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Derivation and Application of a Passive Equivalent Circuit for the Finite Difference Time Domain Algorithm

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Abstract— The widely used finite difference time domain (FDTD) algorithm in its standard form is *conditionally stable*, the condition being the well-known Courant criterion. Much research has focussed on modifying the standard algorithm to improve its characterisation of geometrical detail and curved surfaces; these modified algorithms, however, may easily be *conditionally stable*—there is no value of time step that stabilizes the algorithm. This contribution presents a passive electrical circuit that, by virtue of its formal equivalence with FDTD, provides a criterion by which unconditionally unstable algorithms may be avoided. As an example the passive circuit criterion is used to remove the instability from a contour-path FDTD algorithm.

I. INTRODUCTION

THE widely accepted finite difference time domain (FDTD) algorithm is known to be incapable of efficiently modeling strucures such as electrically small and curved geometries. Techniques that produce an improved characterization of these geometries include, among others, the deformed contour-path (CP) [1] and static field solution (SFS) [2] methods.

A feature common to these useful techniques is that they modify the standard difference algorithm by locally altering the coefficients of the update equations. A consequence of this is that the stability properties of the basic algorithm may be destroyed. More importantly, instability may be inherent in the modified *spatial* discretization and no reduction in the time step will result in stability—the algorithm is *unconditionally unstable*.

To the authors' knowledge, there exists no practical technique for distinguishing between these unstable algorithms and those that, like the standard FDTD algorithm, are stable given a suitable time step. The only technique that is known to be able to assess the stability of an arbitrary difference algorithm is the matrix method [3, ch. 3], which is not a practical stability test and gives no insight into how any instability may be avoided.

In this letter, it is shown that the standard FDTD algorithm is formally equivalent to the time-discretized simulation of a passive electrical circuit. If a modified FDTD algorithm retains this correspondence then given that it satisfies the Courant criterion [4], the algorithm will be stable.
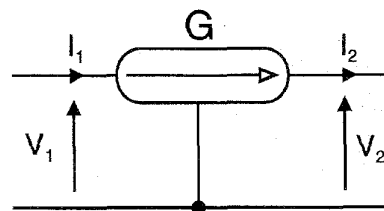
Fig. 1. A gyrator.

Determining the Courant criterion for a modified scheme is a separate problem that is not addressed in this letter. It must be stressed, however, that it is pointless to attempt such a determination unless the *spatial* discretization leads to a stable system. In the authors' experience, once the latter condition is satisfied the algorithms are, without exception, stable at the usual Courant limit.

The equivalent circuit has been successfully used to suggest modifications to the distorted CP algorithm [1] and the SFS method [2], which have completely removed instability problems. In this letter the former application is demonstrated.

II. AN EQUIVALENT PASSIVE CIRCUIT FOR FDTD

In this section it is demonstrated that the FDTD equations are formally identical to those relating the voltage variables in a passive circuit.

Although a passive lumped circuit equivalent to Maxwell's equations has previously been published [5], the form of this circuit does not lend itself easily to the analysis of FDTD schemes. Passive circuits have also been reported for the planar FDTD algorithm [6].

It is emphasized that the circuit presented here is not a lumped equivalent to some discontinuity or other physical feature to be included in the FDTD algorithm—it is an equivalent to the algorithm itself.

It is necessary to define the generalized gyrator shown in Fig. 1 whose symbol includes an arrow that serves to set a reference direction for the currents at ports 1 and 2; with the reference shown, the relationships at ports 1 and 2 are $V_2 = GI_1$ and $V_1 = GI_2$.

Now consider the circuit given by Fig. 2 consisting of a network of gyrators wth capacitors attached to each junction of the circuit (for clarity only one capacitor is shown). Considering power flow in the circuit, it is simple to show that
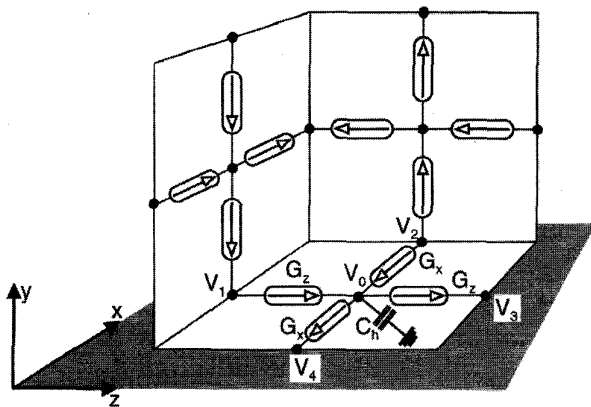
Fig. 2.   Yee algorithm equivalent circuit.

Fig. 3.   Cross section through cylindrical resonator.

Fig. 4.   Modified equivalent circuit.

the circuit is entirely passive. Summing currents at the node labeled V_0 gives, for example

$$C_h \partial_t V_0 = G_x(V_2 - V_4) - G_z(V_3 - V_1). \qquad (1)$$

With the following change of variables: $C_h \to \mu_0 \Delta_x \Delta_y \Delta_z$, $G_x \to \Delta_y \Delta_z$, $G_z \to \Delta_x \Delta_y$, $V_0 \to H_y(i + 0.5, j, k + 0.5)$ and transforming $V_1 \ldots V_4$ to the four corresponding electric components in the algorithm, the FDTD equation for $\partial_t H_y(i + 0.5, j, k + 0.5)$ is recovered.

The relationships for all the fields in the FDTD algorithm can be derived in this manner from the circuit; the only aspect of the FDTD algorithm not represented in the circuit is the discrete time approximation of the continuous time derivative ∂_t.

The main theorem is

• *An FDTD algorithm cannot be unconditionally unstable if it arises from the centred-difference discrete-time approximation to the passive circuit.*

The proof of this intuitive result is given in [7, ch. 5] but is too lengthy to be included here.

It can be seen, therefore, that if the FDTD method is to be modified, the modification should be based on a modification of the passive circuit. In this way unconditionally unstable schemes will not arise and there will always be a value of time step that guarantees stability—this point is illustrated by the following example.

III. APPLICATION OF EQUIVALENT CIRCUIT

As a simple example of a situation where the equivalent circuit provides an invaluable insight into the stability of a modified FDTD algorithm, consider using the deformed Contour method [1] to model the interior of a perfectly conducting cylinder (radius 19 cm) on a two-dimensional uniform mesh (mesh spacing $\Delta = 5$ cm). A portion of this model is shown by Fig. 3; E_x and E_z fields are indicated by arrows, the H_y components by crosses.

The CP method is employed (as described in [1]) to alter the FDTD algorithm close to the curved surface. When calculating $H_y(0.5, 0.5)$, however, the algorithm requires the value of $E_z(1, 0.5)$—which itself cannot be calculated from its sur-
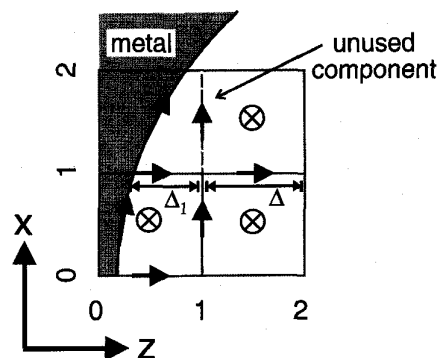
rounding H components. In this case the technique 'borrows' the nearest collinear E field, $E_z(1, 1.5)$.
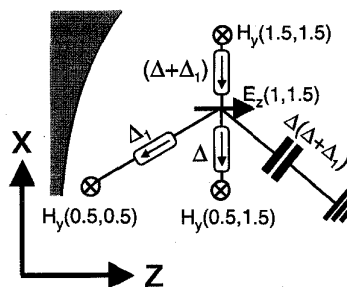
This results in the situation where $H_y(0.5, 0.5)$ is updated from $E_z(1, 1.5)$ but not *vice-versa*. In terms of an equivalent circuit this situation *cannot* be achieved with passive components and must be avoided. Instead, an extra gyrator is introduced between $H_y(0.5, 0.5)$ and $E_z(1, 1.5)$, as shown in Fig. 4. The only change to the algorithm is the updating of $E_z(1, 1.5)$ which *is* now a function of $H_y(0.5, 0.5)$

$$\epsilon_0(\Delta + \Delta_1)\Delta \partial_t E_z(1, 1.5) = (\Delta + \Delta_1)H_y(1.5, 1.5)$$
$$- \Delta H_y(0.5, 1.5) - \Delta_1 H_y(0.5, 0.5). \qquad (2)$$

The CP method involving field borrowing exhibited instability after a few hundred iterations even with greatly reduced values of time step and results were unavailable. The new technique, derived using the equivalent circuit, exhibited no instability even with an unmodified time step of $\Delta/c\sqrt{2}$ and produced the first two TE resonant frequencies of the cylinder within 1% of analytic results.

IV. CONCLUSION

This letter has shown the existence of an equivalent passive circuit for the FDTD method. Such a passive representation ensures that the algorithm (given an appropriate limit on the time-step) is stable.

The equivalent circuit concept is shown to stabilize the field "borrowing" procedure widely used in the contour path method. The usefulness of the equivalent circuit is not limited, however, to this problem; it has also been applied to stabi-

lize other variants of the CP and SFS methods [2]. Further applications of the circuit are under investigation.

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