Transitive and Absorbent Filters of Implicative Almost Distributive Lattices

Berhanu Assaye Alaba, Mihret Alamneh and Tilahun Mekonnen

***Abstract*—In this paper, we introduce the concept of transitive and absorbent filters of implicative almost distributive lattices and studied their properties. A necessary and sufficient condition is derived for every filter to become a transitive filter. Some sufficient conditions are also derived for a filter to become a transitive filter. A set of equivalent conditions is obtained for a filter to become an absorbent filter.**

***Index Terms*—Implicative almost distributive lattices, transitive filters, absorbent filters.**

I. Introduction

In order to research the logical system whose propositional value is given in a lattice, Y. Xu [1] proposed the concept of lattice implication algebras and discussed some of their properties. Y. Xu and K. Y. Qin [2] introduced the notions of a filter and an implicative filter in a lattice implication algebra, and investigate their properties. Y. B. Jun [3], [4] introduced various types of filters in lattice implication algebra and studied their properties. M. Sambasiva Rao [5] introduced the notions of transitive and absorbent filters and studied some of their properties. Venkateswarlu Kolluru and Berhanu Bekele [6] introduced the concept of implicative algebras and obtained certain properties. They also proved that every implicative algebra is a lattice implication algebra. The concept of an Almost Distributive Lattice (ADL) was introduced in 1981 by U. M. Swamy and G. C. Rao [7] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebra. Berhanu Assaye, Mihret Alamneh and Tilahun Mekonnen [8] introduced the concept of Implicative Almost Distributive Lattices (IADLs) as a generalization of implicative algebra in the class of ADLs. We proved some properties and equivalence condition in an implicative almost distributive lattice. We also introduced filter, implicative filter, positive implicative filter and associative filter in an implicative almost distributive lattice [9]. We proved that every positive implicative filter is an implicative filter and hence a filter. We gave example to show that a filter may not be an associative filter. We provided equivalent conditions for both a positive implicative filter and an associative filter. In this paper, the concept of transitive filter is introduced in implicative almost distributive lattices and their properties are studied. A necessary and sufficient condition is obtained for every filter of IADLs to become a transitive filter. Some sufficient conditions are also derived for a filter to become a transitive filter. The concept of absorbent filters are introduced in IADLs and their properties are studied. A set of equivalent conditions is derived among absorbent, implicative and positive implicative filters.

In the following, we give some important definitions and results that will be useful in this study.

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The authors are with the Department of Mathematics, College of Science, Bahir Dar University, Bahir Dar, Ethiopia. E-mail: berhanu_assaye@yahoo.com, mihretmahlet@yahoo.com, tilametekek@yahoo.com

II. Preliminaries

Definition 2.1 ([7]): An algebra $(L,\vee,\wedge,0)$ of type $(2,2,0)$ is called an Almost Distributive Lattice (ADL) with 0 if it satisfies the following axioms:

1) $(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)$
2) $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$
3) $(x\vee y)\wedge y = y$
4) $(x\vee y)\wedge x = x$
5) $x\vee(x\wedge y) = x$
6) $0\wedge x = 0$, for all $x,y,z\in L$.

If $(L,\vee,\wedge,0)$ is an ADL, for any $x,y\in L$, define $x\leq y$ if and only if $x = x\wedge y$ or equivalently $x\vee y = y$, then $\leq$ is a partial ordering on L.

Definition 2.2 ([7]): Let L be an ADL. An element $m\in L$ is called maximal if for any $x\in L$, $m\leq x$ implies $m = x$.

Definition 2.3 ([7]): Let L be an ADL. For any $a\in L$, principal filter of L generated by a is $[a) = \{x\vee a : x\in L\}$.

Definition 2.4 ([7]): A non-empty subset F of an ADL L is called a filter of L if it satisfies

1) $x,y\in F$ implies $x\wedge y\in F$
2) $x\in F$ and $y\in L$ implies $y\vee x\in F$, for all $x,y\in L$.

Theorem 2.5 ([7]): Let F be a filter of an ADL L and $x,y\in L$. Then $x\vee y\in F$ if and only if $y\vee x\in F$.

Definition 2.6 ([7]): An algebra $(L,\rightarrow,\prime,0,1)$ of type $(2,1,0,0)$ is called implicative algebra if it satisfies the following conditions:

1) $x\rightarrow(y\rightarrow z) = y\rightarrow(x\rightarrow z)$
2) $1\rightarrow x = x$
3) $x\rightarrow 1 = 1$
4) $x\rightarrow y = y'\rightarrow x'$
5) $(x\rightarrow y)\rightarrow y = (y\rightarrow x)\rightarrow x$
6) $0' = 1$, for $x,y,z\in L$

Definition 2.7 ([6]): A relation $\leq$ on an implicative algebra L is defined as follows: $x\leq y\Leftrightarrow x\rightarrow y = 1$, for all $x,y\in L$.

Theorem 2.8 ([6]): Let $(L,\rightarrow,',0,1)$ be an implicative algebra. Then $(L,\vee,\wedge,\rightarrow,',0,1)$ is a lattice implication algebra.

Definition 2.9 ([8]): Let $(L,\vee,\wedge,0,m)$ be an ADL with 0 and maximal element m. Then an algebra $(L,\vee,\wedge,\rightarrow,',0,m)$ of type $(2,2,2,1,0,0)$ is called Implicative Almost Distributive Lattice (IADL) if it satisfies the following conditions:

1) $x \vee y = (x \to y) \to y$
2) $x \wedge y = [(x \to y) \to x']'$
3) $x \to (y \to z) = y \to (x \to z)$
4) $m \to x = x$
5) $x \to m = m$
6) $x \to y = y' \to x'$
7) $0' = m$, for all $x, y, z \in L$.

Now we define the relation $\leq$ on an IADL L as follows: $x \leq y \Leftrightarrow x \to y = m$, for all $x, y \in L$.The relation $\leq$ on L is a partial ordering. Thus $(L, \leq)$ is a poset.

Theorem 2.10 ([8]): In an IADL L, for all $x, y, z \in L$ the following conditions hold:

1) $[(x \to y) \to y] \wedge m = [(y \to x) \to x] \wedge m$
2) $[((x \to y) \to x')'] \wedge m = [((y \to x) \to y')'] \wedge m$
3) $x \to x = m$
4) $m' = 0$
5) $(x')' = x$
6) $x' = x \to 0$
7) $0 \to x = m$
8) $x \to y = m = y \to x$ implies x = y.
9) If $x \to y = m$ and $y \to z = m$, then $x \to z = m$
10) $x \leq y$ if and only if $z \to x \leq z \to y$ and $y \to z \leq x \to z$
11) $((x \to y) \to y) \to y = x \to y$
12) $(x \to y) \to ((y \to z) \to (x \to z)) = m$, $(x \to y) \to ((z \to x) \to (z \to y)) = m$
13) $(x \to z) \to (x \to y) = (z \to x) \to (z \to y)$.
14) $x \to y \leq (y \to z) \to (x \to z)$
15) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$
16) $x \leq y$ implies $y' \leq x'$.
17) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
18) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$
19) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
20) $x \to (y \vee z) = (x \to y) \vee (x \to z)$.

Definition 2.11 ([9]): Let L be an IADL.

1) A subset F of L is called a filter of L if it satisfies:
   (F_1) $m \in F$
   (F_2) $x \in F$ and $x \to y \in F$ implies $y \in F$, for all $x, y \in L$.
2) A subset F of L is called implicative filter of L if it satisfies
   (F_1) $m \in F$
   (I) $x \to y \in F$ and $x \to (y \to z) \in F$ implies $x \to z \in F$, for all $x, y, z \in L$, .

Lemma 2.12 ([9]): Let F be a non-empty subset of an IADL L. Then F is a filter of L if and only if it satisfies for all $x, y \in F$ and $z \in L$:

$x \leq y \to z$ implies $z \in F$

Lemma 2.13 ([9]): Every filter F of an IADL L has the following property: $x \leq y$ and $x \in F$ implies $y \in F$.

Definition 2.14 ([9]): A subset F of IADL L is called a positive implicative filter of L if it satifies:

(F_1) $m \in F$

(P_1) $x \in F$ and $x \to ((y \to z) \to y) \in F$ implies $y \in F$, for $x, y, z \in L$.

Theorem 2.15 ([9]): Let F be a filter of IADL L. Then F is a positive implicative filter of L if and only if for all $x, y \in L$,

(F_I) $m \in F$

(P_2) $(x \to y) \to x \in F$ implies $x \in F$.

Theorem 2.16 ([9]): Let F be a non-empty subset of an IADL L. If F is a positive implicative filter of L, then it is an implicative filter of L.

Definition 2.17 ([9]): Let L be an IADL and $x \in L$ be fixed. A subset F of L is called an associative filter of L with respect to x if it satisfies:

(F_1) $m \in F$

(A_1) $x \to y \in F$ and $x \to (y \to z) \in F$ implies $z \in F$, for all $x, y \in L$.

Theorem 2.18 ([9]): Every associative filter of an IADL L is a filter of L.

Theorem 2.19 ([9]): Let F be a filter of an IADL L. Then F is an assocaitive filter of L if and only if it satisfies:

(A_2) $x \to (y \to z) \in F$ implies $(x \to y) \to z \in F$ for all $x, y, z \in L$.

Theorem 2.20 ([9]): Let F be a filter on an IADL L. Then F is an associative filter of L if and only if it satisfies

(A_3) $x \to (x \to y) \in F$ implies $y \in F$ for all $x, y \in L$.

Definition 2.21 ([10]): A subset F of lattice implication algebra L is called a fantastic filter of L if it satisfies: (F_1) $1 \in F$ and

(F_8) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z \in L$.

Theorem 2.22 ([9]): Every positive implicative filter of IADL L is a filter of L.

III. Transitive filters in Implicative Almost Distributive Lattices (IADLs)

In this section, we introduce transitive filters in an Implicative almost distributive (IADL). We discuss some properties of these filters and an equivalent condition is obtained for every filter to become a transitive filter.

Definition 3.1: A subset F of an IADL L is called a transitive filter of L if it satisfies:

(F_1) $m \in F$ where $m \in L$ and

(T_1) $x \to y \in F$, $y \to z \in F$ implies that $x \to z \in F$, for all $x, y, z \in L$.

Example 3.2: The principal filter $F = [m)$ of an IADL L is a transitive filter of L.

Proof: Let L be an IADL. Let $x, y, z \in L$ be such that $x \to y \in [m)$ and $y \to z \in [m)$. Then $m \leq x \to y$ and $m \leq y \to z$. Since m is a maximal element of L, we have $x \to y = m$ and $y \to z = m$. This implies $x \leq y$ and $y \leq z$ (by Theorem 2.9). Hence $x \leq z$, which implies $x \to z = m \in [m)$. Therefore F is a transitive filter of L. ■

Example 3.3: Let $L = \{0, x, y, z, m\}$ be a set. Define the partially ordered relation on L as $0 < x < y < z < m$ and also define $x \wedge y = min\{x, y\}$, $x \vee y$ = max $\{x, y\}$ for all $x, y, z \in L$. Define the unary operation $'$ and binary operation $\to$ as shown in the tables below respectively,

a	a'
0	m
x	z
y	y
z	x
m	0

$\to$	0	x	y	z	m
0	m	m	m	m	m
x	z	m	m	m	m
y	y	z	m	m	m
z	x	y	z	m	m
m	0	x	y	z	m

Then clearly $(L,\vee,\wedge,\rightarrow,',0,m)$ is an IADL. Now consider $F=\{x,y,z,m\}$. It can be easily verified that F is a transitive filter.

In the following, some properties of transitive filters in IADLs are discussed.

Lemma 3.4: Let F be a transitive filter of IADL L. Then, we have the following

1) $x\in F$ and $x\leq y$ implies that $y\in F$
2) $x\in F$ and $x\rightarrow y\in F$ implies that $y\in F$.
3) The intersection of two transitive filters is again a transitive filter.

Proof: Let F and G be a transitive filter of an IADL L and $x,y,z\in L$.

1) Suppose $x\in F$ and $x\leq y$. Then $x\in F$ and $x\rightarrow y=m$. Since $m\rightarrow x=x\in F$ and $x\rightarrow y=m\in F$, we get $y=m\rightarrow y\in F$ (by definition of transitive filter).
2) Let $x\in F$ and $x\rightarrow y\in F$. Then we get that $m\rightarrow x=x\in F$ and $x\rightarrow y\in F$. Since F is transitive filter, it yields $y\in F$.
3) Suppose $x\rightarrow y\in F\cap G$ and $y\rightarrow z\in F\cap G$. We need to show $x\rightarrow z\in F\cap G$. Now $x\rightarrow y\in F\cap G$ and $y\rightarrow z\in F\cap G$ implies $x\rightarrow y\in F$, $y\rightarrow z\in F$, $x\rightarrow y\in G$ and $y\rightarrow z\in G$. Since F and G are transitive filter of L, we get $x\rightarrow z\in F$ and $x\rightarrow z\in G$. This implies $x\rightarrow z\in F\cap G$. Therefore $F\cap G$ is a transitive filter of L.

■

Theorem 3.5: Every transitive filter of an IADL L is an ADL filter of L.

Proof: Let F be a transitive filter of an IADL L. We need to prove F is an ADL filter of L.

1) Let $x,y\in F$. Now $y\leq x\rightarrow y=m\wedge(x\rightarrow y)=(x\rightarrow x)\wedge(x\rightarrow y)=x\rightarrow(x\wedge y)$. From Lemma 3.4 we get $x\rightarrow x\wedge y\in F$. Again by Lemma 3.4, F is a filter and $x\in F$ then it follows that $x\wedge y\in F$.
2) Let $x\in F$ and $y\in L$. Since $x\leq y\vee x$ it follows from Lemma 3.4(1) that $y\vee x\in F$. Hence F is an ADL filter of L.

■

But the converse of Theorem 3.5 is not true. It can be seen in the following example.

Example 3.6: Let $L=\{0,x,y,z,w,m\}$ be the underlining set with partial ordering $\leq=\{$ $(0,w)$, $(0,x)$, $(0,z)$, $(0,y)$, $(0,m)$, (w,x), (w,y), (w,m), (z,y), (z,m), (y,m), $(0,0)$, (w,w), (x,x), (z,z), (y,y), $(m,m)\}$ Define the unary operation $'$ and a binary operation $\rightarrow$ on L as shown in the tables below respectively,

a	a'
0	m
x	z
y	w
z	x
w	y
m	0

$\rightarrow$	0	x	y	z	w	m
0	m	m	m	m	m	m
x	z	m	y	z	y	m
y	w	x	m	w	x	m
z	x	x	w	m	x	m
w	y	m	m	y	m	m
m	0	x	y	z	w	m

Define the binary operation $\vee$ and $\wedge$ on L as follows

$x\vee y=(x\rightarrow y)\rightarrow y$ and $x\wedge y=[(x\rightarrow y)\rightarrow x']'$. Then $(L,\vee,\wedge,\rightarrow,',0,m)$ is an IADL. Clearly $F=\{y,m\}$ be an ADL filter of L but not transitive filter of L, because of $x\rightarrow w=y\in F$ and $w\rightarrow z=y\in F$ but $x\rightarrow z=z\notin F$.

Theorem 3.7: Let F be an ADL filter of IADL L. If $x\wedge(x\rightarrow y)=x\wedge y$ for all $x,y,z\in L$, then F is a transitive filter of L.

Proof: Let F be an ADL filter of an IADL L. Let $x,y,z\in L$ be such that $x\rightarrow y\in F$ and $y\rightarrow z\in F$. Assume $x\wedge(x\rightarrow y)=x\wedge y$. Since $y\rightarrow z\leq x\rightarrow(y\rightarrow z)$ and F is an ADL filter of L, we get $x\rightarrow(y\rightarrow z)\in F$. Now $(x\rightarrow y)\wedge(x\rightarrow z)=x\rightarrow(y\wedge(y\rightarrow z))=(x\rightarrow y)\wedge(x\rightarrow(y\rightarrow z))\in F$(...from theorem 2.9). This implies $(x\rightarrow y)\wedge(x\rightarrow z)\in F$ and F is an ADL filter of L. Thus $x\rightarrow z\in F$. Therefore, F is a transitive filter of L.

From Lemma 3.4 (2) it can be easily observed that every transitive filter is a filter. However, in the following a necessary and sufficient condition is derived for every filter of IADL L to become a transitive filter. ■

Theorem 3.8: Let F be a filter of IADL L. Then F is a transitive filter if and only if for all $x,y,z\in L$,it satisfies the following condition:

(T_2) $x\rightarrow y\in F$, $(x\rightarrow y)\rightarrow(y\rightarrow z)\in F$ implies $(x\rightarrow y)\rightarrow(x\rightarrow z)\in F$

Proof: Let F be a filter of IADL L. Assume that F is a transitive filter of L. Let $x,y,z\in L$ be such that $x\rightarrow y\in F$ and $(x\rightarrow y)\rightarrow(y\rightarrow z)\in F$. Since F is a filter we have $y\rightarrow z\in F$. Now $(y\rightarrow z)\rightarrow[(x\rightarrow y)\rightarrow(x\rightarrow z)]=(x\rightarrow y)\rightarrow[(y\rightarrow z)\rightarrow(x\rightarrow z)]=(x\rightarrow y)\rightarrow[(z\rightarrow y)\rightarrow(x\rightarrow y)]=(z\rightarrow y)\rightarrow m=m\in F$. Since $(x\rightarrow y)\rightarrow(y\rightarrow z)\in F$, $(y\rightarrow z)\rightarrow[(x\rightarrow y)\rightarrow(x\rightarrow z)]\in F$ and F is a transitive filter, we can conclude that $(x\rightarrow y)\rightarrow[(x\rightarrow y)\rightarrow(x\rightarrow z)]\in F$. Since $x\rightarrow y\in F$ and F is a filter, we get that $(x\rightarrow y)\rightarrow(x\rightarrow z)\in F$. Conversely assume that condition (T_2) holds. Let $x\rightarrow y\in F$ and $y\rightarrow z\in F$. We need to prove $x\rightarrow z$. Now $y\rightarrow z\leq(x\rightarrow y)\rightarrow(y\rightarrow z)\in F$. By assumed condition (T_2), we get $(x\rightarrow y)\rightarrow(x\rightarrow z)\in F$. Now $x\rightarrow y\in F$, $(x\rightarrow y)\rightarrow(x\rightarrow z)\in F$ and F is a filter, we get that $x\rightarrow z\in F$. Therefore, F is a transitive filter. ■

In the following, some sufficient conditions are obtained for a filter of IADL to become a transitive filter.

Theorem 3.9: Let F be a filter of IADL L. Then F is a transitive filter of L if it satisfies the following condition: $x\rightarrow(y\rightarrow z)\in F$ implies $(x\rightarrow y)\rightarrow z\in F$ for all $x,y,z\in L$.

Proof: Let F be a filter of IADL L which satisfies the given condition. Let $x,y,z\in L$ be such that $x\rightarrow y\in F$ and $y\rightarrow z\in F$. Since $y\rightarrow z\leq x\rightarrow(y\rightarrow z)$, we can get $x\rightarrow(y\rightarrow z)\in F$ this implies $(x\rightarrow y)\rightarrow z\in F$ (by the given condition). Since $x\rightarrow y\in F$ and F is a filter of L , it yields that $z\in F$, since $z\leq x\rightarrow z$, we get $x\rightarrow z\in F$. Therefore, F is a transitive filter of L. ■

The following corollary is a direct consequence of Theorem 2.19

Corollary 3.10: Let F be a filter of IADL L. If F is an associative filter, then it is a transitive filter of L.

Proof: Let F be a filter of IADL L. Let $x,y,z\in L$ such that $x\rightarrow y\in F$ and $y\rightarrow z\in F$. Assume that F is an associative filter of L. We need to show that F is a transitive filter of L. Since F is an associative filter, $x\rightarrow(y\rightarrow z)\in F$ implies $(x\rightarrow y)\rightarrow z\in F$ (by Theorem 2.19). Since $x\rightarrow y\in F$ and F is a filter of L, we have $z\in F$. Since $z\leq x\rightarrow z$ and F is a filter of L, we have $x\rightarrow\in F$. Therefore, F is a transitive filter of L. ■

IV. Absorbent Filters in Implicative Almost Distributive Lattices

In this section, the concept of absorbent filters is introduced in an implicative almost distributive lattice (IADL) and some properties of absorbent filters are studied. A set of equivalent conditions is obtained for every filter to become an absorbent filter.

Definition 4.1: A subset F of an IADL L is called absorbent filter if it satisfies the following conditions:

(F_1) $m \in F$ where $m \in L$ and

(Ab_1) $(x \to y) \to x \in F$ implies that $x \in F$.

Theorem 4.2: Every associative filter of IADL L is an absorbent filter of L.

Proof: Let F be an associative filter of IADL L. Clearly (F_1) holds. Let $x, y \in L$ be such that $(x \to y) \to x \in F$. Since F is an associative filter, $(x \to y) \to x = (x \to y) \to (m \to x) \in F$ and $(x \to y) \to m = m \in F$ implies $x \in F$. Therefore, F is an absorbent filter of L. ■

Remark 4.3: The converse of Theorem 4.2, is not true as shown in the following example.

Example 4.4: Let $L = \{0, x, y, z, m\}$ be the underlining set with partial ordering $\leq = \{\ (0,x),\ (0,z),\ (0,y),\ (0,m),\ (x,z),\ (x,m),\ (y,z),\ (y,m),\ (z,m),\ (0,0),\ (x,x),\ (y,y),\ (z,z),\ (m,m)\}$. Define the unary operation $'$ and binary operation $\to$ as shown in the tables below respectively,

a	a'
0	m
x	y
y	x
z	z
m	0

$\to$	0	x	y	z	m
0	m	m	m	m	m
x	y	m	z	m	m
y	x	z	m	m	m
z	z	z	z	m	m
m	0	x	y	z	m

Define the binary operation $\vee$ and $\wedge$ by

$x \vee y = (x \to y) \to y$

$x \wedge y = [(x \to y) \to x']'$. Then clearly $(L, \vee, \wedge, \to, ', 0, m)$ is an IADL. Now consider $F = \{m, y, z\}$. Clearly, we can show that F is an absorbent filter. But F is not an associative filter, because of $y \to (z \to x) = y \to z = m \in F$ and $(y \to z) \to x \notin F$.

In general, every filter of IADL L need not be an absorbent filter. From Example 3.5, consider $F = \{m, z\}$. Then clearly F is a filter of L but not an absorbent filter of L. For this consider $y, z \in L$. Then, it is clear that $(y \to z) \to y = w \to y = m \in F$ but $y \notin F$. Therefore, F is not an absorbent filter of L.

In [9], we proved that every positive implicative filter of an IADL L is an implicative filter but not the converse. However, in the following, we derive a set of equivalent conditions for every implicative filter of IADL L to become a positive implicative filter which leads to the characterization of an absorbent filter in an IADL L.

Theorem 4.5: Let F be a filter of IADL L and $x, y, z \in L$. Then, the following conditions are equivalent.

1) F is an absorbent filter
2) $a \in F$ and $(x \to y) \to (a \to x) \in F$ implies $x \in F$
3) F is an implicative filter
4) F is a positive implicative filter.

Proof: Let L be an IADL and $x, y, z \in L$. (1) $\Rightarrow$ (2): Assume that F is an absorbent filter of L. Let $a \in F$ and $(x \to y) \to (a \to x) \in F$. Then $(x \to y) \to (a \to x) = a \to [(x \to y) \to x] \in F$. since $a \in F$ and F is a filter of L, we get that $(x \to y) \to x \in F$. Since F is an absorbent filter of L, we have $x \in F$.

(2) $\Rightarrow$ (3): Assume condition (2) holds. Suppose that $x \to (y \to z) \in F$ and $x \to y \in F$. Now $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to [(x \to (x \to z)]$. Since F is a filter and $x \to (y \to z) \in F$, we get $(x \to y) \to [x \to (x \to z)] \in F$. Since $x \to y \in F$ and F is a filter of L, we get $x \to (x \to z) \in F$. Put $x \to (x \to z) = a$. Then we have the following consequence, $[(x \to z) \to z] \to [a \to (x \to z)] = a \to \{[(x \to z) \to z] \to (x \to z)\} = a \to \{x \to [(x \to z) \to z) \to z]\} = a \to [x \to (x \to z)] = a \to a = m \in F$. Then by assumption we get $x \to z \in F$.

(3) $\Rightarrow$ (4): Assume condition (3) holds. Let $x \in F$ and $x \to [(y \to z) \to y] \in F$. Since $x \in F$ and F is a filter, we get that $[(y \to z) \to y] \in F$. Since $y' \leq y \to z$, we have $(y \to z) \to y \leq y' \to y = y' \to (y' \to 0)$. Since F is a filter in L, we get $y' \to (y' \to 0) \in F$ and $y' \to y' = m \in F$. Since F is an implicative filter, it yields that $y = (y')' = y' \to 0 \in F$.

(4) $\Rightarrow$ (1): Assume that F is a positive implicative filter of L. Suppose $(x \to y) \to x \in F$. Then $m \to [(x \to y) \to x] \in F$. Since F is a positive implicative filter of L, we get $x = m \to x \in F$. Therefore, F is an absorbent filter of L. ■

Definition 4.6: A subset F of an IADL L is called a fantastic filter of L if it satisfies:

(F_1) $m \in F$, where $m \in L$ and

(F_7) $z \in F$ and $z \to (y \to x) \in F$ implies $((x \to y) \to y) \to x \in F$ for all $x, y, z \in F$.

Theorem 4.7: Every positive implicative filter of an IADL L is a fantastic filter of L.

Proof: Let F be a positive implicative filter of an IADL L. Then F is a filter of L (see Theorem 2.22). Let $x, y \in L$ be such that $y \to x \in F$. It is sufficient to show that $((x \to y) \to y) \to x \in F$. Since $x \leq ((x \to y) \to y) \to x$, we get $(((x \to y) \to y) \to x) \to y \leq x \to y$. Putting $a = ((x \to y) \to y) \to x$, we obtain $(a \to y) \to a = ((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = (x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. It follows from Lemma 2.13 that $(a \to y) \to a \in F$ so, from Theorem 2.15, that $a \in F$, i.e., $((x \to y) \to y) \to x \in F$. Hence F is a fantastic filter of L. ■

Corollary 4.8: Let F be a filter of an IADL L. If F is an absorbent filter of L, then it is a fantastic filter of L.

Proof: Let L be an IADL and $x, y, z \in L$. Assume F is an absorbent filter of L. Then by Theorem 4.5, F is a positive implicative filter of L. From Theorem 2.21, F is a fantastic filter of L. ■

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