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Stochastic ordering of bivariate elliptical distributions

Zinoviy Landsman
Department of Statistics
University of Haifa

Andreas Tsanakas*
Lloyd's of London

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Abstract

It is shown that for elliptically distributed bivariate random vectors, the riskiness and dependence strength of random portfolios, in the sense of the univariate convex and bivariate concordance stochastic orders respectively, can be simply characterised in terms of the vector's $\mathbf{\Sigma}$-matrix.

Keywords: Elliptical distributions, convex order, concordance order, dependence, risk management.

*Corresponding author: Andreas Tsanakas, Market Risk and Reserving, Lloyd's, One Lime Street, London EC3M 7HA, UK. Tel: +44 (0)20 7327 5268, fax: +44 (0)20 7327 5658, e-mail: andreas.tsanakas@lloyds.com. The authors would like to thank an anonymous referee for helpful suggestions. The views expressed in this paper are solely those of the authors, and do not necessarily reflect the views of Lloyd's of London.

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1 Introduction

Elliptical probability distributions have gained prominence in recent years as effective tools for multivariate modelling in risk management. Elliptical distributions, introduced by Kelker (1970) and further discussed by Fang et al. (1987) constitute generalisations of the multivariate normal family, which allow for the presence of heavy tails and asymptotic tail dependence. The importance of elliptical distributions for risk management and actuarial science has been highlighted by Embrechts et al. (2002), while the application of elliptical distributions in insurance was further studied by Landsman and Valdez (2002).

Stochastic orders (Shaked and Shanthikumar (1994), Dhaene and Goovaerts (1996), Müller and Stoyan (2002)) provide methods of comparing random variables and vectors. It is shown in the present paper that, among two elliptically distributed risks with equal means, the one with the higher variance (or the higher value of the diagonal elements of the $\mathbf{\Sigma}$-matrix if the variances do not exist) is the riskiest in the stop-loss and convex order senses. This implies that for elliptically distributed random variables the variance presents a comprehensive means for quantifying risk, and is consistent with economic concepts such as utility theory and second order stochastic dominance.

Furthermore the effect of correlation on bivariate elliptical distributions is examined. It is shown that for two bivariate random vectors, belonging to the same elliptical family, the ordering of their correlation coefficients (or corresponding quantities if the covariance matrix does not exist), is equivalent to their being ordered in the concordance order sense. This is equivalent

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to saying that a bivariate elliptical cumulative distribution is increasing in the correlation coefficient. Moreover it is shown that the riskiness (in the stop-loss and convex order senses) of a portfolio of two elliptically distributed risks increases in the (generalised) correlation coefficient, which forms a stronger version of a result obtained by Dhaene and Goovaerts (1996).

2 Elliptical distributions

In this section the class of elliptical distributions is briefly discussed. Let Ψ_n be a class of functions $\psi(t) : [0, \infty) \mapsto \mathbb{R}$ such that the function $\psi(\sum_{i=1}^{n} t_i^2)$ is an $n-$dimensional characteristic function (Fang et al., 1987). It then follows that $\Psi_n \subset \Psi_{n-1} \cdots \subset \Psi_1$.

Definition 1. *Consider an $n-$dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$. The random vector $\mathbf{X}$ has a multivariate elliptical distribution, denoted by $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, \psi)$, if its characteristic function can be expressed as*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T\mu)\psi\left(\frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\right) \tag{1}$$

for some column-vector μ, $n \times n$ positive-definite matrix $\mathbf{\Sigma}$, and for some function $\psi(t) \in \Psi_n$, which is called the characteristic generator.

Besides the multivariate normal family, obtained by $\psi(t) = e^{-t}$, examples of elliptical distributions are the multivariate t, logistic, symmetric stable, and exponential power families.

The mean vector and covariance matrix of $\mathbf{X}$ does not necessarily exist; there are in fact elliptical families with infinite means and variances (Landsman and Valdez, 2003). It can be shown that if the mean exists, then for

$\mathbf{X} \sim \mathbf{E}_n (\mu, \mathbf{\Sigma}, \psi)$ it is $E(\mathbf{X}) = \mu$. If the covariance matrix also exists, it equals $Cov(\mathbf{X}) = -\psi'(0)\mathbf{\Sigma}$. The characteristic generator can be chosen such that $\psi'(0) = -1$ so that the covariance becomes $Cov(\mathbf{X}) = \mathbf{\Sigma}$.

An important property of elliptical distributions is that linear transformations of elliptically distributed vectors are also elliptical, with the same characteristic generator. Specifically, from (1) it follows that if $\mathbf{X} \sim \mathbf{E}_n (\mu, \mathbf{\Sigma}, \psi)$, A is a $m \times n$ dimensional matrix of rank $m \leq n$, and b is an m dimensional vector, then

$$A\mathbf{X} + b \sim \mathbf{E}_m \left(A\mu + b, A\mathbf{\Sigma}A^T, \psi \right). \tag{2}$$

A direct consequence of (2) is that any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator, that is, if the diagonal elements of $\mathbf{\Sigma}$ are $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$, then for $k = 1, 2, ..., n$ it is $X_k \sim \mathbf{E}_1 \left(\mu_k, \sigma_k^2, \psi \right)$.

Finally, let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a matrix such that $\mathbf{A}\mathbf{A}^T = \mathbf{\Sigma}$. Then for vector $\mathbf{X}$ the following stochastic stochastic representation holds (Fang et al., 1987, (2.12)):

$$\mathbf{X} \overset{d}{=} \mu + r \cdot \mathbf{A} \cdot \mathbf{U}, \tag{3}$$

where $r \geq 0$ is a random variable and $\mathbf{U}$ is uniformly distributed on the unit hypersphere $\mathbf{u}^T\mathbf{u} = 1$, $\mathbf{u} \in \mathbb{R}^n$.

3 Stop-loss and convex orders

Consider a set of random variables or risks $\mathcal{X}$. The partial *stop-loss order* on elements of the set of risks $\mathcal{X}$ provides a natural way of comparing the riskiness of probability distributions:

Definition 2. *For random variables $X, Y \in \mathcal{X}$, we say that X is smaller than Y in stop-loss order and write $X \underset{sl}{\leq} Y$, whenever*

$$E[(X-d)_+] \leq E[(Y-d)_+], \ \forall d \in (-\infty, \infty). \tag{4}$$

In insurance terms (2) is interpreted as a comparison of the stop-loss premiums of risks for any given retention (Dhaene and Goovaerts, 1996). If the random variables X, Y have equal means, stop-loss order corresponds to the notion of second order stochastic dominance, well-known in economics (Müller and Stoyan, 2002). For such variables the *convex order* can additionally be defined.

Definition 3. *For random variables $X, Y \in \mathcal{X}$ with equal means, we say that X is smaller than Y in convex order and write $X \underset{cx}{\leq} Y$, whenever*

$$E[v(X)] \leq E[v(Y)], \tag{5}$$

for all convex functions v such that the expectations exist.

Convex ordering of two random variables constitutes a strong statement on the comparison of their variability. There is a close relationship between the stop-loss and convex orders (Shaked and Shanthikumar, 1994):

Lemma 1. $E[X] = E[Y]$ *and* $X \underset{sl}{\leq} Y \ \Leftrightarrow \ X \underset{cx}{\leq} Y.$

Now, simple characterisations of the stop-loss and convex orders in terms of elements of the $\mathbf{\Sigma}$ matrix are obtained.

Theorem 1. *Consider $(X, Y) \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$ with a finite vector of means $\mu = (\mu_X, \mu_Y)$. Define $X' = X - \mu_X$, $Y' = Y - \mu_Y$. Then*

$$\sigma_X^2 \leq \sigma_Y^2 \iff X' \underset{sl}{\leq} Y', \tag{6}$$

where σ_X^2 and σ_Y^2 are the corresponding diagonal elements of the matrix $\mathbf{\Sigma}$.

Proof. From (2) it trivially follows that

$$X' \stackrel{d}{=} \frac{\sigma_X}{\sigma_Y} Y'. \tag{7}$$

Direct application of Theorem 1.15.8 in Müller (2002), yields (6). □

From Theorem 1, the following corollary can be easily obtained:

Corollary 1. *If $\mu_X = \mu_Y$, $\sigma_X^2 \leq \sigma_Y^2 \iff X' \underset{cx}{\leq} Y'$*

Proof: Follows directly from the fact that $X - \mu \underset{sl}{\leq} Y - \mu \iff X \underset{sl}{\leq} Y$ and Lemma 1. □

4 Concordance order and portfolio risk

Let $\stackrel{d}{=}$ signify equality in distribution. Consider risks $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$ with probability distributions F_1, F_2 respectively. The random vectors (X_1, X_2) and (Y_1, Y_2) are then different only in the way that their elements depend on each other. The *Frechet Space* $\mathcal{R}_2(F_1, F_2)$ is defined as the space of two-dimensional random vectors with fixed marginals F_1 and F_2 (see (Dhaene and Goovaerts, 1996)). Elements of $\mathcal{R}_2(F_1, F_2)$ can be compared in terms of their dependence structure via the partial *concordance order*:

Definition 4. *Consider the random vectors $(X_1, X_2), (Y_1, Y_2) \in \mathcal{R}_2(F_1, F_2)$ with joint distributions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$ respectively. We say that (X_1, X_2) is less concordant than (Y_1, Y_2) and write $(X_1, X_2) \underset{conc}{\leq} (Y_1, Y_2)$, whenever*

$$F_{\mathbf{X}}(x_1, x_2) \leq F_{\mathbf{Y}}(x_1, x_2),\ \forall x_1, x_2. \tag{8}$$

Concordance order can also be understood via the following result (Dhaene and Goovaerts (1996), Müller and Stoyan (2002)):

Lemma 2. *Consider random vectors* $(X_1, X_2), (Y_1, Y_2) \in \mathcal{R}_2(F_1, F_2)$. *Then:*

$$(X_1, X_2) \underset{conc}{\leq} (Y_1, Y_2) \Leftrightarrow Cov(h_1(X_1), h_2(X_2)) \leq Cov(h_1(Y_1), h_2(Y_2)) \quad (9)$$

for all increasing functions h_1, h_2 *such that the covariances exist.*

It is now shown that for bivariate elliptically distributed random vectors the concordance ordering is equivalent to the ordering of correlation coefficients. We note that a similar result has been proved just for the normal family by Müller and Stoyan (2002). To cater for cases where the covariance matrix does not exist, a *generalised correlation coefficient* for a bivariate elliptically distributed vector $\mathbf{X} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$, with $\mathbf{\Sigma} = \{\sigma_{ij}\}$ is defined as

$$\rho_{\mathbf{X}} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}. \quad (10)$$

Of course, $\rho_{\mathbf{X}}$ coincides with the usual (Pearson) correlation coefficient if covariance matrix exists.

Lemma 3. *Consider the random vector* $\mathbf{Y} = (Y_1, Y_2)$, *defined by* $\mathbf{Y} = \mathbf{A} \cdot \mathbf{U}$, *where* $\mathbf{U}$ *is uniformly distributed on the unit circle, and*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}. \quad (11)$$

Then $F_{\mathbf{Y}}(s, t; \rho) = \mathbb{P}(Y_1 \leq s, Y_2 \leq t)$ *in increasing in* ρ.

Proof. It is

$$\begin{aligned} \mathbb{P}(Y_1 \leq s, Y_2 \leq t) &= \mathbb{P}(U_1 \leq s, \rho U_1 + \sqrt{1-\rho^2} U_2 \leq t) \\ &= \mathbb{P}(\cos\Phi \leq s, \rho\cos\Phi + \sqrt{1-\rho^2}\sin\Phi \leq t), \end{aligned} \quad (12)$$

where Φ is uniformly distributed in $[0, 2\pi]$.

To evaluate the above joint probabilities we can set $\cos\Phi = x,\ \sin\Phi = y$ and solve the system of inequalities:

$$x \le s, \qquad \rho x + \sqrt{1-\rho^2}y \le t, \qquad x^2 + y^2 = 1. \tag{13}$$

Consider first the case that $t > 0,\ \rho \in (0,1)$. Then the situation is as depicted in figure 1. The solution of (13) consists of all points on the unit circle that are below the line $y = (t - \rho x)/\sqrt{1-\rho^2}$ and to the left of the line $x = s$. This means that

$$\mathbb{P}(\cos\Phi \le s, \rho\cos\Phi + \sqrt{1-\rho^2}\sin\Phi \le t) = \mathbb{P}(\Phi \le \phi), \tag{14}$$

where ϕ is the angle indicated in the figure.

The inequality $\rho x + \sqrt{1-\rho^2}y \le t$ restricts the solution to the part of the circle below and to the left of line ZE, corresponding to the arc defined by angle ψ, as indicated in the figure. From elementary geometry we calculate:

$$\begin{aligned} \left(\tfrac{ZE}{2}\right)^2 &= 1 - (\Delta O)^2 \\ (\Delta O)^2 = ((OA)^{-2} + (OB)^{-2})^{-1} &= \left(\tfrac{\rho^2}{t^2} + \tfrac{1-\rho^2}{t^2}\right)^{-1} = t^2 \end{aligned} \tag{15}$$

Hence the length (ZE) and therefore the angle ψ does not depend on ρ.

Now consider the inequality $x \le s$. If $s < -1$ then the solution of (13) is the empty set and $\mathbb{P}(Y_1 \le s, Y_2 \le t) = 0$ which does not depend on ρ. If $s \ge -1$ but is such that point H is to the right of E, then $\phi = \psi$, the inequality $x \le s$ is redundant, and

$$\mathbb{P}(Y_1 \le s, Y_2 \le t) = \mathbb{P}(\Phi \le \psi) = \frac{\psi}{2\pi}, \tag{16}$$

which is invariant to changes in ρ. If, on the other hand, $s \ge -1$ such that H is to the left of E, an infinitesimal increase in ρ will tend to decrease

$(OA) = t/\rho$ and to increase $(OB) = t/\sqrt{1-\rho^2}$. Consequently point E will move slightly to the left, while Z will move slightly to the right. The infinitesimal movement of E will have no effect on ϕ since E will remain to the right of H. On the other hand, the movement of Z to the right will increase ϕ. Hence

$$\mathbb{P}(Y_1 \leq s, Y_2 \leq t) = \mathbb{P}(\Phi \leq \phi) = \frac{\phi}{2\pi}, \tag{17}$$

also increases in ρ.

The situations $(\rho > 0, t < 0)$, $(\rho < 0, t > 0)$ and $(\rho < 0, t < 0)$ can be dealt with in a similar way. The cases were $t = 0$, $\rho = 0$, $\rho = \pm 1$ are also simple. □

Lemma 3 can now be extended to the more general class of bivariate elliptical distributions.

Theorem 2. [1] *Consider bivariate elliptically distributed vector $\mathbf{X} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$, with generalised correlation coefficient ρ and joint cumulative distribution function $F_{\mathbf{X}}(x_1, x_2; \rho)$. Then $F_{\mathbf{X}}(x_1, x_2; \rho)$ is increasing in ρ.*

Proof. It is enough to prove the theorem for $\mathbf{\Sigma}$ of the form:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \rho \in [-1, 1] \tag{18}$$

and $\mu_1 = \mu_2 = 0$. Then by (3), vector $\mathbf{X}$ can be represented as $\mathbf{X} \stackrel{d}{=} r \cdot \mathbf{A} \cdot \mathbf{U}$, where $\mathbf{U}$ is uniformly distributed on the unit circle, $r \geq 0$, $\mathbf{A}$ as in (11).

[1] After the reviewing process, Prof Abram Kagan informed us that this result, as well as a multivariate version, can also be obtained from Theorem 5.1 in Das Gupta et al (1972). We are very grateful to Prof Kagan for pointing this out.

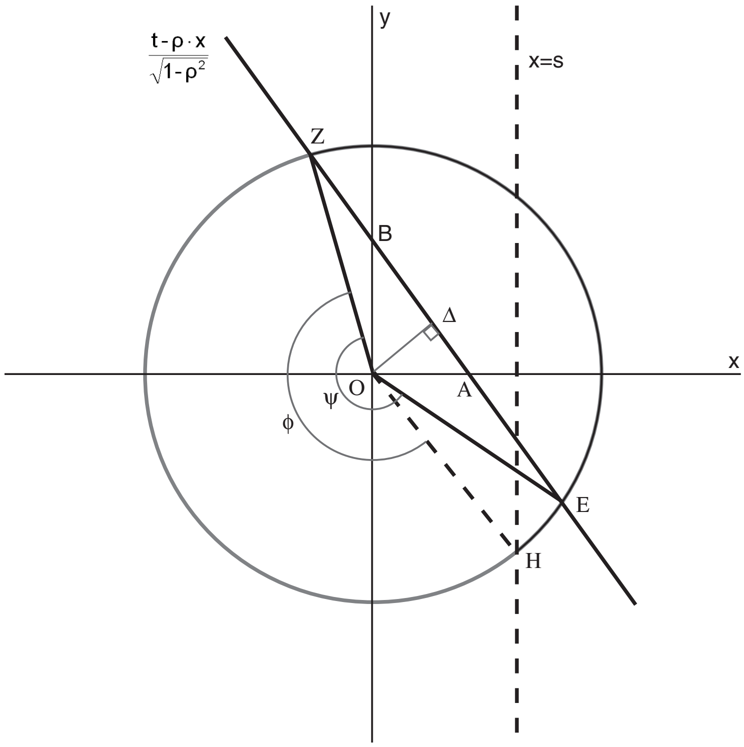

Figure 1: Geometrical proof of Lemma 3.

Denote $\mathbf{Y} = \mathbf{A} \cdot \mathbf{U}$. Now write:

$$
\begin{aligned}
F_{\mathbf{X}}(x_1, x_2; \rho) &= \mathbb{P}(r \cdot Y_1 \leq x_1, r \cdot Y_2 \leq x_2) \\
&= E\left[\mathbb{P}\left(r \cdot Y_1 \leq x_1, r \cdot Y_2 \leq x_2 | r\right)\right] \\
&= E\left[F_{\mathbf{Y}}\left(\tfrac{x_1}{r}, \tfrac{x_2}{r}; \rho\right) \mathbf{1}_{r>0}\right] + A(x_1, x_2)
\end{aligned}
\tag{19}
$$

where $A(x_1, x_2) = 0$ if $x_1 < 0$ or $x_2 < 0$ and $A(x_1, x_2) = \mathbb{P}(r = 0)$ otherwise, and the last equality follows from the independence of r and $\mathbf{Y}$. By Lemma 3, $F_{\mathbf{Y}}(y_1, y_2; \rho)$ is increasing in ρ. Hence, so is $F_{\mathbf{X}}(x_1, x_2; \rho)$ □

From Theorem 2 it immediately follows:

Corollary 2. *Consider bivariate elliptically distributed vectors* $\mathbf{X} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}_{\mathbf{X}}, \psi)$, $\mathbf{Y} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}_{\mathbf{Y}}, \psi)$, *with* $\sigma_{11}^{\mathbf{X}} = \sigma_{11}^{\mathbf{Y}}$, $\sigma_{22}^{\mathbf{X}} = \sigma_{22}^{\mathbf{Y}}$ *and generalised correlation coefficients* $\rho_{\mathbf{X}}, \rho_{\mathbf{Y}}$. *Then:*

$$
\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}} \iff \mathbf{X} \underset{conc}{\leq} \mathbf{Y} \tag{20}
$$

Remark 1: It is apparent from Lemma 2 that concordance order is invariant under monotone transformations of the random variables considered. This implies that concordance order relates only to the random vectors' copulas. Copulas are joint distributions with uniform marginals, which summarise the dependence structure of random vectors (Nelsen, 1999). Random vectors with the same copulas as elliptically distributed ones, such as the 'meta-elliptical' family considered by Fang and Fang (2002), can be obtained by applying monotone transforms on the elements of elliptical vectors. Consequently, Theorem 2 generalises to the case of two bivariate random vectors with the same marginals, which are monotone transforms of elliptical vectors that belong to the same family, but have different $\mathbf{\Sigma}$-matrices.

Remark 2: It is noted that, in the two dimensional case, concordance order is equivalent to the supermodular order (Müller, 2002). Hence, in higher dimensions, the supermodular order can be viewed as a generalisation the concordance order. Monotonicity results for the supermodular order, along the lines of Theorem 2 and Corollary 2, have been obtained in higher dimensions for special cases of elliptical distributions; see Müller (2001) for the case of multivariate normal distributions and Ding and Zhang (2004) for Kotz-type distributions. It remains an interesting open problem whether the results obtained here generalise to the multivariate elliptical distributions of arbitrary dimension.

Dhaene and Goovaerts (1996) showed that a portfolio consisting of two positive random variables becomes more risky in the stop-loss order sense, as the two risks become more concordant. A stronger version of that result can be obtained for elliptically distributed risks:

Theorem 3. *Consider* $\mathbf{X} \sim \mathbf{E}_2(\mu,\ \mathbf{\Sigma}_{\mathbf{X}}, \psi)$, $\mathbf{Y} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}_{\mathbf{Y}}, \psi)$, *with* $\sigma_{11}^{\mathbf{X}} = \sigma_{11}^{\mathbf{Y}}$, $\sigma_{22}^{\mathbf{X}} = \sigma_{22}^{\mathbf{Y}}$ *and respective generalised correlation coefficients* $\rho_{\mathbf{X}}, \rho_{\mathbf{Y}}$. *Then,* $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}} \iff X_1 + X_2 \underset{sl}{\leq} Y_1 + Y_2$.

Proof. By Theorem 1 and property (2) of elliptical distributions we have:

$$X_1 + X_2 \underset{sl}{\leq} Y_1 + Y_2 \iff \sigma_{11}^{\mathbf{X}} + \sigma_{22}^{\mathbf{X}} + 2\sigma_{12}^{\mathbf{X}} \leq \sigma_{11}^{\mathbf{Y}} + \sigma_{22}^{\mathbf{Y}} + 2\sigma_{12}^{\mathbf{Y}}, \tag{21}$$

which is equivalent to $\rho_{\mathbf{X}} \leq \rho_{\mathbf{Y}}$. □

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