OPEN ACCESS

University of BRISTOL

van Tartwijk, GHM., Gray, GR., & Krauskopf, B. (1999). *Aspects of symmetry in lasers with optical feedback.*

Early version, also known as pre-print

Link to publication record in Explore Bristol Research
PDF-document

University of Bristol - Explore Bristol Research
General rights

Aspects of symmetry in lasers with optical feedback

Bernd Krauskopf[a], Guido H.M. van Tartwijk[b], and George R. Gray[c]

[a]Dept of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, UK
[b]Uniphase Netherlands B.V., Prof. Holstlaan 4 (WY 63), 5656 AA Eindhoven, The Netherlands
[c] Seagate Technology, Riverbluff Development Center, 10701 Lyndale Avenue South, Bloomington, MN 55420

ABSTRACT

This paper discusses a fundamental difference between the symmetry properties of a laser with conventional optical feedback and that of a laser with phase-conjugate feedback. As a consequence, the possible dynamics and bifurcations for these two types of laser systems differ considerably, a fact that has not been noted in earlier comparisons between these two systems.

Keywords: (semiconductor) laser, conventional optical feedback, phase-conjugate feedback, symmetry, bifurcations

1. INTRODUCTION

Conventional optical feedback (COF) is unavoidable in many applications of lasers, and semiconductor lasers in particular are very susceptible to external influences because of the low reflectivity of their cleaved facets[1]. For this reason and because of their complicated dynamics, lasers with COF are a topic of much research, both experimental and theoretical[2-5].

Phase conjugate feedback (PCF) on the other hand has recently received a lot of attention because of possible practical applications. A lot of interesting dynamics has been found[6-15], including mode locking, phase and frequency locking, and different routes to chaotic dynamics.

Both the COF laser and the PCF laser are examples of delay systems, and from a mathematical point of view the rate equations describing their dynamics are very similar. The main difference is that the feedback term for COF contains the electric field directly, but that for PCF contains the complex conjugate of the electric field, owing to the phase conjugation of the light during reflection. In the literature PCF has been considered a combination of COF (because of the delay) and optical injection (because of the presence of detuning). A comparison between (semiconductor) lasers with COF and PCF was conducted recently[9] with the aid of bifurcation diagrams.

In this paper we show that the possible dynamics of a COF laser differ fundamentally from that of a PCF laser because of different underlying symmetry properties. The COF laser is invariant under any phase shift of the electric field. Physically this means that the phase of the COF laser is 'not important', a fact that has been noted in the literature[16], but was not stated explicitly as a symmetry property. As a consequence of this, the COF laser cannot phase-lock. On the other hand, the PCF laser is invariant only under a phase shift of π. In particular, it does phase-lock for a suitable feedback strength. What is more, as was recently shown the PCF laser undergoes symmetry breaking and restoring bifurcations of limit cycles and even chaotic attractor[17,18]. This shows that the dynamics and bifurcations of these two systems are fundamentally different. This fact had been overlooked so far.

The paper is structured as follows. In Section 2 we describe the rate equation models for COF and PCF lasers. Sections 3 and 4 discuss their symmetry properties, and Section 5 is a summary of the results.

Further author information (Email addresses) –
B.K.: B.Krauskopf@bristol.ac.uk
G.R.G.: George_R_Gray@notes.seagate.com
G.H.M.T.: tartwijk@nl.uniphase.com

2. SINGLE-MODE RATE EQUATIONS

In the semiclassical approach, the dynamical interaction between the single mode of the electric field and the population inversion in the laser medium is adequately described by the classical Maxwell's equations and the quantum mechanical Bloch equations[19]. Thus one obtains three dynamical equations for the electric field, the medium polarization, and the laser inversion. For many laser materials, the dynamics of the medium polarization can be adiabatically eliminated[20,21]. After applying the slowly-varying envelope and rotating-wave approximations one ends up with first-order ordinary differential equations for the *complex* electric field amplitude E and for the inversion N. The complex amplitude E is related to the true electric field $\mathcal{E}$ by $\mathcal{E} = \frac{1}{2}(E\exp(i\omega_0 t) + c.c.)$, where ω_0 is the center frequency of the single mode under consideration.

Without specifying the details of the laser medium, one can write these two equations in their general form as follows[22]:

$$\frac{dE}{dt} = \frac{1}{2}[\mathcal{G}(N, |E|^2) - \Gamma_0(t)]E, \tag{1a}$$

$$\frac{dN}{dt} = J(t) - \frac{N}{T_1(N)} - \mathrm{Re}[\mathcal{G}(N, |E|^2)]|E|^2 . \tag{1b}$$

Here J is the generation rate of inversion through a pump mechanism, $\mathcal{G}$ is the complex optical gain, Γ_0 is the loss rate, and T_1 is the inversion lifetime. We allow these essential parameters to have various dependencies, to keep the analysis as general as possible. Note that the complex optical gain $\mathcal{G}$ remains largely unspecified, but is only dependent on the inversion N and the electric field intensity $|E|^2$ (and higher powers $|E|^{2k}$ thereof).

When the external resonator is much longer than the laser itself, the effect of feedback can be included in the field equation by means of a difference scheme[23]. For a single mode laser subject to COF one needs to add the following feedback term to the equation for the electric field (1a):

$$\frac{(r_1^2 - 1)}{\tau_{in}r_1^2} \sum_{m=1}^{M} (-r_1 r_3)^m \exp(-im\omega_0\tau)E(t - m\tau). \tag{2}$$

Here r_1 is the amplitude reflection coefficient at the opaque side of the facet facing the external cavity, r_3 the amplitude reflection coefficient of the external resonator. Furthermore, τ_{in} is the round-trip time of the light inside the laser cavity, τ is the round-trip time of the light in the external cavity, and m is an integer denoting multiple round-trip times, where up to M round-trips are taken into account.

In the same way, for a single mode laser subject to PCF from a non-degenerate Four-Wave Mixing (FWM) set-up the following feedback term needs to be added[14] to (1a):

$$\frac{(r_1^2 - 1)}{\tau_{in}r_1} \sum_{m=1}^{M} \left[\left(-r_1\sqrt{R_{PCM}}\right)^{2m-1} \exp\left[(2m-1)i(\varphi'_{PCM} + \delta\tau) - (m+1)i\delta t + i(\varphi_{12} - \pi/2)\right] E^\star(t - (2m-1)\tau) \right.$$

$$\left. + \left(-r_1\sqrt{R_{PCM}}\right)^{2m} \exp\left[-2m(\varphi'_{PCM} + \delta\tau)\right] E(t - 2m\tau) \right] \tag{3}$$

Here R_{PCM} is the reflectivity of the phase conjugate mirror, φ'_{PCM} is its phase shift, δ is the detuning, and φ_{12} expresses the complex nature of the product of the two pump-fields that drive the phase-conjugation. Due to phase conjugation the complex conjugate E^* of the electric field enters exactly for odd numbers of roundtrips.

Note that Eqs. (1a,1b) with the feedback terms (2) or (3) added to (1a) are of a very general form, allowing for different expressions for the gain and for any number M of external round-trips. Since we are interested in deterministic effects, spontaneous-emission Langevin-noise terms are intentionally left out. These equations are examples of delay–differential systems[24,25]. We denote a trajectory from time t_0 to t_1 by

$$(E, N)_{t_0}^{t_1} := \{(E(t), N(t)) \mid t \in [t_0, t_1]\}. \tag{4}$$

The trajectory $(E, N)_{t_0}^{t_1}$ is determined by the values of E and N in the time interval $[t_0 - M\tau, t_0]$. (Recall that M is the number of external round trips taken into account.) In other words, after prescribing $(E, N)_{-M\tau}^{0}$ as initial condition

the trajectory $(E, N)_0^{t_1}$ is determined for any $t_1 \geq 0$, and can be obtained by numerical integration. Formally, the delay system is an operator that describes how the continuous function $(E, N)^0_{-M\tau}$ with values in (E, N)-space is transformed into the continuous function $(E, N)_0^{M\tau}$. The interval $[0, M\tau]$ specifies a new initial condition and can be thought of as being shifted back to $[-M\tau, 0]$. In other words, delay equations define an operator on the space of continuous functions over the interval $[-M\tau, 0]$ with values in (E, N)-space. This infinite dimensional function space is the phase space of this delay-differential system. Physically one usually looks at trajectories in (E, N)-space as defined above, but one should keep in mind that (E, N)-space is not the phase space of the system. Rather one is looking at a particularily useful projection.

For concreteness we introduce now two examples for a semiconductor laser with COF and PCF, respectively, which were also used to produce the figures in this paper. A well-studied set of equations are the famous Lang–Kobayashi equations[1] for a semiconductor laser subject to COF[22,23,26]. The feedback is assumed to be so weak that only one external round-trip needs to be taken into account (M=1). Suitably scaled and normalized, the equations can be written as[14]:

$$\frac{dE}{dt} = \frac{1}{2}(1 + i\alpha)N(t)E(t) + \kappa E(t - \tau) \exp\left(-i\omega_0\tau\right), \tag{5a}$$

$$\frac{dN}{dt} = -\gamma N(t) - A[1 + N(t)](|E(t)|^2 - 1), \tag{5b}$$

where time is measured in units of the cavity photon lifetime Γ_0^{-1}, α is the linewidth enhancement factor, the electric field amplitude E is scaled with respect to its feedback free value $\sqrt{P_0}$, the inversion $N = g_N(n - n_{th})/\Gamma_0$ is the scaled excess carrier density (g_N is the differential gain and n_{th} is the feedback-free threshold carrier density), $A = g_N P_0/\Gamma_0$ accounts for the pump, $\gamma = (\Gamma_0 T_1)^{-1} + g_N P_0/\Gamma_0$ is the dressed carrier relaxation rate (with T_1 the bare carrier life time), and $\kappa = (1 - r_1^2)r_3/(r_1\tau_{in})$ is the feedback rate.

Similarly, lasers with PCF are also usually modeled by taking into account only a single roundtrip[14]. For the case of instantaneous PCF one obtains[17,18]:

$$\frac{dE}{dt} = \frac{1}{2}(1 + i\alpha)N(t)E(t) + \kappa E^*(t - \tau) \exp(2i\delta(t - \tau/2)) \tag{6a}$$

$$\frac{dN}{dt} = -\gamma N(t) - A[1 + N(t)](|E(t)|^2 - 1). \tag{6b}$$

In the feedback term κ is again the feedback rate, δ is the detuning, and τ is the external-cavity round trip time.

3. SYMMETRY OF THE COF LASER

The COF laser, as described by Eqs. (1a,1b) with the feedback term (2) added to (1a), is invariant under the symmetry transformation

$$(E, N) \mapsto (cE, N) \text{ for any } c \in \mathbf{C} \text{ with } |c| = 1. \tag{7}$$

If we write $E = Re^{i\varphi}$ we have invariance under the equivalent symmetry transformation

$$(R, \varphi, N) \mapsto (R, \varphi + \Phi, N) \text{ for any } \Phi \in [0, 2\pi). \tag{8}$$

In other words, the equations are invariant under any phase shift of the electric field E. Because we consider (E, N)–space to be the natural space, we will discuss the symmetry mainly in the form of Eq. (7). In spite of a large number of papers devoted to the dynamics of (semiconductor) lasers subject to optical feedback, this symmetry was never stated explicitly. It is the aim of this paper to investigate this symmetry further, and to contrast it with that of the PCF laser discussed in the next section.

Let us now be more mathematical[27,28]. The rotational symmetries form a group denoted by $S^1 = \{c \in \mathbf{C} \mid |c| = 1\}$. The group S^1 acts on a trajectory by

$$c \circ (E, N)_{t_0}^{t_1} = (cE, N)_{t_0}^{t_1} \text{ for all } c \in S^1. \tag{9}$$

Because of the symmetry of the COF laser, $(cE, N)_{t_0}^{t_1}$ is also a trajectory. Physically this means that if one shifts the phase of a trajectory by any prescribed amount then the resulting set of points is also a trajectory. In other words, if one rotates the prescribed initial condition $(E, N)_{-M_\tau}^0$ to $(cE, N)_{-M_\tau}^0$ and integrates the equations one obtains $(cE, N)_0^{t_1}$ as a trajectory instead of $(E, N)_0^{t_1}$. Consequently, the trajectories for these two initial conditions run entirely parallel, meaning that at any moment in time the only difference between them is the phase-shift $\arg(c)$ of the electric field.

Consider now the *group orbit* of a solution

$$S^1 \circ (E, N)_{t_0}^{t_1} = \{ (cE, N)_{t_0}^{t_1} \mid c \in S^1 \}, \tag{10}$$

the image of the trajectory $(E, N)_{t_0}^{t_1}$ under all rotations in S^1. We choose the two-dimensional half-plane $\Sigma = \{ (E, N) \mid \text{Im}(E) = 0 \text{ and } \text{Re}(E) \geq 0 \}$, which we identify with the (R, N)–(half)plane. Then we can consider the intersection of a group orbit with Σ, which we call the *trace* of the trajectory. The trace gives important information about the trajectory we started off with and is easy to compute in practice. It is by definition the set of intersection points of any rotation of the given trajectory with Σ. Instead of rotating the trajectory one can rotate Σ and keep the trajectory fixed. As a direct result the trace is

$$(R, N)_{t_0}^{t_1} = \{ (R(t), N(t)) \mid t \in [t_0, t_1] \}. \tag{11}$$

Note that no extra computation is necessary to obtain the trace; all one needs to do is plot N versus R.

We now discuss how the rotational symmetry can be used to describe different kinds of dynamics of the COF laser. For concreteness, this is illustrated with trajectories of the Lang–Kobayashi equations (5a,5b).

3.1. CW states and their stability

A CW state is a trajectory with constant intensity and inversion, and a phase which depends linearly on time. In other words, such a trajectory is of the form

$$\left(R_s \exp\left(i\omega_s t \right), N_s \right)_0^{2\pi/\omega_s} \tag{12}$$

for fixed values of R_s and N_s. A CW state is rotationally symmetric, that is, invariant under the group action of S^1. The trace of the CW state in Eq. (12) is simply the single point (R_s, N_s) in the (R, N)-plane (which we identified with Σ).

The CW states for the Lang–Kobayashi equations (5a,5b) are given by[5]:

$$(\omega_s - \omega_0)\tau = -C \sin\left(\omega_s \tau + \arctan \alpha \right), \tag{13a}$$
$$N_s = -2\kappa \cos\left(\omega_s \tau \right), \tag{13b}$$
$$|E_s|^2 = 1 - [\gamma N_s / A(1 + N_s)], \tag{13c}$$

where $C = \kappa\tau\sqrt{1 + \alpha^2}$ is the effective feedback parameter. (Recall that ω_0 is the central frequency of the single mode we are considering.) An important question concerns the stability of the CW states given by (13a,13b,13c). It follows immediately from our symmetry considerations that a perturbation in just the φ-direction does not grow or decay and, hence, is the eigenvector for the eigenvalue zero in the stability analysis[16,23,29,30].

3.2. Periodic trajectories

Suppose we found a trajectory $(E, N)_0^T$ such that $R(0) = R(T)$ and $N(0) = N(T)$, where we assume that T is the smallest number with this property; an example is shown in Fig. 1(a). Note that $\varphi(t)$ is not periodic; see Fig. 1(b). We call this trajectory (R, N)–periodic or simply periodic with period T. Because of periodicity the trace of $(E, N)_0^T$ is the closed curve $(R, N)_0^T$ with period T in Fig. 1(e).
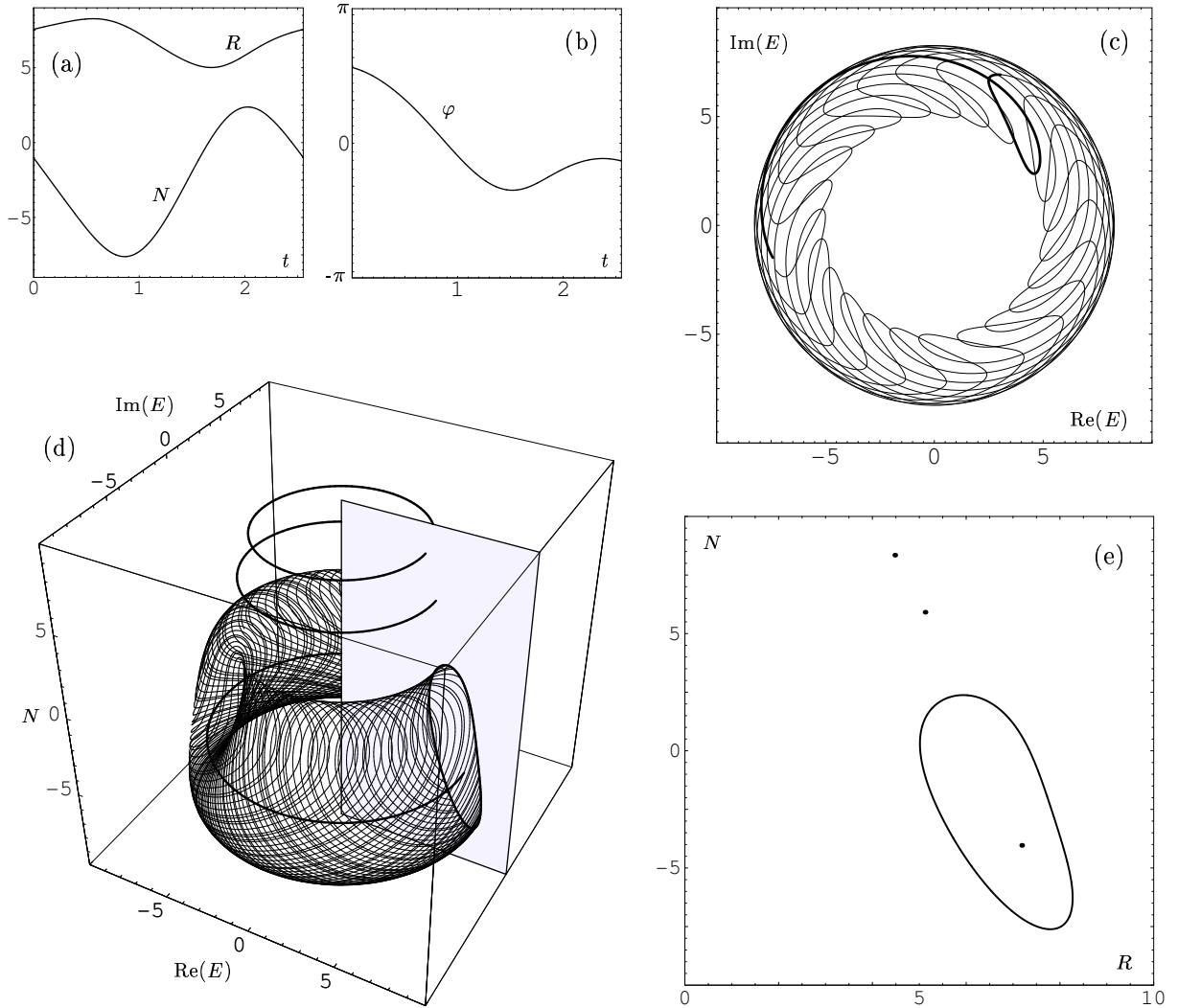
Fig. 1. For a periodic trajectory $(E, N)_0^T$ the intensity R and the inversion N are periodic (a), but the phase φ is not (b). Any orbit $(E, N)_0^t$ can be constructed by glueing together copies of the original orbit that are phase shifted over consecutive multiples of $\varphi(0) - \varphi(T)$ (c). The orbit $(E, N)_0^t$ is a torus (d) that intersects the fixed half-plane $\{Im(E) = 0; Re(Z) > 0\}$ in the trace $(R, N)_0^T$ (e).

The group orbit $S^1 \circ (E, N)_0^T$ is a torus in (E, N)-space; see Fig. 1(d). There are exactly two possibilities for the dynamics on this torus, depending on the phase difference $\Phi_T := \varphi(0) - \varphi(T)$ over the period T; see Fig. 1(b). If $\Phi_T/(2\pi)$ is rational then the torus consists of an infinite collection of closed orbits in (E, N)-space. (The period of each of these closed orbits in (E, N)-space is a multiple of T.) If $\Phi_T/(2\pi)$ is irrational then the dynamics is quasiperiodic, meaning that $(E, N)_0^\infty$ never becomes periodic in (E, N)-space, but lies dense in the torus. There can be no locking on the torus: two trajectories that are rotated with respect to one another will stay parallel on the torus for all times.

The symmetry allows us to construct the orbit $(E, N)_0^{kT}$ for any $k > 0$ as follows. If we let $c_T := e^{i\Phi_T}$ then the rotated trajectory $(c_T E, N)_0^T$ can be 'glued' to the old trajectory $(E, N)_0^T$ because $(R(T), N(T)) = (R(0), N(0))$. In this fashion we obtain $(E, N)_0^{2T}$. Iterating this process gives

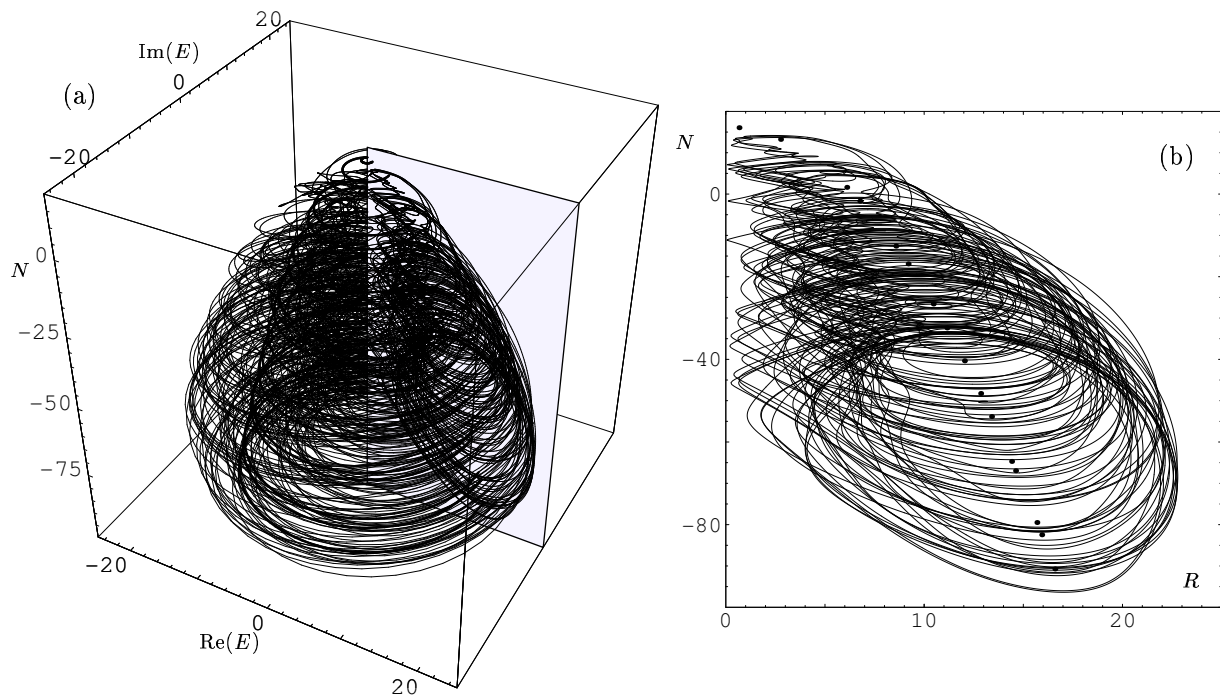$$(E, N)_0^{kT} = \bigcup_{0 \le j \le k} (c_T^j E, N)_0^T. \tag{14}$$

Fig. 2. Low frequency fluctuations in (E, N)-space (a) and their trace in the (R, N)-plane (b).

In other words, it is enough to produce data over one period T. The entire orbit can then be reconstructed as described and further, expensive, integration can be avoided. This is illustrated in Fig. 1(c), where $(E, N)_0^{23T}$ was constructed in this way from the boldface $(E, N)_0^T$.

3.3. Nonperiodic trajectories

Suppose a trajectory $(E, N)_0^{t_1}$ is not (R,N)-periodic, that is, $(E(t), N(t)) \neq (E(0), N(0))$ for all $t \in (0, t_1]$. Then we can still rotate $(E, N)_0^{t_1}$ by any angle, but no such rotation will 'glue' to the original trajectory $(E, N)_0^{t_1}$ to produce a longer trajectory. Nevertheless, the group orbit $S^1 \circ (E, N)_0^{t_1}$ is a collection of 'parallel' orbits of the form $(cE, N)_0^{t_1}$. Each individual orbit may have complicated dynamics, but two rotated orbits run parallel, that is, at any given moment they differ only in a fixed phase shift of the E-field. The key information is again contained in the trace $(R, N)_0^{t_1}$ in the (R, N)-plane. An example of a nonperiodic trajectory in (E, N)-space is shown in Figure 2(a). If it is rotated through Σ it produces the trace in Figure 2(b). This figures shows an example of low frequency fluctiations (LFF) where the trajectory comes very close to the N-axis at irregular intervals, which correspond to the typical intensity dropouts of LFF.

An important open problem is to explain the transition from simple dynamics like in Fig. 1 to LFF like in Fig. 2. Because the COF laser is an infinite dimensional system, the question arises in which framework one should study the bifurcations in this transition. Of particular interest are global bifurcations, in which stable and unstable manifolds of CW states change their relative positions. These manifolds exist in the infinite dimensional space of continuous functions with values in (E, N)-space, the phase space of the system. It appears to be possible to reduce the system to a suitable lower dimensional subspace, but this is beyond the scope of this paper. Any of these manifolds is foliated by images of trajectories under the group S^1. As a consequence, stable and unstable manifolds project down nicely to the (R, N)-plane, which makes this a good space in which to look for global bifurcations.
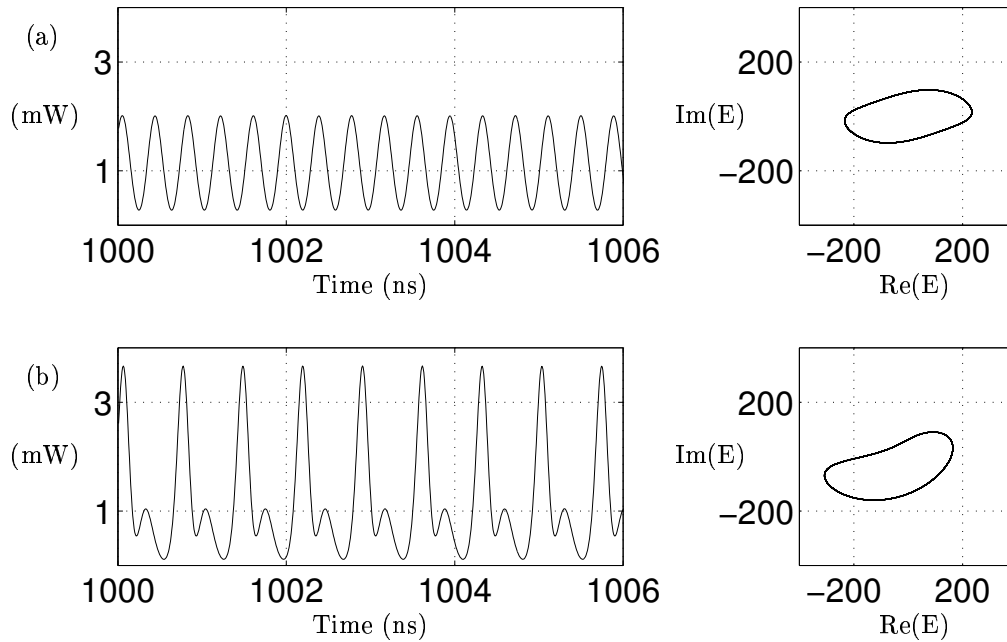
Fig. 3. Symmetry breaking bifurcation of a symmetric limit cycle (a) to a nonsymmetric limit cycle (b). On the left is shown the time series of the power, and on the right the orbit projected onto the complex E-plane. Notice that the frequency of the power is halved in this bifurcation because successive maxima develop different heights.

4. SYMMETRY OF THE PCF LASER

The PCF laser, as described by Eqs. (1a,1b) with the feedback term (3) added to (1a), is invariant under the symmetry transformation

$$(E, N) \mapsto (-E, N).$$ (15)

If we write $E = Re^{i\varphi}$ we have invariance under the equivalent symmetry transformation

$$(R, \varphi, N) \mapsto (R, \varphi + \pi, N).$$ (16)

This means that the PCF laser is invariant under the phase shifts by 0 and π only. This can be explained by the fact that multiplying E with a complex number c of modulus one is a counterclockwise rotation of E, but a clockwise rotation of E^*. The PCF laser is therefore invariant only under rotations over angles that have the same effect when applied clockwise or counterclockwise, which gives the angles 0 and π as the only possibilities.

The group of symmetries is the discrete group of rotations over 0 and π, denoted by $\mathbf{Z}_2 = \{1, -1\}$. Any attractor is either symmetric under a rotation of E by π, or has a symmetric counterpart, which can be found by changing the phase of an appropriate initial condition by π. (The group orbit of an attractor consists either of one or two elements.)

This symmetry allows for the possibility of symmetry breaking and restoring bifurcations. In symmetry breaking, a symmetric attractor becomes unstable, creating two nonsymmetric attractors. In symmetry restoration, two nonsymmetric attractors merge and give rise to a symmetric attractor. These bifurcations have important physical consequences, as we discuss below. Furthermore, this possibility of symmetry breaking and restoring bifurcations fundamentally distinguishes a laser with PCF from a laser with COF.
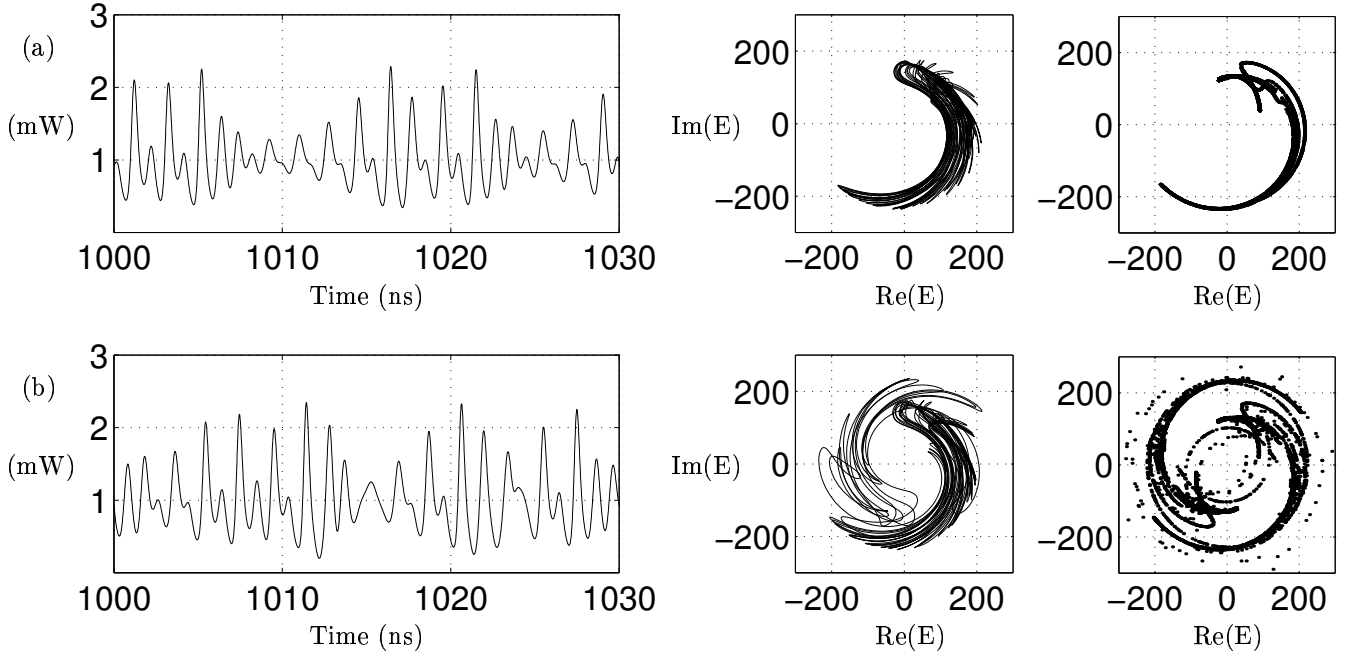
Fig. 4.   Symmetry restoring bifurcation of a nonsymmetric chaotic attractor (a) to a symmetric chaotic attractor (b). From left to right is shown the time series of the power, the orbit projected onto the complex E-plane, and the strange attractor of the Poincaré map in a plane $\{N = const\}$.

4.1. Symmetry breaking of a limit cycle

When the feedback strength is increased or decreased in Eqs. (6a,6b), a symmetric periodic orbit can lose its stability and create two symmetric stable limit cycles[17,18]. This situation is depicted in Fig. 3. We now discuss the physical relevance of this bifurcation. If we denote the period of the symmetric limit cycle by T, then it is of the form $(E(t), N(t))_0^T$. Because the limit cycle is symmetric, it surrrounds the origin in the E-plane, so that the phase is unbounded; see Fig. 3(a). Rotating the symmetric limit cycle by π around the N-axis is equivalent to waiting for half a period. This means that the dynamics on the limit cycle has the spatio-temporal symmetry

$$(E, N, t) \mapsto (-E, N, t - T/2).\tag{17}$$

As a consequence, the time series of the power is periodic with period $T/2$, even though the period of the limit cycle itself is T; see Fig. 3(a).

In symmetry breaking this spatio-temporal symmetry is lost and the time series of the power has the same period T as the limit cycle itself. In this bifurcation consecutive maxima of the power start to differ, which changes the period of the power from $T/2$ to T. As a consequence, symmetry breaking may be mistaken for a period doubling, in particular, when one considers bifurcation diagrams only.

If the feedback strength is decreased through the bifurcation value, then symmetry restoration takes place as follows. Two attracting nonsymmetric limit cycles move closer to each other as they approach a symmetric unstable limit cycle. In the bifurcation the nonsymmetric limit cycles disappear and the symmetric limit cycle becomes stable. Consecutive maxima of the power become equal, leading to a halving of the period of the power.

4.1. Symmetry restoration of a chaotic attractor

After a series of period doublings one can find a nonsymmetric chaotic attractor of Eqs. (6a,6b), which collides with its symmetric counterpart as the feedback strength is increased[17,18] to create a symmetric chaotic attractor. The situation is shown in Fig. 4. The phase of each of the nonsymmetric chaotic attractors is bounded. In the bifurcation the time series of the power practically does not change, still being chaotic. However, the attractor becomes much larger and now visits both parts of the previously distinct attractors. As a result, the phase is not bounded any longer. The sudden merging of the two chaotic attractors can best be seen in the plots of the attractors of the Poincaré map in the right column of Fig. 4. When one follows one of the nonsymmetric attractors through the bifurcation, it suddenly appears to include its symmetric counterpart. This is an example of a symmetry increasing bifurcation of a chaotic attractor.

5. CONCLUSIONS

We have shown that the COF laser is symmetric under any shift of the phase of the electric field, while the PCF laser is symmetric only under a phase shift over π. This implies that the dynamics of these two systems are fundamentally different, because any symmetry imposes restrictions on the possible dynamics and bifurcations. In particular, we gave examples of symmetry breaking and restoring bifurcations in the PCF laser, which do not occur in the COF laser.

The underlying symmetry has direct physical meaning. For the COF laser it explains the fact that there is no phase locking. In the PCF laser the symmetry implies that the time series of the power in the presence of a symmetric limit cycle has exactly half the period of the limit cycle itself. This property is lost in symmetry breaking, which should be clearly observable in experiments.

ACKNOWLEDGEMENTS

REFERENCES

1. R. Lang and K. Kobayashi, *External Optical Feedback Effects on Semiconductor Injection Laser Properties*, IEEE J. Quantum Electron. **QE-16**, 347 (1980).
2. K. Ikeda, K. Otsuka, and K. Matsumoto, *Maxwell-Bloch Turbulence*, Progr. of Theor. Phys. Suppl. **99**, 295 (1989).
3. K. Otsuka, *Self-induced phase turbulence and chaotic itinerancy in coupled laser systems*, Phys. Rev. Lett. **65**, 329 (1990).
4. I. Fischer, O. Hess, W. Elsäßer, and E. O. Göbel, *High-dimensional chaotic dynamics of an external cavity semiconductor laser*, Phys. Rev. Lett. **73**, 2188 (1994).
5. I. Fischer, G. H. M. van Tartwijk, A. M. Levine, W. Elsäßer, E. O. Göbel, and D. Lenstra, *Fast pulsing and chaotic itinerancy with a drift in the coherence collapse of semiconductor lasers*, Phys. Rev. Lett. **76**, 220 (1996).
6. G. R. Gray, D. H. DeTienne, and G. P. Agrawal, *Mode Locking in semiconductor lasers by phase-conjugate optical feedback*, Opt. Lett. **20**, 1295 (1995).
7. G. H. M. van Tartwijk, H. J. C. van der Linden, and D. Lenstra, *Theory of a diode laser with phase-conjugate feedback*, Opt. Lett. **17**, 1590 (1992).
8. G. P. Agrawal and G. R. Gray, *Effect of phase-conjugate feedback on the noise characteristics of semiconductor lasers*, Phys. Rev. A **46**, 5890 (1992).
9. G. R. Gray, D. Huang, and G. P. Agrawal, *Chaotic dynamics of semiconductor lasers with phase-conjugate feedback*, Phys. Rev. A **49**, 2096 (1994).
10. N. Cyr, M. Breton, M. Tetu, and S. Theriault, *Laser-diode frequency control by resonant phase-conjugate reflection from an atomic vapor*, Opt. Lett **16**, 1298 (1991).
11. T. Shimura, M. Tamura, and K. Kuroda, *Injection locking and mode switching of a diode laser with a double phase-conjugate mirror*, Opt. Lett. **18**, 1645 (1993).

12. S. Mailhot and N. McCarthy, *Influence of phase conjugate optical feedback on the emission properties of visible low-power diode lasers*, Can. J. Phys. **71**, 429 (1993).

13. G. P. Agrawal and J. T. Klaus, *Effect of phase-conjugate feedback on semiconductor laser dynamics*, Opt. Lett. **16**, 1325 (1991).

14. G.H.M. van Tartwijk, D. Lenstra, *Semiconductor lasers with optical injection and feedback*, Quant. Semiclass. Opt. **7**, 87 (1995).

15. D. H. DeTienne, G. R. Gray, G. P. Agrawal, and D. Lenstra, *Semiconductor laser coupled to a finite-response time phase-conjugate mirror*, Proc. SPIE **2693**, 689 (1996).

16. B. Tromborg, J.H. Osmundsen, and H. Olesen, *Stability Analysis for a Semiconductor Laser in an External Cavity*, IEEE J. Quantum Electron. **QE-20**(9), 1023 (1984).

17. B. Krauskopf, G.R. Gray, and D. Lenstra *Bifurcations in a semiconductor laser with phase-conjugate feedback*, Proceedings of SPIE **3283**, 510 (1998)

18. B. Krauskopf, G.R. Gray, and D. Lenstra *Semiconductor laser with phase-conjugate feedback: Dynamics and bifurcations* Phys. Rev. E **58**(6), 7190 (1998).

19. G. P. Agrawal and N. K. Dutta, *Long-wavelength Semiconductor Lasers*, 2nd ed. (Van Nostrand Reinhold, New York, 1993).

20. C. O. Weiss and R. Vilaseca, *Dynamics of Lasers*, (Weinheim, New York, 1991).

21. H. Haken, *Light Vol. 2*, (North-Holland, Amsterdam, 1985).

22. G. H. M. van Tartwijk and G. P. Agrawal, *Laser instabilities: a modern perspective*, Progr. Quantum Electron. **22**, 43-122 (1998)

23. G. H. M. van Tartwijk and D. Lenstra, *Semiconductor lasers with optical injection and feedback*, Quantum Semiclass. Opt. **7**, 87 (1995).

24. J. Hale and S.M. Verduyn Lunel *Functional Differential Equations*, Applied Mathematical Sciences **99**, Springer 1993

25. O. Diekmann, S.A.van Gils, S.M.V. Lunel, and H.-O. Walther *Delay Equations*, Applied Mathematical Sciences **110**, Springer 1995

26. C. Masoller and N. B. Abraham, *Stability and dynamical properties of the coexisting attractors of an external-cavity semiconductor laser*, Phys. Rev. A **57**, 1313 (1998).

27. M. Golubitsky and D.G. Schaeffer *Singularities and Groups in Bifurcation Theory I* Applied Mathematical Sciences **51**, Springer 1985

28. M. Golubitsky, I. Stuart, and D.G. Schaeffer *Singularities and Groups in Bifurcation Theory II*, Applied Mathematical Sciences **69**, Springer 1988

29. D. Lenstra, M. van Vaalen, and B. Jaskorzyńska, *On the theory of a single-mode laser with optical feedback*, Physica C **125**, 255 (1984).

30. A. Ritter and H. Haug, *Theroy of laser diodes with waek otical feedback. I. Small-signal analysis and side-mode spectra*, J. Opt. Soc. Am. B **10**(1), 130 (1993).