrounds the origin in the  $E$ -plane, so that the phase is unbounded; see Fig. 2(a). Rotating the symmetric limit cycle by  $\pi$  around the  $N$ -axis is equivalent to waiting for half a period. This means that the dynamics on the limit cycle has the spatio-temporal symmetry

$$(E, N, t) \mapsto (-E, N, t - T/2). \quad (10)$$

As a consequence, the time series of the power is periodic with period  $T/2$ , even though the period of the limit cycle itself is  $T$ ; see Fig. 2(a).

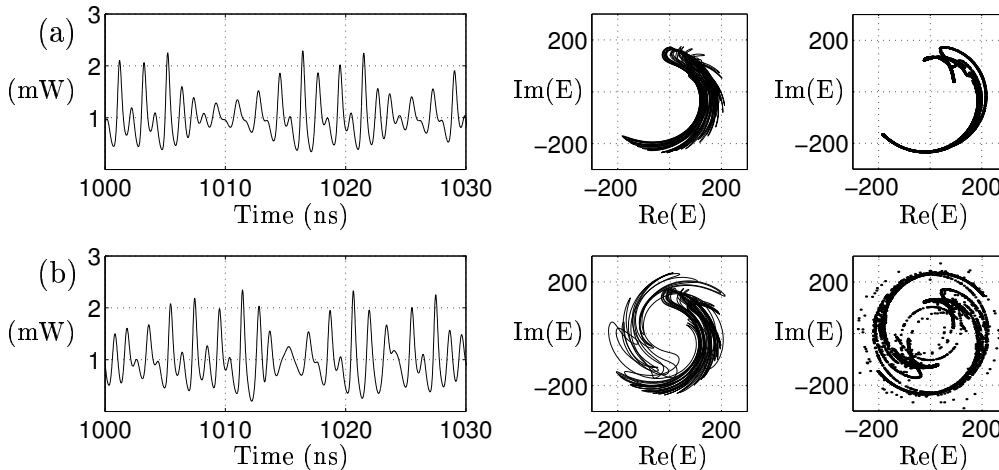


FIG. 3. Symmetry restoring bifurcation of a nonsymmetric chaotic attractor (a) to a symmetric chaotic attractor (b). From left to right is shown the time series of the power, the orbit projected onto the complex  $E$ -plane, and the strange attractor of the Poincaré map in a plane  $\{N = \text{const}\}$ .

In symmetry breaking this symmetry is lost and the time series of the power has the same period  $T$  as the limit cycle itself. Consecutive maxima of the power start to differ, which changes the period of the power from  $T/2$  to  $T$ ; see Fig. 2(b). As a consequence, symmetry breaking may be mistaken for period doubling, in particular, when one considers bifurcation diagrams only. If the feedback strength is decreased through the bifurcation value, then the symmetry is restored through the disappearance of two attracting nonsymmetric limit cycles.

Interestingly, symmetry restoration can also be found for chaotic attractors. A nonsymmetric chaotic attractor colliding with its symmetric counterpart as the feedback strength is increased to create a symmetric chaotic attractor is shown in Fig. 3. The phase of each of the nonsymmetric chaotic attractors is bounded. In the bifurcation the time series of the power practically does not change, still being chaotic. However, the attractor becomes much larger and now visits both parts of the previously distinct attractors. As a result, the phase is not bounded any longer. The sudden merging of the two chaotic attractors can best be seen in plots of the attractors of the Poincaré map in the right column of Fig. 3. When one follows one of the nonsymmetric attractors through the bifurcation, it suddenly appears to include its symmetric counterpart.

#### IV. CONCLUSIONS

We showed that a laser subject to COF and a laser subject to PCF have fundamentally different underlying symmetries, namely  $S^1$ - and  $Z_2$ -symmetry, respectively. This has important consequences for the dynamics and bifurcations that can be observed experimentally and in numerical studies.



We gave examples of symmetry breaking and restoring bifurcations in the PCF laser, which do not occur in the COF laser. The symmetry breaking of a limit cycle that we found should clearly be observable in experiments by a doubling of the period of oscillations of the power. However, the period of the corresponding limit cycle itself does not double, and it is important not to confuse this type of symmetry breaking bifurcation with period-doubling.

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