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Nonlinear eigenvalue problems

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(Dated: January 24, 2014)

This paper presents a detailed asymptotic study of the nonlinear differential equation $y'(x) = \cos[\pi x y(x)]$ subject to the initial condition $y(0) = a$. Although the differential equation is nonlinear, the solutions to this initial-value problem bear a striking resemblance to solutions to the time-independent Schrödinger eigenvalue problem. As x increases from $x = 0$, $y(x)$ oscillates and thus resembles a quantum wave function in a classically allowed region. At a critical value $x = x_{\rm crit}$, where $x_{\rm crit}$ depends on a, the solution $y(x)$ undergoes a transition; the oscillations abruptly cease and $y(x)$ decays to 0 monotonically as $x \to \infty$. This transition resembles the transition in a wave function that occurs at a turning point as one enters the classically forbidden region. Furthermore, the initial condition a falls into discrete classes; in the nth class of initial conditions $a_{n-1} < a < a_n$ ($n = 1, 2, 3, \ldots$), $y(x)$ exhibits exactly n maxima in the oscillatory region. The boundaries a_n of these classes are the analogs of quantum-mechanical eigenvalues. An asymptotic calculation of a_n for large n is analogous to a high-energy semiclassical (WKB) calculation of eigenvalues in quantum mechanics. The principal result of this paper is that as $n \to \infty$, $a_n \sim A\sqrt{n}$, where $A = 2^{5/6}$. Numerical analysis reveals that the first Painlevé transcendent has an eigenvalue structure that is quite similar to that of the equation $y'(x) = \cos[\pi x y(x)]$ and that the nth eigenvalue grows with n like a constant times $n^{3/5}$ as $n \to \infty$. Finally, it is noted that the constant A is numerically very close to the lower bound on the power-series constant P in the theory of complex variables, which is associated with the asymptotic behavior of zeros of partial sums of Taylor series.

PACS numbers: 02.30.Hq, 02.30.Mv, 02.60.Cb

arXiv:1401.6161v1 [math-ph] 23 Jan 2014

I. INTRODUCTION

This paper presents a detailed asymptotic analysis of the nonlinear initial-value problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a. \tag{1}$$

This remarkable and deceptively simple looking differential equation was given as an exercise in the text by Bender and Orszag [1]. Since then, it and closely related differential equations have arisen in a number of physical contexts involving the complex extension of quantum-mechanical probability [2, 3] and the structure of gravitational inspirals [4]. We will see that the properties of solutions to this equation are strongly analogous to those of the time-independent Schrödinger eigenvalue problem.

Recall that the Schrödinger eigenvalue problem has the general form

$$-\psi''(x) + V(x)\psi(x) = E\psi(x), \qquad \psi(\pm\infty) = 0, \tag{2}$$

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where E is the eigenvalue. For simplicity, we assume that the potential $V(x)$ has one local minimum and rises monotonically to ∞ as $x \to \pm\infty$. In general this eigenvalue problem is not analytically solvable except for special potentials [such as the harmonic oscillator potential $V(x) = x^2$]. However, it is possible to find the large-n asymptotic behavior of the nth eigenvalue E_n by using semiclassical (WKB) analysis. To leading order the large-n asymptotic behavior of the eigenvalues of the two-turning-point problem may be obtained from the Bohr-Sommerfeld condition

$$\int_{x_1}^{x_2} dx \sqrt{E_n - V(x)} \sim (n + 1/2)\pi \quad (n \to \infty), \tag{3}$$

where the turning points x_1 and x_2 are real roots of the equation $V(x) = E_n$. This WKB condition determines the eigenvalues implicitly for large n. As an example, for the anharmonic potential $V(x) = x^4$ the large-n asymptotic behavior of the eigenvalues is [5]

$$E_n \sim B n^{4/3} \quad (n \to \infty), \tag{4}$$

where the constant B is given by $B = 3\Gamma(3/4)\sqrt{\pi}/\Gamma(1/4)$.

The quantum eigenfunctions $\psi(x)$ exhibit several well known characteristic features. In the classically allowed region between the turning points ($x_1 < x < x_2$), the eigenfunctions are oscillatory and the eigenfunction corresponding to E_n has n nodes. In the classically-forbidden regions $x > x_2$ and $x < x_1$ the eigenfunctions decay exponentially and monotonically to zero as $|x| \to \infty$. Thus, at the turning points the behavior of the eigenfunctions changes abruptly from rapid oscillation to smooth exponential decay.

The solutions $y(x)$ to the nonlinear differential equation (1) have many features in common with the solutions $\psi(x)$ to the Schrödinger equation (2). For any choice of $y(0) = a$ the initial slope $y'(0)$ is 1. As x increases from 0, $y(x)$ oscillates as shown in Fig. 1. This regime of oscillation is analogous to a classically allowed region in quantum mechanics. Note that the number of maxima of the function $y(x)$ in the oscillatory region increases as $y(0)$ increases. With increasing x the oscillations abruptly cease, and as $x \to \infty$ the function $y(x)$ then decays smoothly and monotonically to 0. This kind of behavior resembles that of $\psi(x)$ in a classically forbidden region.

Figure 1 reveals that in the decaying regime the curves merge into quantized bundles. This large-x asymptotic behavior of $y(x)$ can be explained by using elementary asymptotic analysis. If we seek an asymptotic behavior of the form $y(x) \sim c/x$ ($x \to \infty$) and substitute this *ansatz* into (1), we find that $c = m + 1/2$ ($m = 0, 1, 2, 3, \ldots$). This is just the *leading* term in the asymptotic expansion of $y(x)$ for large x. The full series has the form

$$y(x) \sim \frac{m + 1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty). \tag{5}$$

The first few coefficients are

$$\begin{aligned}
c_1 &= \frac{(-1)^m}{\pi}(m + 1/2), \\
c_2 &= \frac{3}{\pi^2}(m + 1/2), \\
c_3 &= (-1)^m \left[\frac{(m + 1/2)^3}{6\pi} + \frac{15(m + 1/2)}{\pi^3}\right], \\
c_4 &= \frac{8(m + 1/2)^3}{3\pi^2} + \frac{105(m + 1/2)}{\pi^4}, \\
c_5 &= (-1)^m \left[\frac{3(m + 1/2)^5}{40\pi} + \frac{36(m + 1/2)^3}{\pi^3} + \frac{945(m + 1/2)}{\pi^5}\right], \\
c_6 &= \frac{38(m + 1/2)^5}{15\pi^2} + \frac{498(m + 1/2)^3}{\pi^4} + \frac{10395(m + 1/2)}{\pi^6}.
\end{aligned} \tag{6}$$

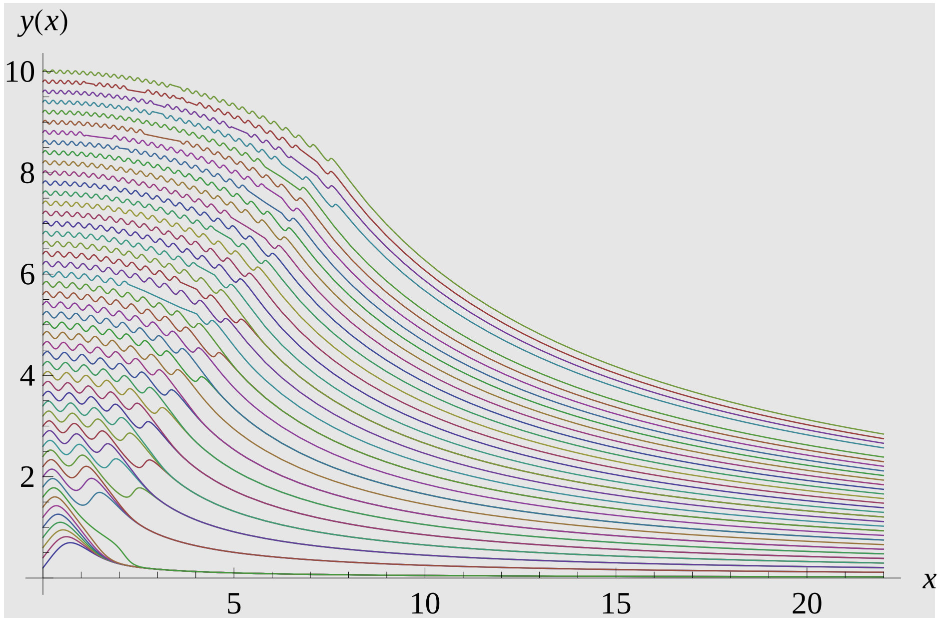

FIG. 1: Numerical solutions $y(x)$ to (1) for $0 \le x \le 24$ with initial conditions $y(0) = 0.2k$ for $k = 1, 2, 3, \ldots, 50$. The solutions initially oscillate but abruptly become smoothly and monotonically decaying. In the decaying regime the solutions merge into discrete quantized bundles.

A. Hyperasymptotic analysis

A close look at Fig. 1 shows a surprising result: Half of the predicted large-x asymptotic behaviors in (5) appear to be missing. The bundles of curves shown in Fig. 1 correspond only to *even* values of m. To explain what has happened to the odd-m bundles, we perform a hyperasymptotic analysis (asymptotics beyond all orders) [6]. Let $y_1(x)$ and $y_2(x)$ represent two different curves in the mth bundle. Even though they are different curves they have exactly the same asymptotic approximation as given in (5). Then $Y(x) \equiv y_1(x) - y_2(x)$ satisfies the differential equation

$$\begin{aligned} Y'(x) &= \cos[\pi x y_1(x)] - \cos[\pi x y_2(x)] \\ &= -2\sin\left[\tfrac{1}{2}\pi x y_1(x) + \tfrac{1}{2}\pi x y_2(x)\right]\sin\left[\tfrac{1}{2}\pi x y_1(x) - \tfrac{1}{2}\pi x y_2(x)\right] \\ &\sim -2\sin\left[\pi\left(m+\tfrac{1}{2}\right)\right]\sin\left[\tfrac{1}{2}\pi x Y(x)\right] \quad (x\to\infty) \\ &\sim -(-1)^m \pi x Y(x) \quad (x\to\infty). \end{aligned} \tag{7}$$

Thus, we conclude that

$$Y(x) \sim K \exp\left[-(-1)^m \pi x^2\right] \quad (x\to\infty), \tag{8}$$

where K is an arbitrary constant. This calculation shows that while two different curves in the same bundle have the same asymptotic expansion for large x, they differ by an exponentially small amount. This result explains why no arbitrary constant appears in the asymptotic expansion (5); the arbitrary constant appears in the beyond-all-orders hyperasymptotic (exponentially small) correction to this asymptotic series.

More importantly, this argument demonstrates that two curves can only be in the same bundle if m is *even.* If m is odd, the two curves *move away from one another* as x increases. Thus, while there is a bundle of infinitely many curves when m is even, we see that there is a unique and *discrete* curve, called a *separatrix*, for the case of odd m. The nth separatrix, whose large-x asymptotic behavior is $(2n - 1/2)/x$ ($n = 1, 2, 3, \ldots$), is *unstable* for increasing x; that is, as x increases, nearby curves $y(x)$ veer away from it and become part of the bundles above or below the separatrix. This explains why there are no curves shown in Fig. 1 when m is odd. Ten separatrix curves are shown in Fig. 2.

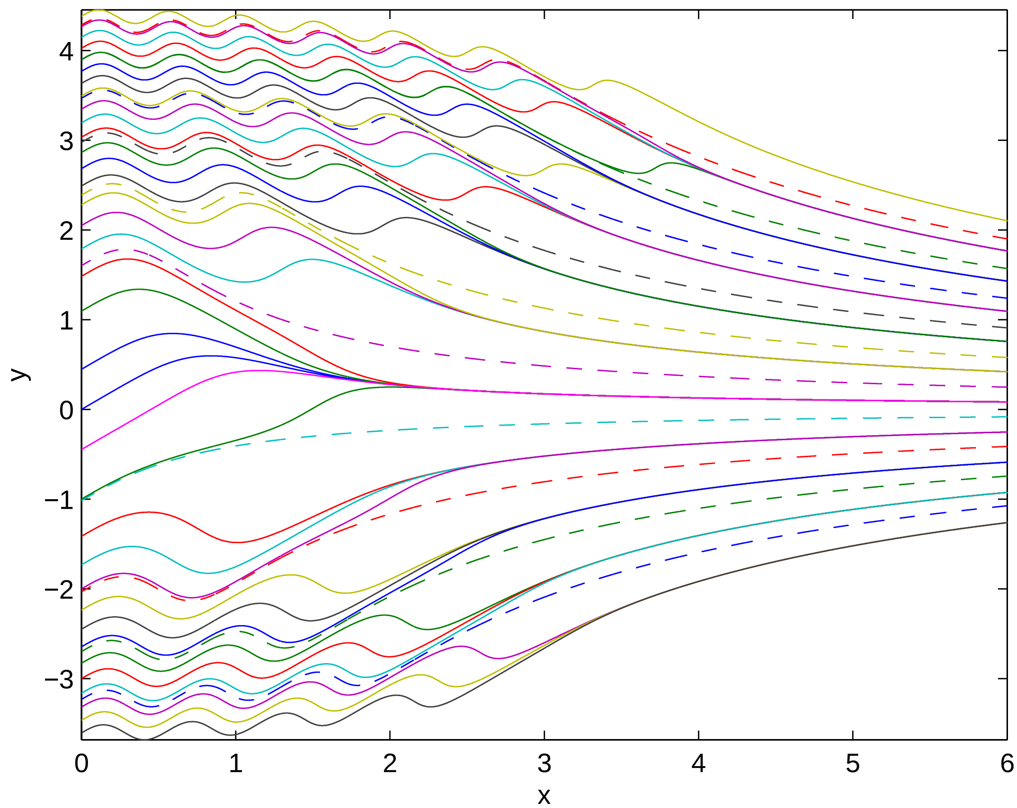

FIG. 2: Numerical solutions to (1) showing ten separatrix curves, which cross the y axis at $a_{-3} = -3.231360$, $a_{-2} = -2.698369$, $a_{-1} = -2.032651$, $a_0 = -1.016702$, $a_1 = 1.602573$, $a_2 = 2.388358$, $a_3 = 2.976682$, $a_4 = 3.467542$, $a_5 = 3.897484$, and $a_6 = 4.284674$.

While the separatrix curves are unstable for increasing x, they are stable for decreasing x and thus it is numerically easy to trace these curves backward from large values of x down to $x = 0$. We treat the discrete point a_n ($n = 1, 2, 3, \ldots$) at which the nth separatrix crosses the y axis as an eigenvalue. The curves $y(x)$, whose initial values $y(0) = a$ lie in the range $a_{n-1} < y(0) < a_n$, have n maxima. Our objective in this paper is to determine analytically the large-n asymptotic behavior of the eigenvalues; we will establish that

$$a_n \sim A\sqrt{n} \quad (n \to \infty), \tag{9}$$

where $A = 2^{5/6}$. The constant A is a nonlinear analog of the WKB constant B in (4).

Hyperasymptotics also plays a crucial role in quantum theory. Because the Schrödinger differential equation for the eigenvalue problem (2) is second order, the asymptotic behavior of $\psi(x)$ as $x \to \infty$ contains two arbitrary constants. However, there is only *one* constant C in the WKB asymptotic approximation

$$\psi(x) \sim C[V(x) - E]^{-1/4} \exp\left[\int^x ds\sqrt{V(s) - E}\right] \quad (x \to \infty). \tag{10}$$

There is a second constant D, of course, but this constant multiplies the subdominant (exponentially decaying) solution, and thus this constant does not appear to any order in the WKB expansion. The constant D remains invisible except at an eigenvalue because only at an eigenvalue does the coefficient C of the exponentially growing solution (10) vanish *to all orders* in the large-x asymptotic expansion, leaving the physically acceptable exponentially decaying solution

$$\psi(x) \sim D[V(x)-E]^{-1/4}\exp\left[-\int^x ds\sqrt{V(s)-E}\right] \quad (x\to\infty). \tag{11}$$

B. Organization of this paper

The principal thrust of the analysis in this paper is an asymptotic study of the separatrices, which for large x are approximated by the formula in (5) with m odd. Thus, we let $m = 2n-1$ and we scale both the independent and dependent variables in (1):

$$x = \sqrt{2n-1/2}\,t, \quad y(x) = \sqrt{2n-1/2}\,z(t), \tag{12}$$

and let

$$\lambda = (2n-1/2)\pi. \tag{13}$$

The resulting equation for $z(t)$ is

$$z'(t) = \cos[\lambda t z(t)]. \tag{14}$$

With these changes of variable, the nth separatrix [which behaves like $(2n-1/2)/x$ as $x\to\infty$] now behaves like $1/t$ as $t\to\infty$. Also, for large λ the turning point (the point at which the oscillations cease and monotone decreasing behavior begins) is located at $t=1$.

In Sec. II we begin by examining the differential equation (1) numerically. We then show numerically that for large λ the solution $z(t)$ to the scaled equation (14) that satisfies the initial condition $z(0) = 2^{1/3}$ is oscillatory until $t=1$, at which point it decays smoothly like $z(t)\sim 1/t$ as $t\to\infty$. We also show that the amplitude of the oscillations is of order $1/\lambda$ for large λ. Hence, in the limit $\lambda\to\infty$ the function $z(t)$ converges to a smooth and nonoscillatory function $Z(t)$ that passes through $2^{1/3}$ at $t=0$ and through 1 at $t=1$. Thus, the nth eigenvalue is asymptotic to $A\sqrt{n}$ as $n\to\infty$, where $A = 2^{5/6}$. In Sec. III we perform an asymptotic calculation of $Z(t)$ correct to order $1/\lambda$ and use this result to obtain the number A in (9). In Sec. IV we suggest that the techniques presented in this paper may apply to many other nonlinear differential equations. As evidence, we present numerical results regarding the first Painlevé transcendent. We also conjecture that the number A in (9) may be related to the power-series constant P, which describes the asymptotic behavior of the zeros of partial sums of Taylor series of analytic functions.

II. NUMERICAL STUDY OF (1) AND (14)

We begin our analysis of (1) by constructing the Taylor series expansion

$$y(x) = \sum_{n=0}^{\infty} b_n x^n \tag{15}$$

of the solution $y(x)$. To find the Taylor coefficients b_n we substitute this expansion into the differential equation and collect powers of x. The first few Taylor coefficients are

$$
\begin{aligned}
b_0 &= y(0) = a, \\
b_1 &= 1, \\
b_2 &= 0, \\
b_3 &= -\tfrac{1}{6}\pi^2 a^2, \\
b_4 &= -\tfrac{1}{4}\pi^2 a, \\
b_5 &= \tfrac{1}{120}\pi^4 a^4 - \tfrac{1}{10}\pi^2, \\
b_6 &= \tfrac{1}{18}\pi^4 a^3, \\
b_7 &= -\tfrac{1}{5040}\pi^6 a^6 + \tfrac{2}{21}\pi^4 a^2, \\
b_8 &= -\tfrac{1}{180}\pi^6 a^5 + \tfrac{31}{480}\pi^4 a, \\
b_9 &= \tfrac{1}{362880}\pi^8 a^8 - \tfrac{161}{6480}\pi^6 a^4 + \tfrac{17}{1080}\pi^4.
\end{aligned} \tag{16}
$$

We then observe that we can reorganize and regroup the terms in the Taylor series. For example, the first terms in b_1, b_3, b_5, b_7, b_9, and so on, give rise to the function

$$\frac{1}{\pi a}\sin s$$

and the first terms in b_4, b_6, b_8, b_{10}, and so on, give rise to

$$\frac{1}{8\pi^2 a^3}\left[2s\sin(2s) + \cos(2s) - 2s^2 - 1\right],$$

where $s = \pi a x$. This partial summation of the Taylor series, a procedure used in multiple-scale perturbation theory to eliminate secular behavior [7], shows that the solution $y(x)$ is approximately a falling parabola with an oscillatory contribution whose amplitude that is of order $1/a$. This is indeed what we observe in Fig. 1. The partial summation of the Taylor series suggests that a and y are both of order $\sqrt{n}$ and motivates the changes of variable in (12) and (13), which give the scaled differential equation (14).

As λ in (14) tends to ∞, the oscillations disappear. (This is demonstrated in Sec. III.) The resulting curve $Z(t)$, which begins at $Z(0) = 2^{1/3}$ and passes through $Z(1) = 1$, is shown as a dashed line (red in the electronic version) in Fig. 3 (upper panel). Also shown are the first four eigencurve (separatrix) solutions to (14) (blue, cyan, magenta, and green in the electronic version), which have one, two, three, and four maxima. Note that these eigensolutions rapidly approach the limiting dashed curve as the number of oscillations increases. The lower panel in Fig. 3 indicates the difference between the dashed curve and the solid curves plotted in the upper panel.

For large values of λ the convergence to the limiting curve is dramatic. In Fig. 4 we plot the limiting curve $Z(t)$ in the upper panel and the difference between the limiting curve and the $n = 500,000$ separatrix curve (eigencurve) in the lower panel. Note that the difference is of order $1/n$ (10^{-6}). On the basis of these numerical calculations we were able to use Richardson extrapolation [8] to calculate the coefficient A accurate to one part in 10^{10} and we conjectured reliably that $A = 2^{5/6}$.

The convergence of $z(t)$ (which is rapidly oscillatory when $0 \le t \le 1$) to $Z(t)$ (which is smooth and nonoscillatory) as $\lambda \to \infty$ strongly resembles the convergence of a Fourier series. Consider, for example, the convergence of the Fourier sine series to the function $f(x) = 1$ on the interval $0 < x < \pi$. The $2N + 1$ partial sum of the Fourier sine series is

$$S_{2N+1}(x) = \frac{4}{\pi}\sum_{n=0}^{N}\frac{\sin[(2n+1)x]}{2n+1}. \tag{17}$$

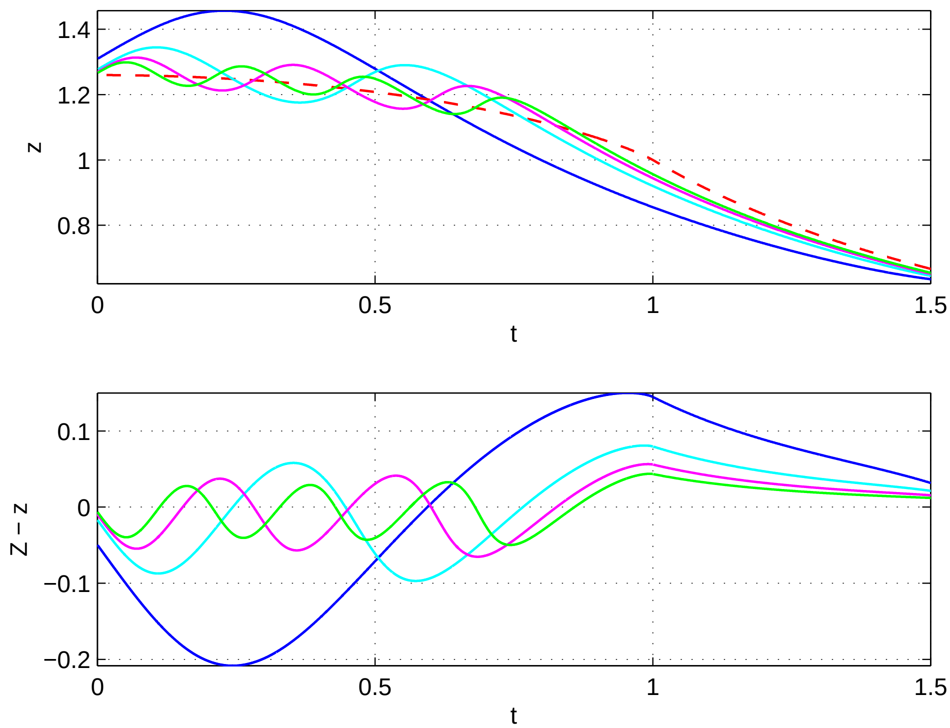

FIG. 3: Upper panel: Numerical plots of the first four separatrix solutions $z(t)$ (eigensolutions) to (14) (blue, cyan, magenta, and green in the electronic version). These solutions have one, two, three, and four maxima. As λ increases, these curves approach the solution to (14) for $\lambda=\infty$ (dashed curve) (red in the electronic version). [The $\lambda=\infty$ curve is called $Z(t)$ and satisfies the differential equation (31).] Lower panel: A plot of the differences between the solid curves and the dashed curve.

As can be inferred from Fig. 5, which displays the partial sums for $N=5$, 20, 80, as N increases, $S_{2N+1}(x)$ approaches 1 (except for values of x near $x=0$ and $x=\pi$) in a highly oscillatory fashion that strongly resembles the approach of $z(t)$ to $Z(t)$ in Fig. 4.

III. ASYMPTOTIC SOLUTION OF THE SCALED EQUATION (14)

The objective of the asymptotic analysis in this section is to solve (14) for large λ and to verify the result in (9); namely, that $A=2^{5/6}$. We begin by converting the differential equation in (14) to the integral equation

$$[z(t)]^2-[z(0)]^2+t^2/2+\eta(t)=\mathrm{O}(1/\lambda)\quad(\lambda\to\infty), \tag{18}$$

where

$$\eta(t)=\int_0^t ds\,s\cos[2\lambda s z(s)]. \tag{19}$$

To obtain this result we multiply (14) by $z(t)+tz'(t)$, integrate from 0 to t, and use the double-angle formula for the cosine function.

The problem is now to calculate $\eta(t)$. To do so, we observe that $\eta(t)$ is just one of an infinite

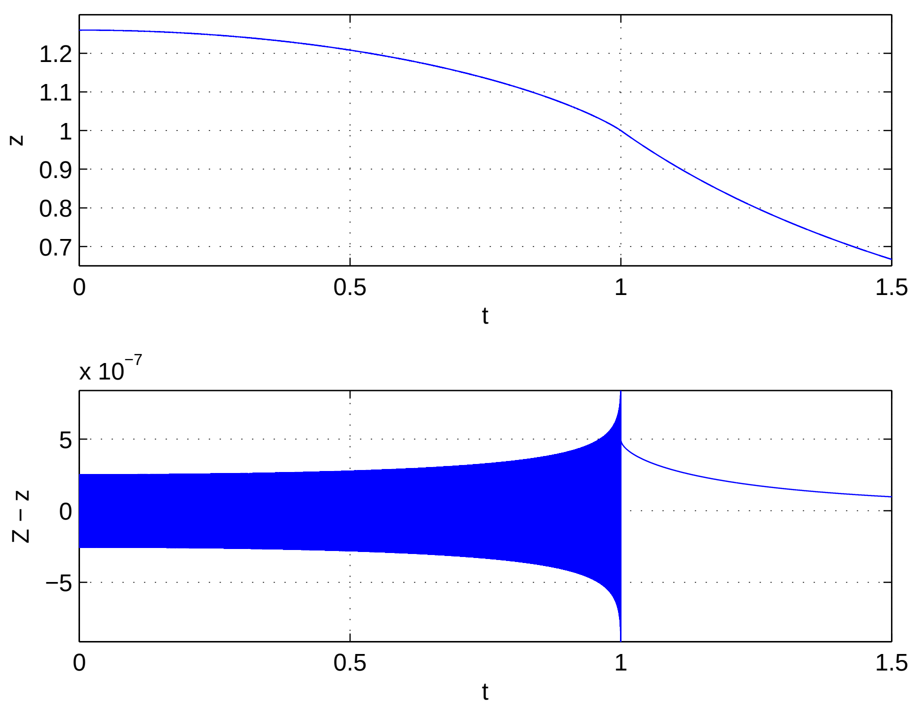

FIG. 4: Upper panel: Numerical solution $z(t)$ to (14) corresponding to $n = 500,000$. No oscillation is visible because the amplitude of oscillation is of order $1/\lambda$ when λ is large. Lower panel: Difference between the $n = 500,000$ eigencurve $z(t)$ and the $\lambda = \infty$ curve $Z(t)$. Note that the difference is highly oscillatory and is of order 10^{-6}.

set of moments $A_{n,k}(t)$, which are defined as follows:

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}. \tag{20}$$

Note that $\eta(t) = A_{2,0}(t)$.

For large λ these moments satisfy the linear difference equation

$$A_{n,k}(t) = -\tfrac{1}{2} A_{n-1,k+1}(t) - \tfrac{1}{2} A_{n+1,k+1}(t) \quad (n \geq 2). \tag{21}$$

To obtain this equation we multiply the integrand of the integral in (20) by

$$\frac{z(s) + s z'(s)}{z(s)} - \frac{s z'(s)}{z(s)}. \tag{22}$$

(Note that this quantity is merely an elaborate way of writing 1.) We then evaluate the first part of the resulting integral by parts and verify that it is negligible as $\lambda \to \infty$ if $t \leq 1$. In the second part of the integral we replace $z'(t)$ by $\cos[\lambda t z(t)]$ and use the trigonometric identity

$$\cos(na)\cos(a) = \tfrac{1}{2}\cos[(n+1)a] + \tfrac{1}{2}\cos[(n-1)a].$$

By using repeated integration by parts it is easy to show that $\eta(t)$ in (19) can be expanded as the series

$$\eta(t) = \sum_{p=0}^{\infty} \alpha_{1,2p+1} A_{1,2p+1}(t), \tag{23}$$

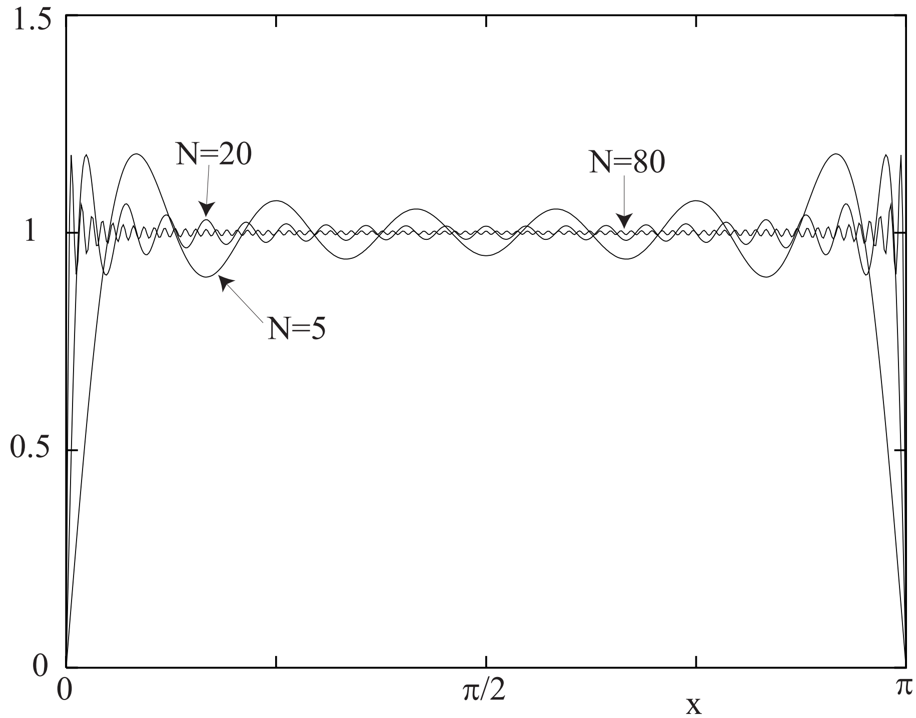

FIG. 5: Convergence of the $N = 5$, 20, and 80 partial sums in (17) of the Fourier sine series for $f(x) = 1$. The partial sums of the Fourier series converge to 1 as $N \to \infty$ in much the same way that $z(t)$ converges to $Z(t)$ as $\lambda \to \infty$. As N increases, the frequency of oscillation increases and the amplitude of oscillation approaches zero.

where the coefficients $\alpha_{n,k}$ are determined by a one-dimensional random-walk process in which random walkers move left or right with equal probability but become static when they reach $n = 1$. The initial condition for the random walk is that $\alpha_{n,0} = 0$ if $n \neq 2$ and $\alpha_{2,0} = 1$. The coefficients $\alpha_{n,k}$ obey the difference equations

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0, \tag{24}$$

$$2\alpha_{2,k} + \alpha_{3,k-1} = 0, \tag{25}$$

$$2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3). \tag{26}$$

(Note that $\alpha_{n,k} = 0$ if one of the subscripts is odd and the other is even.) The difference equations (25) and (26) can be solved in closed form, and we obtain the following exact result for $n \geq 2$:

$$\alpha_{n,k} = \frac{(-1)^n (n-1) k!}{2^k (k/2 + n/2)! (k/2 - n/2 + 1)!}, \tag{27}$$

which holds if n and k are both even or both odd. Finally, we use equation (24) to obtain

$$\alpha_{1,2p+1} = -\tfrac{1}{2}\alpha_{2,2p} = -\frac{(2p)!}{2^{2p+1} p! (p+1)!} = -\frac{\Gamma(p + 1/2)}{2\sqrt{\pi}\,(p+1)!}, \tag{28}$$

where the duplication formula for the Gamma function was used to obtain the last equality.

Thus, the series in (23) for $\eta(t)$ reduces to the series of integrals

$$\eta(t) = -\frac{1}{2\sqrt{\pi}} \sum_{p=0}^{\infty} \frac{\Gamma(p+1/2)}{(p+1)!} \int_0^t ds\, z'(s) \frac{s^{2p+2}}{[z(s)]^{2p+1}},$$

which is valid for $t \leq 1$. This series can be summed in closed form:

$$\eta(t) = \int_0^t ds\, z(s)z'(s)\sqrt{1 - s^2/[z(s)]^2} - \int_0^t ds\, z(s)z'(s). \tag{29}$$

There is no explicit reference to λ in this expression, so we pass to the limit as $\lambda \to \infty$. In this limit the function $z(t)$, which is rapidly oscillatory (see Fig. 4), approaches the function $Z(t)$, which is smooth and not oscillatory. We therefore obtain from (18) an integral equation satisfied $Z(t)$:

$$[Z(t)]^2 - [Z(0)]^2 + \tfrac{1}{2}t^2 - \int_0^t ds\, Z(s)Z'(s) + \int_0^t ds\, Z(s)Z'(s)\sqrt{1 - s^2/[Z(s)]^2} = 0. \tag{30}$$

We differentiate (30) to obtain an elementary differential equation satisfied by $Z(t)$:

$$Z(t)Z'(t) + t + Z'(t)\sqrt{[Z(t)]^2 - t^2} = 0. \tag{31}$$

This differential equation is easy to solve because it is of *homogeneous* type; that is, the equation can be rearranged so that $Z(t)$ is always accompanied by a factor of $1/t$. Such an equation can be solved by making the substitution $Z(t) = tG(t)$ to reduce (31) to a separable differential equation for $G(t)$. The general solution for $G(t)$ is

$$\frac{K}{t^3} = \left(1 + 3[G(t)]^2\right)\left(G(t) + \sqrt{[G(t)]^2 - 1}\right)\frac{\sqrt{[G(t)]^2 - 1} - 2G(t)}{\sqrt{[G(t)]^2 - 1} + 2G(t)}, \tag{32}$$

where K is an arbitrary constant. The condition that $G(1) = 1$ then determines that $K = -4$, and we obtain the exact result that $Z(0) = 2^{1/3}$. We thus conclude that $A = 2^{5/6}$. This establishes the principal result of this paper.

IV. DISCUSSION AND DESCRIPTION OF FUTURE WORK

A. First Painlevé transcendent

We believe that the asymptotic approach developed in this paper may be applicable to many nonlinear differential equations having separatrix structure. One possibility is the differential equation for the first Painlevé transcendent

$$y''(x) = [y(x)]^2 + x. \tag{33}$$

How do solutions to this equation behave as $x \to -\infty$? It is clear that when x becomes large and negative, there can be a dominant asymptotic balance between the positive term $[y(x)]^2$ and the negative term x, which implies that $y(x)$ can have two possible leading asymptotic behaviors:

$$y(x) \sim \pm\sqrt{-x} \quad (x \to -\infty). \tag{34}$$

This asymptotic result is justified because the second derivative of $\sqrt{-x}$ is small compared with x when $|x|$ is large.

This problem is interesting because the asymptotic behavior $y(x) \sim -\sqrt{-x}$ is stable but the asymptotic behavior $y(x) \sim \sqrt{-x}$ is unstable. To verify this, we calculate the corrections to these two asymptotic behaviors. It is easy to show that when x is large and negative, the solution to (33) oscillates about and decays slowly towards the curve $-\sqrt{-x}$ [1]:

$$y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos\left[\tfrac{4}{5}\sqrt{2}(-x)^{5/4} + d\right] \quad (x \to -\infty), \tag{35}$$

where c and d are two arbitrary constants. The differential equation (33) is second order and, as expected, this asymptotic behavior contains two arbitrary constants.

On the other hand, the correction to the $+\sqrt{-x}$ behavior has an exponential form

$$y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp\left[\pm\tfrac{4}{5}\sqrt{2}(-x)^{5/4}\right] \quad (x \to -\infty). \tag{36}$$

Thus, if $c_+ \neq 0$, nearby solutions generally veer away from the curve $\sqrt{-x}$ as $x \to -\infty$. The special solutions that decay exponentially towards the curve $\sqrt{-x}$ form a one-parameter and not a two-parameter class because $c_+ = 0$. The vanishing of the parameter c_+ gives rise to an eigenvalue condition on the choice of initial value of $y'(0)$. For each value of $y(0)$ there is a set of eigencurves (separatrices). These curves correspond to a discrete set of initial slopes $y'(0)$.

We have performed a numerical study of the solutions to (33) that satisfy the initial conditions $y(0) = 1$ and $y'(0) = a$. There is a discrete set of eigencurves whose initial positive slopes are $a_1 = 0.231955$, $a_2 = 3.980669$, $a_3 = 6.257998$, $a_4 = 8.075911$, $a_5 = 9.654843$, $a_6 = 11.078201$, $a_7 = 12.389217$, $a_8 = 13.613878$, $a_9 = 14.769304$, $a_{10} = 15.867511$, $a_{11} = 16.917331$, $a_{12} = 17.925488$. (There is also an infinite discrete set of *negative* eigenvalues.) The first two of these curves are shown in the left panel and the next two of these curves are shown in the right panel of Fig. 6. Note that the separatrix curves do not just exhibit n maxima as do the dashed curves in Fig. 2. Rather, these curves pass through increasingly many double poles. The curve corresponding to a_1 approaches $+\sqrt{-x}$ from above and the curve corresponding to a_2 approaches $+\sqrt{-x}$ from below. The curves corresponding to a_3 and a_4 also approach $+\sqrt{-x}$ from above and below, but these curves both pass through one double pole. Similarly, the curves corresponding to a_5 and a_6 pass through two double poles, and the curves corresponding to a_{2n-1} and a_{2n} pass through n double poles. The key feature of these separatrix curves is that after passing through n double poles, they approach the curve $+\sqrt{-x}$ exponentially fast as $x \to -\infty$. If the value of $y'(0)$ lies in between two eigenvalues, the curve either oscillates about and approaches the stable asymptotic curve $-\sqrt{-x}$ as in the left panel of Fig. 7 or else it lies above the unstable asymptotic curve $+\sqrt{-x}$ and passes through an infinite number of double poles as in the right panel of Fig. 7.

We have used Richardson extrapolation [8] to find the behavior of the numbers a_n for large n, and we obtain a result very similar in structure to that in (9). Specifically, we find that

$$a_n \sim Cn^{3/5} \quad (n \to \infty), \tag{37}$$

where $C = 4.28373$. This number is very close to $\frac{17}{5}2^{1/3}$. The constant C appears to be universal in that it is seems to be the same for all values of $y(0)$. We are currently trying to apply our analytical asymptotic methods to this problem to find an analytic calculation for the number C.

B. Conjectural connection with the power-series constant

In conclusion, we point out a possible connection between this work and the power-series constant P in the theory of complex variables. (The quest to find the value of the power series constant was originated by Hayman [9].) Let $\mathcal{F}$ be the class of functions $f(z)$ that are analytic for $|z| < 1$

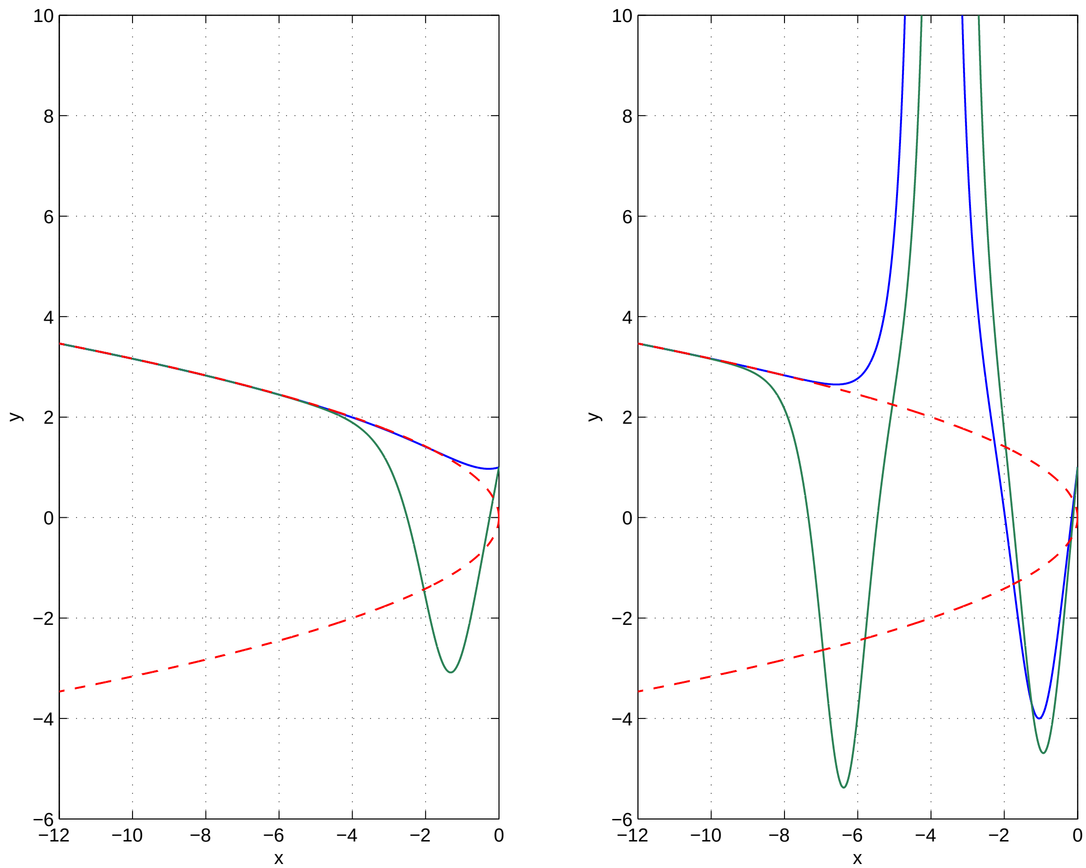

FIG. 6: Eigencurve solutions to the first Painlevé trenscendent. The eigencurves pass through $y(0) = 1$ and the slopes of the curves at $x = 0$ are the eigenvalues a_n. As $x \to -\infty$, the eigencurves approach $+\sqrt{-x}$ exponentially rapidly. Left panel: first two eigencurves corresponding to the eigenvalues $a_1 = 0.231955$ and $a_2 = 3.980669$. The a_1 curve approaches $+\sqrt{-x}$ from above and the a_2 curve approaches $+\sqrt{-x}$ from below. Right panel: The second two eigencurves for the Painlevé transcendent corresponding to the eigenvalues $a_3 = 6.257998$ and $a_4 = 8.075911$. Note that the second pair of eigenvalues passes through one double pole before approaching the curve $+\sqrt{-x}$.

but not analytic for $|z| < r$ for $r > 1$. If $f \in \mathcal{F}$, its power-series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

has a radius of convergence of 1. We denote $\rho_n(f)$ as the largest number r such that the partial sum of $f(z)$,

$$S_n(z) = \sum_{k=0}^{n} a_k z^k, \qquad (38)$$

has at least one zero on $|z| = r$. We define

$$\rho(f) \equiv \lim_{n\to\infty} \inf \rho_n(f). \qquad (39)$$

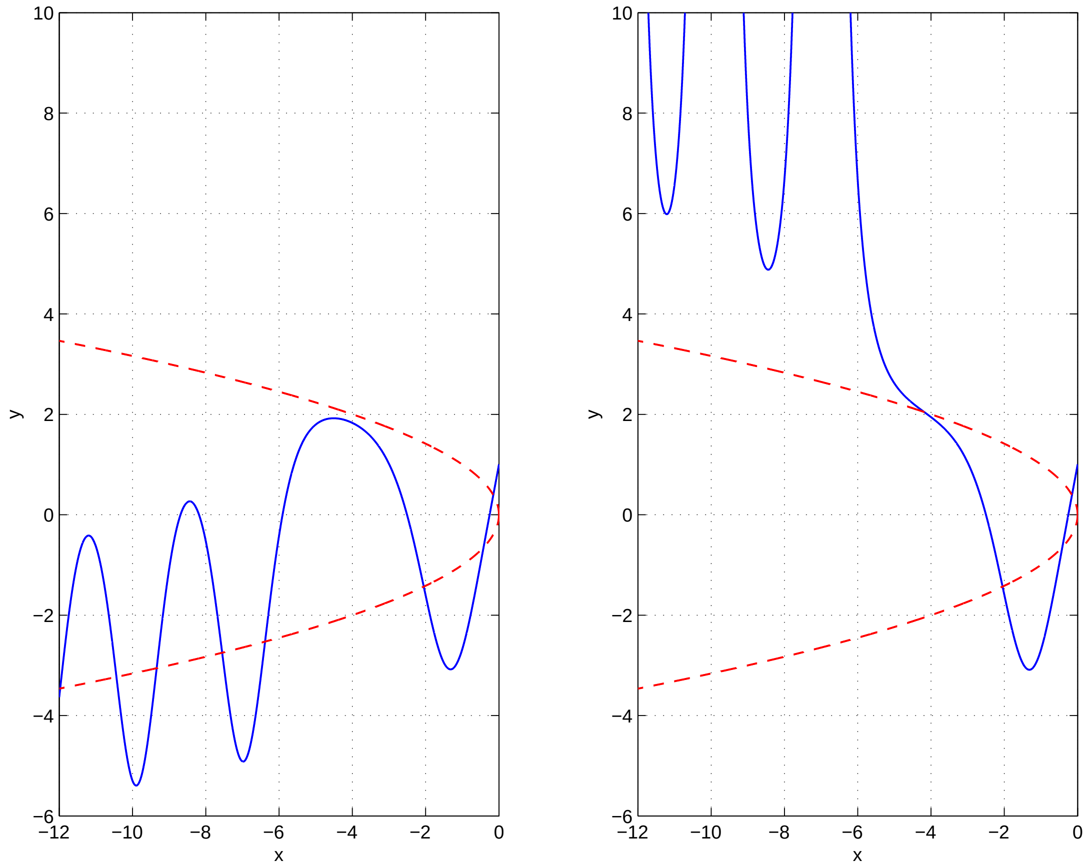

FIG. 7: Non-eigenvalue solutions to the first Painlevé transcendent. If $y(0) = 1$ but $y'(0)$ is not one of the eigenvalues a_n, the curve either oscillates about and approaches the stable asymptotic curve $-\sqrt{-x}$ as in the left panel or else it lies above the unstable asymptotic curve $+\sqrt{-x}$ and passes through an infinite number of double poles as in the right panel.

The power series constant P is then defined as

$$P \equiv \sup_{f \in \mathcal{F}} \rho(f). \tag{40}$$

The precise value for P is not known. However, lower and upper bounds on P have been established. The power series constant was known to lie in the interval $1 \leq P \leq 2$ until Clunie and Erdös [10] sharpened these bounds to $\sqrt{2} \leq P \leq 2$. Buckholtz [11] further sharpened these bounds to $1.7 \leq P \leq 12^{1/4}$, which was optimized by Frank [11] to

$$1.7818 \leq P \leq 1.82. \tag{41}$$

The latter values appear to be the best known values to date. It is astonishing that the value of A in (9) agrees with the best known lower bound for the value of P. We do not know whether our value $2^{5/6}$ coincides exactly with the lower bound. We leave this observation here as coincidence and hope to elaborate on the precise relation in a future paper [12].

As an illustration, let us compute $\rho(f)$ for some specific functions. In some rare cases we can sum the entire series and can therefore compute the value exactly. For instance, for the class of functions

$$f_\tau(z) = \sum_{k=0}^{\infty} \exp[i\pi\tau(k^2+k)]\ z^k \tag{42}$$

we compute

$$f_{1/4}(z) = \frac{1 + iz - iz^2 - z^3}{1 + z^4} \tag{43}$$

with zeros $z_1 = 1$ and $z_{2/3} = -1/2(1+i) \pm \sqrt{2i-4}$, such that $\rho(f_{1/4}) = |z_2| \approx 1.70002$. We can also compute a value closer to P:

$$f_{3/8}(z) = \frac{1 + e^{\frac{3i\pi}{4}} z + e^{\frac{i\pi}{4}} z^2 + iz^3 - iz^4 - e^{\frac{i\pi}{4}} z^5 - e^{\frac{3i\pi}{4}} z^6 - z^7}{1 + z^8}, \tag{44}$$

leading to $\rho\left(f_{3/8}\right) \approx 1.7804$. We must terminate the summation and evaluate $\rho_n(f)$ for a sufficiently large value of n. In Fig. 8 we display our numerical results for $\rho_{50}(f_\tau)$ obtained from the partial sum $S_{50}(z)$. The maximum values are $\rho_{50}(f_{0.3780}) = \rho_{50}(f_{0.8780}) \approx 1.7818$, which coincide with the best known lower bound for P up to the precision of the computation.

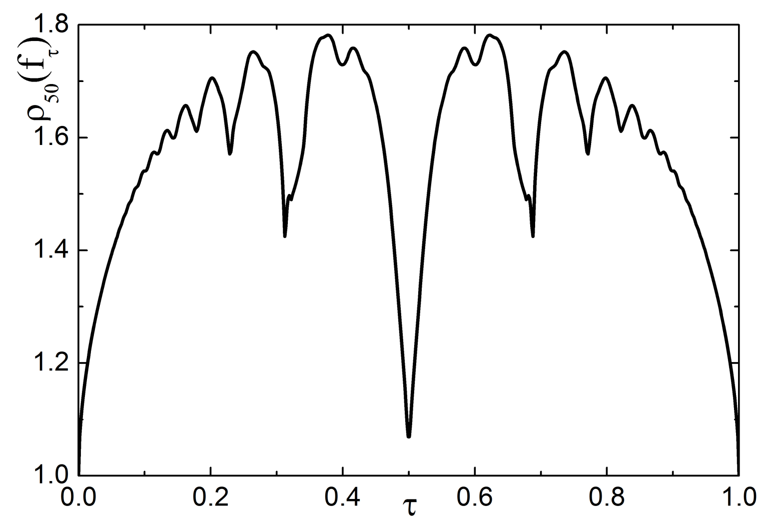

FIG. 8: A plot of $\rho_{50}(f_\tau)$ as a function of τ. Note that at the optimal value of the parameter τ, the maximum of the curve is very close to the value 1.7818.

C. Final comments

In this paper we have focused on separatrix behavior, which is a consequence of instabilities of nonlinear differential equations. We have interpreted separatrices as being eigenfunctions (eigencurves). The corresponding eigenvalues are the initial conditions needed to specify the separatrix curves. For the differential equation $y'(x) = \cos[\pi x y(x)]$, we have shown that the nth eigenvalue grows like $2^{1/3}\sqrt{2n}$ for large n. The number $2^{1/3}$ appears again in the numerical analysis of the first Painlevé transcendent. Moreover, to the currently known precision, the number $2^{5/6}$ appears in another asymptotic context, namely, as the lower bound 1.7818 on the power series constant P. We conjecture that the number $2^{5/6}$ may even be the exact value of P.

We emphasize that in this paper we have been interested in the limit of large eigenvalues. For linear eigenvalue problems this limit is accessible by using WKB theory. In the case of the nonlinear eigenvalue problem solved in this paper the large-eigenvalue limit is accessible because the problem becomes *linear* in this limit; indeed, the large-eigenvalue separatrix curve was found by reducing the problem to a *linear* random walk problem, which can be solved exactly. The strategy of transforming a nonlinear problem to an equivalent linear problem is reminiscent of the Hopf-Cole substitution that reduces the nonlinear Burgers equation to the linear diffusion equation or the inverse-scattering analysis that reduces the nonlinear Korteweg-de Vries equation to a linear integral equation.

There is a plausible argument that the power series constant P is connected with the asymptotic behavior of eigenvalues: On one hand, P is associated with the zero of largest modulus of a polynomial of degree n, which is constructed as the nth partial sum of a Taylor series. On the other hand, a conventional linear eigenvalue problem of the form $H\psi = E\psi$ may be solved by introducing a basis and replacing the operator H by an $N \times N$ matrix H_N. Then, to calculate the eigenvalues numerically we find the zeros of the secular polynomial $\mathrm{Det}(H_N - IE)$. Finding the asymptotic behavior of the high-energy eigenvalues corresponds to finding the behavior of the largest zero of the secular polynomial as N, the degree of the polynomial, tends to infinity.

We believe that the techniques proposed here to find the asymptotic behavior of large eigenvalues may extend to other nonlinear differential equations exhibiting instabilities and separatrix behavior.

Acknowledgments

CMB and JK thank the U.S. Department of Energy for financial support.

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