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Estimates for the Counting Function of the Laplace Operator on Domains with Rough Boundaries

Yuri Netrusov and Yuri Safarov

Abstract We present explicit estimates for the remainder in the Weyl formula for the Laplace operator on a domain Ω, which involve only the most basic characteristics of Ω and hold under minimal assumptions about the boundary $\partial\Omega$.

This is a survey of results obtained by the authors in the last few years. Most of them were proved or implicitly stated in our papers [10, 11, 12]; we give precise references or outline proofs wherever it is possible. The results announced in Subsection 5.2 are new.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain in $\mathbb{R}^n$, and let $-\Delta_{\mathrm{B}}$ be the Laplacian on Ω subject to the Dirichlet ($\mathrm{B} = \mathrm{D}$) or Neumann ($\mathrm{B} = \mathrm{N}$) boundary condition. Further on, we use the subscript B in the cases where the corresponding statement refers to (or result holds for) both the Dirichlet and Neumann Laplacian. Let $N_{\mathrm{B}}(\Omega, \lambda)$ be the number of eigenvalues of Δ_{B} lying below λ^2. If the number of these eigenvalues is infinite or $-\Delta_{\mathrm{B}}$ has essential spectrum below λ^2, then we define $N_{\mathrm{N}}(\Omega, \lambda) := +\infty$. Let

$$R_{\mathrm{B}}(\Omega, \lambda) := N_{\mathrm{B}}(\Omega, \lambda) - (2\pi)^{-n}\, \omega_n\, |\Omega|\, \lambda^n\,,$$

where ω_n is the volume of the n-dimensional unit ball and $|\Omega|$ denotes the volume of Ω. According to the Weyl formula, $R_{\mathrm{B}}(\Omega, \lambda) = o(\lambda^n)$ as $\lambda \to +\infty$. If $\mathrm{B} = \mathrm{D}$, then this is true for every bounded domain [4]. If $\mathrm{B} = \mathrm{N}$, then the Weyl formula holds only for domains with sufficiently regular boundaries. In

Yuri Netrusov
Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK
e-mail: y.netrusov@bristol.ac.uk

Yuri Safarov
Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
e-mail: yuri.safarov@kcl.ac.uk

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the general case, R_{N} may well grow faster than λ^n; moreover, the Neumann Laplacian on a bounded domain may have a nonempty essential spectrum (see, for instance, Remark 6.1 or [6]). The necessary and sufficient conditions for the absence of the essential spectrum in terms of capacities were obtained by Maz'ya [8].

The aim of this paper is to present estimates for $R_{\mathrm{B}}(\Omega,\lambda)$, which involve only the most basic characteristics of Ω and constants depending only on the dimension n. The estimate from below (1.2) for $R_{\mathrm{B}}(\Omega,\lambda)$ and the estimate from above (4.1) for $R_{\mathrm{D}}(\Omega,\lambda)$ hold for all bounded domains. The upper bound (4.2) for $R_{\mathrm{N}}(\Omega,\lambda)$ is obtained for domains Ω of class C, i.e., under the following assumption:

- every point $x\in\partial\Omega$ has a neighborhood U_x such that $\Omega\cap U_x$ coincides (in a suitable coordinate system) with the subgraph of a continuous function f_x.

If all the functions f_x satisfy the Hölder condition of order α, one says that Ω belongs to the class C^α. For domains $\Omega\in C^\alpha$ with $\alpha\in(0,1)$ our estimates $R_{\mathrm{D}}(\Omega,\lambda)=O\left(\lambda^{n-\alpha}\right)$ and $R_{\mathrm{N}}(\Omega,\lambda)=O(\lambda^{(n-1)/\alpha})$ are order sharp in the scale C^α as $\lambda\to\infty$. The latter estimate implies that the Weyl formula holds for the Neumann Laplacian whenever $\alpha>1-\frac{1}{n}$. If $\alpha\leqslant 1-\frac{1}{n}$, then there exist domains in which the Weyl formula for $N_{\mathrm{N}}(\Omega,\lambda)$ fails (see Remark 4.2 for details or [11] for more advanced results).

For domains of class C^∞ our methods only give the known remainder estimate $R_{\mathrm{B}}(\Omega,\lambda)=O(\lambda^{n-1}\log\lambda)$. To obtain the order sharp estimate $O(\lambda^{n-1})$, one has to use more sophisticated techniques. The most advanced results in this direction were obtained in [7], where the estimate $R_{\mathrm{B}}(\Omega,\lambda)=O(\lambda^{n-1})$ was established for domains which belong to a slightly better class than C^1.

Throughout the paper, we use the following notation.

$d(x)$ is the Euclidean distance from the point $x\in\Omega$ to the boundary $\partial\Omega$;
$\Omega_\delta^{\mathrm{b}}:=\{x\in\Omega\,|\,d(x)\leqslant\delta\}$ is the internal closed δ-neighborhood of $\partial\Omega$;
$\Omega_\delta^{\mathrm{i}}:=\Omega\setminus\Omega_\delta^{\mathrm{b}}$ is the interior part of Ω.

1 Lower Bounds

Denote by $\Pi_{\mathrm{B}}(\lambda)$ the spectral projection of the operator $-\Delta_{\mathrm{B}}$ corresponding to the interval $[0,\lambda^2)$. Let $e_{\mathrm{B}}(x,y;\lambda)$ be its integral kernel (the so-called *spectral function*). It is well known that $e_{\mathrm{B}}(x,y;\lambda)$ is an infinitely differentiable function on $\Omega\times\Omega$ for each fixed λ and that $e_{\mathrm{B}}(x,x;\lambda)$ is a nondecreasing polynomially bounded function of λ for each fixed $x\in\Omega$.

By the spectral theorem, the cosine Fourier transform of $\frac{\mathrm{d}}{\mathrm{d}\lambda}e_{\mathrm{B}}(x,y;\lambda)$ coincides with the fundamental solution $u_{\mathrm{B}}(x,y;t)$ of the wave equation in Ω. On the other hand, due to the finite speed of propagation, $u_{\mathrm{B}}(x,x;t)$ is equal to $u_0(x,x;t)$ whenever $t\in(-d(x),d(x))$, where $u_0(x,y;t)$ is the fundamental solution of the wave equation in $\mathbb{R}^n$. By a direct calculation, $u_0(x,x;t)$ is independent of x and coincides with the cosine Fourier transform of the function $n\,(2\pi)^{-n}\omega_n\,\lambda_+^{n-1}$. Applying the Fourier Tauberian theorem proved in [12], we obtain

$$
\begin{aligned}
&|e_{\mathrm{B}}(x,x;\lambda) - (2\pi)^{-n}\omega_n\,\lambda^n| \\
&\qquad\qquad \leqslant \frac{2n(n+2)^2\,(2\pi)^{-n}\omega_n}{d(x)}\left(\lambda + \frac{(n+2)\,\sqrt[n+2]{3}}{d(x)}\right)^{n-1}
\end{aligned}
\tag{1.1}
$$

for all $x \in \Omega$ and $\lambda > 0$ [12, Corollary 3.1]. Since

$$
N_{\mathrm{B}}(\Omega,\lambda) = \int_\Omega e_{\mathrm{B}}(x,x;\lambda)\,\mathrm{d}x \;\geqslant\; \int_{\Omega^{\mathrm{i}}_\delta} e_{\mathrm{B}}(x,x;\lambda)\,\mathrm{d}x
$$

for all $\delta > 0$, integrating (1.1) over $\Omega^{\mathrm{i}}_{\lambda^{-1}}$, we arrive at

$$
R_{\mathrm{B}}(\lambda,\Omega) \geqslant -2n(n+2)^2\,(2\pi)^{-n}\omega_n\big(1+(n+2)\,\sqrt[n+2]{3}\big)^{n-1}\,\lambda^{n-1}\int_{\Omega^{\mathrm{i}}_{\lambda^{-1}}}\frac{\mathrm{d}x}{d(x)}\,.
$$

Estimating constants and taking into account the obvious inequality

$$
\int_{\Omega^{\mathrm{i}}_\delta}\frac{\mathrm{d}x}{d(x)} = \int_\delta^\infty s^{-1}\,\mathrm{d}(|\Omega^{\mathrm{b}}_s|) \leqslant \int_0^{\delta^{-1}} |\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t\,,
$$

we see that

$$
\begin{aligned}
R_{\mathrm{B}}(\lambda,\Omega) &\geqslant -\,C_{n,1}\,\lambda^{n-1}\int_{\lambda^{-1}}^\infty s^{-1}\,\mathrm{d}(|\Omega^{\mathrm{b}}_s|) \\
&\geqslant -\,C_{n,1}\,\lambda^{n-1}\int_0^\lambda |\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t
\end{aligned}
\tag{1.2}
$$

for all $\lambda > 0$, where $C_{n,1} := \dfrac{2\,(n+2)^{n+1}}{\pi^{n/2}\,\Gamma(n/2)}$ and Γ is the gamma-function.

2 Variational Formulas

In order to obtain upper bounds for $R_{\mathrm{B}}(\lambda,\Omega)$, we need to estimate the contribution of $\Omega^{\mathrm{b}}_\delta$. For the Neumann Laplacian

$$
\int_{\Omega^{\mathrm{b}}_\delta} e_{\mathrm{N}}(x,x;\lambda)\,\mathrm{d}x
$$

may well not be polynomially bounded, even if $\varOmega \in C$. In this case, the Fourier Tauberian theorems are not applicable. Instead, we use the variational technique.

The idea is to represent $\varOmega$ as the union of relatively simple domains and estimate the counting function for each of these domains. Then upper bounds for $N_{\mathrm{B}}(\lambda, \varOmega)$ are obtained with the use of the following two lemmas.

Let $N_{\mathrm{N,D}}(\widetilde{\varOmega}, \varUpsilon, \lambda)$ be the counting function of the Laplacian on $\widetilde{\varOmega}$ with Dirichlet boundary condition on $\varUpsilon \subset \partial\widetilde{\varOmega}$ and Neumann boundary condition on $\partial\widetilde{\varOmega} \setminus \varUpsilon$.

Lemma 2.1. If $\{\varOmega_i\}$ is a countable family of disjoint open sets $\varOmega_i \subset \varOmega$ such that $|\varOmega| = |\cup_i \varOmega_i|$, then

$$\sum_i N_{\mathrm{D}}(\varOmega_i, \lambda) \leqslant N_{\mathrm{D}}(\varOmega, \lambda) \leqslant N_{\mathrm{N}}(\varOmega, \lambda) \leqslant \sum_i N_{\mathrm{N}}(\varOmega_i, \lambda)$$

and

$$N_{\mathrm{N}}(\varOmega, \lambda) \geqslant \sum_i N_{\mathrm{N,D}}(\varOmega_i, \partial\varOmega_i \setminus \partial\varOmega, \lambda)\,.$$

Proof. It is an elementary consequence of the Rayleigh–Ritz formula. □

Given a collection of sets $\{\varOmega_j\}$, let us denote by $\aleph\{\varOmega_j\}$ the multiplicity of the covering $\{\varOmega_j\}$, i.e., the maximal number of the sets $\varOmega_j$ containing a common element.

Lemma 2.2. Let $\{\varOmega_j\}$ be a countable family of open sets $\varOmega_j \subset \varOmega$ such that $|\varOmega| = |\cup_j \varOmega_j|$, and let $\aleph\{\varOmega_j\} \leqslant \varkappa < +\infty$. If $\varUpsilon \subset \partial\varOmega$ and $\varUpsilon_j := \partial\varOmega_j \bigcap \varUpsilon$, then

$$N_{\mathrm{N,D}}(\varOmega, \varUpsilon, \varkappa^{-1/2}\lambda) \leqslant \sum_j N_{\mathrm{N,D}}(\varOmega_j, \varUpsilon_j, \lambda).$$

Proof. See [11, Lemma 2.2]. □

Remark 2.1. Lemmas 2.1 and 2.2 remain valid for more general differential operators. This allows one to extend our results to some classes of higher order operators [11].

3 Partitions of $\varOmega$

The following theorem is due to H. Whitney.

Theorem 3.1. There exists a countable family $\{Q_{i,m}\}_{m\in\mathcal{M}_i\,,\,i\in\mathcal{I}}$ of mutually disjoint open n-dimensional cubes $Q_{i,m}$ with edges of length 2^{-i} such that

$$\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \bigcup_{m\in\mathcal{M}_i} \overline{Q_{i,m}} \quad \text{and} \quad Q_{i,m} \subset \left(\Omega^{\mathrm{b}}_{4\delta_i} \setminus \Omega^{\mathrm{b}}_{\delta_i}\right)$$

where $\delta_i := \sqrt{n}\,2^{-i}$, $\mathcal{I}$ is a subset of $\mathbb{Z}$, and $\mathcal{M}_i$ are some finite index sets.

Proof. See, for example, [13, Chapter VI]. □

Lemma 3.1. For every $\delta > 0$ there exists a finite family of disjoint open sets $\{M_k\}$ such that

(i) each set M_k coincides with the intersection of Ω and an open n-dimensional cube with edges of length δ;
(ii) $\Omega^{\mathrm{b}}_{\delta_0} \subset \bigcup_k \overline{M_k} \subset \Omega^{\mathrm{b}}_{\delta_1} \bigcup \partial\Omega$, where $\delta_0 := \delta/\sqrt{n}$ and $\delta_1 := \sqrt{n}\,\delta + \delta/\sqrt{n}$.

Proof. Consider an arbitrary covering of $\mathbb{R}^n$ by cubes with disjoint interiors of size δ and select the cubes which have nonempty intersections with Ω. □

Theorem 3.1 and Lemma 3.1 imply that Ω can be represented (modulo a set of measure zero) as the union of Whitney cubes and the subsets M_k lying in cubes of size δ. This is sufficient to estimate $R_{\mathrm{D}}(\lambda, \Omega)$. However, the condition (i) of Lemma 3.1 does not imply any estimates for $N_{\mathrm{N}}(\lambda, M_k)$. In order to obtain an upper bound for $R_{\mathrm{N}}(\lambda, \Omega)$, one has to consider a more sophisticated partition of Ω.

If Ω' is an open $(d-1)$-dimensional set and f is a continuous real-valued function on the closure $\overline{\Omega'}$, let

- $G_{f,b}(\Omega') := \{x \in \mathbb{R}^n \,|\, b < x_d < f(x'),\ x' \in \Omega'\}$, where b is a constant such that $\inf f > b$;
- $\operatorname{Osc}(f, \Omega') := \sup_{x'\in\Omega'} f(x') - \inf_{x'\in\Omega'} f(x')$;
- $\mathcal{V}_\delta(f, \Omega')$ be the maximal number of disjoint $(n-1)$-dimensional cubes $Q'_i \subset \Omega'$ such that $\operatorname{Osc}(f, Q'_i) \geqslant \delta$ for each i.

If $n = 2$, then, roughly speaking, $\mathcal{V}_\delta(f, \Omega')$ coincides with the maximal number of oscillations of f which are not smaller than δ. Further on,

- $\mathbf{V}(\delta)$ is the class of domains V which are represented in a suitable coordinate system in the form $V = G_{f,b}(Q')$, where Q' is an $(n-1)$-dimensional cube with edges of length not greater than δ, $f : \overline{Q'} \mapsto \mathbb{R}$ is a continuous function, $b = \inf f - \delta$, and $\operatorname{Osc}(f, Q') \leqslant \delta/2$;
- $\mathbf{P}(\delta)$ is the set of n-dimensional rectangles such that the length of the maximal edge does not exceed δ.

Assume that $\Omega \in C$. Then there is a finite collection of domains $\Omega_l \subset \Omega$ such that $\Omega_l = G_{f_l, b_l}(Q'_l) \in \mathbf{V}(\delta_l)$ with some $\delta_l > 0$ and $\partial\Omega \subset \bigcup_{l\in\mathcal{L}} \overline{\Omega_l}$. Let us fix such a collection, and set

- n_Ω is the number of the sets Ω_l;

- $\mathcal{V}_\delta(\Omega) := \max\{1, \mathcal{V}_\delta(f_1, Q_1'), \mathcal{V}_\delta(f_2, Q_2'), \ldots\}$;
- δ_Ω is the largest positive number such that $\Omega^{\mathrm{b}}_{\delta_\Omega} \subset \bigcup_{l\in\mathcal{L}} \Omega_l$, $\delta_\Omega \leqslant \operatorname{diam} Q_l'$, and $2\delta_\Omega \leqslant \inf f_l - b_l$ for all l.

Theorem 3.2. Let $\Omega \in C$. Then for each $\delta \in (0, \delta_\Omega]$ there exist finite families of sets $\{P_j\}$ and $\{V_k\}$ satisfying the following conditions:

(i) $P_j \in \mathbf{P}(\delta)$ and $V_k \in \mathbf{V}(\delta)$;

(ii) $\aleph\{P_j\} \leqslant 4^n\, n_\Omega$ and $\aleph\{V_k\} \leqslant 4^{n-1}\, n_\Omega$;

(iii) $\Omega^{\mathrm{b}}_{\delta_0} \subset \cup_{j,k} \left(\overline{P_j} \bigcup \overline{V_k}\right) \subset \Omega^{\mathrm{b}}_{\delta_1}$, where $\delta_0 := \delta/\sqrt{n}$ and $\delta_1 := \sqrt{n}\,\delta + \delta/\sqrt{n}$;

(iv) $\#\{V_k\} \leqslant 2^{3(n-1)} \left(3^{n-1}\, \mathcal{V}_{\delta/2}(\Omega) + n_\Omega\, \delta^{-n}\, |\Omega^{\mathrm{b}}_{\delta_1}|\right)$ and

$$\#\{P_j\} \leqslant 2^{3n-1} 3^{n-1}\, \delta^{-1} \int_{(2\operatorname{diam}\Omega)^{-1}}^{4/\delta} t^{-2}\, \mathcal{V}_{t^{-1}}(\Omega)\, \mathrm{d}t \;+\; 2^{3n}\, n^{n/2}\, n_\Omega\, \delta^{-n}\, |\Omega^{\mathrm{b}}_{\delta_1}|\,.$$

Proof. The theorem follows from [11, Corollary 3.8]. □

4 Upper Bounds

The counting functions of the Laplacian on Whitney cubes can be evaluated explicitly. For other domains introduced in the previous section the counting functions are estimated as follows.

Lemma 4.1. (i) If $P \in \mathbf{P}(\delta)$, then $N_{\mathrm{N}}(P, \lambda) = 1$ for all $\lambda \leqslant \pi\delta^{-1}$.

(ii) If $V \in \mathbf{V}(\delta)$, then $N_{\mathrm{N}}(V, \lambda) = 1$ for all $\lambda \leqslant (1 + 2\pi^{-2})^{-1/2}\delta^{-1}$.

(iii) If M is a subset of an n-dimensional cube Q with edges of length δ and $\Upsilon := \partial M \bigcap Q$, then

$$N_{\mathrm{N,D}}(M, \Upsilon, \lambda) = 0 \quad \textit{for all } \lambda \leqslant (2^{-1} - 2^{-1}\delta^{-n}|M|)^{1/2}\, \pi\delta^{-1}$$

and

$$N_{\mathrm{N,D}}(M, \Upsilon, \lambda) \leqslant 1 \quad \text{for all } \lambda \leqslant \pi\delta^{-1}.$$

Proof. See [11, Lemma 2.6]. □

Remark 4.1. The first result in Lemma 4.1(iii) is very rough. Much more precise results in terms of capacities were obtained in [9, Chapter 10, Section 1].

Applying Theorem 3.1 and Lemmas 2.1, 2.2, 3.1, 4.1 and putting $\delta = C\lambda^{-1}$ with an appropriate constant C, we obtain

$$R_{\mathrm{D}}(\Omega,\lambda) \leqslant 2^{7n}\, n^{2n}\, \lambda^{n-1} \int_0^{\lambda} |\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t \quad \forall\,\lambda>0\,. \tag{4.1}$$

Similarly, if $\Omega \in C$, then Theorems 3.1, 3.2 and Lemmas 2.1, 2.2, 4.1 imply

$$\begin{aligned} R_{\mathrm{N}}(\Omega,\lambda) \leqslant\ & 2^{7n}\, n_\Omega^{1/2} \lambda \int_{(2\,\mathrm{diam}\Omega)^{-1}}^{C_\Omega\,\lambda} t^{-2}\, \mathcal{V}_{t^{-1}}(\Omega)\,\mathrm{d}t \\ & +\ 2^{8n}\, n^{2n}\, n_\Omega\, \lambda^{n-1} \int_0^{C_\Omega\,\lambda} |\Omega^{\mathrm{b}}_{t^{-1}}|\,\mathrm{d}t \end{aligned} \tag{4.2}$$

for all $\lambda \geqslant \delta_\Omega^{-1}$, where $C_\Omega := 2^{n+3}\, n_\Omega^{1/2}$ (see [11] for details). Note that

$$|\Omega^{\mathrm{b}}_{t^{-1}}| \leqslant 2^{2n-2}\, 3^n\, n_\Omega\, (\mathrm{diam}\Omega)^{d-1} t^{-1} + 2^{3n-3}\, 3^{2n}\, t^{-n}\, \mathcal{V}_{t^{-1}}(\Omega)$$

for all $t>0$ [11, Lemma 4.3]. Therefore, (4.2) implies the estimate

$$R_{\mathrm{N}}(\Omega,\lambda) \leqslant C'_\Omega\, \lambda^{n-1} \left(\log\lambda + \int_{(2\,\mathrm{diam}\Omega)^{-1}}^{C'_\Omega\,\lambda} t^{-n}\, \mathcal{V}_{t^{-1}}(\Omega)\,\mathrm{d}t \right) \tag{4.3}$$

with a constant C'_Ω depending on Ω.

Remark 4.2. Assume that Ω belongs to the Hölder class C^α for some $\alpha \in (0,1)$. Then, by [11, Lemma 4.5], there are constants C'_1 and C'_2 such that

$$\mathcal{V}_{t^{-1}}(\Omega) \leqslant C'_1\, t^{(n-1)/\alpha} + C'_2\,.$$

Now, (1.2) and (4.2) imply that

$$R_{\mathrm{N}}(\Omega,\lambda) = O\big(\lambda^{(n-1)/\alpha}\big)\,, \qquad \lambda\to\infty\,.$$

This estimate is order sharp. More precisely, for each $\alpha\in(0,1)$ there exists a domain Ω with C^α-boundary such that $R_{\mathrm{N}}(\Omega,\lambda) \geqslant c\,\lambda^{(n-1)/\alpha}$ for all sufficiently large λ, where c is a positive constant [11, Theorem 1.10]. The inequalities (1.2) and (4.1) imply the well known estimate

$$R_{\mathrm{D}}(\Omega,\lambda) = O\left(\lambda^{n-\alpha}\right)\,, \qquad \lambda\to\infty\,.$$

It is obvious that $(n-1)/\alpha > n-\alpha$. Moreover, if $\alpha < 1-n^{-1}$, then $(n-1)/\alpha > n$, which means that $R_{\mathrm{N}}(\Omega,\lambda)$ may grow faster than λ^n as $\lambda\to\infty$.

Remark 4.3. In a number of papers, estimates for $R_{\mathrm{D}}(\Omega,\lambda)$ were obtained in terms of the so-called upper Minkowski dimension and the corresponding Minkowski content of the boundary (see, for instance, [2, 3] or [5]). Our formulas (1.2) and (4.1) are universal and imply the known estimates.

5 Planar Domains

In the two-dimensional case, it is much easier to construct partitions of a domain Ω, since the intersection of Ω with any straight line consists of disjoint open intervals. This allows one to refine the above results. Throughout this section, we assume that $\Omega \subset \mathbb{R}^2$.

5.1 The Neumann Laplacian

Consider the domain

$$\Omega = G_\varphi := \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, -1 < y < \varphi(x)\}, \tag{5.1}$$

where $\varphi : (0,1) \mapsto [0,+\infty]$ is a lower semicontinuous function such that $|G_\varphi| < \infty$ (this implies, in particular, that φ is finite almost everywhere). Note that Ω does not have to be bounded; the results of this subsection hold for unbounded domains of the form (5.1).

For each fixed $s > 0$ the intersection of G_φ with the horizontal line $\{y = s\}$ coincides with a countable collection of open intervals. Let us consider the open set $E(\varphi, s)$ obtained by projecting these intervals onto the horizontal axis $\{y = 0\}$,

$$E(\varphi,s) = \{x \in (0,1) \mid (x,s) \in G_\varphi\} = \bigcup_{j \in \Gamma(\varphi,s)} I_j\,,$$

where I_j are the corresponding open disjoint subintervals of $(0,1)$ and $\Gamma(\varphi,s)$ is an index set. It is obvious that $E(\varphi,s_2) \subset E(\varphi,s_1)$ whenever $s_2 > s_1$.

It turns out that the spectral properties of the Neumann Laplacian on G_φ are closely related to the following function, describing geometric properties of G_φ. Given $t \in \mathbb{R}_+$, let us denote

$$n(\varphi,t) = \sum_{k=1}^{+\infty} \#\Big\{j \in \Gamma(\varphi,kt) \mid \mu(I_j) < 2\,\mu\big(I_j \bigcap E(\varphi,kt+t)\big)\Big\}\,,$$

where $\mu(\cdot)$ is the one-dimensional measure of the corresponding set. Note that $n(\varphi,t)$ may well be $+\infty$.

Recall that the first eigenvalue of the Neumann Laplacian is equal to zero and the corresponding eigenfunction is constant. If the rest of the spectrum is separated from 0 and lies in the interval $[\nu^2,\infty)$, then we have the so-called Poincaré inequality

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L_2(\Omega)} \leqslant \nu^{-2}\,\|\nabla u\|^2_{L_2(\Omega)} \quad \forall\, u \in W^{2,1}(\Omega)\,,$$

where $W^{2,1}(\Omega)$ is the Sobolev space.

Theorem 5.1. The Poincaré inequality holds in $\Omega = G_\varphi$ if and only if there exists $t > 0$ such that $n(\varphi, t) = 0$. Moreover, there is a constant $C \geqslant 1$ independent of φ such that

$$C^{-1}(t_0 + 1) \leqslant \nu^{-2} \leqslant C\,(t_0 + 1),$$

where $t_0 := \inf\{t > 0 \mid n(\varphi, t) = 0\}$ and ν^{-2} is the best possible constant in the Poincaré inequality.

Proof. See [10, Theorem 1.2]. □

Theorem 5.2. The spectrum of Neumann Laplacian on G_φ is discrete if and only if $n(\varphi, t) < +\infty$ for all $t > 0$.

Proof. See [10, Corollary 1.4]. □

Theorem 5.3. Let $\Psi : [1, +\infty) \mapsto (0, +\infty)$ be a function such that

$$C^{-1} s^a \leqslant \frac{\Psi(s\,t)}{\Psi(t)} \leqslant C\, s^b \quad \forall s, t \geqslant 1,$$

where $a > 1$, $b \geqslant a$ and $C \geqslant 1$ are some constants. Then the following two conditions are equivalent.

(i) There exist constants $C_1 \geqslant 1$ and $\lambda_* > 0$ such that

$$C_1^{-1}\,\Psi(\lambda) \leqslant R_{\mathrm{N}}(G_\varphi, \lambda) \leqslant C_1\,\Psi(\lambda) \quad \forall\,\lambda \geqslant \lambda_*\,.$$

(ii) There exist constants $C_2 \geqslant 1$ and $t_* > 0$ such that

$$C_2^{-1}\,\Psi(t) \leqslant n(\varphi, t^{-1}) \leqslant C_2\,\Psi(t) \quad \forall t \geqslant t_*\,.$$

Proof. See [10, Theorem 1.6]. □

5.2 The Dirichlet Laplacian

Berry [1] conjectured that the Weyl formula for the Dirichlet Laplacian on a domain with rough boundary might contain a second asymptotic term depending on the fractal dimension of the boundary. This problems was investigated by a number of mathematicians and physicists and was discussed in many papers (see, for instance, [2, 5] and the references therein). To the best of our knowledge, positive results were obtained only for some special classes of domains (such as domains with model cusps and disconnected selfsimilar fractals). The following theorem justifies the conjecture for planar domains of class C.

Theorem 5.4. Let Ω be a planar domain of class C such that

$$|\Omega_\delta^{\mathrm{b}}| = C_1\delta^{\alpha_1} + \cdots + C_m\delta^{\alpha_m} + o(\delta^\beta)\,, \qquad \delta \to 0,$$

where C_j, α_i and β are real constants such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m \leqslant \beta < 1$ and $\beta < (1+\alpha_1)/2$. Then

$$R_{\mathrm{D}}(\Omega,\lambda) = \tau_{\alpha_1} C_1 \lambda^{2-\alpha_1} + \cdots + \tau_{\alpha_m} C_m \lambda^{2-\alpha_m} + o(\lambda^{2-\beta})\,, \qquad \lambda \to \infty,$$

where τ_{α_j} is a constant depending only on α_j for each $j = 1, \ldots, m$.

Recall that the interior Minkowski content of order α of a planar domain Ω is defined as

$$M_\alpha^{\mathrm{int}}(\Omega) \;:=\; c(\alpha) \lim_{\delta \to 0} \delta^{\alpha-2}\, |\Omega_\delta^{\mathrm{b}}| \tag{5.2}$$

provided that the limit exists. Here, $\alpha \in (0,2)$ and $c(\alpha)$ is a normalizing constant. Theorem 5.4 with $m = 1$ and $\alpha_1 = \beta = \alpha$ immediately implies the following assertion.

Corollary 5.1. If Ω is a planar domain of class C and $0 < M_\alpha^{\mathrm{int}}(\Omega) < +\infty$ for some $\alpha \in (1,2)$, then

$$\lim_{\lambda \to +\infty} R_{\mathrm{D}}(\Omega,\lambda)/\lambda^{2-\alpha} = \tau_\alpha\, M_\alpha^{\mathrm{int}}(\Omega)\,,$$

where τ_α is a constant depending only on α.

The proof of Theorem 5.4 consists of two parts, geometric and analytic. The first part uses the technique developed in [10] and the following lemma about partitions of planar domains $\Omega \in C$.

Lemma 5.1. For every planar domain $\Omega \in C$ there exists a finite collection of open connected disjoint subsets $\Omega_i \subset \Omega$ and a set D such that

(i) $\Omega \subset ((\cup_i \Omega_i) \cup D) \subset \overline{\Omega}$;

(ii) D coincides with the union of a finite collection of closed line segments;

(iii) each set Ω_i is either a Lipschitz domain or is obtained from a domain given by (5.1) with a continuous function φ_i by translation, rotation and dilation.

The second, analytic part of the proof involves investigation of some one-dimensional integral operators.

6 Concluding Remarks and Open Problems

Remark 6.1. It is not clear how to obtain upper bounds for $N_{\mathrm{N}}(\Omega,\lambda)$ for general domains Ω. It is not just a technical problem; for instance, the Neumann Laplacian on the relatively simple planar domain Ω obtained from the square $(0,2)\times(0,2)$ by removing the line segments $\frac{1}{n}\times(0,1)$, $n = 1,2,3\ldots$, has a nonempty essential spectrum.

Remark 6.2. It may be possible to extend and/or refine our results, using a combination of our variational approach with the technique developed by Ivrii [7].

Remark 6.3. There are strong reasons to believe that Theorem 5.4 cannot be extended to higher dimensions.

Finally, we draw reader's attention to the following open problems.

Problem 6.1. By Lemma 2.2, $N_{\mathrm{N}}(\Omega,\varkappa^{-1/2}\lambda)\leqslant\sum_j N_{\mathrm{N}}(\Omega_j,\lambda)$ for any finite family $\{\Omega_j\}$ of open sets $\Omega_j\subset\Omega$ such that $|\Omega| = |\cup_j\,\Omega_j|$ and $\aleph\{\Omega_j\}\leqslant\varkappa<+\infty$. It is possible that the better estimate

$$N_{\mathrm{N}}(\Omega,\lambda)\leqslant\sum_j N_{\mathrm{N}}(\Omega_j,\lambda)$$

holds. This conjecture looks plausible and is equivalent to the following statement: if $\Omega_1\subset\Omega$, $\Omega_2\subset\Omega$ and $\Omega\subset\Omega_1\bigcup\Omega_2$, then

$$N_{\mathrm{N}}(\Omega_1,\lambda)+N_{\mathrm{N}}(\Omega_2,\lambda)\geqslant N_{\mathrm{N}}(\Omega,\lambda).$$

Problem 6.2. It would be interesting to know whether the converse statement to Corollary 5.1 is true. Namely, assume that Ω is a planar domain of class C such that

$$R_{\mathrm{D}}(\Omega,\lambda) = C\,\lambda^{2-\alpha}+o(\lambda^{2-\alpha})\,,\qquad \lambda\to\infty\,,$$

with some constant C. Does this imply that the limit (5.2) exists and finite?

Problem 6.3. Is it possible to improve the estimate $R_{\mathrm{B}}(\Omega,\lambda) = O(\lambda^{n-1}\log\lambda)$ for Lipschitz domains? The variational methods are applicable to all domains Ω of class C but do not allow one to remove the $\log\lambda$, whereas Ivrii's technique gives the best possible result $R_{\mathrm{B}}(\Omega,\lambda) = O(\lambda^{n-1})$ but works only for Ω which are "logarithmically" better than Lipschitz domains.

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