Patcharamaneepakorn, P., Armour, S., & Doufexi, A. (2012). On the equivalence between SLNR and MMSE precoding schemes with single-antenna receivers. IEEE Communications Letters, 16(7), 1034-1037. 10.1109/LCOMM.2012.050912.120329

On the Equivalence between SLNR and MMSE Precoding Schemes with Single-antenna Receivers

Piya Patcharamaneepakorn, Simon Armour, and Angela Doufexi, *Member, IEEE*

***Abstract*—This letter considers transmit precoding schemes based on the maximum signal-to-leakage-and-noise ratio (SLNR) in multiuser MIMO systems with single-antenna receivers. The closed-form solution of SLNR design is generally given in the form of the eigenvector associated to the maximum eigenvalue. In this letter, analytic expressions of SLNR-based solutions and resulting SLNR are derived for a generic power allocation (GPA). The solution is shown to be a function of user-allocated power and an arbitrary phase shift. Under equal power allocation (EPA), the SLNR precoding scheme is shown to be equivalent to the minimum mean square error (MMSE) precoding scheme. Several useful implications in terms of the possible extension of existing algorithms and performance analysis are also discussed.**

***Index Terms*—Multiuser MIMO, linear precoding, SLNR, MMSE, performance analysis.**

I. Introduction

MULTIUSER multiple input multiple output (MU-MIMO) schemes enable simultaneous multiplexing of multiuser data streams into the same frequency and time resources, yielding significant gain in system throughput. It is known that the theoretical sum capacity of MU-MIMO can be achieved by dirty paper coding (DPC) [1]. However, its implementation is hampered by nonlinear complexity. Linear precoding techniques are, therefore, often considered in practical MU-MIMO systems. These techniques include zero-forcing (ZF) [2], Block-Diagonalization (BD) [3], minimum mean square error (MMSE) [4] and maximum signal-to-leakage-and-noise ratio (SLNR) [5].

In this letter, the SLNR precoding scheme (SLNR-PS) in MU-MIMO systems with single-antenna users is considered. In contrast to [5] where the closed-form solution of the SLNR-PS is given in terms of the eigenvector associated to the maximum eigenvalue, an explicit formula of SLNR-based solutions for GPA is presented in this letter and is shown to be in a form of a regularised channel inverse [2] shifted by random phase-shifts. The regularisation factors are shown to be a function of user-allocated power to noise ratio. Under EPA, the SLNR-PS can be viewed as a phase-shifted, power-normalised version of the MMSE precoding scheme (MMSE-PS), resulting in the equivalence between both schemes under this power constraint. This leads to several useful implications in terms of performance analysis and the applications of techniques developed for MMSE to SLNR precoding schemes.

Notation: $(\cdot)^T$, $(\cdot)^H$ denote the transpose and Hermitian operations, respectively. $[\cdot]_{kk}$ denotes the (k,k) entry of a matrix. $\|\cdot\|$ is the vector norm and $Q(\cdot)$ is the Q-function.

Manuscript received February 16, 2012. The associate editor coordinating the review of this letter and approving it for publication was Z. Z. Lei.

The authors are with the Centre for Communications Research, University of Bristol, Bristol, BS8 1UB, UK (e-mail: {eezpp, Simon.Armour, A.Doufexi}@bristol.ac.uk).

Digital Object Identifier 00.0000/LCOMM.0000.000000.000000

II. System Model and Precoding Schemes

Consider a downlink MU-MIMO system having M transmit antennas and K single-antenna users. The transmitted signal is given by $\mathbf{x} = \mathbf{W}\mathbf{A}\mathbf{s}$, where $\mathbf{s} = [s_1, s_2, ..., s_K]^T$ is the overall data vector, $s_k \in \mathbb{C}$, $E\{|s_k|^2\} = 1$; $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_K]$ is the transmit precoding matrix, $\mathbf{w}_k \in \mathbb{C}^{M\times 1}$; and $\mathbf{A} = diag(a_1, ..., a_K)$ is the power normalisation matrix, such that $P_k = a_k^2\|\mathbf{w}_k\|^2$, $\sum_k P_k = P$. The composite channel matrix, given by $\mathbf{H}^H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_K]^H \in \mathbb{C}^{K\times M}$, has independent complex Gaussian elements $\mathcal{CN}(0,1)$. The additive noise n_k is an independent complex Gaussian variable with zero mean and equal variance for all users, i.e. $E\{n_k n_k^H\} = \sigma_k^2$, where $\sigma_k^2 = \sigma^2, \forall k$. The system signal to noise ratio (SNR) is defined as P/σ^2. The received signal at user k can be written as

$$y_k = \mathbf{h}_k^H \mathbf{W}\mathbf{A}\mathbf{s} + n_k. \quad (1)$$

User k's estimated data stream, after processing by the receiver filter $g_k^H \in \mathbb{C}$, is given by

$$\hat{s}_k = g_k^H y_k = g_k^H \mathbf{h}_k^H \mathbf{w}_k a_k s_k + g_k^H \mathbf{h}_k^H \sum_{j\neq k} \mathbf{w}_j a_j s_j + n_k. \quad (2)$$

A. MMSE Precoding Scheme

For the single-antenna receiver case, the transmit precoding matrix satisfying MMSE criteria is given by [2] [4]

$$\mathbf{W}_{mmse} = \mathbf{H}(\mathbf{H}^H\mathbf{H} + \frac{K\sigma^2}{P}\mathbf{I})^{-1}. \quad (3)$$

From (3), it can be shown that the MMSE beamforming vector for any user k can be written as

$$\mathbf{w}_{k,mmse} = (\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I})^{-1}\mathbf{h}_k. \quad (4)$$

The receiver filter is given by $g_k^H = 1$, $\forall k$.

B. SLNR Precoding Scheme

For SLNR-PS, the power normalisation is normally assumed such that $a_k = \sqrt{P_k}$ and $\mathbf{w}_k^H\mathbf{w}_k = 1$. The SLNR criterion leads to the following optimisation problem [5]:

$$\mathbf{w}_{k,slnr} = arg \max_{\mathbf{w}_k\in\mathbb{C}^M} \frac{\mathbf{w}_k^H\mathbf{h}_k\mathbf{h}_k^H\mathbf{w}_k}{\mathbf{w}_k^H\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\mathbf{w}_k + \frac{\sigma_k^2}{P_k}} \quad (5)$$
$$\text{subject to } \mathbf{w}_k^H\mathbf{w}_k = 1$$

where $\tilde{\mathbf{H}}_k = [\mathbf{h}_1, ..., \mathbf{h}_{k-1}, \mathbf{h}_{k+1}, ..., \mathbf{h}_K]$. The closed-form solution to (5) can be written as [5]

$$\mathbf{w}_{k,slnr} \propto \max .eigenvector\left(\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\mathbf{h}_k^H\right). \quad (6)$$

The resulting maximum SLNR value corresponds to the maximum eigenvalue, i.e. $slnr_k = \lambda_{max,k}$, and the matched filter, $g_k^H = \frac{\mathbf{w}_k^H\mathbf{h}_k}{\|\mathbf{w}_k^H\mathbf{h}_k\|}$, is deployed as the receiver filter [5].

III. An Explicit Expression of SLNR-based Solutions for Generic Power Allocation

From (6), an SLNR solution lies in the eigenspace associated to the maximum eigenvalue, i.e. any $\mathbf{v}_k$ that satisfies the following eigenvector equation:

$$\left(\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\mathbf{h}_k^H\right)\mathbf{v}_k = \lambda\mathbf{v}_k \qquad (7)$$

$$\lambda\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)\mathbf{v}_k = (\lambda+1)\mathbf{h}_k\mathbf{h}_k^H\mathbf{v}_k. \qquad (8)$$

Observe that, for a nonzero eigenvalue, a nonzero vector $\mathbf{v}_k$ satisfying (8) must not be orthogonal to $\mathbf{h}_k$, i.e. the inner product $\mathbf{h}_k^H\mathbf{v}_k$ results in a nonzero complex scalar, denoted as $\alpha_k e^{j\theta_k}$. Thus, (8) becomes

$$\begin{aligned}\mathbf{v}_k &= (\frac{\lambda+1}{\lambda})(\alpha_k e^{j\theta_k})\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k \\ &= \beta_k e^{j\theta_k}\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k;\ \beta_k = (\frac{\lambda+1}{\lambda})\alpha_k. \qquad (9)\end{aligned}$$

Since a multiplication of an eigenvector with any nonzero complex scalar leads to another eigenvector, (9) satisfies (7) for any arbitrary values of β_k and θ_k, where β_k is nonzero. By imposing the power normalisation constraint, a nonzero solution to (5) can be rewritten as

$$\mathbf{w}_k = \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|} = \frac{1}{\rho_k}e^{j\theta_k}\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k \qquad (10)$$

where $\rho_k = \|\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\|$ is a normalisation factor so that $\mathbf{w}_k^H\mathbf{w}_k = 1$, θ_k is arbitrary. The corresponding receiver filter can also be rewritten as

$$\begin{aligned}g_k^H = \frac{\mathbf{w}_k^H\mathbf{h}_k}{\|\mathbf{w}_k^H\mathbf{h}_k\|} &= e^{-j\theta_k}\frac{\mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k}{\|\mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\|} \\ &= e^{-j\theta_k}. \qquad (11)\end{aligned}$$

Note that the solution (10) is generally not unique. It is in a form of a regularised channel inverse [2], similar to the MMSE solution (4), but with different regularisation factors and additional phase-shifts. While the MMSE solution utilises the same factor equal to the inverse of average SNR ($K\sigma^2/P$) for all users, the SLNR solution sets each user's regularisation factor to the inverse of its individual SNR (σ_k^2/P_k). Arbitrary phase shifts introduced by the SLNR precoder have insignificant impact upon system performance as they are generally cancelled at the receiver. Thus, the solution of SLNR-PS can be chosen such that the receiver filter reduces to a *simple form*, i.e. choosen $\theta_k = 0$ so that $g_k^H = 1$, $\forall k$ (no post-processing is required at the receivers). This solution can be defined as the *basic solution*. It is unique (having zero phase-shifts) and can always be assumed for simplicity in performance analysis and practical implementations, without loss of generality.

The resulting maximum SLNR value $\lambda_{max,k}$ can also be determined from the characteristic equation:

$$\det\left[\lambda\mathbf{I} - \left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\mathbf{h}_k^H\right] = 0$$

$$\det\left[\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right) - (\frac{\lambda+1}{\lambda})\mathbf{h}_k\mathbf{h}_k^H\right] = 0. \qquad (12)$$

Since $\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)$ is invertible, $\det\left[\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right] \neq 0$. Let $\mathbf{A}^{-1} = \left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)$, $\mathbf{u} = -(\frac{\lambda+1}{\lambda})\mathbf{h}_k$, and $\mathbf{v}^H = \mathbf{h}_k^H$ and applying the Matrix Determinant Lemma: $\det(\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^H) = (1 + \mathbf{v}^H\mathbf{A}\mathbf{u})\cdot\det(\mathbf{A}^{-1})$ into (12) results in

$$\left[1 - \frac{\lambda+1}{\lambda}\mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\right] = 0. \qquad (13)$$

As the matrix $\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\mathbf{h}_k^H$ in (7) is rank one, it has only one nonzero eigenvalue that is the maximum SLNR value, which can be obtained from (13) as

$$\lambda_{max,k} = slnr_k = \frac{1}{1 - \mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k} - 1. \qquad (14)$$

IV. The Equivalence Between SLNR and MMSE Precoding Schemes, and its Implications

As shown in Section III, the user-pairwise solutions of SLNR-PS and MMSE-PS are typically different due to different regularisation factors and arbitrary phase-shifts. However, it can be easily seen that they lie in the same direction for every user's pair if and only if the EPA constraint is imposed (provided that all receivers have equal noise variance), i.e. $P_k = P/K$, $\forall k$. In this case, the SLNR beamforming vector and the resulting SLNR value can be given by

$$\begin{aligned}\mathbf{w}_{k,slnr} &= e^{j\theta_k}\frac{\left(\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-1}\mathbf{h}_k}{\|\left(\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-1}\mathbf{h}_k\|} \\ &= e^{j\theta_k}\frac{\mathbf{w}_{k,mmse}}{\|\mathbf{w}_{k,mmse}\|} \qquad (15)\end{aligned}$$

$$\begin{aligned}slnr_k &= \frac{1}{1 - \mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-1}\mathbf{h}_k} - 1 \qquad (16) \\ &= \frac{P}{K\sigma^2\left[\left(\mathbf{H}^H\mathbf{H} + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-1}\right]_{kk}} - 1. \qquad (17)\end{aligned}$$

Clearly seen from (15), under EPA, any solution of SLNR-PS can be obtained by arbitary phase-shifting and power-normalising of the MMSE solution. Due to power normalisation constraint, both schemes only differ by arbitrary phase-shifts. They are, therefore, equivalent under EPA. Similar results are also observed in an independent work [6], where the equivalence is shown by a verification that the MMSE solution (referred to as conventional regulated ZF (RZF) in [6]) is an eigenvector satisfying (7), i.e. a solution of the SLNR scheme. While only EPA is considered and no generic solutions are given in [6], this letter derives a generic form of SLNR-based solutions for GPA and asserts that SLNR-PS and MMSE-PS are equivalent if and only if EPA is imposed. Specifically, under EPA, the MMSE solution is shown to be one (the basic solution) of the infinite solutions of SLNR-PS.

Several useful implications can be drawn from this equivalent viewpoint, e.g. SLNR-PS has identical performance as

MMSE-PS, i.e. outperforming the ZF precoding scheme (ZF-PS), under EPA. Furthermore, previously developed algorithms and analysis of MMSE-PS are generally applicable for SLNR-PS under EPA. Some modifications may be required for GPA. To illustrate this idea, an extension of UL MMSE performance analysis [7] to SLNR-PS under GPA is given in the following subsections. Comparing to possible extensions of DL MMSE analysis (e.g. [2]), this approach provides a better insight of the performance of SLNR-PS with respect to ZF-PS.

A. SLNR analysis when $K \leq M$

Notice that the SLNR expression (17) corresponds to the signal-to-interference-and-noise ratio (SINR) of the UL MMSE equaliser with the composite channel matrix defined by $\mathbf{H}$ (see e.g. [8]), the SLNR maximisation problem in downlink can thus be equivalently viewed as the MMSE optimisation problem in virtual uplink under EPA. This motivates an extension of SINR analysis of UL MMSE [7] to SLNR analysis of SLNR-PS. For $K \leq M$ and GPA, the basic solution of (10) can be decomposed into two orthogonal vectors, i.e.

$$\mathbf{w}_k = \frac{1}{\nu_k}\left[\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k} + \frac{\sigma_k^2}{P_k}\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\right]\mathbf{h}_k. \tag{18}$$

Note that $\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k = \left[\mathbf{I} - \tilde{\mathbf{H}}_k\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k\right)^{-1}\tilde{\mathbf{H}}_k^H\right]\mathbf{h}_k$ is the orthognal projection of $\mathbf{h}_k$ into the null space of $\tilde{\mathbf{H}}_k$, i.e. lies in the direction of ZF beamforming. The extra part $\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k = \tilde{\mathbf{H}}_k\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k\right)^{-1}\tilde{\mathbf{H}}_k^H\mathbf{h}_k$ lies in the column space of $\tilde{\mathbf{H}}_k$, i.e. generating interference to other users. The interference is implicitly well-controlled for GPA as this vector is scaled by $\frac{\sigma_k^2}{P_k}$ (inversely proportional to the allocated power). It can be verified that $\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k$ and $\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k$ are orthogonal and $\nu_k = \sqrt{\|\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k\|^2 + \|\frac{\sigma_k^2}{P_k}\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k\|^2}$.

Using the orthogonality property, i.e. the inner product of $\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k$ and $\mathbf{h}_j$ ($j \neq k$) is zero, the leakage from user k to user j to noise ratio can be given by

$$\frac{L_{k\to j}}{\sigma_k^2} = \frac{P_k}{\sigma_k^2}\|\mathbf{h}_j^H\mathbf{w}_k\|^2 = \frac{P_k}{\sigma_k^2\nu_k^2}\|\mathbf{h}_j^H\left(\frac{\sigma_k^2}{P_k}\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\right)\mathbf{h}_k\|^2. \tag{19}$$

As ν_k is finite, i.e. $\|\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k\| \leq \nu_k \leq \|\mathbf{h}_k\|$, it can be seen from (19) that the leakage from a user to another to noise ratio converges to zero in the low and high SNR regimes. The SLNR expression (14) can also be decomposed as

$$\begin{aligned} slnr_k &= snr_k\mathbf{h}_k^H\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k + \mathbf{h}_k^H\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k \\ &= \gamma_k^{zf} + \gamma_k^{etr} \end{aligned} \tag{20}$$

where $snr_k = \frac{P_k}{\sigma_k^2}$, $\gamma_k^{zf} = \frac{P_k}{\sigma_k^2}\mathbf{h}_k^H\mathbf{P}^{\perp}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k$ corresponds to the SINR of ZF-PS and $\gamma_k^{etr} = \mathbf{h}_k^H\mathbf{P}^{etr}_{\tilde{\mathbf{H}}_k}\mathbf{h}_k$ is an additional gain as a result of the transmission of the extra part in the range of $\tilde{\mathbf{H}}_k$. Following [7], it can be shown that the gain γ_k^{etr} is a nonnegative nondecreasing function of snr_k and statistically independent of γ_k^{zf}. This indicates the superior performance of SLNR-PS compared to ZF-PS. As $snr_k \to \infty$, γ_k^{etr} converges to a scaled $\mathcal{F}$ random variable $\gamma_{k,\infty}^{etr} = \mathbf{h}_k^H\tilde{\mathbf{H}}_k\left(\tilde{\mathbf{H}}_k^H\tilde{\mathbf{H}}_k\right)^{-2}\tilde{\mathbf{H}}_k^H\mathbf{h}_k$ with probability distribution function (pdf) given by

$$\frac{M-K+2}{K-1}\gamma_{k,\infty}^{etr} \sim \mathcal{F}_{2(K-1),2(M-K+2)}. \tag{21}$$

B. SLNR analysis when $K > M$

For $K > M$, the ZF solution is not applicable. Neither is the decomposition in (18). Using (10), the leakage from user k to user j to noise ratio and the SLNR expression (14) can be written as

$$\frac{L_{k\to j}}{\sigma_k^2} = \frac{P_k}{\sigma_k^2}\frac{\|\mathbf{h}_j^H\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\|^2}{\|\left(\mathbf{H}\mathbf{H}^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k\|^2} \tag{22}$$

$$slnr_k = \mathbf{h}_k^H\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H + \frac{\sigma_k^2}{P_k}\mathbf{I}\right)^{-1}\mathbf{h}_k. \tag{23}$$

From (22), it can be seen that the leakage power converges to zero at low SNR but it diverges (increases with snr_k) at high SNR. As $snr_k \to \infty$, the $slnr_k$ converges to a scaled $\mathcal{F}$ random variable $slnr_{k,\infty} = \mathbf{h}_k^H\left(\tilde{\mathbf{H}}_k\tilde{\mathbf{H}}_k^H\right)^{-1}\mathbf{h}_k$, which is independent of snr_k, indicating the saturation of SLNR. Thus, an attempt to multiplex excessive data streams leads to ineffective SLNR solutions, causing inevitable interference at high SNR. The pdf of $slnr_{k,\infty}$ can be given by

$$\frac{K-M}{M}slnr_{k,\infty} \sim \mathcal{F}_{2M,2(K-M)}. \tag{24}$$

C. SLNR as an approximation of SINR

Despite the different definitions between SINR and SLNR, it could be argued that SINR can be well-approximated by SLNR for symmetric channels. To show this, observe the relationship between the interference at user k induced by user j and the associated power leakage (assuming EPA and equal noise variance for simplicity):

$$\begin{aligned} I_{k\leftarrow j} &= \frac{P}{K}\|\mathbf{h}_k^H\mathbf{w}_j\|^2 = \frac{P}{K}\frac{1}{\rho_j^2}\|\mathbf{h}_k^H\left(\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-1}\mathbf{h}_j\|^2 \\ &= \frac{\rho_k^2}{\rho_j^2}\frac{P}{K}\|\mathbf{h}_j^H\mathbf{w}_k\|^2 = \frac{\rho_k^2}{\rho_j^2}L_{k\to j} \end{aligned} \tag{25}$$

where $\rho_n^2 = \left[\mathbf{H}^H\left(\mathbf{H}\mathbf{H}^H + \frac{K\sigma^2}{P}\mathbf{I}\right)^{-2}\mathbf{H}\right]_{nn}, n \in \{j,k\}$. Assuming the channel is symmetric, $E\{\rho_j^2\} = E\{\rho_k^2\}$, and the leakage is small compared to noise, the interference plus noise can be estimated from the leakage plus noise by neglecting the correction factor $\frac{\rho_k^2}{\rho_j^2}$, i.e. $\sum_{j\neq k} I_{k\leftarrow j} + \sigma^2 \approx \sum_{j\neq k} L_{k\to j} + \sigma^2$. Hence, the SINR of SLNR-PS can be approximated by the corresponding SLNR. This is analogous to the approximation of DL SINR by UL SINR in the MMSE scheme, previously observed in [9]. The estimation is generally tight when leakage power is relatively small compared to noise variance. These arguments can also be extended to GPA.

For $K \leq M$, (20) can be used as the approximation of SINR and is analytically tight at low and high SNR as leakage power converges to zero. At high SNR, the ergodic capacity and the uncoded bit error rate (BER) of user k (for quadrature phase-shift keying (QPSK) modulation) can be estimated by

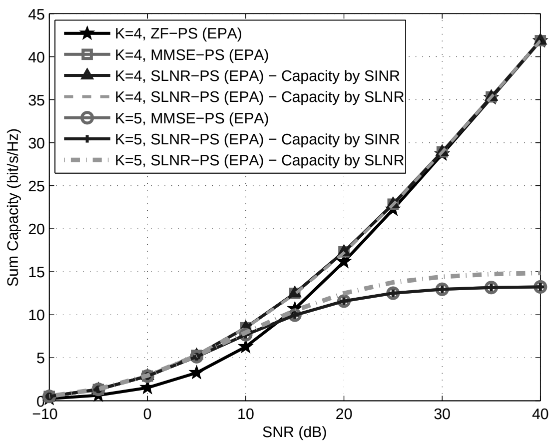

Fig. 1. Ergodic sum capacity when the number of transmit antennas $M = 4$.

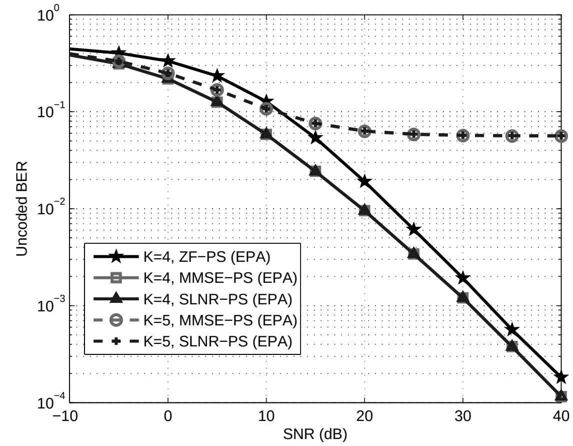

Fig. 2. Average uncoded BER with QPSK modulation when $M = 4$.

$$C_k^{slnr} \approx log_2 \left(1 + \gamma_k^{zf}\left(1 + \frac{\gamma_{k,\infty}^{etr}}{\gamma_k^{zf}}\right)\right) \approx C_k^{zf} \qquad (26)$$

$$ber_k^{slnr} \approx E\left[Q\left(\sqrt{\gamma_k^{zf} + \gamma_{k,\infty}^{etr}}\right)\right] \qquad (27)$$

$$\approx E\{e^{-\gamma_{k,\infty}^{etr}/2}\} \cdot ber_k^{zf} \qquad (28)$$

where $ber_k^{zf} \simeq E\left[Q\left(\sqrt{\gamma_k^{zf}}\right)\right]$. (26) shows that the sum capacity of SLNR-PS converges to that of ZF-PS in the high SNR regime. In contrast, a non-vanishing gap in BER is seen in (28) (since $E\{e^{-\gamma_{k,\infty}^{etr}/2}\}$ is constant and less than unity) as will also be shown in simulation results in Section V.

For $K > M$, leakage power diverges at high SNR. Thus, the SINR estimation using (23) is only tight at low SNR.

V. Simulation Results

This section provides simulation results for the case of EPA and $M = 4$. The number of users is assumed to be $K = 4$ ($K \leq M$) and $K = 5$ ($K > M$). In both cases, SLNR-PS and MMSE-PS provide the same capacity as depicted in Fig. 1 due to their equivalence under EPA. For $K \leq M$, SLNR-PS achieves higher sum capacity compared to ZF-PS in low-to-moderate SNR range and converges to ZF-PS at asymptotic high SNR as suggested by (26). The estimated capacity by using SLNR (20) is tight in the low and high SNR regimes as expected. It also appears to be rather accurate in the moderate SNR range. For $K > M$, the sum capacity of SLNR-PS and MMSE-PS reaches a ceiling at high SNR as discussed in Section IV.B. The SINR approximation using (23) is loose at high SNR; however, it remains tight in the low SNR regime.

The uncoded BER performance is presented in Fig. 2. The equivalence between SLNR-PS and MMSE-PS can again be observed for both $K \leq M$ and $K > M$. For $K \leq M$, SLNR-PS and MMSE-PS outperform ZF-PS over the entire SNR range. Notice the BER gap at high SNR as result of the non-vanishing SINR gain. For $K > M$, the BER performance of SLNR-PS and MMSE-PS converges to an irreducible level at high SNR due to residual interference unavoidable by the precoding schemes. Note that the results of ZF-PS for $K > M$ are left out as ZF solutions cannot be obtained in this case.

VI. Conclusion

This letter derived an explicit expression of SLNR-based solutions and the resulting SLNR for GPA in MU-MIMO systems with single-antenna receivers. It was shown that SLNR-PS can be classified as a regularised channel inversion scheme, with regularisation factors customised per user according to their operating SNR, plus arbitrary phase-shifts. Under the specific constraint of EPA, SLNR-PS was shown to be equivalent to MMSE-PS. In particular, MMSE-PS was shown to be the basic solution of SLNR-PS under EPA. Several useful implications, such as the possible application of analysis and algorithms (e.g. power allocation) from MMSE-PS to SLNR-PS, were also discussed. As an illustration, the performance of SLNR-PS was evaluated for GPA by the extension of MMSE analysis. For $K \leq M$ and any specific power allocation, analytical results show that SLNR-PS is superior to ZF-PS. At high SNR, the sum capacity of SLNR-PS converges to that of ZF-PS, whereas a non-vanishing gap remains to be seen in BER performance.

References

[1] M. Costa, "Writing on Dirty Paper," *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 439-441, May 1983.

[2] C. B. Peel, B. M. Hochwald, and A. L. Swindlehurst, "A Vector-Perturbation Technique for Near-Capacity Multiantenna Multiuser Communication - Part I: Channel Inversion and Regularization," *IEEE Trans. Commun.*, vol. 53, no. 1, pp. 195-202, Jan 2005.

[3] Q. H. Spencer, A. L. Swindlehurst, and M. Haardt, "Zero-Forcing methods for Downlink Spatial Multiplexing in Multiuser MIMO channels," *IEEE Trans. Signal Process.*, vol. 52, no. 2, pp. 461-471, Feb 2004.

[4] M. Joham, W. Utschick, and J. A. Nossek, "Linear transmit processing in MIMO communications systems," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 2700-2712, Aug 2005.

[5] M. Sadek, A. Tarighat, and A. H. Sayed, " Leakage-Based Precoding Scheme for Downlink Multi-User MIMO Channels," *IEEE Trans. Wireless Commun.*, vol. 6, no. 5, May 2007.

[6] F. Yuan and C. Yang, "Equivalence of SLNR Precoder and RZF Precoder in Downlink MU-MIMO Systems," *ArXiv e-prints*, arXiv:1202.1888, Feb 2012. [Online]. Available: http://arxiv.org/abs/1202.1888.

[7] Y. Jiang, M. K. Varanasi, and J. Li, "Performance Analysis of ZF and MMSE Equalizers for MIMO systems: An In-Dept Study of the High SNR Regime," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2008-2026, Apr 2011.

[8] D. J. Love and R. W. Heath, "Limited Feedback Unitary Precoding for Spatial Multiplexing Systems," *IEEE Trans. Inf. Theory*, vol. 51, no. 8, pp. 29672976, Aug 2005.

[9] X. Shao, J. Yuan, and Y. Shao, "Error performance analysis of linear zero forcing and MMSE precoders for MIMO broadcast channels," *IET Commun.*, vol. 1, no. 5, pp. 1067-1074, Oct 2007.