Lie Groups

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Abstract

Este es un trabajo de fin de grado sobre la teoría de Grupos de Lie y Álgebras de Lie. Se centra en dos objetivos: Por un lado, proporciona al lector los conceptos básicos en relación a Grupos de Lie, Algebras de Lie, la conexión entre ellos, y algunas herramientas para entender su estructura; por otro lado, explora el teorema de Lie-Kolchin, no solo las motivaciones detrás de éste, sino también algunos resultados relacionados.

El trabajo está muy centrado al caso matricial, por lo que la mayoría de definiciones y ejemplos están pensados para grupos y álgebras de Lie. Esto se debe parcialmente a que una explicación geométrica completa requeriría incluir algunos conceptos que nos apartarían de los objetivos principales de este trabajo. El otro argumento a favor del caso matricial es el hecho que el objetivo principal es el teorema de Lie-Kolchin, un resultado de teoría de la representación. Debido a que la teoría de la representación busca llevar grupos y álgebras de Lie a grupos lineales generales, conseguir una percepción los efectos de esta teoría en matrices será útil para un mejor entendimiento del teorema de Lie-Kolchin. Este teorema establece que la imagen de un grupo de Lie resoluble que es representado en el grupo general lineal de un espacio vectorial puede ser simultáneamente transformado en un grupo de matrices triangulares superiores mediante conjugación.

En particular, la estructura de este trabajo es como sigue: Primero, se explican los conceptos de grupo de Lie, álgebra de Lie, la aplicación que los une (llamada aplicación exponencial), y la idea de homomorfismos e isomorfismos entre grupos de Lie y entre álgebras de Lie; y segundo, se estudian conceptos como resolubilidad o semisimplicidad, que permiten obtener una idea más especifica de cómo están estructurados las álgebras y los grupos de Lie, y después se usan estas ideas combinadas con teoría de la representación para presentar el teorema de Lie-Kolchin y cuáles son sus implicaciones.

This is a bachelor thesis on the theory of Lie groups and Lie algebras. It focuses of two objectives: On the one hand, it provides the reader with the basic concepts regarding Lie groups, Lie algebras, the connection between them, and some tools to understand their structure; on the other hand, it explores the Lie-Kolchin theorem, not only the motivations behind it and its statement, but also some related results.

This thesis is heavily focused on matrix case, so most of the definitions and examples are meant for matrix Lie groups and algebras. This is partially because a full geometric explanation would require including some concepts that would push us away from the main objectives of this thesis. The other argument in favour of the matrix cases is that the main objective of the thesis is the Lie-Kolchin theorem, which is a representation theory result. Since representation theory attempts to map abstract Lie groups and algebras to general linear groups, getting some insight of the effects of this theory on matrices will be useful for a better understanding of the Lie-Kolchin theorem. This theorem states that the image of a solvable Lie group represented in the general linear group of a vector space can be simultaneously transformed into a group of upper triangular matrices via conjugation.

In particular, the structure of the thesis is as follows; first, we explain the concepts of Lie group, Lie algebra, the map that join them (named exponential map), and the idea of homomorphisms and isomorphisms between Lie groups and between Lie algebras; and second, we study concepts as solvability and semisimplicy, that allow us to get a more specific idea of how Lie algebras and Lie groups are structured, and then we use this ideas combined with representation theory to present the Lie-Kolchin theorem and what are its implications.

Prologue

The theory of Lie groups and Lie algebras was born in the late 19th century by the hand of Sophus Lie, who settled its basis (that is why its is named after him). This theory intertwines the concept of a differential manifold and algebraic ideas, most importantly group theory and abstract algebra.

Thus, the theory of Lie Groups aim to provide a smooth manifold structure to a group, so that both the maps that generates the group structure and the homomorphisms that relate these groups become smooth. On the other hand, the theory connects these groups with some special algebras, the Lie algebras, through an special map called the exponential map.

In this thesis we will only study the correspondence between matrix Lie groups and matrix Lie algebras, but in the general case the Lie algebra of a Lie group can be constructed through the left-invariant vector fields of a Lie group, and then related though an exponential map created ex professo, whereas in this thesis the exponential map is defined before the correspondence is made. With this relations established, the theory of Lie groups and Lie algebras borrows concepts from group theory and differential topology to provide some results of its own. Another theory that is quite frequent to be used in the situation of Lie groups and Lie algebras is representation theory.

Finally, it is worth mentioning that the results that arise from this theory are then applied to many branches of mathematics such as algebra or geometry, and into applied mathematics as well. Furthermore, the theory of Lie groups has proved to be quite useful in mathematical physics, and thus it can be found in many parts of physics.

In this thesis, we will get started with the basic concepts and we will study some of the definitions and results that this theory borrows from group theory, and then we will apply them in representation theory to give a remarkable result: the Lie-Kolchin theorem. The idea of this theorem is to be able to provide some conditions under which a set of matrices can be simultaneously transformed into an upper triangular set of matrices. Some similar results were given in abstract algebra before the creation of this theory, and some generalizations have been proved since then, but the objective of simultaneously transformation into upper triangular matrices is still there due to the tremendous usefulness of the concept.

Without further ado, let us get started into this thesis.

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Chapter 1

First definitions for Lie Groups

1.1 Lie Group

Definition 1.1. A Lie Group is a smooth manifold G with a group structure compatible with its manifold structure, that is, with two maps

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\mapsto gh \end{aligned} \qquad\qquad \begin{aligned} G &\longrightarrow G \\ g &\mapsto g^{-1} \end{aligned}$$

so that the manifold structure makes these maps smooth.

The definition of Lie group is seemingly innocent but it has great implications on both its group and manifold structures. In order to make sure that the first ideas are understood we will define its arguably most important case: The matrix Lie group.

1.2 Matrix Lie group

Consider the set of $n \times n$ complex-valued matrices $M_n(\mathbb{C})$ and its subset of invertible $n \times n$ matrices, $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) |\ det\ A \neq 0\}$. In order to view them as Lie groups, we first need to provide them with a manifold structure. This can be done through the Frobenius (or Hilbert-Schmidt) norm:

Definition 1.2. $M_n(\mathbb{C})$ (and $GL_n(\mathbb{C})$) can be provided with a metric space structure by the norm

$$\|X\| = \sqrt{\sum_{i,j=1}^{n} \left|x_{ij}\right|^2}$$

Lemma 1.3. *This norm is submultiplicative, that is,* $\|AB\| \leq \|A\| \|B\|$.

Proof.

$$\|AB\| = \sqrt{\sum_{i,j=1}^{n} \sum_{k=1}^{n} \left|a_{ik} b_{kj}\right|^2} \leq \sqrt{\sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2 \sum_{k=1}^{n} \left|b_{kj}\right|^2 \right)} \leq \sqrt{\sum_{i,k=1}^{n} |a_{ik}|^2 \sum_{k,j=1}^{n} \left|b_{kj}\right|^2} = \|A\| \|B\|$$

$\square$

As $GL_n(\mathbb{C})$ is a metric space with this norm and since it is homeomorphic to $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$, it can be given a smooth manifold structure through this homeomorphism.

Of the other hand, it is easy to see that $GL_n(\mathbb{C})$ can be given a group structure, with the group multiplication and group inversion being the matrix multiplication and matrix inversion. Furthermore, both matrix multiplication and (thanks to Crammer's rule) matrix inversion are polynomial functions, and thus smooth.

Now that we have seen that $GL_n(\mathbb{C})$ is a Lie group, we can define a matrix Lie group as follows.

Definition 1.4. A matrix Lie Group is a closed subgroup of $GL_n(\mathbb{C})$.

It is easy to see that this definition makes sense since it inherits both the smooth manifold and group structures from $GL_n(\mathbb{C})$.

1.3 First examples

Because of the importance of matrix Lie groups in both the theory of Lie groups and its applications, they have been studied for a long time, and many of the properties obtained have been latter used in other branches not only of mathematics but also of physics and applications. In particular, some of the best known matrix Lie groups are the classical groups. We will introduce some of them as our first examples of Lie groups.

Examples. We will start with some classical groups

- First of all, $GL_n(\mathbb{C})$ itself can be viewed as a matrix Lie group, since it is a subgroup of itself (and trivially closed). Now, since $\mathbb{R}$ is closed in $\mathbb{C}$, $GL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{C}) \mid a_{ij} \in \mathbb{R}\}$ can be considered as a closed subgroup of $GL_n(\mathbb{C})$ and thus a matrix Lie Group.
- The special linear group $SL_n(\mathbb{C}) = \{A \in Gl_n(\mathbb{C}) \mid det\ A = 1\}$ is a closed subgroup of $GL_n(\mathbb{C})$: If $A, B \in SL_n(\mathbb{C})$ then
$$det\ A^{-1} = (det\ A)^{-1} = 1 \text{ and } det\ AB = det\ A\ det\ B = 1$$
Furthermore, $SL_n(\mathbb{R}) = SL_n(\mathbb{C}) \cap GL_n(\mathbb{R})$ is also a matrix Lie group.
- The unitary group $U(n) = \{A \in GL_n(\mathbb{C}) \mid AA^* = 1\}$ (where A^* is the conjugate traspose of A) is also a matrix Lie group: If $A, B \in SL_n(\mathbb{C})$ then
$$A^{-1}(A^{-1})^* = (A^*A)^{-1} = (AA^*)^{-1} = I^{-1} = I \text{ and } (AB)(AB)* = ABB^*A^* = AA^* = I$$
In particular, we can consider the special unitary group $SU(n) = U(n) \cap SL_n(\mathbb{C})$ as a matrix Lie Group.
- In the same way, the orthogonal group $O(n) = \{A \in GL_n(\mathbb{C}) \mid AA^t = 1\} = U(n) \cap GL_n(\mathbb{R})$ and the special orthogonal $SO(n) = O(n) \cap SL_n(\mathbb{R})$ are also matrix Lie groups.

Finally, for an important example which is not one of the classical groups, let us define the Heisenberg group as follows: Let H be the set of matrices in $GL_(\mathbb{C})$ of the form
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in GL_n(\mathbb{C})$$
where $a, b, c \in \mathbb{C}$. Then we have that
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix} \in H$$
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix} \in H$$
so we have a closed subgroup of $GL_n(\mathbb{C})$, that is, a matrix Lie group, known as the Heisenberg group. This group is used in physics for descriptions of quantum mechanics systems.

Chapter 2

Lie Algebras

The other key ingredient in the theory of Lie groups are Lie algebras and their relation with the former, as exploring these relations gives an insight of their structure and behaviour. We will start with the concept of a Lie algebra:

2.1 Lie algebra

Definition 2.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear, skew-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that the Jacobi identity holds, that is

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \ \forall A,B,C \in \mathfrak{g}$$

the map $[\cdot,\cdot]$ is called the Lie bracket.

In the same spirit as in the previous chapter, here we will work mainly with matrix cases, so that the concepts explained may be easier to follow. Thus, we will start considering our first matrix Lie algebra:

Proposition 2.2. *For the vector space $\mathfrak{g} = M_n(\mathbb{C})$ the operator*

$$[X,Y] = XY - YX$$

is a Lie bracket, so that $M_n(\mathbb{C})$ it becomes a Lie algebra.

Proof. First, notice that the map is a well-defined map from $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ to $M_n(\mathbb{C})$. Now, it is skew-symmetric as

$$[Y,X] = YX - XY = -(XY - YX) = -[X,Y]$$

it is also bilinear since for any $\alpha, \beta \in \mathbb{C}$

$$[\alpha X + \beta Y, Z] = (\alpha X + \beta Y)Z - Z(\alpha X + \beta Y) = \alpha XZ + \beta YZ - \alpha ZX - \beta ZY = \alpha[X,Z] + \beta[Y,Z]$$

and linearity on the first term and skew-symmetry implies bilinearity. Finally we have to prove that the Jacobi identity holds.

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = X[Y,Z] - [Y,Z]X + Y[Z,X] - [Z,X]Y + Z[X,Y] - [X,Y]Z$$

$$= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + ZXY - ZYX - XYZ + YXZ = 0$$

□

2.2 The (matrix) exponential map

The idea of the exponential map arises as the connecting map of a Lie group and a Lie algebra. Here, we will begin with the classical definition.

Definition 2.3. Given a matrix $X \in M_n(\mathbb{C})$, its exponential exp$= e^X$ is defined as

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

where it is understood that $X^0 = I$ (thus $e^{0_n} = I$).

Note that exp is absolutely convergent for every $X \in M_n(\mathbb{C})$, as thanks to Lemma 1.3 we have that $\|X^n\| \leq \|X\|^n$, which implies

$$\left\| e^X \right\| = \left\| \sum_{k=0}^{\infty} \frac{X^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|X\|^k}{k!} = e^{\|X\|} < \infty$$

Therefore we have a well-defined map exp: $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ (and in fact we will later prove next that $\exp(M_n(\mathbb{C})) \subset G_n(\mathbb{C})$). Furthermore, since exp can be seen as the limit of the functions $f_n(X) = \sum_{k=0}^{n} \frac{X^k}{k!}$ and these are continuous and bounded by $e^{\|X\|}$, Weierstrass M-test shows that exp is continuous. Now we will present some basic properties of exp:

Proposition 2.4. *Let $X, Y \in M_n(\mathbb{C})$, $g \in Gl_n(\mathbb{C})$, then*

1. $XY = YX \Rightarrow e^{X+Y} = e^X e^Y$
2. *If λ is an eigenvalue of X, e^λ is an eigenvalue of e^X*
3. $det\ e^X = e^{tr\ X}$
4. $e^{gXg^{-1}} = ge^X g^{-1}$
5. $\dfrac{d}{dt} e^{tX} = Xe^{tX} = e^{tX} X$
6. $X = \dfrac{d}{dt} e^{tX} \Big|_{t=0}$

Proof. We will consider $X, Y \in M_n(\mathbb{C})$, $g \in Gl_n(\mathbb{C})$, $\lambda \in \mathbb{C}$.

1. $$e^{X+Y} = \sum_{k=0}^{\infty} \frac{(X+Y)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} X^m Y^{k-m} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{X^m}{m!} \frac{Y^{k-m}}{(k-m)!} = e^X e^Y$$
   notice that in the second step we are allowed to write the binomial formula that way thanks to the fact that $XY = YX$.

2. Let λ be an eigenvalue of X and v an eigenvector of eigenvalue λ, then
   $$e^X = \sum_{k=0}^{\infty} \frac{X^k v}{k!} = \sum_{k=0}^{\infty} \frac{\lambda^k v}{k!} = e^\lambda v$$
   thus v is an eigenvector of e^X with eigenvalue e^λ.

3. Suppose that X has eigenvalues $z_1 \dots z_n$. Then by 2. we have $dete^X = e^{z_1} \dots e^{z_n} = e^{z_1 + \dots + z_n} = e^{tr\ X}$

4. $$e^{gXg^{-1}} = \sum_{k=0}^{\infty} \frac{(gXg^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{gX^k g^{-1}}{k!} = g \sum_{k=0}^{\infty} \frac{X^k}{k!} g^{-1} = ge^X g^{-1}$$

5.
$$\frac{d}{dt}e^{tX} = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{d}{dt}(tX)^k = \sum_{k=1}^{\infty}\frac{1}{k!}kX(tX)^{k-1} = X\sum_{k=1}^{\infty}\frac{(tX)^{k-1}}{(k-1)!} = X\sum_{m=0}^{\infty}\frac{(tX)^m}{m!} = Xe^{tX}$$
and the fact that $Xe^{tX} = e^{tX}X$ comes from the fact that X commutes with X^k $\forall k \in \mathbb{N}\cup\{0\}$.

6. It is a consequence of 4.

$\square$

Remark. Thanks to Proposition 2.4 we have the following consequences

- $(e^X)^{-1} = e^{-X}$ since $e^Xe^{-X} = e^0 = 1$ as X and $-X$ commute.
- $e^{(\alpha+\beta)X} = e^{\alpha X}e^{\beta X}$ with $\alpha,\beta \in \mathbb{C}$ as αX and βX commute.

Now, we will get into more meaningful properties of exp. First, we will study whether the exponential map is invertible.

Proposition 2.5. *The exponential map defines a bijection from a neighbourhood of $0_n \in M_n(\mathbb{C})$ to a neighbourhood of $I \in GL_n(\mathbb{C})$.*

Proof. Since by Proposition 2.4 $\frac{d}{dt}e^{tX}\big|_{t=0} = X$, so $d\ exp_0 = I$ and by the Inverse Function Theorem it follows that exp is a bijection on a sufficiently small neighbourhood of $0_n \in M_n(\mathbb{C})$. $\square$

We know that near $0_n \in M_n(\mathbb{C})$ the exponential map is invertible, but what does this inverse look like?

Proposition 2.6. *Let $U = \{A \in GL_n(\mathbb{C})|\ \|A-I\| < 1\}$, then the map $log : U \longrightarrow M_n(\mathbb{C})$ given by*
$$log\ X = \sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k}(X-I)^k$$
is well-defined on U. Furthermore,
$$e^{log\ X} = X\ \forall X \in U \text{ and } log\ e^X = X\ \forall X \in \{A \in M_n(\mathbb{C})|\ \|A\| < log\ 2\}$$

Proof. First, assume that X is diagonalizable. Hence $X = CDC^{-1}$ and
$$\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k}(X-I)^k = C\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k}(D-I)^k\right)C^{-1} =$$
$$= C\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k}\begin{pmatrix}(z_1-1)^k & & \\ & \ddots & \\ & & (z_n-1)^k\end{pmatrix}\right)C^{-1} = C\begin{pmatrix}log\ z_1 & & \\ & \ddots & \\ & & log\ z_n\end{pmatrix}C^{-1}$$

Since $\|I-D\| = \left\|C^{-1}C(I-D)C^{-1}C\right\| \leq \left\|C^{-1}\right\|\left\|C(I-D)C^{-1}\right\|\|C\|^{-1} = \|X-I\| < 1$ we get that $|z_i-1| < 1\ \forall z_i$ eigenvalue of X, so log converge in U. Now, if $\|X-I\| < 1$ then
$$e^{log\ X} = exp\left(C\begin{pmatrix}log\ z_1 & & \\ & \ddots & \\ & & log\ z_n\end{pmatrix}C^{-1}\right) = C\begin{pmatrix}e^{log\ z_1} & & \\ & \ddots & \\ & & e^{log\ z_n}\end{pmatrix}C^{-1} = CDC^{-1} = X\ \forall X \in U$$

Conversely, if $\|X\| < log\ 2$ then $\left\|e^X\right\| < 2$ and $\left\|e^X - I\right\| \leq \left|\left\|e^X\right\| - \|I\|\right| = |2-1| = 1$, so $e^X \in U$ and

$$log\ e^X = log\ e^{CDC^{-1}} = log\ \left(Ce^DC^{-1}\right) = C\left(log\begin{pmatrix} e^{z_1} & & \\ & \ddots & \\ & & e^{z_n}\end{pmatrix}\right)C^{-1} = CDC^{-1} = X$$

Finally, if X is not diagonalizable, take a sequence of diagonalizable matrices $X_n \to X$ and appeal to the continuity of exp and log as power series to show that the limit of the image is the image of the limit. □

The inverse map is denoted *log* due to the analogy between exp and the usual exponential function in $\mathbb{R}$. Next, we will give a property of e^{X+Y} which is very useful in the case when $X, Y \in M_n(\mathbb{C})$ do not commute:

Theorem 2.7. *(Lie product formula) Let $X, Y \in M_n(\mathbb{C})$, then*

$$e^{X+Y} = \lim_{n\to\infty}(e^{X/n}e^{Y/n})^n$$

Proof. To begin with notice that if n is big enough both X/n and Y/n are sufficiently small so that $e^{X/n}e^{Y/n}$ falls in a neighbourhood of I for which *log* is defined. Thus if we multiply the series we get

$$e^{X/n}e^{Y/n} = I + X/n + Y/n + O(1/n^2)$$

applying *log*

$$log(e^{X/n}e^{Y/n}) = X/n + Y/n + O\left(\left\|X/n + Y/n + \frac{1}{n^2}\right\|^2\right) = X/n + Y/n + O(1/n^2)$$

and getting this term again through exp

$$e^{X/n}e^{Y/n} = e^{X/n+Y/n+O(1/n^2)} \Rightarrow (e^{X/n}e^{Y/n})^n = e^{X+Y+O(1/n)}$$

This implies that $\lim_{n\to\infty}(e^{X/n}e^{Y/n})^n = e^{X+Y}$. □

With these understanding of the exponential map, it is time to see the concept that will allow us to define the Lie algebra of a Lie group: one-parameter subgroups.

2.3 One-parameter subgroups

Definition 2.8. Let G be a matrix Lie group. A one-parameter subgroup of G is a continuous map $\gamma : \mathbb{R} \longrightarrow \mathbb{G}$ such that $\gamma(0) = I$ and $\gamma(t+s) = \gamma(t)\gamma(s)$.

Interestingly enough this conditions are sufficient to characterize one-parameter subgroups:

Proposition 2.9. *If $\gamma(t)$ is a one-parameter subgroup of G, then $\gamma(t) = e^{tZ}$ for some $Z \in M_n(\mathbb{C})$.*

Proof. First, we claim that that if $B_\varepsilon = \{A \in M_n(\mathbb{C}) | \ \|A\| < \varepsilon\}$ with $\varepsilon < log2$ and if U =exp$(B_{\varepsilon/2})$ then $\forall B \in U$ there exists a unique square root $C = e^{log(B)/2}$ on U. Indeed, C is clearly a square root of B in U. Now let C' be another square root of B in U, then there exists $Y \in B_{\varepsilon/2}$ such that $e^Y = C'$, and thus

$$e^{2Y} = C'^2 = B = e^{log(B)}$$

Therefore since $2Y \in B_\varepsilon$, $log\ B \in B_{\varepsilon/2} \subset B_\varepsilon$ and since by Proposition 2.5 exp is bijective in B_ε we get that $2Y = log\ B \Longleftrightarrow Y = log(B)/2 \Longleftrightarrow C' = e^Y = e^{log(B)/2} = C$.

Now, since $\gamma(0) = I$, there exists a $t_0 \neq 0$ small enough so that $\gamma(t_0) \in U$. We define $Z = \frac{1}{t_0} log(\gamma(t_0))$ so that $\gamma(t_0) = e^{t_0 Z}$. Now, since $\gamma(t+s) = \gamma(t)\gamma(s)$ we have that $\gamma(\alpha t) = \gamma(t)^{\alpha}$ $\forall \alpha \in \mathbb{N}$. Thus, $\gamma(t_0/2)$ equals to the unique square root of $e^{t_0 Z}$ in U, that is,

$$\gamma(\frac{t_0}{2}) = e^{\frac{t_0}{2}\,Z} \Rightarrow \gamma(\frac{t_0}{2^k}) = e^{\frac{t_0}{2^k}\,Z} \Rightarrow \gamma(m\frac{t_0}{2^k}) = \gamma(\frac{t_0}{2^k})^m = (e^{\frac{t_0}{2^k}\,Z})^m = e^{m\frac{t_0}{2^k}\,Z}\ \forall k,m \in \mathbb{N}$$

Thus $\gamma(m\frac{t_0}{2^k}) = e^{m\frac{t_0}{2^k}\,Z}$ $\forall k,m \in \mathbb{N}$, and by continuity of exp necessarily $\gamma(t) = e^{tZ}$ $\forall t \in \mathbb{R}$. □

It is worth noting that a one-parameter subgroup e^{tZ} is usually identified with its image, the set $\{e^{tZ} |\ t \in \mathbb{R}\}$ for a given $Z \in M_n(\mathbb{C})$. Now, we will give some properties of the one-parameter subgroups.

Proposition 2.10. *Let $X,Y \in M_n(\mathbb{C})$, then*

1. $[X,Y] = \dfrac{d}{dt} e^{tX} Y e^{-tX}\Big|_{t=0}$
2. $e^{tX} = e^{tY}$ $\forall t \in \mathbb{R} \Rightarrow X = Y$

Proof. For the first one we have that

$$\frac{d}{dt} e^{tX} Y e^{-tX}\big|_{t=0} = Xe^{tX}Ye^{-tX} - e^{tX}Ye^{-tX}X\big|_{t=0} = XY - YX = [X,Y]$$

As for the second, if $e^{tX} = e^{tY}$ $\forall t \in \mathbb{R}$, then we can take t_0 small enough so that we can use log, and

$$tX = log(e^{tX}) = log(e^{tY}) = tY\ \forall t < t_0 \Rightarrow X = Y$$

□

2.4 Lie algebra of a Lie group

Now we are in a position to explain which is the role does the exponential map play on the relation between Lie groups and Lie algebras.

Definition 2.11. The Lie algebra of a Lie group G is $Lie(G) = \{X \in M_n(\mathbb{C}) |\ e^{tX} \in G\ \forall t \in \mathbb{R}\}$.

As we can see, one-parameter subgroups play a vital role on this relation. The first question that arises is whether $Lie(G)$ is actually a Lie algebra as we have defined on Definition 2.1.

Theorem 2.12. *Let G be a matrix Lie Group. Then $Lie(G)$ is a Lie algebra with the usual Lie bracket $[X,Y] = XY - YX$.*

Proof. First, if $X \in Lie(G)$ and $\alpha \in \mathbb{R}$ then $e^{t(\alpha X)} = e^{(t\alpha)X} \in G$ $\forall t \in \mathbb{R}$ as $t\alpha \in \mathbb{R}$, so $\alpha X \in Lie(G)$. Now, given $X,Y \in Lie(G)$ we have that

$$e^{t(X+Y)} = \lim_{n\to\infty} (e^{tX/m} e^{tY/m})^m \in G\ \forall t \in \mathbb{R}$$

since $X/m, Y/m \in Lie(G)$ so $e^{tX/m}e^{tY/m} \in G$, and since G is closed in $GL_n(\mathbb{C})$ the limit is also in G. Finally, using property 1 of the Remark in section 2.2 we get that $e^{tX}Ye^{-tX} \in Lie(G)$ $\forall t \in \mathbb{R}$ as

$$e^{s(e^{tX}Ye^{-tX})} = e^{e^{tX}(sY)e^{-tX}} = e^{tX}e^{sY}e^{-tX} \in G\ \forall s \in \mathbb{R}$$

so thanks to Proposition 2.10 we have that

$$[X,Y] = \frac{d}{dt} e^{tX}Ye^{-tX}\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\left(e^{tX}Ye^{-tX} - Y\right) \in Lie(G)$$

since the limit is contained in G as it is closed. □

2.5 Examples

Now that we have seen how a Lie group defines a Lie algebra, it is time to illustrate this relation through some examples.

Examples. Lets consider the Lie algebras of the examples given in section 1.3:

- We know that $e^{tX} \in GL_n(\mathbb{C})$ for every $X \in M_n(\mathbb{C})$, so $M_n(\mathbb{C}) \subseteq Lie(GL_n(\mathbb{C})) \subseteq M_n(\mathbb{C})$, and thus

$$\mathfrak{gl}_n(\mathbb{C}) := Lie(GL_n(\mathbb{C})) = M_n(\mathbb{C})$$

- Now let $X \in Lie(SL_n(\mathbb{C}))$, then $e^{tX} \in SL_n(\mathbb{C})\ \forall t \in \mathbb{R}$, that is, $det\ e^{tX} = 1\ \forall t \in \mathbb{R}$. By Proposition 2.4 this implies that $tr\ tX = 0\ \forall t \in \mathbb{R} \Rightarrow tr\ X = 0$. Conversely, if $A \in M_n(\mathbb{C})$ has $tr\ A = 0$, then $det\ e^{tA} = 1 \forall t \in \mathbb{R}$, and thus $A \in \mathfrak{sl}_n(\mathbb{C})$. We conclude that

$$\mathfrak{sl}_n(\mathbb{C}) := Lie(SL_n(\mathbb{C})) = \{X \in M_n(\mathbb{C}) |\ tr\ X = 0\}$$

- For $U(n)$, consider $X \in Lie(U(n))$, then $e^{tX}(e^{tX})^* = I$, so $(e^{tX})^* = (e^{tX})^{-1}$, that is, $e^{t(X^*)} = e^{-tX}$. Since this holds for every $t \in \mathbb{R}$, we have that $X^* = -X$, so X is skew-hermitian. Conversely if $A^* = -A$, then $(e^{tA})^* = (e^{tA})^{-1}$ and thus $e^{tA} \in U(n)\ \forall t \in \mathbb{R}$, so we get that

$$\mathfrak{u}(n) := Lie(U(n)) = \{X \in M_n(\mathbb{C}) |\ X^* = -X\}$$

  in the same spirit we can consider

$$\mathfrak{su}(n) := Lie(SU(n)) = \{X \in M_n(\mathbb{C}) |\ X^* = -X, tr\ X = 0\}$$

- Following the same steps as with $\mathfrak{u}(n)$, we can deduce that

$$\mathfrak{o}(n) := Lie(O(n)) = \{X \in M_n(\mathbb{R}) | X^t = -X\}$$

  and

$$\mathfrak{so}(n) := Lie(SO(n)) = \{X \in M_n(\mathbb{R}) | X^t = -X, tr\ X = 0\}$$

- Finally, we will compute the Lie algebra of H, the Heisenberg group. We are looking for a matrix $X \in M_n(\mathbb{C})$ such that $e^{tX} \in H\ \forall t \in \mathbb{R}$. Since e^{tX} has to be upper triangular matrix, and looking at the shape of the exponential map, we can deduce that X also has to be upper triangular. Furthermore, the eigenvalues of e^{tX} are the elements in its diagonal, which we know that are 1. Thus, the eigenvalues of X (and its diagonal values) have to be 0, so that $e^0 = 1$. With all this, we can deduce that

$$\mathfrak{h} := Lie(H) = \left\{ X \in M_n(\mathbb{C}) \middle| X = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{pmatrix} \alpha, \beta, \delta \in \mathbb{C} \right\}$$

  $\mathfrak{h}$ is obviously closed under addition and product by scalars. Furthermore, since the product of two strictly upper triangular matrices is another strictly upper triangular matrix, we have that $\mathfrak{h}$ is closed under the Lie bracket, and thus it is a Lie algebra, the Lie algebra of H.

2.6 BCH formula

To finish with this chapter, we will include one useful tool to find $C \in Lie(G)$ so that $e^A e^B = e^C$ where G is a matrix Lie group and $A, B \in Lie(G)$ are such that $\|A\|, \|B\| < r$ for $r \in \mathbb{R}$ small enough. In this situation, $e^A e^B$ falls in U, the neighbourhood of the identity where exp is a bijection, so there exists $C = C(A,B) \in log(U)$ such that $e^A e^B = e^{C(A,B)}$. Furthermore:

Proposition 2.13. $C(A,B) = A + B + \frac{1}{2}[A.B] + O(r^3) \in log(U)$.

Proof. Suppose that $C(A,B) = C_1 + O(r^2)$ where $C_1 = O(r)$ then computing the exp power series up to order 1 we get

$$e^{C(A,B)} = I + C_1 + O(r^2)$$

On the other hand

$$e^A e^B = (I + A + O(r^2))(I + B + O(r^2)) = I + A + B + O(r^2)$$

Thus

$$0 = e^{C(A,B)} - e^A e^B = C_1 - A + B + O(r^2) \Rightarrow C_1 = A + B + O(r^2)$$

This argument can be repeated to obtain the next order coefficient: If $C(A,B) = A + B + C_2 + O(r^3)$, then

$$e^{C(A,B)} = I + A + B + C_2 + \frac{1}{2}(A + B + C_2)^2 + O(r^3) = I + A + B + C_2 + \frac{1}{2}(A+B)^2 + O(r^3)$$

and we can repeat the argument computing

$$e^A e^B = (I + A + \frac{1}{2}A^2 + O(r^3))(I + B + \frac{1}{2}B^2 + O(r^3)) = I + A + B + AB + \frac{1}{2}(A^2 + B^2) + O(r^3)$$

Hence we get

$$C_2 + \frac{1}{2}(A+B)^2 = AB + \frac{1}{2}(A^2 + B^2) + O(r^3) \Rightarrow$$

$$\Rightarrow C_2 + \frac{1}{2}(AB + BA) = AB + O(r^3) \Rightarrow C_2 = \frac{1}{2}(AB - BA) + O(r^3) = \frac{1}{2}[A,B] + O(r^3)$$

Therefore $C(A,B) = A + B + \frac{1}{2}[A,B] + O(r^3)$. □

Furthermore, the same process can be repeated to get higher order terms of $C(A,B)$:

Proposition 2.14.

$$C(A,B) = I + A + B + \sum_{k=2}^{\infty} C_k(A,B)$$

where the $C_k(A,B)$ can be expressed in terms of the Lie brackets.

Remark. As the previous proposition stated, we have that

$$log(e^X e^Y) = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}\Big([X,[X,Y]] + [Y,[Y,X]]\Big) - \tfrac{1}{24}[Y,[X,[X,Y]]] + \dots$$

Although we will not need to explicitly describe these $C_k(A,B)$ here, it is remarkable that the $C(A,B)$ can be described with the Baker–Campbell–Hausdorff formula. To show this formula, we will use some useful notation.

Definition 2.15. $[X,\cdot]$ can be also considered as a map,

$$\begin{aligned} ad_X : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\mapsto [X,Y] \end{aligned}$$

named the adjoint map.

The adjoint map notation is very convenient since it allow us to write $ad_x ad_y ad_x ad_x(Y)$ instead of $[X,[Y,[X,[X,Y]]]]$.

Theorem 2.16. *(Baker–Campbell–Hausdorff formula) Let $X,Y \in Lie(G)$ with $\|X\|,\|Y\|$ sufficiently small (see [2], pg.119), then*

$$log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{t\ ad_Y})(Y)dt$$

where

$$g(z) = \frac{logz}{1-z^{-1}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}(z-1)^n$$

Chapter 3

Homomorphisms

3.1 Lie group and Lie algebra homomorphisms

Now that Lie groups and Lie algebras have been defined, it is time to study how do they interact with other Lie groups and Lie algebras. In group theory, groups are related through homomorphisms. In differential topology, smooth manifolds are related through smooth maps. With the next definition, we are looking for a notion that joins both concepts:

Definition 3.1. Let G, H be Lie groups. A Lie group homomorphism $\Phi : G \to H$ is a group homomorphism so that the manifold structures of G and H make ϕ smooth. In addition, if Φ is bijective, then it is called a Lie group isomorphism, and it is said that G is isomorphic to H (or that G and H are isomorphic, since it is easy to check that this relation is symmetric).

Of course, we also want to relate Lie algebras through a similar map.

Definition 3.2. Let $\mathfrak{g} \longrightarrow \mathfrak{h}$ be Lie algebras. A Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a linear map compatible with the Lie bracket of the Lie algebras, i.e., such that $\phi[X,Y] = [\phi(X), \phi(Y)]$. In addition, if ϕ is bijective, then it is called a Lie algebra isomorphism, and it is said that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ (or that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic).

3.2 Group to algebra homomorphism correspondence

Now that the concepts of Lie group homomorphism and Lie algebra have been defined, the next question that naturally arises is how do these definitions interact with the relations between Lie groups and Lie algebras we have studied before. This interaction can be express through the exponential map:

Theorem 3.3. *Let, G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, and let $\Phi : G \longrightarrow H$ be a Lie group homomorphism. Then there exists a unique Lie algebra homomorphism $d\Phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that*

$$e^{d\Phi(X)} = \Phi(e^X)$$

Proof. The idea of this theorem is to find a Lie algebra homomorphism $d\Phi$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, dashed, "d\Phi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

For every $X \in \mathfrak{g}$ we have that the map $\gamma : t \mapsto \Phi(e^{tX})$ with $t \in \mathbb{R}$ is a one-parameter subgroup, thus because of 2.9 we know that there exists $Z(X) \in \mathfrak{h}$ such that $\Phi(e^{tX}) = e^{tZ(X)} \in H$ $\forall t \in \mathbb{R}$. It is easy to see that $Z(X)$ is uniquely defined. Therefore, we define $d\Phi(X) := Z(X)\ \forall X \in \mathfrak{g}$. Furthermore, this definition implies that $d\Phi(\alpha X) = \alpha\, d\Phi(X)$ for $\alpha \in \mathbb{R}$ as

$$e^{d\Phi(\alpha X)} = \Phi(e^{\alpha X}) = \Phi((e^X)^\alpha) = \Phi(e^X)^\alpha = (e^{d\Phi(X)})^\alpha = e^{\alpha\, d\Phi(X)}$$

We also have that

$$e^{d\Phi(X+Y)} = \Phi(e^{X+Y}) = \lim_{n\to\infty} \Phi((e^{X/n})^n(e^{Y/n})^n) = \lim_{n\to\infty} (e^{d\Phi(X/n)})^n(e^{d\Phi(X/n)})^n = e^{d\Phi(X)+d\Phi(Y)}$$

so $d\Phi(X+Y) = d\Phi(X) + d\Phi(Y)$, and we have that $d\Phi$ is linear. The only thing left to prove is that the map respects the Lie bracket. Notice that $d\Phi(gXg^{-1}) = \Phi(g)d\Phi(X)\Phi(g)^{-1}$ since

$$e^{\Phi(g)d\Phi(X)\Phi(g)^{-1}} = \Phi(g)e^{d\Phi(X)}\Phi(g)^{-1} = \Phi(ge^Xg^{-1}) = \Phi(e^{gXg^{-1}}) = e^{d\Phi(gXg^{-1})}$$

Then combining this with Proposition 2.10 we get that

$$d\Phi([X,Y]) = d\Phi\Big(\frac{d}{dt}e^{tX}Ye^{-tX}\big|_{t=0}\Big) = \frac{d}{dt}d\Phi(e^{tX}Ye^{-tX})\big|_{t=0} =$$

$$= \frac{d}{dt}\Phi(e^{tX})d\Phi(Y)\Phi(e^{-tX})\big|_{t=0} = \frac{d}{dt}e^{td\Phi(X)}d\Phi(Y)e^{-td\Phi(X)}\big|_{t=0} = [d\Phi(X), d\Phi(Y)]$$

□

Definition 3.4. Let $\Phi : G \longrightarrow H$ be a Lie group homomorphism. We say that a Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is the differential of Φ if $\Phi(e^X) = e^{\phi(X)}$ for every $X \in \mathfrak{g}$.

This theorem is quite relevant, not only for the relation between Lie group homomorphisms and Lie group algebras, but also for the relation between its Lie groups and Lie algebras:

Corollary 3.5. *If G and H are isomorphic, then their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are also isomorphic.*

It comes without saying that this corollary is very interesting, as it allow us to study the Lie algebra of a Lie group by studying the Lie algebra of a Lie group isomorphic to the former.

3.3 Simple connection

Of course, the next question we have to ask is whether this relation works the other way around: if we have a Lie algebra isomorphism, does it define a unique Lie group homomorphism? If the Lie algebras of two Lie groups are isomorphic, does that mean that these Lie groups have to be isomorphic? The answer, in general, is no. Nevertheless, we can include one more condition that allows us to answer affirmatively both questions: Simple connection.

Definition 3.6. A Lie group G is said to be connected if for any $g_1, g_2 \in G$ there exist a path connecting g_1 and g_2, and it is said to be simply connected if any two paths connecting g_1 and g_2 are homotophic.

Observe that since G has a manifold structure, then G is connected (as a manifold) if and only if it is path-connected. Therefore it is useful to define the notion of being connected for Lie groups with such a handy concept as path-connectedness.

3.4 Algebra to group homomorphism correspondence

As for now, we will focus in the task at hand: showing that under the simple connection condition there is a correspondence from Lie algebras to Lie groups.

Theorem 3.7. *Let G,H be Lie groups with Lie algebras $\mathfrak{g},\mathfrak{h}$ such that G is simply connected, and let $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a unique Lie group homomorphism $\Phi : G \longrightarrow H$ such that $e^{\phi(X)} = \Phi(e^X)$.*

Proof. Let U be a neighbourhood of the identity of G sufficiently small, such that both the BCH formula holds in U and exp defines a bijection between U and its image. Now, we define

$$f = \exp \circ \phi \circ log$$

where $log := \exp^{-1}$. Therefore we have a map on U

$$\begin{tikzcd} U \arrow[r, dashed, "f"] & H \\ log(U) \arrow[u, "\exp"] \arrow[r, "\phi"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Notice also that f is an homomorphism: If $A_1,A_2 \in U$, then there exist $B_1,B_2 \in log(U)$ such that $A_1 = e^{B_1}$ and $A_2 = e^{B_2}$. Furthermore, by Proposition 2.13 there exists $C = C(A,B) \in log(U)$ so that $A_1A_2 = e^{B_1}e^{B_2} = e^C$, and the BCH formula implies that C is a linear combination of monomials which are Lie brackets in B_1,B_2. As ϕ is a Lie algebra homomorphism it preserves those brackets and we can deduce that $\phi(C)$ is the result of applying the BCH formula to $\phi(B_1)\phi(B_2)$. Thus $e^{\phi(B_1)}e^{\phi(B_2)} = e^{\phi(C)}$ and

$$f(A_1)f(A_2) = e^{\phi(log(A_1))}e^{\phi(log(A_1))} = e^{\phi(B_1)}e^{\phi(B_2)} = e^{\phi(C)} = e^{\phi(log(A_1A_2))} = f(A_1A_2)$$

Now we want to extend this homomorphism f to all G. Since G is connected, for any $g \in G$ there is a path α from the identity I to g. Let $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ be a partition of $[0,1]$ such that $\alpha(t_i)\alpha(t_{i-1})^{-1} \in U \ \ \forall i = 1,\dots k$. Therefore, since log is defined in U, we can find $X_i \in log(U)$ such that $\alpha(t_i)\alpha(t_{i-1})^{-1} = e^{X_i} \ \forall i = 1\dots k$. Then, we have that

$$g = e^{X_k}e^{X_{k-1}}\dots e^{X_2}e^{X_1}$$

Since the map we are looking for should be a Lie homomorphism and $e^{X_i} \in U \ \forall i = 1\dots k$, we would like to define a map

$$\Phi(g) = f(e^{X_k})f(e^{X_{k-1}})\dots f(e^{X_2})f(e^{X_1})$$

but so far we do not know if this value depends on the choice of path α or the partition $t_0 < \cdots < t_k$. Therefore, we must make sure that this map is well defined.

First, we will modify our path α to create another path α' by adding an element $s \in (t_{i-1},t_i)$ into our partition so that $\alpha'(t_i)\alpha'(s)^{-1}$ and $\alpha'(s)\alpha'(t_{i-1})^{-1}$ fall in U. Then we can find $Y_1,Y_2 \in log(U)$ such that $\alpha'(t_i)\alpha'(s)^{-1} = e^{Y_1}$ and $\alpha'(s)\alpha'(t_{i-1})^{-1} = e^{Y_2}$. Now, in order to prove that the value of $\Phi(g)$ does not change, we need

$$f(e^{X_i}) = f(e^{Y_1})f(e^{Y_2})$$

But since $e^{X_i} = \alpha(t_i)\alpha(t_{i-1})^{-1} = \alpha(t_i)\alpha(s)\alpha(s)^{-1}\alpha(t_{i-1})^{-1} = e^{Y_1}e^{Y_2}$ and f is an homomorphism, we have that

$$f(e^{Y_1})f(e^{Y_2}) = f(e^{Y_1}e^{Y_2}) = f(e^{X_i})$$

so α and α' give the same value for $\Phi(g)$. Furthermore, since G is simply connected, any two paths α and α'' connecting I with g are homotopic, so there exists $\psi : [0,1]^2 \longrightarrow G$ such that $\psi(0,\cdot) = \alpha$, $\psi(1,\cdot) = \alpha''$, and $\psi(t,\cdot)$ is a path from I to g. Then, we can find elements $p_n = \psi(t,s)$ so that we can modify our path α into α'' in the same way as we just did.

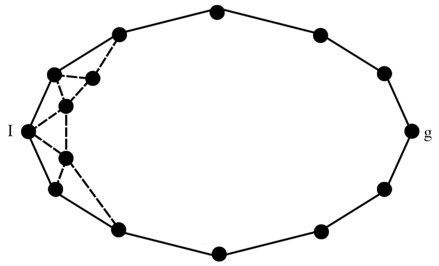

Figure 3.1: A path α can be altered into α' through these modifications.

Thus we have defined a map

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & H \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \end{array}$$

such that $d\Phi = \phi$, so the only thing left to prove is that it is a Lie group homomorphism. Since it is clearly smooth by definition, we have to prove that it is an homomorphism. To do so, let α be as above, and for an element $h \in G$ let β be a path from I to h with a partition $s_0 < \cdots < s_m$ as above. Then, we define

$$\gamma(t) = \begin{cases} \beta(t/2),\ t \in [0,1/2] \\ \alpha(2t-1)h\ t \in [1/2,1] \end{cases}$$

and we want to prove that $0 = s_0/2 < \cdots < s_m/2 = 1/2 = 1/2 + t_0/2 < \ldots 1/2 + t_k/2 = 1$ is a partition of γ with the same properties the previous partitions had. In fact, if $\beta(s_i)\beta(s_{i-1})^{-1} = e^{Y_i} \in U$ then

$$\gamma(s_i/2)\gamma(s_{i-1}/2)^{-1} = \beta(s_i)\beta(s_{i-1})^{-1} = e^{Y_i} \in U$$

$$\gamma(1/2+t_i/2)\gamma(1/2+t_{i-1}/2)^{-1} = \alpha(t_i)hh^{-1}\alpha(t_{i-1})^{-1} = \alpha(t_i)\alpha(t_{i-1})^{-1} = e^{X_i} \in U$$

So we get that

$$\Phi(gh) = f(\gamma(1/2+t_k/2)\gamma(1/2+t_{k-1}/2)^{-1})\ldots f(\gamma(s_2/2)\gamma(s_1/2)^{-1})f(\gamma(s_1/2)) =$$

$$f(e^{X_k})f(e^{X_{k-1}})\ldots f(e^{X_1})f(e^{Y_m})f(e^{Y_{m-1}})\ldots f(e^{Y_0}) = \Phi(g)\Phi(h)$$

□

Chapter 4

Solvable and semisimple algebras.

In this chapter we will give some useful tools to understand the behaviour of a Lie algebra by studying some special subalgebras of it (and analogously with Lie groups).

4.1 Solvable, nilpotent and abelian algebras

This first section consist on adapting some concepts of algebra and group theory into our theory of Lie algebras. Indeed, some of this definitions are actually the same for discrete groups (i.e. groups without a topology).

First, consider a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{a} \subset \mathfrak{g}$. Then, the quotient vector space $\mathfrak{g}/\mathfrak{a}$ can be given a Lie group structure by setting $[X+\mathfrak{a}, Y+\mathfrak{a}] := [X,Y]+\mathfrak{a}$. This algebra is called the quotient Lie algebra of $\mathfrak{g}$ and $\mathfrak{a}$.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra over a field K. We define the commutator of $\mathfrak{g}$ as

$$[\mathfrak{g},\mathfrak{g}] = \operatorname{span}\{[X,Y] \in \mathfrak{g} |\ X,Y \in \mathfrak{g}\}$$

It is easy to see that $[\mathfrak{g},\mathfrak{g}]$ is an ideal, since $\forall X \in \mathfrak{g}, [Y,Z] \in [\mathfrak{g},\mathfrak{g}]$ we have that $[X,[Y,Z]] \in [\mathfrak{g},\mathfrak{g}]$.

Proposition 4.2. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, then $\mathfrak{h}$ is abelian (i.e. $[\mathfrak{h},\mathfrak{h}] = \{0\}$).*

Proof. Let $A,B \in \mathfrak{g}$ then $[A+\mathfrak{g}, B+\mathfrak{g}] \in \mathfrak{h}$ and $[A+[\mathfrak{g},\mathfrak{g}], B+[\mathfrak{g},\mathfrak{g}]] = [A,B]+[\mathfrak{g},\mathfrak{g}] = [\mathfrak{g},\mathfrak{g}]$, therefore $[\mathfrak{h},\mathfrak{h}] = \{0\}$. □

Now, out of this concept of $[\mathfrak{g},\mathfrak{g}]$, we will build the following chains of ideals:

Definition 4.3. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}^1 = \mathfrak{g}_1 = \mathfrak{g}$, and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k] \qquad\qquad \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}]$$

A Lie algebra $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^n = \{0\}$ for some $n \in \mathbb{N}$, and it is said to be nilpotent if $\mathfrak{g}_n = \{0\}$ for some $n \in \mathbb{N}$. A Lie group G is said to be solvable if its Lie algebra is solvable, and it is said to be nilpotent if its Lie algebra is nilpotent. In this case G is solvable (resp. nilpotent) as a group in the classical sense.

Of course, the names of these definitions arise from the analogous concepts in group theory (notice that if $\mathfrak{g} \subseteq GL_n(\mathbb{C})$ is abelian then $XY = YX\ \forall X,Y \in \mathfrak{g}$). As the similarities between their definitions suggest, these three concepts are closely related.

Proposition 4.4. *If $\mathfrak{g}$ is abelian then it is nilpotent. If $\mathfrak{g}$ is nilpotent then it is solvable.*

Proof. Clearly $[\mathfrak{g}^k, \mathfrak{g}^k] \subseteq [\mathfrak{g}, \mathfrak{g}^k]$ since $\mathfrak{g}^k \subseteq \mathfrak{g}$, so if $\mathfrak{g}$ is nilpotent $\mathfrak{g}_n = \{0\}$, so $\mathfrak{g}^n \subseteq \mathfrak{g}_n = \{0\}$ implies $\mathfrak{g}^n = \{0\}$, that is, $\mathfrak{g}$ is solvable. Now, if $\mathfrak{g}$ is abelian, then $\mathfrak{g}_2 = [\mathfrak{g},\mathfrak{g}] = \{0\}$, and $\mathfrak{g}$ is nilpotent. □

Now we will show a result intertwining solvable Lie algebras an quotient Lie algebras, that will be useful later on.

Theorem 4.5. *Let* $\mathfrak{g}$ *be a Lie group and let* $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$*. If both* $\mathfrak{a}$ *and* $\mathfrak{g}/\mathfrak{a}$ *are solvable, then* $\mathfrak{g}$ *is solvable.*

Proof. Let $q : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$ be the quotient homomorphism. If $\mathfrak{g}/\mathfrak{a}$ is solvable, there exists $j \in \mathbb{N}$ such that $(\mathfrak{g}/\mathfrak{a})^j = \{0\}$. Since q is surjective we have that $q(\mathfrak{g}^k) = (\mathfrak{g}/\mathfrak{a})^k\ \forall k \in \mathbb{N}$. In particular $q(\mathfrak{g}^j) = \{0\}$, that is, $\mathfrak{g}^j \subseteq \mathfrak{a}$. But if $\mathfrak{a}$ is solvable, then $\mathfrak{a}^m = \{0\}$ for some $m \in \mathbb{N}$, so $\mathfrak{g}^{j+m} = (\mathfrak{g}^j)^m \subseteq \mathfrak{a}^j = \{0\}$. Therefore $\mathfrak{g}$ is solvable. □

Solvable and nilpotent Lie algebras are frequent concepts in theory of Lie algebras, as some results based on these are later used in other parts of the theory. In particular, the next three examples happen to be very relevant.

Examples. To illustrate these concepts, we will consider again Lie algebras in $M_n(\mathbb{C})$:

1. The set of strictly upper triangular $n \times n$ matrices $\mathfrak{a}$, besides being obviously a Lie algebra, is a nilpotent Lie algebra. If $A, B \in \mathfrak{a}$ are strictly upper triangular, then

$$[A,B]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} - b_{ik}a_{kj} = \sum_{k=i+1}^{j-1} a_{ik}b_{kj} - b_{ik}a_{kj}$$

since $a_{ij}, b_{ij} = 0\ \forall i < j.$. So in particular

$$[A,B]_{i,i+1} = \sum_{k=1}^{n} a_{ik}b_{k,i+1} - b_{ik}a_{k,i+1} = \sum_{k=i+1}^{i} a_{ik}b_{k,i+1} - b_{ik}a_{k,i+1} = 0\ \forall i = 1,\dots n$$

Now, let $D = [E,F] \in \mathfrak{a}_2$, where $E \in \mathfrak{a}_1, F \in \mathfrak{a}$, then $D_{i,i+1} = 0\ \forall i = 1,\dots n$ since $\mathfrak{a}_2 \subseteq \mathfrak{a}_1$. Furthermore,

$$D_{ij} = \sum_{k=i+2}^{j-1} e_{ik}f_{kj} - \sum_{k=i+1}^{j-2} e_{ik}f_{kj}$$

since $e_{ij} = 0\ \forall i+1 < j$ and $f_{ij} = 0\ \forall i < j$. In particular

$$D_{i,i+2} = \sum_{k=i+2}^{i+1} e_{ik}f_{kj} - \sum_{k=i+1}^{i} e_{ik}f_{kj} = 0\ \forall i = 1,\dots n$$

Since $\mathfrak{a}_2$ is the span of the element of the form of D, we get that $D'_{i,i+2}\ \forall D' \in \mathfrak{a}_2$. Inductively we get that if $C \in \mathfrak{a}_k$, then $c_{i,i+k} = 0\ \forall i < k < n-i$, so $\mathfrak{a}_n = \{0\}$.

Notice that if $X \in \mathfrak{a}$, then $X^n = 0_{n\times n}$, so the exp series is finite, and thus log always converges in $\exp(\mathfrak{a})$, so exp is a bijection from $\mathfrak{a}$ to $\exp(\mathfrak{a})$. Furthermore, this strictly upper triangular matrices are precisely the Lie algebras of the matrix Lie groups of unitriangular matrices (i.e. triangular matrices where its diagonal elements are all 1). In particular, this is a generalization of the Heisenberg group and its Lie algebra.

2. The set of upper triangular matrices $\mathfrak{b}$ is a solvable Lie algebra. First, it is again trivially a Lie algebra, and if $A, B \in \mathfrak{b}$, then

$$[A,B]_{i,j} = \sum_{k=i}^{j} a_{ik}b_{kj} - b_{ik}a_{kj}$$

so $[A,B]_{i,i} = a_{i,i}b_{i,i} - b_{i,i}a_{i,i} = 0\ \forall i = 1,\dots n$. Therefore $[\mathfrak{b},\mathfrak{b}]$ is nilpotent, and thus solvable. Furthermore, thanks to Theorem 4.5, since both $[\mathfrak{b},\mathfrak{b}]$ and $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ are solvable (notice that $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is abelian by Proposition 4.2), so we get that $\mathfrak{b}$ is solvable. In a similar way as the previous example, these are the Lie algebras of the matrix Lie groups of triangular matrices.

3. An example of an abelian Lie algebra is the set of diagonal matrices $\{A \in M_n(\mathbb{C}) | A = \lambda I, \lambda \in \mathbb{C}\}$, since for all $A = tI, B = sI$ in this set $(t, s \in \mathbb{C})$ it is easy to see that $\alpha A + \beta B = (\alpha t + \beta s)I$, which is again diagonal, and $[A, B] = (ts - st)I = 0$, so it is a Lie algebra and abelian.

To finish with this section, we will show a crucial example of a solvable Lie algebra: The radical of a Lie algebra. We will begin by establishing a well known result for algebras:

Theorem 4.6. *(The second Isomorphism Theorem) Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathfrak{g}$. Then there exists a canonical isomorphism*

$$(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$$

The proof of the theorem is the same as with the classical Second Isomomorphism theorem. Next, we will introduce an important result for the definition of the radical.

Corollary 4.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two solvable ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{a}+\mathfrak{b}$ is also a solvable ideal.*

Proof. Since $\mathfrak{b}$ is solvable there exists $n \in \mathbb{N}$ such that $\mathfrak{b}^n = \{0\}$, and since $\mathfrak{h} = \mathfrak{b}/\mathfrak{a}\cap\mathfrak{b}$ is a quotient of $\mathfrak{b}$, then $\{0\} = q(\{0\}) = q(\mathfrak{b}^n) = q(\mathfrak{b})^n = \mathfrak{h}^n$, so $\mathfrak{h}$ is also solvable. Furthermore, by Theorem 4.6 $\mathfrak{h}$ is isomorphic to $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$, so $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ is solvable. Therefore by Theorem 4.5 we have that $\mathfrak{a}+\mathfrak{b}$ is solvable since both $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ and $\mathfrak{a}$ are solvable. □

Thanks to last theorem, for every two solvable ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ we can construct another solvable ideal $\mathfrak{a}+\mathfrak{b}$ containing both. Thus we can construct the following ideal:

Proposition 4.8. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a unique solvable ideal containing all solvable ideals.*

Proof. Uniqueness is clear, since if two ideals $\mathfrak{h}(1)$ and $\mathfrak{h}(2)$ satisfy the above conditions, then $\mathfrak{h}(1) \subseteq (2)$ and $\mathfrak{h}(2) \subseteq \mathfrak{h}(1)$, so $\mathfrak{h}(1) = \mathfrak{h}(2)$. Now, if $\mathfrak{a}$ and $\mathfrak{b}$ are two solvable ideals of $\mathfrak{g}$, then we know that $\mathfrak{a}+\mathfrak{b}$ is also a solvable ideal. Now, we can construct a relation between two solvable ideals $\mathfrak{a} \leq \mathfrak{c}$ if $\mathfrak{c}$ contains $\mathfrak{a}$ which induces a partial order. Since $\mathfrak{g}$ is finite-dimensional, every chain of this relation has an upper bound, so we can use Zorn's Lemma to obtain a maximal element on every chain. Now, if $\mathfrak{h}'$ and $\mathfrak{h}''$ are two maximal ideals, then necessarily $\mathfrak{h}' \subseteq \mathfrak{h}''$ and $\mathfrak{h}'' \subseteq \mathfrak{h}'$, so $\mathfrak{h}' = \mathfrak{h}'' = \mathfrak{h}$ is the desired unique solvable ideal containing all others. □

Definition 4.9. Let $\mathfrak{g}$ is a finite-dimensional Lie algebra, the radical of $\mathfrak{g}$ is defined as the unique solvable ideal containing all solvable ideas of $\mathfrak{g}$, and it is denoted $rad\mathfrak{g}$.

4.2 Semisimple algebras

Now that we have seen solvability, it is time to introduce the other ingredient to understand Lie algebras through its subalgebras:

Definition 4.10. A Lie algebra is said to be simple if it is non-abelian and its only ideals are $\{0\}$ and itself. A Lie group is said to be simple if its lie algebra is simple.

A Lie algebra is said to be semisimple if it is a direct sum of ideals which are simple algebras. A Lie group is said to be semisimple if its Lie algebra is semisimple.

Proposition 4.11. *Let $\mathfrak{g}$ be a simple Lie algebra. Then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. Notice that since the commutator $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, necessarily $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ or $[\mathfrak{g}, \mathfrak{g}] = \{0\}$. But $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ would imply that $\mathfrak{g}$ is abelian, so the only possibility is $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. □

This definition is just one of the many possible definitions that one finds in the literature. Here, we will illustrate the versatility of the concept by giving some equivalent definitions for it, some of which we will be uses in next theorems.

Proposition 4.12. *The following properties are equivalent:*

1. $\mathfrak{g}$ *is semisimple.*
2. $rad\,\mathfrak{g} = \{0\}$*, that is, it has no non-zero solvable ideals.*
3. $\mathfrak{g}$ *has no non-zero abelian ideals.*
4. *(Cartan's criterion) The Killing form* $(B(X,Y) = tr(ad_X ad_Y))$ *is non-degenerate on* $\mathfrak{g}$.

Proof. Here, we will prove just some of the implications. For the implications on the Cartan's criterion, read [4], pg. 480. The idea for the $1 \to 2$ implication is to use 4 to construct a a simple ideal $\mathfrak{a}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$, and then proceed by induction. (For a full proof check [6], pg.55).

- 1 implies 2: Let $\mathfrak{g} = \bigoplus_{i_1}^{m} \mathfrak{g}(i)$ be a sum of simple Lie algebras, and let $\mathfrak{a}$ be a solvable ideal of $\mathfrak{g}$. Then, since $[\mathfrak{a}, \mathfrak{g}(i)]$ is an ideal of $\mathfrak{g}(i)$, either $[\mathfrak{a}, \mathfrak{g}(i)] = \{0\}$ or $[J, \mathfrak{g}(i)] = \mathfrak{g}(i)$ $\forall i = 1 \ldots m$. Without loss of generality we can assume that $[\mathfrak{a}, \mathfrak{g}(i)] = \mathfrak{g}(i)$ $\forall i \leq k$, and $[\mathfrak{a}, \mathfrak{g}(i)] = \{0\}$ $\forall i > k$ for some $0 \leq k \leq m$.

  This decomposition implies that $\forall x \in \mathfrak{a}$ there exists $x_i \in \mathfrak{g}(i)$ such that $x = x_1 + \ldots x_m$. We claim that $x_i = 0$ $\forall i > k$. Otherwise suppose that there exists $m \geq j > k$ such that $x_j = 0$, then $x_j \notin Z(\mathfrak{g}(\mathfrak{j})) = \{0\}$ as $\mathfrak{g}(j)$ is simple. Thus there exists $y \in \mathfrak{g}(j)$ such that $[x_j, y] \neq 0$. But

  $$0 = [x,y] = [x_1,y] + \cdots + [x_j,y] + \cdots + [x_m,y] = [x_j,y] \neq 0$$

  a contradiction. Thus $x_i = 0$ $\forall i > k$ and $\mathfrak{a} = \bigoplus_{i=1}^{k} \mathfrak{g}(\mathfrak{i})$. Now,

  $$[\mathfrak{a},\mathfrak{a}] = [\bigoplus_{i=1}^{k} \mathfrak{g}(\mathfrak{i}), \bigoplus_{i=1}^{k} \mathfrak{g}(\mathfrak{i})] = \bigoplus_{i=1}^{k} [\mathfrak{g}(\mathfrak{i}), \mathfrak{g}(\mathfrak{i})] = \bigoplus_{i=1}^{k} \mathfrak{g}(\mathfrak{i}) = \mathfrak{a}$$

  so necessarily $\mathfrak{a} = \{0\}$ since $\mathfrak{a}$ is assumed to be solvable.

- 2 implies 3: If $\mathfrak{g}$ has no non-zero solvable ideals and $\mathfrak{a}$ is an abelian ideal, then it is solvable, so $\mathfrak{a} = \{0\}$.

- 3 implies 2: If $\mathfrak{g}$ has no non-zero abelian ideals and $\mathfrak{a}$ is non-zero solvable ideal of $\mathfrak{g}$, then there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n = 0$, that is, $[\mathfrak{a}^{n-1}, \mathfrak{a}^{n-1}] = \{0\}$, This implies that $\mathfrak{a}^{n-1}$ is abelian, a contradiction. Thus we have that necessarily $\mathfrak{a} = \{0\}$.

□

In the next section. these concepts will allow us to state the Levi decomposition of a Lie algebra (Theorem 5.5). As for now, we will show a closely related Lemma.

Lemma 4.13. *Let* $\mathfrak{g}$ *be a finitely-dimensional Lie algebra. Then* $\mathfrak{g}/rad\,\mathfrak{g}$ *is semisimple.*

Proof. Let $q : \mathfrak{g} \longrightarrow \mathfrak{g}/rad\,\mathfrak{g}$ be the quotient homomorphism, and let $\mathfrak{h}$ be a solvable ideal of $\mathfrak{g}$. Then $q^{-1}(\mathfrak{h}) := \mathfrak{a}$ is an ideal of $\mathfrak{g}$. Furthermore, $\mathfrak{a}/ker\, q|_{\mathfrak{a}} \cong q(\mathfrak{a}) = \mathfrak{h}$ is solvable, and $ker\, q|_{\mathfrak{a}}$ is also solvable since is contained in $rad\,\mathfrak{g}$. Therefore $\mathfrak{a}$ is also solvable, so $\mathfrak{a} \subseteq rad\,\mathfrak{g}$, and $\mathfrak{h} = q(\mathfrak{a}) = \{0\}$. Thus $\mathfrak{g}/rad\,\mathfrak{g}$ is semisimple. □

Chapter 5

Representations and Lie-Kolchin theorem

In this chapter we will explore representation theory of Lie groups. The idea is to study Lie groups by transforming them into matrix Lie groups, allowing us to apply the concepts we have studied regarding the latter. Furthermore, we will give some interesting results for the representation of solvable and semisimple Lie algebras, including the Lie-Kolchin theorem.

5.1 Representations

Definition 5.1. A representation of a Lie group G is a Lie group homomorphism $\Pi : G \longrightarrow GL(V)$ where V is a finite-dimensional vector space over a field K. In the same way, if $\mathfrak{g}$ is a Lie algebra, a representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = Lie(GL(V))$. A representation is said to be faithful if it is one-to-one.

G can be viewed as acting on V through Π. Whenever there is no confusion over the representation Π, by abuse of notation we will write $\Pi(g)(v)$ as $g \cdot v$ (where $v \in V, g \in G$). In this case we say that V is a G-module and we say that a subspace W of V is a submodule if it is G-invariant.

We will be considering V as vector space over the fields $\mathbb{C}$ or $\mathbb{R}$. It is also worth noticing that in virtue of the previous chapter, a representation Π of a Lie group G induces a representation $d\Pi$ of the Lie algebra $\mathfrak{g}$ of G.

Examples. Here we will show some representations for some matrix Lie algebras. Although trivially the identity map $id : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ can be considered as a representation for any Lie algebra, choosing the right representation can provide us a different insight of the Lie algebra.

1. Consider the Lie algebra $\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) |\ X^* = -X, tr\, X = 0\}$. This is an interesting example because it is the first case of a Lie group that is non-abelian and compact (as a manifold, with the topology given in Chapter 1). It can be given a representation by

$$g = \begin{pmatrix} ic & -b+ia \\ b+ia & -ic \end{pmatrix} = a\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} =$$

   $= au_1 + bu_2 + cu_3\ \ \forall g \in \mathfrak{su}(2)$ and $a,b,c \in \mathbb{R}$, where u_1,u_2,u_3 is a basis of G, such that $[u_1,u_2] = 2u_3$, $[u_2,u_3] = 2u_1$ and $[u_3,u_1] = 2u_2$. This particular representation is interesting since it has a physical use, as it used in quantum physics to describe the spin of a particle.

2. Now consider $\mathfrak{so}(3) = \{X \in M_3(\mathbb{C}) |\ X^t = -X, tr\ X = 0\}$. This is an interesting example, because its Lie group, $SO(3)$ is understood in physics as the group of 3D rotations. $\mathfrak{so}(3)$ admits a representation

$$g = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$= as_1 + bs_2 + Cs_3$ $\forall g \in \mathfrak{so}(3)$ and $a,b,c \in \mathbb{R}$. Same as before, we have that $[s_1,s_2] = s_3$, $[s_2,s_3] = s_1$ and $[s_3,s_1] = s_2$.

As a note, the relation between this two Lie algebras (and specifically between the Lie groups $SU(2)$ and $SO(3)$) is a very interesting one with physical meaning. We will not get into it but the idea is that there exists a two-to-one homomorphism from $SU(3)$ onto $SO(3)$, so that $SU(3)$ is a covering group of $SO(3)$.

5.2 Existence of representations

Now, we need to be sure that we can find Lie algebras outside of the matrix case, so that we can apply what we know in the matrix case to them. But thanks to the next result we are able to faithfully represent finite-dimensional Lie algebras and thus consider matrix Lie algebras.

Theorem 5.2. *(Ado's theorem) Any finite-dimensional Lie algebra over a field K of characteristic 0 has at least one faithful representation.*

While this theorem is quite important for Lie algebra representation, it does not imply that a Lie group G must have a faithful representation, but rather that its Lie algebra $\mathfrak{g}$ does. Of course, if G is simply connected we are able to use Theorem 3.7 to get a representation of G through its Lie algebra representation.

As a side note, Ado's theorem can be used to prove Lie's third theorem, which states that every finite-dimensional Lie algebra is the Lie algebra of a Lie group.

But even if we do not meet the conditions to apply Ado's theorem, we are always able to represent any Lie group or Lie algebra, although it might not always be faithful. This can be done though the adjoint representation.

Example. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra (although this representation also works of Lie algebras of infinite dimension), then the map $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ such that $g \mapsto ad_g$ (defined in section 2.6) is a representation of $\mathfrak{g}$. This representation is not always faithful, since it always annihilate the center.

Consider now the adjoint representation of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}(i)$, where $\mathfrak{g}(i)$ are semisimple algebras of dimension k_i so that $dim\ \mathfrak{g} = \sum_{i=1}^m k_i$. Then, then we may form a representation π of $\mathfrak{g}$ by summing the adjoint representations of the $g(i)$, so that for every element $g \in \mathfrak{g}$

$$\pi(g) = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix}$$

where each A_i is a $(k_i \times k_i)$ matrix. If we consider $g = g_1 + g_2 + \cdots + g_m$ where $g_i \in \mathfrak{g}(i)$ then A_i can be viewed as the image of g_i through the adjoint representation of $\mathfrak{g}(i)$.

5.3 The Lie-Kolchin theorem

While representation of Lie groups is a complex topic with many aspects, here we will be using all the ideas we have been exploring to give an important result, the Lie-Kolchin theorem.

Theorem 5.3. *(Lie–Kolchin theorem) Let G be a connected solvable Lie group, and let $\Pi : G \longrightarrow GL(V)$ be a representation of G where V is a finite-dimensional vector space over an algebraically closed field K. Then there exists $g \in GL(V)$ such that $g\Pi(G)g^{-1}$ is contained in the group of upper triangular matrices.*

Before starting with the proof, notice that the matrix $g \in GL(V)$ does not depend on any element in $\Pi(G)$. This mean that all the matrices in $\Pi(G)$ can be simultaneously transformed into upper triangular matrices: hence the relevance of this theorem.

Proof. Consider $\pi = d\Pi : Lie(G) \longrightarrow \mathfrak{gl}(V)$. We will prove first that there exists $g \in GL(V)$ such that if $\mathfrak{h} = \pi(Lie(G))$ then $g\mathfrak{h}g^{-1}$ is upper triangular.
We claim that there exists an $\mathfrak{h}$-invariant one-dimensional subspace L of V. Then, this will allow us to find a matrix $g_1 \in GL(V)$ such that for every $Y \in \mathfrak{h}$

$$g_1 Y g_1^{-1} = \left(\begin{array}{c|ccc} \lambda_Y & \dots & \dots & \dots \\ \hline 0 & & & \\ \vdots & & B_Y & \\ 0 & & & \end{array}\right)$$

where $B_Y \in M_{n-1}(K)$. Now, by considering the quotient space $V/L = V_1$, π induces a representation $\pi_1 : Lie(G) \longrightarrow \mathfrak{gl}(V_1)$, where $B_Y = \pi_1(Y)$. Thus, by induction over $n = dim\ Lie(G)$ we can find $g = g_n g_{n-1} \dots g_1$ such that gYg^{-1} is upper triangular $\forall Y \in Lie(G)$.

We prove this claim by induction over $dim\mathfrak{h}$. For the first step of induction, if $dim\ \mathfrak{h} = 1$, then $\mathfrak{h}_0 = KH$ for $H \in \mathfrak{h}$, and since K is algebraically closed, H must have an eigenvector v. Now for the induction step, since $\mathfrak{h}$ is solvable, necessarily $[\mathfrak{h},\mathfrak{h}] \neq \mathfrak{h}$. Thus there exists a subspace $\mathfrak{h}_0$ of codimension 1 containing $[\mathfrak{h},\mathfrak{h}]$, and $X \in \mathfrak{h}$ such that $\langle X, \mathfrak{h}_0\rangle = \mathfrak{h}$. Therefore, by our induction assumption there must exist a vector $\overline{0} \neq v \in V$ such that $\mathfrak{h}_0 v \subset Kv$. Therefore $Yv = \lambda(Y)v\ \forall Y \in \mathfrak{h}_0$. Now, let $v_i = X^i v$ (where $v_0 = v$), then

$$Yv_i = YXv_{i-1} = XYv_{i-1} + [Y,X]v_{i-1}\ \forall Y \in \mathfrak{h}_0$$

From this we can deduce that $\langle v_0, \dots v_i\rangle$ is $\mathfrak{h}_0$-invariant by induction. For the first step, notice that if $Y \in \mathfrak{h}_0$ then $Yv_0 = \lambda(Y)v_0 \in \langle v_0\rangle$. Now assume that $\langle v_0, \dots v_{i-1}\rangle$ is $\mathfrak{h}_0$-invariant. Then for $Y \in \mathfrak{h}_0$ we have

$$Y\sum_{j=0}^{i} a_j v_j = \sum_{j=0}^{i} a_j Y v_j = a_0\lambda(Y)v_0 + \sum_{j=1}^{i} a_j(XYv_{j-i} + [Y,X]v_{j-1}) = d_0v_0 + \sum_{j=0}^{i-1} a_{j+1}(XYv_j + [Y,X]v_j)$$

Here is when we apply our assumption, since $Y, [Y,X] \in \mathfrak{h}_0$ (as $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}_0$). Thus

$$d_0v_0 + \sum_{j=0}^{i-1} a_{j+1}(XYv_j + [Y,X]v_j) = d_0v_0 + \sum_{j=0}^{i-1} a_{j+1}\left(X\sum_{k=0}^{j} b_{j,k}v_k + \sum_{k=0}^{j} c_{j,k}v_k\right) =$$

$$= d_0v_0 + \sum_{j=0}^{i-1} a_{j+1}\left(\sum_{k=0}^{j} b_{j,k}v_{k+1} + \sum_{k=0}^{j} c_{j,k}v_k\right) = \sum_{j=0}^{i} d_j v_j \in \langle v_0 \dots v_i\rangle$$

Furthermore, again by induction we get that $Yv_i - \lambda(Y)v_i \in \langle v_0, \dots v_{i-1}\rangle$. For the first step it is clear that $Yv_0 - \lambda(Y)v_0 = 0 \in \langle v_0\rangle$. Now assume that $Yv_{i-1} - \lambda(Y)v_{i-1} \in \langle v_0, \dots v_{i-2}\rangle$. Then

$$Yv_i - \lambda(Y)v_i = XYv_{i-1} - \lambda(Y)v_i = X\left(\lambda(Y)v_{i-1} + \sum_{j=0}^{i-2} a_j v_j\right) - \lambda(Y)v_i =$$

$$= \lambda(Y)v_i + \sum_{j=0}^{i-2} a_j v_{j+1} - \lambda(Y)v_i = \sum_{j=1}^{i-1} a_{j-1} v_j \in \langle v_0 \dots v_{i-1} \rangle$$

Therefore $Yv_i - \lambda(Y)v_i \in \langle v_0, \dots v_{i-1} \rangle$. Now, let $W = \langle v_0, \dots v_i, \dots \rangle$, and notice that since V is finite-dimensional there exists $m \in \mathbb{N}$ such that v_{m+1} linearly dependent of $v_0 \dots v_m$. Then by induction $W = \langle v_0 \dots v_m \rangle$, if we assume that $v_{m+k-1} \in \langle v_0 \dots v_m \rangle$, then

$$v_{m+k} = Xv_{m+k-1} = X\sum_{j=0}^{m} \alpha_j v_j = \sum_{j=0}^{m} \alpha_j v_{j+1} = \sum_{j=1}^{m+1} \alpha_{j-1} v_j = \sum_{j=1}^{m} \alpha_{j-1} v_j + \alpha_m v_{m+1} = \sum_{j=0}^{m} \beta_j v_j$$

as $v_{m+1} \in \langle v_0 \dots v_m \rangle$. Thus, since W is $\mathfrak{h}_0$-invariant, we have that the matrix of $[Y,X]|_W$ in this basis will be upper diagonal (since $[Y,X] \in \mathfrak{h}_0$, so it will map every v_j to a linear combination if the previous elements of the base). Furthermore, its diagonal elements will be $\lambda([Y,X])$ we get the relation $tr([Y,X]|_W) = \lambda([Y,X])\ dim\ W$. But $tr([Y,X]|_W) = tr(X|_W Y|_W - Y|_W X|_W) = 0$, so $\lambda([Y,X]) = 0$ for every $Y \in \mathfrak{h}_0$.

From this we deduce by induction that $[\mathfrak{h}_0, X]$ acts trivially on W and $Yv_i = \lambda(Y)v_i\ \forall i = 0, \dots m$ and $Y \in \mathfrak{h}_0$. For the first step of induction, notice that $[Y,X]v_0 = \alpha v_0 = 0$ for every $Y \in \mathfrak{h}_0$, as $tr([Y,X]|_W) = 0$ implies that the coefficient of v_0 of $[Y,X]v_0$ is 0. Also, we already know that $Yv_0 = \lambda(Y)v_0$. Now we will assume that $[Y,X]v_{i-1} = 0\ \forall Y \in \mathfrak{h}_0$ and $Yv_{i-1} = \lambda(Y)v_{i-1}$, then

$$Yv_i = XYv_{i-1} + [Y,X]v_{i-1} = X\lambda(Y)v_{i-1} = \lambda(Y)v_i\ \forall Y \in \mathfrak{h}_0$$

In particular $[Y,X]v_i = \lambda([Y,X])v_i = 0$. Therefore $Yw_i = \lambda(Y)w_i$ for every $w_i \in \langle v_0, \dots v_i \rangle$ and $Y \in \mathfrak{h}_0$ (as it is a linear combination of the v_j). This prove our claim that W is $[\mathfrak{h}_0.X]$-invariant, and that $Yw = \lambda(Y)w$ for every $w \in W$.

And this is what we were looking for, because now as K is algebraically closed we may choose an eigenvector u of X and if $M \in \mathfrak{h}$ then $M = \alpha X + Y$ for some $Y \in \mathfrak{h}_0$ and $\alpha \in K$, so if β is the eigenvalue of X related to v, we have that $Mu = \alpha Xu + Yu = (\alpha\beta + \lambda(Y))u$. This implies that $L = \langle u \rangle$ is a one-dimensional $\mathfrak{h}$-invariant subspace of V, which proves the claim.

This proves that for every Lie group G, there exists $g \in G$ such that $g\pi(Lie(G))g^{-1}$ is upper triangular, but we are yet to prove that $g\Pi(G)g^{-1}$ is upper triangular. First, since exp is a bijection on an open neighbourhood U of the identity element in $\mathfrak{gl}(V)$, we have that $e^{g\Pi(log(U))g^{-1}} = ge^{log(U)}g^{-1} = gUg^{-1}$ is upper triangular as well, so the subgroup generated by gUg^{-1}, which we denote $\langle gUg^{-1} \rangle$, is also upper triangular. We claim that $\langle gUg^{-1} \rangle = \Pi(G)$. Since gUg^{-1} is open, $H = \langle gUg^{-1} \rangle$ is an open subgroup of G, so we can give a left coset decomposition $G = \bigsqcup_{g \in G/H} gH$. But G is connected, so the only way that this is possible is if G/H has only one element, that is, $G = H$, and $g\Pi(G)g^{-1}$ is upper triangular.

□

A proof of this theorem for Lie algebras was given by Sophus Lie in 1876 [7], and later Kolchin would give a version for Lie groups in 1948 [8]. Some versions only state the existence of the $\pi(G)$-invariant one-dimensional space, while this version of the theorem takes it further away by applying this result inductively.

The Lie-Kolchin theorem was generalized after by Mal'cev in 1951 [9], stating that if V is a n-dimensional vector space over an algebraically closed field K and G is a solvable subgroup of $GL(V)$, then it has a normal triangularizable subgroup of finite dimension. Furthermore, there is a similar result for polycyclic groups [10].

To show this theorem in action, we will provide an example of a matrix Lie algebra which is simultaneously conjugated into an upper triangular matrix Lie algebra.

Example. Consider the group of affine transformations on the plane, that is, a rotation following a translation. It can be expressed as a 3×3 matrix

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right) \text{ where } v\in\mathbb{R}^2 \text{ and } A=\left(\begin{array}{c|c} cos\theta & -sin\theta \\ \hline sin\theta & cos\theta \end{array}\right),\ \theta\in[-\pi,\pi]$$

These matrices form a matrix Lie group

$$SE(2)=\left\{\left(\begin{array}{ccc} cos\theta & -sin\theta & a \\ sin\theta & cos\theta & b \\ 0 & 0 & 0 \end{array}\right)\in GL_3(\mathbb{R})\,\middle|\,\theta\in[-\pi,\pi],a,b\in\mathbb{R}\right\}$$

it is easy to check that

$$Lie(SE(2))=\mathfrak{se}(2)=\left\{\left(\begin{array}{ccc} 0 & -\theta & \alpha \\ \theta & 0 & \beta \\ 0 & 0 & 0 \end{array}\right)\in M_3(\mathbb{R})\,\middle|\,\theta,\alpha,\beta\in\mathbb{R}\right\}$$

this is easy to see since by brute-force computing we have

$$\exp\left(\begin{array}{ccc} 0 & -\theta & \alpha \\ \theta & 0 & \beta \\ 0 & 0 & 0 \end{array}\right)=\left(\begin{array}{ccc} cos(\theta t) & -sin(\theta t) & a(\alpha,\beta,\theta) \\ sin(\theta t) & cos(\theta t) & b(\alpha,\beta,\theta) \\ 0 & 0 & 1 \end{array}\right)$$

Now, $\mathfrak{se}(2)$ is a solvable Lie algebra since

$$\left(\begin{array}{ccc} 0 & -\theta_1 & \alpha_1 \\ \theta_1 & 0 & \beta_1 \\ 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc} 0 & -\theta_2 & \alpha_2 \\ \theta_2 & 0 & \beta_2 \\ 0 & 0 & 0 \end{array}\right)-\left(\begin{array}{ccc} 0 & -\theta_2 & \alpha_2 \\ \theta_2 & 0 & \beta_2 \\ 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc} 0 & -\theta_1 & \alpha_1 \\ \theta_1 & 0 & \beta_1 \\ 0 & 0 & 0 \end{array}\right)=$$

$$\left(\begin{array}{ccc} -\theta_1\theta_2 & 0 & -\theta_1\beta_2 \\ 0 & -\theta_1\theta_2 & \theta_1\alpha_2 \\ 0 & 0 & 0 \end{array}\right)-\left(\begin{array}{ccc} -\theta_1\theta_2 & 0 & -\theta_2\beta_1 \\ 0 & -\theta_1\theta_2 & \theta_2\alpha_1 \\ 0 & 0 & 0 \end{array}\right)=\left(\begin{array}{ccc} 0 & 0 & -(\theta_1\beta_2+\theta_2\beta_1) \\ 0 & 0 & \theta_1\alpha_2+\theta_2\alpha_1 \\ 0 & 0 & 0 \end{array}\right)$$

so we have that

$$\mathfrak{se}(2)^2=[\mathfrak{se}(2),\mathfrak{se}(2)]=\left\{\left(\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array}\right)\in M_3(\mathbb{R})\,\middle|\,x,y\in\mathbb{R}\right\}$$

and it is clear multiplicating two matrices in $\mathfrak{se}(2)^2$ gives $0_{n\times n}$, so $\mathfrak{se}(2)^3=[\mathfrak{se}(2)^2,\mathfrak{se}(2)^2]=\{0\}$. Now, we are looking for a matrix $g\in GL_3(\mathbb{R})$ such that gSg^{-1} is upper triangular. But this cannot be done, since we know from linear algebra that the matrix

$$B=\left(\begin{array}{cc} 0 & -\theta \\ \theta & 0 \end{array}\right)$$

cannot be diagonalized by conjugation with a matrix with real coefficients, since its eigenvalues are $i\theta$ and $-i\theta$. Here is here we need the field to be algebraically closed.

Now, consider $SE(2)$ as a matrix Lie group in $GL_3(\mathbb{C})$, and $\mathfrak{se}(2)$ as its matrix Lie algebra in $M_n(\mathbb{C})$ (we still consider its coefficients to be real). Then we are actually able to find such a matrix $g\in M_3(\mathbb{C})$, as

$$\left(\begin{array}{ccc} \frac{-i}{2} & \frac{1}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc} 0 & -\theta & \alpha \\ \theta & 0 & \beta \\ 0 & 0 & 0 \end{array}\right)\left(\begin{array}{ccc} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)=\left(\begin{array}{ccc} i\theta & 0 & \frac{1}{2}(b-ia) \\ 0 & -i\theta & \frac{1}{2}(b+ia) \\ 0 & 0 & 0 \end{array}\right)$$

so we have found that taking

$$g = \begin{pmatrix} \frac{-i}{2} & \frac{1}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } g^{-1} = \begin{pmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the properties of Theorem 5.3 are satisfied.

5.4 Levi decomposition

On this last section we want to give an interesting result for both semisimplicity and solvability, that even relates to representation theory. To do so, first we will show one more theorem:

Theorem 5.4. *(Weyl's complete reducibility theorem) Let $\mathfrak{g}$ be a semisimple Lie algebra over a field K of characteristic zero and let V be a finite-dimensional module over $\mathfrak{g}$. Then if W is a submodule of V has a complementary submodule U, that is, there exists a submodule U of V such that $V = W \oplus U$.*

Now we have all the ingredients to state the Levi decomposition.

Theorem 5.5. *(Levi decomposition) Let $\mathfrak{g}$ be a finitely-dimensional Lie algebra. Then it can be decomposed as the vector space direct sum $\mathfrak{g} = \mathfrak{h} \oplus rad\mathfrak{g}$ where $\mathfrak{h}$ is a semisimple subalgebra of $\mathfrak{g}$.*

Proof. First, we will assume that there is no non-zero ideal of $\mathfrak{g}$ contained in $rad\mathfrak{g}$. Otherwise, let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ strictly inside $rad\mathfrak{g}$ and maximal in these conditions, and assume the theorem is true for the case where $rad\mathfrak{g}$ contain no non-zero ideals of $\mathfrak{g}$.

Then we have that $\mathfrak{g}/\mathfrak{a} = \mathfrak{h}/\mathfrak{a} \oplus (rad\mathfrak{g})/\mathfrak{a}$, where $\mathfrak{h}/\mathfrak{a}$ is semisimple, so necessarily $\mathfrak{a} = rad\mathfrak{h}$. Thus by induction over $dim\mathfrak{g}$ (as $dim\mathfrak{h} < dim\mathfrak{g}$), we have $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}'$, so

$$\mathfrak{g} = rad\mathfrak{g} + \mathfrak{h} = rad\mathfrak{g} + \mathfrak{a} \oplus \mathfrak{h}' = rad\mathfrak{g} \oplus \mathfrak{h}'$$

where $\mathfrak{h}'$ is as required.Furthermore, we will also assume that $rad\mathfrak{g}$ is abelian, since otherwise $(rad\mathfrak{g})^2$ is an ideal of $rad\mathfrak{g}$, and thus of $\mathfrak{g}$, so we are on the previous situation.

Finally, we will also assume that $[\mathfrak{g}, rad\mathfrak{g}] = rad\mathfrak{g}$. Indeed, $[\mathfrak{g}, rad\mathfrak{g}]$ is an ideal of $rad\mathfrak{g}$, and by the first assumption either $[\mathfrak{g}, rad\mathfrak{g}] = rad\mathfrak{g}$ or $[\mathfrak{g}, rad\mathfrak{g}] = \{0\}$. But in the latter situation $rad\mathfrak{g}$ is the kernel of the adjoint representation ad, thus $\mathfrak{g}$ is a $\mathfrak{g}/rad\mathfrak{g}$-module and $rad\mathfrak{g} \subset \mathfrak{g}$ is $\mathfrak{g}/rad\mathfrak{g}$-invariant. Hence by Weyl theorem (as $\mathfrak{g}/rad\mathfrak{g}$ is semisimple by Lemma 4.13) there is a complementary $\mathfrak{g} = \mathfrak{h} \oplus rad\mathfrak{g}$. Therefore from now on we will assume $[\mathfrak{g}, rad\mathfrak{g}] = rad\mathfrak{g}$.

The idea of the proof is to provide a candidate for a subalgebra complementary to $rad\mathfrak{g}$. To do so, we will consider the adjoint representation $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, as

$$X \cdot \varphi = [ad(X), \varphi] = ad(X) \circ \varphi - \varphi \circ ad(X) \; \forall X \in \mathfrak{g}, \; \varphi \in \mathfrak{gl}(\mathfrak{g})$$

so that

$$(X \cdot \varphi)(Y) = [X, \varphi(Y)] - \varphi([X,Y]) \; \forall X, Y \in \mathfrak{g}, \; \varphi \in \mathfrak{gl}(\mathfrak{g})$$

In order to define this subalgebra, we will also consider the spaces

$$A_1 = \{\varphi \in \mathfrak{gl}(\mathfrak{g}) | \; \varphi(\mathfrak{g}) \subset rad\mathfrak{g} \text{ and } \varphi|_{rad\mathfrak{g}} \text{ is a multiplication by a scalar}\}$$

$$A_2 = \{\varphi \in \mathfrak{gl}(\mathfrak{g}) | \; \varphi(\mathfrak{g}) \subset rad\mathfrak{g} \text{ and } \varphi(rad\mathfrak{g}) = \{0\}\}$$

Notice that A_1, A_2 are $\mathfrak{g}$-submodules with $A_2 \subset A_1$. We claim that $\mathfrak{g} \cdot A_1 \subset A_2$. Indeed, if $\varphi \in A_1$ is such that $\varphi|_{rad\mathfrak{g}} = c \; id_{rad\mathfrak{g}}$, then

$$(X \cdot \varphi)(Y) = [X, cY] - c[X,Y] = 0 \; \forall X \in \mathfrak{g}, \; Y \in rad\mathfrak{g}$$

so $X \cdot \varphi \in A_2$. Now let us define

$$A_3 = \{ad(X) |\, X \in rad\mathfrak{g}\}$$

so that A_1, A_2, A_3 are $\mathfrak{g}$-submodules such that $A_3 \subset A_2 \subset A_1$. We now claim that $rad\mathfrak{g} \cdot A_1 \subset A_3$. To prove it, let $Y \in \mathfrak{g}$, $X \in rad\mathfrak{g}$, and $\varphi \in A_1$. Since $[X, \varphi(Y)] \in [rad\mathfrak{g}, rad\mathfrak{g}] = \{0\}$ because we assumed that $rad\mathfrak{g}$ is abelian, then

$$(X \cdot \varphi)(Y) = -\varphi([X,Y]) = [-cX, Y] = ad(-cX)(Y)$$

thus $X \cdot \varphi = ad(-cX) \in A_3$. The claims $\mathfrak{g} \cdot A_1 \subset A_2$ and $rad\mathfrak{g} \cdot A_1 \subset A_3$ imply that A_1/A_3 is a $\mathfrak{g}/rad\mathfrak{g}$-submodule and $\mathfrak{g}$ acts trivially on A_1/A_2. Also A_2/A_3 is $\mathfrak{g}$-invariant and as $\mathfrak{g}/rad\mathfrak{g}$ is semisimple by Lemma 4.13 Weyl theorem implies that there is $D/A_3 \subseteq A_1/A_3$ with $A_1/A_3 = D/A_3 \oplus A_2/A_3$ and D/A_3 is a $\mathfrak{g}/rad\mathfrak{g}$-submodule. In fact, using a Lie algebra theory version of the Third isomorphism theorem

$$(D/A_3) \cong (D/A_3 \oplus A_2/A_3)/(A_2/A_3) = (A_1/A_3)/(A_2/A_3) \cong A_1/A_2$$

so $\mathfrak{g}$ acts trivially on D/A_3. Therefore (multiplicating by a scalar if necessary) there exists an element $\phi \in A_1$ such that $\phi_{rad\mathfrak{g}} = id_{rad\mathfrak{g}}$ and $\mathfrak{g} \cdot \phi$ is contained in A_3. Finally, let

$$\mathfrak{h} = \{X \in \mathfrak{g} |\, X \cdot \phi = 0\}$$

It is a subalgebra of $\mathfrak{g}$, as $X, Y \in \mathfrak{h}$ imply $\alpha X + \beta Y$ and $[X,Y] \in$ since $(\alpha X + \beta Y) \cdot \phi = 0$ and $([X,Y])\phi = 0$. Furthermore, $\mathfrak{h} \cap rad\mathfrak{g} = \{0\}$ since if $0 \neq X \in \mathfrak{h} \cap rad\mathfrak{g}$ then $X \cdot \phi = ad(-X)$, so $ad(X) = 0$ because by construction $X \cdot \phi = X$, and therefore $[\mathfrak{g}, X] = 0$. Finally, $\mathfrak{g} = \mathfrak{h} + rad\mathfrak{g}$: If $X \in \mathfrak{g}$, then $X \cdot \phi \in A_3$, so $X \cdot \phi = ad(Y)$ for some $\in rad\mathfrak{g}$. Since $ad(Y) = -Y \cdot \phi$, then

$$X \cdot \phi - ad(Y) = X \cdot \phi - (-Y \cdot \phi) = (X+Y) \cdot \varphi = 0$$

that is, $X + Y \in \mathfrak{h}$. Hence $X = (X+Y) - Y$ is the sum of $\mathfrak{h}$ and $rad\mathfrak{g}$.

□

We have proved the Levi decomposition (Theorem 5.5), where a finite-dimensional Lie algebra can be split as a vector space direct sum between a semisimple subalgebra and its radical (a solvable subalgebra), and we have showed that we can represent a solvable finitely-dimensional algebra as a triangular matrix in the Lie-Kolchin theorem (Theorem 5.3). Now, we want to give an idea about how a semisimple algebra is represented, so that give to give a hint of how any finite-dimensional Lie algebra can be represented.

Example. We know that $\mathfrak{g}$ is any finite-dimensional Lie algebra, by Theorem 5.5 (Levi decomposition) it can be expressed as a vector space direct sum of a semisimple subalgebra $\mathfrak{h}$ and $rad\mathfrak{g}$ (which is solvable). Suppose that both $\mathfrak{h}$ and $rad\mathfrak{g}$ are not trivial (we already know how to represent these cases), and furthermore suppose that $\mathfrak{h}$ is also an ideal. If we add this to our previous arguments, we can get that there exists a representation π on a vector space V over a field K such that

$$g = \left(\begin{array}{ccc|cccc} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_m & & & & \\ \hline & & & b_{11} & b_{12} & \dots & b_{1n} \\ & & & & b_{22} & \ddots & \vdots \\ & & & & & \ddots & \\ & & & & & & b_{kk} \end{array}\right)$$

where the first block corresponds to its semisimple subalgebra, and the second part, to its solvable part. Notice that we need the assumption that $\mathfrak{h}$ is an ideal, otherwise the upper right block might not be zero.

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