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Structured pseudospectra for matrix functions

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Abstract

In this paper we introduce structured pseudospectra for analytic matrix functions and derive computable formulae. The results are applied to the sensitivity analysis of the eigenvalues of a second-order system arising from structural dynamics and of a time-delay system arising from laser physics. In the former case, a comparison is made with the results obtained in the framework of random eigenvalue problems.

Key words: pseudospectra, robustness, structured singular values

1 Introduction

Pseudospectra have recently found application in analysing the sensitivity of eigenvalues of a system [6,14]. Principally, pseudospectra are sets in the complex plane to which the eigenvalues of a system can be shifted, under a random perturbation of a given size. In this way, one can classify the degree of sensitivity of the system's eigenvalues. Moreover, for robust stability, the pseudospectra identify the minimum size of a random perturbation required to shift an eigenvalue such that stability is lost. In this case, one may directly compare the size of the perturbation with the stability radius of the system [9].

Mathematically, in the simplest setting, given a matrix $A \in \mathbb{C}^{n \times n}$ one can investigate the sensitivity of its eigenvalues under additive perturbations by considering the pseudospectra (or spectral value sets)

$$\Lambda_\varepsilon(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + P) \text{ for some } P \in \mathbb{C}^{n \times n} \text{ with } \|P\| < \varepsilon\}$$
$$= \{\lambda \in \mathbb{C} : \left\|(A - \lambda I_n)^{-1}\right\| > 1/\varepsilon\},$$

where I_n denotes the $n \times n$-identity matrix [13,14].

In a number of problems the matrix A has a certain structure, for example, a block-structure, which should be respected in the sensitivity analysis. For this, perturbations of the form $A + DPE$ are considered in Ref. [5], where the fixed matrices D and E describe the perturbation structure and P is a complex perturbation matrix . This approach has been further developed in Ref. [15] for perturbations of the form $A + \sum D_i P_i E_i$, which, in particular, allow one to deal with element-wise perturbations.

On the other hand, specific classes of systems, like higher order systems or systems with time-delays, lead to the study of the zeros of *matrix functions* of the form

$$F(\lambda) := \sum_{i=1}^{m} A_i p_i(\lambda), \tag{1}$$

where $p_i, \ i = 1, \ldots, m$, are entire functions. For example, the characteristic matrix of the second order system $A_3 \ddot{x}(t) + A_2 \dot{x}(t) + A_1 x(t) = 0$ is given by $A_3 \lambda^2 + A_2 \lambda + A_1$ and the characteristic matrix of the time-delay system $\dot{x}(t) = A_1 x(t) + A_2 x(t - \tau)$ by $\lambda I - A_1 - A_2 e^{-\lambda \tau}$. Although such systems can usually be rewritten in a first order form, it is advantageous to exploit the structure of the governing equation. Pseudospectra for polynomials matrices were introduced in Ref. [12]. A general theory for matrix functions of the form (1) has been presented in Ref. [9]. The latter reference deals with the distribution of zeroes of $\sum_{i=1}^{m}(A_i + \delta A_i) p_i(\lambda)$, where the δA_i are complex, unstructured perturbation matrices, and a suitable joint norm for these perturbation matrices is used in the definitions of pseudospectra.

The goal of this study is to combine the above two approaches for exploiting a system's structure. In light of this, we define pseudospectra for the matrix function (1) and derive computable formulae, where, in addition to exploiting the form of the matrix function, a particular structure can be imposed on the perturbations of the individual coefficient matrices A_i. The motivation stems from the fact that in a lot of applications the coefficient matrices have a certain structure that should be respected in a sensitivity analysis, as unstructured perturbations may lead to irrelevant or non-physical effects. An example is discussed in [3], where the emergence of unbounded pseudospectra of a delay system in certain directions is explained by non-physical perturbations that destroy an intrinsic property, namely the singular nature, of one of the coefficient matrices. Other motivating examples from application areas will be discussed in Section 3.

The main mathematical tool to arrive at computable formulae is a reformulation of the sensitivity problem in terms of *structured singular values* (ssv). See the appendix, or Ref. [6,10] for more details. For a broad class of perturbation structures a general computable expression for the corresponding pseudospectra is derived. This involves the calculation of appropriately defined structured singular values. It is outlined in which cases such ssv can be computed exactly or how bounds can be derived otherwise. Next, it is illustrated how relaxing the perturbation structure may lead to exact and more efficient computable formulae, by following the approach of Ref. [9]. This allows one to weigh the advantages of imposing structure versus computational complexity, which is relevant from an application point of view.

The structure of the paper is as follows. In Section 2 structured pseudospectra for matrix functions are defined and computable formulae are derived. Section 3 describes practical applications from structural mechanics and laser physics. Section 4 contains the conclusions. The appendix is devoted to some background material on the structured singular value.

2 Structured pseudospectra for matrix functions

Following the work of Ref. [9], we are interested in general matrix functions of the form (1), where $A_i \in \mathbb{C}^{n \times n}$ and $p_i : \mathbb{C} \to \mathbb{C}$ is an entire function, for all $i = 1, \ldots, m$. In what follows, we call $F(\lambda)$ the *characteristic matrix* and refer to the zeros of $\det(F(\lambda)) = 0$ as the *eigenvalues* of F. We denote the *spectrum* of F as

$$\Lambda := \{\lambda \in \mathbb{C} : \ \det(F(\lambda)) = 0\} . \tag{2}$$

A definition for the ε-*pseudospectrum* of the matrix function (1) is given in Ref. [9] as

$$\Lambda_\varepsilon(F) := \left\{\lambda \in \mathbb{C} : \ \det\left(\sum_{i=1}^m (A_i + \delta A_i) p_i(\lambda)\right) = 0, \text{ for some } \delta A_i \in \mathbb{C}^{n \times n}\right.$$
$$\left. \text{with } w_i \|\delta A_i\|_2 < \varepsilon, \ 1 \leq i \leq m\right\}, \tag{3}$$

where $w_i > 0$ are weights and $\| \cdot \|_2$ denotes the 2-norm of a matrix. Denoting the largest singular value of a matrix by $\bar{\sigma}$ we have $\| \cdot \|_2 = \bar{\sigma}(\cdot)$. We observe that the perturbations δA_i considered in (3) lead to an additive uncertainty on the characteristic matrix (1) given by

$$\delta F(\lambda) := \sum_{j=1}^m \delta A_j \ p_j(\lambda). \tag{4}$$

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Although the structure of the expression (1) is explicitly taken into account in the definition (3), the perturbations δA_i applied to the different matrices A_i are unstructured. In other words, the element-wise structure of A_i is not taken into account when using the corresponding perturbation δA_i.

The goal of this section is to present a framework for the definition and computation of pseudospectra, in which various types of structure on the perturbation matrices can also be imposed. For this, we assume a more general additive uncertainty on (1) than what (4) allows. This uncertainty takes the form:

$$\delta F(\lambda) := \sum_{j=1}^{f} D_j(\lambda)\Delta_j E_j(\lambda) + \sum_{j=1}^{s} d_j G_j(\lambda) H_j(\lambda). \tag{5}$$

In this expression $\Delta_j \in \mathbb{C}^{k_j \times k_j}$ and $d_j \in \mathbb{C}$, denote the underlying unstructured perturbations, and $D_j \in \mathbb{C}^{n \times k_j}$, $E_j \in \mathbb{C}^{k_j \times n}$, $G_j \in \mathbb{C}^{n \times l_j}$ and $H_j \in \mathbb{C}^{l_j \times n}$ are appropriate shape matrices, whose elements are entire functions. We further assume that $l_j \geq 2$ and that G_j has full column rank, for all $j = 1, \ldots, s$. The structured ε-pseudospectrum $\Lambda_\varepsilon^s(F)$ of F with respect to the uncertainty (5) can then be defined as follows:

$$\Lambda_\varepsilon^s(F) := \{\lambda \in \mathbb{C} : \det(F(\lambda) + \delta F(\lambda)) = 0, \text{ for some } \delta F \text{ of the form (5)}$$
$$\text{with } \|\Delta_j\|_2 < \varepsilon, \ 1 \leq j \leq f \text{ and } |d_j| < \varepsilon, \ 1 \leq j \leq s\}. \tag{6}$$

To arrive at computational formulae for Λ_s^ε we reformulate (6) in terms of *structured singular values*; see the Appendix for a short introduction. This leads to the following general result:

Theorem 1 *Considering the characteristic matrix (1) with additive uncertainty (5). We define the uncertainty set $\boldsymbol{\Delta}$ as*

$$\boldsymbol{\Delta} := \Big\{ \mathrm{diag}(\Delta_1, \ldots, \Delta_f, d_1 I_{l_1}, \ldots, d_s I_{l_s}) : \ \Delta_i \in \mathbb{C}^{k_i \times k_i}, \ d_j \in \mathbb{C}, \tag{7}$$
$$1 \leq i \leq f, \ 1 \leq j \leq s\},$$

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where $\mathrm{diag}(\cdot)$ *represents a block diagonal matrix, and let*

$$
T(\lambda) := \begin{bmatrix} E_1(\lambda) \\ \vdots \\ E_f(\lambda) \\ H_1(\lambda) \\ \vdots \\ H_s(\lambda) \end{bmatrix} F(\lambda)^{-1} \left[D_1(\lambda) \cdots D_f(\lambda)\; G_1(\lambda) \cdots G_s(\lambda) \right]. \tag{8}
$$

Then

$$
\Lambda_\varepsilon^s(F) = \Lambda \cup \left\{ \lambda \in \mathbb{C} : \; \mu_{\boldsymbol{\Delta}}(T(\lambda)) > \frac{1}{\varepsilon} \right\}, \tag{9}
$$

where $\mu_{\boldsymbol{\Delta}}(\cdot)$ is the structured singular value with respect to the uncertainty set (8).

Proof: If $\det(F(\lambda)) \neq 0$ we have the following equivalence

$$\det(F(\lambda) + \delta F(\lambda)) = 0$$

$\Updownarrow$

$$
\det\left(I + F(\lambda)^{-1} \left[D_1(\lambda) \cdots D_f(\lambda)\; G_1(\lambda) \cdots G_s(\lambda) \right] \underline{\Delta} \begin{bmatrix} E_1(\lambda) \\ \vdots \\ E_f(\lambda) \\ H_1(\lambda) \\ \vdots \\ H_s(\lambda) \end{bmatrix} \right) = 0 \tag{10}
$$

$\Updownarrow$

$$\det\left(I + T(\lambda)\, \underline{\Delta} \right) = 0,$$

for some matrix $\underline{\Delta} = \mathrm{diag}(\Delta_1, \ldots, \Delta_f, d_1 I, \ldots, d_s I) \in \boldsymbol{\Delta}$.

Furthermore,

$$
\begin{aligned}
&\|\underline{\Delta}\|_2 < \varepsilon \\
&\Leftrightarrow \|\Delta_j\|_2 < \varepsilon,\; 1 \leq j \leq f \text{ and } |d_j| < \varepsilon,\; 1 \leq j \leq s.
\end{aligned} \tag{11}
$$

Considering (10) and (11) with the definition of Λ^s_ε, it follows that if $\lambda \in \Lambda^s_\varepsilon$, then either $\lambda \in \Lambda$ or the following holds:

$$\exists \underline{\Delta} \in \boldsymbol{\Delta} \text{ with } \|\underline{\Delta}\|_2 < \varepsilon, \text{ such that } \det(I + T(\lambda)\underline{\Delta}) = 0$$

Hence,

$$\min\{\|\Delta\|_2 : \ \Delta \in \boldsymbol{\Delta} \text{ and } \det(I + T(\lambda)\Delta) = 0\} < \varepsilon,$$

which implies $\mu_{\boldsymbol{\Delta}}(T(\lambda)) > \varepsilon^{-1}$. $\hfill\square$

Subsequently, from (9) the boundaries of structured ε-pseudospectra can be determined as level sets of the function

$$\mu_{\boldsymbol{\Delta}}(T(\lambda)), \quad \lambda \in \mathbb{C}. \tag{12}$$

In general the ssv of a matrix with respect to the uncertainty set (8) cannot be computed exactly. However, lower and upper bounds on the ssv can be obtained by solving eigenvalue optimisation problems. These bounds are sharp in many cases. If the additional restriction $f + 2s \leq 3$ holds for the uncertainty set (8), then an exact computation of $\mu_{\boldsymbol{\Delta}}(\cdot)$ is always possible; see the Appendix, Refs. [10,16] and the references therein. In some cases the particular structure of $T(\lambda)$ can be exploited when evaluating (12). This is illustrated with the following result, which slightly generalises one of the assertions of Theorem 1 of Ref. [9] and is also related to Prop. 3.4 of Ref. [11]:

Proposition 2 *We consider the characteristic matrix (1) with uncertainty (5). Furthermore, we assume that $s = 0$, and that there exist analytic matrix functions D and E and functions $q_j : \ \mathbb{C} \to \mathbb{C}$ such that*

$$D_j(\lambda) = D(\lambda),$$
$$E_j(\lambda) = E(\lambda) \, q_j(\lambda), \quad 1 \leq j \leq f.$$

By defining $T(\lambda)$ and $\boldsymbol{\Delta}$ as in Theorem 1, the following holds:

$$\mu_{\boldsymbol{\Delta}}(T(\lambda)) = \left\|E(\lambda)F^{-1}(\lambda)D(\lambda)\right\|_2 \left(\sum_{j=1}^f |q_j(\lambda)|\right). \tag{13}$$

Proof: If $\det(F(\lambda)) \neq 0$, then

$$\det(F(\lambda) + \delta F(\lambda)) = 0$$
$$\Leftrightarrow \det\left(I + E(\lambda)F(\lambda)^{-1}D(\lambda) \sum_{j=1}^f \Delta_j q_j(\lambda)\right) = 0, \tag{14}$$

and we can proceed as in the proof of Ref. [9, Theorem 1]. $\hfill\square$

We note that, in addition to the availability of a directly computable formula, the dimensions of $E(\lambda)F^{-1}(\lambda)D(\lambda)$ are f times smaller than the dimensions of $T(\lambda)$. This is one of the main contributions of the approach of Ref. [9].

To conclude this section, we detail how different types of perturbations can be written as an additive uncertainty on (1) of the form (5), where illustrative examples are given in the next section.

- Let $s = 0$, $D_j(\lambda) = D_j$, and $E_j(\lambda) = \sum_{i=1}^m E_{ij}\, p_i(\lambda)$ in (5), where D_i and E_{ij} are constant matrices. Then the perturbed characteristic matrix (1) and (5) reduces to

$$\sum_{i=1}^m \left(A_i + \sum_{j=1}^f D_j \Delta_j E_{ij} \right) p_i(\lambda). \tag{15}$$

  This corresponds to the perturbation structure used in Ref. [11] in the context of stability radii for polynomial matrices. If, in addition, $f = m$, $E_{ij} = 0$ for $i \neq j$ and D_j and E_{jj} are multiples of the unity matrix, then the unstructured case considered in Ref. [9] is obtained. The shape matrices D_j and E_{ij} in (15) can be used to perturb only a sub-matrix of A_i, to assign weights to perturbations of rows, columns or elements of each A_i, and to weight the perturbations applied to the matrices $A_1, \ldots, A_m$ with respect to each other. $i = 1, \ldots, m$,
- Assume that the characteristic matrix of an uncertain system is given by $\sum_{i=1}^m \tilde{A}_i p_i(\lambda)$, where the matrices $\tilde{A}_i$ linearly depend on a number of uncertain scalar parameters, say

$$\tilde{A}_i = A_i + \sum_j \theta_j P_{ij},$$

  with $\theta_j \in \mathbb{C}$ describing the uncertainties on these parameters. Furthermore, assume that we wish to investigate the possible positions of the eigenvalues when $|\theta_j| \leq \varepsilon$, $\forall j$. It follows that we are in the framework of (1), (5) and (6), as we can express

$$\begin{aligned}
\sum_{i=1}^m \tilde{A}_i p_i(\lambda) &= F(\lambda) + \sum_j \theta_j \left(\sum_{i=1, P_{ij} \neq 0}^m U_{ij} V_{ij}^*\, p_i(\lambda) \right) \\
&= F(\lambda) + \sum_j \theta_j \left[\cdots U_{ij} \cdots \right] \left[\cdots V_{ij}\, \bar{p}_i(\lambda) \cdots \right]^*,
\end{aligned} \tag{16}$$

  where each U_{ij} has full column rank and U_{ij} and V_{ij} can be computed for instance from a singular value decomposition of P_{ij}. Notice that (16) leads to $s > 0$ in the general expression (5) if and only if one of the matrices P_{ij} has rank larger than one, or if one of the parameters explicitly appears in different matrices $\tilde{A}_i$.

  Furthermore, weighted combinations of uncertain scalar parameters and matrix valued perturbations can be considered, provided the characteristic matrix depends linearly on the uncertainty.

- Finally, we observe that a nonlinear dependence on the uncertainty can sometimes be removed by a model transformation. As an illustration, the uncertain system

$$\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)(C + \delta C)x(t - \tau)$$

can be rewritten in a descriptor form as

$$\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)y(t),$$
$$0 = (C + \delta C)x(t - \tau) - y(t).$$

It has a nominal characteristic matrix

$$F(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ Ce^{-\lambda\tau} & -I \end{bmatrix},$$

to which we may apply structured perturbations.

It is worthwhile to mention that from a conceptual point of view it is possible to further refine the structure of the allowable perturbations (5) and to characterise the resulting pseudospectra using appropriately defined structured singular values as in Theorem 1. For example, an extension to uncertainty sets which include repeated non-scalar blocks, non-rectangular blocks, or only real elements might be of interest in applications. From a computational point of view, however, such a transformation to a structured singular value problem makes sense only if the corresponding structured singular value can be computed or well approximated. In light of this, the choice of (5) stems from a trade-off between both the generality of the matrix function (1) and the extend to which structure can be imposed on the uncertainty, and the availability and effectiveness of computational schemes.

3   Applications

We now use the theory developed in Section 2 to analyse the sensitivity of eigenvalues in two physical systems. The first example, from structural dynamics, is of an undamped spring-mass system [1]. This leads to studying structured pseudospectra of a second order system. Our second example, from laser physics, is of a semiconductor laser subject to optical feedback [7], leading to a study of structured pseudospectra of delay differential equations.

3.1 An example from structural dynamics

In Ref. [1] the effect of random perturbations on the eigenvalues of a second-order system is studied. The authors consider the three degrees of freedom undamped spring-mass system shown in Fig. 1.

It is described by the second-order differential equation

$$M\ddot{x}(t) + Kx(t) = 0, \tag{17}$$

where the mass matrix M and the stiffness matrix K have the following structure:

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_2 + k_4 + k_5 & -k_5 \\ -k_6 & -k_5 & k_3 + k_5 + k_6 \end{bmatrix}.$$

In this example, we assume that all mass and stiffness parameters, m_i and k_i, are constant but uncertain. Specifically,

$$\begin{aligned} m_i &= \bar{m}_i(1 + \epsilon_m x_i), \quad i = 1, \ldots, 3 \\ k_i &= \bar{k}_i(1 + \epsilon_k x_{i+3}), \quad i = 1, \ldots, 6, \end{aligned} \tag{18}$$

where $\bar{m}_i$ and $\bar{k}_i$ are the expected values and x_i are complex random variables, whose real and imaginary parts are uncorrelated Gaussian random variables with zero mean and standard deviation one. In the numerical experiments that follow, the parameter values are taken as $\bar{m}_i = 1$, $i = 1, \ldots, 3$, $\bar{k}_i = 1$, $i = 1, \ldots 5$, $\bar{k}_6 = 1.275$ and the degree of uncertainty is described by

$$\epsilon_m = \epsilon_k = 0.15;$$

see the second example of Ref. [1]. The eigenvalues of (17) are the zeros of the random matrix polynomial $P(\lambda) := M\lambda^2 + K$. The characteristic matrix, obtained by taking the expectation of the parameters,

$$P_0(\lambda) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 3.275 & -1 & -1.275 \\ -1 & 3 & -1 \\ -1.275 & -1 & 3.275 \end{bmatrix}, \tag{19}$$

has eigenvalues

$$\lambda_{\pm 1} = \pm i, \quad \lambda_{\pm 2} = \pm 2i, \quad \lambda_{\pm 3} = \pm 2.1331 i.$$

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To investigate the effect of the uncertainty on the parameters given by (18) we first perform Monte Carlo simulations. The eigenvalues of 2000 simulations are shown in Fig. 2. The eigenvalues $\lambda_{\pm 2}$ and $\lambda_{\pm 3}$ appear to be most sensitive to perturbation. Furthermore, a clear separation between the perturbations of $\lambda_{\pm 2}$ and $\lambda_{\pm 3}$ cannot be observed.

We now perform a rigorous sensitivity analysis using structured pseudospectra. Starting from the characteristic matrix (19), we express all uncertainty as an additive perturbation of the form (5), as follows:

$$
\delta P(\lambda) = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{D_1(\lambda)} \delta m_1 \underbrace{[1\ 0\ 0]\lambda^2}_{E_1(\lambda)} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta m_2 [0\ 1\ 0]\lambda^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta m_3 [0\ 0\ 1]\lambda^2
$$

$$
+ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta k_1 [1\ 0\ 0] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta k_2 [0\ 1\ 0] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta k_3 [0\ 0\ 1]
$$

$$
+ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \delta k_4 [1\ -1\ 0] + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \delta k_5 [0\ 1\ -1] + \underbrace{\begin{bmatrix} 1.275 \\ 0 \\ -1.275 \end{bmatrix}}_{D_9(\lambda)} \delta k_6 \underbrace{[1\ 0\ -1]}_{E_9(\lambda)}
$$

Observe that the weights entering the shape matrices D_i and E_i are chosen according to the distribution (18). In this way pseudospectra can be computed from Theorem 1, where $\boldsymbol{\Delta}$ reduces to the set of complex 9×9 diagonal matrices and

$$
T(\lambda) = \begin{bmatrix} \lambda^2 I_3 \\ \hline I_3 \\ \hline \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{matrix} \end{bmatrix} P_0(\lambda)^{-1} \left[I_3 \middle| I_3 \middle| \begin{matrix} 1 & 0 & 1.275 \\ -1 & 1 & 0 \\ 0 & -1 & -1.275 \end{matrix} \right]. \tag{20}
$$

The computation of the structured pseudospectra is performed using the MATLAB Routine `mussv`, contained in the Robust Control Toolbox, [8]. We compute $\mu_{\boldsymbol{\Delta}}(T(\cdot))$ on a 300×300 grid over a region of the complex plane. A

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contour plot then yields the boundaries of the structured pseudospectra. Note that, for the perturbation structure under consideration, only upper and lower bounds on the structure singular value can be computed. Along the grid the maximum relative difference between the bounds, obtained by the function `mussv`, is of order 10^{-3}.

Figure 3(a) shows the boundaries of structured ε-pseudospectra for $\varepsilon/0.15 = 10^{-1.5}$, 10^{-1}, $10^{-0.5}$, 1, and $10^{0.5}$. We find a good qualitative agreement with the simulations, in the sense that the eigenvalues furthest from the real axis are the most sensitive to perturbation.

To illustrate the importance of taking the structure of the perturbations into account, let us compare the results with unstructured pseudospectra of P_0 in the sense of Ref. [12]. This corresponds to definition (3). The weights of the perturbations of M and K were chosen as the 2-norm of the matrices obtained by taking the standard deviation element-wise, namely $w_M = 1/0.15$ and $w_K = 1/0.8081$. The contours of the computed pseudospectra Λ_ε are shown in Fig. 3(b), for $\varepsilon/0.15 = 10^{-1.5}$, 10^{-1}, $10^{-0.5}$, and 1. In contrast to Fig. 3(a) and the simulation results shown in Fig. 2, the eigenvalues closest to the real axis appear as the most sensitive. This indicates that unstructured pseudospectra do not adequately describe the sensitivity of eigenvalues in this problem.

Finally, we interpret the structured pseudospectra in a quantitative way by relating the corresponding ε-values with the uncertainty measures $\epsilon_{m,k}$ in (18). In particular, the $\varepsilon = 0.15$ contour fits well with the simulation results shown in Fig. 2 (for $\epsilon_{m,k} = 0.15$). This correspondence is again illustrated in Fig. 4 (a), where we display both the pseudospectrum contour for $\varepsilon = 0.15$ and the eigenvalues of 2000 random simulations. Thus indicating that for the system under consideration the relation $\varepsilon = \epsilon_{m,k}$ leads to a good qualitative and quantitative agreement between both approaches. Note that ϵ_m and ϵ_k are the standard deviation of the normalized uncertain parameters, which have a Gaussian distribution, whereas ε bounds the allowable perturbations on the mean values of these parameters in the definition of the ε-pseudospectrum. This explains why some eigenvalues lie outside the pseudospectrum contour in Fig. 4 (a). For comparison, Fig. 4 (b) shows the boundary of the $\varepsilon = 0.15$-pseudospectrum and the results of 2000 simulations, where it is assumed that m_i and k_i satisfy (18) but with the x_i being uniformly distributed over the complex unit circle. All the eigenvalues obtained from the simulations are now inside the pseudospectrum contour, as expected. Observe also that the pseudospectrum contour is hardly approached. As pseudospectrum contours are related to a *worst-case* behaviour of the eigenvalues subjected to bounded perturbations, it seems unlikely to generate perturbations that push eigenvalues close to the boundary. Such an observation has also been made in Ref. [13].

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In Ref. [3] pseudospectra have been applied to the analysis of the robust stability of a model for a semiconductor laser subject to optical feedback. For certain fixed model parameters, the problem leads to the study of the delay differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1), \tag{21}$$

where

$$A_0 = \begin{bmatrix} -0.84982 & 0.14790 & 44.373 \\ 0.0037555 & -0.28049 & -229.23 \\ -0.17537 & 0.022958 & -0.36079 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.28 & 0 & 0 \\ 0 & -0.28 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

The stability of the zero solution of (21) is inferred from the eigenvalues, which are the zeros of the characteristic matrix

$$F(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda}. \tag{23}$$

As a characteristic of delay equations of retarded type, there are infinitely many eigenvalues, yet the number of eigenvalues in any right-half plane is finite, [4]. Figure 5 shows the rightmost eigenvalues of (21)-(22), computed with the software package `DDE-BIFTOOL` [2]. Notice the typical shape with a *tail* of eigenvalues to the left.

In this example we investigate the effect which an uncertainty on specific elements of A_0 and A_1 has on the eigenvalues by computing structured pseudospectra. From physical considerations an important requirement on the uncertainty is that in A_1 only the elements on positions (1,1) and (2,2) are nonzero and remain opposite to each other. Physically, these elements describe the feedback process of the laser; see Ref. [7] for full details. We can take this structure into account by considering perturbations on A_1 of the form $\text{diag}(\delta a, -\delta a, 0)$, with $\delta a \in \mathbb{C}$, in addition to unstructured perturbations on A_0. The resulting additive uncertainty on F has the general form (5), namely

$$\delta F(\lambda) = \underbrace{-I_3}_{D_1(\lambda)} \delta A_0 \underbrace{I_3}_{E_1(\lambda)} + \delta a \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{G_1(\lambda)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{H_1(\lambda)} e^{-\lambda}. \tag{24}$$

An application of Theorem 1 yields

$$
\Lambda_\varepsilon^s(F) = \left\{ \lambda \in \mathbb{C} \ : \ \mu_{\boldsymbol{\Delta}} \left(\left[\begin{array}{c|cc} & I_3 & \\ \hline e^{-\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \end{array} \right] F(\lambda)^{-1} \left[\begin{array}{c|cc} & -1 & 0 \\ -I_3 & 0 & 1 \\ & 0 & 0 \end{array} \right] \right) > \frac{1}{\varepsilon} \right\},
$$

where $\boldsymbol{\Delta}$ is the set of complex block-diagonal 5×5 matrices with one full 3×3 block and one repeated scalar 2×2 block. For this type of uncertainty structure ($f = s = 1$), the structured singular value can be computed exactly as the solution of a convex optimisation problem; see the Appendix. We have once again combined the `mussv` routine of MATLAB with a contour plotter to visualise the structured pseudospectra and the results are shown in Fig. 6(a).

For comparison, unstructured pseudospectra of (23) in the sense of Ref. [3] are shown in Fig. 6(b). This corresponds to

$$
\delta F(\lambda) = \delta A_0 + \delta A_1 e^{-\lambda},
$$

where δA_0 and δA_1 are unstructured. This allows to combine Theorem 1 and Proposition 2 to:

$$
\Lambda_\varepsilon = \left\{ \lambda \in \mathbb{C} : \ \|F(\lambda)^{-1}\|_2 \left(1 + \left| e^{-\lambda} \right| \right) > \frac{1}{\varepsilon} \right\}.
$$

As a significant qualitative difference, the ε-pseudospectra stretch out infinitely far along the negative real axis, even for arbitrarily small values of ε. In Ref. [9, Section 3.3], this phenomenon is related to the behaviour of eigenvalues, which are introduced by perturbations that make the matrix A_1 nonsingular. Such perturbations are, however, non-physical and, as we have shown, can be excluded by applying the novel structured uncertainty (24).

4 Conclusions

We have presented a general theory for computing structured pseudospectra of analytic matrix functions. Our novel method allows one to direct perturbations to specific elements (or, indeed, groups of elements) of the individual matrices of a corresponding eigenvalue problem.

As an illustration, we first applied these methods to an example from structural dynamics. In this case the eigenvalue problem was of second-order. We showed how structured perturbations could be directly compared to probabilistic uncertainties on the parameters. The pseudospectra were used to derive

bounds on the position of the eigenvalues obtained through a computationally intensive Monte Carlo simulation.

Our second example involved an infinite-dimensional eigenvalue problem obtained from the modeling of a feedback laser using delay differential equations. Here, structured perturbations were applied in order to preserve the structure of the matrix associated with the delayed variable. Specifically, in the governing system this matrix was singular. With the structured approach we could allow physically realistic perturbations only, which have the property of maintaining the singularity of the matrix. This leads to pseudospectra which are quantitatively and qualitatively different from the case where unstructured perturbations are allowed. This stems from the fact that the latter generically increase the rank of the matrix.

A   The structured singular value

In this appendix, we introduce the concept of structured singular values of matrices and outline the main principles behind the standard computational schemes, based on the review paper [10] and Chapter 11 of Ref. [16].

A classical result from robust control theory, which lays the basis for the celebrated small gain theorem, relates the largest singular value $\bar{\sigma}(G)$ of a matrix $G \in \mathbb{C}^{N \times M}$ to the solutions of the equation

$$\det(I + G\Delta) = 0, \tag{25}$$

in the following way:

$$\bar{\sigma}(G) = \begin{cases} 0, & \text{if } \det(I + G\Delta) \neq 0, \ \forall \Delta \in \mathbb{C}^{M \times N}, \\ \left(\min \left\{ \bar{\sigma}(\Delta) : \ \Delta \in \mathbb{C}^{M \times N} \text{ and } \det(I + G\Delta) = 0 \right\} \right)^{-1}, & \text{otherwise.} \end{cases} \tag{26}$$

We refer to Δ as the 'uncertainty'. As in a robust control framework, (25) typically originates from a feedback interconnection of a nominal transfer function and an uncertainty block.

Next we reconsider the solutions of equation (25), where Δ is restricted to having a particular structure by imposing $\Delta \in \boldsymbol{\Delta}$, with $\boldsymbol{\Delta}$ a closed subset of $\mathbb{C}^{N \times N}$. In analogy with (26) one defines the *structured singular value* of the matrix G with respect to the uncertainty set $\boldsymbol{\Delta}$ as

$$\mu_{\boldsymbol{\Delta}}(G) := \begin{cases} 0, & \text{if } \det(I + G\Delta) \neq 0, \ \forall \Delta \in \boldsymbol{\Delta}, \\ \left(\min \left\{ \bar{\sigma}(\Delta) : \ \Delta \in \boldsymbol{\Delta} \text{ and } \det(I + G\Delta) = 0 \right\} \right)^{-1}, & \text{otherwise.} \end{cases}$$

In what follows we restrict ourselves for simplicity to square matrices, $G \in \mathbb{C}^{N \times N}$, and to uncertainty sets of the form (8), with $\sum_{i=1}^{f} k_i + \sum_{i=1}^{s} l_i = N$ (see the book [6] for a general theory). In this way, we always have

$$\rho(G) \leq \mu_{\mathbf{\Delta}}(G) \leq \bar{\sigma}(G), \tag{27}$$

where $\rho(\cdot)$ is the spectral radius. For this, we note that $\bar{\sigma}(G)$ equals the structured singular value corresponding to the least structured uncertainty set of the form (8) (1 full block, $f = 1, s = 0$) and that $\rho(G)$ equals the structured singular value corresponding to the most structured set (1 repeated diagonal block, $f = 0, s = 1$). With the sets $\mathcal{U}$ and $\mathcal{D}$ defined as

$$\mathcal{U} := \{U \in \mathbf{\Delta} : \ U^* U = I\},$$
$$\mathcal{D} := \left\{\mathrm{diag}(a_1 I_{k_1}, \ldots, a_f I_{k_f}, D_1, \ldots, D_s) : \ a_i > 0, \ D_i \in \mathbb{C}^{l_i \times l_i}, \ D_i^* = D_i > 0\right\},$$

the following invariance property holds:

$$\mu_{\mathbf{\Delta}}(G) = \mu_{\mathbf{\Delta}}(GU) = \mu_{\mathbf{\Delta}}(DGD^{-1}), \quad \forall D \in \mathcal{D}, \forall U \in \mathcal{U}. \tag{28}$$

Most computation schemes for $\mu_{\mathbf{\Delta}}$ rely on the fact that this invariance property is not generally valid for the functions $\rho(\cdot)$ and $\bar{\sigma}(\cdot)$, which can be exploited to tighten the bounds in (27). Namely, by combining (27) and (28) one obtains

$$\max_{U \in \mathcal{U}} \rho(GU) \leq \mu_{\mathbf{\Delta}}(G) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DGD^{-1}). \tag{29}$$

Therefore, optimisation algorithms can be used to compute improved estimates for $\mu_{\mathbf{\Delta}}$. Moreover, one can show that the lower bound in (29) is in fact an equality, that is,

$$\mu_{\mathbf{\Delta}}(G) = \max_{U \in \mathcal{U}} \rho(GU). \tag{30}$$

However, the objective function on the right-hand side of (30) may have several local maxima and, for this, a local optimisation algorithm may get stuck in a local maximum which is not global. On the other hand, the computation of the upper-bound in (29) can be recast into a standard *convex* optimisation problem. However, in general $\mu_{\mathbf{\Delta}}$ is not equal to the upper-bound. An exception to this holds if the number of blocks in the uncertainty set $\mathbf{\Delta}$ satisfies $f + 2s \leq 3$.

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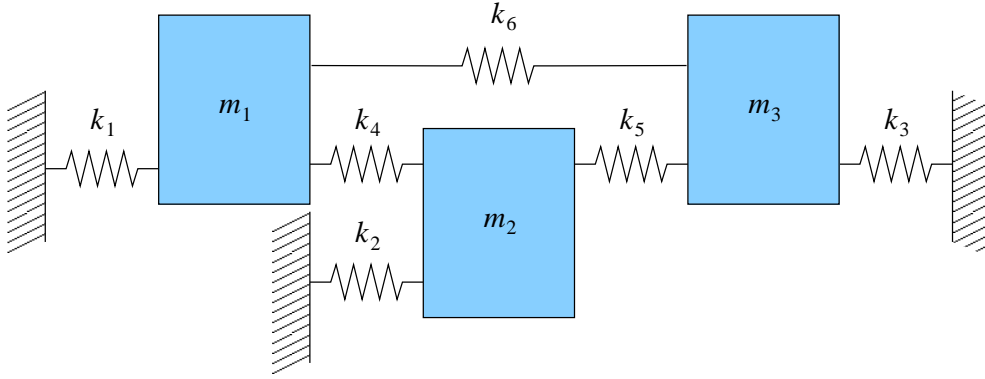
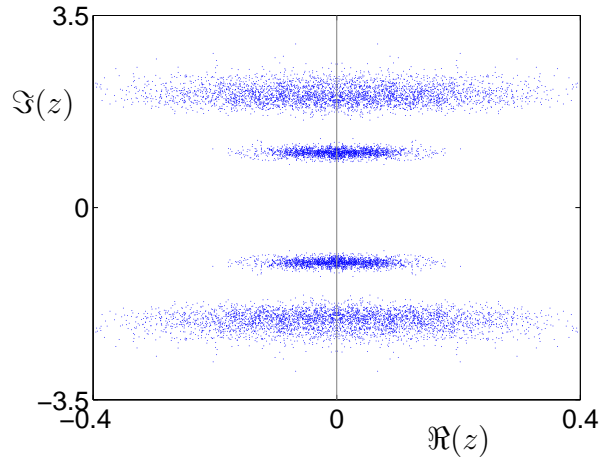Fig. 1. A three degrees-of-freedom spring-mass system, taken from Ref. [1].

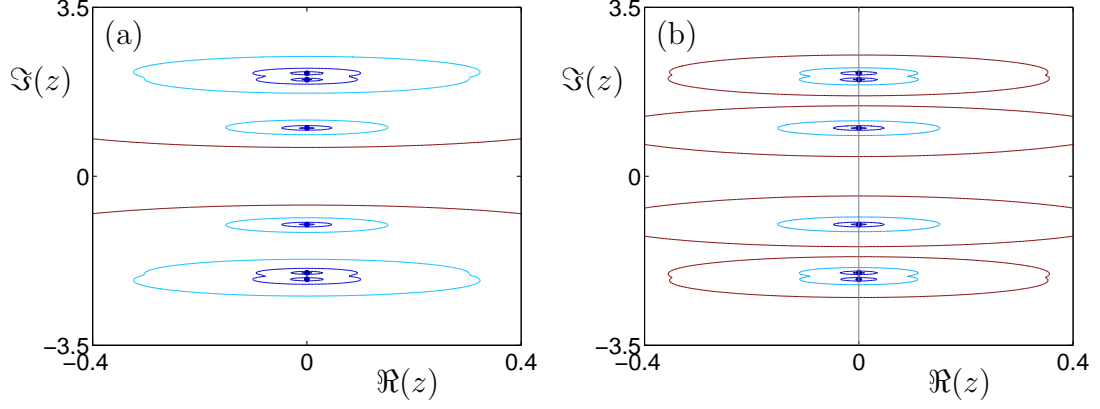Fig. 2. Eigenvalues of 2000 simulations of the random 2nd order system (17).

Fig. 3. Structured (a) and unstructured (b) pseudospectra of the matrix polynomial $M_0\lambda^2 + K_0$.
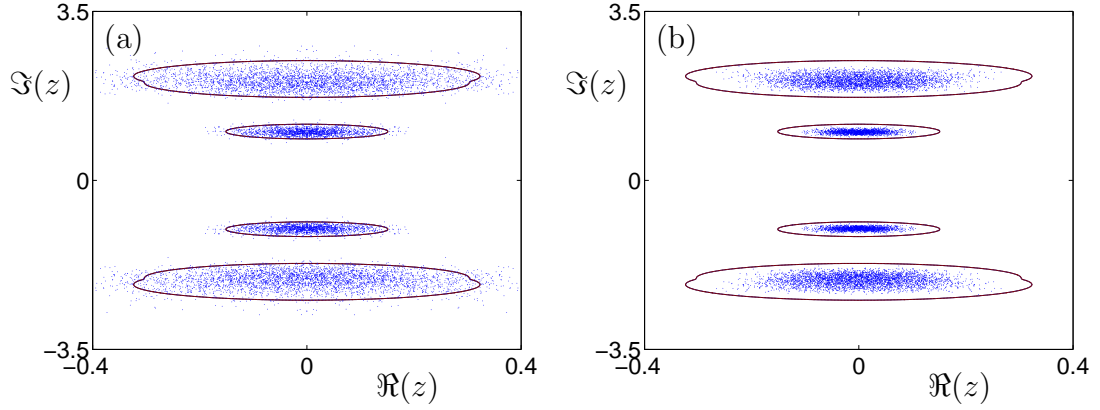

Fig. 4. Comparison of the structured pseudospectrum for $\varepsilon = 0.15$ and corresponding simulation results for normally distributed perturbations (a) and uniformly distributed perturbations (b) (see text for details).
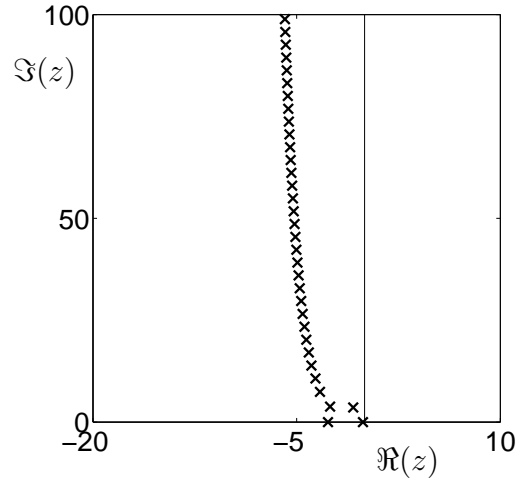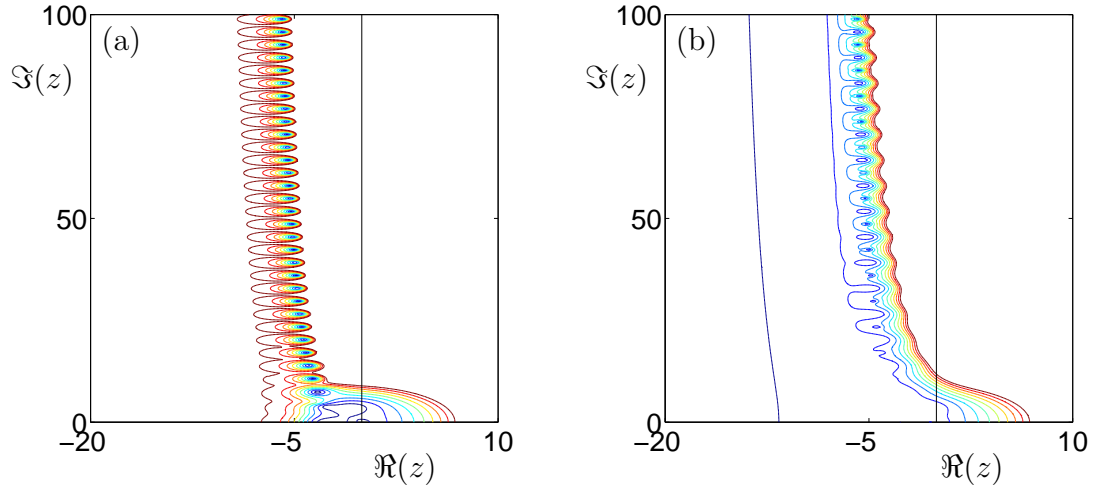
Fig. 5. Roots of (23) in the complex plane.

Fig. 6. Structured (a) and unstructured (b) pseudospectra of the delayed characteristic $F(\lambda)$, given by (23).

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