Free resolutions of algebras

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Abstract

Given an algebra A, presented by generators and relations, i.e. as a quotient of a tensor algebra by an ideal, we construct a free algebra resolution of A, i.e. a differential graded algebra which is quasi-isomorphic to A and which is itself a tensor algebra. The construction rests combinatorially on the set of bracketings that arise naturally in the description of a free contractible differential graded algebra with given generators..

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1. Introduction

Let S be any ring. We write $\otimes$ for $\otimes_S$, and, for any S,S-bimodule V, we write V^n for $V^{\otimes n}$. Further, we write

$$T_S^+(V) = \bigoplus_{n\geq 1} V^n, \qquad V^{(m)} = \bigoplus_{n\geq m} V^n, \tag{1.1}$$

for the non-unital tensor algebra of 'free words' in V over S and for the ideal of words of length at least m, respectively.

Our objective in this paper is to construct a free algebra resolution of an arbitrary (non-unital) S-algebra A presented in terms of generators and

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relations, that is,

$$A = T_S^+(V)/I,$$

for some S,S-bimodule V and ideal I. As a first step, in Section 2, we work towards describing the case where $I \subseteq V^{(2)}$, i.e. the generators are 'minimal', and I is homogeneous (Theorem 2.2). This case is controlled by the by-now-familiar A_∞ combinatorics of rooted trees or the corresponding set of bracketings (see, for example, Figure 1). By extending this set of bracketings in various ways, we find, in Sections 3 and 4, that the essential nature of the proof becomes more transparent in the general case (Theorem 4.1).

As a typical motivating example, consider a quiver Q and field k. Set $S = k^{Q_0}$, the semi-simple algebra spanned by the vertex idempotents, and $V = k^{Q_1}$, the S,S-bimodule spanned by the arrows. Then the augmented unital algebra $S \oplus T_S^+(V)$ is the path algebra kQ and $S \oplus A$ is a 'quiver algebra', i.e. is presented by a quiver with relations. Such augmentation allows one to move easily between unital and non-unital S-algebras; we find it notationally simpler to work in the slightly less familiar non-unital context in this paper.

Giving a free resolution of A as an S-algebra is, almost tautologically, the same as giving a 'locally finite' A_∞-coalgebra over S

$$K_\bullet = \bigoplus_{n\geq 1} K_n,$$

where each K_n is an S,S-bimodule and such that A is quasi-isomorphic to the differential graded (dg) algebra

$$\operatorname{Cobar} K_\bullet = \big(T_S^+(K_\bullet[1]), d\big). \tag{1.2}$$

Here $K_\bullet[1]$ denotes the shifted complex, defined by $K[1]_n = K_{n+1}$. In general, the quasi-isomorphism should be induced by some S,S-morphism

$$\theta\colon K_1 \to A$$

such that $\theta(K_1)$ generates A. In our case, since the generators V of A are explicitly specified, we will suppose that $K_1 = V$ and that $\theta\colon V \to A$ is the specifying map. Then the degree zero term in $\operatorname{Cobar} K_\bullet$ is $T_S^+(V)$ and the quotient map $T_S^+(V) \to A$ will be the quasi-isomorphism.

The cobar construction here is 'almost tautological' in that the differential d on $T_S^+(K_\bullet[1])$ may be considered simply as an efficient way of encoding the

A_∞-coalgebra structure on $K_\bullet$. More precisely, the A_∞-coalgebra structure consists of S, S-morphisms

$$\Delta_k \colon K_n \to K_{m_1} \otimes \cdots \otimes K_{m_k},$$

for each $k \geq 1$ and $\sum_{i=1}^k m_i = n - 2 + k$, satisfying the conditions

$$\sum_{r+s+t=n} (-1)^{r+st} \big(1^{\otimes r} \otimes \Delta_s \otimes 1^{\otimes t}\big) \circ \Delta_{r+1+t} = 0$$

for all $n \geq 1$ (cf. [5, Definition 1.2.1.8]). These may be packaged into the single condition that the endomorphism d of $T_S^+ (K_\bullet[1])$, determined by

$$d = -[1]^{\otimes n} \circ \Delta_n \circ [-1] : K_\bullet[1] \to K_\bullet[1]^{\otimes n} \tag{1.3}$$

and the (graded) Leibniz rule (see (3.2)), satisfies $d^2 = 0$ (cf. [5, §1.2]). Note also that, in evaluating $[1]^{\otimes n}$, we use the standard Koszul sign rule; for example

$$[1] \otimes [1] := (-1)^m : K_m \otimes K_n \to K_m \otimes K_n.$$

Remark 1.1. The condition that $K_\bullet$ is 'locally finite' is simply the requirement that (1.3) does define a differential on $T_S^+ (K_\bullet[1])$, which is defined as an infinite direct sum, and not just on its completion, i.e. the corresponding direct product. In other words, for any element $x \in K_\bullet$, the coproducts $\Delta_n(x)$ are non-zero for only finitely many n. It will be clear that all coalgebras we will encounter have this property and we will not explicitly mention it again.

2. First examples

We begin by discussing the sort of construction we are looking for in the case that $I = V^{(2)}$ and so $A = T_S^+ (V) / I$ is just V with trivial multiplication.

As a warm-up, we first observe that the classical candidate for $K_\bullet$, in this case, is the free coassociative coalgebra generated by V,

$$B_\bullet(V) = \bigoplus_{n \geq 1} V^n, \tag{2.1}$$

which is what the usual (unaugmented) bar construction yields. The comultiplication $\Delta \colon B_{m+n} \to B_m \otimes B_n$ is tautological, i.e. is the natural identification $\tau \colon V^{m+n} \to V^m \otimes V^n$. This gives the A_∞-coproduct Δ_2, with all other coproducts vanishing.

As one should expect, the dg algebra $\mathcal{A}_\bullet = \operatorname{Cobar} B_\bullet(V)$ is a free resolution of A. Indeed, we may explicitly write

$$\mathcal{A}_\bullet = \bigoplus_{\pi\in\Pi} V^\pi, \tag{2.2}$$

where Π is the set of all finite sequences $\pi = (\pi_1, \dots, \pi_r)$ of positive integers, while

$$V^\pi = V^n, \quad \text{for } n = \sum_k \pi_k,$$

which is in homological degree d, i.e. is a summand of $\mathcal{A}_d$, for

$$d = |\pi| = \sum_k (\pi_k - 1).$$

Multiplication in $\mathcal{A}_\bullet$ corresponds to concatenation of sequences, i.e. is given by the tautological maps $\tau\colon V^\pi \otimes V^\eta \to V^{\pi\eta}$, while the non-trivial components of the differential $d\colon V^\pi \to V^{\pi'}$ are $\pm\tau$ whenever π' is obtained from π by splitting some term in the sequence into two. The sign is $(-1)^{|(\pi'_1,\dots,\pi'_p)|}$ when the p-th term of π is split.

To see explicitly that $\mathcal{A}_\bullet$ is quasi-isomorphic to the algebra A, we notice first that, for each $n \geq 0$, there is precisely one sequence of degree 0 summing to n, namely $\pi = (1, \dots, 1)$, and this yields $\mathcal{A}_0 = T^+_S(V)$. On the other hand, for each $n \geq 2$, the part of $\mathcal{A}_\bullet$ consisting of summands equal to V^n is given by $V^n \otimes_{\mathbb{Z}} C^{aug}_\bullet(\sigma_{n-2})[1]$ the (shifted and augmented) chain complex of the $(n-2)$-simplex, which is exact, as required.

From the point-of-view of this paper, a more natural, but rather bigger, candidate for $K_\bullet$ is the free A_∞-coalgebra generated by V,

$$B^\infty_\bullet(V) = \bigoplus_{\beta\in\mathcal{B}^{[2]}} V^\beta. \tag{2.3}$$

Here $\mathcal{B}^{[2]}$ is the set of all closed non-degenerate bracketings (or equivalently those that correspond to 'rooted trees') and the bracketed tensor product $V^\beta = V^n$ if β has n inputs and is in homological degree d if β has $d-1$ pairs of brackets. Note that "closed" means that the whole expression is enclosed in an outer bracket, while "non-degenerate" means that each inner pair of brackets encloses at least two inputs. For example, one summand of $B^\infty_4(V)$ would be

$$V^\beta = \texttt{[[}V \otimes V\texttt{]} \otimes \texttt{[}V \otimes V \otimes V\texttt{]]} = V^5$$

for $\beta = \texttt{[[}\bullet\bullet\texttt{][}\bullet\bullet\bullet\texttt{]]}$ or $\texttt{[[2][3]]}$, a 3-fold bracketing of 5 inputs.

The differential Δ_1 has non-zero components

$$(-1)^{m-1}\tau\colon V^{\beta} \to V^{\beta(\widehat{m})},$$

where $\beta(\widehat{m})$ is obtained from β by removing the mth internal left bracket $\texttt{[}$ (counted from the left) together with its matching right bracket $\texttt{]}$. Precisely one higher coproduct Δ_k, for some $k \geq 2$, is defined on each V^{β} and corresponds to removing the outer bracket and writing what is inside as a concatenation of k closed bracketings, up to a sign. For example, there is a component

$$\Delta_4\colon V^{\texttt{[[2]2[3]]}} \to V^{\texttt{[2]}} \otimes V \otimes V \otimes V^{\texttt{[3]}}.$$

For the general component

$$\Delta_k\colon V^{\texttt{[}\beta_1\ldots\beta_k\texttt{]}} \to V^{\beta_1} \otimes \ldots \otimes V^{\beta_k}$$

the sign we choose (see Remark 3.1 for an explanation) is

$$-(-1)^{\sum_{i=1}^{k}(k-i)d_i}, \tag{2.4}$$

where d_i is the homological degree of V^{β_i} in $B^{\infty}_{\bullet}(V)$.

By convention, the only 0-fold closed bracketing is 1 and so $B^{\infty}_1(V) = V$. On the other hand, the 1-fold closed bracketings are $\texttt{[}k\texttt{]}$, for each $k \geq 2$, and hence $B^{\infty}_2(V) = V^{(2)}$. As a further example, the possible bracketings of 4 inputs are 3, 2 or 1-fold, as listed in Figure 1 together with their corresponding rooted trees. These are well-known to correspond to the 0,1 and 2 dimensional cells of a pentagon. Furthermore, the restriction of the differential on $B^{\infty}_{\bullet}$ corresponds to the coboundary map on the cochain complex of the pentagon (with some care needed over signs).

More generally, writing $B^{\infty}_{n,k}(V)$ for the sum over all $(n-1)$-fold bracketings of k inputs, we have for $k \geq 2$,

$$B^{\infty}_{n,k}(V) \cong V^k \otimes_{\mathbb{Z}} C^{k-n}(\mathcal{S}_{k-2}), \tag{2.5}$$

where $C^{\bullet}(\mathcal{S}_m)$ is the cochain complex of the mth associahedron (or Stasheff polytope, introduced in [7]). In addition, the differential on $C^{\bullet}(\mathcal{S}_{k-2})$ induces the differential on $B^{\infty}_{\bullet,k}(V)$. Since the associahedra are all contractible, this means that the homology of $B^{\infty}_{\bullet,k}(V)$ is just V^k in degree k. In other words,

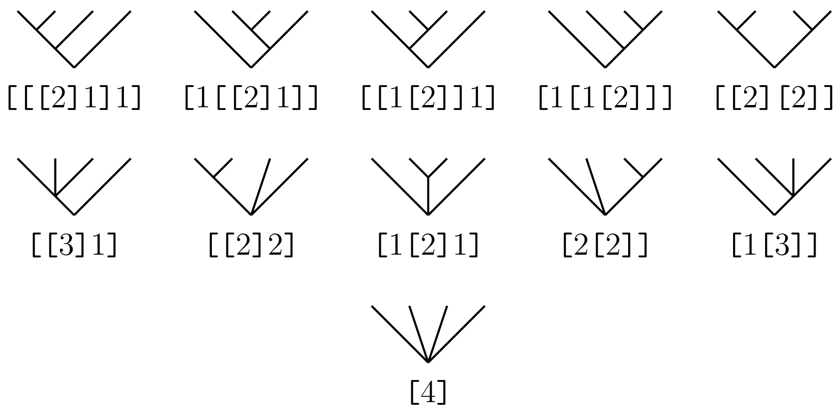

Figure 1: Bracketings with four inputs.

the kernel of d restricted to $B^{\infty}_{k,k}(V)$ is isomorphic to $B_k(V)$ and the induced map

$$\eta\colon B_{\bullet}(V) \to B^{\infty}_{\bullet}(V), \tag{2.6}$$

is a quasi-isomorphism of A_∞ coalgebras (essentially because the A_∞ operad resolves the associative operad).

Using Corollary 6.3 from the Appendix, this is actually sufficient to prove the following result, but we will also prove it directly as a special case of our main result Theorem 4.1 in Section 4.

Theorem 2.1. *Cobar* $B^{\infty}_{\bullet}(V) \simeq V$.

More generally, suppose A is a graded algebra, generated by V in degree 1, so that $A = T^+_S(V)/I$, where

$$I = \bigoplus_{n\geq 2} R_n$$

is a homogeneous ideal in $V^{(2)}$. Then, generalising (2.3), we define

$$B^{\infty}_{\bullet}(V, I) = \bigoplus_{\beta\in\mathcal{B}^{[2]}} (V, I)^{\beta}, \tag{2.7}$$

where $(V, I)^{\beta}$ is obtained from V^{β} by replacing every occurrence of an inner bracketed $[V^n]$ by $[R_n]$. The A_∞ coalgebra structure on $B^{\infty}_{\bullet}(V, I)$ is defined, as for $B^{\infty}_{\bullet}(V)$, by tautological maps (with the appropriate sign), except for the component of the differential corresponding to the removal of an inner bracket. Such a component is induced by one of the two maps

$$V^a \otimes \mathtt{[}R_n\mathtt{]} \otimes V^b \to R_{n+a+b},$$
$$V^a \otimes \mathtt{[}R_n\mathtt{]} \otimes V^b \to V^{n+a+b},$$

depending on whether the domain is enclosed by matching brackets `[..]` or non-matching ones, e.g. `]..]`. The first map exists because I is an ideal, while the second is the composite of the first with the inclusion $R_{n+a+b} \subseteq V^{n+a+b}$. This inclusion also gives the degree 2 component $\Delta_k\colon \mathtt{[}R_k\mathtt{]} \to V^k$.

Another special case of Theorem 4.1 is then the following.

Theorem 2.2. *Cobar* $B^{\infty}_{\bullet}(V, I) \simeq A$.

Example 2.3. We look more closely at the case when S is a separable k-algebra, so that $\otimes$ is exact and hence $B^{\infty}_{\bullet}(V, I) \subseteq B^{\infty}_{\bullet}(V)$. Now Theorem 2.2 implies (by Remark 4.3 and Proposition 5.1, or directly from Proposition 6.4) that the degree k part of $\mathrm{Tor}^A_n(S, S)$ is isomorphic to the homology of d at $B^{\infty}_{n,k}(V, I)$, so this degree k part vanishes for $k < n$, since there are no bracketings in this case. Since $\mathrm{Ext}^n_A(S, S) \cong \mathrm{Hom}_S(\mathrm{Tor}^A_n(S, S), S)$, this also starts in degree n (cf. [1, Lemma 2.1.2]).

Observe further that all inner brackets in the bracketings that contribute to $B^{\infty}_{n,n}(V)$ are `[2]` (as for example in the top row of Figure 1). Hence, under the embedding $\eta\colon B_{\bullet}(V) \to B^{\infty}_{\bullet}(V)$ of (2.6), the kernel of d restricted to $B^{\infty}_{n,n}(V, I)$ is identified with the familiar Koszul term $B_n(V, R_2) \subseteq B_n(V)$ (cf. [1, §2.6]), defined by

$$B_n(V, R_2) = \bigcap_{p+q=n-2} V^p \otimes R_2 \otimes V^q.$$

Thus, as is also well-known from Koszul theory (cf. [1, Thm 2.6.1]), the degree n part of $\mathrm{Tor}_n(S, S)$ is isomorphic to $B_n(V, R_2)$. Note also that $B_{\bullet}(V, R_2)$ is a subcoalgebra of $B_{\bullet}(V)$.

But now, if $\mathrm{Ext}^n_A(S, S)$, and hence $\mathrm{Tor}^A_n(S, S)$, is concentrated in degree n, which is one characterisation of A being Koszul (cf. [1, Prop 2.1.3]), then necessarily $I = (R_2)$ and we also deduce that $B_{\bullet}(V, R_2) \simeq B^{\infty}_{\bullet}(V, I)$ and hence, by Corollary 6.3, that $\mathrm{Cobar}\, B_{\bullet}(V, R_2) \simeq A$, which is the algebra incarnation of the Koszul resolution for A.

3. A free contractible dg algebra

We now develop the machinery that will enable us to generalise the constructions behind Theorems 2.1 and 2.2. In the process, we are naturally led to consider bracketings of a slightly more general form than in Section 2.

For any ring S and any S,S-bimodule V, we can construct a free contractible dg S-algebra

$$\mathcal{F}_\bullet = \mathcal{F}_\bullet(V) = \bigoplus_{n\geq 0} \mathcal{F}_n(V)$$

that is 'freely' generated by V in degree 0 and a contracting homotopy $h\colon \mathcal{F}_\bullet \to \mathcal{F}_\bullet$ of degree 1, satisfying $dh + hd = \mathrm{id}$ and $h^2 = 0$. The differential d, of degree -1, is then determined recursively by the two conditions

$$d(hv) = v - h(dv), \tag{3.1}$$

$$d(v\cdot w) = dv\cdot w + (-1)^{\deg v} v\cdot dw, \tag{3.2}$$

starting from $dv = 0$ for $v \in V$. Notice that (with these signs) we deduce inductively that $d^2 = 0$, by observing that

$$d^2hv = dv - dhdv = dv - (dv - hd^2v) = hd^2v$$

and

$$\begin{aligned} d^2(v\cdot w) &= d\big(dv\cdot w + (-1)^{\deg v} v\cdot dw\big) \\ &= d^2v\cdot w + (-1)^{\deg dv} dv\cdot dw + (-1)^{\deg v} dv\cdot dw + v\cdot d^2w \\ &= d^2v\cdot w + v\cdot d^2w, \end{aligned}$$

since $\deg(dv) = (\deg v) - 1$.

We may describe $\mathcal{F}_\bullet$ more explicitly using an extended notion of bracketed tensor products, similar to Section 2,

$$\mathcal{F}_\bullet(V) = \bigoplus_{\beta\in\mathcal{B}^{(1)}} V^\beta, \tag{3.3}$$

where, if β is a d-fold bracketing of n inputs, then V^β is V^n and is in homological degree d, i.e. is a summand of $\mathcal{F}_d$. Note that this degree is different from (2.3), because we are not describing an A_∞-coalgebra here, but its cobar construction directly.

We define the set $\mathcal{B}^{(1)}$ of bracketings recursively as follows, noting that each such bracketing is, in the first instance, a word in the symbols "$\bullet$" (representing an input), "$\mathtt{[}$" and "$\mathtt{]}$", with the later two balanced in the usual way of brackets:

(i) $\mathcal{B}^{(1)}$ contains $\bullet$,
(ii) if $\beta \in \mathcal{B}^{(1)}$ and $\beta \neq \mathtt{[}\alpha\mathtt{]}$, for some $\alpha \in \mathcal{B}^{(1)}$, then $\mathtt{[}\beta\mathtt{]} \in \mathcal{B}^{(1)}$,
(iii) if $\alpha, \beta \in \mathcal{B}^{(1)}$, then their concatenation $\alpha\beta \in \mathcal{B}^{(1)}$

Note that, in contrast to $\mathcal{B}^{[2]}$ in (2.3), bracketings in $\mathcal{B}^{(1)}$ may be open, i.e. without an outer bracket, and degenerate, i.e. with only one input in an inner bracket.

As before, we abbreviate a sequence of n uninterrupted $\bullet$'s by "n". Thus, for example, $\mathtt{[}\bullet\bullet\mathtt{]}\mathtt{[}\bullet\bullet\mathtt{[}\bullet\mathtt{]]}$ becomes $\mathtt{[}2\mathtt{]}\mathtt{[}2\mathtt{[}1\mathtt{]]}$. In particular, there are just two bracketings 1 and $\mathtt{[}1\mathtt{]}$ with one input and eight with two inputs

$$2,\ \mathtt{[}2\mathtt{]},\ \mathtt{[}1\mathtt{]}1,\ 1\mathtt{[}1\mathtt{]},\ \mathtt{[[}1\mathtt{]}1\mathtt{]},\ \mathtt{[}1\mathtt{[}1\mathtt{]]},\ \mathtt{[}1\mathtt{]}\mathtt{[}1\mathtt{]},\ \mathtt{[[}1\mathtt{]}\mathtt{[}1\mathtt{]]}.$$

Now the product in $\mathcal{F}_\bullet$ has non-zero components consisting of the tautological maps

$$\tau\colon V^{\alpha} \otimes V^{\beta} \to V^{\alpha\beta},$$

while the contracting homotopy $h\colon \mathcal{F}_\bullet \to \mathcal{F}_\bullet$ has non-zero components given by the tautological maps

$$\tau\colon V^{\beta} \to V^{\mathtt{[}\beta\mathtt{]}}, \quad \text{for each } \beta \neq \mathtt{[}\alpha\mathtt{]}, \tag{3.4}$$

that is, τ is the identity map $V^k \to V^k$, where k is the number of inputs in β (and in $\mathtt{[}\beta\mathtt{]}$).

On the other hand, the differential $d\colon \mathcal{F}_n \to \mathcal{F}_{n-1}$ has n non-zero components

$$(-1)^{m-1}\tau\colon V^{\beta} \to V^{\beta(\widehat{m})}, \quad \text{for } m = 1, \dots, n, \tag{3.5}$$

where $\beta(\widehat{m})$ is obtained from β by removing the mth "$\mathtt{[}$" from the left, together with its matching "$\mathtt{]}$". It is straightforward to check that this does give a differential satisfying (3.1) and (3.2), noting that we count "$\mathtt{[}$"s from the left, because (3.2) is a 'left' Leibniz rule. Thus h is a contracting homotopy and so $\mathcal{F}_\bullet$ is a contractible dg algebra.

Remark 3.1. Forgetting the differential d and the contracting homotopy h, $\mathcal{F}_\bullet$ is the free graded S-algebra generated by

$$K_\bullet(V)[1] = V \oplus h(\mathcal{F}_\bullet) = \bigoplus_{\beta \in \mathcal{B}^{[1]}} V^\beta, \tag{3.6}$$

where $\mathcal{B}^{[1]} \subseteq \mathcal{B}^{(1)}$ is the set of closed bracketings. Note that the convention that "1" is a closed bracketing is precisely to get the initial summand V here.

In other words, $\mathcal{F}_\bullet = \operatorname{Cobar} K_\bullet(V)$. The differential and coproducts on the A_∞-coalgebra $K_\bullet(V)$ are determined by applying (1.3) in reverse and one can check that they are given explicitly by the same rules as those of $B_\bullet^\infty(V)$ in the paragraphs following (2.3). In particular, this explains the choice of sign in (2.4).

4. Main theorem

Note that $\mathcal{F}_0(V) = T_S^+(V)$ and, furthermore, that we have many other copies of $T_S^+(V)$ 'embedded' in $\mathcal{F}_\bullet$ for every pair of inner brackets. To make this explicit, we may introduce a new symbol $\mathtt{[}\,*\,\mathtt{]}$ with the meaning

$$V^{\mathtt{[}*\mathtt{]}} = \bigoplus_{m \geq 1} V^{\mathtt{[}m\mathtt{]}} = \mathtt{[}T_S^+(V)\,\mathtt{]}$$

and more generally

$$V^{\alpha\mathtt{[}*\mathtt{]}\beta} = \bigoplus_{m \geq 1} V^{\alpha\mathtt{[}m\mathtt{]}\beta} = V^\alpha \otimes \mathtt{[}T_S^+(V)\,\mathtt{]} \otimes V^\beta,$$

for matching partial bracketings α, β. Thus we can define contracted sets $\mathcal{B}^{[*]}$ of closed bracketings and $\mathcal{B}^{(*)}$ of open bracketings, in which the innermost brackets are all $\mathtt{[}\,*\,\mathtt{]}$, and so that we can write

$$\mathcal{F}_\bullet(V) = \bigoplus_{\beta \in \mathcal{B}^{(*)}} V^\beta, \qquad K_\bullet(V) = \bigoplus_{\beta \in \mathcal{B}^{[*]}} V^\beta, \tag{4.1}$$

where the degrees of terms in the 2nd equation are shifted compared to (3.6). Now, for any ideal $I \subseteq T_S^+(V)$ we can define a new dg algebra

$$\mathcal{F}_\bullet(V, I) = \bigoplus_{\beta \in \mathcal{B}^{(*)}} (V, I)^\beta, \tag{4.2}$$

where $(V, I)^\beta$ is obtained from V^β by replacing $\texttt{[}T_S^+(V)\texttt{]}$ by $\texttt{[}I\texttt{]}$, for each occurrence of $\texttt{[}*\texttt{]}$ in β. Because I is an ideal, there is a well-defined differential $d\colon \mathcal{F}_\bullet(V, I) \to \mathcal{F}_\bullet(V, I)$ given by the same rule (3.5) as the differential $d\colon \mathcal{F}_\bullet(V) \to \mathcal{F}_\bullet(V)$. We can not quite define a contracting homotopy in the same way, but simply because we have not replaced $\mathcal{F}_0(V) = T_S^+(V)$ by I.

Indeed, the image of $d\colon \mathcal{F}_1(V, I) \to \mathcal{F}_0(V, I)$ is $I \subseteq T_S^+(V)$ and hence there is a dg morphism

$$\rho\colon \mathcal{F}_\bullet(V, I) \to T_S^+(V)/I, \tag{4.3}$$

where the codomain here is just an algebra concentrated in degree 0.

Theorem 4.1. *The map ρ in (4.3) is a quasi-isomorphism, that is, the dg algebra $\mathcal{F}_\bullet(V, I)$ in (4.2) is a free resolution of the algebra $A = T_S^+(V)/I$.*

Proof. Because ρ is surjective, we just want to show that $\ker \rho$ is contractible. For this, we observe that $\ker \rho$ is the dg ideal

$$\mathcal{F}'_\bullet(V, I) \subseteq \mathcal{F}_\bullet(V, I) \tag{4.4}$$

obtained by also replacing $\mathcal{F}_0(V)$ by I. Then, the contracting homotopy h can be defined on $\mathcal{F}'_\bullet(V, I)$ by the rule (3.4) as it was on $\mathcal{F}_\bullet(V)$ and so we can use it to deduce that $\mathcal{F}'_\bullet(V, I)$ is contractible, as required. □

Remark 4.2. In some cases, e.g. when S is a separable k-algebra (so that $\otimes$ is exact), we note that $\mathcal{F}_\bullet(V, I)$ is a sub-dg-algebra of $\mathcal{F}_\bullet(V)$, and so the equations $d^2 = 0$, for $\mathcal{F}_\bullet(V, I)$, and $dh + hd = \mathrm{id}$, for $\mathcal{F}'_\bullet(V, I)$, follow by restriction from $\mathcal{F}_\bullet(V)$. However, in general, they follow rather because the combinatorial structure of $\mathcal{F}_\bullet(V, I)$ is identical to that of $\mathcal{F}_\bullet(V)$.

Remark 4.3. Just as $\mathcal{F}_\bullet(V) = \operatorname{Cobar} K_\bullet(V)$, as in Remark 3.1, we have $\mathcal{F}_\bullet(V, I) = \operatorname{Cobar} K_\bullet(V, I)$, where

$$K_\bullet(V, I) = \bigoplus_{\beta \in \mathcal{B}^{[*]}} (V, I)^\beta, \tag{4.5}$$

with $(V, I)^\beta$ now in homological degree d when β has $d-1$ pairs of brackets.

In particular, when $I = V^{(2)}$, we have $K_\bullet(V, I) = B_\bullet^\infty(V)$ from (2.3), so that Theorem 4.1 yields Theorem 2.1. Further, for a homogeneous ideal $I \subseteq V^{(2)}$, we have $K_\bullet(V, I) = B_\bullet^\infty(V, I)$ from (2.7), so that Theorem 4.1 yields Theorem 2.2.

5. Homological application

Here we restict to the case when S is a separable k-algebra over a field k and V is an S,S-bimodule over k, i.e. the S,S-action on V factors through $S \otimes_k S^{\mathrm{op}}$.

Note that S is a left and right A-module, with A acting trivially, and we would expect (in good cases) that we could use an A_∞-coalgebra $K_\bullet$ with $\operatorname{Cobar} K_\bullet \simeq A$ to compute the (positive) Tor-groups of S and hence, by duality, its Ext groups. We observe that the case in hand, with $K_\bullet = K_\bullet(V,I)$ as in (4.5), is a good one.

Proposition 5.1. *If $A = T_S^+(V)/I$, then we have an isomorphism*

$$H(K_\bullet(V,I)) \cong \operatorname{Tor}_\bullet^A(S,S)$$

of graded coalgebras.

Proof. The A_∞-coalgebra $K_\bullet(V)$ is cocomplete; e.g. the filtration given by

$$K_\bullet(V)(i) = \bigoplus_{\beta \in \mathcal{B}^{[1]}, |\beta| \leq i} V^\beta$$

is admissible (see Appendix for definition). Since S is a separable k-algebra, $K_\bullet(V,I)$ is a A_∞-sub-coalgebra of $K_\bullet(V,I)$ and is therefore also cocomplete. Since Theorem 4.1 tells us that $\mathcal{F}_\bullet(V,I) = \operatorname{Cobar} K_\bullet(V,I)$ is a dg resolution of A, we deduce from Proposition 6.4 that $H(K_\bullet(V,I)) \cong \operatorname{Tor}^A(S,S)$ as graded coalgebras. □

It follows that $\operatorname{Hom}_S(K_\bullet(V,I),S)$ is an A_∞-algebra whose cohomology is $\operatorname{Ext}_A^\bullet(S,S)$, in positive degrees.

6. Appendix

Again we restrict to the case where S is a separable k-algebra over a field k. We extend part of the bar-cobar formalism for dg algebras and cocomplete dg coalgebras over S to cocomplete A_∞-coalgebras, following [5], [3] and [4]. As a change of notation, we write $\Omega_\infty C$ for the cobar construction of an A_∞-coalgebra C. This coincides with the more classical cobar construction ΩC, in the special case that C is a dg coalgebra. The bar construction of a dg algebra A is denoted BA.

A *filtration* on an A_∞-coalgebra C is a sequence $C(0) \subseteq C(1) \subseteq \ldots \subseteq C$ of graded sub-bimodules such that for all $n \geq 1$, the coproduct $\Delta_n : C \to C^n$ is compatible with the (induced) filtrations on C and C^n. A filtration is *admissible* if $C(0) = 0$ and $C = \cup C(i)$. We say C is *cocomplete* if it supports an admissible filtration; this is equivalent to the usual definition if C is a dg coalgebra. See [6, §9.3, Remark] for the related notion of conilpotent (curved) A_∞-coalgebra.

A morphism $f = (f_i) : C' \to C$ between A_∞-coalgebras is a *quasi-isomorphism* if f_1 is a quasi-isomorphism of the underlying complexes of C' and C, and a *weak equivalence* if the morphism $\Omega_\infty f : \Omega_\infty C' \to \Omega_\infty C$ of cobar constructions is a quasi-isomorphism of dg algebras. A morphism $f : C' \to C$ between cocomplete A_∞-coalgebras is a *filtered quasi-isomorphism* if admissible filtrations can be chosen on C' and on C so that the maps $f_i : C' \to C^i$ are compatible with the (induced) filtrations on C' and C^i, and the morphism $\mathrm{Gr}(f_1) : \mathrm{Gr}(C) \to \mathrm{Gr}(C')$ induced by f_1 is a quasi-isomorphism. Note that a filtered quasi-isomorphism is a quasi-isomorphism.

For any dg algebra A, the classical bar-cobar resolution gives a canonical morphism $\varepsilon_A \colon \Omega_\infty BA \to A$ of dg algebras. Taking $A = \Omega_\infty C$ for an A_∞-coalgebra C, we obtain a corresponding map $\epsilon_C : B\Omega_\infty C \to C$ of A_∞-coalgebras. In fact, ϵ_C is a weak equivalence, because ε_A is a quasi-isomorphism (see, e.g. [5, Lemme 1.3.2.3(b)]), but we can say more if C is cocomplete.

Lemma 6.1.
1. *If $A \to A'$ is a quasi-isomorphism of dg algebras, then the induced morphism $BA \to BA'$ is a filtered quasi-isomorphism.*
2. *If C is a cocomplete A_∞-coalgebra, then $\epsilon_C : B\Omega_\infty C \to C$ is a filtered quasi-isomorphism.*

Proof. The induced morphism $BA \to BA'$ is a filtered quasi-isomorphism with respect to the primitive filtrations of BA and BA'. For the second statement, equip $B\Omega_\infty C$ with the admissible filtration induced by a given admissible filtration on C. Then $\epsilon_C : B\Omega_\infty C \to C$ is filtered, and it suffices to show that $\mathrm{Gr}_i((\epsilon_C)_1) : \mathrm{Gr}_i(B\Omega_\infty C) \to \mathrm{Gr}_i(C)$ is a quasi-isomorphism for all $i \geq 1$. Put $W = C[1]$, so that

$$B\Omega_\infty C = T^+ \left(T^+(W)[-1]\right)$$

Since the filtration on C is admissible, $\mathrm{Gr}_i(C^j) = 0$ if $j > i$. Equip

$$\mathrm{Gr}_i \left(B\Omega_\infty C\right) = \mathrm{Gr}_i \left(\bigoplus_{i_1 + \ldots + i_k \leq i} W^{i_1}[-1] \otimes \ldots \otimes W^{i_k}[-1] \right)$$

with the filtration

$$F_l = \mathrm{Gr}_i\left(\bigoplus_{i+1-l\le i_1+\ldots+i_k\le i} W^{i_1}[-1]\otimes\ldots\otimes W^{i_k}[-1]\right),\quad l\ge 0.$$

Then $\mathrm{Gr}_i((\epsilon_C)_1)$ is a surjective map of complexes with kernel F_{i-1}, and, for each $1\le l\le i-1$, the subquotient complex

$$F_l/F_{l-1} = \bigoplus_{i_1+\ldots+i_k=i+1-l} W^{i_1}[-1]\otimes\ldots\otimes W^{i_k}[-1]$$

is acyclic, with a contracting homotopy vanishing on components with $i_1 = 1$ and given by isomorphisms

$$W^{i_1}[-1]\otimes\ldots\otimes W^{i_k}[-1]\to W[-1]\otimes W^{i_1-1}[-1]\otimes W^{i_2}[-1]\otimes\ldots\otimes W^{i_k}[-1]$$

otherwise. □

Lemma 6.2. *Let $f : C'\to C$ be a morphism of cocomplete A_∞-coalgebras.*

1. *If f is a filtered quasi-isomorphism, then it is a weak equivalence.*
2. *If f is a weak equivalence, then it is a quasi-isomorphism.*

Proof. For the first part, the proof given by Lefevre [5, Lemma 1.3.2.2] for filtered quasi-isomorphisms of cocomplete dg coalgebras goes through without change. For the second, consider the commutative diagram

$$\begin{tikzcd} C' \arrow[r, "f"] & C \\ B\Omega_\infty C' \arrow[u, "\epsilon_{C'}"] \arrow[r, "B\Omega_\infty f"] & B\Omega_\infty C \arrow[u, "\epsilon_C"'] \end{tikzcd}$$

By Lemma 6.1, ϵ_C, $\epsilon_{C'}$ and $B\Omega_\infty f$ are quasi-isomorphisms. Hence f is a quasi-isomorphism. □

Corollary 6.3 (Keller [4])**.** *Let $f\colon C'\to C$ be a morphism of A_∞-coalgebras which is also compatible with strictly positive gradings on C' and C. If f is a quasi-isomorphism, then it is a weak equivalence.*

Proof. The additional positive gradings give rise to admissible filtrations on C' and C in an obvious way. If f is a quasi-isomorphism, then it is automatically a filtered quasi-isomorphism, and then Lemma 6.2 applies. □

Proposition 6.4. *Suppose that C is a cocomplete A_∞-algebra and we have a quasi-isomorphism $\Omega_\infty C \to A$ of dg algebras. Then C is weakly equivalent to BA. In particular we have an isomorphism of graded coalgebras*

$$H(C) \cong \mathrm{Tor}^A(S, S).$$

Proof. By Lemma 6.1, we have filtered quasi-isomorphisms $B\Omega_\infty C \to BA$ and $B\Omega_\infty C \to C$. In particular C and BA are weakly equivalent, and thus quasi-isomorphic, by Lemma 6.2. Finally, recall that $H(BA) \cong \mathrm{Tor}^A(S, S)$. $\square$

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