A Concept for Exploring Western Music Tonality in Physical Space

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ABSTRACT

Musical theory about the structure and morphology of Western tonality is quite difficult to teach to young children, due to the relatively complex mathematical concepts behind tonality. Children usually grasp the concepts of musical harmony intuitively through listening to music examples. Placing the 12 notes of the well-tempered scale into a spatial arrangement, in which the proximity of these notes represents their mutual harmonic relationship, would allow to link physical motion through a spatial area with the exploration of music tonality. Music theorists have postulated the Circle of Fifth, the "Spiral Array", and the "Tonnetz" as paradigms for spatial arrangements of music notes which allow mapping the distance between notes onto their "mutual consonance". These approaches mostly have been of qualitative nature, leaving the actual numeric parameters of the spatial description undetermined. In this paper, these parameters have been determined, leading to a concrete numerical description of the planar Tonnetz. This allows the design of a physical space in which the music notes are distributed in space according to their musical consonance. Set up in an outdoor area, handheld devices (e.g. PDA) with integrated Global Positioning System can be used to play these notes at their actual physical location. This makes it possible for children to explore this musical space by moving through the real spatial area and experience the relationships of the notes through their proximity. Defining a range for each note as a circular area around each note location, consonant chords can be produced in those areas where those circles overlap. Using this concept, games can be developed in which the listeners have to perform certain tasks related to this musical space. This appears to be a promising approach for the music education of young children who can intuitively learn about music morphology without being explicitly taught about the complex theoretical mathematical background.

Categories and Subject Descriptors

J.4.3 [**Social and Behavioral Science**]: Psychology. H.5.5 [**Sound and Music Computing**]: Methodologies and Techniques. K.3.1 [**Computer Uses in Education**]. Computer managed instruction. K8 [**Personal Computing**]: Games.

General Terms

Algorithms, Experimentation, Human Factors, Theory.

Keywords

Keywords: music, tonality, musical consonance, harmony, learning, playing.

1. INTRODUCTION

Playing in an outdoor space is an activity that is essential to childhood [1]: children can learn about their environment through motion and physical activity within a spatial context, and in games which involve motion through physical space, many motoric skills of the developing child are being exercised and strengthened, navigation skills and spatial awareness are being learned intuitively and "naturally". This leads to the question if this natural learning process can be linked to learning about more complex issues in a completely different domain, by linking the physical space to certain elements of that other domain. Western music tonality, for example, has been described by music theorists through spatial arrangement of music notes in certain ways which reveal the connection and relationship of these music notes. The mathematical background of music tonality, consonance, and harmony is difficult to grasp by young children [2]. Exploring this musical space by employing the metaphor of a real physical space has the potential to enable children to learn about the connection between music notes in an intuitive playful way.

A big obstacle in using this metaphor for actual practical teaching has been that the numeric description of the music morphology space had been not completely determined: in many theories only quantitative descriptions of the relation between the notes have been postulated, leaving several variables open to arbitrary values and preventing a unique mapping into physical space.

We have investigated the numeric parametrisation of the "Tonnetz", which is a planar arrangement of music notes according to their "consonance", and have been able to determine all numeric parameters describing this model. This allows now to design a physical space, where locations are associated with musical notes, being able to be explored by children.

In chapter 2 an overview will be given on the theoretical models. Chapter 3 contains the derivation of the numerical metric for the Tonnetz model, and Chapter 4 contains suggestions for using this in a learn-by-play environment for teaching children about the musical morphology of Western tonal music.

2. THEORETICAL BACKGROUND

2.1 Well-Tempered 12-Tone Music Space

Music in the Western tradition has evolved into using scales with 12 discrete tones. Originally in Antiquity, these tones have been defined by the ratios of their frequencies and resulted in the “just intonation” of the diatonic scale. If musical instruments are tuned according to this type of scale, music played on these instruments cannot transition between scales, as the frequency ratios between the notes in an interval are different after transposition. In the past three centuries, the well-tempered scale has been adopted by slightly shifting tones in the scales until the ratio between each set of two neighboring tones is $2^{1/12}$. This scale allows each tone in the scale to become the base tone of a new scale, as the ratio between the scale tones remains invariant in this transposition. This has allowed Western music to modulate between different scales, often using ambiguities of chords which belong to different scales as the links between those scales. The Musical Instrument Digital Interface (MIDI) standard has incorporated this well-tempered 12-tone scale and has defined a set of integer numbers to represent the notes in this scale.

2.2 Consonant and Dissonant Intervals

It is of interest for musical analysis to sort the notes of the well-tempered scale in such a way that “consonant” notes are close to each other, and “dissonant” notes are further apart from each other. The reason for such an order is to have an immediate measure for the dissonance, which then allows quantifying dissonance in a music piece. In order to achieve this, one needs first to define the terms “consonant” and “dissonant”, as the meaning of these terms has been somewhat blurred in the 20th century classical music.

The definition of musical closeness in traditional music conventions can be derived from the harmonics of a string as integer multiples of the base frequency (see Table 1): the first few upper harmonics (<8) are perceived to be more consonant than the higher upper harmonics. This indicates that the octave is the most consonant interval, followed by the Fifth (Quint), the Quart, the Major Third and the Minor Third.

Table 1. The harmonics of a string and the related intervals.

	Base	1st harmonic	2nd harmonic	3rd harmonic	4th harmonic	5th harmonic	6th harmonic
Multiple of f	1 x f	2 x f	3 x f	4 x f	5 x f	6 x f	7 x f
Example key: C major	C1	C2	G2	C3	E3	G3	A#3
Interval		Octave	Quint	Quart	Major Third	Minor Third	Minor Third

In general, the intervals between two tones are more “consonant” if the integer numbers in the ratio of the frequencies are smaller. The combination of three notes with those intervals leads to the traditional tri-chords of Western classical music. When just considering the notes without the octave transposition, then Quint (Fifth) and Quart (Fourth) actually are equivalent, representing the same two notes. Similar, Major and Minor Third can be considered as equivalent in terms of their consonance, as inserting the 4th harmonic (e.g. E) into the Fifth interval (e.g. C-G) automatically creates both the Major and the Minor Third.

2.3 The Circle of Fifths

The most elementary graphical representation of the 12 notes of the well-tempered scale is the Circle of Fifths. In this graph the 12 notes of a scale (modulo octave transpositions) are placed onto a circle: neighboring notes are separated by a fifth interval, hereby showing the two most consonant intervals. For the sake of simplicity, from here on the C key scale is assumed as the base scale in this paper. The results for any other scale can easily be achieved by transposition.

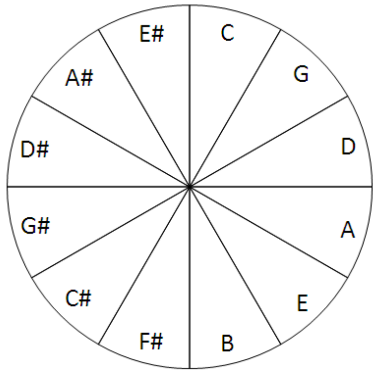

Figure 1. The Circle of Fifth.

The Quint circle (Circle of Fifth) representation in Figure 1 shows correctly the proximity of two notes separated by a Fifth and also represents the Tritone (C-F#) as the furthest apart note pair. However, after two consecutive transformations by a Fifth transposition (C→G→D), the resulting interval is a Second. When interpreting the distance of the notes as a measure of consonance, the graphical representation indicates that a Second would be closer in tonality than a Third (which comes after a 3rd transposition (C→G→D→A). This is not in accordance with the definition of tonality in 2.2.

2.4 The Spiral Array

In her PhD thesis, Elaine Chew [3] proposed a modification of this Circle of Fifth, using the 3rd dimension to overcome this problem of the Third being further apart than the Second: the circle is bent into a spiral in such a way that one full 360 deg rotation of the spiral contains 4 consecutive notes from the Circle of Fifths. This brings the Third interval notes closer together than the Second, as shown in Figure 2. For example, the interval C-D is now represented by a larger distance than the intervals C-G or C-E. The morphology of this visualization means that the full Circle of Fifth is completed in 3 spiral loops (C-G-D-A-E, E-B-F#-C#-G#, G#-D#-A#-E#-C).

Introducing h as the spiral height of one full rotation (in Chew’s work, $4h$ is chosen for this height) and r as its radius, the distance between the notes of a Major Third (M3) is h. The distance of a Fifth which is the smallest distance between two notes, becomes $\sqrt{h^2/16+2r^2}$. The distance of a Minor Third (m3) becomes $\sqrt{9h^2/16+2r^2}$

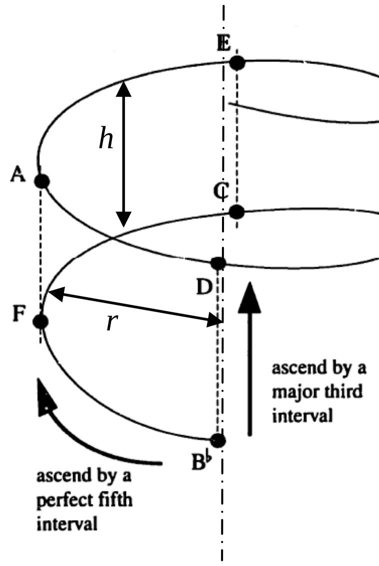

Figure 2. Chew's Spiral Array (adapted from [3]).

2.5 The Minor and Major Third

The Spiral Array model has been proven to be a valuable tool for qualitative analysis of music in terms of structure, key chords, tonality etc. There are, however, limitations of this model, when a precise quantification of the tonality relations is sought. Especially the exact value of the ratio *h/r* needs to be determined, in order to evaluate the precise spatial relationships of the notes. By applying musical theorems, Chew could determine boundaries for this ratio. For example, the Fifth has a smaller distance than M3, which leads to a boundary condition for the ratio $\frac{h}{r} > \sqrt{32/15}$. Chew put forward more constraints and obtained an upper limit for $\frac{h}{r} < \sqrt{32/7}$. One of those theorems is that the Minor Third interval is further apart than the Major Third interval. There is, however, no strong argument for them being different. Under the assumption that M3 and m3 are equivalent (in the same way as Fifth and Fourth are equivalent) as assumed in this paper here, the ratio becomes:

$$\frac{h}{r} = \sqrt{32/7} \qquad \textbf{(1)}$$

The spatial arrangements of the notes on the spiral would actually have to be mapped onto a torus. Chew proposed to wrap the surface of the spiral onto a torus, which would represent better the invariance to the octave transposition of notes: the surface needs to be closed so that the periodicity is represented properly. This means that after 3 spiral rotations the surface of the spiral is closed. For a correct quantification of the distances between the notes, these distances would have to be computed in the 3D torus model.

2.6 The Tonnetz

In 1739, the mathematician Leonard Euler developed a planar arrangement of the 12 musical notes called the "Ton-Netz" (tone net) [4]. In the 19th century, the music-mathematician Hugo Riemann [5] has further developed this model. In this note arrangement, the proximity of the notes indicates their closeness in consonance. Again, the closest distance between notes is the Fifth. Basically, the Ton-Netz is a series of linearized Circles-of-Fifth, put next to each other. Implicitly, through the invariance towards octave transposition, this representation contains the Spiral Array. In recent years it has inspired further research into music morphology (e.g. [7][9]), as this model appears to represent well the human listener [8].

The Tonnetz can be considered as a planar structure built from a repetitive cell pattern, where the unit cell is being defined as a parallelogram with the same note on each of its 4 corners. An example of such a unit cell with the note C is shown in Figure 3. This unit cell contains all 12 notes.

Much work has been done regarding describing and studying the qualitative aspects of this structure and its meaning for music (e.g. [6]) by defining transformations and applying graph theory, but the specific numerical values have not been determined. Without loss of generality, the distance between Fifth (along the Fifth axis) can be set as the unit distance to be $f = 1$. This needs to be the smallest distance between two neighboring notes. The construction of the Tonnetz based on the set of parallel Fifth axes introduces two more parameters: d = distance between two adjacent Fifth lines, and s = shift between these two Fifth axes. The distance d needs to be set so that the unit distance still is the smallest distance between two notes. The Fifth axes are placed in such a way that the tones of one axis are neighbors to the Third notes on the closest parallel axis. This leaves the shift s only to indicate how the relation between Major and Minor Thirds are: a value of $s = 0$ will place the Major Third as the shortest distance (C across to E), a value of 1 will place the Minor Third as the shortest distance (C across to A). A value of $s = 0.5$ indicates the same distance of Major and Minor Third. This is what is assumed in this paper, as discussed earlier. A further parameter is introduced here: t = the distance of the Third.

$$t = \sqrt{\tfrac{1}{4} + d^2} \quad > 1 \qquad \textbf{(2)}$$

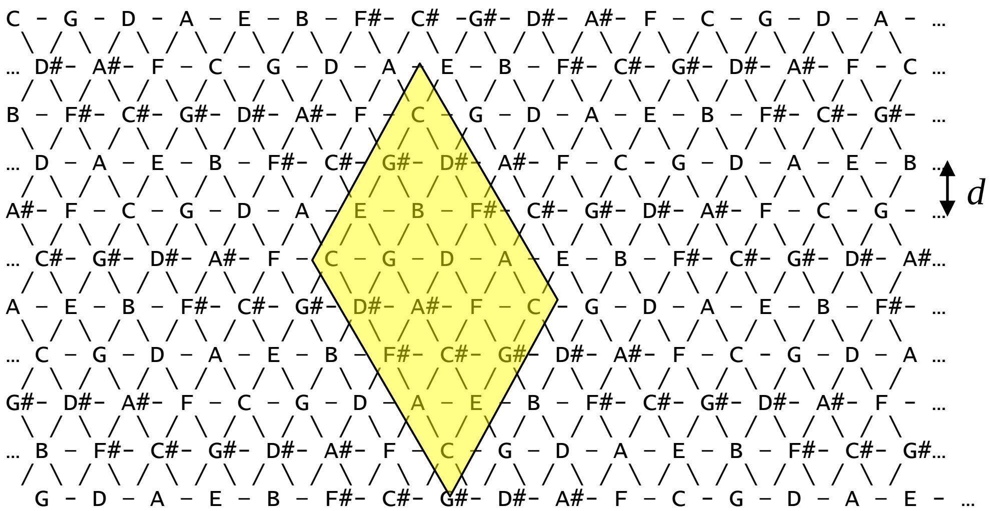

Figure 3. Riemann's Tonnetz [5]. A "unit cell" with the note C at its corners is highlighted.

3. METRIC

3.1 Torus Model

The unit cell in Figure 3 forms the surface of a torus, as it is a continuous closed surface. This is the result from the invariance towards transposition by an octave: the notes after octave transposition are assumed to be identical. In order to fulfill the condition that this parallelogram can be turned into a torus surface, the individual note locations at the cell borders need to match up: this leads to the requirement that the spatial note arrangement must be in such a way that a rectangle must be able to be placed onto the surfaces so that its four corners are coinciding with identical notes (e.g. C). When trying to fit a rectangle onto the Tonnetz surface with its corners at the note C, the underlying Tonnetz needs to be deformed until the notes C match with the corners of this rectangle. This is done by varying the distance of the lines of Fifths (C-G-D-A-…), i.e. the distance between the Thirds. In Figure 4 the unit cell rectangle is shown (with the sides as the Third axes) together with the parallelogram which is a result of this transformation. The parallelogram needs to be created by a rotation of the notes array around one corner point C. This will increase the distance C-B, but will reduce the distance G#-D# (along the lines of Fifth). The sides and therefore the Third distance *t* (e.g. C-D#) remain constant.

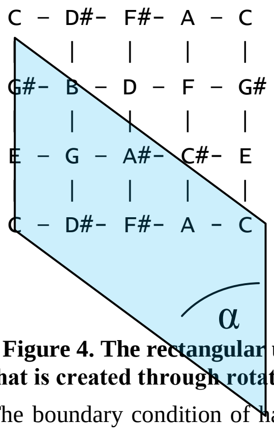

Figure 4. The rectangular unit cell and the parallelogram that is created through rotation of the "long" axis around C.

The boundary condition of having the Fifth distance being the shortest distance is violated in the case of the straight rectangle: for example, the interval C-G is on the diagonal path and therefore is longer than each of the two bordering sides C-D# and C-E. This means that the plain rectangular shape of the unit cell with borders along notes is not permissible for the torus structure.

The resulting parallelogram has its two sides 3t and 4t long. This determines the angle of the parallelogram: $\alpha = \arccos(3/4) = 41.41°$. Further geometric considerations lead to the Fifth distance $f = 2t \sin \alpha/2 = 0.707\ t$. This can also be obtained without explicit angle calculation and results in the value for $f = t/\sqrt{2}$. Having assumed this Fifth interval $f = 1$, the resulting value for $t = \sqrt{2}$. For the spacing of the Fifth axes, equation (1) provides the result:

$$d = \frac{\sqrt{7}}{2} \qquad \textbf{(3)}$$

This now determines the full spatial structure of the quantitative Tonnetz and allows quantitative analysis of music. It is interesting that another outcome of this particular arrangement of notes is that the distance between a Second interval has the value 2. The most relevant intervals are shown in **Error! Reference source not found.**.

Table 2. Interval distances in the quantified Tonnetz.

Interval	Fifth	Third	Second
distance	1	$\sqrt{2}$	2

3.2 Chord Consonance

Each note in the Tonnetz can be assigned a "range", at which the sound of the note can be heard. This leads to overlapping circles in the Tonnetz. These overlapping regions (Figure 5) are the areas where chords can be heard. Due to the nature of the note arrangement in the Tonnetz, where "consonant" notes are close to each other, these chords are harmonic in the sense defined in section 2.2. The radius of those circles determines the range where the note can be heard. It needs to be chosen so that the Fifth and the Third interval produce an overlap, resulting in harmonious chords. Where two circles overlap, a duo-chord can be heard. In the region where 3 circles overlap, a tri-chord can be heard, consisting of a Fifth and the Third in the middle (either minor or Major).

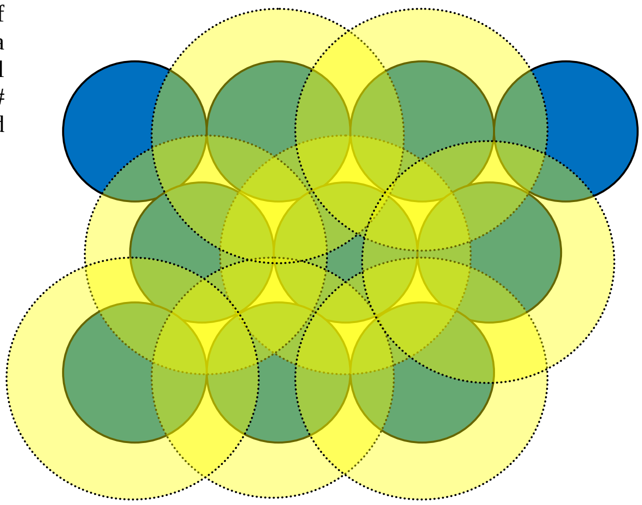

Figure 5. Overlap of note sound. Concentric circles are shown with radius = 0.5 (inner circle) and radius = 1 (outer circles).

4. LEARNING GAMES

4.1 System Implementation

To put the concept of the planar Tonnetz into a learning game, a mobile device is needed for each pupil which has a location sensing mechanism and a tool for replaying audio. Suitable devices for such an implementation are PDAs and mobile phones with integrated Global Positioning System (GPS). The precision of GPS is about 1-3m, which means that the Tonnetz grid needs to be large enough to be tolerant to this location inaccuracy. Also, non-kinematic GPS provides only updates once per second, which means that the note grid has to be set up large enough, to prevent that a fast using user accidentally skips a note. The Tonnetz would need to be setup not just as one unit cell, but as a tiled pattern of unit cells, allowing different kinds of user motions across the note arrays without boundary limits.

Hewlett Packard (HP) has developed the generic development environment MSCAPE for setting up location-based applications (Mediascapes) on a Windows Mobile powered device with GPS. It supports triggering events when the device reaches certain locations. The code is based on Java and allows a flexible reaction to user motion within physical space. We have recorded a set of 12 wavefiles, one for each tone. The array of tones has then been set up as a Tonnetz, using the geometry relation that has been derived in chapter 3. One variable parameter can be set externally: the distance between the Fifth *f* determines the extent of the Tonnetz unit cell. This parameter can be adapted to the dimension of the available space in which the users can move about. When the device is within the range and vicinity of a note, its wave file is being played. If the user is within the vicinity of several notes, they are all being played simultaneously, providing consonant tri-chords.

4.2 Games Ideas

One possible game could be that the pupils need to reach each note only once: when reaching a unique note, the user receives reward points. When a note is reached a second time, points are subtracted. This playful spatial exploration has the potential to sharpen the perception of atonal 12-tone music in which no single tone has the preference for being a base tone.

Another game would be to “play” a melody, a sequence of note in a given order, by walking across the spatial Tonnetz. Points are collected for each correctly reached note location, points are subtracted for false notes. This can sharpen the child’s perception of note relationships, as the spatial arrangement does not relate to notes which are neighboring in the pitch scale, but are neighboring in the functional sense.

The motion across the Tonnetz also can highlight the meaning of music cadences: these are sequences of closely related chords, which are in fact closely spaced in the Tonnetz representation. Simply moving across the space makes the user experience the relationship of notes and chords within the Western tonality space.

This spatial music exploration tool can also be used for creative purposes: children can be asked to “compose” a piece of music by simply moving across the Tonnetz in a pattern they choose. The mobile device can record the notes, and in the end each child can reply their recorded music work.

5. CONCLUSION AND OUTLOOK

A concept has been shown for linking playful spatial outdoor games with an understanding of musical relationships of Western music tonality. Mathematical boundary conditions have led to an exact numeric quantification of the spatial parameters of the Tonnetz, which then in turn can be mapped onto a real physical space. Mobile phone or PDAs with integrated GPS can be used as devices for exploring this physical space, as they can play the notes linked to the nodes of the Tonnetz.

This concept has not yet been tested with children. It would be interesting to conduct such a pilot study with an actual class of young children and assess the potential and effectiveness of this approach for music education. This can be done in partnership with local schools.

It needs to be noted that the well-tempered 12-tone musical space is something that is only attributable to the Western cultural domain. There are other possible music morphology concepts which are based on different intervals. One would need to set up a spatial implementation of those concepts, which will result in different spatial notes arrangements.

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