MEROMORPHIC FUNCTIONS

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ABSTRACT: In this paper, we have given the applications of homogeneous differential polynomials to the Nevanlinna's theory of meromorphic functions in the finite complex plane and given some generalizations by these polynomials.

Key words: *Meromorphic function, homogeneous dıfferential polynomial and finite complex plane.*

ÖZET: Bu çalışmada, homojen diferansiyel polinomlar Nevanlinna kuramına uygulandı ve bu homojen polinomlarla bazı genelleştirmeler verildi.

Anahtar kelimeler: *Meromorfik fonksiyon, homojen dıferansiyel polinom ve sonlu karmaşık düzlem.*

1. INTRODUCTION

In this work, we are going to use the usual notations of the Nevanlinna theory of meromorphic functions as explained in (Hayman,1968,1-20), (Nevanlinna,1974,10-25) and (Wittich, 1968, 5-30) such as m(r,f), N(r,f), m(r,a), $\overline{N}$r,a), T(r,f), $\delta(a,f)=\delta(a)$, $\bar{\delta}(a)$ and $\Delta(a)$. By a meromorphic function we shall always mean that a function is meromorphic in the finite complex plane.

If f is a non-constant meromorphic function we shall denote by S(r,f) any quantity satisfying S(r,f)=o[T(r,f)] as $r\to\infty$ through all values if f is of finite order and $r\to\infty$ possibly outside a set of finite linear measure if f is of infinite order. Also, we shall always denote a(z), $a_0(z)$, $a_1(z)$, $a_2(z)$, etc. meromorphic functions satisfying

$$T[r,a(z)]=S(r,f) \text{ and } T[r,a_i(z)]=S(r,f).$$

We shall be concerned with meromorphic functions P which are polynomials in the meromorphic function f and the derivatives of f with coefficients of the form a(z).

Let

$$F_k=a(f)^{t_0}[f^{(1)}]^{t_1}[f^{(2)}]^{t_2}\dots[f^{(m)}]^{t_m}$$

and

$$P=\sum_{k=1}^{N}F_k$$

where $f^{(1)}$, $f^{(2)}$..., $f^{(m)}$ are the successive derivatives of f and t_0, t_1,..., t_m are non-negative integers.

Definition 1. If $t_0+t_1,+\dots,+\ t_m$ for a fixed positive integer in every term of P, then Pis called a homogeneous differential polynomial in f of degree n.

2. LEMMAS

Lemma 1. If P is a homogeneous differential polynomial in f of degree n ≥ 1, then we have

$$m\left(r,\frac{P}{f^n}\right)=S(r,f)$$

(Gopalakrishna, 1973, 330).

Lemma 2. Let P be a homogeneous differential polynomial in f of degree n and suppose that P does not involve f. That is, P is a homogenous polynomial of degree n in $f^{(1)}$, $f^{(2)}$,..., $f^{(m)}$ with coefficients of the form a(z) satisfying T[r,a(z)] = S(r,f).

If P is not a constant and $a_1, a_2, \ldots, a_q$ are distinct elements of C where q is any positive integer, then we have

$$n\sum_{i=1}^{q} m\left(r,\frac{1}{f-a_i}\right) \le T(r,P) - N\left(r,\frac{1}{P}\right) + S(r,f) \tag{1}$$

or

$$nqT(r,f) \le T(r,P) + n\sum_{i=1}^{q} N\left(r,\frac{1}{f-a_i}\right) - N\left(r,\frac{1}{P}\right) + S(r,f) \tag{2}$$

(Gopalakrishna, 1973, 329-335).

3. THEOREMS

Theorem 1. Let P be a homogeneous differential polynomial in f of degree n and $a \neq b$. If f is a non-constant meromorphic function in the finite complex plane, then we have the following inequality

$$T(r,f) \le N(r,P) + N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) - N(r,f) - N\left(r,\frac{1}{P}\right) + S(r,f).$$

Proof. Since $a \neq b$ we can write

$$\frac{1}{f-b} = \left(\frac{P}{f-b} - \frac{P}{f-a}\right)\left(\frac{f-a}{P}\right)\frac{1}{b-a}.$$

If we take absolute values, positive logarithms and mean values of the both sides of this equality we have

$$\left.,\frac{1}{f-b}\right) \le m\left(r,\frac{P}{f-b}\right) + m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{f-a}{P}\right) + O(1)$$

$$\le m\left(r,\frac{P}{f-b}\right) + m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{P}{f-a}\right) + N\left(r,\frac{P}{f-a}\right)$$

$$- N\left(r,\frac{f-a}{P}\right) + O(1)$$

$$\le N(r,P) + N\left(r,\frac{1}{f-a}\right) - N\left(r,\frac{1}{P}\right) - N(r,f) + S(r,f) \tag{3}$$

where

$$m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{P}{f-b}\right) = S(r,f)$$

and

$$\left.\frac{}{-a}\right) - N\left(r,\frac{f-a}{P}\right) = N(r,P) + N\left(r,\frac{1}{f-a}\right) - N\left(r,\frac{1}{P}\right) - N(r,f).$$

If we add the term $N\left(r,\frac{1}{f-b}\right)$ on both sides of the inequality (3), we get

$$T(r,f) \le N(r,P) + N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) - N(r,f) - N\left(r,\frac{1}{P}\right) + S(r,f). \qquad (4)$$

If we restrict P = f′(z), the inequality (4) becomes

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f)$$

which is one of the Nevanlinna's results.

Theorem 2. Let P be a homogeneous differential polynomial in f of degree n and $b \neq 0$. If f is a non-constant meromorphic function in the finite complex plane, we have the following inequality

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{P-b}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f). \qquad (5)$$

Proof. Since $b \neq 0$ we can write

$$\frac{1}{f-a} = \left(\frac{P}{f-a} - \frac{P'}{f-a}\,\frac{P-b}{P'}\right)\frac{1}{b}.$$

The mean values of this equality give

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\le m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{P'}{f-a}\right) + m\left(r,\frac{P-b}{P'}\right) + O(1) \\ &\le N\left(r,\frac{P'}{P-b}\right) - N\left(r,\frac{P-b}{P'}\right) + S(r,f) \\ &\le N(r,P') + N\left(r,\frac{1}{P-b}\right) - N\left(r,\frac{1}{P'}\right) - N(r,P) + S(r,f) \\ &\le \overline{N}(r,P') + N\left(r,\frac{1}{P-b}\right) - N\left(r,\frac{1}{P'}\right) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{P-b}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f) \end{aligned}$$

or

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{P-b}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f). \qquad (6)$$

If we restrict $P = f^{(k)}(z)$, the inequality (6) becomes

$$T(r,f) \le \overline{N}(r,f) + N\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f)$$

which is the one of Milloux's results (Dönmez, 1979, 203-207).

Theorem 3. Let P be a homogeneous differential polynomial in f of degree n. If f is a non-constant meromorphic function in the finite complex plane, we have

$$T(r,f) \le N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{P-b}\right) + N\left(r,\frac{1}{P-c}\right) - N_1(r,P) + S(r,f) \tag{7}$$

where

$$N_1(r,P) = 2N(r,P) - N(r,P') + N\left(r,\frac{1}{P'}\right)$$

and non-negative.

Proof. It is easy to write

$$\frac{1}{f-a} = \frac{1}{P}\frac{P}{f-a}.$$

The mean values of this equality give

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\le m\left(r,\frac{1}{P}\right) + m\left(r,\frac{P}{f-a}\right) \\ &\le m\left(r,\frac{1}{P}\right) + S(r,f) \\ &\le T(r,P) - N\left(r,\frac{1}{P}\right) + S(r,f). \end{aligned} \tag{8}$$

We know that Nevanlinna's second fundamental theorem is the following in terms of P

$$T(r,P) \le N\left(r,\frac{1}{P}\right) + N\left(r,\frac{1}{P-b}\right) + N\left(r,\frac{1}{P-c}\right) - N_1(r,P) + S(r,P).$$

If we use the second fundamental theorem in the inequality (8), we can write

$$;,f) \le N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{P}\right) + N\left(r,\frac{1}{P-b}\right) + N\left(r,\frac{1}{P-c}\right) - N\left(r,\frac{1}{P}\right) - N_1(r,P) + S(r,f)$$

or

$$T(r,f) \le N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{P-b}\right) + N\left(r,\frac{1}{P-c}\right) - N_1(r,P) + S(r,f).$$

If we restrict $P = f^{(k)}(z)$, the inequality (7) becomes

$$T(r,f) \le N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f^{(k)}-b}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N_1(r,f^{(k)}) + S(r,f)$$

which is the one of Hiong's results (Dönmez, 1979, 203-207).

Theorem 4. If P is a homogeneous differential polynomial in f of degree n, then we have

$$nqT(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{P-b}\right) + n\sum_{i=1}^{q} N\left(r,\frac{1}{f-a_i}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f). \quad (9)$$

Proof. The Nevanlinna's second fundamental theorem can be written in terms of the homogenous differential polynomial P as the following,

$$T(r,P) \le \overline{N}(r,P) + N\left(r,\frac{1}{P}\right) + \overline{N}\left(r,\frac{1}{P-b}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,P).$$

On the other hand, it is easy to write $\overline{N}(r,P) \le \overline{N}(r,f) + S(r,f)$. If we use the inequality (2), we can write

$$nqT(r,f) \le \overline{N}(r,P) + N\left(r,\frac{1}{P}\right) + \overline{N}\left(r,\frac{1}{P-b}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f) + n\sum_{i=1}^{q} N\left(r,\frac{1}{f-a_i}\right) - N\left(r,\frac{1}{P}\right) + S(r,f)$$

or

$$nqT(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{P-b}\right) + n\sum_{i=1}^{q} N\left(r,\frac{1}{f-a_i}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f).$$

If n = 1 and q = 1 the inequality (9) gives the inequality (5). That is, the inequality (9) is the generalization of the inequality (5).

Theorem 5. If P is a homogeneous differential polynomial in f of degree n and s = 2,3,4,... then

$$(s-1)nqT(r,f) \le \overline{N}(r,f) + (s-1)n\sum_{i=1}^{q} N\left(r,\frac{1}{f-a_i}\right) + \sum_{j=1}^{s} N\left(r,\frac{1}{P-b_j}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f). \quad (10)$$

If s = 3,4,5,... then we have

$$(s-2)nqT(r,f) \le (s-2)n\sum_{i=1}^{q} \overline{N}\left(r,\frac{1}{f-a_i}\right) + \sum_{j=1}^{s} \overline{N}\left(r,\frac{1}{P-b_j}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f). \quad (11)$$

Proof. The Nevanlinna's second fundamental theorem can be written in terms of the homogeneous differential polynomial P as the following

$$(s-1)T(r,P) \le \overline{N}(r,f) + \sum_{j=1}^{s} \overline{N}\left(r,\frac{1}{P-b_j}\right) - N_0\left(r,\frac{1}{P'}\right) + S(r,f) \quad (12)$$

and

$$(s-2)T(r,P) \le \sum_{j=1}^{s} \overline{N}\left(r,\frac{1}{P-b_j}\right) - N_1(r,P) + S(r,P) \quad (13)$$

where $N_1(r,P) = 2N(r,P) - N(r,P') + N\left(r,\frac{1}{P'}\right)$ and non-negative. If we use the inequalities (12) and (13) in the equality (10), we obtain the inequality (11).

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