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Open Circular Billiards and the Riemann Hypothesis

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A comparison of escape rates from one and from two holes in an experimental container (e.g., a laser trap) can be used to obtain information about the dynamics inside the container. If this dynamics is simple enough one can hope to obtain exact formulas. Here we obtain exact formulas for escape from a circular billiard with one and with two holes. The corresponding quantities are expressed as sums over zeros of the Riemann zeta function. Thus we demonstrate a direct connection between recent experiments and a major unsolved problem in mathematics, the Riemann hypothesis.

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Billiard systems, in which a point particle moves freely except for specular reflections from rigid walls, permit close connections between rigorous mathematics and experimental physics. Very general physical situations, in which particles or waves are confined to cavities or other homogeneous regions, are related to well understood billiard dynamical systems, directly for particles and via semiclassical (short wavelength) theories for waves. Precise billiard experiments have used microwaves in metal [1,2] and superconducting [3] cavities and with wave guides [4], visible light reflected from mirrors [5], phonons in quartz blocks [6], electrons in semiconductors confined by electric potentials [7,8], and atoms interacting with laser beams [9–11]. Experiments employ both two and three [12] dimensional geometries, and both closed and open systems. For instance, escape of cold atoms from a laser trap with a hole was studied in [9,10]. Closed systems exhibit energy level distributions and scarring of wave functions as predicted by semiclassical [13] and random matrix [14] theories. Open systems exhibit unexpected and incompletely understood phenomena such as fractal conductance fluctuations [8,15–17]. For both closed and open systems, the behavior depends crucially on the classical dynamics, which can be tuned to be integrable, chaotic, or mixed [9,10,15,18]. The escape rate is a characteristic of open billiards which is both experimentally accessible [9,10] and important for transport properties of many systems. In this Letter we propose to ask what may be understood about the intrinsic dynamics of billiards using only this experimental escape information, specifically by comparing systems with one and two holes. The two hole escape rate is not measured in the experimental literature but presents no fundamental difficulties. We naturally begin with the simplest of shapes, the circle, and find remarkable exact expressions based on the most famous unsolved mathematical problem, the Riemann hypothesis.

At long times, the probability of a particle remaining in an integrable billiard with a hole is well-known to exhibit power law decay, in contrast to exponential decay from strongly chaotic billiards [19]; however the coefficient of the power (“escape rate”) in the integrable case has not been computed exactly to our knowledge. Numerical simulations can be misleading; for example, a power law decay at long times can be masked by an exponential term at short times.

Here we consider the circle billiard, which is integrable due to angular momentum conservation. Some three dimensional cases, namely, the cylinder and sphere, can be treated analogously. The circle billiard of unit radius has collisions defined by the angle around the circumference $\beta \in (-\pi, \pi]$ and the angle between the outward trajectory and the normal $\psi \in (-\pi/2, \pi/2)$. The billiard map is then $(\beta, \psi) \to (\beta + \pi - 2\psi, \psi)$ where angles are taken modulo 2π as usual. The time between collisions is $T = 2\cos\psi$.

The dynamical evolution is of two types depending on the value of ψ. For $\psi = \psi_{m,n} \equiv \pi/2 - m\pi/n$ with $m < n$ coprime integers, the trajectory has period n and the β values are equally spaced at intervals of $2\pi/n$. For ψ,

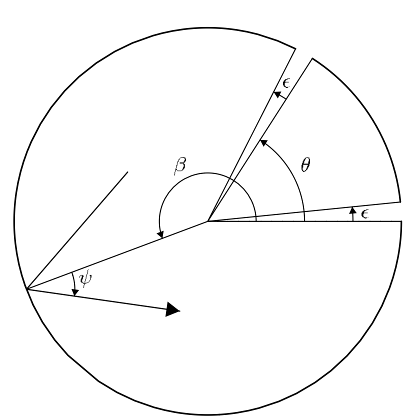

FIG. 1. Geometry of the billiard.

which are irrational multiples of π, the trajectory is uniformly distributed in β.

For the escape problem the billiard is filled with a uniform density of particles normalized to unity given by $\cos\psi d\beta d\psi/(4\pi)$ at the boundary. Two (possibly overlapping) holes are placed at the boundary at $\beta \in [0, \epsilon]$ and $\beta \in [\theta, \theta + \epsilon]$; the one hole problem is simply $\theta = 0$. The number of collisions to escape is some function $N(\beta_0, \psi_0)$ (possibly infinite) of the initial conditions and the time to escape is $t = NT = 2\cos\psi N(\beta_0, \psi_0)$. See Fig. 1.

Now we compute the density remaining near periodic orbits (β_0, m, n) at long time t. Such a long-lived trajectory has $\psi = \psi_{m,n} + \eta$ for $\eta \ll \epsilon$. A prime indicates values taken modulo $2\pi/n$ and lying in $[0, 2\pi/n)$. Thus the dynamics is now $\beta' \to \beta' - 2\eta$ and escape takes place when this passes one of the values ϵ or $\epsilon + \theta'$ for $\eta > 0$, or 0 or θ' for $\eta < 0$. The set of initial conditions for which escape takes at least time t; hence $t/(2\cos\psi_{m,n})$ collisions is thus

$$\beta_0' \in \left(\epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \theta'\right) \bigcup \left(\theta' + \epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \frac{2\pi}{n}\right) \quad (1)$$

for $\eta > 0$ and similar expressions for $\eta < 0$. Integrating over this region, including the $\cos\psi$ weighting for the invariant measure and a factor n to account for the equivalence modulo $2\pi/n$, and summing over all nonescaping periodic orbits gives

$$P(t) \sim \frac{1}{4\pi}\sum_{m,n} \frac{n[g(\frac{2\pi}{n} - \theta' - \epsilon) + g(\theta' - \epsilon)]}{t} \sin^2\frac{\pi m}{n},$$
$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \le 0, \end{cases} \quad (2)$$

where the sum is restricted to $0 \le m < n < 2\pi/\epsilon$ for coprime m, n, and faster decaying terms in t have been neglected. Note that $n = 1$ is included but gives no contribution. Now we write the sin functions as sums of exponentials and apply the Ramanujan identity [20]

$$\sum_{\substack{m=0 \\ \gcd(m,n)=1}}^{n-1} e^{2\pi i m/n} = \mu(n), \quad (3)$$

where μ is the Möbius function, defined by $\mu(1) = 1$, $\mu(p) = -1$ for primes p, and $\mu(mn) = \mu(m)\mu(n)$ if $\gcd(m, n) = 1$; otherwise $\mu(mn) = 0$. The result is

$$P_\infty \equiv \lim_{t\to\infty} tP(t) = \frac{1}{8\pi}\sum_{n=1}^{\infty} n[\phi(n) - \mu(n)]\left[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)\right]. \quad (4)$$

Here $\phi(n)$ is the Euler totient function, giving the number of positive integers $m \le n$ with $\gcd(m, n) = 1$. Conventionally $\phi(1) = 1$. Of course P_∞ depends on both ϵ and θ, but this is suppressed for brevity.

P_∞ is a finite sum, piecewise smooth when considered as a function of ϵ and/or θ with the number of terms unbounded in the limit of small ϵ. For small holes we can extract the asymptotic behavior using Mellin transforms. Writing $\tilde{P}(s) = \int_0^\infty P_\infty \epsilon^{s-1} d\epsilon$ and then $P_\infty = \frac{1}{2\pi i} \times \int_{c-i\infty}^{c+i\infty} \epsilon^{-s}\tilde{P}(s)ds$ gives the required asymptotic series as a sum of residues. Interchanging the sum and the first integral, integrating over ϵ and substituting $\theta' = f(n\theta/(2\pi))2\pi/n$, where f indicates the fractional part, we find

$$P_\infty = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{ds\epsilon^{-s}(2\pi)^{s+1}}{2s(s+1)(s+2)} \times \sum_{n=1}^{\infty} \frac{\phi(n) - \mu(n)}{n^{s+1}}\left\{\left[1 - f\left(\frac{n\theta}{2\pi}\right)\right]^{s+2} + f\left(\frac{n\theta}{2\pi}\right)^{s+2}\right\}. \quad (5)$$

Now we consider the case of rational angles $\theta = 2\pi r/q$, where $\gcd(r, q) = 1$; the $r = 0$, $q = 1$ case gives a single hole. In this case the sum over n splits into conjugacy classes modulo q and the fractional parts are known rational numbers which depend only on the conjugacy class.

To evaluate the sum over n we first divide all quantities through by $b = \gcd(a, q)$, writing $n' = n/b$, $a' = a/b$, and $q' = q/b$. Now we consider Dirichlet characters [20] $\chi(n')$, defined as follows. The conjugacy classes modulo q' which are coprime to q' form an Abelian group under multiplication, of order $\phi(q')$. Since the group is Abelian and finite, there are $\phi(q')$ irreducible representations χ in which each n' coprime to q' is represented by a complex root of unity $\chi(n')$ satisfying $\chi(m')\chi(n') = \chi(m'n')$; $\chi(n') = 0$ if n' and q' have a common factor.

Inserting the orthogonality relation for characters [20] into the sum allows n' to be summed over all integers and decomposed into prime factors $n' = \prod_p p^{\alpha_p}$,

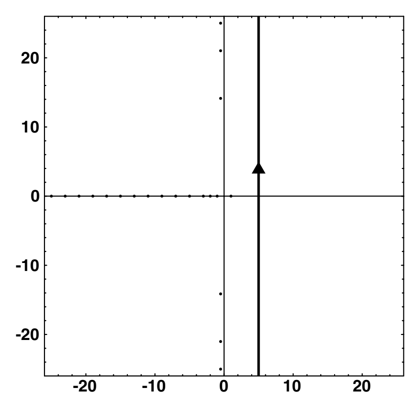

FIG. 2. Poles of $\tilde{P}(s)$ [Eq. (9)] for $q = 1$ (one hole) at -2, odd integers less than or equal to 1, and nontrivial values with real part $-1/2$ assuming the Riemann hypothesis. The contour of Eq. (5) is given.

$$\sum_{n\equiv a\ (\mathrm{mod}\, q)} \frac{\phi(n)-\mu(n)}{n^{s+1}} = \sum_{n'\equiv a'\ (\mathrm{mod}\, q')} \frac{\phi(bn')-\mu(bn')}{(bn')^{s+1}} = \frac{1}{\phi(q')}\sum_{\chi}\bar{\chi}(a')\sum_{n'=1}^{\infty}\chi(n')\frac{\phi(bn')-\mu(bn')}{(bn')^{s+1}}; \tag{6}$$

$$\chi(n') = \prod_p \chi(p)^{\alpha_p}, \qquad \phi(bn') = \phi(b)\prod_{p|n', p\nmid b}(1-p^{-1}), \qquad \mu(bn') = \begin{cases} \mu(b)\prod_p(-1)^{\alpha_p} c^2\nmid bn' & \text{for all } c>1 \\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Here, the bar indicates complex conjugation. The condition on μ is taken into account by setting $\alpha_p = 0$ if $p|b$ and summing $\alpha_p = 0, 1$ otherwise. The sum for ϕ is given by the $\alpha = 0$ term together with a geometric series. The result is

$$\sum_{n\equiv a\ (\mathrm{mod}\, q)} \frac{\phi(n)-\mu(n)}{n^{s+1}} = \frac{1}{b^{s+1}\phi(q')}\sum_{\chi}\frac{\bar{\chi}(a')[\phi(b)L(s,\chi)-\mu(b)]}{L(s+1,\chi)\prod_{p|b}[1-\chi(p)p^{-s-1}]}, \qquad L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \prod_p\left(1-\frac{\chi(p)}{p^s}\right), \tag{8}$$

where characters are taken modulo q' and $L(s, \chi)$ is the Dirichlet L function, which in the case $q' = 1$ [i.e., $\chi(n) = 1$ for all n] reduces to the Riemann zeta function. Our first main result is the exact expression for the probability $P(t)$ of remaining in the unit circular billiard with two holes $[0, \epsilon]$ and $[2\pi r/q, 2\pi r/q + \epsilon]$

$$\lim_{t\to\infty} tP(t) = \sum_j \mathrm{res}_{s=s_j}\tilde{P}(s)\epsilon^{-s},$$
$$\tilde{P}(s) = \frac{(2\pi)^{s+1}}{2s(s+1)(s+2)}\sum_{a=1}^{q}\frac{[1-f(\frac{ap}{q})]^{s+2}+f(\frac{ap}{q})^{s+2}}{b^{s+1}\phi(q')}\sum_{\chi}\frac{\bar{\chi}(a')[\phi(b)L(s,\chi)-\mu(b)]}{L(s+1,\chi)\prod_{p|b}[1-\chi(p)p^{-s-1}]}, \tag{9}$$

where, as above, $b = \gcd(a, q)$, $a' = a/b$, $q' = q/b$, and the characters are taken modulo q'. In performing the contour integral (Fig. 2) we recall that the contour lies to the right of all poles of the integrand ($c > 1$) and that a semicircular arc to the left may be added that avoids the poles and for which the integral vanishes in the limit of infinite radius.

We now consider specific values of q; recall that the angle separating the holes is $\theta = 2\pi r/q$. Thus $q = 1$ is a single hole and $q = 2$ is two opposite holes. Experimentalists could especially notice a contrast between small values of q, in which the angles are simple rational multiples of π and the escape rate is expressed in terms of only a few L functions (in fact, only the Riemann zeta function for $q = 1, 2, 3, 4, 6$), and angles which are not close to rational numbers with small denominators. Particular values of $\tilde{P}(s)$ and its residues are given in Table I. Some points to note are that the leading behavior at $s = 1$, that is, of order ϵ^{-1}, is purely given by the total size of the holes. The $q = 1$ case is twice as large since

TABLE I. The function $\tilde{P}(s)(q/2\pi)^{s+1}$ [see Eq. (9)] and some of the residues of $\tilde{P}(s)$ for $q = 1, 2, 3, 4$ and $r = 1$.

q	$\tilde{P}(s)(q/2\pi)^{s+1}$	1	-1	-2
1	$\frac{\zeta(s)-1}{2s(s+1)(s+2)\zeta(s+1)}$	2	$-\frac{13}{12}$	$\frac{3}{2\pi}$
2	$\frac{\zeta(s)}{s(s+1)(s+2)\zeta(s+1)}$	1	$-\frac{1}{6}$	0
3	$\frac{3^s\{7\zeta(s)+2^{s+2}[\zeta(s)-1]+2\}-\zeta(s)(2^{s+2}+1)}{2s(s+1)(s+2)(3^{s+1}-1)\zeta(s+1)}$	1	$-\frac{1}{4}-\frac{5\ln 2}{9\ln 3}$	$\frac{3}{4\pi}$
4	$\frac{2^s\{13\zeta(s)+3^{s+2}[\zeta(s)-1]+3\}-\zeta(s)(3^{s+2}+5)}{4s(s+1)(s+2)(2^{s+1}-1)\zeta(s+1)}$	1	$-\frac{1}{3}-\frac{11\ln 3}{16\ln 2}$	$\frac{3}{\pi}$

there is only a single hole in this case. For these values of q the second to leading order terms, which are not given in the table, come from the nontrivial zeroes of the $\zeta(s + 1)$, and for multiplicity m are of order $\sqrt{\epsilon}(\ln\epsilon)^{m-1}$ if all zeros of $\zeta(s)$ lie in $\mathrm{Re}s \leq 1/2$. The latter is a statement of the Riemann hypothesis [21]. An alternative formulation stating that all the nontrivial zeros of $\zeta(s)$ have $\mathrm{Re}s = 1/2$ is easily shown to be equivalent using the functional equation [21] relating $\zeta(s)$ and $\zeta(1 - s)$. Thus this celebrated unsolved problem is equivalent to either of

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta-1/2}[tP_1(t) - 2/\epsilon] = 0, \tag{10}$$

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta-1/2}[tP_1(t) - 2tP_2(t)] = 0 \tag{11}$$

for every $\delta > 0$, and the subscript indicates the one or symmetric two hole problem. This is our second main result: The difference between the escape of the one and two hole problems is determined to leading order in the small hole limit by the Riemann hypothesis. The generalized Riemann hypothesis is the equivalent statement for L functions and implies that corrections are of order $\epsilon^{\delta+1/2}$ in the rational two hole case. For irrational θ the above analysis breaks down; presumably the poles on the critical line become dense, blocking analytic continuation. However, the leading order (in ϵ) term can be shown to be θ independent for $\theta > 0$, as for the rational values given above, as follows. For the leading order behavior, the sum over n can be approximated by an integral, with parts of the summand replaced by "mean field" averages $\langle\rangle$. If θ is

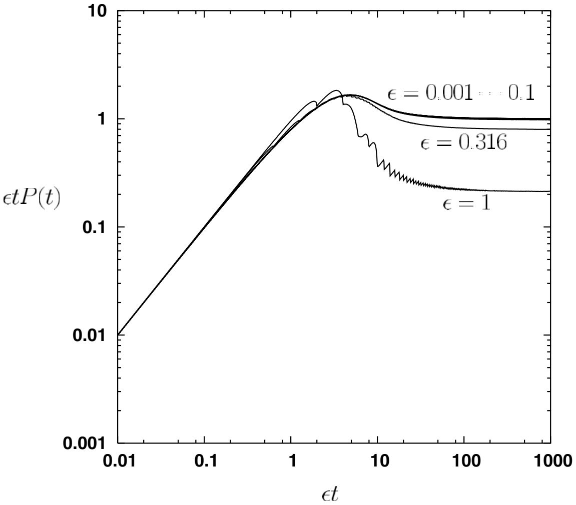

FIG. 3. The scaled survival probability $\epsilon t P(t)$ [see Eq. (4)] appears to approach a limiting function as $\epsilon \to 0$ with ϵt held constant. Here we use 10^8 random initial conditions, $\theta = 1$ (an irrational multiple of π) and the curves are $\epsilon = 10^{-n/2}$ with $n = 0 \cdots 6$. At large ϵt the function converges to unity, consistent with Eq. (13).

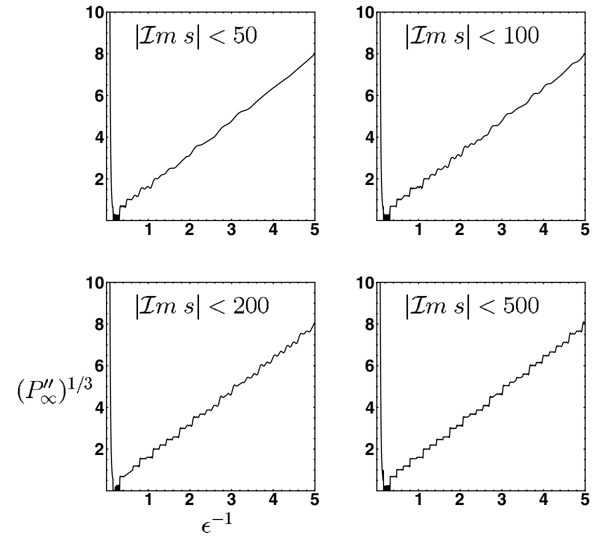

FIG. 4. Numerical computation of $P''_\infty(\epsilon)$ for the single hole case $q = 1$, using Eq. (9) and varying the number of poles considered on the critical line; real poles are considered for $s > -10$ in all plots. From (4) it is seen that the second derivative of P_∞ is a function with uniformly spaced steps and constant average gradient in the variables shown on the axes. Convergence is evident except for very small ϵ^{-1} for which more real poles would be required; large ϵ^{-1} require more critical poles for the steps to be visible.

irrational, the fractional parts are uniformly distributed, so that we compute

$$\left\langle g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)\right\rangle = \frac{n}{3\pi}\left(\frac{2\pi}{n} - \epsilon\right)^3. \quad (12)$$

We also use $\langle \phi(n) \rangle = 6n/\pi^2$ and $\langle \mu(n) \rangle = 0$, so that

$$tP(t) \approx \frac{1}{24\pi^2}\int_0^{2\pi/\epsilon} n^2 \frac{6n}{\pi^2}\left(\frac{2\pi}{n} - \epsilon\right)^3 dn = \frac{1}{\epsilon} \quad (13)$$

as required.

We now present some numerical tests. The main results of this Letter are exact. However, there are some related questions to do with rates of convergence of various limits which are of great importance to numerical or experimental extensions of this work. The rate of convergence of $tP(t)$ as $t \to \infty$ is considered in Fig. 3. The convergence of Eq. (9) with the number of residues is tested in Fig. 4.

In conclusion, we computed the escape from a circular billiard with one hole and related it to the Riemann hypothesis, computed escape for two holes (rational case), and related it to the generalized Riemann hypotheses and obtained the leading order behavior for the irrational two hole case. Many interesting questions remain concerning one or two hole escape from pseudointegrable, chaotic or mixed billiards, and to the quantum/wave signature of these systems [22]. A fuller understanding of open quantum billiards should have important practical benefits, for example, in the design of microlasers [23].

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