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Quantum resolution of the nonlinear super-Schrödinger equation

V. Caudrelier * and E. Ragoucy[†]

Laboratoire de Physique Théorique LAPTH[‡]

LAPP, BP 110, F-74941 Annecy-le-Vieux Cedex, France.

Abstract

We introduce a \mathbb{Z}_2 -graded version of the nonlinear Schrödinger equation that includes one fermion and one boson at the same time. This equation is shown to possess a supersymmetry which proves to be itself part of a super-Yangian symmetry based on $gl(1|1)$. The solution exhibits a super version form of the classical Rosales solution. Then, we second quantize these results, and give a Lax pair formulation (based on $gl(2|1)$) for the model.

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*caudreli@lapp.in2p3.fr

[†]ragoucy@lapp.in2p3.fr

[‡]UMR 5108 du CNRS, associée à l'Université de Savoie.

1 Introduction

The nonlinear Schrödinger (NLS) equation (for a review, see e.g. [1]) is one of the most studied system in quantum integrable systems, and its simplest version played an important role in the development of the Quantum Inverse Scattering Method (QISM) [2]. It is known [3] that the quantum NLS model with spin $\frac{1}{2}$ fermions and repulsive interaction on the line has a Yangian symmetry $Y(sl(2))$. More generally, its vectorial version, based on N -component bosons or fermions, was shown to possess an $Y(gl(N))$ symmetry [4].

It was thus natural to seek a supersymmetric version of these models. Indeed, different versions of such a generalization were proposed, from the simple boson-fermion systems related to NLS [5, 6], or superfields formulation [7] of NLS, up to more algebraic studies of these models [8, 9]. The difficulty with such generalizations is to keep the fundamental notion of integrability while allowing for the existence of supersymmetry. Even when some of the suggested supersymmetric systems were shown to pass some integrability conditions [10], the status of such models remained not clearly established, and one is still looking for e.g. their Lax presentation.

Another \mathbb{Z}_2 -graded version of NLS has been introduced by Kulish [11]. The fields are matrix valued and only the finite interval was studied, using the Thermodynamical Bethe Ansatz (see also [12]).

The aim of this article is to present a vectorial version (close to the matricial version introduced by Kulish) of the NLS model on the infinite line which includes both a boson and a fermion field. It is integrable and admits a Lax presentation without using a superfield formalism. We will construct the classical and quantum solutions of the model under consideration, and exhibit a symmetry superalgebra containing fermionic operators which close on the impulsion operator. However, this supersymmetry algebra is different from the ones already proposed, and is actually embedded into a super-Yangian based on $gl(1|1)$.

The article is organized as follows: in section 2, we review basic results on the classical version of the NLS equation, and present the (classical) supersymmetric version we will deal with. Then, in section 3, we will define the formalism needed for the quantization of our model, and which appears to rely essentially on the notion of Zamolodchikov-Faddeev (ZF) algebras. In section 4, we show how to construct canonical quantum fields starting from the previously introduced ZF algebra as well as the quantum version of our model, and in section 5 we propose a Lax construction for our model.

2 Classical approach

We first review very briefly some of the standard results for the classical nonlinear Schrödinger (CNLS) equation and then develop a super-version to describe bosons and fermions at the same time.

2.1 The nonlinear Schrödinger equation and classical results

The CNLS equation

$$(i\partial_t + \partial_x^2) \Phi(x, t) = 2g|\Phi(x, t)|^2\Phi(x, t) \quad \text{with } g > 0 \quad (2.1)$$

is obtained via a Hamiltonian formalism as follows. We first need a Poisson bracket defined over the space of functionals $F(\Phi, \bar{\Phi})$ where the classical field $\bar{\Phi}$ is the conjugate of Φ and these two fields are regarded as independent.

If F and G are two such functionals, we define their Poisson brackets by

$$\{F, G\} = i \int_{-\infty}^{\infty} dx \left(\frac{\delta F}{\delta \Phi(x)} \frac{\delta G}{\delta \bar{\Phi}(x)} - \frac{\delta F}{\delta \bar{\Phi}(x)} \frac{\delta G}{\delta \Phi(x)} \right) \quad (2.2)$$

The time dependence is omitted until we explicitly need it.

This provides the usual canonical Poisson brackets for the basic functionals $\Phi(x)$ and $\bar{\Phi}(x)$:

$$\{\Phi(x), \Phi(y)\} = \{\bar{\Phi}(x), \bar{\Phi}(y)\} = 0, \quad \{\Phi(x), \bar{\Phi}(y)\} = i\delta(x - y) \quad (2.3)$$

Now, given a Hamiltonian $H(\Phi, \bar{\Phi})$, one gets the Hamiltonian equation of motion for a functional F :

$$\partial_t F = \{H, F\}$$

With $F = \Phi(x, t)$ and the (time-independent) CNLS Hamiltonian given by

$$H(\Phi, \bar{\Phi}) = \int_{-\infty}^{\infty} dx \left(\partial_x \bar{\Phi}(x) \partial_x \Phi(x) + g \bar{\Phi}^2(x) \Phi^2(x) \right) \quad (2.4)$$

one recovers the CNLS equation (2.1).

The important feature of the CNLS is that it is a completely integrable system and Rosales in [13] found an explicit solution of the form

$$\Phi(x, t) = \sum_{n=0}^{\infty} (-g)^n \Phi^{(n)}(x, t), \quad g > 0 \quad (2.5)$$

where

$$\Phi^{(n)}(x, t) = \int_{\mathbb{R}^{2n+1}} d^n \mathbf{p} d^{n+1} \mathbf{q} \bar{\lambda}(p_1) \dots \bar{\lambda}(p_n) \lambda(q_n) \dots \lambda(q_0) \frac{e^{i\Omega_n(x, t; \mathbf{p}, \mathbf{q})}}{Q_n(\mathbf{p}, \mathbf{q}, 0)} \quad (2.6)$$

$$\Omega_n(x, t; \mathbf{p}, \mathbf{q}) = \sum_{j=0}^n (q_j x - q_j^2 t) - \sum_{i=1}^n (p_i x - p_i^2 t) \quad (2.7)$$

$$Q_n(\mathbf{p}, \mathbf{q}, \varepsilon) = \prod_{i=1}^n (p_i - q_{i-1} + i\varepsilon)(p_i - q_i + i\varepsilon) \quad (2.8)$$

$$d^n \mathbf{p} d^{n+1} \mathbf{q} = \prod_{\substack{i=1 \\ j=0}}^n \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} \quad (2.9)$$

where we have denoted $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_0, \dots, q_n)$.

This solution is well-defined (*i.e.* the integral (2.6) exists and the series (2.5) converges) for a large class of functions containing at least the space of Schwartz test functions on \mathbb{R} , $\mathcal{S}(\mathbb{R})$ for all x as long as g is sufficiently small (see e.g. [14]).

As was noted in [15], the Rosales solution is of first importance since its structure is preserved upon quantization and directly provides the solution of the quantum nonlinear Schrödinger (QNLS) equation, formally replacing λ , $\bar{\lambda}$ by their quantized counterparts. We also remind the reader that λ and $\bar{\lambda}$ are related to the so-called *scattering data* of the inverse scattering method (see [16, 15, 17]). We shall see below that this situation extends to our \mathbb{Z}_2 -graded formalism.

2.2 Extension to the classical nonlinear super-Schrödinger equation and solution

In this section, we introduce a graded formalism allowing us to deal with a classical field containing one bosonic and one fermionic component. We define

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad x \in \mathbb{R} \quad (2.10)$$

which we rewrite as

$$\Phi(x) = \phi_i(x) e_i, \quad \text{where } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.11)$$

and summation is understood for repeated indices. ϕ_1 and ϕ_2 are the bosonic and fermionic components respectively. Similarly, we define

$$\boldsymbol{\lambda}(x) = \begin{pmatrix} \lambda_1(x) \\ \lambda_2(x) \end{pmatrix} = \lambda_i(x) e_i, \quad x \in \mathbb{R} \quad (2.12)$$

We shall also need adjoints of these quantities

$$\Phi^\dagger(x) = (\bar{\phi}_1(x), \bar{\phi}_2(x)) = \bar{\phi}_i(x) e_i^\dagger, \quad x \in \mathbb{R} \quad (2.13)$$

$$\boldsymbol{\lambda}^\dagger(x) = (\bar{\lambda}_1(x), \bar{\lambda}_2(x)) = \bar{\lambda}_i(x) e_i^\dagger, \quad x \in \mathbb{R} \quad (2.14)$$

with

$$e_1^\dagger = (1, 0) \quad \text{and} \quad e_2^\dagger = (0, 1) \quad (2.15)$$

Here and below, the vectors e_i , e_i^\dagger , and the matrices E_{ij} which enter into the formalism of auxiliary spaces are \mathbb{Z}_2 -graded:

$$[e_i] = [e_i^\dagger] = [i]; \quad [E_{ij}] = [i] + [j] \quad \text{with} \quad [1] = 0 \quad \text{and} \quad [2] = 1$$

Accordingly, the tensor product of auxiliary spaces will also be \mathbb{Z}_2 -graded, e.g.

$$(\mathbb{I} \otimes e_i)(E_{jk} \otimes \mathbb{I}) = (-1)^{[i]([j]+[k])} E_{jk} \otimes e_i$$

We will consider even objects in the following sense: $v = v_i e_i$ and $M = M_{ij} E_{ij}$ are even iff $[v_i] = [i]$ and $[M_{ij}] = [i] + [j]$. For example, the field Φ in (2.11) is even.

The bosonic or fermionic aspect of the components is then encoded by a graded commutation relation as follows: if we consider $\boldsymbol{\lambda}(x)$ with components $\lambda_i(x)$, we have

$$\lambda_i(x) \lambda_j(y) = (-1)^{[i][j]} \lambda_j(y) \lambda_i(x) \quad (2.16)$$

For $i = j = 2$, we recover the fermionic nature of our classical field and λ_2 is a Grassmann-valued function. This arises naturally when using a graded formalism in auxiliary spaces. If we consider $\boldsymbol{\lambda}_1(x)$ and $\boldsymbol{\lambda}_2(y)$ where, by definition,

$$\boldsymbol{\lambda}_1(x) = \boldsymbol{\lambda}(x) \otimes \mathbb{I} \quad \text{and} \quad \boldsymbol{\lambda}_2(x) = \mathbb{I} \otimes \boldsymbol{\lambda}(x) \quad (2.17)$$

the \mathbb{Z}_2 -graded commutativity (2.16) is gathered into:

$$\boldsymbol{\lambda}_1(x) \boldsymbol{\lambda}_2(y) = \boldsymbol{\lambda}_2(y) \boldsymbol{\lambda}_1(x) \quad (2.18)$$

This discussion extends to the various objects that we will use throughout this article and we will switch from the global fields to the components to emphasize its strength: formally, there is no difference between the classical results and our global formalism while all the novelties spring when translating into components.

Note also that, when dealing with tensor product of auxiliary spaces, one has to be careful not to confuse (even) objects like $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda} \otimes \mathbb{I} = \sum_{i=1}^2 \lambda_i e_i \otimes \mathbb{I}$ with their (\mathbb{Z}_2 -graded) components λ_i , $i = 1, 2$. For clarity, we will use boldface letters for the even objects, and ordinary letters for their components.

Our next task is to generalize the Hamiltonian formalism described in the previous section. We first introduce the usual \mathbb{Z}_2 -graded Poisson bracket over the space

of functionals $\mathcal{F}(\Phi, \Phi^\dagger)$. For two such functionals, their super-Poisson bracket is given by

$$\{\mathcal{F}, \mathcal{G}\} = i \sum_{\ell=1}^2 \int_{-\infty}^{\infty} dx (-1)^{[\mathcal{F}][\ell]} \left((-1)^{[\ell]} \frac{\delta \mathcal{F}}{\delta \phi_\ell(x)} \frac{\delta \mathcal{G}}{\delta \bar{\phi}_\ell(x)} - \frac{\delta \mathcal{F}}{\delta \bar{\phi}_\ell(x)} \frac{\delta \mathcal{G}}{\delta \phi_\ell(x)} \right) \quad (2.19)$$

This bracket is a graded Poisson bracket *i.e.* it has the following properties (proved by direct calculation):

- i) $\{\mathcal{F}, \mathcal{G}\}$ is bilinear.
- ii) $[\{\mathcal{F}, \mathcal{G}\}] = [\mathcal{F}] + [\mathcal{G}] \pmod{2}$.
- iii) $\{\mathcal{F}, \mathcal{G}\} = -(-1)^{[\mathcal{F}][\mathcal{G}]} \{\mathcal{G}, \mathcal{F}\}$: graded antisymmetry.
- iv) $\{\mathcal{F}, \mathcal{G}\mathcal{H}\} = \{\mathcal{F}, \mathcal{G}\}\mathcal{H} + (-1)^{[\mathcal{F}][\mathcal{G}]} \mathcal{G}\{\mathcal{F}, \mathcal{H}\}$: graded Leibniz rule.
- v) $\{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}\} = \{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + (-1)^{[\mathcal{F}][\mathcal{G}]} \{\mathcal{G}, \{\mathcal{F}, \mathcal{H}\}\}$: graded Jacobi identity.

One can also associate to the graded Poisson bracket, a “global” Poisson bracket for even functionals \mathbf{F} and \mathbf{G} . Their bracket is given by

$$\{\mathbf{F}_1, \mathbf{G}_2\} = \{F_i, G_j\}(e_i \otimes e_j) \quad (2.20)$$

Besides bilinearity, it has the following properties:

- i) $\{\mathbf{F}_1, \mathbf{G}_2\} = -\{\mathbf{G}_2, \mathbf{F}_1\}$: antisymmetry.
- ii) $\{\mathbf{F}_1, \mathbf{G}_2 \mathbf{H}_3\} = \{\mathbf{F}_1, \mathbf{G}_2\} \mathbf{H}_3 + \mathbf{G}_2 \{\mathbf{F}_1, \mathbf{H}_3\}$: Leibniz rule.
- iii) $\{\mathbf{F}_1, \{\mathbf{G}_2, \mathbf{H}_3\}\} + \{\mathbf{H}_3, \{\mathbf{F}_1, \mathbf{G}_2\}\} + \{\mathbf{G}_2, \{\mathbf{H}_3, \mathbf{F}_1\}\} = 0$: Jacobi identity.

The reader clearly realizes now that our formalism is totally transparent at the “global” level but nevertheless contains all the information about the various components encoded in the graded calculus on the auxiliary spaces. As we shall see, this entails the conservation of the form of the equations and solutions “globally” at the classical level as well as at the quantum level.

Finally, in order to apply our formalism to derive the classical nonlinear super-Schrödinger (CNLSS) equation, we need to introduce the “global” Kronecker symbol:

$$\delta_{12} = \delta^{ij}(e_i \otimes \mathbb{1})(\mathbb{1} \otimes e_j^\dagger) = (e_i \otimes e_i^\dagger) \quad (2.21)$$

and, accordingly

$$\delta_{21} = (-1)^{[i]}(e_i^\dagger \otimes e_i) \quad (2.22)$$

Using the expression (2.19), one immediately computes that the canonical Poisson brackets for the basic fields $\Phi(x)$, $\Phi^\dagger(y)$ with corresponding components $\phi_i(x)$, $\bar{\phi}_j(y)$ take the following form

$$\{\Phi_1(x), \Phi_2^\dagger(y)\} = i\delta_{12}\delta(x-y) \quad (\text{globally}) \quad (2.23)$$

$$\{\phi_j(x), \bar{\phi}_k(y)\} = i\delta_{jk}\delta(x-y) \quad (\text{in components}) \quad (2.24)$$

We now proceed with the derivation of the equation of motion, globally and in components, by introducing the generalization of the Hamiltonian (2.4):

$$H(\Phi, \Phi^\dagger) = \int_{-\infty}^{\infty} dx \left(\partial_x \Phi^\dagger(x) \partial_x \Phi(x) + g (|\Phi(x)|^2)^2 \right) \quad (2.25)$$

The field $\Phi(x, t)$ of components $\phi_i(x, t)$ satisfies the following Hamiltonian equation of motion which we call the Classical Nonlinear super-Schrödinger equation:

$$i\partial_t \Phi(x, t) = -\partial_x^2 \Phi(x, t) + 2g|\Phi(x, t)|^2 \Phi(x, t) \quad (\text{globally}) \quad (2.26)$$

$$i\partial_t \phi_j(x, t) = -\partial_x^2 \phi_j(x, t) + 2g(\bar{\phi}_k(x, t)\phi_k(x, t))\phi_j(x, t) \quad (\text{in components}) \quad (2.27)$$

These equations are simply derived from the Hamiltonian equations of motion

$$\partial_t \Phi(x, t) = \{H, \Phi(x, t)\}$$

using the Hamiltonian (2.25). We remind the reader of the component form for H :

$$H(\Phi, \Phi^\dagger) = \int_{-\infty}^{\infty} dx \left(\partial_x \bar{\phi}_i(x) \partial_x \phi_i(x) + g \bar{\phi}_j(x) \bar{\phi}_k(x) \phi_k(x) \phi_j(x) \right)$$

The important feature in the equations (2.26-2.27) is that they both are exactly the same as the usual one. This means that the solution *à la Rosales* (2.5), (2.6) is still valid in our case, as one can check explicitly.

2.3 Supersymmetry of the CNLSS

Although the present equation is not supersymmetric in the sense studied in [5]-[10], one can, owing to the presence of both bosons and fermions, construct fermionic operators which generate a supersymmetry in the following sense.

As a first step, we introduce the fermionic operator

$$\mathcal{Q} = \int_{\mathbb{R}} dx \left(\phi_1(x) \frac{\delta}{\delta \phi_2(x)} - \phi_2(x) \frac{\delta}{\delta \phi_1(x)} + \bar{\phi}_1(x) \frac{\delta}{\delta \bar{\phi}_2(x)} + \bar{\phi}_2(x) \frac{\delta}{\delta \bar{\phi}_1(x)} \right) \quad (2.28)$$

one can compute

$$\mathcal{Q} \phi_1(x) = -\phi_2(x) ; \quad \mathcal{Q} \phi_2(x) = \phi_1(x) ; \quad \mathcal{Q} \bar{\phi}_1(x) = \bar{\phi}_2(x) ; \quad \mathcal{Q} \bar{\phi}_2(x) = \bar{\phi}_1(x)$$

which shows that $\mathcal{Q}H = 0$.

The form of \mathcal{Q} implies that on functionals $F(\Phi, \bar{\Phi})$, one has $\mathcal{Q}^2 = \mathcal{N}$, where \mathcal{N} is the particle number operator

$$\mathcal{N} \phi_1(x) = -\phi_1(x) ; \quad \mathcal{N} \phi_2(x) = -\phi_2(x) ; \quad \mathcal{N} \bar{\phi}_1(x) = \bar{\phi}_1(x) ; \quad \mathcal{N} \bar{\phi}_2(x) = \bar{\phi}_2(x)$$

Note that using the PB, one gets for

$$Q = - \int_{\mathbb{R}} dx \left(\bar{\phi}_1(x) \phi_2(x) + \bar{\phi}_2(x) \phi_1(x) \right) = - \int_{\mathbb{R}} dx \bar{\Phi}(x) \sigma \Phi(x) \quad \text{with} \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the following identity for any functional: $\mathcal{Q}F(\Phi, \bar{\Phi}) = i\{Q, F(\Phi, \bar{\Phi})\}$. It is then easy to see that

$$\{Q, H\} = 0$$

One also has

$$\{Q, Q\} = -2iN \quad \text{with} \quad N = - \int_{\mathbb{R}} dx \left(\bar{\phi}_1(x) \phi_1(x) + \bar{\phi}_2(x) \phi_2(x) \right) = - \int_{\mathbb{R}} dx \bar{\Phi}(x) \Phi(x)$$

N satisfies $\mathcal{N}F(\Phi, \bar{\Phi}) = i\{N, F(\Phi, \bar{\Phi})\}$ as well as

$$\{N, H\} = 0$$

At the end of the first step, we get two functionals Q and N which Poisson-commute with the Hamiltonian H . Although it is fermionic, Q is not a supersymmetry generator because it does not close on the impulsion P . However, one can make a second step in the construction, introducing a second fermionic functional $Q_{(2)}$, given by

$$Q_{(2)} = i \int_{\mathbb{R}} dx \bar{\Phi}(x) \sigma \partial_x \Phi(x) + \frac{ig}{2} \int_{\mathbb{R}} dx dy sg(y-x) \left(\bar{\Phi}(x) \sigma \Phi(y) \right) \left(\bar{\Phi}(y) \Phi(x) \right)$$

where $sg(x)$ is the sign function. This additional functional satisfies

$$\{Q_{(2)}, N\} = 0 \quad \{Q_{(2)}, H\} = 0$$

together with

$$\{Q_{(2)}, Q\} = -2iP$$

where P is associated to the impulsion operator. It is given by

$$P = i \int_{\mathbb{R}} dx \left(\bar{\phi}_1(x) \partial_x \phi_1(x) + \bar{\phi}_2(x) \partial_x \phi_2(x) \right) = i \int_{\mathbb{R}} dx \bar{\Phi}(x) \partial_x \Phi(x)$$

and acts as

$$\begin{aligned} \mathcal{P} \phi_1(x) &= \partial_x \phi_1(x) ; & \mathcal{P} \phi_2(x) &= \partial_x \phi_2(x) \\ \mathcal{P} \bar{\phi}_1(x) &= \partial_x \bar{\phi}_1(x) ; & \mathcal{P} \bar{\phi}_2(x) &= \partial_x \bar{\phi}_2(x) \end{aligned}$$

with as above $\mathcal{P}F(\Phi, \bar{\Phi}) = \{P, F(\Phi, \bar{\Phi})\}$. Again, one can define the operator $\mathcal{Q}_{(2)}F(\Phi, \bar{\Phi}) = \{Q_{(2)}, F(\Phi, \bar{\Phi})\}$, and, at the end of the second step, we get a new fermionic operator such that

$$\mathcal{Q} \mathcal{Q}_{(2)} + \mathcal{Q}_{(2)} \mathcal{Q} = \mathcal{P} \quad (2.29)$$

In that sense, one can say that we have a supersymmetry algebra which is symmetry of our model (since it commutes with H). In fact, one can compute the remaining PB:

$$\{P, Q\} = 0 \quad \{P, Q_{(2)}\} = 0 \quad \{P, N\} = 0$$

The above PB are also valid, as (anti-)commutators, for the corresponding operators \mathcal{N} , \mathcal{Q} , ... We have chosen the normalisation in such a way that

$$\bar{N} = N ; \quad \bar{P} = P ; \quad \bar{H} = H ; \quad \bar{Q} = Q ; \quad \bar{Q}_{(2)} = Q_{(2)}$$

Note that N , P and H are central in the above algebra.

It is clear that one can repeat this procedure as much as needed, with, at each step, a new fermionic generator $Q_{(n)}$ and a new (central) bosonic operator. In such a way, one produces an infinite dimensional superalgebra which is a symmetry of the CNLSS and generates supersymmetry in the sense mentioned above. This superalgebra is related to the super-Yangian $Y(gl(1|1))$ (see section 5).

Let us also remark that similar towers of supersymmetry operators have been constructed in [6]. However, the underlying algebras are different, as can be seen by looking for instance at the scaling dimension of the operator content: indeed, in [6], the scaling dimension of the bosonic and fermionic fields are respectively 1 and $\frac{1}{2}$, while here they both have dimension 1. Consequently, the operators $\mathcal{Q}_{(n)}$ have dimension $n - 1$, $n \in \mathbb{Z}_+$, while they have dimension $n + \frac{1}{2}$ in [6].

3 ZF algebra and super-formalism

3.1 Graded ZF algebra

We start from the ZF algebra [18] and write a graded version using auxiliary spaces and entities containing one bosonic and one fermionic component which will be identified as the quantum versions of λ , λ^\dagger . With the same notations as before these entities read

$$\mathbf{A}(k) = \begin{pmatrix} a_1(k) \\ a_2(k) \end{pmatrix} = a_i(k)e_i \quad \text{and} \quad \mathbf{A}^\dagger(k) = (a_1^\dagger(k), a_2^\dagger(k)) = a_i^\dagger(k)e_i^\dagger \quad (3.1)$$

Definition 3.1 *The graded ZF algebra reads*

$$\mathbf{A}_1(k_1)\mathbf{A}_2(k_2) = R_{21}(k_2 - k_1)\mathbf{A}_2(k_2)\mathbf{A}_1(k_1) \quad (3.2)$$

$$\mathbf{A}_1^\dagger(k_1)\mathbf{A}_2^\dagger(k_2) = \mathbf{A}_2^\dagger(k_2)\mathbf{A}_1^\dagger(k_1)R_{21}(k_2 - k_1) \quad (3.3)$$

$$\mathbf{A}_1(k_1)\mathbf{A}_2^\dagger(k_2) = \mathbf{A}_2^\dagger(k_2)R_{12}(k_1 - k_2)\mathbf{A}_1(k_1) + \delta_{12}\delta(k_1 - k_2) \quad (3.4)$$

where

$$\mathbf{A}_1(k) = \mathbf{A}(k) \otimes \mathbb{I} \quad \text{and} \quad \mathbf{A}_2(k) = \mathbb{I} \otimes \mathbf{A}(k)$$

and

$$R_{12}(u) = \frac{u\mathbb{I} \otimes \mathbb{I} - igP_{12}}{u + ig}$$

is the R -matrix for the super-Yangian $Y(gl(1|1)) \equiv Y(1|1)$.

$R_{21}(x) = P_{12} R_{12}(x) P_{12}$, and P_{12} is the super-permutation operator:

$$P_{12} = \sum_{i,j=1}^2 (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (3.5)$$

Note that for even vectors u, v and even matrices B, C (as defined in section 2.2), one has $P_{12}(u \otimes v) = v \otimes u$ and $P_{12}(B \otimes C) P_{12} = C \otimes B$.

The R -matrix has the following useful properties

$$R_{12}(p_1 - p_2) R_{21}(p_2 - p_1) = \mathbb{I} \otimes \mathbb{I} \quad (3.6)$$

$$R_{12}^\dagger(p_1 - p_2) = R_{21}(p_2 - p_1) \quad (3.7)$$

In terms of components, we shall see below that this graded algebra contains both commutation and anticommutation relations for the bosonic and fermionic oscillators $a_1(k)$, $a_1^\dagger(k)$ and $a_2(k)$, $a_2^\dagger(k)$ respectively.

For quantities of definite \mathbb{Z}_2 -grade, we define their super-commutator by

$$[[B, C]] = BC - (-1)^{|B||C|} CB \quad (3.8)$$

Then, the component version of the ZF algebra reads ($j, k = 1, 2$):

$$[[a_j(k_1), a_k(k_2)]] = \frac{-ig}{k_2 - k_1 + ig} \left(a_j(k_2) a_k(k_1) + (-1)^{[j][k]} a_k(k_2) a_j(k_1) \right) \quad (3.9)$$

$$[[a_j^\dagger(k_1), a_k^\dagger(k_2)]] = \frac{-ig}{k_2 - k_1 + ig} \left(a_j^\dagger(k_2) a_k^\dagger(k_1) + (-1)^{[j][k]} a_k^\dagger(k_2) a_j^\dagger(k_1) \right) \quad (3.10)$$

$$[[a_j(k_1), a_k^\dagger(k_2)]] = \frac{-ig}{k_1 - k_2 + ig} \left((-1)^{[j][k]} a_k^\dagger(k_2) a_j(k_1) + \delta_{jk} \sum_{\ell=1}^2 a_\ell^\dagger(k_2) a_\ell(k_1) \right) + \delta_{jk} \delta(k_1 - k_2) \quad (3.11)$$

3.2 Fock representation

The previous algebra can be represented on a Fock space, which is most useful for our quantization of CNLSS, and we follow here the basic ideas developed in e.g. [19] and [14]. This Fock space \mathcal{F}_R has the following properties

1. $\mathcal{F}_R = \bigoplus_{n=0}^{\infty} \mathcal{H}_R^n$ where $\mathcal{H}_R^0 = \mathbb{C}$, $\mathcal{H}_R^1 = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \equiv 2L^2(\mathbb{R})$, *i.e.*

$$\mathcal{H}_R^1 = \{ \boldsymbol{\varphi}(p) = \varphi_j(p) e_j \text{ s.t. } \varphi_j \in L^2(\mathbb{R}), j = 1, 2 \}$$

and for $n \geq 2$, $\mathcal{H}_R^n \subset 2^n L^2(\mathbb{R}^n) \equiv \underbrace{L^2(\mathbb{R}^n) \oplus \dots \oplus L^2(\mathbb{R}^n)}_{2^n}$ is given by:

$$\begin{aligned} \mathcal{H}_R^n = \{ & \boldsymbol{\varphi}_{1\dots n}(p_1, \dots, p_n) = \sum_{i_1, \dots, i_n=1}^2 \varphi_{i_1, \dots, i_n}(p_1, \dots, p_n) (e_{i_1} \otimes \dots \otimes e_{i_n}) \\ & \text{s.t. } \varphi_{i_1, \dots, i_n} \in L^2(\mathbb{R}^n), i_1, \dots, i_n = 1, 2 \text{ and} \\ & \boldsymbol{\varphi}_{1\dots i, i+1\dots n}(p_1, \dots, p_i, p_{i+1}, \dots, p_n) = R_{i, i+1}(p_i - p_{i+1}) \boldsymbol{\varphi}_{1\dots i+1, i\dots n}(p_1, \dots, p_{i+1}, p_i, \dots, p_n) \} \end{aligned}$$

2. The generators $\mathbf{A}(k)$, $\mathbf{A}^\dagger(k)$ are operator-valued distributions acting on a common domain \mathcal{D} dense in \mathcal{F}_R .
3. There exists a (vacuum) vector $\Omega \in \mathcal{D}$ which is cyclic with respect to $\mathbf{A}^\dagger(k)$ and annihilated by $\mathbf{A}(k)$.
4. The scalar product which we define below on \mathcal{H}_R^n provides the usual L^2 topology and \mathcal{F}_R is the completed vector space over \mathbb{C} for this topology: \mathcal{F}_R is a Hilbert space. This last point will be most useful since we will first regard our operators as bilinear forms on \mathcal{F}_R and deduce their properties using the non-degeneracy of the scalar product.

The sesquilinear form $\langle \cdot, \cdot \rangle$ defined on $\mathcal{H}_R^n \times \mathcal{H}_R^n$, $n \geq 1$ by

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle = \int_{\mathbb{R}^n} d^n p \boldsymbol{\varphi}_{1\dots n}^\dagger(p_1, \dots, p_n) \boldsymbol{\psi}_{1\dots n}(p_1, \dots, p_n) \quad (3.12)$$

$$\boldsymbol{\varphi}_{1\dots n}^\dagger(p_1, \dots, p_n) = (-1)^{\sum_{k=1}^{n-1} ([i_1] + \dots + [i_k])[i_{k+1}]} \bar{\varphi}^{i_1 \dots i_n} (e_{i_1}^\dagger \otimes e_{i_2}^\dagger \otimes \dots \otimes e_{i_n}^\dagger) \quad (3.13)$$

is a (hermitian) scalar product.

Indeed, from the identity $\boldsymbol{\varphi}_{1\dots n}^\dagger \boldsymbol{\psi}_{1\dots n} = \bar{\varphi}^{i_1 \dots i_n} \psi_{i_1 \dots i_n}$ one realizes that (3.12) is nothing but the usual L^2 -scalar product restricted to \mathcal{H}_R^n .

Let $\mathcal{F}_R^0 \subset \mathcal{F}_R$ be the finite particle space spanned by the sequences $(\varphi, \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_{1\dots n}, \dots)$ with $\boldsymbol{\varphi}_{1\dots n} \in \mathcal{H}_R^n$ and $\boldsymbol{\varphi}_{1\dots n} = 0$ for n large enough. As (3.12) is defined for all n , it extends naturally to \mathcal{F}_R^0 . In this context, the vacuum state is $\Omega = (1, 0, \dots, 0, \dots)$, so that it is normalized to 1.

We are now able to define the action of the creation and annihilation operators $\{A(\mathbf{f}), A^\dagger(\mathbf{f}) \text{ for } \mathbf{f} \in \mathcal{H}_R^1\}$ on \mathcal{F}_R^0 through their action on each \mathcal{H}_R^n :

$$A(\mathbf{f})\Omega = 0$$

$$A(\mathbf{f}) : \begin{cases} \mathcal{H}_R^{n+1} & \rightarrow \mathcal{H}_R^n \\ \varphi_{0\dots n} & \mapsto [A(\mathbf{f})\varphi]_{1\dots n} \end{cases}$$

$$\text{with } [A(\mathbf{f})\varphi]_{1\dots n}(p_1, \dots, p_n) = \sqrt{n+1} \int_{\mathbb{R}} dp_0 \mathbf{f}_0^\dagger(p_0) \varphi_{0\dots n}(p_0, p_1, \dots, p_n) \quad (3.14)$$

$$A^\dagger(\mathbf{f}) : \begin{cases} \mathcal{H}_R^n & \rightarrow \mathcal{H}_R^{n+1} \\ \varphi_{1\dots n} & \mapsto [A^\dagger(\mathbf{f})\varphi]_{0\dots n} \end{cases}$$

$$\text{with } [A^\dagger(\mathbf{f})\varphi]_{0\dots n}(p_0, \dots, p_n) = \frac{1}{\sqrt{n+1}} \varphi_{1\dots n}(p_1, \dots, p_n) f_0(p_0) \quad (3.15)$$

$$+ \frac{1}{\sqrt{n+1}} \sum_{k=1}^n R_{k-1,k}(p_{k-1} - p_k) \dots R_{0k}(p_0 - p_k) \varphi_{0\dots\hat{k}\dots n}(p_0, \dots, \hat{p}_k, \dots, p_n) \mathbf{f}_k(p_k)$$

where the hatted symbols are omitted.

It is easily checked that (3.14) and (3.15) are indeed elements of \mathcal{H}_R^n and \mathcal{H}_R^{n+1} respectively. Therefore, we have operators acting on \mathcal{F}_R^0 (linearity in φ obvious) with the additional property that they are bounded (*i.e.* continuous) on each finite particle sector \mathcal{H}_R^n with the estimates

$$\forall \varphi \in \mathcal{H}_R^n, \quad \|A(\mathbf{f})\varphi\| \leq \sqrt{n} \|\mathbf{f}\| \|\varphi\|, \quad \|A^\dagger(\mathbf{f})\varphi\| \leq \sqrt{n+1} \|\mathbf{f}\| \|\varphi\| \quad (3.16)$$

where $\|\cdot\|$ is the norm associated to the scalar product (3.12). Another essential feature is the adjointness of these operators with respect to $\langle \cdot, \cdot \rangle$

$$\forall \varphi \in \mathcal{H}_R^n, \quad \forall \psi \in \mathcal{H}_R^{n+1}, \quad \forall \mathbf{f} \in \mathcal{H}_R^1, \quad \langle \varphi, A(\mathbf{f})\psi \rangle = \langle A^\dagger(\mathbf{f})\varphi, \psi \rangle \quad (3.17)$$

At this stage, the Fock representations $\mathbf{A}(p)$, $\mathbf{A}^\dagger(p)$ of the generators of the ZF algebra appear as operator-valued distributions through the definition

$$A(\mathbf{f}) = \int_{\mathbb{R}} dp \mathbf{f}^\dagger(p) \mathbf{A}(p), \quad A^\dagger(\mathbf{f}) = \int_{\mathbb{R}} dp \mathbf{A}^\dagger(p) \mathbf{f}(p) \quad (3.18)$$

where \mathbf{f} is from now on restricted to live in the space of Schwartz test functions $2\mathcal{S}(\mathbb{R}) \equiv \mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R}) \subset \mathcal{H}_R^1$. It is readily shown from these definitions that $\mathbf{A}(p)$ and $\mathbf{A}^\dagger(p)$ satisfy the exchange relations (3.2-3.4) thus providing the desired representation. The explicit action in this representation reads

$$\mathbf{A}_0(p_0)\Omega = 0$$

$$\forall \varphi \in \mathcal{H}_R^n, \quad [\mathbf{A}_1(p_1)\varphi]_{2\dots n}(p_2, \dots, p_n) = \sqrt{n} \varphi_{12\dots n}(p_1, \dots, p_n)$$

$$\begin{aligned} \forall \varphi \in \mathcal{H}_R^{n-1}, \quad [\mathbf{A}_{n+1}^\dagger(p_{n+1})\varphi]_{1\dots n}(p_1, \dots, p_n) &= \frac{1}{\sqrt{n}} \varphi_{2\dots n}(p_2, \dots, p_n) \delta(p_1 - p_{n+1}) \boldsymbol{\delta}_{1,n+1} \\ + \frac{1}{\sqrt{n}} \sum_{k=2}^n R_{k-1,k}(p_{k-1} - p_k) \dots R_{1k}(p_1 - p_k) \varphi_{1\dots\hat{k}\dots n}(p_1, \dots, \hat{p}_k, \dots, p_n) &\delta(p_k - p_{n+1}) \boldsymbol{\delta}_{k,n+1} \end{aligned}$$

so that $\left[\mathbf{A}_2^\dagger(p_2)\Omega \right]_1(p_1) = \delta(p_1 - p_2)\boldsymbol{\delta}_{12}$. One can notice that, while \mathcal{F}_R^0 is stable under $\mathbf{A}(p)$, $\mathbf{A}^\dagger(p)$ takes φ out of \mathcal{F}_R^0 because of the appearance of a δ -function.

It remains to show that Ω is cyclic with respect to $\mathbf{A}^\dagger(p)$ *i.e.*

$$\forall \varphi \in \mathcal{H}_R^n, n \geq 1, \left(\forall p_i, i = 1, \dots, n, \langle \varphi, \mathbf{A}_1^\dagger(p_1)\dots\mathbf{A}_n^\dagger(p_n)\Omega \rangle = 0 \right) \Rightarrow \varphi = 0 \quad (3.19)$$

We want to emphasize that, strictly speaking, $\langle \cdot, \cdot \rangle$ is not defined in (3.19) since $\mathbf{A}_1^\dagger(p_1)\dots\mathbf{A}_n^\dagger(p_n)\Omega$ is not in \mathcal{F}_R^0 . However, maintaining the definition for $\langle \cdot, \cdot \rangle$, one easily computes

$$\langle \varphi, \mathbf{A}_1^\dagger(p_1)\dots\mathbf{A}_n^\dagger(p_n)\Omega \rangle = \sqrt{n!} \varphi_{1\dots n}^\dagger(p_1, \dots, p_n)$$

which shows the result. We just note that when evaluating $\langle \cdot, \cdot \rangle$ on $\mathbf{A}_1^\dagger(p_1)\dots\mathbf{A}_n^\dagger(p_n)\Omega$, it is no longer a *scalar* product but it produces an element of \mathcal{F}_R^0 . Bearing that in mind, we will indifferently use both concepts in what follows.

We now have all the ingredients to deduce results for the whole Fock space \mathcal{F}_R while working on smaller and more intuitive spaces dense in \mathcal{F}_R , using the continuity of the operators. Keeping that in mind, it is interesting to introduce the equivalent of a state space, a basis of which is usually denoted by $|k_1, \dots, k_n\rangle = a^\dagger(k_1)\dots a^\dagger(k_n)|0\rangle$, $k_1 > \dots > k_n$. In our case, this is not directly obtained since $\mathbf{A}^\dagger(k)\Omega$ is not an element of \mathcal{H}_R^1 (it contains a δ -function) and one has to define such a state space $\mathcal{D} \subset \mathcal{F}_R$ *in the sense of distributions* as follows

$$\mathcal{D}^0 = \mathbb{C}, \quad (3.20)$$

$$\mathcal{D}^n = \left\{ \int_{\mathbb{R}^n} d^n p \mathbf{A}_1^\dagger(p_1)\dots\mathbf{A}_n^\dagger(p_n)\Omega \mathbf{f}(p_1, \dots, p_n); \mathbf{f} \in 2^n \mathcal{S}(\mathbb{R}^n), n \geq 1 \right\} \quad (3.21)$$

and \mathcal{D} is spanned by the sequences $\boldsymbol{\chi} = (\chi, \boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_{1\dots n}, \dots)$, where $\boldsymbol{\chi}_{1\dots n} \in \mathcal{D}^n$ and $\boldsymbol{\chi}_{1\dots n} = 0$ for n large enough.

We can go further in the analogy with the state space by restricting \mathbf{f} in (3.21) to be of the form

$$\mathbf{f}_{1\dots n}(p_1, \dots, p_n) = \mathbf{f}_1(p_1) \otimes \mathbf{f}_2(p_2) \otimes \dots \otimes \mathbf{f}_n(p_n), \quad \mathbf{f}_i \in 2\mathcal{S}(\mathbb{R}), i = 1, \dots, n \quad (3.22)$$

Anticipating the next section, we define therefore

$$\mathcal{D}_0^0 = \mathbb{C}, \quad \mathcal{D}_0^n = \left\{ \tilde{\mathbf{A}}_1^\dagger(\mathbf{f}_1, t)\dots\tilde{\mathbf{A}}_n^\dagger(\mathbf{f}_n, t)\Omega, \mathbf{f}_1 \succ \dots \succ \mathbf{f}_n \right\} \subset \mathcal{H}_R^n, n \geq 1 \quad (3.23)$$

where

$$\tilde{\mathbf{A}}^\dagger(\mathbf{f}, t) = \int_{\mathbb{R}} dx \tilde{\mathbf{A}}^\dagger(x, t)\mathbf{f}(x), \quad \mathbf{f} \in 2\mathcal{S}(\mathbb{R}) \quad (3.24)$$

$$\tilde{\mathbf{A}}^\dagger(x, t) = \int_{\mathbb{R}} dp \mathbf{A}^\dagger(p)e^{ipx - ip^2 t}, \quad x, t \in \mathbb{R} \quad (3.25)$$

and the space \mathcal{D}_0 is the linear span of sequences $\boldsymbol{\chi} = (\chi, \boldsymbol{\chi}_1, \dots, \boldsymbol{\chi}_{1\dots n}, \dots)$, where $\boldsymbol{\chi}_{1\dots n} \in \mathcal{D}_0^n$ and $\boldsymbol{\chi}_{1\dots n} = 0$ for n large enough. We also introduced the following partial ordering relation on $2\mathcal{S}(\mathbb{R})$

$$\mathbf{f} \succ \mathbf{g} \Leftrightarrow \forall i, j = 1, 2, \forall x \in \text{supp}(f_i), \forall y \in \text{supp}(g_j), x > y$$

which is just the extension of the ordering of the momenta k_i in the definition of a state space basis $|k_1, \dots, k_n\rangle$.

Then, one shows that \mathcal{D} and \mathcal{D}_0 are dense in \mathcal{F}_R (see the line of argument given in [14]).

Summarizing, we have constructed a graded ZF algebra and its Fock representation \mathcal{F}_R and, inspired by earlier works [16, 17, 15, 20], we shall see that this allows to construct the quantum version of CNLSS and its solution.

4 Quantizing CNLSS

4.1 Quantization of the fields

Following [15] and [20], we simply write the quantum version of $\phi_j^{(n)}(x, t)$ as

$$\begin{aligned} \phi_j^{(n)}(x, t) &= \int_{\mathbb{R}^{2n+1}} d^n \mathbf{p} d^{n+1} \mathbf{q} \sum_{k_1, \dots, k_n=1}^2 a_{k_1}^\dagger(p_1) \dots a_{k_n}^\dagger(p_n) a_{k_n}(q_n) \dots a_{k_1}(q_1) a_j(q_0) \\ &\quad \times \frac{e^{i\Omega_n(x, t; \mathbf{p}, \mathbf{q})}}{Q_n(\mathbf{p}, \mathbf{q}, \varepsilon)} \end{aligned} \quad (4.1)$$

using the same notations as in (2.6) and an $i\epsilon$ contour prescription. And then the global field reads

$$\Phi(x, t) = \sum_{n=0}^{\infty} (-g)^n \Phi^{(n)}(x, t) \quad \text{with} \quad \Phi^{(n)}(x, t) = \phi_j^{(n)}(x, t) e_j \quad (4.2)$$

As such, we know that $\Phi(x, t)$ is ill-defined because of the nature of $\mathbf{A}(p)$, $\mathbf{A}^\dagger(p)$ but this is easily cured by regarding $\Phi(x, t)$ as bilinear form on \mathcal{D} . Actually, for the rest of this section, we follow the constructions given in [15, 20], and implemented later in [14] (in a different context): we refer to these articles for detailed proofs. Our aim is to define properly the fields $\Phi(x, t)$ and $\Phi^\dagger(x, t)$ and to show that they are canonical fields for the quantum theory satisfying the canonical commutation relations (CCR).

Let $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{D}$, then the function $(x, t) \mapsto \langle \boldsymbol{\varphi}, \Phi^{(n)}(x, t) \boldsymbol{\psi} \rangle$ is C^∞ for all n .

Therefore, $\Phi(x, t)$ is also a bilinear form on \mathcal{D} smooth in (x, t) (since \mathcal{D} contains only finite particle vectors, the sum in (4.2) is actually finite). And the same holds for $\Phi^\dagger(x, t)$ defined by

$$\forall \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{D}, \quad \langle \boldsymbol{\varphi}, \Phi^\dagger(x, t) \boldsymbol{\psi} \rangle = \overline{\langle \boldsymbol{\psi}, \Phi(x, t) \boldsymbol{\varphi} \rangle} \quad (4.3)$$

From (3.17), we deduce

$$\Phi^\dagger(x, t) = \sum_{n=0}^{\infty} (-g)^n \Phi^{\dagger(n)}(x, t) \quad (4.4)$$

$$\begin{aligned} \text{with } \Phi^{\dagger(n)}(x, t) &= \int_{\mathbb{R}^{2n+1}} d^n \mathbf{p} d^{n+1} \mathbf{q} \mathbf{A}^\dagger(q_0) \mathbf{A}_1^\dagger(q_1) \dots \mathbf{A}_n^\dagger(q_n) \mathbf{A}_n(p_n) \dots \mathbf{A}_1(p_1) \\ &\times \frac{e^{-i\Omega_n(x, t; \mathbf{p}, \mathbf{q})}}{Q_n(\mathbf{p}, \mathbf{q}, -\varepsilon)} \end{aligned} \quad (4.5)$$

Just like we dealt with $A(\mathbf{f})$ and $A^\dagger(\mathbf{f})$, we are naturally led to introduce

$$\Phi(\mathbf{f}, t) = \int_{\mathbb{R}} \mathbf{f}^\dagger(x) \Phi(x, t), \quad \Phi^\dagger(\mathbf{f}, t) = \int_{\mathbb{R}} \Phi^\dagger(x, t) \mathbf{f}(x), \quad \mathbf{f} \in 2\mathcal{S}(\mathbb{R}) \quad (4.6)$$

Again following the case of NLS, one then shows that for $\varphi, \psi \in \mathcal{D}$, one has

1. for $\mathbf{f} \succ \mathbf{g}$

$$\langle \varphi, \Phi^\dagger(\mathbf{g}, t) \tilde{A}^\dagger(\mathbf{f}, t) \psi \rangle = \langle \varphi, \tilde{A}^\dagger(\mathbf{f}, t) \Phi^\dagger(\mathbf{g}, t) \psi \rangle \quad (4.7)$$

2. for $\mathbf{g} \succ \mathbf{f}_i$, $i = 1, \dots, n$

$$\langle \varphi, \Phi^\dagger(\mathbf{g}, t) \tilde{A}^\dagger(\mathbf{f}_1, t) \dots \tilde{A}^\dagger(\mathbf{f}_n, t) \Omega \rangle = \langle \varphi, \tilde{A}^\dagger(\mathbf{g}, t) \tilde{A}^\dagger(\mathbf{f}_1, t) \dots \tilde{A}^\dagger(\mathbf{f}_n, t) \Omega \rangle \quad (4.8)$$

3. for any $\mathbf{f}_1 \succ \mathbf{f}_2 \succ \dots \succ \mathbf{f}_n$

$$\langle \varphi, \Phi(\mathbf{g}, t) \tilde{A}^\dagger(\mathbf{f}_1, t) \dots \tilde{A}^\dagger(\mathbf{f}_n, t) \Omega \rangle = \sum_{j=1}^n \langle \mathbf{g}, \mathbf{f}_j \rangle \langle \varphi, \tilde{A}^\dagger(\mathbf{f}_1, t) \dots \widehat{\tilde{A}^\dagger(\mathbf{f}_j, t)} \dots \tilde{A}^\dagger(\mathbf{f}_n, t) \Omega \rangle \quad (4.9)$$

We remind that hatted symbols are omitted.

The next step is to show that $\Phi(\mathbf{f}, t)$ and $\Phi^\dagger(\mathbf{f}, t)$ are indeed well-defined operators on a common invariant domain which turns out to be \mathcal{D}_0 . Still following the NLS case, one has the estimate

$$\forall \varphi \in \mathcal{D}_0^n, \forall \psi \in \mathcal{D}_0^{n+1}, \forall \mathbf{f} \in 2\mathcal{S}(\mathbb{R}), \quad |\langle \varphi, \Phi(\mathbf{f}, t) \psi \rangle| \leq (n+1) \|\mathbf{f}\| \|\varphi\| \|\psi\| \quad (4.10)$$

which shows that $\Phi(\mathbf{f}, t)$, considered so far as a bilinear form, is bounded on $\mathcal{D}_0^n \times \mathcal{D}_0^{n+1}$ for each n . Using the usual continuity argument, this gives rise to a bounded operator $\Phi(\mathbf{f}, t) : \mathcal{H}_R^{n+1} \mapsto \mathcal{H}_R^n$ for any n . Thus, by linearity $\Phi(\mathbf{f}, t) : \mathcal{F}_R^0 \mapsto \mathcal{F}_R^0$ is a linear operator with the following properties

- $\Phi(\mathbf{f}, t) \Omega = 0, \quad \Phi(\mathbf{f}, t) : \mathcal{H}_R^{n+1} \mapsto \mathcal{H}_R^n, \quad n \geq 0$
- $\forall \varphi, \psi \in \mathcal{F}_R^0, \quad (\mathbf{f}, t) \mapsto \langle \varphi, \Phi(\mathbf{f}, t) \psi \rangle$ is antilinear and continuous (for the topology of $\|\cdot\|$) in $\mathbf{f} \in 2\mathcal{S}(\mathbb{R})$ and continuous in $t \in \mathbb{R}$.

- $\forall \varphi, \psi \in \mathcal{D}, (\mathbf{f}, t) \mapsto \langle \varphi, \Phi(\mathbf{f}, t)\psi \rangle$ is smooth in $t \in \mathbb{R}$

Of course, analogous results hold for the adjoint $\Phi^\dagger(\mathbf{f}, t)$:

- $\Phi^\dagger(\mathbf{f}, t)\Omega = \tilde{A}^\dagger(\mathbf{f}, t)\Omega, \quad \Phi^\dagger(\mathbf{f}, t) : \mathcal{H}_R^n \mapsto \mathcal{H}_R^{n+1}, \quad n \geq 0$
- $\forall \varphi, \psi \in \mathcal{F}_R^0, \quad \langle \varphi, \Phi(\mathbf{f}, t)\psi \rangle = \langle \Phi^\dagger(\mathbf{f}, t)\varphi, \psi \rangle$

Now that the nature of $\Phi(\mathbf{f}, t), \Phi^\dagger(\mathbf{f}, t)$ is clear, we can proceed to show that they are canonical (non-relativistic) quantum fields. The first requirement deals with the cyclicity of Ω with respect to $\Phi^\dagger(\mathbf{f}, t)$. From (4.7-4.8), one deduces

$$\text{for } \mathbf{f}_1 \prec \dots \prec \mathbf{f}_n, \quad \Phi^\dagger(\mathbf{f}_1, t)\dots\Phi^\dagger(\mathbf{f}_n, t)\Omega = \tilde{A}^\dagger(\mathbf{f}_n, t)\dots\tilde{A}^\dagger(\mathbf{f}_1, t)\Omega \quad (4.11)$$

so the first requirement is satisfied. We now turn to the second requirement embodied in the following theorem

Theorem 4.1 *The quantum fields $\Phi(\mathbf{f}, t), \Phi^\dagger(\mathbf{g}, t)$ satisfy the equal time canonical commutation relations as operators on \mathcal{F}_R^0*

$$[\Phi(\mathbf{f}, t), \Phi(\mathbf{g}, t)] = [\Phi^\dagger(\mathbf{f}, t), \Phi^\dagger(\mathbf{g}, t)] = 0 \quad (4.12)$$

$$[\Phi(\mathbf{f}, t), \Phi^\dagger(\mathbf{g}, t)] = \langle \mathbf{f}, \mathbf{g} \rangle \quad (4.13)$$

for any $\mathbf{f}, \mathbf{g} \in 2\mathcal{S}(\mathbb{R})$

Proof: the proof is the same as in the ordinary NLS equation: it uses extensively (4.7-4.9) and the non-degeneracy of $\langle \cdot, \cdot \rangle$ to get non-bracketed terms. ■

The real novelty now appears when writing the equal time CCR in components for the operator-valued distributions $\phi_j(x, t), \bar{\phi}_k(y, t)$:

$$[[\phi_j(x, t), \phi_k(y, t)]] = [[\bar{\phi}_j(x, t), \bar{\phi}_k(y, t)]] = 0 \quad (4.14)$$

$$[[\phi_j(x, t), \bar{\phi}_k(y, t)]] = \delta_{jk}\delta(x - y) \quad (4.15)$$

where for $j, k = 2$, the above CCR correspond to anticommutator.

4.2 Time evolution

We first wish to emphasize that the form of the Hamiltonian (2.25) cannot be reproduced here owing to the nature of the fields (products of distributions are not defined). Fortunately, the power of the ZF algebra and the quantum inverse method (leading to (4.1-4.2)) rescues us by delivering a simple, free-like Hamiltonian in terms of oscillators. Indeed, one easily checks that the Hamiltonian defined by

$$H = \int_{\mathbb{R}} dp p^2 \mathbf{A}^\dagger(p) \mathbf{A}(p) \quad (4.16)$$

is self-adjoint, *i.e.* $H^\dagger = H$. Moreover,

$$\forall \varphi \in \mathcal{D}, \quad [H\varphi]_{1\dots n}(p_1, \dots, p_n) = (p_1^2 + \dots + p_n^2)\varphi_{1\dots n}(p_1, \dots, p_n) \quad (4.17)$$

which shows that \mathcal{D} is also an invariant domain for H and that this operator has the correct eigenvalues. Finally, H generates the time evolution of the field:

$$\Phi(f, t) = e^{iHt}\Phi(f, 0)e^{-iHt} \quad (4.18)$$

Therefore, H , so defined, is the Hamiltonian of our quantum system.

Note that (4.17) and (4.18) have to be understood as operator equalities and must be evaluated on \mathcal{D} .

The free-like expression for H in terms of creation and annihilation oscillators may be surprising at first glance but it is actually a mere consequence of the rather complicated exchange relations (3.2-3.4). One can say that the effect of the non-linear term has been encoded directly in the oscillators instead of the Hamiltonian (or equivalently the Lagrangian) of the field theory, yielding a (possibly misleading) simple expression for H . One may finally wonder about the coupling constant which seems to disappear. Once again, it is actually present through the R -matrix in the exchange relations.

4.3 Quantum equation of motion

We follow here the line of argument developed for the NLS equation, focusing on the nonlinear term $|\Phi(x, t)|^2\Phi(x, t)$ which has to be normal-ordered. In the normal-ordering of products involving Φ and Φ^\dagger , all creation operators $\mathbf{A}^\dagger(p)$ should be placed to the left of all the annihilation operators $\mathbf{A}(p)$ with the further requirement that the original order of the creation operators be preserved as well as the original order of two annihilation operators if they belonged to the same Φ or Φ^\dagger . Applying this procedure, the classical nonlinear term becomes $:\Phi\Phi^\dagger\Phi:(x, t)$. Besides, the quantum nonlinear super-Schrödinger equation holds in the following form:

$$\forall \varphi, \psi \in \mathcal{D}, \quad (i\partial_t + \partial_x^2)\langle \varphi, \Phi(x, t)\psi \rangle = 2g\langle \varphi, : \Phi\Phi^\dagger\Phi : (x, t)\psi \rangle \quad (4.19)$$

5 Lax pairs

As in the ordinary NLS equation, one can produce a Lax pair for CNLSS. We define the Lax even super-matrix

$$L(\lambda; x) = \frac{i\lambda}{2}\Sigma + \Omega(x) \quad \text{with} \quad \Sigma = E_{11} + E_{22} - E_{33} \quad (5.1)$$

and

$$\Omega(x) = i\sqrt{g} \left(\phi_1(x)E_{13} + \phi_2(x)E_{23} - \bar{\phi}_1(x)E_{31} - \bar{\phi}_2(x)E_{32} \right) \quad (5.2)$$

Let us stress that, as above, the elementary matrices E_{jk} (with 1 at position j, k) are \mathbb{Z}_2 -graded, with $[E_{jk}] = [j] + [k]$, $[1] = [3] = 0$ and $[2] = 1$. As a consequence, the above super-matrix is based on $gl(2|1)$, with the fermionic entries on the first minor diagonals.

Using the PB of the ϕ 's, it is easy to compute that

$$\{L_1(\lambda; x), L_2(\mu; y)\} = i\delta(x - y) [r_{12}(\lambda - \mu), L_1(\lambda; x) + L_2(\mu; y)] \quad (5.3)$$

where we have introduced

$$r_{12}(\lambda - \mu) = \frac{g}{\lambda - \mu} \Pi_{12} \quad \text{with} \quad \Pi_{12} = \sum_{i,j=1}^3 (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (5.4)$$

$$\{L_1(\lambda; x), L_2(\mu; y)\} = \sum_{j,k,l,m=1}^3 \{L_{jk}(\lambda; x), L_{lm}(\mu; y)\} E_{jk} \otimes E_{lm} \quad (5.5)$$

$$(5.6)$$

Now, we introduce the transition matrix by

$$\partial_x T(\lambda; x, y) = L(\lambda; x)T(\lambda; x, y), \quad x > y \quad (5.7)$$

One shows that its PB is given by

$$\{T_1(\lambda; x, y), T_2(\mu; x, y)\} = [r_{12}(\lambda - \mu), T(\lambda; x, y) \otimes T(\mu; x, y)] \quad (5.8)$$

$T(\lambda; x, y)$ obeys to the iterative equation

$$T(\lambda; x, y) = E(\lambda; x - y) + E(\lambda; x) \int_y^x dz \Omega(z) E(\lambda; z) T(\lambda; z, y) \quad (5.9)$$

Like in the usual NLS equation, one now introduces the monodromy matrix $T(\lambda)$ as the following well-defined limit

$$T(\lambda) = \lim_{\substack{x \rightarrow \infty \\ y \rightarrow -\infty}} E(\lambda; -x) T(\lambda; x, y) E(\lambda; y) \quad (5.10)$$

Still following what has been done for the usual NLS (see e.g. [21, 22, 1] and ref. therein), one computes

$$\{T_1(\lambda), T_2(\mu)\} = r_+(\lambda - \mu) T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu) r_-(\lambda - \mu)$$

with

$$\begin{aligned}
r_+(\lambda - \mu) &= \frac{g}{\lambda - \mu} (P_{12} + E_{3,3} \otimes E_{3,3}) \\
&\quad + i\pi g \delta(\lambda - \mu) \sum_{j=1}^2 (E_{j,3} \otimes E_{3,j} - (-1)^{[j]} E_{3,j} \otimes E_{j,3}) \quad (5.11)
\end{aligned}$$

$$\begin{aligned}
r_-(\lambda - \mu) &= \frac{g}{\lambda - \mu} (P_{12} + E_{3,3} \otimes E_{3,3}) \\
&\quad + i\pi g \delta(\lambda - \mu) \sum_{j=1}^2 ((-1)^{[j]} E_{3,j} \otimes E_{j,3} - E_{j,3} \otimes E_{3,j}) \quad (5.12)
\end{aligned}$$

where P_{12} is the super-permutation in the space of 2×2 matrices, given in (3.5).

Introducing $t(\lambda)$, the 2×2 sub-matrix of $T(\lambda)$ with the third row and column removed, and $D(\lambda) = T_{33}(\lambda)$, one finally computes for $\lambda \neq \mu$:

$$\{t_1(\lambda), t_2(\mu)\} = \frac{g}{\lambda - \mu} [P_{12}, t(\lambda) \otimes t(\mu)] \quad (5.13)$$

$$\{D(\lambda), t(\mu)\} = 0 \quad (5.14)$$

(5.13) shows that $t(\lambda)$ defines a classical version of the super-Yangian $Y(gl(1|1))$. Moreover, one can show that $D(\lambda)$ generates the Hamiltonians of the NLSS hierarchy, the first ones being N , P and H . Thus, (5.14) proves that $Y(gl(1|1))$ is a symmetry of this hierarchy.

A detailed analysis of this symmetry, and of its quantum version is currently under investigation [23].

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