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Market value margin via mean-variance hedging

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Abstract

We use mean-variance hedging in discrete time in order to value an insurance liability. The prediction of the insurance liability is decomposed into claims development results, that is, yearly deteriorations in its conditional expected values until the liability is finally settled. We assume the existence of a tradeable derivative with binary pay-off written on the claims development result and available in each development period. General valuation formulas are stated and, under additional assumptions, these valuation formulas simplify to resemble familiar regulatory cost-of-capital-based formulas. However, adoption of the mean-variance framework improves upon the regulatory approach by allowing for potential calibration to observed market prices, inclusion of other tradeable assets, and consistent extension to multiple periods. Furthermore, it is shown that the hedging strategy can also lead to increased capital efficiency.

Keywords: Market value margin, mean-variance hedging, market consistent valuation, cost-of-capital, Solvency II.

1   Introduction

Market consistent valuation of insurance liabilities is a fundamental feature of new regulatory directives, as exemplified by the Swiss Solvency Test [23] and Solvency II, see [10]. Broadly speaking, regulatory valuation techniques distinguish between liabilities that can be replicated in deep, liquid and transparent markets and liabilities for which this is not possible. For the former type of liabilities, following standard financial arguments, the market values equal the initial costs of the replicating portfolios. For the latter, the market values are postulated as the sum of the expected present value of the liabilities and a *market value margin* that is set using cost-of-capital arguments. Thus, an explicit link is induced between capital assessment and valuation for regulatory purposes.

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The application of the above regulatory principle is not straightforward. First, the cost-of-capital rate used is a rather arbitrary, exogenously specified constant figure. Second, liabilities cannot be readily classified as perfectly replicable or completely non-replicable. It is usually the case that a liability can only be partly hedged and it is not entirely clear how the regulatory valuation approach should proceed in this case. A recent effort to reconcile cost-of-capital principles with replication arguments is given by Möhr [16], who obtains Solvency II valuation formulas as a special case. Third, given the long-term nature of many insurance liabilities, it is not clear what a multi-period extension of cost-of-capital valuation principles should be. Salzmann and Wüthrich [20] and Wüthrich and Merz [26] investigate alternative multi-period versions of cost-of-capital valuation and show that conceptually consistent approaches can become computationally very expensive.

In this paper, valuation via mean-variance hedging of liabilities in discrete time is proposed; in particular, we use the solution in terms of sequential regression of Černý and Kallsen [3]. Mean-variance hedging identifies the self-financing trading strategy that minimizes the quadratic deviation between the investment portfolio and the insurance liability at maturity. The general theory of mean-variance hedging is surveyed by Schweizer [21]. For more recent developments we refer the reader to Černý and Kallsen [2] and references therein. Insurance applications of mean-variance hedging have been more common in life insurance where products demonstrate a higher dependence on instruments traded in financial markets, see Thomson [24], Dahl and Møller [5], and Delong [7]. Application to non-life insurance (see Delong and Gerrard [7]) is less frequent due to the greater difficulty in identifying suitable tradeable instruments.

However, the development of markets in insurance-linked securities, such as cat-bonds and weather derivatives, generates the possibility of at least partially hedging (non-life) insurance liabilities that are exposed to specific risks, such as those arising from natural catastrophes. We do not review the large literature on such securities and their markets here, but refer to Doherty [9] and Cummins [4]. Indicatively we mention the progress that has been made in understanding the statistical behavior of observed cat-bond prices (Papachristou [19]) and attempts to derive reinsurance prices that are consistent with them (Haslip and Kaishev [12]).

Mean-variance hedging is related to other incomplete market pricing methods that have been applied to insurance. In particular, the discrete-time risk-minimization approach of Föllmer and Schweizer [11] determines locally optimal trading strategies. However, these are not necessarily self-financing as they allow the injection of capital at fixed times. For insurance applications of risk-minimizing hedging strategies in continuous time see Møller [17, 18]. An alternative approach has been to derive, via indifference arguments, market-consistent versions of traditional actuarial premium calculations rules, such as the variance and standard deviation premium principles, see Schweizer [22].

We adopt a mean-variance hedging framework that considers a terminal liability (ultimate claim). Motivated by stochastic claims reserving methods in non-life insurance, the liability is

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decomposed into its expected value and *claims development results* representing yearly deteriorations (or improvements) in the liability's prediction using conditional expected values based on the latest information available (Merz and Wüthrich [14]). We then assume that the liability can be partially replicated by a tradeable instrument that pays 1 monetary unit in each period in the case that the claims development result exceeds a given threshold. Thus, investing in the derivative is a form of buying protection, loosely equivalent to issuing simple 1-year bonds that are subordinated to the insurance liabilities. In addition, we allow for the possibility of investing in a risk-free asset and in a number of stocks. The analysis thus predicates on the assumption that a derivative as described above can be traded or indeed (its complement) issued by the liability holder. While buying the former will not always be a feasible option, issuing the latter is a possibility for several insurance operators. Furthermore, the valuation formulas can be seen as worst-case scenario valuation over the set of similar derivatives written on risks that are only partially correlated with the liability at hand (e.g. index-triggered cat-bonds).

General valuation formulas are derived for multiple time periods, multiple tradeable assets and general asset dynamics. Simplifying assumptions lead to more transparent and practical formulas: if (a) asset returns have state-independent one-period Sharpe ratios, and (b) asset returns and claims development results are uncorrelated across different time periods, the market value of the liability is decomposed into valuations of the individual claims development results. Each of those can be written as a weighted sum of a TVaR measure applied to the claims development result and of CAPM-type terms corresponding to the other tradeable assets. If it is further assumed that only the derivative is tradeable, the valuation formula becomes very similar to multi-period cost-of-capital formulas found in Salzmann and Wüthrich [20].

While, in their simplest form, the valuation formulas obtained bear a strong similarity to the regulatory approach, their interpretation is different. Thus, we illustrate a potential set of alternative assumptions on which (regulatory) valuation may be based. The benefit of this approach is, firstly, to allow in a consistent manner the extension of liability valuation formulas to include multiple assets, partial replication, and multiple time periods. Secondly, by making explicit the assumptions needed to obtain simple and practically useful valuation formulas, it becomes transparent what the price paid for such simplification is.

In Section 2 the simple single-period and single-asset case is introduced and the corresponding valuation formulas are derived. We also show that the hedging approach used may lead to a more efficient use of capital, which is a positive side effect of the replication strategy. In Section 3, general results for the multi-period and multi-asset case are presented and valuation formulas are derived for specific cases. Simple numerical examples illustrate the analysis. Finally, brief conclusions are given in Section 4.

Throughout the paper we assume that the (conditional) second moments of all random variables considered exist.

2 Single-period and single-asset case

2.1 Preliminaries

We start with a toy model to illustrate the key ideas of the paper. A single-period is considered with two time points $t = 0$ and $t = 1$. There is an insurance liability $H \geq 0$ that has to be met at time $t = 1$. At time $t = 0$, the insurer of H invests a total amount of v monetary units in order to replicate H as closely as possible. All assets and liabilities are considered in discounted units and there are two tradeable assets: a risk-free asset with price 1 at time $t = 0$ and payoff 1 at time $t = 1$ and a risky asset with price S_0 at $t = 0$ and payoff S_1 at $t = 1$. The risky asset's excess return is denoted by $X_1 = S_1/S_0 - 1$.

Capital ϑ_1 is invested in the risky asset (ϑ_1/S_0 units are bought) at time $t = 0$ and the remainder of the initial wealth $v - \vartheta_1$ is invested in the risk-free asset. This asset portfolio generates value at time 1 given by $v + \vartheta_1 X_1$. The optimal initial wealth V_0 and investment ξ_1 in the risky asset with respect to a quadratic loss function are calculated by minimizing the quadratic deviation between the liability and the asset portfolio's pay-off. That is,

$$(V_0, \xi_1) = \arg \min_{(v, \vartheta_1)} \mathbb{E}\left((v + \vartheta_1 X_1 - H)^2\right). \tag{1}$$

Since V_0 corresponds to the initial cost of replicating H as closely as possible w.r.t. the quadratic loss, we will throughout this paper identify V_0 with the market consistent value of H at time 0.

Standard arguments yield the solution to optimization problem (1) which reads as

$$\xi_1 = \frac{\text{Cov}(X_1, H)}{\text{Var}(X_1)}, \tag{2}$$

$$V_0 = \mathbb{E}(H) - \frac{\text{Cov}(X_1, H)}{\text{Var}(X_1)} \mathbb{E}(X_1). \tag{3}$$

In this simple setting, the value of V_0 reflects the capital asset pricing model (CAPM) price of H, where the risky asset with pay-off S_1 plays the role of the market portfolio.

In the sequel, the risk measures Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) are used extensively. For a random variable Z with continuous distribution function and a security level $\alpha \in (0, 1)$ they are defined in the common way, see for instance McNeil et al. [13],

$$\text{VaR}_\alpha(Z) = \inf\left\{z \in \mathbb{R} : \mathbb{P}(Z \leq z) \geq \alpha\right\}, \tag{4}$$

$$\text{TVaR}_\alpha(Z) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_\beta(Z)d\beta = \mathbb{E}\left(Z \,|\, Z \geq \text{VaR}_\alpha(Z)\right). \tag{5}$$

For the latter identity to hold true we need to assume that Z has a continuous distribution function, see Lemma 2.16 in McNeil et al. [13].

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2.2 Valuation formulas

Now, a particular choice for the risky asset is made. For given random variable Z_1 and threshold d_1 we define

$$S_1 = \mathbf{1}_{D_1} \qquad \text{with} \qquad D_1 = \{Z_1 \geq d_1\}. \tag{6}$$

Assume $p_1 = \mathbb{E}(S_1) = \mathbb{P}(D_1) \in (0,1)$ and $S_0 = q_1 \in (p_1, 1)$ which implies $X_1 = \mathbf{1}_{D_1}/q_1 - 1$.

There are two ways of considering such pay-offs in an insurance market. First, S_1 may be the pay-off from an index-linked insurance derivative (such as a weather derivative or an industry loss-triggered cat-bond) with Z_1 playing the role of the relevant index. The derivative considered will be such that Z_1 is a reasonable proxy for the liability H, hence we will require that Z_1 is positively correlated with H. In particular, the risky asset S_1 pays a positive return on the event D_1 associated with a large loss in H. Probability p_1 then is the real-world probability of such an event and q_1 is its risk-neutral probability implied by market prices. The condition $q_1 > p_1$ is explained by S_1 playing the role of reinsurance for large losses. While $q_1 \leq p_1$ would theoretically be possible, in practice the assumption generally holds that the cost of protection against an adverse event is higher than its expected value, due to risk-aversion of the risk bearer.

Alternatively, consider the case that the holder of the liability H sponsors a catastrophe bond, with D_1 being the triggering event. The bond structure is such that the holder of H pays 1 monetary unit at time 1 if D_1^c takes place and 0 units if D_1 occurs. Let $1 - q_1$ be the price of the bond. Then, if the sponsor issues ϑ_1/q_1 bonds, the gains from the trade are $-\frac{\vartheta_1}{q_1}\left(\mathbf{1}_{D_1^c} - (1-q_1)\right) = \vartheta_1 X_1$. The solution to problem (1) now gives the optimal level of debt ϑ_1 that should be issued. The bond will be constructed so as to maximize the correlation between the trigger Z_1 and the liability H. In particular, if an indemnity trigger is used, perfect positive correlation between Z_1 and H can be achieved (see Papachristou [19]); otherwise so-called basis-risk remains.

For the tradeable asset (6), equation (3) and the identity $\text{Var}(S_1) = p_1(1 - p_1)$ yield

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1 - p_1}\left[\mathbb{E}\left(H|Z_1 \geq d_1\right) - \mathbb{E}(H)\right]. \tag{7}$$

Formula (7) can be further refined by choosing the indemnity-based trigger $Z_1 = H - \mathbb{E}(H)$ (the threshold d_1 can now be interpreted as $d_1 = \text{VaR}_{1-p_1}(H) - \mathbb{E}(H)$). Assuming that H has a continuous distribution function, formula (7) becomes

$$V_0 = \mathbb{E}(H) + \frac{q_1 - p_1}{1 - p_1}\left[\text{TVaR}_{1-p_1}(H) - \mathbb{E}(H)\right]. \tag{8}$$

Therefore valuation takes place according to a simple rule: "expected value plus a percentage of the excess of TVaR over the expected value".

Even when the derivative is not indemnity-triggered (i.e. H and Z_1 are not perfectly correlated), (8) can still be interpreted as a conservative upper bound on the value given in (7). To see this observe that the vectors $(H, \mathbf{1}_{\{H \geq \mathbb{E}(H)+d_1\}})$ and $(H, \mathbf{1}_{\{Z_1 \geq d_1\}})$ have the same marginals,

but the elements of the former are comonotonic. Hence $\mathbb{E}(H\mathbf{1}_{\{H \geq \mathbb{E}(H)+d_1\}}) \geq \mathbb{E}(H\mathbf{1}_{\{Z_1 \geq d_1\}})$ implies $\mathrm{TVaR}_{1-p_1}(H) \geq \mathbb{E}(H|Z_1 \geq d_1)$, where the first inequality follows from Proposition 6.2.6 in Denuit et al. [6]. With this in mind, from now on we will always identify Z_1 with $H - \mathbb{E}(H)$.

Formula (8) bears a close resemblance to valuation formulas used under solvency regimes such as Solvency II and the Swiss Solvency Test, where the market consistent value of a liability is set equal to its expected value plus a risk loading deriving from a cost-of-capital charge, see European Commission [10] and Swiss Solvency Test [23]. If the regulator prescribes a translation invariant regulatory risk measure ρ to support adverse events, the market consistent value under the cost-of-capital method equals

$$V_0^{\mathrm{CoC}} = \mathbb{E}(H) + \lambda\left[\rho(H) - \mathbb{E}(H)\right], \tag{9}$$

where λ is the *cost-of-capital rate*. In Solvency II jargon, the quantity $\lambda\left[\rho(H) - \mathbb{E}(H)\right]$ is termed *market value margin*.

The valuation formulas (8) and (9) are structurally similar, both satisfying the rather pragmatic criterion of making an allowance for extreme tail events in the value of liability H via a (tail) risk measure. However, they are derived using different economic arguments and thus are different in significant ways. In (9) λ is an exogenously given constant, while $\frac{q_1-p_1}{1-p_1}$ in (8) is a potentially observable and market-sensitive quantity. In (9) $\rho(H)$ corresponds to the capital requirement for H, while the risk measure $\mathrm{TVaR}_{1-p_1}(H)$ is not associated with the assets held. In particular, for the optimal portfolio held, the investment in the risky asset would also be reflected in the regulatory capital requirement; this is a point to which we return in Section 2.3.

Example 1. The findings of Papachristou [19], who performed statistical analysis of catastrophe bond spreads at the time of issue, allow us to get a feeling for the potential range of the quantity $\frac{q_1-p_1}{1-p_1}$. In particular, the behavior of the "multiple" is studied, that is, the ratio of the spread to the annualized expected loss which in our simple model can be identified with the probability p_1. It is found that the multiple tends to decrease in p_1, reflecting a higher risk premium for protection against extreme events. Furthermore, it is shown how the multiple changes with time and responds to insurance events, e.g. a rise is observed after Hurricane Katrina in 2006. For the period of 2003-2008 studied and the sample of bonds considered, the multiple for bonds with $p_1 = 1\%$ has tended to fluctuate between about 4 and 8. This implies that the spread $r_B = \frac{\mathbb{E}[\mathbf{1}_{D_1^c}]}{1-q_1} - 1 = \frac{1-p_1}{1-q_1} - 1 > 0$ ranges from 4% to 8% and, consequently, $\frac{q_1-p_1}{1-p_1}$ varies from 3.8% to 7.4%. Interestingly, the range contains the cost-of-capital rate $\lambda = 6\%$ favored by regulators, see TP.5.25 in [10]. $\qquad\square$

2.3 Hedging and capital efficiency

The arguments presented above focused on deriving a risk-sensitive valuation formula as an alternative to what is proposed in current insurance regulation. However, we did not consider

the change in portfolio risk after investment in the derivative with pay-off $S_1 = \mathbf{1}_{D_1}$. This is an issue worth considering since the buyer of such a derivative would be interested in reducing the risk on the book and thus freeing up economic capital.

Let the solvency capital requirement be determined by a translation invariant risk measure ρ, such that $\rho(H - v) = \rho(H) - v$ for all $v \in \mathbb{R}$ and all random variables H under consideration. Denote by G_1 the value of the optimal investment portfolio (V_0, ξ_1) of (1) at time 1, i.e. $G_1 = V_0 + \xi_1 X_1$. Then, trading in the derivative frees up capital as long as the cost V_0 of the trading strategy and the capital requirement for the hedged loss $H - G_1$ add up to less than the capital requirement for the unhedged loss H:

$$V_0 + \rho(H - G_1) \leq \rho(H) \qquad \Leftrightarrow \qquad \rho(H - (G_1 - V_0)) \leq \rho(H). \tag{10}$$

Noting that the portfolio that generates G_1 has initial price V_0, (10) states that investment in the portfolio can reduce the solvency capital requirement under the risk measure ρ, which looks similar to the indifference price for G_1 in an expected utility framework.

Nonetheless, it is by no means obvious that inequality (10) will generally be satisfied, since the trading strategy is formulated to replicate the liability H as closely as possible in a quadratic norm and not specifically to minimize the capital requirement described by the risk measure ρ. Some situations where (10) holds are characterized in Proposition 1 stated below.

Proposition 1. *Assume H has a continuous and strictly increasing distribution function on $\mathbb{R}_+$ and let the risk measure ρ be either VaR_α or TVaR_α at security level $\alpha \in (0, 1)$. Define $k = \frac{1}{1-p_1}\mathrm{TVaR}_{1-p_1}(H - \mathbb{E}(H))$. We have $k > 0$, and inequality (10) holds if and only if*

$$q_1 k \leq \rho(H) - \rho(H - \mathbf{1}_{D_1} k). \tag{11}$$

In particular, the following hold:

    i) Assume that $\mathrm{VaR}_{1-p_1}(H) - k > 0$. If $\alpha < 1 - p_1$ is small enough, such that $\mathrm{VaR}_\alpha(H) \leq \mathrm{VaR}_{1-p_1}(H) - k$, then:

        - *For $\rho \equiv \mathrm{VaR}_\alpha$, there is no $q_1 \in (0, 1)$ such that inequality (10) holds.*

        - *For $\rho \equiv \mathrm{TVaR}_\alpha$, inequality (10) holds for all $q_1 \in (p_1, \frac{p_1}{1-\alpha})$.*

    ii) If $\alpha > 1 - p_1$ is large enough, such that $\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_{1-p_1}(H) \geq k$, then, for either of $\rho \equiv \mathrm{VaR}_\alpha$ and $\rho \equiv \mathrm{TVaR}_\alpha$, inequality (10) holds for all $q_1 \in (p_1, 1)$ and the freed-up capital equals

$$\rho(H) - V_0 - \rho(H - G_1) = (1 - q_1)k. \tag{12}$$

Proof. Note that both VaR and TVaR are translation invariant and that $k > 0$ by the properties of TVaR (e.g. see Property 2.4.5 in Denuit et al. [6]). We have

$$G_1 - V_0 = \xi_1 X_1 = \frac{\mathrm{Cov}(X_1, H)}{\mathrm{Var}(X_1)} X_1 = \frac{\mathbf{1}_{D_1} - q_1}{1 - p_1}\mathrm{TVaR}_{1-p_1}(H - \mathbb{E}(H)) = (\mathbf{1}_{D_1} - q_1)k.$$

It follows that $\rho(H - (G_1 - V_0)) = \rho(W) + q_1 k$, where $W = H - \mathbf{1}_{D_1} k$. Thus, the freed-up capital can be written as $\rho(H) - \rho(H - (G_1 - V_0)) = \rho(H) - \rho(W) - q_1 k$, so that to satisfy inequality (10) we need requirement (11).

Let $d = d_1 + \mathbb{E}(H)$ which implies $D_1 = \{H \geq d\}$ and $\mathrm{VaR}_{1-p_1}(H) = d$. We have for $w \in \mathbb{R}$

$$
\begin{aligned}
\mathbb{P}(W \leq w) &= \mathbb{P}(W \leq w, D_1) + \mathbb{P}(W \leq w, D_1^c) \\
&= \mathbb{P}(d \leq H \leq w + k) + \mathbb{P}(H \leq \min\{w, d\}).
\end{aligned}
$$

Continuity of the distribution function F of H and $k > 0$ immediately imply

$$
\mathbb{P}(W \leq w) = \begin{cases} F(w), & w \leq d - k, \\ F(w + k) - F(d) + F(w), & d - k < w < d, \\ F(w + k), & d \leq w. \end{cases}
$$

It is easily seen that the distribution of W is also continuous and strictly increasing. Moreover, when $\mathrm{VaR}_\alpha(H) \leq d - k$ (corresponding to Case i)), it is $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H)$. On the other hand, when $\mathrm{VaR}_\alpha(H) \geq d + k$ (corresponding to Case ii)), we have $\mathbb{P}(W \leq \mathrm{VaR}_\alpha(H) - k) = \mathbb{P}(H \leq \mathrm{VaR}_\alpha(H)) = \alpha$, such that $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H) - k$. We now deal with the two cases separately.

Case i) Assume $d - k > 0$ and choose $\mathrm{VaR}_\alpha(H) \leq d - k$.

1. First let $\rho \equiv \mathrm{VaR}_\alpha$. Then $\mathrm{VaR}_\alpha(H - \mathbf{1}_{D_1} k) = \mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H)$ such that condition (11) cannot be satisfied for any $q_1 > 0$ (note that $k > 0$).

2. Now let $\rho \equiv \mathrm{TVaR}_\alpha$. Observe that the vectors $(H, \mathbf{1}_{\{H \geq \mathrm{VaR}_\alpha(H)\}})$ and $(H, \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}})$ have the same marginals, but the elements of the former are comonotonic. This implies inequality $\mathbb{E}(H \mathbf{1}_{\{H \geq \mathrm{VaR}_\alpha(H)\}}) \geq \mathbb{E}(H \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}})$, which follows from Proposition 6.2.6 in Denuit et al. [6]. Consider now, using $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H)$,

$$
\begin{aligned}
\mathrm{TVaR}_\alpha(H) - \mathrm{TVaR}_\alpha(W) &= \frac{1}{1 - \alpha} \left[\mathbb{E}(H \mathbf{1}_{\{H \geq \mathrm{VaR}_\alpha(H)\}})) - \mathbb{E}(W \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}}) \right] \\
&\geq \frac{1}{1 - \alpha} \left[\mathbb{E}(H \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}})) - \mathbb{E}(W \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}}) \right] \\
&= \frac{k}{1 - \alpha} \mathbb{E}(\mathbf{1}_{D_1} \mathbf{1}_{\{W \geq \mathrm{VaR}_\alpha(W)\}}) \\
&= \frac{k}{1 - \alpha} \mathbb{P}(H \geq \max\{d, k + \mathrm{VaR}_\alpha(H)\}) \\
&= \frac{k}{1 - \alpha} \mathbb{P}(H \geq d) = \frac{p_1}{1 - \alpha} k.
\end{aligned}
$$

Hence, by (11) it is sufficient to have $q_1 \leq \frac{p_1}{1-\alpha}$ for inequality (10) to hold.

Case ii) Assume $\mathrm{VaR}_\alpha(H) \geq d + k$ which gives $\mathrm{VaR}_\alpha(W) = \mathrm{VaR}_\alpha(H) - k$.

1. First let $\rho \equiv \mathrm{VaR}_\alpha$. The freed up capital equals $\mathrm{VaR}_\alpha(H) - \mathrm{VaR}_\alpha(W) - q_1 k = k - q_1 k$, which proves the stated result.

2. Now consider $\rho \equiv \mathrm{TVaR}_\alpha$. For $\beta \in [\alpha, 1)$ monotonicity implies $\mathrm{VaR}_\beta(H) \geq \mathrm{VaR}_\alpha(H) \geq d + k$. Therefore $\mathrm{VaR}_\beta(W) = \mathrm{VaR}_\beta(H) - k$ for all $\beta \in [\alpha, 1)$ and by the integral identity for TVaR given

in (5) the freed up capital statement immediately follows from $\text{VaR}_\beta(H) - \text{VaR}_\beta(W) - q_1 k = k - q_1 k$. $\qquad\square$

Case i) of Proposition 1 refers to the case where the security level α is so low that the risk reduction effected by the derivative is not reflected in the VaR measure, due to the risk measure's insensitivity to the extreme tail of the distribution of H. Thus, investing any amount in the derivative incurs a cost with no apparent benefit. When TVaR is used, the extreme tails are reflected in the risk measurement and the benefit from investing in the derivative is recognized, as long as the derivative is not too expensive (q_1 is not too high). On the other hand, Case ii) refers to the situation where the security level α is very high such that under all scenarios considered by the risk measure, the derivative produces a pay-off of 1 monetary unit, which is always higher than the price $q_1 < 1$. Consequently, a capital saving is always produced. However, the freed-up capital, as seen in equation (12), depends on the price q_1. Thus, if q_1 is close to its lowest level p_1, there is no market risk premium for the derivative and the freed up capital is maximized. On the contrary, if q_1 is close to 1, the market considers the event of the derivative paying as nearly certain, such that the derivative becomes very expensive, and investing in it produces only a small capital reduction.

Of course, in many cases it will be $-k < \text{VaR}_\alpha(H) - \text{VaR}_{1-p_1}(H) < k$, a case not fully characterized in Proposition 1. The following numerical example shows that for realistic parameter choices, investment in the derivative will tend to be capital efficient.

Example 2. Let H be log-normally distributed such that $\mathbb{E}(H) = 100$, $\text{Var}(H) = 20^2$. We consider two cases of the derivative, with $p_1 = 0.01$ and $p_1 = 0.05$. For illustrative purposes, we follow again Papachristou [19], choosing for $p_1 = 0.01$ (resp. $p_1 = 0.05$) a multiple of 6 (resp. 4), leading to $q_1 = 0.066$ (resp. $q_1 = 0.208$).

Table 1: Valuation of a log-normal liability with $\mathbb{E}(H) = 100$, $\text{Var}(H) = 20^2$.

p_1	0.01	0.05
q_1	0.066	0.208
$\text{TVaR}_{1-p_1}(H)$	166.56	147.95
$\frac{q_1 - p_1}{1 - p_1}$	0.057	0.167
V_0	103.77	107.99
k	67.23	50.47

Table 1 summarizes the quantities needed for the valuation of the liability H according to (8), as well as the value $k = \frac{1}{1-p_1} \text{TVaR}_{1-p_1}(H - \mathbb{E}(H))$ appearing in Proposition 1. While for $p_1 = 0.05$ the risk measure $\text{TVaR}_{1-p_1}(H)$ is substantially lower, this is compensated by a higher ratio $\frac{q_1 - p_1}{1 - p_1}$, such that the liability H has a higher market value V_0 for $p_1 = 0.05$.

It is easy to check that for all security levels $\alpha \in [0.99, 0.999]$ (a plausible range for regulatory risk measurement) we have $\text{VaR}_\alpha(H) - \text{VaR}_{1-p_1}(H) \in (0, k)$. From equation (11), the maximum
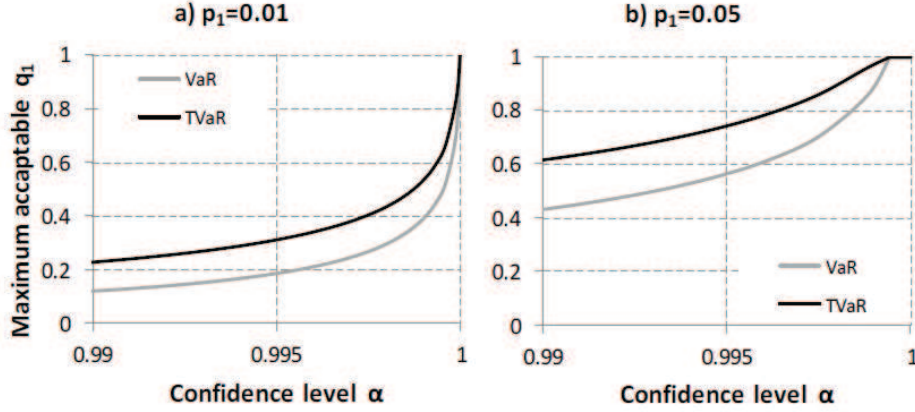
a) $p_1 = 0.01$

b) $p_1 = 0.05$

Figure 1: Maximum q_1 such that inequality (10) is satisfied.

value of the price q_1 that leads to freeing up capital is given by the relation $q_1 \leq \big(\rho(H) - \rho(H - \mathbf{1}_{D_1} k)\big)/k$, where $\rho \equiv \mathrm{VaR}_\alpha$ or $\rho \equiv \mathrm{TVaR}_\alpha$. The maximum such level of q_1 is plotted in Figure 1 against the security level α of the regulatory risk measure used, for $\alpha \in [0.99, 1)$. It is seen that in each case the value of q_1 is well below the plotted curves, such that for the plausible range of security levels α, investment in the derivative indeed frees up capital. $\qquad \square$

3 The multi-period and multi-asset case

3.1 Preliminaries

We extend the previous setup to a model with several assets traded over multiple time periods. We consider a finite time horizon $T \in \mathbb{N}$ and a finite set of trading dates $\mathcal{T} = \{0, 1, \ldots, T\}$. The filtered probability space is denoted by $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$ with finite and discrete time filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$. The corresponding conditional expectations, variances, and covariances are denoted by $\mathbb{E}_t(X) = \mathbb{E}(X|\mathcal{F}_t)$, $\mathrm{Var}_t(X) = \mathbb{E}_t(X^2) - \mathbb{E}_t(X)^2$, $\mathrm{Cov}_t(X, Y) = \mathbb{E}_t(XY) - \mathbb{E}_t(X)\mathbb{E}_t(Y)$ for $t \in \mathcal{T}$.

The insurance liability is represented by a non-negative, $\mathcal{F}_T$-measurable, square-integrable random variable $H \in L^2(\mathbb{P})$. We assume that we have $n \in \mathbb{N}$ tradeable risky assets with price processes represented by the n-dimensional, $\mathbb{F}$-adapted stochastic process $(S_t)_{t \in \mathcal{T}}$. Denote the elements of S_t by $S_t^{(i)}$, $i = 1, \ldots, n$ and let $S_t^{(i)} > 0$. X_t is then the vector of one-period excess returns with elements $X_t^{(i)} = S_t^{(i)}/S_{t-1}^{(i)} - 1$. We assume $\mathbb{E}_t(X_{t+1}^{(i)} X_{t+1}^{(j)}) < \infty$ for all i, j and $t < T$, and that the returns of traded assets are linearly independent such that the matrices $\big\{\mathbb{E}_t(X_{t+1}^{(i)} X_{t+1}^{(j)})\big\}_{1 \leq i, j \leq n}$ have full rank. For any vector $y \in \mathbb{R}^n$, y' denotes the transpose of y.

An $\mathcal{F}_0$-measurable initial endowment v is given. A trading strategy $\vartheta = (\vartheta_t)_{t \in \mathcal{T} \setminus \{0\}}$ is an n-dimensional and $\mathbb{F}$-previsible process. The value at time $t > 0$ of an investment portfolio with

initial endowment v and trading strategy ϑ is

$$G_t^{v,\vartheta} = v + \sum_{k=1}^{t} \vartheta_k' X_k. \tag{13}$$

By its construction, the portfolio (13) is self-financing. Only strategies such that $G_T^{v,\vartheta} \in L^2(\mathbb{P})$ are admitted; for a detailed technical discussion of admissibility see Černý and Kallsen [3].

Directly extending the discussion in Section 2, the aim is to derive the optimal initial endowment and trading strategy such that the quadratic deviation between the final portfolio value $G_T^{v,\vartheta}$ and the liability H is minimized. In other words, we need to solve optimization problem

$$\arg \min_{(v,\vartheta)} \mathbb{E}_0\big((G_T^{v,\vartheta} - H)^2\big). \tag{14}$$

The solution to problem (14) is provided by Theorem 8.7 in Černý and Kallsen [3]:

Theorem 2. *The process given by the recursion $L_T = 1$ and for $0 < t \le T$*

$$L_{t-1} = \mathbb{E}_{t-1}(L_t) - \mathbb{E}_{t-1}(L_t X_t') \left(\mathbb{E}_{t-1}(L_t X_t X_t')\right)^{-1} \mathbb{E}_{t-1}(L_t X_t)$$

is $(0,1]$-valued and the probability measure $\mathbb{P}^*$, defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \prod_{t=1}^{T} \frac{L_t}{\mathbb{E}_{t-1}(L_t)},$$

is well defined. Let $\mathbb{E}_{t-1}^*(\cdot)$ denote conditional expectations under $\mathbb{P}^*$. The following processes for $0 < t \le T$ are well defined:

$$a_t^* = \mathbb{E}_{t-1}^*(X_t') \left(\mathbb{E}_{t-1}^*(X_t X_t')\right)^{-1},$$
$$b_t^* = a_t^* \, \mathbb{E}_{t-1}^*(X_t),$$
$$V_{t-1}^* = \mathbb{E}_{t-1}^* \left(\frac{1 - a_t^* X_t}{1 - b_t^*} V_t^*\right), \quad V_T^* = H,$$
$$\xi_t^* = \mathbb{E}_{t-1}^*((V_t^* - V_{t-1}^*)X_t)' \left(\mathbb{E}_{t-1}^*(X_t X_t')\right)^{-1}.$$

For initial endowment v define the trading strategy $\phi(v) = (\phi_t(v)_t)_{t \in \mathcal{T} \setminus \{0\}}$ iteratively by

$$\phi_t(v) = \xi_t^* + a_t^* \left(V_{t-1}^* - G_{t-1}^{v,\phi(v)}\right).$$

Then, the pair $(V_0^*, \phi(V_0^*))$ solves the optimization problem (14).

The probability measure $\mathbb{P}^*$ is termed the *opportunity-neutral measure*. The opportunity-neutral measure $\mathbb{P}^*$ is not a martingale measure. Switching to $\mathbb{P}^*$ is necessary in the case that asset returns are not independent in order to compensate for one-period Sharpe ratios at a given time not being the same in all states (see Černý and Kallsen [3]). In the case $\mathbb{P}^* = \mathbb{P}$ we can keep the same notation as in Theorem 2, after dropping the superscripts *

from all variables. The probability measure $\mathbb{P}^*$ reduces to $\mathbb{P}$ if and only if the product of $b_t = \mathbb{E}_{t-1}(X_t') \left(\mathbb{E}_{t-1}(X_t X_t') \right)^{-1} \mathbb{E}_{t-1}(X_t)$ over all t is constant (see Cerný and Kallsen [2], Proposition 3.28). A sufficient condition for this is to require that the maximal one-period Sharpe ratio for each time step is known at time zero, equivalently b_t is $\mathcal{F}_0$-measurable. Independence of asset returns is a substantially stronger condition; one can for example achieve constant Sharpe ratios in stochastic volatility models so that the returns are not i.i.d. but the b_t's remain deterministic. Independence is a sufficient (but not necessary) condition for *both* a_t and b_t to be $\mathcal{F}_0$-mensurable, that is, state-independent. It also noted that the more general form of Theorem 2 is given in terms of price increments rather than returns; we use the current form (requiring $S_t > 0$) for practical reasons, as the dynamics of asset returns, rather than prices, are typically specified.

3.2 Valuation of an insurance liability

We work towards deriving multi-period valuation formulas, generalizing those of Section 2. First, we decompose the $\mathcal{F}_T$-measurable liability $H \in L^2(\mathbb{P})$ as

$$H = \mathbb{E}_0(H) + Y_1 + \cdots + Y_T \qquad \text{with} \qquad Y_t = \mathbb{E}_t(H) - \mathbb{E}_{t-1}(H), \tag{15}$$

where Y_t is termed the *claims development result*, see Merz and Wüthrich [14]. The notion of the claims development result is based on the understanding that insurance companies need to close their books after every period. At time t they will book the so-called *best-estimate liability* $\mathbb{E}_t(H)$, updating the previous prediction $\mathbb{E}_{t-1}(H)$. The resulting adjustment of the best estimate produces a claims development result of Y_t in period t, which may be a gain or a loss. Essentially, Y_t corresponds to the single-period risk exposure of the holder of H and the regulator asks for a risk measure to support possible shortfalls in Y_t in period t. Since the time series $Y_1, \ldots, Y_T$ is formed by the innovations of a martingale, its elements are uncorrelated and have zero mean. For the rest of the paper we will assume that Y_t has a continuous and strictly increasing distribution.

For such a decomposition of the liability H, direct application of Theorem 2 gives a general valuation formula.

Proposition 3. *For H as in* (15), *the optimal initial endowment of Theorem 2 becomes*

$$V_0^* = \mathbb{E}_0(H) + \sum_{t=1}^{T} \mathbb{E}_0^* \left(\prod_{i=1}^{t} \frac{1 - a_i^* X_i}{1 - b_i^*} Y_t \right).$$

Proof. From Theorem 2 we have (noting that $\mathbb{E}^*_{t-1}\left((1 - a^*_t X_t)/(1 - b^*_t)\right) = 1$ and $V^*_T = H$),

$$V^*_{T-1} = \mathbb{E}^*_{T-1}\left(\frac{1 - a^*_T X_T}{1 - b^*_T} H\right) = \mathbb{E}_0(H) + \sum_{t=1}^{T-1} Y_t + \mathbb{E}^*_{T-1}\left(\frac{1 - a^*_T X_T}{1 - b^*_T} Y_T\right),$$

$$V^*_{T-2} = \mathbb{E}^*_{T-2}\left(\frac{1 - a^*_{T-1} X_{T-1}}{1 - b^*_{T-1}} V_{T-1}\right)$$

$$= \mathbb{E}_0(H) + \sum_{t=1}^{T-2} Y_t + \mathbb{E}^*_{T-2}\left(\frac{1 - a^*_{T-1} X_{T-1}}{1 - b^*_{T-1}} Y_{T-1}\right) + \mathbb{E}^*_{T-2}\left(\frac{1 - a^*_{T-1} X_{T-1}}{1 - b^*_{T-1}}\frac{1 - a^*_T X_T}{1 - b^*_T} Y_T\right).$$

Iterating the process yields the required result for V^*_0. $\qquad\square$

Now we assume the existence of a traded insurance derivative which we identify with the first traded risky asset. The derivative is written at each time $t - 1$ and pays 1 unit at time t, if the claims development result Y_t exceeds a given high threshold d_t. Specifically,

$$X^{(1)}_t = \frac{\mathbf{1}_{D_t}}{q_t} - 1, \tag{16}$$

where $D_t = \{Y_t \geq d_t\}$, $\mathbb{E}_{t-1}(\mathbf{1}_{D_t}) = p_t$, and q_t is the $\mathcal{F}_{t-1}$-measurable price at time $t - 1$ with $p_t < q_t < 1$. In fact, much of the following analysis remains unchanged if we assume, similarly to Section 2, that the event $D_t = \{Z_t \geq d_t\}$, where Z_t is an (index) variable closely correlated to Y_t. For the sake of simplicity, we do not pursue this route here.

Additional assumptions give rise to formulas generalizing those of Section 2.

Proposition 4. *Let $X^{(1)}_t$ be as in (16). Assume that a_t and b_t are state-independent and that for any $1 \leq i, j \leq n$ and $0 < t < t + s \leq T$ it is*

$$\mathbb{E}_{t-1}(X^{(i)}_t X^{(j)}_{t+s} Y_{t+s}) = \mathbb{E}_{t-1}(X^{(i)}_t)\mathbb{E}_{t-1}(X^{(j)}_{t+s} Y_{t+s}), \quad \mathbb{E}_{t-1}(X^{(i)}_t Y_{t+s}) = 0.$$

Then the optimal initial endowment of Theorem 2 becomes

$$V_0 = \mathbb{E}_0(H) + \sum_{t=1}^T \mathbb{E}_0\left(\frac{1 - a_t X_t}{1 - b_t} Y_t\right)$$

$$= \mathbb{E}_0(H) - \sum_{t=1}^T \frac{a^{(1)}_t}{1 - b_t}\mathbb{E}_0\left(\frac{p_t}{q_t}\,\mathrm{TVaR}_{1-p_t,t-1}(Y_t)\right) - \sum_{t=1}^T \sum_{i=2}^n \frac{a^{(i)}_t \mathrm{Cov}_0(X^{(i)}_t, Y_t)}{1 - b_t},$$

where $\mathrm{TVaR}_{1-p_t,t-1}$ is the risk measure calculated with respect to information $\mathcal{F}_{t-1}$ at time $t-1$.

Proof. The uncorrelatedness assumption and normalization imply

$$\mathbb{E}_{T-2}\left(\frac{1 - a_{T-1} X_{T-1}}{1 - b_{T-1}}\frac{1 - a_T X_T}{1 - b_T} Y_T\right) = \mathbb{E}_{T-2}\left(\frac{1 - a_T X_T}{1 - b_T} Y_T\right).$$

The proof of the first statement then follows from Proposition 3 working backwards in time. The second formula derives from

$$\mathbb{E}_0(X^{(1)}_t Y_t) = \mathbb{E}_0\left(\frac{1}{q_t}\mathbf{1}_{D_t} Y_t\right) = \mathbb{E}_0\left(\frac{p_t}{q_t}\frac{1}{p_t}\mathbb{E}_{t-1}\left(\mathbf{1}_{D_t} Y_t\right)\right) = \mathbb{E}_0\left(\frac{p_t}{q_t}\,\mathrm{TVaR}_{1-p_t,t-1}(Y_t)\right).$$

$\qquad\square$

The conditions of Proposition 4 correspond, loosely speaking, to the assumption that the conditional expected performance of assets over each time period is already known at time $t = 0$ and that assets and liabilities are uncorrelated across time periods. Then, the market value V_0 of the liability H equals its expected value plus a number of terms producing valuations of the individual claims development results Y_t. Each of the latter terms can be written as a weighted sum of the expected value of a TVaR measure applied to Y_t at time $t-1$, scaled by p_t/q_t, and $n-1$ CAPM-type terms corresponding to the other tradeable assets. Thus, the valuation formula of Proposition 4 bears a formal similarity to commonly used multi-period cost-of-capital formulas termed the *split of total uncertainty approach* in Salzmann and Wüthrich [20] or *expected risk margin* in Möhr [16]. At the same time, it generalizes them by including further tradeable assets via standard valuation arguments.

If we do not consider any tradeable assets except the derivatives on Y_t ($n = 1$) a further simplification arises. It is easily shown that $a_t = \frac{p_t - q_t}{q_t^2 + p_t - 2q_t p_t}$ and $b_t = \frac{(p_t - q_t)^2}{q_t^2 + p_t - 2q_t p_t}$. Moreover, if a_t, b_t are state independent, so are p_t, q_t, such that a single-asset and multi-period valuation formula, directly generalizing (8), is obtained:

$$V_0 = \mathbb{E}_0(H) + \sum_{t=1}^{T} \frac{q_t - p_t}{1 - p_t} \mathrm{TVaR}_{1-p_t}(Y_t). \tag{17}$$

A comment is due relating to the uncorrelatedness assumption of Proposition 4. For simplicity, consider the single-asset case. Then, the proposition requires $\mathbb{E}_{t-2}\left(\mathbf{1}_{\{Y_{t-1} \geq d_{t-1}\}} Y_t\right) = 0$ and $\mathbb{E}_{t-2}\left(\mathbf{1}_{\{Y_{t-1} \geq d_{t-1}\}} \mathbf{1}_{\{Y_t \geq d_t\}} Y_t\right) = \mathbb{E}_{t-2}\left(\mathbf{1}_{\{Y_{t-1} \geq d_{t-1}\}}\right) \mathbb{E}_{t-2}\left(\mathbf{1}_{\{Y_t \geq d_t\}} Y_t\right)$. While the random variables Y_{t-1} and Y_t are uncorrelated (due to the martingale property) this does not necessarily imply that the pairs of random variables $(\mathbf{1}_{\{Y_{t-1} \geq d_{t-1}\}}, Y_t)$ and $(\mathbf{1}_{\{Y_{t-1} \geq d_{t-1}\}}, \mathbf{1}_{\{Y_t \geq d_t\}} Y_t)$ are also uncorrelated. Stronger assumptions on the joint distribution of the vector $(Y_1, \ldots, Y_T)$ are thus required, for instance it is sufficient to assume that the martingale innovations are independent.

Two numerical examples are now presented. In Example 3, a direct application of Proposition 4 is given for the case of two assets and several time periods. In Example 4 we discuss the case where a_t, b_t are not state-independent. In particular, we assume a scenario where, though the random variables Y_t are independent of each other, markets take a different view such that a high level of Y_{t-1} is associated with a high market price q_t for the payoff $\mathbf{1}_{\{Y_t \geq d_t\}}$.

Example 3. In this example we consider a long-term $\mathcal{F}_T$-measurable liability H with $\mathbb{E}_0(H) = 100$, $T = 10$ years and two tradeable assets in each period. These are a derivative with price at time $t - 1$ of q_t and pay-off at time t of $\mathbf{1}_{\{Y_t \geq d_t\}}$ and a stock with price process $S_t^{(2)}$ and excess return $X_t^{(2)}$. We assume that the claims development results $Y_1, \ldots, Y_T$ are mutually independent and so are their derivative returns $X_1^{(1)}, \ldots, X_T^{(1)}$. Moreover, the pair $(Y_t, X_t^{(2)})$ is defined via a bivariate log-normal model, such that

$$Y_t = \exp\left(\mu_t + \sigma_t Z_t^{(1)}\right) - \exp\left(\mu_t + \sigma_t^2/2\right) \qquad \text{and} \qquad X_t^{(2)} = \exp\left(m + s Z_t^{(2)}\right) - 1,$$
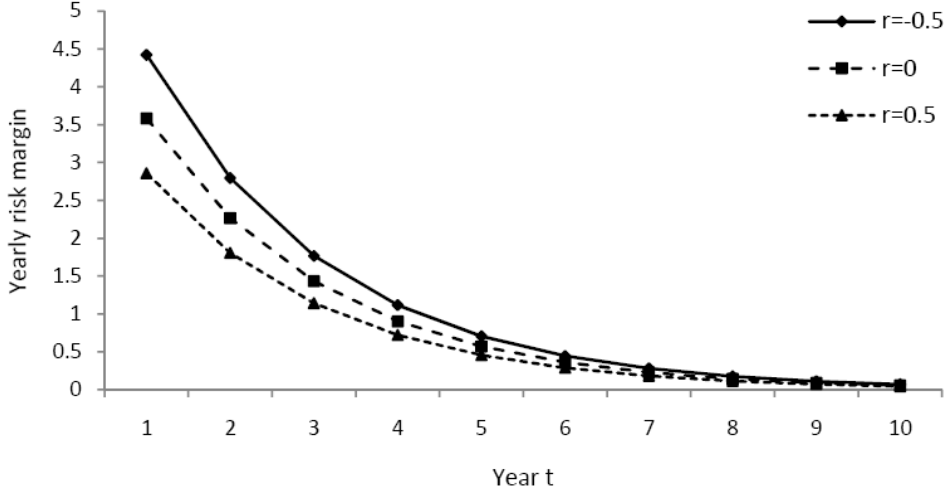
Figure 2: Yearly risk margins for different levels of correlation parameter r.

where $(Z_t^{(1)}, Z_t^{(2)})$ follow a bivariate standard normal distribution with correlation r. This implies that we can write $Z_t^{(2)} = rZ_t^{(1)} + \sqrt{1-r^2}W_t$, where $(Z_t^{(1)}, W_t)$ are independent standard normal variables. Note that, as required, $\mathbb{E}_{t-1}(Y_t) = 0$. The model for Y_t used here is illustrative, as in a more realistic application one would need to derive the dynamics of Y_t from a stochastic reserving model, see for instance Merz et al. [15].

For the derivative we use parameters $p_t = 0.01$ and $q_t = 0.066$ for all t, implying that the threshold d_t is always set at the 99th percentile of Y_t and that the derivative price in future periods is assumed constant. For the claims development results we use $\mu_t = 0.4586(T - t + 1)$ and $\sigma_t = 0.198$ for all t, such that the standard deviation of Y_t reduces over time, reflecting that uncertainty decreases with increasing information. For the stock we use $m = 0.15$ and $s = 0.2$. Finally, r is allowed to vary in the range $(-1, 1)$. A positive (negative) correlation corresponds to the situation when stock prices tend to increase (decrease) at times of high claims development results (motivated by economically driven claims inflation).

We proceed by applying Proposition 4. The necessary calculations are somewhat tedious and are documented in the Appendix. For the correlation parameter values $r \in \{-0.5, 0, 0.5\}$, market values of H equaling $V_0(r) = \{111.90, 109.64, 107.69\}$, respectively, are obtained. In Figure 2, we plot the market risk margin applied for each year of the liability's run-off, that is, the quantities $\sum_{i=1}^2 \frac{a_t^{(i)} \mathrm{Cov}_0(X_t^{(i)}, Y_t)}{1 - b_t}$, $t = 1, \ldots, T$.

We observe that the case $r = 0$ is equivalent to the absence of the stock such that V_0 is given by expression (17). This means that no risk in the claims development result can be mitigated by the asset stock price process. When $r = 0.5$, long positions in the stock produce a natural hedging effect for the liability as investment returns pay for claims development results. This situation, which is desirable for the holder of the liability, *decreases* V_0 in relation to the case

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$r = 0$. Conversely, when $r = -0.5$, short positions in the stock are taken. Thus, in order to hedge the liability, negative expected stock returns are incurred. This adverse situation, analogous to the liability being subject to systematic risk, *increases* V_0 in relation to the case $r = 0$. It can be seen from Figure 2 that the annual contributions to the market value of H decrease with time. This is explained by the decay of the standard deviation of Y_t in our model as t increases. $\qquad\square$

Example 4. To avoid computational issues, we now consider a shorter term liability H, with $T = 2$ and $\mathbb{E}_0(H) = 100$. In this example there is no stock correlated with claims development results such that the only tradeable asset is the derivative on Y_1 and Y_2. Again, we assume that the claims development results are mutually independent and

$$Y_t = \exp\left(\mu_t + \sigma_t Z_t\right) - \exp\left(\mu_t + \sigma_t^2/2\right), \quad t = 1, 2,$$

where Z_1, Z_2 are independent standard normals. The parameters of the claims development results are $\mu_1 = 4.586$, $\mu_2 = 4.127$, $\sigma_1 = \sigma_2 = 0.198$.

We now consider a derivative with a higher probability of a pay-off than in the previous example such that $p_1 = p_2 = 0.05$ and $q_1 = 0.21$. However, q_2 is no longer known at time $t = 0$, but is instead dependent on Y_1. If the derivative produces a pay-off the market price of the derivative increases in the next period (and vice versa). Specifically, we define q_2 by:

$$q_2 = \begin{cases} \underline{q} \leq q_1, & \text{if } Y_1 < d_1, \\ \overline{q} \geq q_1, & \text{if } Y_1 \geq d_1. \end{cases}$$

To aid comparisons, we let $\mathbb{E}_0(q_2) = (1 - p_1)\underline{q} + p_1\overline{q} = q_1$. The sensitivity of q_2 on past performance of the derivative is studied by considering three cases: (i) $\overline{q}/\underline{q} = 1$ giving $\underline{q} = \overline{q} = 0.21$; (ii) $\overline{q}/\underline{q} = 2$ giving $\underline{q} = 0.2$, $\overline{q} = 0.4$; and $\overline{q}/\underline{q} = 4$ giving $\underline{q} = 0.183$, $\overline{q} = 0.730$.

To calculate the market value V_0^* we use Theorem 2. In particular, we have $L_2 = 1$ and

$$a_2^* = a_2 = \frac{\mathbb{E}_1(X_2)}{\mathbb{E}_1(X_2^2)} = \frac{p_2 - q_2}{p_2 + q_2^2 - 2p_2q_2},$$

$$b_2^* = b_2 = \frac{(\mathbb{E}_1(X_2))^2}{\mathbb{E}_1(X_2^2)} = \frac{(p_2 - q_2)^2}{p_2 + q_2^2 - 2p_2q_2},$$

$$V_1^* = V_1 = \mathbb{E}_1\left(\frac{1 - a_2 X_2}{1 - b_2}H\right) = \mathbb{E}_0(H) + Y_1 + \mathbb{E}_1\left(\frac{1 - a_2 X_2}{1 - b_2}Y_2\right)$$

$$= \mathbb{E}_0(H) + Y_1 + \frac{q_2 - p_2}{1 - p_2}\text{TVaR}_{1-p_2}(Y_2).$$

Hence V_1^* can be explicitly calculated as a function of q_2, which is in turn a function of Y_1. To derive V_0^*, we need to calculate $L_1 = 1 - b_2$ and, observe that $d\mathbb{P}_0^* = L_1/\mathbb{E}_0[L_1]\,d\mathbb{P}_0$,

$$a_1^* = \frac{\mathbb{E}_0(L_1 X_1)}{\mathbb{E}_0(L_1 X_1^2)} \quad \text{and} \quad b_1^* = \frac{\mathbb{E}_0(L_1 X_1)^2}{\mathbb{E}_0(L_1 X_1^2)\mathbb{E}_0(L_1)},$$

$$V_0^* = \frac{1}{\mathbb{E}_0(L_1)}\mathbb{E}_0\left(L_1\frac{1 - a_1^* X_1}{1 - b_1^*}V_1^*\right).$$

These calculations of the market value V_0^* can be easily done by Monte-Carlo simulation. Using a simulated sample of $5 \cdot 10^6$ from Y_1, we obtain that (i) for $\overline{q}/\underline{q} = 1$ it is $V_0^* = 113.2$; (ii) for $\overline{q}/\underline{q} = 2$ it is $V_0^* = 114.2$; and for $\overline{q}/\underline{q} = 4$ it is $V_0^* = 116.0$.

Hence, with increasing sensitivity of q_2 to the outcome of Y_1 the market value of the liability increases. Intuitively, this is clear that the uncertainty is increased by increasing price sensitivity in q_2 in terms of Y_1. This case may be more realistic in comparison to a scenario where derivative prices are unaffected by observed losses, that is, where q_2 does not depend on the outcome of Y_1, because investors react sensitively based on past observations. However, at least for this short-tail example, the increase is not particularly dramatic. □

3.3 Hedging and capital efficiency

In Section 2.3 the issue of capital efficiency was discussed in relation to the single-period model. The relation between hedging and capital efficiency becomes rather convoluted in the multi-period case. The reason for this is structural. While capital requirements in insurance are typically calculated with respect to a 1-year time horizon, the optimal investment strategy is formulated to minimize a quadratic error calculated at the time horizon T. In particular, the trading strategy in each period will also reflect the performance of the portfolio to-date, which introduces path-dependency.

Consider the simplest possible case, where $Y_1, \ldots, Y_T$ are independent, a_t, b_t are $\mathcal{F}_0$-measurable, and the only traded asset is the derivative on Y_t. Then, from Theorem 2 it is seen that the optimal trading strategy for initial endowment V_0 is given by

$$\phi_t(V_0) = \xi_t + a_t \left(V_{t-1} - G_{t-1}^{V_0, \phi(V_0)} \right), \qquad \text{where } \xi_t = \frac{\mathbb{E}_{t-1}((V_t - V_{t-1})X_t)}{\mathbb{E}_{t-1}(X_t^2)}.$$

Straightforward but tedious calculations then yield $\xi_t = \frac{\text{Cov}_0(X_t, Y_t)}{\text{Var}_0(X_t)}$. Hence the trading strategy $\phi_t(V_0)$ consists of two parts: ξ_t, the values of which in this simple setting are known at time 0, and $a_t(V_{t-1} - G_{t-1}^{V_0, \phi(V_0)})$, which reflects the value of the investment portfolio at time t. Note that ξ_t is essentially identical to ξ_1 in (2). Let $\delta_t = V_{t-1} - G_{t-1}^{V_0, \phi(V_0)}$ represent the difference between the value of the liability and the value of the investment portfolio at time $t - 1$. Then, since $\mathbb{E}_{t-1}(X_t) \leq 0 \implies a_t \leq 0$, in the multi-period case we adjust the trading strategy such that, if the shortfall is $\delta_t > 0$, less is invested in the risky asset and vice versa.

Analogously to what was discussed in Section 2.3, a plausible re-formulation of the capital efficiency condition (10) at time $t - 1$ is

$$\rho_{t-1}(Y_t - (G_t^{V_0, \phi(V_0)} - G_{t-1}^{V_0, \phi(V_0)})) \leq \rho_{t-1}(Y_t), \tag{18}$$

where ρ_{t-1} is the regulatory risk measure evaluated given the information $\mathcal{F}_{t-1}$ available at time $t - 1$. The principle here is that the liability in respect to which capital needs to be held during $(t - 1, t]$ is the corresponding claims development result Y_t. Inequality (18) represents

the condition that the capital required to support Y_t, minus the net gains from trading over the same interval, is less than the capital required to support Y_t, assuming that all funds are invested in the risk free asset. The left hand side of inequality (18) can be written as

$$\rho_{t-1}(Y_t - \phi_t(V_0)X_t) = \rho_{t-1}\left(Y_t - \xi_t X_t - a_t \delta_t X_t\right). \tag{19}$$

Define $\tilde{k}_t = a_t \delta_t + \frac{1}{1-p_t}\text{TVaR}_{t-1,1-p_t}(Y_t)$. Then, retracing the first steps in the proof of Proposition 1, it follows that the condition for inequality (18) to hold is, analogously to (11),

$$q_t \tilde{k}_t \le \rho_{t-1}(Y_t) - \rho_{t-1}(Y_t - \tilde{k}_t \mathbf{1}_{D_t}). \tag{20}$$

Finally, we remark that ξ_t corresponds exactly to the investment in the stock under the (non-self-financing) local risk-minimizing hedging strategy of Föllmer and Schweizer [11]. Therefore, under such a trading strategy with explicit one-period optimization targets, the present discussion of capital efficiency would be much simplified.

4   Concluding remarks

We discussed the problem of valuing insurance liabilities in discrete time through mean-variance hedging. Key features of the proposed approach are the decomposition of the terminal liability into claims development results and the presence of a derivative on the claims development result in each period. In simple cases, the resulting valuation formulas become structurally very similar to regulatory cost-of-capital based formulas. However, adoption of the mean-variance framework improves upon such formulas, by introducing sensitivity to observed market prices, the inclusion of other tradeable assets, and the consistent extension to multiple periods.

The similarity between the formulas derived here and the ones used in regulation should not obscure the very different interpretations underlying them. In our approach the market value margin obtained (difference between market consistent and expected values) does *not* correspond to the cost-of-capital, but reflects the cost of a replication portfolio. Hence, it is conceivable that a cost-of-capital loading may be *added* to the market consistent value that we obtain, since investors need to be compensated for the frictional costs that holding capital incurs (see e.g. the discussions in Zanjani [27] and Venter [25]). The analysis of Section 2.3 shows that the mean-variance hedging approach may also deliver a reduction in such capital costs.

It is then useful to distinguish between the possible constituent parts of the value of a liability. Thus, if a cost-of-capital loading is added to the (partial) replication cost that our valuation formulas reflect, this should only represent frictional capital costs. In particular, it should not be further increased to act as a proxy for replication costs, as current regulatory valuation approaches implicitly do. Finally, besides the cost of replication and the frictional cost of capital, it is plausible that an additional risk load is applied via a performance measure, purely to reward investors for risk taking. This need not be related to a tail risk measure like

VaR or TVaR; for example, mean-variance hedging approaches can be adjusted to deliver a pre-specified minimal level of Sharpe ratio (Černý [1], Section 13.2).

Appendix: Calculations in Example 3

For the calculations shown here it is convenient to use excess returns X_t rather than price increments, as discussed in Section 3.1. To determine V_0 we first need to calculate all terms in

$$a_t = \mathbb{E}_{t-1}(X_t')\,\mathbb{E}_{t-1}(X_t X_t')^{-1} \qquad \text{and} \qquad b_t = \mathbb{E}_{t-1}(X_t')\,\mathbb{E}_{t-1}(X_t X_t')^{-1}\,\mathbb{E}_{t-1}(X_t).$$

Model assumption $Y_t = \exp(\mu_t + \sigma_t Z_t^{(1)}) - \exp\left(\mu_t + \sigma_t^2/2\right)$ provides returns

$$X_t^{(1)} = \frac{1}{q_t}\mathbf{1}_{\{Y_t \geq d_t\}} - 1, \qquad X_t^{(2)} = \exp\left(m + sZ_t^{(2)}\right) - 1,$$

where $Z_t^{(2)} = rZ_t^{(1)} + \sqrt{1-r^2}W_t$ and $(Z_t^{(1)}, W_t)$ are independent standard normals. Therefore, to apply Proposition 4 we need to calculate the first and second moments of X_t as well as the covariances $\mathrm{Cov}_{t-1}\left(X_t^{(i)}, Y_t\right)$ for $i = 1, 2$. The first moments of X_t are given by

$$\mathbb{E}_{t-1}\left(X_t^{(1)}\right) = \frac{p_t}{q_t} - 1 \quad \text{and} \quad \mathbb{E}_{t-1}\left(X_t^{(2)}\right) = \exp\left(m + s^2/2\right) - 1.$$

The second moments of X_t are given by

$$\mathbb{E}_{t-1}\left((X_t^{(1)})^2\right) = \frac{1}{q_t^2}\,p_t(1-p_t) + \left(\frac{p_t}{q_t} - 1\right)^2,$$

$$\mathbb{E}_{t-1}\left((X_t^{(2)})^2\right) = 1 - 2\exp\left(m + s^2/2\right) + \exp\left(2m + 2s^2\right),$$

$$\mathbb{E}_{t-1}\left(X_t^{(1)} X_t^{(2)}\right) = 1 - \exp\left(m + s^2/2\right) - \frac{p_t}{q_t} + \mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{Y_t \geq d_t\}}\exp\left(m + sZ_t^{(2)}\right)\right).$$

Let $\tilde{d}_t = d_t + \exp\left(\mu_t + \sigma_t^2/2\right) = \exp\left(\mu_t + \sigma_t\Phi^{-1}(1 - p_t)\right)$, where Φ is the standard normal distribution. Then $\mathbf{1}_{\{Y_t \geq d_t\}} = \mathbf{1}_{\{\exp(\mu_t + \sigma_t Z_t^{(1)}) \geq \tilde{d}_t\}}$, such that

$$\mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{Y_t Y \geq d_t\}}\exp(m + sZ_t^{(2)})\right) = \frac{1}{q_t}(\exp\left(m + s^2(1-r^2)\right) \cdot g(r),$$

where have defined $g(r) = \mathbb{E}_{t-1}\left(\mathbf{1}_{\{Z_t^{(1)} \geq (\log \tilde{d}_t - \mu_t)/\sigma_t\}}\exp\left(srZ_t^{(1)}\right)\right)$. From the definition of $\tilde{d}_t$ we obtain for $r = 0$ the value $g(0) = p_t$. Denote $k = (\log \tilde{d}_t - \mu_t)/\sigma_t$. If $r > 0$, we have $g(r) = \exp(s^2r^2/2)\Phi\left(\frac{s^2r^2 - srk}{sr}\right)$, using the properties of the log-normal distribution. If $r < 0$, we have $g(r) = \exp(s^2r^2/2)\left[1 - \Phi\left(\frac{s^2r^2 - srk}{-sr}\right)\right]$.

Finally, we move to the calculation of the covariances. They are given by

$$\mathrm{Cov}_{t-1}(X_t^{(1)}, Y_t) = \mathbb{E}_{t-1}\left(\frac{1}{q_t}\mathbf{1}_{\{\exp(\mu_t + \sigma_t Z_t^{(1)}) \geq \tilde{d}_t\}}\exp\left(\mu_t + \sigma_t Z_t^{(1)}\right)\right) - \frac{p_t}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right)$$

$$= \frac{1}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right)\Phi\left(\frac{\mu_t + \sigma_t^2 - \log \tilde{d}_t}{\sigma_t}\right) - \frac{p_t}{q_t}\exp\left(\mu_t + \sigma_t^2/2\right),$$

and

$$\mathrm{Cov}_{t-1}(X_t^{(2)}, Y_t) = \mathbb{E}_{t-1}\left(\exp\left(m + \mu_t + (sr + \sigma_t)Z_t^{(1)} + s\sqrt{1 - r^2}W_t\right)\right)$$
$$- \exp\left(m + \mu_t + (s^2 + \sigma_t^2)/2\right)$$
$$= \exp\left(m + \mu_t + (sr + \sigma_t)^2/2 + s^2(1 - r^2)/2\right)$$
$$- \exp\left(m + \mu_t + (s^2 + \sigma_t^2)/2\right).$$

This completes the required calculations. □

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