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The stability analysis of systems with nonlinear feedback expressed by a quadratic program

Guang Li*, William P. Heath* and Barry Lennox*

Abstract—We consider the stability of the feedback connection of a stable linear time invariant (LTI) plant with a static nonlinearity expressed by a certain class of quadratic program (QP). We establish quadratic constraints from the Karush-Kuhn-Tucker (KKT) conditions that may be used to construct a piecewise quadratic Lyapunov function via the S-procedure. The approach is based on existing results in the literature, but gives a more parsimonious Linear Matrix Inequality (LMI) criterion. Our approach can be extended to Model Predictive Control (MPC), and gives equivalent results to those in the literature but with a much lower dimension LMI criterion.

I. INTRODUCTION

The stability analysis of a closed loop system consisting of an LTI plant in feedback with a static nonlinearity has been studied for a long time, e.g. [3]. Primbs [9] and Primbs and Giannelli [11] observed that an important subclass of such nonlinearities can be represented as the solution of a convex QP—see Fig. 1. They developed a new approach to derive stability by showing that a candidate Lyapunov function is decreasing subject to the plant dynamics and constraints determined by the KKT conditions [2] for the QP. This approach is implemented by applying the S-procedure [12] [13], which leads to the stability conditions in terms of an LMI [1]. It is shown [11] that the test outperforms the circle criterion if a piecewise quadratic Lyapunov function in $[x^T, u^T]$ is constructed (as opposed to a function in x alone). It is also suggested [11] that the approach may outperform the Zames-Falb multiplier method [14] for reducing the conservatism of stability criterion in some cases.

One acknowledged drawback of the method is that “*a priori*, it is not clear how effective a constraint will be” [11]. The inclusion of redundant constraints leads to an LMI with large dimension, which both increases the computational burden and reduces the numerical accuracy, especially for a high order system. In the case of saturation constraint [11] ten constraints are chosen for the S-procedure.

In this paper we are concerned with the further subclass where the nonlinearity $u = \phi(y)$ may be expressed as a convex QP with constraints taking the form

$$Lu \preceq b \text{ and } Mu = 0 \quad (1)$$

for some fixed L , M and $b \succeq 0$. Here “ \preceq ” and “ \succeq ” signifies term by term inequality. This includes saturation

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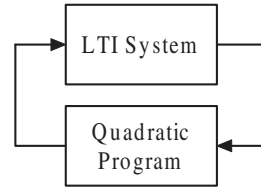


Fig. 1. The system connected with a nonlinearity expressed as a QP

and input-constrained MPC (the two main examples in [9] and [11] respectively). For the continuous case we establish two quadratic equalities satisfied by u , y , \dot{u} and \dot{y} . For the case of saturation these equalities, together with a sector bound inequality, are sufficient to establish Primbs’ stability criterion without any loss of conservatism. Although the class we consider excludes other important applications of Primbs’ method, the methodology may be extended in many such cases; we state the equivalent result for a dead zone.

The paper is structured as follows. In section II the continuous time case is considered. Three constraints for the QP are derived from the KKT conditions and the corresponding stability criterion is proposed. We show that under fairly general conditions it gives a more parsimonious LMI stability condition than Primbs’ method without loss of conservatism. We consider a saturation nonlinearity as an illustrative example. We also state the corresponding constraints for a deadzone. In section III we derive results for the discrete time case that correspond to those of section II. In section IV we use the same simulation example in [9], which is an MPC problem with uncertainty and disturbance. The results demonstrate again that our reduction can give an equivalent result, but with a much lower dimension LMI criterion.

II. CONTINUOUS TIME CASE

A. Problem setup

Consider a stable continuous time multivariable plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (2)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$. The input and output are assumed to have the same dimension without loss of generality. This plant has a feedback connection with a nonlinearity expressed by a QP

$$\begin{aligned} u(t) = \phi(y(t)) &= \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T H \tilde{u} + \tilde{u}^T y(t) \\ &\text{subject to } Lu(t) \preceq b \text{ and } Mu(t) = 0 \end{aligned} \quad (3)$$

with the Hessian matrix $H = H^T \geq 0$ and $b \geq 0$.

B. Main results

Result 1 (QP properties—continuous time case): The constrained QP (3) has the following properties

$$u^T(Hu + y) \leq 0 \quad (4)$$

$$\dot{u}^T(Hu + y) = 0 \text{ where } \dot{u} \text{ exists} \quad (5)$$

$$\dot{u}^T(H\dot{u} + \dot{y}) = 0 \text{ where } \dot{u} \text{ exists} \quad (6)$$

Proof: See Appendix. \square

Remark: The first condition is the sector bound condition, which has been found and used in stability establishment by Heath, *et al.* [6], [4]; the second is for the description of the saturation property of the QP; the third is the slope restricted condition (cf [5]).

The three quadratic constraints (4), (5) and (6) may be used to establish stability:

Corollary 1 (stability criterion—continuous time case): Consider a continuous time system (2) in feedback with a nonlinearity expressed as a QP (3). Then the system is stable if there exists a symmetric positive definite matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \quad (7)$$

such that the following LMI is satisfied

$$\Pi_0 + \sum_{i=1}^3 r_i \Pi_i \leq 0 \quad (8)$$

where

$$\Pi_0 = \begin{bmatrix} A^T P_{11} + P_{11} A & A^T P_{12} + P_{11} B & P_{12} \\ P_{12}^T A + B^T P_{11} & P_{12}^T B + B^T P_{12} & P_{22} \\ P_{12}^T & P_{22} & 0 \end{bmatrix} \quad (9)$$

$$\Pi_1 = \begin{bmatrix} 0 & -C^T & 0 \\ -C & -2H & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Pi_2 = \begin{bmatrix} 0 & 0 & -C^T \\ 0 & 0 & -H^T \\ -C & -H & 0 \end{bmatrix}$$

$$\Pi_3 = \begin{bmatrix} 0 & 0 & -(CA)^T \\ 0 & 0 & -(CB)^T \\ -CA & -CB & -2H \end{bmatrix} \quad (10)$$

and scalars $r_1 \geq 0$, $r_2, r_3 \in \mathbb{R}$.

Proof: See Appendix. \square

Remark: The LMI (8) requires that A is Hurwitz. The same requirement holds for Corollary 2.

Remark: Following the definition of stability and asymptotic stability given in [8], we may also say that the system is asymptotically stable if (8) holds with the strict inequality.

The stability test is constructed in a similar manner to those proposed by Primbs and Giannelli [11] but with fewer constraints. We now show that under fairly general conditions it yields an equivalent stability criterion.

Result 2 (reduction for the general case): Consider the QP (3) without the equality constraint $Mu = 0$. The following

constraints can be derived from the KKT conditions (with the usual caveats about the existence of \dot{u} and $\dot{\lambda}$):

$$Hu + y + L^T \lambda = 0 \quad (11)$$

$$H\dot{u} + \dot{y} + L^T \dot{\lambda} = 0 \quad (12)$$

$$\lambda^T L u \geq 0 \quad (13)$$

$$\lambda^T L \dot{u} = 0 \quad (14)$$

$$\dot{\lambda}^T L \dot{u} = 0 \quad (15)$$

Suppose $L \in \mathbb{R}^{n_c \times n_u}$ with $n_c \geq n_u$ and $\text{rank}(L) = n_u$. When using the S-procedure to establish stability, the above constraints can be further reduced to the three constraints (4), (5) and (6) without increasing the conservatism.

Proof: See Appendix. \square

Remark: It is straightforward to apply the results to the case where there are also equality constraints of the form $Mu = 0$, as these can be represented as the combined inequalities $Mu \leq 0$ and $-Mu \leq 0$. Specifically, for this case relations (11)-(15) also hold, and once again may be reduced to (4)-(6) without increasing the conservatism of the stability analysis.

1) *Example 1—Saturation nonlinearities:* The benefits of the reduction are best illustrated by an example. Primbs and Giannelli [11] consider the stability analysis of a SISO plant

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

with a saturation nonlinearity:

$$u(t) = \text{sat}(y(t)) = \frac{y(t)}{\max\{1, |y(t)|\}}$$

which can be expressed by an optimization problem as

$$u(t) = \arg \min_{\tilde{u}} \frac{1}{2} (\tilde{u} - y(t))^2$$

s.t. $|u(t)| \leq 1$

Ten conditions are derived from the KKT conditions (with appropriate caveats about the existence of derivatives):

$$\begin{aligned} u - y + \lambda_1 - \lambda_2 = 0 & \quad \lambda_1 u \geq 0 & \quad \dot{u} \dot{\lambda}_1 = 0 & \quad \lambda_1 \dot{u} = 0 \\ \dot{u} - \dot{y} + \dot{\lambda}_1 - \dot{\lambda}_2 = 0 & \quad -\lambda_2 u \geq 0 & \quad \dot{u} \dot{\lambda}_2 = 0 & \quad \lambda_2 \dot{u} = 0 \\ \lambda_1 \lambda_2 = 0 & \quad \dot{\lambda}_1 \dot{\lambda}_2 = 0 & & \end{aligned} \quad (16)$$

By applying the S-procedure on (16) and the first derivative of a candidate Lyapunov function, a sufficient stability condition for the system is the satisfaction of the LMI $E_{\perp}^T \Omega E_{\perp} \leq 0$, which corresponds to $\varphi = [x^T, u, \dot{u}, \lambda_1, \lambda_2, \dot{\lambda}_1, \dot{\lambda}_2]^T$. Here E is formed by the coefficients of the equalities and the rows of E_{\perp}^T span the null space of the space spanned by the rows of E . Ω comes from all the other constraints and the first derivative of the Lyapunov function. For this particular case, we have the following result.

Result 3 (reduction for a saturation): Given a SISO plant interconnected with a saturation function. We have the facts

- 1) If the candidate Lyapunov function is $V(x) = x^T P x$, the original ten conditions (16) can be replaced by

the sector bound condition $u(u - y) \leq 0$ without influencing the final result.

- 2) If the candidate Lyapunov function is $V(x, u) = [x^T, u]^T P [x^T, u]$, the ten conditions (16) can be replaced by the three conditions (with the usual caveats about the existence of \dot{u}):

$$u(u - y) \leq 0 \quad \dot{u}(u - y) = 0 \quad \dot{u}(\dot{u} - \dot{y}) = 0 \quad (17)$$

without influencing the final result.

Proof: See Appendix. \square

Remark: From the proof of Result 3, we can see that the conditions $\lambda_1 \lambda_2 = 0$ and $\dot{\lambda}_1 \dot{\lambda}_2 = 0$ are not actually useful in reducing the conservatism. Hence if we delete them the other conditions in (16) can be included by conditions (11)-(15), and Result 3 can be viewed as a corollary of Result 2.

Remark: Using (17) to establish the stability criterion requires an LMI with the same dimension as the vector $[x^T, u, \dot{u}]^T$ and with three multipliers; using the conditions (16) proposed by Primbs requires an LMI with the same dimension as the vector $[x^T, u, \dot{u}, \lambda_1, \dot{\lambda}_1, \lambda_2, \dot{\lambda}_2]^T$ and with ten multipliers.

2) *Example 2—Extension to deadzone nonlinearities:* We have restricted our analysis to constraints of the form (3). However similar results may be found for nonlinearities that do not fall into this category. Consider, for example, a deadzone given by

$$u = \begin{cases} y + 1 & \text{for } y < -1 \\ 0 & \text{for } -1 \leq y \leq 1 \\ y - 1 & \text{for } y > 1 \end{cases} \quad (18)$$

which may be expressed as

$$u = \arg \min_{\tilde{u}} \frac{1}{2} \tilde{u}^T \tilde{u} \quad (19)$$

subject to $|u - y| \leq 1$

Although this does not fall into the category of (3), it is straightforward to derive the following, which we state without proof or further analysis:

Result 4 (QP properties for a deadzone): Given a SISO plant interconnected with a deadzone (18). Then, with the usual caveats about the existence of \dot{u} , we have the three conditions: $u(u - y) \leq 0$, $u(\dot{u} - \dot{y}) = 0$, $\dot{u}(\dot{u} - \dot{y}) = 0$. \square

III. DISCRETE TIME CASE

A. Problem setup

Given a stable discrete time MIMO plant

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad (20)$$

Suppose this plant has a feedback connection with a nonlinearity or controller expressed by a discrete QP:

$$\begin{aligned} u_k = \phi(y_k) &= \arg \min_u \frac{1}{2} u^T H u + u^T y_k \\ &\text{subject to } Lu_k \preceq b \text{ and } Mu_k = 0 \end{aligned} \quad (21)$$

with the Hessian matrix $H = H^T \geq 0$ and $b \succeq 0$.

B. Main results

Result 5 (QP properties—discrete time case): The constrained QP proposed above has the following properties

$$u_k^T (Hu_k + y_k) \leq 0 \quad (22)$$

$$\Delta u_{k+1}^T (Hu_k + y_k) \geq 0 \quad (23)$$

$$\Delta u_{k+1}^T (Hu_{k+1} + y_{k+1}) \leq 0 \quad (24)$$

with $\Delta u_{k+1} = u_{k+1} - u_k$.

Proof: See Appendix. \square

Corollary 2 (stability criterion—discrete time case): Consider a discrete time system (20) in feedback with a nonlinearity expressed as a QP (21). Then the system is stable if there is a symmetric positive definite matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \quad (25)$$

such that the following LMI is satisfied:

$$\Pi_0 + \sum_{i=1}^3 r_i \Pi_i \leq 0 \quad (26)$$

where

$$\Pi_0 = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} & \tilde{P}_{13} \\ \tilde{P}_{21} & \tilde{P}_{22} & \tilde{P}_{23} \\ \tilde{P}_{31} & \tilde{P}_{32} & \tilde{P}_{33} \end{bmatrix} \quad (27)$$

with

$$\begin{aligned} \tilde{P}_{11} &= A^T P_{11} A - P_{11} \\ \tilde{P}_{21} &= \tilde{P}_{12}^T = B^T P_{11} A + P_{12}^T A - P_{12}^T \\ \tilde{P}_{22} &= B^T P_{11} B + P_{12}^T B + B^T P_{12} \\ \tilde{P}_{31} &= \tilde{P}_{13}^T = P_{12}^T A \\ \tilde{P}_{32} &= \tilde{P}_{23}^T = P_{12}^T B + P_{22} \\ \tilde{P}_{33} &= P_{22} \end{aligned}$$

$$\begin{aligned} \Pi_1 &= \begin{bmatrix} 0 & -C^T & 0 \\ -C & -2H & 0 \\ 0 & 0 & 0 \end{bmatrix} & \Pi_2 &= \begin{bmatrix} 0 & 0 & C^T \\ 0 & 0 & H \\ C & H & 0 \end{bmatrix} \\ \Pi_3 &= \begin{bmatrix} 0 & 0 & -(CA)^T \\ 0 & 0 & -(CB + H)^T \\ -CA & -(CB + H) & -2H \end{bmatrix} \end{aligned} \quad (28)$$

Here $r_1 \geq 0$, $r_2 \geq 0$ and $r_3 \geq 0$.

Proof: See Appendix. \square

IV. NUMERICAL EXAMPLE

We consider the MPC example used by Primbs [9]. The extension from our results to the application of MPC can be achieved by following the similar procedure in [9].

The plant with the structured uncertainty is expressed as

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k \quad (29)$$

$$p_k = Cx_k + D_u u_k + D_w w_k \quad (30)$$

$$w_k = \Delta p_k \quad (31)$$

with Δ satisfying $\|\Delta\|_2 \leq 1$ and the state space matrices

$$\begin{aligned} A &= \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} & B_u &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & B_w &= \begin{bmatrix} \theta \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix} & D_u &= 0 & D_w &= 0 \end{aligned}$$

where θ is a fixed value for the size of uncertainty. This system is subject to $|u| \leq 1$.

Suppose the cost function is

$$J_k = x_{k+N}^T P x_{k+N} + \sum_{i=0}^{N-1} [x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}]$$

with the horizon $N = 3$ and the parameters Q and R as

$$Q = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 3/2 \end{bmatrix} \quad R = 1$$

The disturbance is assumed to be constant at each sampling time k , i.e., $w_k = w_{k+1} = \dots = w_{k+N}$. If just using the sector bound constraint and the nominal Lyapunov function to establish the stability criterion, the sufficient condition of θ for the system to be stable is $0 \leq \theta \leq 0.03$; if using the three constraints we proposed and the Lyapunov function in $[x_k^T, u_k^T]^T$, the range of θ for the system to be stable is $0 \leq \theta \leq 0.19$, which is same with the result achieved by Primbs [9]. From this example we can see that our result is no worse than Primbs', but our reduction is much easier to implement and the LMI criterion has a much lower dimension compared with Primbs'. The benefits of such a reduction become especially important for high order systems with a long prediction horizon.

V. CONCLUSION

We have considered Primbs' method for assessing the stability of a closed-loop system with a static nonlinearity that may be expressed as the solution of a class of QP. This includes both simple nonlinearities, such as saturation functions, and MPC applications. We have proposed a set of constraints that lead to a concise and parsimonious application of the S-procedure. For continuous time systems we have shown analytically that the results are no worse than those of Primbs for a fairly broad class of nonlinearity, and considered a saturation nonlinearity as an example. For discrete time systems we have demonstrated a similar phenomenon by simulation example.

VI. APPENDIX

Proof of Result 1: We first write out the KKT conditions of the QP and their corresponding derivatives (where they exist); then two prerequisite conditions are proven, which will be used in the proof of the properties (5) and (6); the three properties are proven finally. Note that result (4) is given in [6] and many of the equations below may be found in [11] (cf also [5]). Nevertheless we include a full derivation of (4)-(6) for completeness.

1) The KKT conditions [2] for the QP problem are

$$Hu + y + L^T \lambda + M^T \mu = 0 \quad (32)$$

$$Lu + s = b \quad (33)$$

$$\lambda^T s = 0 \quad (34)$$

$$Mu = 0 \quad (35)$$

with $\lambda \geq 0$, $s \geq 0$ and $\lambda, s \in \mathbb{R}^l$. Their first derivatives are

$$H\dot{u} + \dot{y} + L^T \dot{\lambda} + M^T \dot{\mu} = 0 \quad (36)$$

$$L\dot{u} + \dot{s} = 0 \quad (37)$$

$$\dot{\lambda}^T s + \dot{s}^T \lambda = 0 \quad (38)$$

$$M\dot{u} = 0 \quad (39)$$

2) From (34), we have $\sum_{i=1}^l \lambda_i s_i = 0$. Since $\lambda_i \geq 0$, $s_i \geq 0$ for all $i = 0, \dots, l$, we have $\lambda_i s_i = 0$, whose first derivative is

$$\dot{\lambda}_i s_i + \dot{s}_i \lambda_i = 0 \quad (40)$$

Multiplying (40) with $\dot{\lambda}_i s_i$, we have $(\dot{\lambda}_i s_i)^2 + \dot{s}_i \lambda_i \dot{\lambda}_i s_i = 0$. Since $\lambda_i s_i = 0$, the second term disappears, which leads to $(\dot{\lambda}_i s_i)^2 = 0$. Hence

$$\dot{\lambda}_i s_i = 0 \quad (41)$$

Substituting (41) into (40) gives $\dot{s}_i \lambda_i = 0$. Hence

$$\dot{s}^T \lambda = 0 \quad (42)$$

3) Multiplying (41) by \dot{s}_i gives $\dot{s}_i \dot{\lambda}_i s_i = 0$. Since $s_i \geq 0$, there are two cases: if $s_i > 0$ then $\dot{s}_i \dot{\lambda}_i = 0$; if $s_i = 0$ then $\dot{s}_i = 0$, hence $\dot{s}_i \dot{\lambda}_i = 0$, so we have $\dot{s}_i \dot{\lambda}_i = 0$. Hence

$$\dot{s}^T \dot{\lambda} = 0 \quad (43)$$

4) Premultiplying (32) by u^T yields

$$\begin{aligned} u^T H u + u^T y &= -u^T L^T \lambda - u^T M^T \mu \\ &= -u^T L^T \lambda \quad \text{from (35)} \\ &= s^T \lambda - b^T \lambda \quad \text{from (33)} \\ &= -b^T \lambda \quad \text{from (34)} \\ &\leq 0 \quad \text{from } \lambda \geq 0 \text{ and } b \geq 0 \end{aligned}$$

Hence (4).

5) Premultiplying (32) by \dot{u}^T yields

$$\begin{aligned} \dot{u}^T H u + \dot{u}^T y &= -\dot{u}^T L^T \lambda - \dot{u}^T M^T \mu \\ &= -\dot{u}^T L^T \lambda = \dot{s}^T \lambda = 0 \quad \text{from (39), (37) and (42)} \end{aligned}$$

Hence (5).

6) Premultiplying (36) by \dot{u}^T yields

$$\begin{aligned} \dot{u}^T H \dot{u} + \dot{u}^T \dot{y} &= -\dot{u}^T L^T \dot{\lambda} - \dot{u}^T M^T \dot{\mu} \\ &= -\dot{u}^T L^T \dot{\lambda} = \dot{s}^T \dot{\lambda} = 0 \quad \text{from (39), (37) and (43)} \end{aligned}$$

Hence (6). \square

Proof of Corollary 1: Consider a candidate piecewise quadratic Lyapunov function $V(x, u) = \frac{1}{2}[x^T, u^T]P[x^T, u^T]^T$ with P as (7). Although \dot{u} does not exist everywhere, continuity conditions ensure the legitimacy of such a candidate Lyapunov function [10],[7].

Introducing $\varphi(t) = [x(t)^T, u(t)^T, \dot{u}(t)^T]^T$, we can express the first derivative of the candidate Lyapunov function (where it exists) as $\sigma_0 = \varphi^T \Pi_0 \varphi$ with Π_0 as (9). We can express the three constraints (4), (5) and (6) separately as $\sigma_1 = \varphi^T \Pi_1 \varphi \geq 0$, $\sigma_2 = \varphi^T \Pi_2 \varphi = 0$ and $\sigma_3 = \varphi^T \Pi_3 \varphi = 0$ with Π_1 , Π_2 and Π_3 as (10). A sufficient condition for the system to be stable is that there exists a matrix $P = P^T > 0$ such that $\sigma_0 \leq 0$ subject to the constraints $\sigma_1 \geq 0$, $\sigma_2 = 0$ and $\sigma_3 = 0$. Using the S-procedure, this implication can be expressed as the LMI (8). \square

Proof of Result 2:

1) The first condition (11) is one of the KKT conditions, and the second one is the first derivative of (11). Using the condition (34), i.e. $\lambda^T s = 0$ and the equations (42) and (43) derived in the proof of Result 1, the conditions (13)-(15) can be derived easily.

2) Introducing $\varphi = [x^T, u^T, \dot{u}^T, \lambda^T, \dot{\lambda}^T]^T$, the matrix formed by the coefficients of the linear equalities (11) and (12) is

$$E = \begin{bmatrix} C & H & 0 & L^T & 0 \\ CA & CB & H & 0 & L^T \end{bmatrix} \quad (44)$$

We form the matrix

$$E_{\perp}^T = \begin{bmatrix} I & 0 & 0 & -C^T L^{\dagger} & -(CA)^T L^{\dagger} \\ 0 & I & 0 & -H^T L^{\dagger} & -(CB)^T L^{\dagger} \\ 0 & 0 & I & 0 & -H^T L^{\dagger} \\ 0 & 0 & 0 & (L^T)_{\perp} & 0 \\ 0 & 0 & 0 & 0 & (L^T)_{\perp}^T \end{bmatrix} \quad (45)$$

which is in its row echelon form with $L^{\dagger} = (L^T L)^{-1} L^T$ and $L(L^T)_{\perp} = 0$, so that $EE_{\perp} = 0$.

If the candidate Lyapunov function is chosen as $V(x) = [x^T, u^T]^T P [x^T, u^T]$. The matrix Ω formed from $\nabla V(x, u) = 0$ and the equations (13), (14) and (15) is

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{22} \end{bmatrix}$$

where

$$\Omega_{11} = \begin{bmatrix} A^T P_{11} + P_{11}^T A & A^T P_{12} + P_{11} B & P_{12} \\ P_{12}^T A + B^T P_{11} & P_{12}^T B + B^T P_{12} & P_{22} \\ P_{12}^T & P_{22} & 0 \end{bmatrix}$$

$$\Omega_{12} = \begin{bmatrix} 0 & 0 \\ r_1 L^T & 0 \\ r_2 L^T & r_3 L^T \end{bmatrix} \quad \Omega_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $r_1 \geq 0$, $r_2, r_3 \in \mathbb{R}$. Hence the final LMI is

$$\tilde{E}_{\perp}^T \Omega \tilde{E}_{\perp} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \leq 0$$

where

$$F_{11} = A^T P_{11} + P_{11} A$$

$$F_{21} = F_{12}^T = P_{12}^T A + B^T P_{11} - r_1 C$$

$$F_{22} = P_{12}^T + B^T P_{12} - 2r_1 H$$

$$F_{31} = F_{13}^T = P_{12}^T - r_2 C - r_3 C A$$

$$F_{32} = F_{23}^T = P_{22} - r_2 H - r_3 C B$$

$$F_{33} = -2r_3 H$$

which takes the same LMI form with the one by using the three constraints (4)-(6) directly. This means the constraints (11)-(15) can be replaced by (4)-(6) without influencing the final result in establishing the stability criterion. In a similar way, it can be shown that when the candidate Lyapunov function is chosen as $V(x) = x^T P x$, the constraints (11)-(15) can be replaced by (4) without influencing the final result. \square

Proof of Result 3: The matrix E is

$$E = \begin{bmatrix} -C & 1 & 1 & -1 & 0 & 0 & 0 \\ -CA & -CB & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

and its corresponding E_{\perp}^T in row echelon form is

$$E_{\perp}^T = \begin{bmatrix} 1 & 0 & 0 & -C^T & 0 & 0 & -(CA)^T \\ 0 & 1 & 0 & 1 & 0 & 0 & -(CB)^T \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

1) When the candidate Lyapunov function is chosen as $V(x) = x^T P x$, The matrix Ω is

$$\begin{bmatrix} A^T P + P A & P B & 0 & 0 & 0 & 0 & 0 \\ B^T P & 0 & r_{11} & -r_{12} & 0 & 0 & 0 \\ 0 & r_{11} & 0 & r_{23} & r_{24} & 0 & 0 \\ 0 & -r_{12} & r_{23} & 0 & r_{25} & 0 & 0 \\ 0 & 0 & r_{24} & r_{25} & 0 & r_{21} & r_{22} \\ 0 & 0 & 0 & 0 & r_{21} & 0 & r_{26} \\ 0 & 0 & 0 & 0 & r_{22} & r_{26} & 0 \end{bmatrix}$$

with $r_{ij} \in \mathbb{R}$ associated with (16). We further require $r_{11} \geq 0$ and $r_{12} \geq 0$. Hence the final LMI is

$$E_{\perp}^T \Omega E_{\perp} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \leq 0 \quad (46)$$

with

$$Q = \begin{bmatrix} A^T P A & P B + r_{12} C^T \\ B^T P + r_{12} C & -2r_{12} \end{bmatrix}$$

$$S = \begin{bmatrix} -r_{23} C & r_{11} + r_{23} - r_{12} \\ -r_{25} C - r_{22} C A & r_{25} - r_{22} C B \\ -r_{26} C A & -r_{26} C B \end{bmatrix}$$

and

$$R = \begin{bmatrix} 2r_{23} & r_{24} + r_{25} & 0 \\ r_{24} + r_{25} & 2r_{22} & r_{21} + r_{22} + r_{26} \\ 0 & r_{21} + r_{22} + r_{26} & 2r_{26} \end{bmatrix}$$

We can set $r_{21} = 0$, $r_{22} = 0$, $r_{23} = 0$, $r_{26} = 0$, $r_{11} = r_{12}$ and $r_{24} = -r_{25}$, so the LMI $Q \leq 0$ is necessary and sufficient for (46). But $Q \leq 0$ is precisely the LMI obtained when the S-procedure is applied to the sector bound condition $u(u - y) \leq 0$. In fact the criterion $Q \leq 0$ corresponds to the circle criterion.

2) When the candidate Lyapunov function is chosen as $\varphi^T P \varphi \geq 0$ with $\varphi = [x^T, u^T]^T$, the matrix Ω is

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^T & \Omega_{22} \end{bmatrix}$$

with

$$\Omega_{11} = \begin{bmatrix} A^T P_{11} + P_{11} A & A^T P_{12} + P_{11} B & P_{12} \\ P_{12}^T A + B^T P_{11} & P_{12}^T B + B^T P_{12} & P_{22} \\ P_{12} & P_{22} & 0 \end{bmatrix}$$

$$\Omega_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_{11} & -r_{12} & 0 & 0 \\ r_{24} & r_{25} & r_{21} & r_{22} \end{bmatrix}$$

$$\Omega_{22} = \begin{bmatrix} 0 & r_{23} & 0 & 0 \\ r_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{26} \\ 0 & 0 & r_{26} & 0 \end{bmatrix}$$

and the r_{ij} is defined as before.

Following the same procedure, the final LMI is

$$E_{\perp}^T \Omega E_{\perp} = \begin{bmatrix} Q & S^T \\ S & R \end{bmatrix} \leq 0 \quad (47)$$

where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

with

$$Q_{11} = A^T P_{11} + P_{11} A$$

$$Q_{21} = Q_{12}^T = P_{12}^T A + B^T P_{11} + r_{12} C$$

$$Q_{22} = P_{12} B + B^T P_{12} - 2r_{12}$$

$$Q_{31} = Q_{13}^T = P_{12}^T - r_{25} C - r_{22} C A$$

$$Q_{32} = Q_{23}^T = P_{22} + r_{25} - r_{22} C B$$

$$Q_{33} = 2r_{22}$$

$$S = \begin{bmatrix} -r_{23} C & r_{11} + r_{23} - r_{12} & r_{24} + r_{25} \\ -r_{26} C A & -r_{26} C B & r_{21} + r_{26} + r_{22} \end{bmatrix}$$

and

$$R = \begin{bmatrix} 2r_{23} & 0 \\ 0 & r_{26} \end{bmatrix}$$

We can set $r_{11} = r_{12}$, $r_{21} = -r_{22}$, $r_{24} = -r_{25}$, $r_{23} = 0$ and $r_{26} = 0$ so the LMI $Q \leq 0$ is necessary and sufficient for (47). Note that $Q \leq 0$ is precisely the LMI obtained when the S-procedure is applied to the three constraints (17). \square

Proof of Result 5:

- 1) The KKT conditions for the discrete QP problem at time instant k take the form (32)-(35) with $u = u_k$, $y = y_k$, $\lambda = \lambda_k$ and $s = s_k$. The first inequality (22) follows immediately from (4).

Premultiplying (32) at time k by Δu_{k+1}^T yields

$$\begin{aligned} & \Delta u_{k+1}^T (H u_k + y_k) \\ &= -\Delta u_{k+1}^T L^T \lambda_k - \Delta u_{k+1}^T M^T \mu_k \\ &= -\Delta u_{k+1}^T L^T \lambda_k \text{ from (35) at } k, k+1 \\ &= (s_{k+1} - s_k)^T \lambda_k \text{ from (33) at } k \text{ and } k+1 \\ &= s_{k+1}^T \lambda_k \text{ from (34) at } k \\ &\geq 0 \text{ from } s_{k+1} \geq 0 \text{ and } \lambda_{k+1} \geq 0 \end{aligned}$$

Hence (23).

Premultiplying (32) at time $k+1$ by Δu_{k+1}^T yields

$$\begin{aligned} & \Delta u_{k+1}^T (H u_{k+1} + y_{k+1}) \\ &= -\Delta u_{k+1}^T L^T \lambda_{k+1} - \Delta u_{k+1}^T M^T \mu_{k+1} \\ &= -\Delta u_{k+1}^T L^T \lambda_{k+1} \text{ from (35) at } k \text{ and } k+1 \\ &= (s_{k+1} - s_k)^T \lambda_{k+1} \text{ from (33) at } k \text{ and } k+1 \\ &= -s_k^T \lambda_{k+1} \leq 0 \end{aligned}$$

Hence (24). \square

Proof of Corollary 2: Consider a candidate piecewise quadratic Lyapunov function $V(x, u) = \frac{1}{2} [x_k^T, u_k^T]^T P [x_k^T, u_k^T]$ with P as (25). Introducing $\varphi_k = [x_k^T, u_k^T, \Delta u_{k+1}^T]^T$, we can express the first difference of the Lyapunov function as $\sigma_0 = \varphi_k^T \Pi_0 \varphi_k$ with Π_0 as (27). We also express the three constraints (22), (23) and (24) separately in quadratic forms as $\sigma_1 = \varphi^T \Pi_1 \varphi \geq 0$, $\sigma_2 = \varphi^T \Pi_2 \varphi \geq 0$ and $\sigma_3 = \varphi^T \Pi_3 \varphi \geq 0$ with Π_1 , Π_2 and Π_3 as (28).

The sufficient condition for the system to be stable is that there is a matrix $P = P^T > 0$ such that $\sigma_0 \leq 0$ subject to the constraints $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and $\sigma_3 \geq 0$. Using the S-procedure, this implication can be expressed in the LMI (26). \square

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