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On the stability of solitary wave solutions of the 5th-order KdV equation

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Abstract

The Korteweg-de Vries equation with a fifth-order-derivative dispersive perturbation has been used as a model for a variety of physical phenomena including gravity-capillary water waves. It has recently been shown that this equation possesses infinitely many multi-pulsed stationary solitary wave solutions. Here it is argued based on the asymptotic theory of Gorshkov & Ostrovsky (*Physica D*, **3** (1981) 428-438) that half of the two-pulses are stable. Comparison with numerically obtained two-pulses shows that the asymptotic theory is remarkably accurate, and time integration of the full partial differential equations confirms the stability results.

1 Introduction

The Korteweg de Vries (KdV) equation with an additional fifth-order dispersion term

$$u_t + 6uu_x + u_{xxx} + u_{xxxxx} = 0 \tag{1.1}$$

has been used as a model for gravity-capillary waves on a shallow layer [16], a chain of coupled nonlinear oscillators [8] and magneto-sound propagation in plasmas [13]. Of particular interest are solitary waves; $u(x,t) = v(z)$, $z = x - ct$, $v(z) \to 0$ as $z \to \pm\infty$. Hence, after an integration, $v(x)$ is a homoclinic solution of

$$u'''' + u'' - cu + 3u^2 = 0. \tag{1.2}$$

A scaled version of the ODE (1.2) arises also in structural mechanics [10, 11] and homoclinic solutions of it have recently been considered by [5, 2, 1]. By a mixture of analytic and numerical results it is known that for $-\infty < c < -1/4$, corresponding to the eigenvalues of equation linearized at the origin being complex, there are infinitely many homoclinic solutions with oscillatory tails. Only two solutions persist for all c in this range. One of these persistent solutions is the primary pulse consisting of a single deep trough with decaying oscillatory tails. The other consists of a pair of troughs separated by a single positive maximum (referred to as a **2**(2)-solution in the notation of [1], where the first digit refers to the two troughs and the second is a measure of their separation). The small amplitude bifurcation of these two solutions at $c = -1/4$ can be described by normal form theory [12], see [9].

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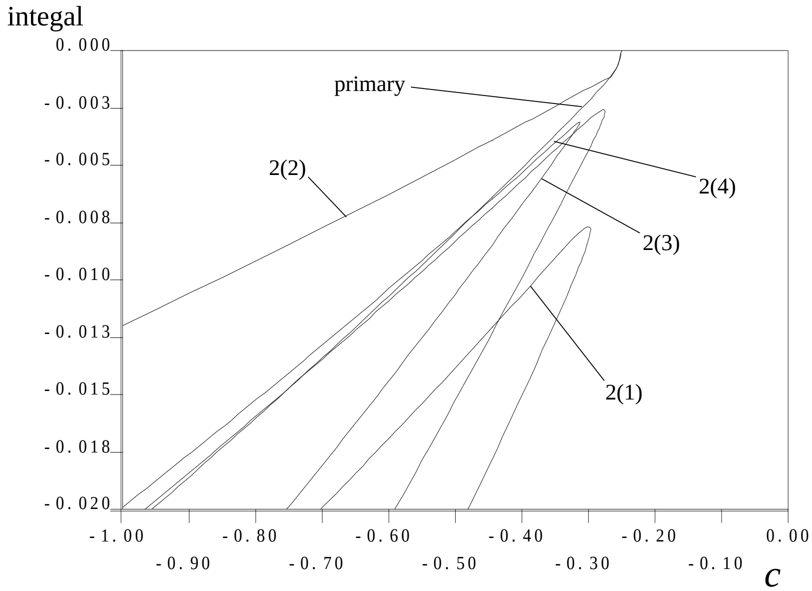

Figure 1: Bifurcation diagram with c of the simplest solitary wave solutions to (1.2). The ordinate is the $\frac{1}{2}\int u(z)dz$, divided by the length of the interval of computation
.

In addition (1.2) admits an infinite family of multi-pulse solutions labeled by arbitrary stings of integers $\mathbf{n}(m_1, m_2 \ldots m_{n-1})$, $\mathbf{n} > 1$, $m_i > 0$, in keeping with the theory of [6]. Note that in general, unlike the usual 3rd-order KdV equation, these solitary waves cannot be expressed in closed form (except for the primary pulse at a single value of c). It has been observed numerically that branches of each solution cease to exist at a c-value strictly less than $-1/4$, whereupon they either turn around at a *coalescence* point or die in a bifurcation from a symmetric solution [1].

In this paper, for simplicity, we consider only two-pulse solitary waves. The bifurcation diagram of the primary and the first few two-pulse solutions to (1.2) is depicted in Figure 1. It should be noted that all these solutions $u(z)$ are even functions of z.

2 The asymptotic theory

The method of Gorshkov & Ostrovsky [7], approximates a two-pulse solitary wave as a bound state of two copies of the primary pulse. This approximation is valid in the limit of large separation of the two troughs (i.e. for $\mathbf{2}(m)$ solutions for m sufficiently large). The two primary pulses are then regarded like independent particles with mass

$$M = -\frac{1}{2}\frac{\partial}{\partial c}\int_{-\infty}^{\infty} u(z)^2 \mathrm{d}z$$

(the minus sign was missing in [7] which, having rechecked the calculations carefully, appears to be a misprint in that paper) and interaction potential

$$U(s) = \frac{1}{2}\int_{-\infty}^{\infty}\left(u^2(z)u(z-s) + u^2(z-s)u(z)\right)\mathrm{d}z \tag{2.3}$$

where s is the separation of the two pulses. For this analogy to work we require $M > 0$ which is obvious from the shape of the primary curve in Figure 1. Within this analogy, that ensures that the primary

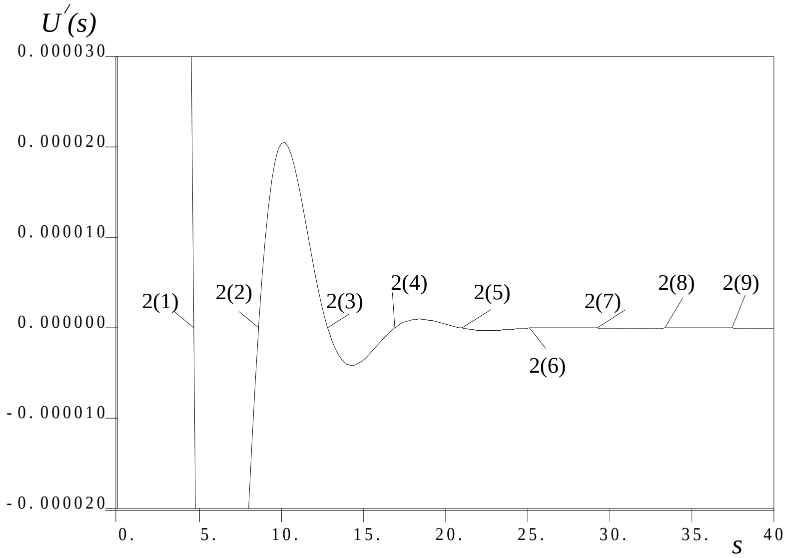

Figure 2: The function $U'(s)$ evaluated numerically along the primary orbit. The label of each zero is the label of the corresponding two-pulse orbit

pulse is stable. Bound states are then given in this approximation by extrema of $U(s)$, with maxima representing stable solutions and minima representing unstable solutions.

Figure 2 represents the function $U'(s)$ for $c = -4/9$ which was obtained numerically by computing two copies of the primary orbit with their separation s allowed to vary. Actually, for computational efficiency we used the evenness of the primary orbit to compute two solutions $u(x)$ and $u(-x)$ up to their points of symmetry over an interval $(0, s)$. The boundary conditions at the other end place the solution in the linearized unstable (or stable) manifold of the origin (see [4]). Thus only the large central part of the integral (2.3) was evaluated. The missing portions were found to be insignificant.

Table 1 shows the s-values of zeros of $U'(s)$ depicted in Figure 2. These are compared with the true distances between the two troughs of two-pulse solutions of (1.2) obtained using accurate numerics (as in [4]).

Note from the table that the asymptotic theory is accurate in describing two-pulse solutions at this c-value even when the separation between the troughs is small (e.g. the **2**(2)). The consistent error (in the third decimal place) between the numerical and asymptotic values of s for the greater separation distances may be accounted for by the truncation mentioned above in the computation of the integral $U'(s)$.

3 Numerical simulations

We have investigated the dynamics of slightly perturbed solitary waves using direct numerical simulations of (1.1) based on a split-step Fourier method (see [15]) with periodic boundary conditions. The analysis is technically very close to that in [3] where the stability of multi-hump solitary waves of the generalized NLS equation was studied. Some results of our numerical modeling, for $c = -4/9$ are presented in Figure 3.

Final conclusions about the stability of multi-hump stationary solitary waves have been made after

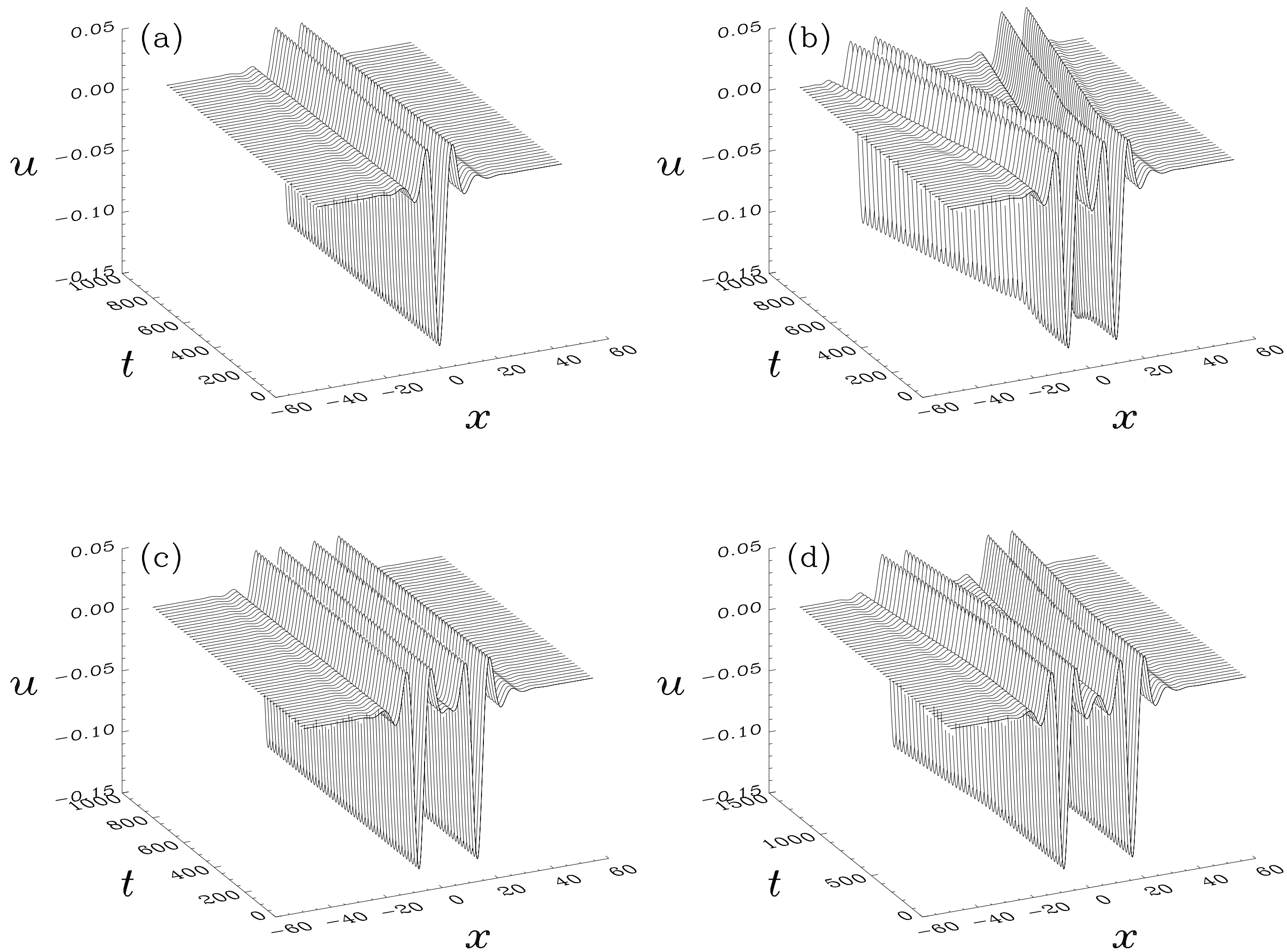

Figure 3: This Figure shows the propagation of several slightly perturbed solutions of equation (1.1). For all cases $c = -4/9$ and the amplitude of an asymmetric perturbation is about 0.01% of the solitary wave amplitude. (a) Stable evolution of the primary solution. (b) Decay of the $\mathbf{2}(4)$ solution into two primary solutions. (c) Stable evolution of the $\mathbf{2}(5)$ solution. (d) Decay of the $\mathbf{2}(6)$ solution into two primary solutions.

solution	numerical	asymptotic	$U''(s)$
$\mathbf{2}(1)$	5.17246	4.66964	-ve
$\mathbf{2}(2)$	8.62103	8.61820	+ve
$\mathbf{2}(3)$	12.72918	12.82627	-ve
$\mathbf{2}(4)$	16.91598	16.90344	+ve
$\mathbf{2}(5)$	21.02929	21.03099	-ve
$\mathbf{2}(6)$	25.14375	25.13911	+ve
$\mathbf{2}(7)$	29.25092	29.72074	-ve
$\mathbf{2}(8)$	33.36998	33.36694	+ve
$\mathbf{2}(9)$	37.48316	37.48048	-ve

Table 1: Showing the values of s (third column) and sign of $U''(s)$ at each zero of the asymptotic function $U'(s)$. Also given in the second column are the distances between the two large troughs of the corresponding numerically computed two-pulse.

a numerical spectral analysis of the corresponding linearized equations (which are obtained by linearization of (1.1) about the stationary solutions of interest). The existence of an exponentially growing mode in the spectrum of linearized problem indicates the instability of the corresponding solitary wave. We also repeated the numerical simulations for the corresponding solutions at $c = -16/49$. In all cases our results were fully consistent with the predictions of Section 2.

4 Conclusion

In this note we have demonstrated the surprising accuracy of the asymptotic method of Gorshkov & Ostrovsky in determining the existence and stability of two-pulse solitary wave solutions of the 5th-order KdV equation. Moreover, these results, together with the direct numerical simulation of the PDE strongly suggest that half of the two-pulse solutions are stable. The other half develop a mode of instability that causes the wave to split into two simpler waves traveling at different speeds (see Figure 3(b),(c)). The stable two-pulse solutions can be characterized by those whose label is $\mathbf{2}(n)$ where n is odd. These solutions have the property that their symmetric point is either a negative maximum of the graph $u(z)$ or is a positive minimum.

We have not investigated the stability of solitary waves consisting of more than two pulses, nor have we investigated the stability implications of the coalescence and symmetry-breaking bifurcation points known to occur for the traveling wave ODE. These issues are left for future work.

Finally, we mention some recent numerical results by Malomed & Vanden-Broek [14] on the collision of multi-pulse solitons for equation (1.1). They took the primary and $\mathbf{2}(\mathbf{1})$ and found them to be reasonably stable under interactions with themselves and each-other, apart from the emission of a small amount of radiation. In contrast, two three-pulse solutions were found to break up into simple waves under collision. It would be interesting to see whether some of the other stable two-pulse solutions described here are similarly approximately stable under collision.

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