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Champneys, AR., & Malomed, BA. (1999). *Embedded solitons in a three-wave system*.

Early version, also known as pre-print

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Embedded Solitons in a Three-Wave System

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ABSTRACT

We report a rich spectrum of isolated solitons residing inside (*embedded* into) the continuous radiation spectrum in a simple model of three-wave spatial interaction in a second-harmonic-generating planar optical waveguide equipped with a quasi-one-dimensional Bragg grating. An infinite sequence of fundamental embedded solitons are found, which differ by the number of internal oscillations. Branches of these zero-walkoff spatial solitons give rise, through bifurcations, to several secondary branches of finite-walkoff solitons. The structure of the bifurcating branches suggests a multistable configuration of spatial optical solitons, which may find applications in photonics.

Recent studies have revealed a novel class of *embedded solitons* (ESs) in various nonlinear-wave systems. An ES is a solitary wave which exists despite having its internal frequency in resonance with linear (radiation) waves. ESs may exist as *codimension-one* solutions, i.e., at discrete values of the frequency, provided that the spectrum of the corresponding linearized system has (at least) two branches, one corresponding to exponentially localized solutions, the other one to delocalized radiation modes. In such systems, quasilocalized solutions (or "generalized solitary waves" [1]) in the form of a solitary wave resting on top of a small-amplitude continuous-wave (cw) background are generic [2]. However, at some special values of the internal frequency, the amplitude of the background may exactly vanish, giving rise to an isolated soliton embedded into the continuous spectrum. Examples of such embedded solitons are available in water-wave equations [3] and in several nonlinear-optical models, including a Bragg grating incorporating wave-propagation terms [4] and second-harmonic generation in the presence of the self-defocusing Kerr nonlinearity [5] (the latter model with competing nonlinearities was introduced earlier in a different context [6]).

ESs are interesting because they naturally appear when higher-order (singular) perturbations are added to the system, which may completely change its soliton spectrum. Optical ESs have considerable potential for various applications, due to the very fact that they are isolated solitons, rather than occurring in continuous families. The stability problem for ESs was solved in some generality in Ref. [5]. There it was demonstrated that an ES is a *semi*-stable object which is stable to linear approximation, but subject to a slowly growing (sub-exponential) one-sided nonlinear instability. In the pre-

viously studied models, only a few branches of ESs were found, and only after careful numerical searching, which suggest they may be hard to observe in an experiment. The present work investigates ESs in a recently introduced model of a three-wave interaction [7]. It will be found that ESs occur in abundance in this model, hence it may be much easier to observe them experimentally.

The model describes *spatial solitons* produced by the second-harmonic generation in a planar waveguide, in which two components of the fundamental harmonic (FH), v_1 and v_2, are linearly coupled by Bragg reflection on a grating in the form of a system of scores parallel to the propagation direction z (for a more detailed description of the model see [7]):

$$i(v_1)_z + i(v_1)_x + v_2 + v_3 v_2^* = 0, \qquad (1)$$
$$i(v_2)_z - i(v_2)_x + v_1 + v_3 v_1^* = 0, \qquad (2)$$
$$2i(v_3)_z - q v_3 + D(v_3)_{xx} + v_1 v_2 = 0. \qquad (3)$$

Here v_3 is the second-harmonic (SH) field, x is a normalized transverse coordinate, q is a mismatch parameter, and D is an effective diffraction coefficient (the diffraction terms in the FH equations may be neglected, as the artificial diffraction induced by the Bragg scattering is much stronger, while the SH, propagating *parallel* to the grating, undergoes no reflection).

Experimental techniques for generation and observation of spatial solitons in planar waveguides are now well elaborated ([8]), and the waveguide carrying parallel scores can be easily fabricated. Hence the present system provides a medium in which experimental observation of ESs are most plausible (simple estimates [7] show the necessary size of the experimental sample is just a few cm). As mentioned above, the observation of ES in this system should be further facilitated by the fact that it

supports a multitude of distinct ES states, see below.

Eqs. (1)–(3) have three dynamical invariants: the momentum and Hamiltonian, which will not be used below, and the energy flux (norm)

$$E \equiv \int_{-\infty}^{+\infty} \left[|v_1(x)|^2 + |v_2(x)|^2 + 4|v_3|^2 \right] dx \,. \qquad (4)$$

The norm played a crucial role in the analysis of the ES stability carried out in [5].
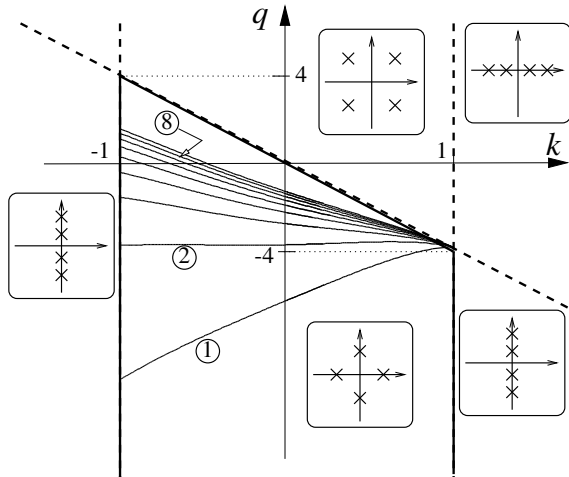
FIG. 1. The (k, q) parameter plane of the three-wave model (1)–(3). The linear analysis (the results of which are summarized in the inset boxes) shows that ES with $c = 0$ may occur only in the region inside the solid bold lines. The bundle of curves emanating from the point $(k = 1, q = -4)$ are branches of embedded-soliton solutions with $c = 0$.

Soliton solutions to (1)–(3) are sought in the form

$$v_{1,2}(x,z) = \exp(ikz) u_{1,2}(\xi), \quad v_3(x,z) = \exp(2ikz) u_3 \,, \qquad (5)$$

where $\xi \equiv x - cz$, with c being the *walkoff* (slope) of the spatial soliton's axis relative to the light propagation direction z. The substitution of (5) into Eqs. (1)–(3) leads to an 8th-order system of ordinary differential equations (ODEs) for the real and imaginary parts of $v_{1,2,3}$ (primes standing for $d/d\xi$):

$$-ku_1 + i(1-c)u_1' + u_2 + u_3 u_2^* = 0, \qquad (6)$$

$$-ku_2 - i(1+c)u_2' + u_1 + u_3 u_1^* = 0, \qquad (7)$$

$$-(4k+q)u_3 + Du_3'' - 2icu_3' + u_1 u_2 = 0. \qquad (8)$$

Before looking for ES solutions to the full nonlinear equations, it is necessary to investigate the eigenvalues λ of their linearized version, in order to isolate the region in which ESs *may* exist. Substituting $u_1, u_2 \sim \exp(\lambda\xi)$, $u_3 \sim \exp(2\lambda\xi)$ into Eqs. (6)–(8) and linearizing, one finds that the FH and SH equations obviously decouple. The FH equations give rise to the biquadratic characteristic equation,

$$(1 - c^2)^2 \lambda^4 + 2 \left[(1+c^2)k^2 - (1-c^2) \right] \lambda^2 + (k^2 - 1)^2 = 0 \,, \qquad (9)$$

and the SH equation produces another four eigenvalues given by

$$\left[D\lambda^2 - (4k + q) \right]^2 + 4c^2 \lambda^2 = 0. \qquad (10)$$

A necessary condition for the existence of ESs is that the eigenvalues given by Eq. (9) should have non-zero real parts - this is necessary for the existence of exponentially localized solutions - while the eigenvalues from Eq. (10) should be purely imaginary (otherwise, one will have ordinary, rather than embedded, solitons). This discrimination between the two sets of the eigenvalues is due to the fact that Eqs. (6) and (7) for the FH are always linearizable, while the SH equation (8) may be *nonlinearizable* which opens the possibility of existence of ESs [5]. From (9) and (10), these two conditions imply

$$k^2 + c^2 < 1; \ 4k + q < c^2/D \,. \qquad (11)$$

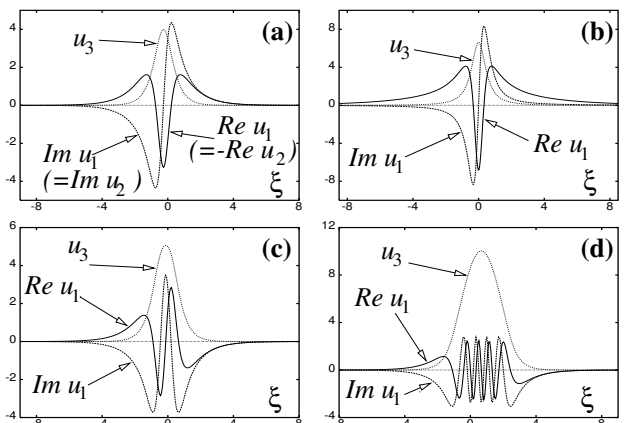For the case $c = 0$, the parametric region defined by the inequalities (11) is displayed in Fig. 1.

FIG. 2. Typical examples of the fundamental embedded solitons with the zero walkoff: (a) the ground-state for $k = 0$; (b) the same solution for $k = -0.95$; (c,d) the first and eight "excited states" for $k = 0$.

In Ref. [7], numerous regular (*gap* [9]) soliton solutions to the present model were found by means of a numerical shooting method. To construct ES solutions, we applied the same method to Eqs. (6), (7) and (8), allowing just one parameter to vary. From each such solution, branches were continued in the parameters k, q and c, by means of the software package AUTO [10]. Note that $c = 0$ solutions admit an invariant reduction $u_2 = -u_1^*$, $u_3 = u_3^*$, which reduces the system to a 4th-order ODE system, thus making numerical shooting feasible. We confine ourselves to *fundamental* solitons, which implies that the SH component u_3 has a single-humped shape

(a distinctive feature of gap solitons in the same system is that not only fundamental, but also certain double-humped bound states appear to be stable [7]). Multi-humped ESs must exist too as per a theorem in Ref. [11], but leaving them aside, we still find a rich structure of fundamental ESs.
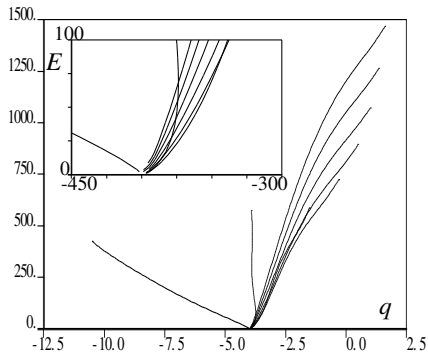

FIG. 3. A diagram of the $c = 0$ embedded solitons on the (energy-flux, mismatch) plane. The inset zooms the most interesting part of the diagram.

We begin with a description of the results from the reduced case $c = 0$, when an additional scaling allows us to set $D \equiv 1$ without loss of generality. The results are displayed in Figs. 1–3. There is strong evidence for existence of an infinite "fan" of fundamental ES branches. The *ground-state* soliton has the simplest internal structure (Fig. 2a). The next "first excited state" differs by adding one (spatial) oscillation to the FH field (Fig. 2c). Adding each time an extra oscillation, we obtain an indefinitely large number of "excited states" (as an example, see the 8th state in Fig. 2d). In Fig. 1, the first nine states (branches) are shown in the (k, q) parametric plane. Note that the whole bundle of branches originates from the point ($k = 1, q = -4$), which is precisely the intersection of the two lines which limit the ES-existence region (see Eq. (11) with $c = 0$). At this degenerate point, the linearization gives four zero eigenvalues. More branches than those depicted have been found, the numerical results clearly pointing towards the existence of infinitely many branches, accumulating on the ES-region border $q + 4k = 0$. In the accumulation process, each u_3 component is successively wider and the $u_{1,2}$ have more and more internal oscillations.

Since k is an arbitrary propagation constant, from physical grounds, the results obtained for the $c = 0$ solutions are better summarized in terms of energy flux E vs. mismatch q (Fig. 3). Note that all the branches shown in Fig. 3 really terminate at their edge points, which correspond to hitting the boundary $k = -1$, see Fig. 1. It is also noteworthy that all the solutions are exponentially localized, except at the edge point $k = -1$, where a straightforward consideration of Eqs. (6)–(8)

demonstrates that, in this case, ES are weakly localized as $|x| \to \infty$ (cf. Fig 2b):

$$u_1 \approx \sqrt{-(4k + q)}|x|^{-1}, \ u_2 \approx (1/2)\sqrt{-(4k + q)}|x|^{-2},$$
$$u_3 \approx x^{-2}.$$

Finally, observe from Figs. 1 and 3 that the first "excited-state" branch has the remarkable property that it corresponds to a nearly constant value of q. This means that while, generally, ES are isolated (*codimension-one*) solutions for fixed values of the physical parameters, this branch is *nearly generic*, existing in a narrow interval of q-values between -4.0 and -3.74.

Now we turn to *walking* ESs, i.e., those with $c \neq 0$. These were sought for systematically by returning to the full 8th-order-ODE model and allowing AUTO to detect bifurcations (of *pitchfork* type), while moving along branches of the $c = 0$ solutions. It transpires that all bifurcating branches have $c \neq 0$, i.e. they are *walking* ESs. Such solutions are of *codimension-two* in the parameter space (i.e. the solutions can be represented by curves $k(q)$, $c(q)$), which can be established by a simple counting argument after noting that the 8th-order linear system has two pairs of pure imaginary eigenvalues. We present results only for the walking solutions which bifurcate from the ground and first excited $c = 0$ states.

It was found that the ground-state branch has exactly two bifurcation points, giving rise to two distinct walking solutions (up to symmetry). These new branches are shown in terms of the $c(q)$ and $E(q)$ dependences in Fig. 4. Note that they, eventually, coalesce and disappear. As the inset to Fig. 4b shows, this process is reminiscent of a tangent (fold) bifurcation.

The first excited state has three bifurcation points. One of them gives rise to a short branch of walking solutions that terminates, while two others appear to extend to $q = -\infty$ (their apparent "merger" in Fig. 5 is an artefact of plotting). It is known that, in the large-mismatch limit $q \to -\infty$, the present three-wave model with the quadratic nonlinearity goes over into a modified Thirring model with cubic nonlinear terms [12]. This suggests that the latter model may also support ES. However, consideration of this issue is beyond the scope of the present work.
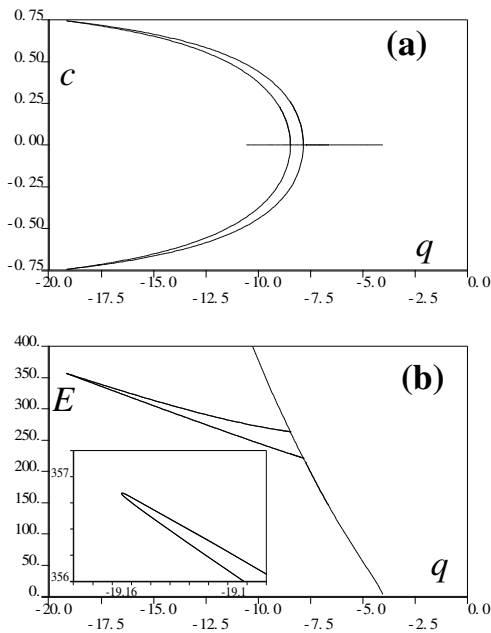
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FIG. 4. Two branches of "walking" ($c \neq 0$) embedded solitons bifurcating from the ground-state $c = 0$ branch: (a) the walkoff c, and (b) the energy flux E vs. the mismatch q. The horizontal segment in (a) shows the $c = 0$ branch. The inset in (b) shows that the two branches meet and disappear via a typical tangent bifurcation.
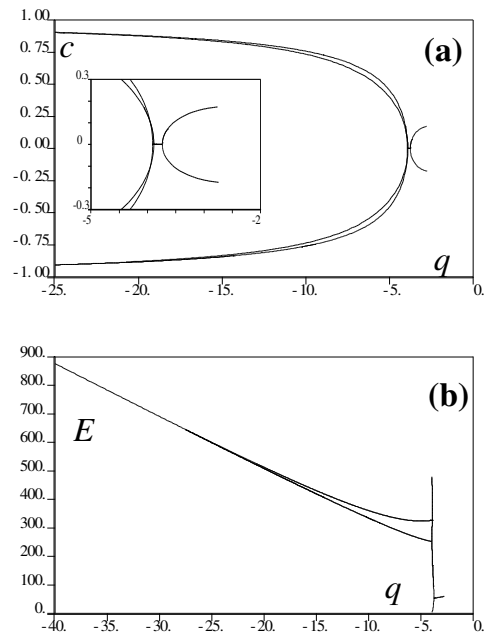

FIG. 5. Three branches of walking ($c \neq 0$) embedded solitons bifurcating from the $c = 0$ branch corresponding to the first "excited state", depicted similarly to Fig. 4 The inset in (a) shows in detail the central part of the diagram.

Fig. 4 clearly shows that, in a certain interval of the mismatch parameter q, the system gives rise to a multistability; i.e. coexistence of different types of spatial solitons in the planar optical waveguide (for instance, taking account of the fact that each $c \neq 0$ branch has symmetric parts with the opposite values of c, we conclude that there are *five* coexisting solutions at q taking values between about -8 and -11). This situation is of obvious interest for applications to photonics, especially in terms of all-optical switching [8]. Indeed, switching from a state with a larger value of the energy flux to a neighboring one with a smaller flux can be easily initiated by a small perturbation, in view of the above-mentioned *one-sided* semistability of ES, shown in a general form in [5]. Switching between the two branches with $c \neq 0$ can be quite easy to realize too, due to the small energy-flux and walkoff differences between them, see Fig. 4.

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