On the Inverse Scattering Method for Integrable PDEs on a Star Graph

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Abstract

We present a framework to solve the open problem of formulating the inverse scattering method (ISM) for an integrable PDE on a star-graph. The idea is to map the problem on the graph to a matrix initial-boundary value (IBV) problem and then to extend the unified method of Fokas to such a matrix IBV problem. The nonlinear Schrödinger equation is chosen to illustrate the method. The framework unifies all previously known examples which are recovered as particular cases. The case of general Robin conditions at the vertex is discussed: the notion of linearizable initial-boundary conditions is introduced. For such conditions, the method is shown to be as efficient as the ISM on the full-line.

Keywords: inverse scattering method, star-graph, integrable PDE, unified Fokas method, nonlinear Schrödinger equation

1 Introduction

For decades now, integrable partial differential equations (PDEs), and more generally integrable systems, have fuelled research and important discoveries in Mathematics and Physics, and still do. Comparatively more recently, graphs and dynamical systems on (quantum) graphs have emerged as a successful framework to model a large variety of (complex) systems. It is therefore not surprising to see a fast growing interest in developing a theory of integrable systems on graphs, which would combine the power of integrable systems with the flexibility of graphs to model more realistic situations. The review [1] for instance gives a flavour and references for this fast growing area in the context of nonlinear Schrödinger (NLS) equations (not restricted to integrable cases).

Originally, integrable PDEs were treated as initial value problems for functions of one space variable $x \in \mathbb{R}$ and one time variable $t \geq 0$. The invention of the inverse scattering method (ISM) [2] and its refinements [3, 4] through the systematic use of a Lax pair [5] represents a cornerstone of modern integrable PDEs. The first departure from this setup to solve an initial-boundary value (IBV) problem for an integrable PDE on the half-line [6, 7] or a finite interval [8] can be viewed in retrospect as the beginning of the study of integrable PDEs on metric graphs. Indeed, a half-line is nothing but a half-infinite edge attached to a vertex and a finite interval is a finite edge connecting two vertices. The next big step in this natural evolution was the study of integrable PDEs on the line with a defect/impurity at a fixed site (or possibly several such defects). The vast literature on this problem[1] [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] shows both its interest and its difficulty. To date however, despite some impressive results on the behaviour of certain solutions [19, 20], the general problem of formulating an ISM for a problem with defects is still open.

It is the purpose of this paper to bring an answer to the more general question of formulating the ISM on a star-graph *i.e.* a single vertex connected to a finite number N of half-infinite edges. The case $N = 2$ will then take care of the situation of a defect/impurity on the line. To be more concrete, we choose to present the framework on the example of the cubic Nonlinear Schrödinger (NLS) equation

$$i\partial_t q + \partial_x^2 q - 2g|q|^2 q = 0 \quad , \quad g \in \mathbb{R} \, . \tag{1.1}$$

This is motivated by the fact that it is one of the most famous and studied example of integrable PDEs. It is also the model that has been most studied on various simple graphs, hence allowing us to show how our method encompasses all known results. But it will be clear to the reader that our framework applies equally well to any integrable PDE that admits a Lax pair formulation.

In the next section, we introduce the model to solve: NLS on a star-graph. In section 3, we show how the problem can be mapped to an IBV problem of a certain matrix form. Section 4 then goes on to exploit this mapping in combination with the unified method of Fokas [21] to provide an ISM for NLS on a star-graph. In Section 5, we show how previous studies fit within our framework. Finally, in Section 6, we illustrate our approach in the general case of a star-graph with the vertex boundary conditions classified in [22]. Based on this example, we introduce the notion of linearizable initial-boundary conditions whereby one can reduce the problem on the star-graph to a scalar linearizable IBV problem. Conclusions are gathered in the last section, where future directions are also pointed out.

[1] Given the large literature, we have tried to give a representative selection of papers related to classical systems with defects. Most of the authors cited here have contributed many more papers on the subject.

2 The model to solve

When formulated on the real line, (1.1) is solved via the ISM for initial data $q(x,0) = q_0(x)$ typically of sufficiently fast decay (see e.g. [23] and references therein). Of course, other classes of initial data have been considered over the years (see e.g. [24] and references therein). In this paper, we want to focus on the framework and not on the technicalities related to functional spaces. We assume for simplicity that all our data is of appropriate smoothness and decay for our purposes.

We consider the NLS equation on a star-graph with N half-infinite edges. We introduce N copies of (1.1) for functions q^α, $\alpha = 1, \ldots, N$. Each q^α lives on edge α, is a function of $x \geq 0$ and $t \geq 0$ and is connected to the other edges via some boundary conditions at $x = 0$. The problem therefore reads, for $\alpha = 1, \ldots, N$,

$$i\partial_t q^\alpha + \partial_x^2 q^\alpha - 2g|q^\alpha|^2 q^\alpha = 0 \quad , \quad x,t > 0\,, \tag{2.1}$$

$$q^\alpha(x,0) = q_0^\alpha(x)\,, \tag{2.2}$$

$$q^\alpha(0,t) = g_0^\alpha(t) \quad , \quad \partial_x q^\alpha(0,t) = g_1^\alpha(t)\,, \tag{2.3}$$

where q_0^α, g_0^α and g_1^α are the initial-boundary data. For each $\alpha = 1, \ldots, N$, (2.1) is the compatibility condition $\partial_{xt}\mu = \partial_{tx}\mu$ of [3]

$$\begin{cases} \partial_x \mu^\alpha + ik[\sigma_3, \mu^\alpha] = W^\alpha\, \mu^\alpha\,, \\ \partial_t \mu^\alpha + 2ik[\sigma_3, \mu] = P^\alpha\, \mu^\alpha\,, \end{cases} \tag{2.4}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad W^\alpha(x,t) = \begin{pmatrix} 0 & q^\alpha(x,t) \\ g\,\bar{q}^\alpha(x,t) & 0 \end{pmatrix}\,, \tag{2.5}$$

$$P^\alpha(x,t,k) = 2k\, W^\alpha - i\partial_x W^\alpha\, \sigma_3 - i(W^\alpha)^2\, \sigma_3\,. \tag{2.6}$$

At this stage, we have to make several important remarks to clarify the role of the following section and to understand why the system of equations (2.1)-(2.3) is indeed relevant to decribe NLS on a star-graph.

First notice that (2.1) is a particular case of the general (square) matrix NLS for a $N \times N$ square matrix-valued function $Q(x,t)$

$$i\partial_t Q + \partial_x^2 Q - 2gQQ^\dagger\, Q = 0\,, \tag{2.7}$$

where Q is chosen to be the diagonal matrix with entries $q^1, \ldots, q^n$

$$Q(x,t) = \begin{pmatrix} q^1(x,t) & & \\ & \ddots & \\ & & q^N(x,t) \end{pmatrix}\,. \tag{2.8}$$

This simple observation is the basis of the mapping of (2.1)-(2.3) to a diagonal matrix IBV problem discussed in the next section. In particular, the model we consider *is not*

$$i\partial_t q^\alpha + \partial_x^2 q^\alpha - 2g\left(\sum_{\beta=1}^{N} |q^\beta|^2\right) q^\alpha = 0 \quad , \quad x,t > 0 \quad , \quad \alpha = 1, \ldots, N\,, \tag{2.9}$$

which would be the vector NLS equation (or general Manakov model [25]) on the half-line. This would correspond to choosing Q as a row matrix of the form

$$Q(x,t) = \begin{pmatrix} q^1(x,t) & \dots & q^N(x,t) \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} . \tag{2.10}$$

This model was studied in [26, 27] for integrable boundary conditions. There, the superscript α refers to an internal degree of freedom (light polarization in the historical case $N = 2$) and there is a global $U(N)$ symmetry. In the present case, the superscript α has the meaning of a discrete spatial coordinate (assuming some embedding of the graph into $\mathbb{R}^2$ for instance) and the global symmetry is only $U(1)^N$.

Second, it is important to realize that the interactions between the edges are mediated through the central vertex via the boundary conditions (2.3). In our context, the presentation of the problem (2.1)-(2.3) in the standard form of an IBV problem could give the misleading impression that one is in fact always dealing with N disconnected half-lines. This would defeat the purpose of this paper which is to deal with equations on a star-graph. In sections 5 and 6, we show that our formulation does indeed include important examples of genuine star-graphs where the half-lines are connected nontrivially. Of course, if one was to give oneself the complete set of boundary data g_0^α, g_1^α (either directly or through Dirichlet-to-Neumann maps), then one could simply apply N times the unified transform of Fokas to solve the problem on each half-line independently. The point is that in situations of interest for problems on a star-graph, only special combinations of g_0^α, g_1^α are supposed to be given and hence one has to consider the problem as a whole and cannot split it into N disconnected problems. This is reminiscent of the well-known case on the half-line where, generically, one would present the problem as

$$i\partial_t q + \partial_x^2 q - 2g|q|^2 q = 0 \quad , \quad x,t > 0 \,, \tag{2.11}$$

$$q(x,0) = q_0(x) \,, \tag{2.12}$$

$$q(0,t) = g_0(t) \quad , \quad \partial_x q(0,t) = g_1(t) \,, \tag{2.13}$$

but then, one would restrict their attention to the case of Robin boundary condition where only the combination $\eta q(0,t) + \partial_x q(0,t)$ is supposed to be given. In other words, only $g_1(t) + \eta g_0(t) \equiv g(t)$ forms the boundary data. The particular case $g(t) = 0$ is the well-known integrable Robin boundary condition for scalar NLS on the half-line [29]. In the same spirit, we will see in the examples of Sections 5 and 6 that only certains combinations of g_0^α, g_1^α are supposed to be given. Such examples will represent typical situations of a star-graph where the half-lines are connected through the vertex.

3 Mapping the problem to a matrix IBV problem

The observation in the previous section that the set of equations (2.1) is a particular case of a general matrix NLS equation suggests that a natural way to deal with the problem on a star-graph is to consider matrix-valued functions of a certain type. Indeed, defining

$$\Sigma_3 = \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{pmatrix} \quad , \quad W(x,t) = \begin{pmatrix} 0 & Q(x,t) \\ g\,\overline{Q}(x,t) & 0 \end{pmatrix} , \tag{3.1}$$

$$P(x,t,k) = 2k\, W - i\partial_x W\, \Sigma_3 - iW^2\, \Sigma_3 \,. \tag{3.2}$$

with Q given by (2.8), then the set of equations (2.1) is the compatibility condition of the following auxiliary problem

$$\begin{cases} \partial_x \mu + ik[\Sigma_3, \mu] = W\,\mu\,, \\ \partial_t \mu + 2ik^2[\Sigma_3, \mu] = P\,\mu\,, \end{cases} \tag{3.3}$$

where all the objects are now $2N \times 2N$ matrices. So let us denote by $\mathcal{M}_p$ the algebra of $p \times p$ matrices over $\mathbb{C}$ and $\{E_{ij}\}_{i,j=1}^p$ its canonical basis. When dealing with different values of p in a single expression, it is understood that the basis matrices E_{ij} have the appropriate size given by the range of their indices. Given $M \in \mathcal{M}_{2N}$, define

$$M_d = \sum_{k,\ell=1}^{2} \sum_{i=1}^{N} M_{i+(k-1)N,i+(\ell-1)N} E_{k\ell} \otimes E_{ii} \quad , \quad M_a = \sum_{k,\ell=1}^{2} \sum_{1\le i\ne j\le N} M_{i+(k-1)N,j+(\ell-1)N} E_{k\ell} \otimes E_{ij}\,. \tag{3.4}$$

We denote $\mathcal{M}_d = \{M_d, M \in \mathcal{M}_{2N}\}$ and $\mathcal{M}_a = \{M_a, M \in \mathcal{M}_{2N}\}$ the corresponding sets. A matrix in $\mathcal{M}_d$ looks like $\left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right)$ where each M_j is an $N \times N$ diagonal matrix. The point is that Σ_3, W and P in (3.3) are all $\mathcal{M}_d$-valued functions. To implement our formalism, detailed in the next section, the crucial fact is that the fundamental solutions of (3.3), properly normalised, will also lie in $\mathcal{M}_d$ for all values of x, t, k where they are defined. This fact, together with the following simple lemma, will then ensure that all the objects appearing in the ISM of a system on a star-graph will also lie in $\mathcal{M}_d$. As a consequence, the original IBV problem (2.1)-(2.3) representing NLS on a star-graph will be solvable by implementing a generalization of Fokas method to the case of $\mathcal{M}_d$-valued functions.

Lemma 3.1 *$\mathcal{M}_d$ and $\mathcal{M}_a$ are vector subspaces of $\mathcal{M}_{2N}$ and the direct sum decomposition $\mathcal{M}_{2N} = \mathcal{M}_d \oplus \mathcal{M}_a$ holds. Moreover, $\mathcal{M}_d$ is a subalgebra of $\mathcal{M}_{2N}$ which is isomorphic to the direct product $\prod_{i=1}^{N} \mathcal{M}_2$ as algebras.*

The last point is easily seen using the following isomorphism

$$\begin{array}{cccc} \theta: & \prod_{i=1}^{N} \mathcal{M}_2 & \to & \mathcal{M}_d \\ & (M^1, \dots, M^N) & \mapsto & M = \sum_{\alpha=1}^{N} M^\alpha \otimes E_{\alpha\alpha} \end{array}, \tag{3.5}$$

where the algebra structure of $\prod_{i=1}^{N} \mathcal{M}_2$ is defined by the pointwise operations.

In the case $N = 1$, the study of the analytic properties of the solutions to Eqs. (3.3) treated simultaneaously forms the basis of the unified method developed by Fokas for IBV problems [21]. We refer to the $N = 1$ case as the *scalar* case. For our purposes, the following theorem is a key result of this paper. As explained above, it shows that analyzing the problem on a star-graph is the same as analyzing a matrix IBV problem in $\mathcal{M}_d$.

Theorem 3.2 *Let $(x_0, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and let μ be the fundamental solution of (3.3) normalised by $\mu(x_0, t_0, k) = \mathbb{1}_{2N}$. Then $\mu(x, t, k) \in \mathcal{M}_d$ wherever it is defined.*

Proof: Denote $\phi(x,t,k) = kx + 2k^2 t$. Eqs. (3.3) are equivalent to the equation

$$d\left(e^{i\phi\Sigma_3}\mu\, e^{-i\phi\Sigma_3}\right) = e^{i\phi\Sigma_3}\left(W\,\mu\, dx + P\,\mu\, dt\right) e^{-i\phi\Sigma_3}\,, \tag{3.6}$$

ensuring that the right-hand side is an exact 1-form. Fix $(x_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ and define the solution, for $x > 0$ and $t > 0$,

$$\mu(x,t,k) = \mu_0(k) + \int_{(x_0,t_0)}^{(x,t)} e^{i\phi(y-x,\tau-t,k)\Sigma_3}\left(W(y,\tau)\mu(y,\tau,k)dy + P(y,\tau,k)\mu(y,\tau,k)d\tau\right) e^{-i\phi(y-x,\tau-t,k)\Sigma_3} \tag{3.7}$$

Using the linearity of the Volterra integral equation (3.7) and Lemma 3.1 to project it on $\mathcal{M}_d$ and $\mathcal{M}_a$, we obtain that both μ_d and μ_a satisfy (3.7). To formulate the ISM, one uses fundamental solutions of (3.3) defined by $\mu_0(k) = \mathbb{1}_{2N}$. Therefore, for such a solution $(\mu_0(k))_a = 0$. By uniqueness of the solution of (3.7), we deduce that $\mu_a = 0$ identically. Hence, any fundamental solution of (3.3) is an $\mathcal{M}_d$-valued function, which concludes the proof. Note that all the ingredients of the ISM being derived from fundamental solutions by algebraic operations and $\mathcal{M}_d$ being an algebra, we deduce that ISM for the problem (2.1)-(2.3) can be entirely formulated in $\mathcal{M}_d$. ■

Remark 1: The values of $k \in \mathbb{C}$ for which a fundamental solution is defined or has given analytic properties depend on (x_0,t_0). This is what we mean in the previous theorem by "wherever it is defined". This will become clear below when we apply Fokas's method in our context.

Remark 2: In practice, the theorem ensures that the fundamental solutions of Eqs. (3.3) and all the associated spectral functions $\{a(k), b(k)\}$ and $\{A(k), B(k)\}$ (in the notations of [21]) appearing in the ISM can be split into N 2×2 matrices whenever this is more convenient than their form in $\mathcal{M}_d$. One simply uses the inverse of the isomorphism θ.

4 Inverse scattering method on a star graph

Equipped with Theorem 3.2, we formulate the ISM on a star graph by extending the unified method of Fokas to matrices in $\mathcal{M}_d$. For full details on the method in the scalar case, we refer to the book [21]. In particular, in the following, we will heavily rely on Chapter 16 of [21]. The approach is of analysis/synthesis nature. Under the assumption that $Q(x,t)$ exists, the analysis part allows one to introduce the relevant scattering data, the so-called *global relation* and to formulate an appropriate Riemann-Hilbert problem which is at the basis of the inverse part of the ISM for IBV problems. Equipped with all this, the synthesis part consists in formulating the direct and inverse parts of the ISM, assuming that the global relation holds, and then check that the obtained solution $Q(x,t)$ indeed satisfies the PDE together with the initial and boundary conditions. Here, we present directly the main results in $\mathcal{M}_d$ and focus on the synthesis part. We only give the key steps of the analysis part, pointing out the main differences with the scalar case.

4.1 Spectral analysis

In Fokas's method for an IBV problem, one needs to define three fundamental solutions $\mu_j(x,t,k)$ as in (3.7), corresponding to the three points $(x_0,t_0) = (0,T)$ for $j=1$, $(x_0,t_0) = (0,0)$ for $j=2$ and $(x_0,t_0) = (\infty, t)$ for $j=3$. Any pair of these solutions is related by a matrix independent of

x, t. So one defines the two scattering matrices $S(k)$ and $T(k)$ for $k \in \mathbb{R}$ by

$$\mu_3(x,t,k) = \mu_2(x,t,k) e^{-i\phi(x,t,k)\Sigma_3}\, S(k)\, e^{i\phi(x,t,k)\Sigma_3}\,, \tag{4.1}$$
$$\mu_1(x,t,k) = \mu_2(x,t,k) e^{-i\phi(x,t,k)\Sigma_3}\, T(k)\, e^{i\phi(x,t,k)\Sigma_3}\,. \tag{4.2}$$

Using the same symmetry of the potential W as in the scalar case and the fact that our matrices lie in $\mathcal{M}_d$, we can show that S and T have the general form

$$S(k) = \begin{pmatrix} \overline{a(\bar{k})} & b(k) \\ g\,\overline{b(\bar{k})} & a(k) \end{pmatrix} \quad , \quad T(k) = \begin{pmatrix} \overline{A(\bar{k})} & B(k) \\ g\,\overline{B(\bar{k})} & A(k) \end{pmatrix}\,, \tag{4.3}$$

where the scattering coefficients $a(k), b(k), A(k), B(k)$ are $N \times N$ diagonal matrices. The entries of the matrices $a(k), b(k), A(k), B(k)$ extend to functions on the complex plane with certain analytic properties (see Propositions 16.1 and 16.2 in [21]). An important role is played by the zeros of the entries of $a(k)$ and of the matrix $d(k) = a(k)\,\overline{A(\bar{k})} - g\,b(k)\,\overline{B(\bar{k})}$. Following [21], we make some assumptions on these zeros. This is where the present matrix case requires extra care compared to the scalar case. Hence, we assume

(C1) For each $\alpha \in \{1, \dots, N\}$, $a_\alpha(k)$ has K^α simple zeros $\{\kappa_j^\alpha\}_{j=1}^{K^\alpha}$ with $K^\alpha = K_1^\alpha + K_2^\alpha$, $\arg\kappa_j^\alpha \in (0, \frac{\pi}{2})$, $j = 1, \dots, K_1^\alpha$; $\arg\kappa_j^\alpha \in (\frac{\pi}{2}, \pi)$, $j = K_1^\alpha + 1, \dots, K^\alpha$.

(C2) For each $\alpha \in \{1, \dots, N\}$, $d_\alpha(k)$ has Λ^α simple zeros $\{\lambda_j^\alpha\}_{j=1}^{\Lambda^\alpha}$ with $\arg\lambda_j^\alpha \in (\frac{\pi}{2}, \pi)$, $j = 1, \dots, \Lambda^\alpha$.

(C3) None of the zeros of $a_\alpha(k)$ with $\arg k \in (\frac{\pi}{2}, \pi)$ coincides with a zero of $d_\alpha(k)$.

In general, one can have coinciding zeros for different α and β *i.e.* $\kappa_j^\alpha = \kappa_\ell^\beta$ for some $\alpha \neq \beta$ and j, ℓ. Similarly, one could have $\lambda_j^\alpha = \lambda_\ell^\beta$ for some $\alpha \neq \beta$ and j, ℓ. Finally, one can also have $\kappa_j^\alpha = \lambda_\ell^\beta$ for some $\alpha \neq \beta$ and j, ℓ. Such instances correspond in general to the situation where the different half-lines are non trivially connected.

Another important ingredient is the global relation. The same derivation as in [21] goes over to our matrix case. From the integration of the exact form (3.6) around the boundary of the domain $\{0 < x < \infty, 0 < t < T_0\}$, one arrives at

$$\mathbb{I}_{2N} - T^{-1}(k)\, S(k) - \int_0^\infty e^{i\phi(x,T_0,k)\Sigma_3}\, W(x,T_0)\, \mu_3(x,T_0,k)\, e^{-i\phi(x,T_0,k)\Sigma_3}\, dx = 0\,, \tag{4.4}$$

for $k \in (\overline{D}_3 \cup \overline{D}_4, \overline{D}_1 \cup \overline{D}_2)$. The block (12) of this relation yields the *global relation*

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T_0}c(k) \quad , \quad k \in \overline{D}_1 \cup \overline{D}_2\,, \tag{4.5}$$

where $c(k) = \int_0^\infty e^{2ikx}\, Q(x,T_0)\,(\mu_3(x,T_0,k))_{22}\, dx$ is analytic for $k \in D_1 \cup D_2$ and of order $\frac{1}{k}$ as $k \to \infty$. In the case $T_0 = \infty$, this boils down to

$$a(k)B(k) - b(k)A(k) = 0 \quad , \quad k \in \overline{D}_1\,. \tag{4.6}$$

4.2 Synthesis: direct and inverse transforms of ISM

4.2.1 Direct part

Consider the initial-boundary data q_0^j, g_0^j, g_1^j, $j = 1, \ldots, N$ from (2.2)-(2.3). Denote $Q_0(x) = diag\left(q_0^1(x), \ldots, q_0^N(x)\right)$ and $H_\ell(t) = diag\left(g_\ell^1(t), \ldots, g_\ell^N(t)\right)$, $\ell = 1, 2$. Also, let

$$W_0(x) = \begin{pmatrix} 0 & Q_0(x) \\ g\overline{Q}_0(x) & 0 \end{pmatrix} \quad , \quad G_\ell(x) = \begin{pmatrix} 0 & H_\ell(t) \\ g\overline{H}_\ell(t) & 0 \end{pmatrix} \quad , \quad \ell = 1, 2 \,. \tag{4.7}$$

Now, define $\varphi(x,k)$, $\Phi(t,k)$ as the $2N \times N$ matrix-valued functions satisfying

$$\begin{cases} \partial_x \varphi + 2ik \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} \varphi = W_0\, \varphi \quad , \quad 0 < x < \infty \quad , \quad k \in \overline{D}_1 \cup \overline{D}_2 \,, \\ \partial_t \Phi + 4ik^2 \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} \Phi = (2kG_0 - iG_1\Sigma_3 - iG_0^2\Sigma_3)\, \Phi \quad , \quad 0 < t < \infty \quad , \quad k \in \overline{D}_1 \cup \overline{D}_3 \,, \\ \lim\limits_{x\to\infty} \varphi(x,k) = \begin{pmatrix} 0 \\ \mathbb{I}_N \end{pmatrix} \quad , \quad \lim\limits_{t \to T_0} \Phi(t,k) = \begin{pmatrix} 0 \\ \mathbb{I}_N \end{pmatrix} . \end{cases} \tag{4.8}$$

We now define the scattering coefficients by

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \varphi(0,k) \quad \text{and} \quad \begin{pmatrix} B(k) \\ A(k) \end{pmatrix} = \Phi(0,k) \,. \tag{4.9}$$

We also assume that Q_0, H_0 and H_1 are such that: $Q_0(0) = H_0(0)$, $Q_{0x}(0) = H_1(0)$ and conditions $(C1) - (C3)$ hold as well as the global relation.

4.2.2 Inverse part

Given the scattering coefficients $a(k), b(k), A(k), B(k)$ together with the zeros as in $(C1) - (C3)$, define the matrix J by $J(k) = J_\ell$ when $\arg k = \frac{\ell\pi}{2}$, where

$$J_1 = \begin{pmatrix} \mathbb{I}_N & 0 \\ \Gamma(k)\, e^{2i\phi(x,t,k)} & \mathbb{I}_N \end{pmatrix} \quad , \quad J_4 = \begin{pmatrix} \mathbb{I}_N & -\gamma(k)\, e^{-2i\phi(x,t,k)} \\ g\bar{\gamma}(k)\, e^{2i\phi(x,t,k)} & \mathbb{I}_N - g|\gamma(k)|^2 \end{pmatrix} , \tag{4.10}$$

$$J_3 = \begin{pmatrix} \mathbb{I}_N & -g\overline{\Gamma(\bar{k})}\, e^{-2i\phi(x,t,k)} \\ 0 & \mathbb{I}_N \end{pmatrix} \quad , \quad J_2 = J_3 J_4^{-1} J_1 \,, \tag{4.11}$$

and

$$\gamma(k) = b(k)\, \bar{a}^{-1}(k) \,, k \in \mathbb{R} \quad , \quad \Gamma(k) = g\overline{B(\bar{k})} a^{-1}(k) d^{-1}(k) \,, k \in \mathbb{R}^- \cup i\mathbb{R}^+ \,, \tag{4.12}$$

$$d(k) = a(k)\, \overline{A(\bar{k})} - g\, b(k)\, \overline{B(\bar{k})} \,, k \in \mathbb{R}^- \cup i\mathbb{R}^+ \,. \tag{4.13}$$

Then, define $M(x,t,k)$ as the solution of the following $2N \times 2N$ matrix Riemann-Hilbert problem

- M is meromorphic for $k \in \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$;
- $M_-(x,t,k) = M_+(x,t,k)\, J(x,t,k) \,, \quad k \in \mathbb{R} \cup i\mathbb{R}$ where $M = M_-$ for k in the second or fourth quadrant, $M = M_+$ for k in the first or third quadrant and J is defined in terms of a, b, A, B as in (4.10)-(4.13);
- $M(x,t,k) = \mathbb{I}_{2N} + O\left(\frac{1}{k}\right) \,, \quad k \to \infty$;

- Dropping the x, t dependence for conciseness and denoting $M(k) = ([M]_1(k), [M]_2(k))$ the splitting of M into the first and last N columns, the following residue conditions hold at the possible zeros of a and d

$$
\begin{aligned}
\underset{\kappa_j^\alpha}{Res}[M]_1 &= e^{2i\phi(\kappa_j^\alpha)}\,[M]_2(\kappa_j^\alpha)\,b^{-1}(\kappa_j^\alpha)\ \underset{\kappa_j^\alpha}{Res}\,a^{-1}\,, &(4.14)\\
\underset{\bar\kappa_j^\alpha}{Res}[M]_2 &= e^{-2i\phi(\bar\kappa_j^\alpha)}\,[M]_1(\bar\kappa_j^\alpha)\,g^{-1}\bar b^{-1}(\kappa_j^\alpha)\ \underset{\bar\kappa_j^\alpha}{Res}\,\bar a^{-1}\,, &(4.15)\\
\underset{\lambda_j^\alpha}{Res}[M]_1 &= g e^{2i\phi(\lambda_j^\alpha)}\,[M]_2(\lambda_j^\alpha)\,\overline{B(\bar\lambda_j^\alpha)}\,a_0^{-1}(\lambda_j^\alpha)\ \underset{\lambda_j^\alpha}{Res}\,d^{-1}\,, &(4.16)\\
\underset{\bar\lambda_j^\alpha}{Res}[M]_2 &= g e^{2i\phi(\bar\lambda_j^\alpha)}\,[M]_1(\bar\lambda_j^\alpha)\,B(\bar\lambda_j^\alpha)\,\bar a_0^{-1}(\lambda_j^\alpha)\ \underset{\bar\lambda_j^\alpha}{Res}\,\bar d^{-1}\,, &(4.17)
\end{aligned}
$$

  where $\underset{\kappa_j^\alpha}{Res}\,a^{-1}$ is the diagonal matrix whose only nonzeros entries are for those β's such that κ_j^α is a zero of $a_\beta(k)$, in which case the element reads $\frac{1}{\dot a_\beta(\kappa_j^\alpha)}$, and similarly for $\underset{\bar\kappa_j^\alpha}{Res}\,\bar a^{-1}$, $\underset{\lambda_j^\alpha}{Res}\,d^{-1}$ and $\underset{\bar\lambda_j^\alpha}{Res}\,\bar d^{-1}$.

The fundamental result of ISM for NLS on a star graph is then the following

Theorem 4.1 *$M(x, t, k)$ exists and is unique. Moreover, if we set*

$$
Q(x,t) = 2i \lim_{k\to\infty} (kM(x,t,k))_{12}\,, \tag{4.18}
$$

then $Q(x, t)$ solve the NLS equation on a star-graph with initial condition $Q(x, 0) = Q_0(x)$ and boundary conditions $Q(0, t) = H_0(t)$, $\partial_x Q(0, t) = H_1(t)$. The index "12" in (4.18) means that we take the block (12) in the natural decomposition of matrices in $\mathcal{M}_d$.

Proof: Using the isomorphism θ, we can map the proof of this theorem to the proof of N copies of the analogous theorem for the scalar case. Indeed, let $(M^1, \dots, M^N)$ be the preimage of M by θ then each M^α is defined as the solution of the 2×2 Riemann-Hilbert problem analogous to the one presented above but based on the scattering data $a_\alpha(k)$, $b_\alpha(k)$, $A_\alpha(k)$, $B_\alpha(k)$ corresponding to $q_0^\alpha(x)$, $h_j^\alpha(t)$, $j = 1, 2$. In this case, it is known that M^α then exists and is unique [28]. Now, setting $Q(x, t) = 2i \lim_{k\to\infty} (kM(x, t, k))_{12}$ is equivalent to setting $q^\alpha(x, t) = 2i \lim_{k\to\infty} (kM^\alpha(x, t, k))_{12}$ for $\alpha = 1, \dots, N$. In the last equation, the index 12 is simply the entry in position (12) in the 2×2 matrix kM^α. Again, it is a consequence of the results in [28] that q^α is then solution of NLS on the half-line which satisfies the initial condition $q^\alpha(x, 0) = q_0^\alpha(x)$ and boundary conditions $q^\alpha(0, t) = h_0^\alpha(t)$, $\partial_x q^\alpha(0, t) = h_1^\alpha(t)$. This means that $Q(x, t)$ satisfies NLS on a star-graph with initial-boundary data Q_0, H_0, H_1. ■

5 Comparison with previous results

Obivously, the scalar case $N = 1$ boils down to NLS on the half-line which has been the object of numerous studies [29, 30, 7, 21]. For $N \geq 2$, the apparent simplicity of the proof of the main theorem of the previous section when one uses the map θ is both beautiful and potentially misleading. One may erroneously infer that we are simply dealing with N disconnected copies of the half-line problem. It is the object of this section to show that this is not so and that our approach actually unifies and encompasses all previous studies (known to the author) of the NLS equation on more complicated supports than the full line. In the last subsection, we also show our approach can tackle the most general problem arising from the study of self-adjoint extensions of the Laplacian on a star-graph [22] and adapted to the present situation.

5.1 Case $N = 2$: problem on the line with a defect/impurity

5.1.1 Recovering the problem on the line

The simplest way to check that our formalism does describe connected half-lines is to show how it reproduces the problem on the full line[2]. The latter can be seen as the problem on two half-lines connected in such a way that there is no reflection and trivial transmission. In fact, the crux of the matter can already been seen for the linear case *i.e.* when the coupling constant $g = 0$ in NLS. The point is that the boundary conditions encoded in the functions $H_0 = diag(g_0^1, g_0^2)$ and $H_1 = diag(g_1^1, g_1^2)$ must "disappear" from the reconstruction formula for the function on the full line and only the initial condition must play a role.

Linear case. It is very instructive to look at the details in the linear case for $N = 2$ first. From the half-line problem

$$i\partial_t Q(x,t) + \partial_x^2 Q(x,t) = 0 \quad , \quad Q(x,0) = Q_0(x) \quad , \quad Q(0,t) = H_0(t) \quad , \quad \partial_x Q(0,t) = H_1(t)\,, \tag{5.1}$$

we want to solve the full line problem

$$i\partial_t u(x,t) + \partial_x^2 u(x,t) = 0 \quad , \quad u(x,0) = u_0(x)\,. \tag{5.2}$$

This goes as follows. We use the two functions $q_j(x,t)$, $j = 1, 2$ contained in $Q(x,t)$ and defined on each half-line to form $u(x,t) = \theta(x)q_1(x,t) + \theta(-x)q_2(-x,t)$. The "transparent" boundary conditions are obtained for $g_0^1(t) = g_0^2(t) \equiv g_0(t)$ and $g_1^1(t) = -g_1^2(t) \equiv g_1(t)$ where g_j^α are the boundary data in (2.3). This corresponds to $H_0(t) = g_0(t)\mathbb{1}_2$ and $H_1(t) = g_1(t)\sigma_3$. To compare (5.1) and (5.2) more efficiently, let us define

$$Q^{line}(x,t) = \sigma_3\left(\theta(x)Q(x,t) + \theta(-x)\sigma Q(-x,t)\sigma\right) = \begin{pmatrix} u(x,t) & 0 \\ 0 & -u(-x,t) \end{pmatrix}, \tag{5.3}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.4}$$

Obviously, one simply extracts the (11) entry of Q^{line} to get u. So, for convenience, we perform the analysis directly on Q^{line}. The prefactor σ_3 is purely conventional here but will turn out to be useful when we go over to the nonlinear case. The usual Fourier transform method applied to Q^{line} yields

$$Q^{line}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{Q}^{line}(k,t)\, e^{-ikx}\, dk\,, \tag{5.5}$$

where

$$\begin{aligned} \widehat{Q}^{line}(k,t) &= \int_{-\infty}^{\infty} Q^{line}(x,t)\, e^{ikx}\, dx \\ &= \sigma_3 \int_0^\infty \left(Q(x,t)e^{ikx} + \sigma Q(x,t)\sigma e^{-ikx}\right) dx \\ &= \sigma_3\left(\widehat{Q}(k,t) + \sigma\widehat{Q}(-k,t)\sigma\right), \end{aligned} \tag{5.6}$$

[2] We are grateful to N. Crampé for this useful observation.

where $\widehat{Q}(k,t)$ is the (half) Fourier transform of $Q(x,t)$. Now the unified method applied to the problem (5.1) provides the solution for $Q(x,t)$ by deriving the global relation

$$\widehat{Q}(k,t) = e^{-ik^2t}\widehat{Q}(k,0) + e^{-ik^2t}\int_0^t e^{ik^2\tau}\left(iH_1(\tau) + kH_0(\tau)\right)d\tau \tag{5.7}$$

and then the inverse tranform

$$Q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{Q}(k,t)\,e^{-ikx}\,dk\,. \tag{5.8}$$

The key is the global relation. Here, since $H_0(t) = g_0(t)\mathbb{I}_2$ and $H_1(t) = g_1(t)\sigma_3$, we find from (5.7) that

$$\widehat{Q}(k,t) + \sigma\widehat{Q}(-k,t)\sigma = e^{-ik^2t}\left(\widehat{Q}(k,0) + \widehat{Q}(-k,0)\right)\,. \tag{5.9}$$

Therefore, H_0 and H_1 have been eliminated from the reconstruction formula and we find

$$Q^{line}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{Q}^{line}(k,0)\,e^{-ikx-ik^2t}\,dk\,, \tag{5.10}$$

as we should, where $\widehat{Q}^{line}(k,0)$ is determined only from the initial condition $Q^{line}(x,0)$.

NLS case. The nonlinear case is technically more difficult but the main steps follow the same principle. We only present the key results here for conciseness. From the half-line problem,

$$\begin{aligned}&i\partial_t Q(x,t) + \partial_x^2 Q(x,t) = 2g\overline{Q}Q^2(x,t)\\ &Q(x,0) = Q_0(x)\quad,\quad Q(0,t) = g_0(t)\mathbb{I}_2\quad,\quad \partial_x Q(0,t) = g_1(t)\sigma_3\,,\end{aligned} \tag{5.11}$$

we want to solve the full line problem

$$i\partial_t u(x,t) + \partial_x^2 u(x,t) = 2g|u|^2u(x,t)\quad,\quad u(x,0) = u_0(x)\,. \tag{5.12}$$

We define Q^{line} as in (5.3) and from it W^{line} and P^{line} as in (3.1). As is well-known, the solution for Q^{line} is obtained through the spectral analysis of the two fundamental solutions $\mu_\pm^{line}(x,t,k)$ of

$$\begin{cases}\partial_x\mu^{line} + ik[\Sigma_3,\mu^{line}] = W^{line}\,\mu^{line}\,,\\ \partial_t\mu^{line} + 2ik^2[\Sigma_3,\mu^{line}] = P^{line}\,\mu^{line}\,,\end{cases} \tag{5.13}$$

normalised as

$$\lim_{x\to\pm\infty} e^{ikx+2ik^2t}\mu_\pm^{line}(x,t,k)e^{-ikx-2ik^2t} = \mathbb{I}_4\quad,\quad k\in\mathbb{R}\,. \tag{5.14}$$

The first observation is that, if $\mu^{line}(x,t,k)$ is a solution of (5.13), then so is $\Sigma\,\mu^{line}(-x,t,-k)\,\Sigma$ where $\Sigma = \mathbb{I}_2\otimes\sigma$. This is a consequence of the symmetry[3]

$$\Sigma\,W^{line}(-x,t)\,\Sigma = -W^{line}(x,t)\,. \tag{5.15}$$

In particular, by uniqueness of normalized solutions, we obtain that

$$\mu_-^{line}(x,t,k) = \Sigma\,\mu_+^{line}(-x,t,-k)\,\Sigma\,, \tag{5.16}$$

[3] This is where the prefactor σ_3 in the definition of Q^{line} is important.

and as a consequence,

$$\mu_+^{line}(x,t,k) = \Sigma\,\mu_+^{line}(-x,t,-k)\,\Sigma\,e^{-ikx-2ik^2t}\,S^{line}(k)\,e^{ikx+2ik^2t}\,, \tag{5.17}$$

where $S^{line}(k)$ is the scattering matrix of the problem on the line. For the problem on the half-line, as we discussed in detail in Section 4, one must consider three fundamental solutions $\mu_j(x,t,k)$ which allow one to define the initial and boundary scattering matrices $S(k)$ and $T(k)$ as in (4.1) and (4.2). In particular, we know that $S(k) = \mu_3(0,0,k)$. The second observation is that, for $x > 0$, $I_3\,\mu_3(x,t,k)\,I_3$ and $\mu_+^{line}(x,t,k)$ satisfy the same equations and have the same normalization as $x \to \infty$. Hence, for $x > 0$,

$$I_3\,\mu_3(x,t,k)\,I_3 = \mu_+^{line}(x,t,k) \quad , \quad I_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}. \tag{5.18}$$

Taking $t = 0$ and the limit as x tends to 0 of this relation and inserting in (5.17), we find

$$I_3\,S(k)\,I_3 = \Sigma I_3\,S(-k) I_3 \Sigma\,S^{line}(k)\,. \tag{5.19}$$

This is the nonlinear analog of (5.6) (at $t = 0$) and plays the same crucial role. Indeed, in the limit where the coupling g tends to 0, one has

$$S(k) \to \begin{pmatrix} \mathbb{I}_2 & \widehat{Q}(k,0) \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad , \quad S^{line}(k) \to \begin{pmatrix} \mathbb{I}_2 & \widehat{Q}^{line}(k,0) \\ 0 & \mathbb{I}_2 \end{pmatrix}, \tag{5.20}$$

and the block (12) of (5.19) yields exactly (5.6). The final step consists in deriving the (nonlinear) global relation and the symmetry of the boundary data encoded in $T(k)$. These two ingredients put together are the nonlinear analogs of (5.7) and of the property $H_1(t) + \sigma\,H_1(t)\,\sigma = 0 = H_0(t) - \sigma\,H_0(t)\,\sigma$ of the boundary data which allowed us to eliminate it in the linear case. The global relation reads

$$\mathbb{I}_4 - T^{-1}(k)\,S(k) - \int_0^\infty e^{i\phi(x,t,k)\Sigma_3}\,W(x,t)\,\mu_3(x,t,k)\,e^{-i\phi(x,t,k)\Sigma_3}\,dx = 0\,. \tag{5.21}$$

Finally, we note that both $\mu_1(0,t,k)$ and $\mu_2(0,t,k)$ are solutions of

$$\partial_t \mu + 2ik^2[\Sigma_3, \mu] = P(0,t,k)\,\mu\,, \tag{5.22}$$

where

$$P(0,t,k) = \begin{pmatrix} -igH_0(t)\overline{H}_0(t) & 2kH_0(t) + iH_1(t) \\ g(2k\overline{H}_0(t) - i\overline{H}_1(t)) & igH_0(t)\overline{H}_0(t) \end{pmatrix}, \tag{5.23}$$

contains the boundary data $H_0(t)$ and $H_1(t)$. Therefore, it satisfies $P(0,t,k) = \Sigma_3\,\Sigma\,P(0,t,-k)\,\Sigma\,\Sigma_3$ and this implies the same symmetry on $\mu_1(0,t,k)$ and $\mu_2(0,t,k)$. In turn, this implies the following symmetry for the boundary scattering matrix $T(k)$

$$T(k) = \Sigma_3\,\Sigma\,T(-k)\,\Sigma\,\Sigma_3\,. \tag{5.24}$$

Although the details are much more technical than in the free and will be omitted here, the main steps identified above allow one to eliminate the boundary data from the reconstruction formula for $Q^{line}(x,t)$.

5.1.2 NLS with a δ potential/impurity

The δ potential is the most famous member of a family of singular point potentials (see e.g. [31]) and is characterized by one real parameter η. In our setting, it corresponds to boundary data H_0 and H_1 satisfying

$$\sigma\, H_0\, \sigma = H_0 \quad , \quad \sigma\, H_1\, \sigma + H_1 = \eta\, H_0\,. \tag{5.25}$$

Of course, the case $\eta = 0$ is known to correspond to the purely transmitting δ impurity *i.e.* the system on the full line. We have discussed this case in detail in the previous subsection. To our knowledge, the first analytical study of this problem was performed in [11] for the defocusing NLS using a Rosales type expansion of the solution [32]. The latter can be seen to arise as the Neumann series solution of the Gelfand-Levitan-Marchenko equations appearing in the inverse part of the usual ISM method for NLS. The key idea was to formulate appropriate conditions on the initial condition $Q_0(x)$ directly in Fourier space by imposing certain relations on the scattering data appearing in the Rosales expansion. In turn, these relations were inspired by the situation in the quantum case where the Reflection-Transmission algebras [33] play a role.

Then, in [19], important results were obtained for the focusing NLS with a repulsive ($\eta > 0$) delta impurity. The initial condition corresponds to a single soliton localised on one half-line and the main result concerns the long-time asymptotic behaviour of the solution. The study uses a clever and intricate combination of functional analysis estimates method combined with methods of integrable systems like the nonlinear steepest method [34]. It is shown that, for high enough velocity, the soliton splits into a reflected and transmitted soliton plus radiation. It is our plan to investigate the same problem (and similar ones studied by Holmer and collaborators later on) using the method presented here and to compare the results of the two approaches in a future paper.

Finally, in [20], the focusing NLS with δ impurity at small coupling is studied for a special initial condition which has the property of being an even function. This allows to map the problem to a scalar problem on the half-line with integrable Robin boundary conditions and use the full power of integrable techniques. In our setting, this would mean that we choose $Q_0(x)$ such that

$$\sigma\, Q_0(x)\, \sigma = Q_0(x)\,. \tag{5.26}$$

We will show below that this a special case of the notion of linearizable initial-boundary conditions that we introduce in the general $N \geq 2$ case.

5.1.3 NLS with a "jump" defect

In the quest for defect/impurity boundary conditions that would preserve the integrability of the model, a privileged class was obtained in [9]. The original approach was based on a lagrangian formalism but a key observation was that the obtained defect conditions were frozen Bäcklund transformations at the defect location. Using this, in [13], the author obtained general results on defect conditions and associated generating functionals for the conserved quantities for all integrable PDEs in the AKNS scheme [4]. In particular, for NLS, the defect conditions read, in our present notations,

$$\begin{cases} g_1^2(t) + g_1^1(t) = -i\alpha(g_0^2(t) - g_0^1(t)) - (g_0^2(t) + g_0^1(t))\Omega(t)\,, \\ \partial_t(g_0^2(t) - g_0^1(t)) = \alpha(g_1^2(t) + g_1^1(t)) - i(g_1^2(t) + g_1^1(t))\Omega(t) + i(g_0^2(t) - g_0^1(t))(|g_0^1(t)|^2 + |g_0^2(t)|^2) \end{cases} \tag{5.27}$$

where $\Omega(t) = \sqrt{\beta^2 + 2g|g_0^2(t) - g_0^1(t)|^2}$ and $\alpha, \beta \in \mathbb{R}$ are two defect parameters. These defect conditions look very complicated and highly nonlinear. But their origin as Bäcklund transformations of NLS ensure that specific solitonic solutions can be cosntructed explicitely. This was done in [10] by direct ansatz on the one and two soliton solutions. Using these solutions with $t = 0$ and $x = 0$ as input for the initial and boundary data, our approach provides the scheme to compute exactly the scattering data and implementing the inverse part of the method explicitely. The final result will of course reproduce the original solutions for $t > 0$.

5.2 Case $N = 3$

In [35], an important generalization of the study in [19] to the case of focusing NLS on three half-infinite edges connected to a single vertex by specific boundary conditions was performed using the same tools and with essentially the same conclusions concerning the splitting of an initial soliton profile localised on one of the edges. The boundary conditions used there are part of a general family of boundary conditions that were classified in the context of quantum graphs, for instance in [22]. We show in the next section how these boundary conditions are implemented in our setup for general $N \geq 2$. Hence, the particular problem studied in [35] fits in the present approach. As already mentioned, a quantitative comparison of their results with results that can be obtained solely by using our approach is an important task that we will return to in the future.

6 $N \geq 2$ case with general Robin boundary conditions

In [22], a classification of boundary conditions giving to self-adjoint extensions of the Laplacian on a metric graph was derived. In the case of the star-graph, these point potentials are well known to induce reflection and transmission between the various edges of the graph. In our context, it is natural to try to implement these point potentials as models of local scatterers for nonlinear waves on a star-graph. The starting point is the linear limit of NLS ($g = 0$) which corresponds precisely to the setting of the Laplace operator on a star-graph. Collecting the functions $q^\alpha(x,t)$ $\alpha = 1, \dots, N$ in a column vector $R(x,t)$, the family of boundary conditions obtained in [22] is parametrized by $U(N)$, the group of $N \times N$ unitary matrices, as follows

$$(U - \mathbb{I}_N)R(0,t) + i(U + \mathbb{I}_N)\partial_x R(0,t) = 0 \quad , \quad U \in U(N)\,. \tag{6.1}$$

In the case $N = 1$, with $U = e^{2i\alpha}$ this is just $\sin\alpha\, r(0,t) + \cos\alpha\, \partial_x r(0,t) = 0$ *i.e.* the Robin boundary condition together with its two limits, the Dirichlet and Neumann boundary conditions. We will then call the conditions (6.1) general Robin boundary conditions.

To transfer this to the nonlinear case, we need to rewrite this condition equivalently in the case where the functions q^α, $\alpha = 1, \dots, N$ are collected in a diagonal matrix $Q(x,t)$. Of course, in the linear limit, using $R(x,t)$ or $Q(x,t)$ is equivalent but in the nonlinear case, this changes dramatically the form of the interaction term in NLS, and hence the nature of the system, as pointed out in the introduction. To achieve this, we need the following simple lemma.

Lemma 6.1 *Let $M \in \mathcal{M}_N$ and let*

$$K = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad , \quad K^N = \mathbb{I}_N\,. \tag{6.2}$$

Then, there exists a unique decomposition of M on powers of K as

$$M = \sum_{j=0}^{N-1} M_j\, K^j\,, \tag{6.3}$$

where M_j is a diagonal matrix for each $j = 0, \ldots, N-1$.

<u>Proof:</u> It suffices to note that

$$M_j = \text{diag}(M_{1,j+1}, M_{2,j+2}, \ldots, M_{N-j,N}, M_{N-j+1,1}, \ldots, M_{N,j})\,. \tag{6.4}$$

■

Let us denote by $\mathcal{D}_N$ the space of $N \times N$ diagonal matrices over $\mathbb{C}$. There is a natural isomorphism between $\mathbb{C}^N$ and $\mathcal{D}_N$. Thanks to the previous lemma, we lift this isomophism to an isomorphism between $End(\mathbb{C}^N) \cong \mathcal{M}_N$ and $End(\mathcal{D}_N)$ by defining

$$\begin{array}{rccc} \mathcal{I}: & \mathcal{M}_N & \to & End(\mathcal{D}_N) \\ & M & \mapsto & \widehat{M} \end{array} \tag{6.5}$$

where, for all $N \in \mathcal{D}_N$

$$\widehat{M} N = \sum_{j=0}^{N-1} M_j\, K^j N K^{-j}\,, \tag{6.6}$$

and the M_j's are the diagonal matrices appearing in the decomposition of M in powers of K. We can now define the NLS equation on a star graph with general Robin boundary conditions as the following problem

$$i\partial_t Q + \partial_x^2 Q - 2g\overline{Q}Q^2 = 0 \quad , \quad 0 < x < \infty\,, \quad 0 < t < T\,, \tag{6.7}$$

$$Q(x,0) = Q_0(x) \quad , \quad (\widehat{U - \mathbb{I}_N})Q(0,t) + i(\widehat{U + \mathbb{I}_N})\partial_x Q(0,t) = 0\,. \tag{6.8}$$

This is a particular case of the general setup when H_0 and H_1 are required to satisfy $(\widehat{U - \mathbb{I}_N})H_0(t) + i(\widehat{U + \mathbb{I}_N})H_1(t) = 0$.

Linearizable initial-boundary conditions. We want to use the problem with general Robin conditions to introduce and illustrate the notion of linearizable initial-boundary conditions. In Fokas method, there exist the so-called linearizable boundary conditions which take their name from the fact that they allow for a solution of the global relation by algebraic means only, hence rendering the unified method for IBV problems just as powerful as the ISM for IV problems in linearizing the problem in Fourier space. In practice, linearizable boundary conditions correspond to integrable boundary conditions that could be found by other methods before the advent of the unified method, like the Bäcklund transformation method initiated by Habibullin [36]. For non-linearizable boundary conditions, solving the global relation is much more involved and remains essentially a nonlinear problem.

The way to idenfity linearizable boundary conditions is to exploit natural symmetries of the global relation. The latter represents in Fourier space a strong relation between the initial-boundary data and the integrable bulk dynamics. So far, symmetries of the global relation have been used in such a way as to identify boundary conditions which would render the problem amenable to solutions for arbitrary initial conditions of the same type as for the problem on the full line. Performing the same reasoning on a star-graph essentially leads to trivial linearizable boundary conditions corresponding for instance to disconnected half-lines with Robin boundary conditions. However, the example of [20] shows that such a restriction on the boundary conditions can be relaxed by restricting one's attention to initial data with a specific symmetry (an even function in that case). Indeed, in general the boundary conditions of the δ impurity are not linearizable but when combined with an even initial condition, one can map the problem to a linearizable one.

A posteriori, this is very natural from the point of view of the global relation: one can trade off freedom on the initial data to gain more flexibility on boundary conditions leading to IBV problems that can be solved as efficiently as IV problems. In the case of the δ impurity, the boundary conditions (5.25) are invariant by the action of σ and so is the bulk dynamics. Therefore, it is natural to split the set of initial data into the two eigenspaces of σ *i.e.* to consider initial data satisfying

$$\sigma\, Q_0(x)\, \sigma = \pm Q_0(x)\,. \tag{6.9}$$

By choosing the initial data in the + subspace, [20] were able to reduce the problem to a scalar linearizable one, with Robin boundary condition. Note that choosing to use the − subspace would also lead to a scalar linearizable problem but with Dirichlet boundary condition.

We now illustrate this idea for the general Robin conditions above. In this general class, there are distinct representatives that have the additional symmetry property that they are invariant by the action of K. This is the case for instance for the generalisation of the δ impurity conditions which read

$$K\, H_0\, K^{-1} = H_0 \quad , \quad \sum_{j=0}^{N-1} K^j\, H_1\, K^{-j} = \eta H_0\,. \tag{6.10}$$

More generally, the class of boundary conditions within the general Robin boundary conditions that are such that U appearing in (6.8) is a circulant matrix, *i.e.* commutes with K, allow us to use this extra symmetry. In that case, it is natural to split the initial data space into the eigenspaces of K. The latter is known to have N distinct eigenvalues ω^j, $j = 0, \ldots, N-1$ where $\omega = e^{\frac{2i\pi}{N}}$ is the N-th root of unity. Therefore, if the initial data satisfies the following symmetry condition

$$K\, Q_0\, K^{-1} = \omega^p\, Q_0\,, \tag{6.11}$$

for some $p \in \{0, \ldots, N-1\}$, then the problem can be mapped to a scalar linearizable problem. Note that (6.11) is a natural generalization of (5.26). We show how this works for the generalized δ boundary conditions. They are obtained by choosing $U = \frac{2}{N+i\alpha} J - \mathbb{1}_N$ where J is the $N \times N$ matrix with 1 in every entry and α is a real number related to η and representing the coupling. The matrix U is circulant and decomposes as

$$U = \sum_{j=0}^{N-1} u_j\, K^j \quad , \quad u_0 = \frac{2}{N+i\alpha} - 2 \quad , \quad u_j = \frac{2}{N+i\alpha} \quad , \quad j \neq 0\,. \tag{6.12}$$

Therefore, for the problem with symmetry (6.11), one obtains the following scalar linearizable problem with Robin boundary condition

$$(2i\alpha - 2(N-1) + \beta_p)\, q(0,t) + i(2+\beta_p)\, \partial_x q(0,t) = 0\,, \tag{6.13}$$

where $\beta_p = 2(N-1)$ if $p = 0$ and -2 otherwise, in which case one actually obtains the Dirichlet boundary condition. Here, the function q represents any one of the entries of Q. This generalizes the setup of [20] and therefore all the methods used there apply here directly. In practice, it means that one can simply study the problem on one of the half-lines of the star-graph and the full solution on the complete graph can be reconstructed by applying the symmetry.

Conclusions

We introduced a general method that solves the problem of formulating an inverse scattering method for integrable nonlinear equations on a star-graph. The key is to map the problem to a matrix IBV problem that can be analysed by a suitable matrix generalization of the unified transform developed by Fokas for scalar IBV problems. Although the method was presented for NLS, it is clear that it allows to tackle any integrable nonlinear equations that can be analysed by Fokas method, that is any nonlinear equation for which a Lax pair is known.

Our results provides a unifying framework in which one can analyse in great detail the long-time asymptotic behaviour of solutions on a star-graph. This is due to the fact that the Riemann-Hilbert approach and the associated nonlinear steepest descent method that are so powerful in Fokas method, naturally extend to our framework. As mentioned before, the next natural step is to implement this program in detail in order to compare with the results obtained in [19] and [35] and to go beyond them hopefully.

In the discussion of the general Robin conditions on the star-graph with N edges, we identified the new concept of *linearizable initial-boundary conditions* which appear as the natural generalization of Fokas linearizable boundary conditions in the scalar case. The latter turn out to coincide with what was called before "integrable boundary" conditions in all known cases. Therefore, the notion of linearizable initial-boundary conditions can be taken as the generalization of the notion of integrable boundary conditions in the context of an integrable PDE on a star-graph. The key feature is that one applies the constraints arising from integrability not only on the boundary conditions but also on the space of initial conditions. The net result is that certain boundary conditions which would have been declared "non integrable" in the usual approach, such as the δ impurity on the line, can in fact be studied via ISM just as effectively as the traditional integrable boundary conditions provided one works with smaller functional spaces.

In the longer term, the present results open the way to a theory of inverse scattering on arbitrary finite connected graphs. This is because any such graph can be viewed as a collection of star-graphs connected together by finite edges. Therefore, to complete this program, one will have to combine the present approach with the unified transform method applied to finite intervals (see e.g. [37]).

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