ON THE BRAIDS FOR 8_{10} KNOT

Hakan ŞİMŞEK
Atatürk Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü

Ahmet KÜÇÜK
Atatürk Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü

SUMMARY: This paper is concerned with 8_{10} knots and its braids. The braids structure plays a very important role in Knots Theory. In view of this structure, we obtain braids for that knot, give the representations of Artin and examine Garside Word problem. Then we examine the positivity structure for these knots.

Key words: Braids, Positive word, Representations of Braids.

ÖZET: Bu makale 8_{10} düğümü ve onun örgüsü ile ilgilidir. Düğüm teorisinde önemli kavramlardan birisi olan örgü kavramını kullanarak 8_{10} düğümünün örgüsünün Artin ve Garside temsilini elde edeceğiz. Daha sonra pozitiflik kavramı irdelenecektir.

Anahtar Kelimeler: Örgü, Pozitif kelime, Örgü Temsili.

INTRODUCTION: The word problem in B_n was solved by Artin in (Artin, E., 1925). His solution was based on his knowledge of structure of the kernel of the map ϕ from from B_n to the symetric group Σ_n which sends the generator δ_i to the transposition $(i,i+1)$. The Conjugacy problem in B_n was also posed in (Artin, E., 1925), also its importance for the problem of recognizing knots and links algorithmicaly was noted, however it took 43 years before progress was made. In a different, but equally foundational manuscript (Garside, F.A., 1969), he discovered a new solution to the word problem which then led him to a related solution to the conjugacy problem (and also [Birman, J., 1998]).

A somewhat different question is the shortest word problem, to find a representative of the word class which has shortest length in the Artin generators. It was proved in (Paterson, M.S., 1991) that this problem in B_n is at last as hard as an NP-complete problem. Garside and Thurston and Birman, using new generators, will be able to solve the word problem.

We gave the Braid which is related to Artin and Garside generators and investigated positivity structure for 8_{10} knot.

An n-braid is a very particular example of an (n)- tangle. On the top and base of a cube, B, mark out n points, $A_1, A_2, \cdots, A_n$ and $A'_1, A'_2, \cdots, A'_n$ respectively.

These points may be arbitrarily placed, however we shall express them in terms of specific coordinates.

Firstly, the coordinates for B in R^3 are,

$$B = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$$

Let us choose $A_1, A_2, \cdots, A_n$, $A_1', A_2', \cdots, A_n'$ as follows,

$$A_1 = (\frac{1}{2}, \frac{1}{n+1}, 1), \cdots, A_n = (\frac{1}{2}, \frac{n}{n+1}, 1)$$

$$A_1' = (\frac{1}{2}, \frac{1}{n+1}, 1), \cdots, A_n' = (\frac{1}{2}, \frac{n}{n+1}, 0)$$

By the construction each A_i' is directly below the corresponding A_i Figure 1. Now join the $A_1, A_2, \cdots, A_n$, to $A_1', A_2', \cdots, A_n'$ by means of n curves in B. As usual, they are joined in such a way that these curves (including the end points) do not mutually intersect each other. We will call these polygonal arcs strings.

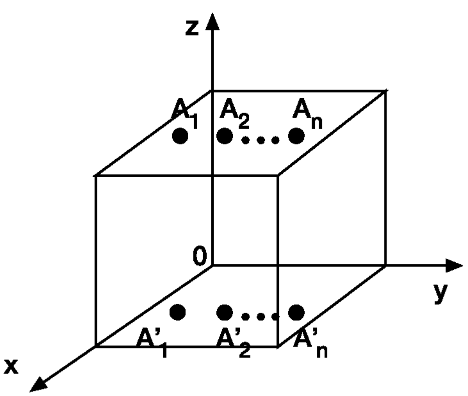


Figure 1

Suppose that we divide the cube into two parts by an arbitrary plane E that is paralel to base of the cube B. Then, if E intersects each string at one and only one point, we say that these n strings in B are an n-braid.

Suppose that B_n is the set of all n-braids (to be more precise all the equivalence class of these braids). For two elements in B_n , i.e., for two n-braids α and β, it is possible to define a product for two n braids. First, glue the base of the cube that contains α to the top face of the cube that contains β. The gluing together of the two cubes produces a rectengular solid in which there exists a braid that has been created from vertical juxtaposition of α and β. Figure 2.

So far we have described a set B_n a product in this set, and also that the associativity holds in the set. The unit e is simply the trivial braid and irrespective of the braid α, $\alpha e = \alpha$ and similarly, $e\alpha = \alpha$.

In order to find an inverse for an arbitrary α , let us consider the mirror image , $\alpha^{\bullet}$ of α. Then we may write that $\alpha\alpha^{\bullet} = e$ and $\alpha^{\bullet}\alpha = e$. Therefore we now have all the essentials for B_n to be a group.

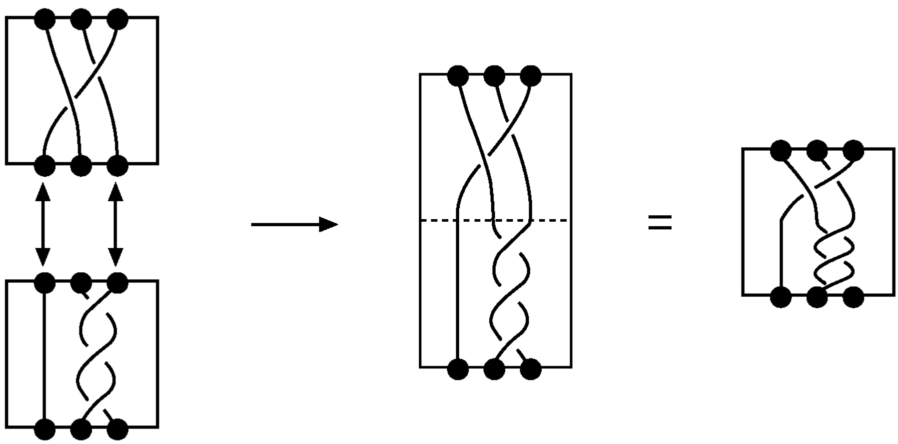


Figure 2

A fundemental result on the braid group B_n is that it has only the following two types of relations called the fundemental relations:

$$(1)\ \sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j|\geq 2)$$
$$(2)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (i = 1,2,\cdots,n-2).$$

Collecting together the various relations that we have discussed so far, we may write B_n in terms of its generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and these fundemantal relations,

$$B_n = \left(\begin{array}{l} \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \mid i-j \mid\geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} (i = 1,2,\cdots,n-2) \end{array} \right)$$

where the right hand side is said to be a presentation of B_n (Artin generators).

In Garside genarators, i.e., positive braids, i.e., braids which are positive powers of the generators, Garside introduced the fundemantal braid Δ:

$$\Delta = (\delta_1 \delta_2 \cdots \delta_{n-1})(\delta_1 \delta_2 \cdots \delta_{n-2}) \cdots (\delta_1 \delta_2)\delta_1. \tag{1}$$

He showed that every element $W \in B_n$ can be represented algoritmically by a word W of the form $\Delta^r P$, where r is an integer and P is a positive word, and r is a maximal for all such representations (Birman, J., 1998).

Lemma 1: The word $\Delta^{\#}$ in S_n has the following properties:

(i) For each word $V \in S_n$, $\Delta^{\#} V = \widehat{V} \Delta^{\#}$.

(ii) For each j $(1 \leq j \leq n-1)$, $D(\Delta^{\#})$ contains a word with initial letter s_j and word with final letter s_j (Birman, J., 1974).

From now on the semigroup S_n will be identified with its image $e(S_n)$ in B_n.

Lemma 2: For each j ($(1 \leq j \leq n-1)$, there is a positive word X_j such that $\sigma_j^{-1} = \Delta^{-1} X_j$. Also, $\Delta \sigma_j^{-1} = \sigma_{n-j}^{-1} \Delta$ (which, together with Lemma 1, implies that $\Delta^{-1} V = \widehat{V} \Delta^{-1}$ for every braid word V) (Birman, J., 1974).

Theorem 1 (Garside Solution to the Word Problem in $\mathbf{B}_n$): If $\beta \in B_n$ then β is represented by unique word of the form $\Delta^m \overline{P}$, where the integers m and the positive word $\overline{P}$ are computed from any representative $\sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$ of the word β in the following manner:

(i) List the positive word $X_1, \cdots, X_{n-1}$ whose existence is established by Lemma 1.

(ii) Replace every letter $\sigma^{-1}{}_{\mu_i}$ which occurs in the braid word $\sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$ by $\Delta^{-1} X_{\mu_i}$.

(iii) Using the property $\Delta^{-1} V = \widehat{V} \Delta^{-1}$ (Lemma 2) collect all Δ^{-1}'s introduced in (ii) at the left, so that β is represented by a word of the form $\Delta^k P_0$, where P_0 is positive. Note that $k \leq 0$.

(iv) Construct $D(P_0)$

(v) In $D(P_0)$, choose a word $\Delta^h P$ such that h is maximal. Let m = h + k (Note that $h \geq 0$)

(vi) Construct D(P). Let $\overline{P}$ be base of D(P).

Proof: See (Birman, J., 1974).

Now, let α be a braid and let us connect, by a set of parallel arcs that lie outside the square, the points $A_1, A_2, \cdots, A_n$ on the top of a rectengular diagram of a braid α to the points $A'_1, A'_2, \cdots, A'_n$ respectively, on the bottom of the same diagram. Then in a natural way we form regular diagram of knot or link from a braid. A knot that has been created in this way is said to be a knot, K, created from the braid α.

Theorem 2 (Alexander's Theorem): Given arbitrary (oriented) knot (or link), then it is equivalent (with orientation) to a knot (or link) that has been formed from a braid (Murasugi, K., 1996).

Proof: Let D be an oriented regular diagram of a knot K. Firstly cut the D at a point (not a crossing point) P_0, and then pull the loose ends apart so that we now have a (1,1) tangle. Figure 3.

We shall show that we can change this tangle into a braid α. The knot, in a sense induced as described previously from the braid, is equivalent to K. If the tangle T has m local maxima, then it also has m local minima. In the case m = 0, T is a 1-braid and so no proof is required.

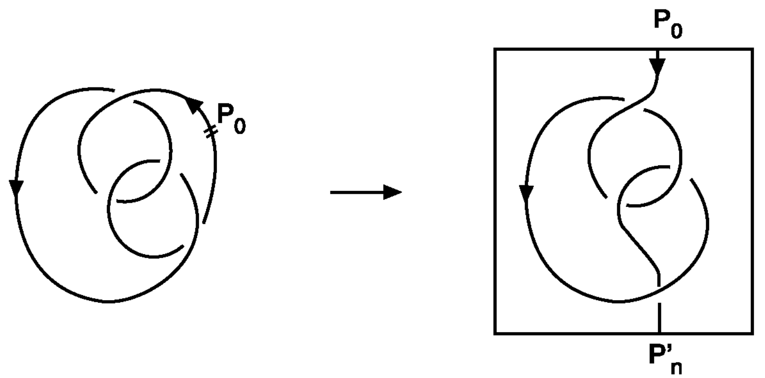


Figure 3

So suppose that m > 0, then there exists an arc ab in T, which we may say "is rising upwards", connecting a local minimum to a local maximum b. Figure 4.

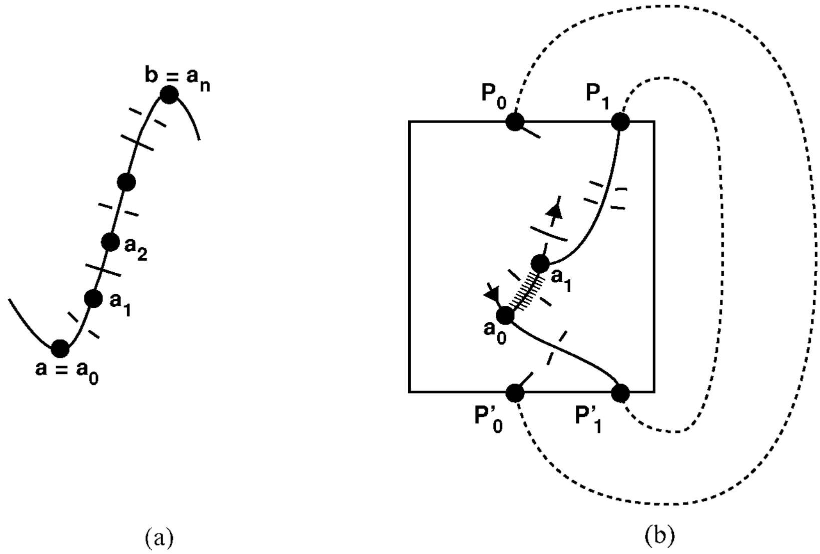


Figure 4

Further we may assume that ab intersects with the other parts of the tangle at n places. Let us now mark n+1 points on ab, i.e., $a = a_0, a_1, \cdots, a_n = b$, such that the arc $a_i a_{i+1}$ intersects only one part of the tangle, see Figure 4(b). Next replace the arc $a_0 a_1$ by the much larger arc $a_0 P_1' P_1 a_1$. The large arc $P_1 P_1'$ lies outside the tangle T, and the arcs $a_0 P_1'$ and $a_1 P_1$ are selected in such a way that if $a_0 a_1$ passes over (or under) the other segment, then they also pass over (or under) all the other segments. The result of the above manipulations is a (2,2) tangle. Figure 4(b).

It follows immediately that the oriented knot obtained by joining (outside the square) the four endpoints of this (2,2) tangle by curves is equivalent to original knot. By using the same methods as above, with regard to the arcs $a_1 a_2, a_2 a_3, \cdots, a_{n-1} a_n$ we will eventually form a $(n+1, n+1)$ tangle T that does not have a local minimum and maximum at a and b respectively, and further has at most $m-1$ local maxima anda minima. Continuing this procedure, we shall finally create a tangle that has no local maxima and minima. This tangle is our required braid (Murasugi, K., 1996).

Definition: A knot is called positive, if it has a positive diagram, i.e., diagram with all crossings positive as follows or its braids has no σ_i^{-1} elements.

Lemma 3: If a positive n-string braid β involves a single occurence of some braid generators δ_j $(1 \le j \le n-1)$, then it has same closure as some positive $(n-1)$ string braid (Birman, J., 1974).

Lemma 4: If the closure of a positive n-braid is a knot or non-separated link, then its Conway polynomial is of degree c-n+1, where c is its number of braid crossing, n braids string (Birman, J., 1974).

Lemma 5: If K is a prime positive knot which has a Conway polynomial of degree 2m, then a minimal string positive braid p which closes to K has at most $4m-3$ crossing (Buskirk, J.V., 1983).

Now, we will find braids for 8_{10} knots. Regular diagram of 8_{10} is given in Figure 5.

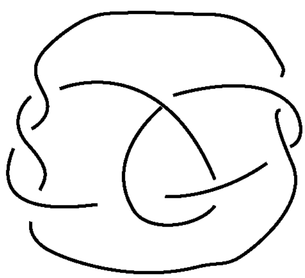
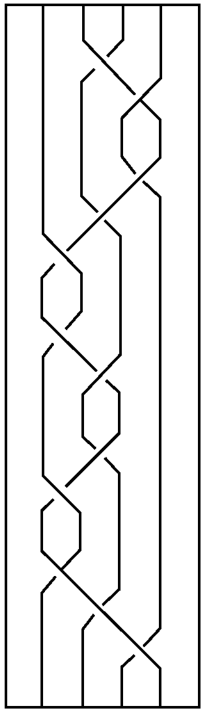


Figure 5

Then Theorem 2 is applied for the knot. Then we got 4-braids for this knot. 4 braid is given in Figure 6.



We get β-braid for 8_{10} knot, i.e.,

$$\beta = \sigma_2^{-1}\sigma_3^{2}\sigma_2\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}.$$

At the same time, we will write the Garside generators for this braid. We applied Theorem 1 to β braid which is shown in Figure 6. Then we write:

$$\beta = \Delta^{-7} A\sigma_3^{2}\sigma_s BC\sigma_2^{2} BCA$$

where,

$$A = \sigma_3\sigma_1\sigma_2\sigma_3\sigma_1,\ B = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2,\ C = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2.$$

These equations are the representation of 8_{10} knots (Garside Rep.).

Now we examine the positivity structure. Lemma 4 gives us that β braid's Conway polynomial is of degree 7. By Figure 6, c = 12, n = 4 . But its Conway polynomial is of degree 6. Also, Lemma 5 gives us its braids at most 9 crossings. But its braids has 12 crossings. Then 8_{10} knot is not positive.

REFERENCES

ARTIN, E. (1925).Theorie der Zopfe Hamburg Abh. 4, 47-72.

BIRMAN, J., K.H. KO and S.J. LEE (1998). “A New Aproach To The Word And Conjugacy Problems In The Braid Group”, *Advances In Mathematics*, vol. 139, (issue 2), 322-353.

BIRMAN, J. (1974). *Braids,Links and Mapping Class Groups*, Princeton Universty Press.

BUSKIRK, J.V. (1983). “Positive Knots Have Positive Conway Polynomials”, *Springer Lecture Notes in Mathematics*, 1144, 146-159.

GARSIDE, F.A. (1969). “The Braid Group and Other Groups”, *Quarterly Journal of Mathematics,* 20 No 78, 235-254.

MURASUGI., K. (1996). *Knots Theory and Its Applications*, Birkhauser: Boston.

PATERSON, M. S., and A.A. RASBOROV (1991). “The Set Minimal Braids is co_NP-Complete”, *Journal of Algorithms*, 12, 393-408.