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EVEN ARTIN GROUPS

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Contents

Resumen

Los grupos de Artin de ángulo recto conforman una familia muy interesante tanto desde un punto de vista algebraico como topológico. Hay muchas propiedades importantes conocidas para los grupos de Artin de ángulo recto: por ejemplo, se sabe que son poli-libres, localmente indicables, ordenables a derecha y residualmente finitos. Además, varios problemas muy importantes están completamente resueltos para esta familia de grupos, como el problema de la palabra, el problema de rigidez, la conjetura $K(\pi, 1)$ o el problema de Serre.

En esta tesis vamos a estudiar alguna de estas propiedades para una subfamilia más grande de grupos de Artin: los grupos de Artin pares. Generalizaremos muchas de estas propiedades bien para la familia completa de grupos de Artin pares o para algunas subfamilias grandes e interesantes.

En particular, probamos que los grupos de Artin pares de tipo FC y de tipo large son poli-libres (lo cual veremos que implicará que son localmente indicables y ordenables a derecha) y que los grupos de Artin pares de tipo FC y los grupos de Artin basados en el grafo de un bosque son residualmente finitos. Finalmente, resolveremos el problema de Serre para la familia completa de grupos de Artin pares.

Abstract

Right-angled Artin groups form an interesting family of groups both from an algebraic and a topological point of view. There are a lot of well-known properties of right-angled Artin groups: for example they are poly-free, locally indicable, right orderable and residually finite. Besides, also many important problems are well understood for these groups such as the word problem, the rigidity problem, Serre's question or the $K(\pi, 1)$ conjecture.

In this thesis, we will study some of these properties for a bigger and interesting subfamily of Artin groups: even Artin groups. We generalize many of these properties either for even Artin groups in full genarility or for some big and interesting subfamilies.

In particular, we prove that even Artin groups of FC type and large even Artin groups are poly-free (which, as we will see, implies that they are also locally indicable and right orderable) and that even Artin groups of FC type and general Artin groups based on trees are residually finite. Finally, we answer Serre's question for the whole family of even Artin groups.

Chapter 1

Introduction and background

Artin groups were first introduced by Tits [74] as an extension of Coxeter groups. They begin to gain fame and importance later on in the works of Brieskorn [17],[16], [18], Brieskorn and Saito [19] and Deligne [36]. In all these works the object of study was the specific family of spherical Artin groups (which is the family associated to finite Coxeter groups), although some results involving general Artin groups can be found in [19].

Nowadays, the family of Artin groups has a huge importance both in group theory and in algebraic geometry since in their study there has been an important interaction of algebraic and geometric techniques. However, there are very few works dealing with general Artin groups (one of them is due to Charney and Paris [29]) and the knowledge of Artin groups mainly consists in the study of more or less general subfamilies such as spherical Artin groups, right-angled Artin groups or Artin groups of FC type. Moreover, the answers to several basic questions are not known in general for Artin groups, for example regarding the word problem or the rigidity problem.

Right-angled Artin groups, usually denoted by RAAGs, are one of the most studied families of Artin groups. RAAGs were first studied in the 1970's by Baudisch [7] and gained importance on the work by Droms (although he called them "graph groups") [42], [43], [44]. This subfamily is important because of many reasons, among them we should remark their applications in homology groups (see Bestvina-Brady [8] or in low dimensional topology (see Haglund-Wise [51] and Agol [1]). To understand in depth the family of RAAGs as well to their main properties we refer to [27].

The objective of this thesis is to study some algebraic and geometric properties which had already been studied to RAAGs and extend them to bigger families of Artin groups, focusing in the family of even Artin groups. As far as we are aware, there are some results in the literature about even Coxeter groups (see for example [2]) but not about even Artin groups, although they are mentioned in [5]. However, we think that this family of groups deserves more attention because they share some remarkable properties with RAAGs, as we will see in Section 1.2, but these properties are lost when we have a relation of "odd" type.

This thesis is structured in the following way. In this first chapter we will give several basic definitions and see the background on the topic of Artin groups. Besides we will explain the importance and interest of the family of even Artin groups.

In the following chapters we will focus on different properties and study them for some families of even Artin groups. In this way, in Chapter 2 we will study some background of poly-freeness and we will study this property for the family of even Artin groups of FC type, in Chapter 3 we will study the property of poly-freeness for large even Artin groups. Later on, in Chapter 4 residually finiteness and in Chapter 5 quasi-projectivity. The structure of each of these sections will be similar: first we will define the property we are dealing with and give some background focusing in the results known for right-angled Artin groups and finally we will explain the results that we have obtained in relation to the corresponding property.

Lastly, in Chapter 5.5 we will see a summary of the results in the rest of the thesis and we will briefly see the projects in which I am working right now and I am planning to work in the future.

1.1 Basic Definitions

Definition 1.1. *Let S be a finite set. A* Coxeter matrix *over S is a square matrix $M = (m_{s,t})_{s,t\in S}$ indexed by the elements of S, with coefficients in $\mathbb{N} \cup \{\infty\}$, and satisfying $m_{s,s} = 1$ for all $s \in S$ and $m_{s,t} = m_{t,s} \geq 2$ for all $s,t \in S$, $s \neq t$.*

We will represent such a Coxeter matrix M by a labelled graph Γ, whose set of vertices is S, and where two distinct vertices $s,t \in S$ are linked by an edge labelled with $m_{s,t}$ if $m_{s,t} \neq \infty$. We will also often use the notation $V(\Gamma)$ to denote the set of vertices of Γ (i.e., $V(\Gamma) = S$), when the graph Γ is clear we shall omit Γ and denote this set of vertices by V.

Remark 1.2. *The labelled graph Γ defined above is not the Coxeter graph of M as defined in Bourbaki [14]. It is another fairly common way to represent a Coxeter matrix.*

If a, b are two letters and m is an integer ≥ 2, we set $\langle a,b\rangle^m = (ab)^{\frac{m}{2}}$ if m is even, and $\langle a,b\rangle^m = (ab)^{\frac{m-1}{2}}a$ if m is odd. In other words, $\langle a,b\rangle^m$ denotes the word $aba\cdots$ of length m.

Definition 1.3. *The* Artin group *associated to* Γ*,* $\mathcal{A} = \mathcal{A}_\Gamma$*, is the group defined by the presentation:*

$$\mathcal{A}_\Gamma = \langle v; v \in V \mid \langle uv \rangle^{m_e} = \langle vu \rangle^{m_e}, e = \{u, v\} \in E \rangle. \tag{1.1}$$

Right-angled Artin groups (RAAGs) *are defined as Artin groups in which all the edges of the graph have label 2.*

Definition 1.4. *The* Coxeter group *associated to* Γ*,* $W = W_\Gamma$*, is the quotient of* $\mathcal{A}_\Gamma$ *by the relations* $v^2 = 1$*,* $v \in V$*. That is, it is defined by the presentation:*

$$W_\Gamma = \langle v; v \in V \mid v^2 = 1, \langle uv \rangle^{m_e} = \langle vu \rangle^{m_e}, e = \{u, v\} \in E \rangle. \tag{1.2}$$

Now, we are going to define some interesting subfamilies that will appear along the text.

For $T \subset S$, we denote by $\mathcal{A}_T$ (resp. W_T) the subgroup of $\mathcal{A}$ (resp. W) generated by T, and by Γ_T the full subgraph of Γ spanned by T. Here we mean that each edge of Γ_T is labelled with the same number as its corresponding edge of Γ. By Bourbaki [14], the group W_T is the Coxeter group of Γ_T, and, by van der Lek [75], $\mathcal{A}_T$ is the Artin group of Γ_T.

Definition 1.5. *The group* $\mathcal{A}_T$ *(resp.* W_T*) is called a* standard parabolic subgroup *of* $\mathcal{A}$ *(resp. of* W*).*

Definition 1.6. *We say that* $\mathcal{A} = \mathcal{A}_\Gamma$ *is of* spherical type *if its associated Coxeter group,* W_Γ*, is finite.*

A subset T of S is called *free of infinity* if $m_{s,t} \neq \infty$ for all $s, t \in T$.

Definition 1.7. *We say that* $\mathcal{A}$ *is of* FC type *if* $\mathcal{A}_T$ *is of spherical type for every free of infinity subset* T *of* S*.*

Definition 1.8. *We say that* $\mathcal{A}_\Gamma$ *is* large *if* $m_{s,t} \geq 3$ *for every label in the graph.*

Finally, we will define the main object of study of this thesis.

Definition 1.9. *We say that* $\mathcal{A}$ *is* even *if any finite* $m_{s,t}$ *is even.*

1.2 Importance and interest of even Artin groups

As remarked in the introduction, the family of even Artin groups hasn't been studied a lot for the moment but we think that they deserve special attention since they have particularly interesting properties as we shall see next. Assume that $\mathcal{A}$ is even.

(1) Let $s, t \in S$, $s \neq t$. If we set $m_{s,t} = 2k_{s,t}$, then the Artin relation $\langle s, t\rangle^{m_{s,t}} = \langle t, s\rangle^{m_{s,t}}$ becomes $(st)^{k_{s,t}} = (ts)^{k_{s,t}}$. This form of relation is less innocuous than it seems as we will see in Chapter 2.

(2) Let T be a subset of S. Then the inclusion map $\mathcal{A}_T \hookrightarrow \mathcal{A}$ always admits a retraction $\pi_T : \mathcal{A} \to \mathcal{A}_T$ which sends s to s if $s \in T$, and sends s to 1 if $s \notin T$.

(3) Moreover, the $K(\pi, 1)$ conjecture, which is one of the most important problems related with Artin groups, is known to be true for this family of groups (see Corollary 1.17 below).

Before stating that conjecture, we will review some basic notions which are necessary to understand its statement.

Definition 1.10. A space V is said to be **aspherical** if its only non-trivial homotopy group is the first homotopy group (i.e. the fundamental group).

Definition 1.11. Given a vector space V and a field $\mathbb{K}$, a **reflection** with respect to a bilinear map $B : V \times V \longrightarrow K$ is a linear map

$$r_u(v) = v - 2B(u, v)u$$

for some unit vector u.

Definition 1.12. Let π be a group. A topological space is a $K(\pi, 1)$ if it is aspherical and has first homotopy group the group π.

Any Coxeter group (W) can be represented as a discrete reflection group, i.e. a discrete group of linear transformations of a finite dimensional vector space V with the generators s_i acting as reflections with respect to some bilinear form B. If W is a finite Coxeter Group and $r \in W$ is a reflection, we define H_r as the hyperplane consisting of the set of fixed points. This way, complexifying the action defined before one obtains a finite arrangement of complex hyperplanes $\mathbb{C}H_r$ in $\mathbb{C}^n$ such that W acts freely in the complement, $Y_W = \mathbb{C}^n - (\cup \mathbb{C}H_r)$.

For infinite Artin groups we can define an analogous hyperplane complement Y_W in $\mathbb{C} \otimes V$.

At this point, we can state the $K(\pi, 1)$-Conjecture:

Conjecture 1.13. *(The $K(\pi, 1)$-Conjecture) Let W be a Coxeter Group and $\mathcal{A}$ the associated Artin Group. Then Y_W/W is aspherical with fundamental group $\mathcal{A}$. That is, Y_W/W is a $K(\mathcal{A}, 1)$ space.*

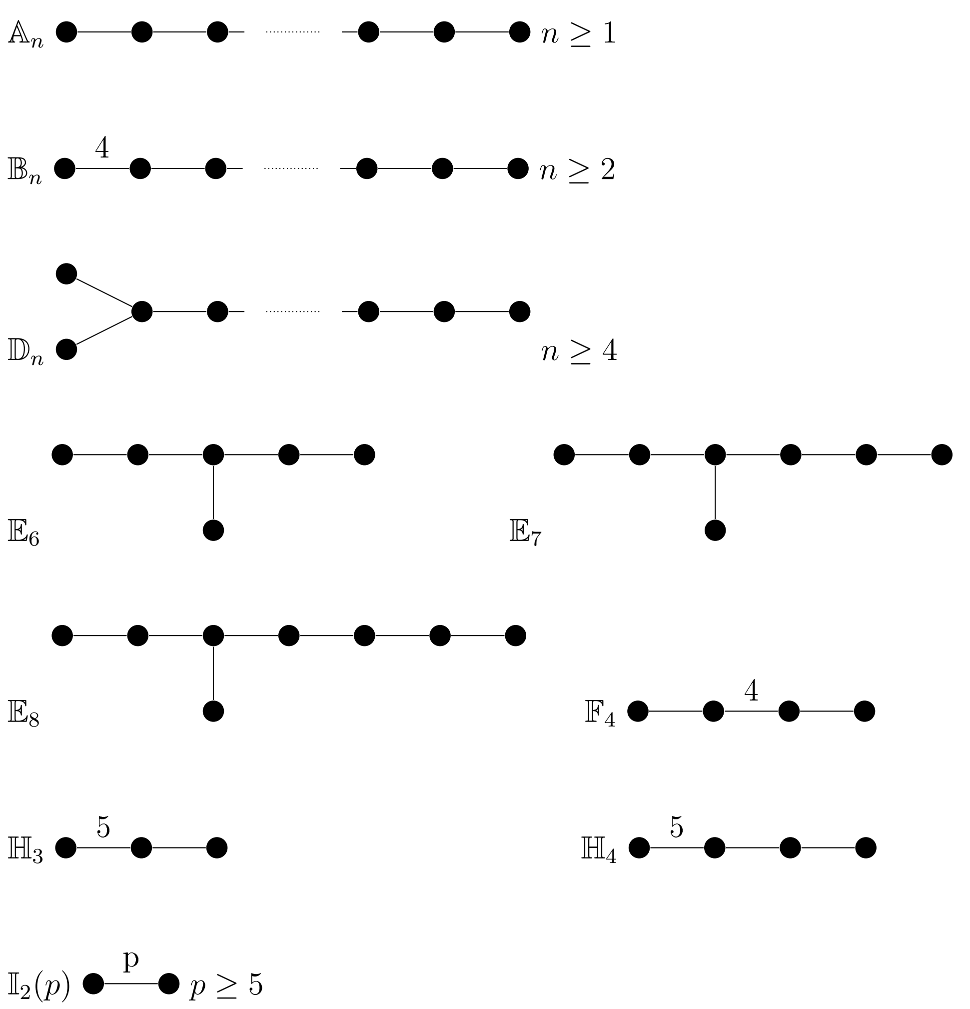
The conjecture is known to be true in several particular cases, including the case when W is a finite Coxeter Group, and the case when W is a right-angled Coxeter group (i.e. the associated Artin Group $\mathcal{A}$ is right-angled). A proof for the general case is still unknown.

The proof for right-angled Coxeter groups is very extense and technical and can be found in [28]. Essentially, the proof consists on constructing what is called the modified Deligne complex and showing that it is contractible.

Using an already known result due to Charney it is easy to see that for even Artin groups the conjecture is also true.

First we need to determine which are the finite even Coxeter groups.

The classification of finite Coxeter groups is well-known. The irreducible finite Coxeter groups are the ones defined by the following Coxeter graphs [35], [34]:



Remark 1.14. *Notice, that here we are using the Coxeter graphs notation due to Bourbaki [14] instead of the one we have defined before. The reason for this change is to make easier the representation of the graphs and because this is the most common notation for spherical Artin groups.*

In the Coxeter graphs two commutative vertices are not joined, if two vertices are joined with an edge without label there is a relation of label 3 *between them. In other case, we label the edge between the two vertices with the same number as in the original graph.*

Theorem 1.15. *Finite even Coxeter groups are direct product of dihedral groups or copies of the cyclic group of two elements* C_2.

Proof. If we look at the classification of irreducible finite Coxeter groups, the only ones wich are even are:

$$\mathbb{A}_1, \mathbb{B}_2, \mathbb{I}_2(p); \quad p = 2k, k \geq 3$$

which gives us the groups $C_2, D_2, D_p; p \geq 3$ respectively (where D_n, $n \in \mathbb{Z}$, is the dihedral group of degree n).

So any finite even Coxeter groups must be a direct product with factors C_2 or $D_k, k \geq 2$. □

Now, using the following proposition due to Charney we will get our desired result:

Proposition 1.16. *[26] Let* (W, S) *be a Coxeter system such that every irreducible, finite subgroup* W_T *(based on a subgraph* T *of* Γ*) is either* $Sym_4, \mathbb{Z}_2$*, or a dihedral group, and let* $\mathcal{A}$ *be the associated Artin group. Then the Deligne complex* $\mathcal{D}_{\mathcal{A}}$ *is CAT(0), and the hyperplane complement* H_W/W *is a* $K(\mathcal{A}, 1)$ *space.*

So, from Theorem 1.15 and Proposition 1.16 we obtain as an immediate consequence:

Corollary 1.17. *Even Artin groups verify the* $K(\pi, 1)$ *conjecture.*

Chapter 2

Poly-freeness: Background and even FC type Artin groups

In this chapter we will study the concept of poly-freeness for some families of even Artin groups. In section 2.1 we will define the notion of poly-freeness and we will see why it is an important property, in Section 2.2 we will recall some previously known results regarding the poly-freeness of RAAGs. Later on, in section 2.3 we will prove that even Artin groups of FC type are poly-free, this result has been published in [11].

2.1 Importance of poly-freeness

Definition 2.1. A group G is **poly-free** if there exists a tower of normal subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_N = G$$

such that every quotient G_{i+1}/G_i is free and $G_i \trianglelefteq G$.

The least natural number, N, such that it exist a tower like this for G is the **poly-free length of** G and we will denote it by $pfl(G)$.

This property is important because it implies the property of being locally indicable:

Definition 2.2. A group G is **locally indicable** if each of its finitely generated subgroups maps homomorphically onto $\mathbb{Z}$.

We will need to use a lemma about locally indicable groups:

Lemma 2.3. *[37] If a group G contains a normal subgroup G^* such that G^* and G/G^* are both locally indicable then G is also locally indicable.*

Proof. Let H be a finitely generated subgroup of G. Then, $HG^* \leq G$ and $H \cap G^* \trianglelefteq H$.

Now, if $G^* \lneqq HG^*$, then $1 \neq HG^*/G^* \leq G/G^*$ is finitely generated and since G/G^* is locally indicable HG^*/G^* can be mapped homomorphically to $\mathbb{Z}$. We have $H/H \cap G^* \simeq HG^*/G^*$, so $H/H \cap G^*$ can be mapped homomorphically to $\mathbb{Z}$, and composing with the natural epimorphism $\phi : H \longrightarrow H/H \cap G^*$ we get that also H can be mapped homomorphically to $\mathbb{Z}$.

Otherwise, if $G^* = HG^*$ then $H \leq G^*$ and since G^* is locally indicable and H is finitely generated, we have that H can be mapped homomorphically to $\mathbb{Z}$. So, G is locally indicable. $\square$

Now, let's see that poly-freeness implies locally indicability. To see this, first we will see that free groups are locally indicable:

Theorem 2.4. *Every free group is locally indicable.*

Proof. It is easy to see that every free group maps homomorphically to $\mathbb{Z}$. And it is known that every subgroup of a free group is also free.

So, from this two facts, we get that every finitely generated subgroup of a free group maps homomorphically to $\mathbb{Z}$, i.e. every free group is locally indicable. $\square$

Theorem 2.5. *Every poly-free group is locally indicable*

Proof. It suffices to iterate the use of Lemma 2.3 and Theorem 2.4. $\square$

Finally, it is possible to see that beging locally indicable implies the existence of a right order:

Definition 2.6. A pair $(G, <)$ (where G is a group and $<$ is a total order) is a **right ordered group** if the order $<$ is right invariant, i.e

$$g < h \Rightarrow gk < hk, \qquad \forall g, h, k \in G.$$

Example 2.7. The group $\mathbb{Z}$ with its usual order is right ordered.

We have the following useful characterization of right orderability:

Theorem 2.8. *[25][13] G is right orderable if and only if every finitely generated non-trivial subgroup of G has a non-trivial quotient which is also right orderable.*

And from this we can deduce our theorem:

Theorem 2.9. *[73] Every locally indicable group is right orderable*

Proof. Since $\mathbb{Z}$ is clearly right-orderable, the result follows from 2.8. $\square$

2.2 Poly-freeness for right-angled Artin groups

In this section we follow [58], [45] and [53].

Definition 2.10. The **chromatic number** of a graph Γ, $chr(\Gamma)$, is the minimum number of colours which are necessary to paint the vertices of the graph Γ in the way that two vertices which are joined by an edge have different colours.

Definition 2.11. A **clique** is a complete graph (i.e. each pair of vertices are joined by an edge).

Definition 2.12. The **clique number** of a graph Γ, $clq(\Gamma)$, is the number of vertices of the greater clique which is a subgraph of Γ.

Definition 2.13. Let ϕ be an homomorphism $\phi : \mathcal{A}_\Gamma \longrightarrow F_q$ from a right-angled Artin group to a free group. The set $D = \{v \in V(\Gamma) \mid \phi(v) = 1\}$ is the **set of dead vertices**, the set $L = V(\Gamma) - D$ is the **set of living vertices** and the smallest subgraph of Γ containing L, Γ_L, is the **living subgraph of** Γ.

Definition 2.14. Let G be a group generated by a set S. A word w of length k over $S \cup S^{-1}$ is a **geodesic word** if no other word over $S \cup S^{-1}$ of length strictly less than k represents the same element in G as w does.

A total order defined over $S \cup S^{-1}$ defines a total order, called **shortlex order**, on all words over $S \cup S^{-1}$ in which shorter words always precede longer ones and the words of the same length are ordered lexicographically according to the order defined on $S \cup S^{-1}$.

A shortlex representative of an element $g \in G$ is the smallest word in the shortlex order that represents g, so it is by definition geodesic.

Now we will prove some results which will be useful during our proof.

Proposition 2.15. *If G is poly-free of length N and $H \leq G$ then H is poly-free of length $\leq N$.*

Proof. Given a poly-free tower for G:

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_N = G.$$

We consider the tower:

$$1 = G_0 \cap H \trianglelefteq G_1 \cap H \trianglelefteq ... \trianglelefteq G_N \cap H = H$$

which is a poly-free tower for H (since every subgroup of a free group is free). $\square$

Proposition 2.16. *If G has a free normal subgroup H and the quotient G/H is poly-free with $pfl(G/H) = N$, then G is poly-free with $pfl(G) \leq N + 1$.*

Proof. Let $\phi : G \longrightarrow G/H$ be the canonical homomorphism.

Consider a poly-free tower:

$$1 = Q_0 \trianglelefteq Q_1 \trianglelefteq ... \trianglelefteq Q_N = G/H.$$

Then the tower:

$$1 \trianglelefteq \phi^{-1}(Q_0) = H \trianglelefteq \phi^{-1}(Q_1) \trianglelefteq ... \trianglelefteq \phi^{-1}(Q_N) = G$$

is a poly-free tower for G. □

Lemma 2.17. *A group G is poly-free if and only if*

$$G = (...(F_{n_1} \rtimes F_{n_2}) \rtimes ...) \rtimes F_{n_k}.$$

Proof. The $\Leftarrow$ implication is obvious.

For the $\Rightarrow$ impication, we know that G is poly-free, i.e there exists a tower:

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_N = G$$

such that G_i/G_{i-1} is free. Then, $F := G/G_{N-1}$ is free. We consider the projection epimorphism $\tau : G \longrightarrow F$ and by the universal property of free groups, there exists $\phi : F \longrightarrow G$ such that $\tau\phi = 1_F$ and $F \simeq \phi(F) \leq G$. So, $G = G_{N-1} \rtimes \phi(F)$. And by induction we get the desired result. □

Lemma 2.18. *The group $\mathbb{Z}^n$ has poly-free length equal to n.*

Proof. Since $\mathbb{Z}^n$ is a direct product of n copies of the free group $\mathbb{Z}$, $\mathbb{Z}^n$ is poly-free with length at most n.

To show that $\mathbb{Z}^n$ cannot have poly-free length less than n, assume that $\mathbb{Z}^n = (...(F_{n_1} \rtimes F_{n_2}) \rtimes ...) \rtimes F_{n_k}$, for some (finite or infinite) n_i, $i = 1, ..., k$.

Since $\mathbb{Z}^n$ is abelian, we must have $n_1 = ... = n_k = 1$. Thus $\mathbb{Z}^n$ can be generated by k elements (one of each F_{n_i}). However, $\mathbb{Z}^n$ cannot be generated by less than n elements, which shows $n \leq k$. □

Given a graph Γ with chromatic number > 1, let D be the set of vertices of one of the colours and $L = V(\Gamma) - D$ the vertices of the remaining colours.

We consider Γ_L the full subgraph of Γ generated by the vertices in L.

Let ϕ be the homomorphism $\phi : \mathcal{A}_\Gamma \longrightarrow \mathcal{A}_{\Gamma_L}$ given by $\phi(d) = 1$ if $d \in D$ and $\phi(a) = a$ if $a \in L$. Then D is the set of the dead vertices and $\mathcal{A}_{\Gamma_L}$ is the living subgraph. This homomorphism is well defined because of the following well known result:

Proposition 2.19. *Given two groups $G_1 = \langle X \mid R \rangle, G_2$, $\phi : G_1 \longrightarrow G_2$ defines a homomorphism if and only if $\phi(r) = 1\ \forall r \in R$.*

In the following proof we will build a free group F (isomorphic to $Ker(\phi)$) and an action of $\mathcal{A}_{\Gamma_L}$ on F and we will see that $\mathcal{A}_\Gamma$ is isomorphic to the semidirect product $F \rtimes \mathcal{A}_{\Gamma_L}$.

This result was proved independently by Duchamp-Krob [45], Howie [58] and Hermiller-Šunić [53].

Theorem 2.20. *[45][58][53] Let Γ be a finite graph, or more generally a graph with finite chromatic number $chr(\Gamma)$ and finite clique number $clq(\Gamma)$.*

The right-angled Artin group $\mathcal{A}_\Gamma$ is poly-free. Moreover:

$$clq(\Gamma) \leq pfl(\Gamma) \leq chr(\Gamma)$$

and there exists a poly-free tower of length $chr(\Gamma)$.

Proof. To prove that $\mathcal{A}_\Gamma$ is poly-free and the upper bound on the poly-free length we argue by induction on $chr(\Gamma)$.

If $chr(\Gamma) = 1$, then Γ is totally disconnected, so it is poly-free of length 1. Let us suppose that $chr(\Gamma) \geq 2$, and that for every graph Γ' with $chr(\Gamma') < chr(\Gamma)$. Then the group $\mathcal{A}'_\Gamma$ is poly-free and has a poly-free tower of length $chr(\Gamma')$.

We choose a coloration of Γ with $chr(\Gamma)$ different colours, one of which will be grey. Let D be the set of grey vertices in $V = V(\Gamma)$, $L = V - D$ the vertices of different colours, Γ_L the subgraph induced by L and $\mathcal{A}_{\Gamma_L}$ the corresponding right-angled Artin group. Then, $chr(\Gamma_L) \leq chr(\Gamma) - 1$, so by induction hypothesis there exists a poly-free tower for $\mathcal{A}_{\Gamma_L}$ of length $chr(\Gamma) - 1$.

We consider the maps induced as follows:

$$\begin{aligned} \pi : \mathcal{A}_\Gamma &\longrightarrow \mathcal{A}_{\Gamma_L} \\ x &\mapsto \begin{cases} x & \text{if } x \in L \\ 1 & \text{if } x \in D \end{cases} \end{aligned}$$

$$\begin{aligned} \iota : \mathcal{A}_{\Gamma_L} &\longrightarrow \mathcal{A}_\Gamma \\ x &\mapsto x \quad x \in L \end{aligned}$$

(note that π, ι are well-defined by Proposition 2.19).

As $\pi \circ \iota = 1_d$ we get the semidirect product: $Ker(\pi) \rtimes \mathcal{A}_{\Gamma_L} = \mathcal{A}_\Gamma$. We only need to see that $Ker(\pi)$ is a free group.

Consider the following sets:

$$T = \{\alpha x \alpha^{-1} \mid x \in D, \alpha \in \mathcal{A}_{\Gamma_L}\} \subset \mathcal{A}_\Gamma$$

$$\hat{T} = \{T_\beta \mid \beta \in T\}$$

$\mathcal{A}_{\Gamma_L}$ acts on $\hat{T}$ in the obvious way:

$$\alpha(T_\beta) = T_{\alpha\beta\alpha^{-1}}$$

Therefore, $\mathcal{A}_{\Gamma_L}$ acts on the free group $F(\hat{T})$ and we define G as:

$$G = F(\hat{T}) \rtimes \mathcal{A}_{\Gamma_L}$$

We consider the following applications:

$$\begin{aligned} f_T : F(\hat{T}) &\longrightarrow \mathcal{A}_\Gamma \\ T_\beta &\mapsto \beta \end{aligned}$$

$$\begin{aligned} f : G &\longrightarrow \mathcal{A}_\Gamma \\ (T_w, \alpha) &\mapsto f_T(T_w)\alpha \end{aligned}$$

Obviously, $\alpha f_T(T_w)\alpha^{-1} = f_T(\alpha(T_w))$. Note that f_T induces a group homomorphism that we also denote f_T and that f is also a group homomorphism (notice that $f((T_{w_1}, \alpha_1), (T_{w_2}, \alpha_2)) = w_1\alpha_1 w_2 \alpha_2 = f(T_{w_1}, \alpha_1) f(T_{w_2}, \alpha_2))$.

Now, we define:

$$\begin{aligned} g : \mathcal{A}_\Gamma &\longrightarrow G \\ x \in L &\mapsto g(x) = x \\ x \in D &\mapsto g(x) = T_x \end{aligned}$$

We claim that g induces a well defined group isomorphism such that $g(F(\hat{T})) = \ker(\pi)$. As $F(\hat{T})$ is free, this implies that $\ker(\pi)$ is free. Let $x, y \in V$ be such that $xy = yx$, we have two options: either $x, y \in L$ or $x \in L$ and $y \in D$ (since two vertices of the same colour cannot be joined, we cannot have $x, y \in D$).

1. $x, y \in L$.
   $g(x)g(y) = xy = yx = g(y)g(x)$.
2. $x \in L, y \in D$.
   $g(x)g(y)g(x^{-1}) = xT_y x^{-1} = T_{xyx^{-1}} = T_y = g(y)$.

Then g induces a group homomorphism. So, we only have to check that $f \circ g = Id$ and $g \circ f = Id$. Since both are well-defined, it is enough to see it for the generators:

1. $f \circ g = Id$

   (a) $x \in L$
   $(f \circ g)(x) = f(x) = x$.

   (b) $x \in D$
   $(f \circ g)(x) = f(T_x) = x$.

2. $g \circ f = Id$

   (a) $x \in L$
   $(g \circ f)(x) = g(x) = x$.

   (b) $\beta \in T$. $\beta = \alpha x \alpha^{-1}$ with $\alpha \in \mathcal{A}_{\Gamma_L}$ and $x \in D$ ($T_\beta = \alpha T_x \alpha^{-1}$).
   $(g \circ f)(T_\beta) = (g \circ f)(\alpha T_x \alpha^{-1}) = g(\alpha x \alpha^{-1}) = (\alpha T_x \alpha^{-1}) = T_\beta$.

Thus, $\mathcal{A}_\Gamma \simeq G = F(\hat{T}) \rtimes \mathcal{A}_{\Gamma_L}$ and notice that the isomorphism sends $F(\hat{T})$ to $\ker(\pi)$, so $F(\hat{T}) \simeq \ker(\pi)$. By induction $\mathcal{A}_{\Gamma_L}$ has poly-free tower of length $chr(\Gamma) - 1$, thus by Proposition 2.16 $\mathcal{A}_\Gamma$ has a poly-free tower of length $chr(\Gamma)$, and $pfl(\Gamma) \leq chr(\Gamma)$.

Finally, we consider the lower bound of the poly-free tower. Let $m = clq(\Gamma)$ and let $\hat{\Gamma}$ be a clique of Γ with m vertices. Then $\hat{\Gamma}$ is a complete graph and the subgroup $\mathcal{A}_{\hat{\Gamma}}$ corresponding to $\hat{\Gamma}$ is isomorphic to $\mathbb{Z}^m$. By Lemma 2.18, $m = pfl(\hat{\Gamma})$, and by Proposition 2.15, $pfl(\hat{\Gamma}) \leq pfl(\Gamma)$. □

Remark 2.21. *We want to remark that it is also known that every one relator Artin group is poly-free. This result is due to Mullholland and Rolfsen and can be found in Theorems 3.6, 3.9 and 3.18 in [68].*

2.3 Poly-freeness on even Artin groups of FC type

In this section we prove that even Artin groups of FC type are poly-free, this section is based in [11].

Recall that an Artin group $\mathcal{A}_\Gamma$ is called of **FC type** if for every complete subgraph Ω of Γ, the associated Coxeter group W_Ω is finite (i.e. $\mathcal{A}_\Omega$ is spherical).

2.3.1 Preliminaries

Britton's lemma

Let G be a group generated by a finite set S. We denote by $(S \cup S^{-1})^*$ the free monoid over $S \cup S^{-1}$, that is, the set of words over $S \cup S^{-1}$, and we denote by $(S \cup S^{-1})^* \to G$, $w \mapsto \bar{w}$, the map that sends a word to the element of G that it represents. Recall that a set of *normal forms* for G is a subset $\mathcal{N}$ of $(S \cup S^{-1})^*$ such that the map $\mathcal{N} \to G$, $w \mapsto \bar{w}$, is a one-to-one correspondence.

Let G be a group with two subgroups $A, B \leq G$, and let $\varphi : A \to B$ be an isomorphism. A useful consequence of Britton's lemma yields a set of normal forms for the HNN-extension $G*_\varphi = \langle G, t \mid t^{-1}at = \varphi(a), a \in A\rangle$ in terms of a set $\mathcal{N}$ of normal forms for G and sets of representatives of the cosets of A and B in G (see Lyndon–Schupp [64]). Explicitly, choose a set T_A of representatives of the left cosets of A in G containing 1, and a set T_B of representatives of the left cosets of B in G also containing 1.

Proposition 2.22 (Britton's normal forms)**.** *Let $\tilde{\mathcal{N}}$ be the set of words of the form $w_0 t^{\varepsilon_1} w_1 \cdots t^{\varepsilon_m} w_m$, where $m \geq 0$, $\varepsilon_i \in \{\pm 1\}$ and $w_i \in \mathcal{N}$ for all i, such that:*

(a) $\bar{w}_i \in T_A$ if $\varepsilon_i = -1$, for $i \geq 1$,

(b) $\bar{w}_i \in T_B$ if $\varepsilon_i = 1$, for $i \geq 1$,

(c) there is no subword of the form $t^\varepsilon t^{-\varepsilon}$.

Then $\tilde{\mathcal{N}}$ is a set of normal forms for the HNN-extension $G*_\varphi$.

Variations of the even Artin-type relations

For a, b in a group G, we denote by $b^a = a^{-1}ba$ the conjugate of b by a. The aim of this subsection is to illustrate how the Artin relations in even Artin groups can be expressed in terms of conjugates. This observation will be a key point in our proof of Theorem 2.41.

Lemma 2.23. *Let s, t be two generators of an Artin group $\mathcal{A}$ such that $m_{s,t} = 2k$ is finite and even. Then*

$$\begin{aligned} t^{s^{-1}} &= t^{-1}(t^s)^{-1} \cdots (t^{s^{k-2}})^{-1} t^{s^{k-1}} t^{s^{k-2}} \cdots t^s t\,, \\ t^{s^k} &= t^{s^{k-1}} \cdots t^s t (t^s)^{-1} \cdots (t^{s^{k-1}})^{-1}\,. \end{aligned}$$

In particular, $t^{s^i} \in \langle t, t^s, \ldots, t^{s^{k-1}}\rangle$ for all $i \in \mathbb{Z}$.

Proof. It is easily proved by induction that $(st)^\ell = s^\ell t^{s^{\ell-1}} \cdots t^s t$ for all $\ell \geq 1$. Then by conjugating this equality by s we have $(ts)^\ell = s^\ell t^{s^\ell} \cdots t^{s^2} t^s$. From the equality $(st)^k = (ts)^k$ follows $t^{s^{k-1}} \cdots t^s t = t^{s^k} \cdots t^{s^2} t^s$. This implies on the one hand that $t^{s^k} = t^{s^{k-1}} \cdots t^s t (t^s)^{-1} \cdots (t^{s^{k-1}})^{-1}$ and, on the other hand, that $t = (t^s)^{-1} \cdots (t^{s^{k-1}})^{-1} t^{s^k} t^{s^{k-1}} \cdots t^s$. By conjugating this last equality by s^{-1} we obtain $t^{s^{-1}} = t^{-1} \cdots (t^{s^{k-2}})^{-1} t^{s^{k-1}} t^{s^{k-2}} \cdots t$. Finally it is easily shown by induction that $t^{s^{-1-\ell}}, t^{k+\ell} \in \langle t, t^s, \ldots, t^{s^{k-1}} \rangle$ for all $\ell \geq 0$, hence $t^{s^i} \in \langle t, t^s, \ldots, t^{s^{k-1}} \rangle$ for all $i \in \mathbb{Z}$. $\square$

2.3.2 Poly-freeness of even Artin groups of FC type

In this section we prove that even Artin groups of FC type are poly-free (Theorem 2.41). We begin with a characterization of these groups in terms of their defining graphs.

Lemma 2.24. *Let $\mathcal{A}_\Gamma$ be an even Artin group. Then $\mathcal{A}_\Gamma$ is of FC type if and only if every triangular subgraph of Γ has at least two edges labelled with* 2.

Proof. We say that Γ is *reducible* if there exists a partition $S = X \sqcup Y$ such that $X \neq \emptyset$, $Y \neq \emptyset$ and $m_{s,t} = 2$ for all $s \in X$ and $t \in Y$. We say that Γ is *irreducible* otherwise. There exists a unique partition $S = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_\ell$ such that $X_i \neq \emptyset$ and Γ_{X_i} is irreducible for all $i \in \{1, \ldots, \ell\}$, and $m_{s,t} = 2$ for all $s \in X_i$, $t \in X_j$, $i, j \in \{1, \ldots, \ell\}$, $i \neq j$. In this case we have $\mathcal{A}_\Gamma = \mathcal{A}_{X_1} \times \mathcal{A}_{X_2} \times \cdots \times \mathcal{A}_{X_\ell}$ and $W_\Gamma = W_{X_1} \times W_{X_2} \times \cdots \times W_{X_\ell}$. In particular, $\mathcal{A}_\Gamma$ is of spherical type if and only if $\mathcal{A}_{\Gamma_{X_i}}$ is of spherical type for all $i \in \{1, \ldots, \ell\}$. Then we say that Γ is the *orthogonal sum* of $\Gamma_{X_1}, \Gamma_{X_2}, \ldots, \Gamma_{X_\ell}$.

Suppose that $\mathcal{A}_\Gamma$ is even. From the classification of finite irreducible Coxeter groups (see Section 1.2, [35], [34]) follows that $\mathcal{A}_\Gamma$ is irreducible of spherical type if and only if Γ has at most two vertices and, if $|S| = 2$, then the two vertices of Γ are connected (by an edge labelled with an even number ≥ 4). It follows that $\mathcal{A}_\Gamma$ is of spherical type if and only if Γ is a complete graph and is the orthogonal sum of irreducible graphs with 1 or 2 vertices. This condition is clearly equivalent to Γ is a complete graph and every triangular subgraph of Γ has at least two edges labelled with 2.

Suppose that $\mathcal{A}_\Gamma$ is even and of FC type. Let Ω be a triangular subgraph of Γ. Since $\mathcal{A}_\Gamma$ is of FC type and Ω is a complete subgraph, $\mathcal{A}_\Omega$ is of spherical type and therefore, by the above, Ω has at least two edges labelled with 2. Suppose that $\mathcal{A}_\Gamma$ is even and every triangular subgraph of Γ has at least two edges labelled with 2. Let Ω be a complete subgraph of Γ. Then every triangular subgraph of Ω has at least two edges labelled with 2, hence, by the above, $\mathcal{A}_\Omega$ is of spherical type. So, $\mathcal{A}_\Gamma$ is of FC type. $\square$

The proof of Theorem 2.41 is based on the following.

Proposition 2.25. *Assume that $\mathcal{A}_\Gamma$ is even and of FC type. Then there is a free group F such that $\mathcal{A}_\Gamma = F \rtimes \mathcal{A}_1$, where $\mathcal{A}_1$ is an even Artin group of FC type based on a proper subgraph of Γ.*

We proceed now with some notations needed for the proof of Proposition 2.25. Fix some vertex z of Γ. Recall that the *link* of z in Γ is the full subgraph $\mathrm{lk}(z, \Gamma)$ of Γ with vertex set $V(\mathrm{lk}(z, \Gamma)) = \{s \in S \mid s \neq z \text{ and } m_{s,z} \neq \infty\}$. As ever, we see $\mathrm{lk}(z, \Gamma)$ as a labelled graph, where the labels are the same as in the original graph Γ. We set $L = \mathrm{lk}(z, \Gamma)$, and we denote by Γ_1 the full subgraph of Γ spanned by $S \setminus \{z\}$. We denote by $\mathcal{A}_1$ and $\mathcal{A}_L$ the subgroups of $\mathcal{A}_\Gamma$ generated by $V(\Gamma_1)$ and $V(L)$, respectively. Recall from Section 1.1 that $\mathcal{A}_1$ and $\mathcal{A}_L$ are the Artin groups associated with the graphs Γ_1 and L, respectively (so this notation is consistent).

As pointed out in Section 1.2, since $\mathcal{A}_\Gamma$ is even, the inclusion map $\mathcal{A}_1 \hookrightarrow \mathcal{A}$ has a retraction $\pi_1 : \mathcal{A}_\Gamma \to \mathcal{A}_1$ which sends z to 1 and sends s to s if $s \neq z$. Similarly, the inclusion map $\mathcal{A}_L \hookrightarrow \mathcal{A}_1$ has a retraction $\pi_L : \mathcal{A}_1 \to \mathcal{A}_L$ which sends s to s if $s \in V(L)$, and sends s to 1 if $s \notin V(L)$. It follows that $\mathcal{A}_\Gamma$ and $\mathcal{A}_1$ split as semi-direct products $\mathcal{A}_\Gamma = \mathrm{Ker}(\pi_1) \rtimes \mathcal{A}_1$ and $\mathcal{A}_1 = \mathrm{Ker}(\pi_L) \rtimes \mathcal{A}_L$.

For $s \in V(L)$ we denote by k_s the integer such that $m_{z,s} = 2k_s$. Lemma 2.24 implies the following statement. This will help us to describe $\mathcal{A}_L$ as an iterated HNN extension.

Lemma 2.26. *Let s, t be two linked vertices of L.*

(1) Either $k_s = 1$, or $k_t = 1$.

(2) If $k_s > 1$, then $m_{s,t} = 2$.

Let L_1 be the full subgraph of L spanned by the vertices $s \in V(L)$ such that $k_s = 1$. Lemma 2.26 implies that $L \setminus L_1$ is totally disconnected. We set $V(L \setminus L_1) = \{x_1, \ldots, x_n\}$. Again, from Lemma 2.26, we deduce that, for each $i \in \{1, \ldots, n\}$, if the vertex x_i is linked to some vertex $s \in V(L_1)$, then the label of the edge between x_i and s must be 2. For each $i \in \{1, \ldots, n\}$ we set $S_i = \mathrm{lk}(x_i, L)$, and we denote by X_i the full subgraph of L spanned by $\{x_1, \ldots, x_i\} \cup V(L_1)$ and $X_0 = L_1$. Note that S_i is a subgraph of L_1 and, therefore, is a subgraph of X_i. The subgraphs of Γ that we have defined so far are sitting as follows inside Γ

$$S_i \subseteq L_1 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = L \subseteq \Gamma_1 \subseteq \Gamma$$

where $i \in \{1, \ldots, n\}$ The defining map of each of the HNN extensions will be the identity in the subgroup generated by the vertices commuting with

x_i, that is, $\varphi_i = \mathrm{Id} : \mathcal{A}_{S_i} \to \mathcal{A}_{S_i}$. Then, writing down the associated presentation, we see that $\mathcal{A}_{X_i} = (\mathcal{A}_{X_{i-1}})*_{\varphi_i}$ with stable letter x_i. So, we get the following.

Lemma 2.27. *We have* $\mathcal{A}_L = ((\mathcal{A}_{L_1} *_{\varphi_1}) *_{\varphi_2} \cdots) *_{\varphi_n}$.

Now, fix a set $\mathcal{N}_1$ of normal forms for $\mathcal{A}_{L_1}$ (for example, the set of shortlex geodesic words with respect to some ordering in the standard generating system). We want to use Britton's lemma to obtain a set of normal forms for $\mathcal{A}_L$ in terms of $\mathcal{N}_1$. To do so, first, for each $i \in \{1, \dots, n\}$, we need to determine a set of representatives of the right cosets of $\mathcal{A}_{S_i}$ in $\mathcal{A}_{L_1}$. The natural way to do it is as follows. Consider the projection map $\pi_{S_i} : \mathcal{A}_{L_1} \to \mathcal{A}_{S_i}$ which sends $s \in V(L)$ to s if $s \in V(S_i)$ and sends s to 1 otherwise. Observe that $\mathcal{A}_{L_1} = \mathcal{A}_{S_i} \ltimes \mathrm{Ker}(\pi_{S_i})$. Then, $\mathrm{Ker}(\pi_{S_i})$ is a well-defined set of representatives of the right cosets of $\mathcal{A}_{S_i}$ in $\mathcal{A}_{L_1}$.

In our next result we will use this set of representatives together with Britton's lemma to construct a set $\mathcal{N}_L$ of normal forms for $\mathcal{A}_L$. More precisely, $\mathcal{N}_L$ denotes the set of words of the form

$$w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m \,,$$

where $w_j \in \mathcal{N}_1$ for all $j \in \{0, 1, \dots, m\}$, $\alpha_j \in \{1, \dots, n\}$, $\bar{w}_j \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$ and $\varepsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, m\}$, and there is no subword of the form $x_\alpha^\varepsilon x_\alpha^{-\varepsilon}$ with $\alpha \in \{1, \dots, n\}$.

Lemma 2.28. *The set* $\mathcal{N}_L$ *is a set of normal forms for* $\mathcal{A}_L$.

Proof. For $i \in \{0, 1, \dots, n\}$, we denote by $\mathcal{N}_{L,i}$ the set of words of the form

$$w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m \,, \tag{2.1}$$

where $w_j \in \mathcal{N}_1$ for all $j \in \{0, 1, \dots, m\}$, $\alpha_j \in \{1, \dots, i\}$, $\bar{w}_j \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$ and $\varepsilon_j \in \{\pm 1\}$ for all $j \in \{1, \dots, m\}$, and there is no subword of the form $x_\alpha^\varepsilon x_\alpha^{-\varepsilon}$ with $\alpha \in \{1, \dots, i\}$. We prove by induction on i that $\mathcal{N}_{L,i}$ is a set of normal forms for $\mathcal{A}_{X_i}$. Since $\mathcal{A}_L = \mathcal{A}_{X_n}$, this will prove the lemma.

The case $i = 0$ is true by definition since $\mathcal{A}_{L_1} = \mathcal{A}_{X_0}$ and $\mathcal{N}_{L,0} = \mathcal{N}_1$. So, we can assume that $i \geq 1$ plus the inductive hypothesis. Recall that $\mathcal{A}_{X_i} = (\mathcal{A}_{X_{i-1}})*_{\varphi_i}$, where φ_i is the identity map on $\mathcal{A}_{S_i}$. By induction, $\mathcal{N}_{L,i-1}$ is a set of normal forms for $\mathcal{A}_{X_{i-1}}$. We want to apply Proposition 2.22, so we also need a set T_i of representatives of the right cosets of $\mathcal{A}_{S_i}$ in $\mathcal{A}_{X_{i-1}}$. Since $\mathcal{A}_{L_1} = \mathcal{A}_{S_i} \ltimes \mathrm{Ker}(\pi_{S_i})$, we see that we may take as T_i the set of elements of $\mathcal{A}_{X_{i-1}}$ whose normal forms, written as in Equation 2.1, satisfy $\bar{w}_0 \in \mathrm{Ker}(\pi_{S_i})$. Now, take $g \in \mathcal{A}_{X_i}$ and use Proposition 2.22 with the set $\mathcal{N}_{L,i-1}$ of normal forms and the set T_i of representatives to write a uniquely determined expression for g. The set of these expressions is clearly $\mathcal{N}_{L,i}$. □

Given $g \in \mathcal{A}_L$, we denote by $n(g)$ the normal form of g in $\mathcal{N}_L$. We denote by T_0^* the set of $g \in \mathcal{A}_L \setminus \{1\}$ such that $n(g)$ is of the form $n(g) = x_{\alpha_1} w_1 x_{\alpha_2}^{\varepsilon_2} w_2 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$ (that is, $w_0 = 1$ and $\varepsilon_1 = 1$) and $n(g)$ does not start with $x_{\alpha_1}^{k_{\alpha_1}}$. We set $T_0 = T_0^* \cup \{1\}$ and $T = T_0 \operatorname{Ker}(\pi_L)$. We take an abstract set $B = \{b_g \mid g \in T\}$ in one-to-one correspondence with T and we denote by $F = F(B)$ the free group freely generated by B. This is the free group that appears in the statement of Proposition 2.25.

Recall that $\pi_1 : \mathcal{A}_\Gamma \to \mathcal{A}_1$ is the projection that sends z to 1 and sends s to s for all $s \in S \setminus \{z\}$ and that we have $\mathcal{A}_\Gamma \simeq \operatorname{Ker}(\pi_1) \rtimes \mathcal{A}_1$. The following lemma is not needed to prove Proposition 2.25 but it helps to understand our approach and explains where the set T comes from.

Lemma 2.29. *The group* $\operatorname{Ker}(\pi_1)$ *is generated by* $\{z^g \mid g \in T\}$.

Proof. It follows from the definition of π_1 that $\operatorname{Ker}(\pi_1)$ is the normal closure of $\{z\}$, hence $\operatorname{Ker}(\pi_1)$ is generated by $\{z^g \mid g \in \mathcal{A}_\Gamma\}$. Let $g \in \mathcal{A}_\Gamma$ that we write $g = g_0 z^{\varepsilon_1} g_1 \cdots z^{\varepsilon_m} g_m$ with $\varepsilon_i \in \{\pm 1\}$ and $g_i \in \mathcal{A}_1$ for all i. Let $h_i = g_i g_{i+1} \cdots g_m$ for $i \in \{0, 1, \dots, m\}$. Then

$$z^g = (z^{h_m})^{-\varepsilon_m} \cdots (z^{h_1})^{-\varepsilon_1} z^{h_0} (z^{h_1})^{\varepsilon_1} \cdots (z^{h_m})^{\varepsilon_m} .$$

So, $\operatorname{Ker}(\pi_1)$ is generated by $\{z^g \mid g \in \mathcal{A}_1\}$.

Recall that we have the decomposition $\mathcal{A}_1 = \mathcal{A}_L \ltimes \operatorname{Ker}(\pi_L)$, hence each element $g \in \mathcal{A}_1$ has a unique decomposition of the form $g = hk$ with $h \in \mathcal{A}_L$ and $k \in \operatorname{Ker}(\pi_L)$. Let H be the subgroup of $\operatorname{Ker}(\pi_1)$ generated by $\{z^g \mid g \in T\}$. We take $h \in \mathcal{A}_L$ and $k \in \operatorname{Ker}(\pi_L)$ and we turn to prove that $z^{hk} \in H$ by induction on the length of $n(h)$.

If $h = 1$, then $hk = k \in T$, hence $z^{hk} \in H$. So, we can assume that $h \neq 1$ plus the inductive hypothesis. Let $n(h) = w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$ be the normal form of h. We have $m_{s,z} = 2$ for all $s \in V(L_1)$, hence each element of $\mathcal{A}_{L_1}$ commutes with z, and therefore $z^{hk} = z^{h'k}$, where $h' = x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$. So, we can assume that $w_0 = 1$. We write $n(h) = x_\alpha^t w_1 x_{\alpha_2}^{\varepsilon_2} w_2 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$, where $t \in \mathbb{Z} \setminus \{0\}$, and either $w_1 \neq 1$, or $\alpha_2 \neq \alpha$, or $m = 1$. For $i \in \{0, 1, \dots, k_\alpha - 1\}$ we set $h_i = x_\alpha^i w_1 x_{\alpha_2}^{\varepsilon_2} w_2 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$. If $1 \le i \le k_\alpha - 1$, then $h_i \in T_0$, hence $h_i k \in T$, and therefore $z^{h_i k} \in H$. If $i = 0$, then $z^{h_i k} \in H$ by induction. By Lemma 2.23 we have $z^{hk} \in \langle z^{h_i k} \mid 0 \le i \le k_\alpha - 1 \rangle$, hence $z^{hk} \in H$. $\square$

So, by Lemma 2.29 we have a surjective homomorphism $F \to \operatorname{Ker}(\pi_1)$ which sends b_g to z^g for all $g \in T$. It remains to show that this homomorphism is injective. This will not be done directly. Our strategy is to prove that $\mathcal{A}_1$ admits an action on F that mimics the action of $\mathcal{A}_1$ on $\operatorname{Ker}(\pi_1)$ (see Lemma 2.37), and then to prove that the semi-direct product $F \rtimes \mathcal{A}_1$ obtained via this action is isomorphic to $\mathcal{A}_\Gamma$.

We set $B_0 = \{b_g \mid g \in T_0\}$ and we denote by $F_0 = F(B_0)$ the free group freely generated by B_0. We start by defining an action of $\mathcal{A}_L$ on F_0. This will be later extended to the desired action of $\mathcal{A}_1$ on F.

Given $h \in T_0$, we denote by $\mathrm{supp}(h)$ the set of $x_i \in V(L \setminus L_1)$ which appear in the normal form $n(h)$. We will need the following technical result.

Lemma 2.30. *Let $s \in V(L)$ and $h \in T_0$.*

(1) If $s \in V(L_1)$ and $s \in V(S_i)$ for every $x_i \in \mathrm{supp}(h)$ (including the case $h = 1$), then $hs \notin T_0$ and $hs^{-1} \notin T_0$, but $shs^{-1}, s^{-1}hs \in T_0$.

(2) If $s = x \in V(L \setminus L_1)$ and $h = x^{k_x - 1}$, then $hs \notin T_0$ and $hs^{-1} \in T_0$.

(3) If $s = x \in V(L \setminus L_1)$ and $h = 1$, then $hs \in T_0$ and $hs^{-1} \notin T_0$.

(4) We have $hs, hs^{-1} \in T_0$ in all the other cases.

Proof. We take $h \in T_0$ and $s \in V(L)$. If $h = 1$ and $s = x \in V(L \setminus L_1)$, then $hs = hx = x \in T_0$ and $hs^{-1} = hx^{-1} = x^{-1} \notin T_0$. If $h = 1$ and $s \in V(L_1)$, then $hs = s \notin T_0$, $hs^{-1} = s^{-1} \notin T_0$ and $s^{-1}hs = shs^{-1} = 1 \in T_0$. So, we can assume that $h \neq 1$. Then the normal form of h is written $x_{\alpha_1} w_1 x_{\alpha_2}^{\varepsilon_2} w_2 \cdots x_{\alpha_m}^{\varepsilon_m} w_m$ where $m \geq 1$, $w_j \in \mathcal{N}_1$, $\alpha_j \in \{1, \dots, n\}$, $\bar{w}_j \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$ and $\varepsilon_j \in \{\pm 1\}$ for all j, there is no subword of the form $x_\alpha^\varepsilon x_\alpha^{-\varepsilon}$, and this word does not begin with $x_{\alpha_1}^{k_{\alpha_1}}$.

Suppose that $s = x \in V(L \setminus L_1)$. If either $w_m \neq 1$, or $w_m = 1$ and $x_{\alpha_m}^{\varepsilon_m} \neq x^{-1}$, then the normal form of $hs = hx$ is $x_{\alpha_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m x$. Then $hx \in T_0$ unless $hx = x^{k_x}$ which means that $h = x^{k_x - 1}$. If $w_m = 1$ and $x_{\alpha_m}^{\varepsilon_m} = x^{-1}$, then the normal form of $hs = hx$ is $x_{\alpha_1} w_1 \cdots x_{\alpha_{m-1}}^{\varepsilon_{m-1}} w_{m-1}$, hence $hx \in T_0$. If either $w_m \neq 1$, or $w_m = 1$ and $x_{\alpha_m}^{\varepsilon_m} \neq x$, then the normal form of $hs^{-1} = hx^{-1}$ is $x_{\alpha_1} w_1 \cdots x_{\alpha_m}^{\varepsilon_m} w_m x^{-1}$, hence $hx^{-1} \in T_0$. If $w_m = 1$ and $x_{\alpha_m}^{\varepsilon_m} = x$, then the normal form of $hs^{-1} = hx^{-1}$ is 1 if $m = 1$ and is $x_{\alpha_1} w_1 \cdots x_{\alpha_{m-1}}^{\varepsilon_{m-1}} w_{m-1}$ if $m \geq 2$, hence $hx^{-1} \in T_0$.

Suppose that $s \in V(L_1)$ and $s \in V(S_{\alpha_j})$ for all $j \in \{1, \dots, m\}$. Let $j \in \{1, \dots, m\}$. Since $s \in V(S_{\alpha_j})$, by Lemma 2.26, the vertices s and x_{α_j} are connected by an edge labelled with 2, hence s and x_{α_j} commute. Moreover, since $\bar{w}_j \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$, we have $s^{-1}\bar{w}_j s, s\bar{w}_j s^{-1} \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$. We denote by w_j' (resp. w_j'') the normal form of $s^{-1}\bar{w}_j s$ (resp. $s\bar{w}_j s^{-1}$). By the above the normal form of hs is $s x_{\alpha_1} w_1' \cdots x_{\alpha_m}^{\varepsilon_m} w_m'$ and the normal form of $s^{-1}hs$ is $x_{\alpha_1} w_1' \cdots x_{\alpha_m}^{\varepsilon_m} w_m'$, hence $hs \notin T_0$ and $s^{-1}hs \in T_0$. Similarly, the normal form of hs^{-1} is $s^{-1} x_{\alpha_1} w_1'' \cdots x_{\alpha_m}^{\varepsilon_m} w_m''$ and the normal form of shs^{-1} is $x_{\alpha_1} w_1'' \cdots x_{\alpha_m}^{\varepsilon_m} w_m''$, hence $hs^{-1} \notin T_0$ and $shs^{-1} \in T_0$.

Suppose that $s \in V(L_1)$ and there exists $k \in \{1, \dots, m\}$ such that $s \notin V(S_{\alpha_k})$. We choose such a k so that $s \in V(S_{\alpha_j})$ for all $j \geq k+1$. As before, we have $s^{-1}\bar{w}_j s, s\bar{w}_j s^{-1} \in \mathrm{Ker}(\pi_{S_{\alpha_j}})$ and s commutes with x_{α_j} if $j \geq k+1$. Moreover, since $s \notin V(S_{\alpha_k})$, we have $\pi_{S_{\alpha_k}}(s) = 1$, hence $\bar{w}_k s, \bar{w}_k s^{-1} \in \mathrm{Ker}(\pi_{S_{\alpha_k}})$. We set $w'_j = w_j$ if $j \leq k-1$, we denote by w'_k the normal form of $\bar{w}_k s$ and, for $j \geq k+1$, we denote by w'_j the normal form of $s^{-1}\bar{w}_j s$. Then $x_{\alpha_1} w'_1 x_{\alpha_2}^{\varepsilon_2} w'_2 \cdots x_{\alpha_m}^{\varepsilon_m} w'_m$ is the normal form of hs, hence $hs \in T_0$. We set $w''_j = w_j$ if $j \leq k-1$, we denote by w''_k the normal form of $\bar{w}_k s^{-1}$ and, for $j \geq k+1$, we denote by w''_j the normal form of $s\bar{w}_j s^{-1}$. Then $x_{\alpha_1} w''_1 x_{\alpha_2}^{\varepsilon_2} w''_2 \cdots x_{\alpha_m}^{\varepsilon_m} w''_m$ is the normal form of hs^{-1}, hence $hs^{-1} \in T_0$. $\square$

Hidden in the proof of Lemma 2.30 is the fact that, if $g_1, g_2 \in \mathcal{A}_{L_1}$ and $h \in T_0$, then $g_1 h g_2 \in \mathcal{A}_{L_1} T_0$. Moreover, by Lemma 2.28, every element of $\mathcal{A}_{L_1} T_0$ is uniquely written in the form gh with $g \in \mathcal{A}_{L_1}$ and $h \in T_0$. In this case we set $u(gh) = h$. So, by Lemma 2.30, if $s \in V(L_1)$ and $h \in T_0$, then $u(hs) = s^{-1}hs$ if $s \in V(S_i)$ for every $x_i \in \mathrm{supp}(h)$, and $u(hs) = hs$ otherwise. If $s = x \in V(L \setminus L_1)$, then $u(hx)$ is not defined if $h = x^{k_x - 1}$, and $u(hx) = hx$ otherwise.

We turn now to define the action of $\mathcal{A}_L$ on F_0. We start with the action of the generators. Let $s \in V(L)$. For $h \in T_0$ we set

$$b_h * s = \begin{cases} b_{u(hs)} & \text{if } hs \in \mathcal{A}_{L_1} T_0\,, \\ b_{x^{k_x-1}} \cdots b_x\, b_1\, b_x^{-1} \cdots b_{x^{k_x-1}}^{-1} & \text{if } s = x \in V(L \setminus L_1) \text{ and } h = x^{k_x-1}\,. \end{cases}$$

Then we extend the map $B \to F_0$, $b_h \mapsto b_h * s$, to a homomorphism $F_0 \to F_0$, $f \mapsto f * s$.

Lemma 2.31. *The above defined homomorphism $*s : F_0 \to F_0$ is an automorphism.*

Proof. The result will follow if we show that the map has an inverse. Our candidate to inverse will be the map $*s^{-1} : F_0 \to F_0$ defined by

$$b_h * s^{-1} = \begin{cases} b_{u(hs^{-1})} & \text{if } hs^{-1} \in \mathcal{A}_{L_1} T_0\,, \\ b_1^{-1}\, b_x^{-1} \cdots b_{x^{k_x-2}}^{-1}\, b_{x^{k_x-1}}\, b_{x^{k_x-2}} \cdots b_x\, b_1 & \text{if } s = x \in V(L \setminus L_1) \\ & \text{and } h = 1\,. \end{cases}$$

Assume first that either $s \in V(L_1)$, or $h \notin \{1, x^{k_x-1}\}$ with $s = x \in V(L \setminus L_1)$. In this case we only have to check that $u(u(hs)s^{-1}) = h = u(u(hs^{-1})s)$. Observe that the condition $s \in V(L_1)$ and $s \in S_i$ for every $x_i \in \mathrm{supp}(h)$ is equivalent to $s \in V(L_1)$ and $s \in S_i$ for every $x_i \in \mathrm{supp}(u(hs))$, thus, if that condition holds, we have $u(hs) = s^{-1}hs$ and $u(u(hs)s^{-1}) = u((s^{-1}hs)s^{-1}) = ss^{-1}hss^{-1} = h$. If the condition fails, then $u(hs) = hs$ and $u(u(hs)s^{-1}) = u((hs)s^{-1}) = hss^{-1} = h$. Analogously, one checks that $h = u(u(hs^{-1})s)$.

Now, we assume that $s = x \in V(L \setminus L_1)$ and either $h = 1$ or $h = x^{k_x-1}$. If $h = x^{k_x-1}$, then

$$\begin{aligned}(b_h * x) * x^{-1} &= (b_{x^{k_x-1}} \cdots b_x\, b_1\, b_x^{-1} \cdots b_{x^{k_x-1}}^{-1}) * x^{-1}\\ &= b_{x^{k_x-2}} \cdots b_1\, (b_1 * x^{-1})\, b_1^{-1} \cdots b_{x^{k_x-2}}^{-1}\end{aligned}$$
$$= b_{x^{k_x-2}} \cdots b_1\, b_1^{-1}\, b_x^{-1} \cdots b_{x^{k_x-2}}^{-1}\, b_{x^{k_x-1}}\, b_{x^{k_x-2}} \cdots b_x\, b_1\, b_1^{-1} \cdots b_{x^{k_x-2}}^{-1} = b_{x^{k_x-1}} = b_h\,.$$

On the other hand,

$$(b_h * x^{-1}) * x = b_{x^{k_x-2}} * x = b_{x^{k_x-1}} = b_h\,.$$

The case when $h = 1$ is analogous. □

To show that this action, defined just for the generators of $\mathcal{A}_L$, yields an action of $\mathcal{A}_L$ on F_0, we need to check that it preserves the Artin relations. We do it in the next two lemmas.

Lemma 2.32. *Let $s, t \in V(L)$ such that $m_{s,t} = 2$. Then $(g*s)*t = (g*t)*s$ for all $g \in F_0$.*

Proof. Since s and t are linked in $V(L)$, Lemma 2.24 implies that at least one of the vertices s, t lies in $V(L_1)$. Without loss of generality we may assume $s \in V(L_1)$, i.e., $k_s = 1$. Let $h \in T_0$. Then $b_h * s = b_{u(hs)}$.

Assume first that either $ht \in T_0$, or $t \in S_i$ for every $x_i \in \operatorname{supp}(h)$. In this last case, we have $t \in S_i$ for any $x_i \in \operatorname{supp}(u(hs))$. Therefore

$$(b_h * s) * t = b_{u(hs)} * t = b_{u(u(hs)t)} \quad \text{and} \quad (b_h * t) * s = b_{u(ht)} * s = b_{u(u(ht)s)}\,.$$

Depending on whether hs lies in T_0 or not we have $u(hs) = hs$ or $u(hs) = s^{-1}hs$, and the same for t. So, we have four cases to consider. If $hs, ht \in T_0$, then $hst = hts \in T_0$ and

$$u(u(hs)t) = u((hs)t) = hst = hts = u((ht)s) = u(u(ht)s)\,.$$

If $hs \in T_0$ and $ht \notin T_0$, then $hst \notin T_0$ but $t^{-1}hts \in T_0$, hence

$$u(u(hs)t) = u((hs)t) = t^{-1}hst = t^{-1}hts = u((t^{-1}ht)s) = u(u(ht)s)\,.$$

Similarly, if $hs \notin T_0$ and $ht \in T_0$, then $u(u(hs)t) = u(u(ht)s)$. If $hs, ht \notin T_0$, then $s^{-1}hst \notin T_0$ and $t^{-1}hts \notin T_0$, thus

$$u(u(hs)t) = u((s^{-1}hs)t) = t^{-1}s^{-1}hst = s^{-1}t^{-1}hts = u((t^{-1}ht)s) = u(u(ht)s)\,.$$

We are left with the case where $t = y \in V(L \setminus L_1)$ and $h = y^{k_y-1}$. Then, since s and y are linked, for every $\alpha \in \{0, 1, \ldots, k_y - 1\}$ we have $u(y^\alpha s) = s^{-1} y^\alpha s = y^\alpha$, thus

$$(b_h * s) * y = b_h * y = b_{y^{k_y-1}} \cdots b_y\, b_1\, b_y^{-1} \cdots b_{y^{k_y-1}}^{-1}\,,$$

$$(b_h*y)*s = (b_{y^{k_y-1}} \cdots b_y\, b_1\, b_y^{-1} \cdots b_{y^{k_y-1}}^{-1})*s = b_{y^{k_y-1}} \cdots b_y\, b_1\, b_y^{-1} \cdots b_{y^{k_y-1}}^{-1}\,.$$ □

If w is a word over $\{s, t\}$, where $s, t \in V(L)$, and if $h \in T_0$, we define $b_h * w$ by induction on the length of w by setting $b_h * 1 = b_h$ and $b_h * (ws) = (b_h * w) * s$.

Lemma 2.33. *Let $s, t \in V(L)$ such that $m_{s,t} = 2k > 2$. Then $b_h * ((st)^k) = b_h * ((ts)^k)$ for all $h \in T_0$.*

Proof. Note that, since s and t are linked and the edge between them is labelled with $2k > 2$, Lemma 2.24 implies that $s, t \in V(L_1)$. Take $h \in T_0$. Then, in a similar way as in the proof of Lemma 2.32, we have $hs \in T_0$ if and only if $u(hstst \cdots t)s \in T_0$ and this is also equivalent to $u(htsts \cdots t)s \in T_0$. The same thing happens for t. So, we may distinguish essentially the same cases as in the first part of the proof of Lemma 2.32 and get the following. If $hs, ht \in T_0$, then

$$(b_h) * (st)^k = b_{h(st)^k} = b_{h(ts)^k} = (b_h) * (ts)^k .$$

If $hs \in T_0$ and $ht \notin T_0$, then

$$(b_h) * (st)^k = b_{t^{-k}h(st)^k} = b_{t^{-k}h(ts)^k} = (b_h) * (ts)^k .$$

Similarly, if $hs \notin T_0$ and $ht \in T_0$, then $(b_h) * (st)^k = (b_h) * (ts)^k$. If $hs, ht \notin T_0$, then

$$(b_h) * (st)^k = b_{(st)^{-k}h(st)^k} = b_{(ts)^{-k}h(ts)^k} = (b_h) * (ts)^k .$$ □

All the previous discussion implies the following.

Lemma 2.34. *The mappings $*s$, $s \in V(L)$, yield a well-defined right-action $F_0 \times \mathcal{A}_L \to F_0$, $(u, g) \mapsto u * g$.*

Moreover, this action behaves as one might expect. To show this, we will need the following technical lemma.

Lemma 2.35. *Let $g \in \mathcal{A}_{L_1} T_0$, and let $n(g) = w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_n}^{\varepsilon_n} w_n$ be its normal form. Then any prefix of $n(g)$ also represents an element in $\mathcal{A}_{L_1} T_0$.*

Proof. The only case where it is not obvious is when the prefix is of the form $w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_j}^{\varepsilon_j} u_j$ with u_j a prefix of w_j. Let h be the element of $\mathcal{A}_L$ represented by $w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_j}^{\varepsilon_j} u_j$, and let h' be the element represented by $w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_j}^{\varepsilon_j}$. It is clear that $h' \in \mathcal{A}_{L_1} T_0$. Moreover, $h = h' \bar{u}_j$, hence, as pointed out after the proof of Lemma 2.30, we have $h \in \mathcal{A}_{L_1} T_0$. □

Lemma 2.36. *For every $g \in \mathcal{A}_{L_1} T_0$ we have $b_1 * g = b_{u(g)}$.*

Proof. Set again $n(g) = w_0 x_{\alpha_1}^{\varepsilon_1} w_1 \cdots x_{\alpha_n}^{\varepsilon_n} w_n$. Let vs^ε be a prefix of $n(g)$ with s a vertex and $\varepsilon \in \{\pm 1\}$. Note that Lemma 2.35 implies that vs^ε and v represent elements in $\mathcal{A}_{L_1} T_0$. We are going to prove that, if $b_1 * \bar{v} = b_{u(\bar{v})}$, then also $b_1 * \bar{v}s^\varepsilon = b_{u(\bar{v}s^\varepsilon)}$. Note that this will imply the result. As $\bar{v}s^\varepsilon$ lies in $\mathcal{A}_{L_1} T_0$, the element $u(\bar{v}s^\varepsilon)$ is well-defined. Observe that $u(u(\bar{v})s^\varepsilon)$ is also well-defined. We have $b_1 * (\bar{v}s^\varepsilon) = b_{u(\bar{v})} * s^\varepsilon = b_{u(u(\bar{v})s^\varepsilon)}$. So, we only need to show that $u(u(\bar{v})s^\varepsilon) = u(\bar{v}s^\varepsilon)$. Set $\bar{v} = qh$ with $h \in T_0$ and $q \in \mathcal{A}_{L_1}$. Then $u(\bar{v}) = h$, thus, by Lemma 2.30, $u(u(\bar{v})s^\varepsilon) = u(hs^\varepsilon) = u(qhs^\varepsilon) = u(\bar{v}s^\varepsilon)$. $\square$

Our next objective is to extend the action of $\mathcal{A}_L$ on F_0 to an action of $\mathcal{A}_1$ on F. Recall that $T = T_0 \mathrm{Ker}(\pi_L)$ and that any $h \in T$ can be written in a unique way as $h = h_0 u$ with $h_0 \in T_0$ and $u \in \mathrm{Ker}(\pi_L)$. Taking this into account we set $b_h = b_{h_0 u} = b_{h_0} \cdot u$. We extend this notation to any element $\omega = \prod b_{h_i}^{\varepsilon_i} \in F_0$ by setting $\omega \cdot u = \prod b_{h_i u}^{\varepsilon_i}$.

Now, let $g \in \mathcal{A}_1$ and $h \in T$. We write $h = h_0 u$ with $h_0 \in T_0$ and $u \in \mathrm{Ker}(\pi_L)$. So, with the previous notation, we have $b_h = b_{h_0} \cdot u$. Then we set

$$b_h * g = (b_{h_0} * \pi_L(g)) \cdot (\pi_L(g)^{-1} u g)\,.$$

We can also write this action as follows. Let $\omega = \prod b_{h_i}^{\varepsilon_i} \in F_0$ and let $u \in \mathrm{Ker}(\pi_L)$. Then

$$(\omega \cdot u) * g = (\omega * \pi_L(g)) \cdot (\pi_L(g)^{-1} u g)\,. \tag{2.2}$$

Lemma 2.37. *The above defined map $F \times \mathcal{A}_1 \to F$, $(\omega, g) \mapsto \omega * g$, is a well-defined right-action of $\mathcal{A}_1$ on F.*

Proof. The lemma is essentially a consequence of the fact that the action of $\mathcal{A}_L$ on F_0 is well-defined. Let $g_1, g_2 \in \mathcal{A}_1$ and let $b_h = b_{h_0} \cdot u \in B$, where $h_0 \in T_0$ and $u \in \mathrm{Ker}(\pi_L)$. Then, using Equation 2.2,

$$\begin{aligned}(b_h * g_1) * g_2 &= ((b_{h_0} \cdot u) * g_1) * g_2 = \big((b_{h_0} * \pi_L(g_1)) \cdot (\pi_L(g_1)^{-1} u g_1)\big) * g_2 \\ &= ((b_{h_0} * \pi_L(g_1)) * \pi_L(g_2)) \cdot (\pi_L(g_2)^{-1} (\pi_L(g_1)^{-1} u g_1) g_2) \\ &= (b_{h_0} * \pi_L(g_1 g_2)) \cdot (\pi_L(g_1 g_2)^{-1} u g_1 g_2) = b_h * g_1 g_2\,. \end{aligned}$$ $\square$

Recall the homomorphism $\varphi : F \to \mathrm{Ker}(\pi_1)$ that sends b_h to z^h for all $h \in T$. We consider the semi-direct product $G = \mathcal{A}_1 \ltimes F$ associated with the above action, and we turn to define an extension of φ to G.

Lemma 2.38. *The map $G \to \mathcal{A}$, $(g, \omega) \mapsto g\,\varphi(\omega)$, is a well-defined homomorphism.*

Proof. We have to check that, for all g_1, $g_2 \in \mathcal{A}_1$ and all ω_1, $\omega_2 \in F$, we have

$$\begin{gathered}\varphi(g_1,\omega_1)\varphi(g_2,\omega_2) = g_1 g_2 \varphi(\omega_1)^{g_2}\varphi(\omega_2)\\ = \varphi(g_1 g_2, (\omega_1 * g_2)\omega_2) = g_1 g_2 \varphi((\omega_1 * g_2)\omega_2)\,.\end{gathered}$$

Since the restriction of φ to F is a group homomorphism, this is equivalent to show that $\varphi(\omega_1)^{g_2} = \varphi(\omega_1 * g_2)$. It is enough to prove this for the group generators. So, we can assume that $\omega_1 = b_h$ for some $h \in T$, and that $g_2 = s$ for some vertex $s \in V(\Gamma)$, $s \neq z$. We have $\varphi(b_h)^s = s^{-1}z^h s = s^{-1}h^{-1}zhs$, and we need to check that this is equal to $\varphi(b_h * s)$. To see it we set $h = h_0 u$, with $h_0 \in T_0$ and $u \in \mathrm{Ker}(\pi_L)$, so that $b_h = b_{h_0} \cdot u$.

Observe first that, if $s \notin V(L)$, then $\pi_L(s) = 1$, thus $b_h * s = b_{h_0} \cdot (us) = b_{h_0 us} = b_{hs}$, and therefore

$$\varphi(b_h * s) = \varphi(b_{hs}) = z^{hs} = \varphi(b_h)^s\,.$$

So, from now on, we will assume that $s \in V(L)$. Then $\pi_L(s) = s$, thus $b_h^s = (b_{h_0} * s) \cdot (s^{-1}us)$, and therefore

$$\varphi(b_h * s) = \varphi(b_{h_0} * s)^{s^{-1}us} = s^{-1}\varphi(b_{h_0} * s)^{s^{-1}u}s\,.$$

So, we only have to prove that $\varphi(b_{h_0} * s) = z^{h_0 s}$. We distinguish three different cases. If $h_0 s \in T_0$, then $b_{h_0} * s = b_{h_0 s}$, thus

$$\varphi(b_{h_0} * s) = \varphi(b_{h_0 s}) = z^{h_0 s}\,.$$

If $h_0 s \notin T_0$, $s \in V(L_1)$ and $s \in S_i$ for every $x_i \in \mathrm{supp}(h_0)$, then $b_{h_0} * s = b_{s^{-1}h_0 s}$, thus

$$\varphi(b_{h_0} * s) = \varphi(b_{s^{-1}h_0 s}) = z^{s^{-1}h_0 s} = z^{h_0 s}\,.$$

Finally, if $s = x \in V(L \setminus L_1)$ and $h_0 = x^{k_x - 1}$, then

$$b_{h_0} * x = b_{x^{k_x-1}} \cdots b_x\, b_1\, b_x^{-1} \cdots b_{x^{k_x-1}}^{-1}\,,$$

thus

$$\begin{gathered}\varphi(b_{h_0} * x) = \varphi(b_{x^{k_x-1}} \cdots b_x\, b_1\, b_x^{-1} \cdots b_{x^{k_x-1}}^{-1})\\ = z^{x^{k_x-1}} \cdots z^x\, z\, (z^x)^{-1} \cdots (z^{x^{k_x-1}})^{-1} = z^{x^{k_x}} = z^{h_0 x}\,.\end{gathered}$$

□

Now, we want to define the inverse map of φ. We do it by giving the images of the Artin generators of $\mathcal{A}$, that is, the vertices of Γ.

Lemma 2.39. *There is a well-defined homomorphism $\psi : \mathcal{A} \to G$ that sends s to s for all $s \in V(\Gamma) \setminus \{z\}$, and sends z to b_1.*

Proof. We have to check that the Artin relations are preserved by ψ. Note that it suffices to check it for the Artin relations that involve z and some $s \in V(\Gamma) \setminus \{z\}$. If $s \notin V(L)$, then there is nothing to check because, in that case, s and z are not linked in Γ, hence there is no relation between them. If $s \in V(L)$, then we can rewrite the Artin relation as

$$z^{s^{k_s}} = z^{s^{k_s-1}} \cdots z^s \, z \, (z^s)^{-1} \cdots (z^{s^{k_s-1}})^{-1} .$$

We include here the case $s \in V(L_1)$, where we have $k_s = 1$ and the above formula is $z^s = z$. Applying ψ to the left hand side of this equation we get

$$\psi(z^{s^{k_s}}) = s^{-k_s} \, b_1 \, s^{k_s} = s^{-1} \, b_{s^{k_s-1}} \, s = b_{s^{k_s-1}} \cdots b_s \; b_1 \, b_s^{-1} \cdots b_{s^{k_s-1}}^{-1} ,$$

which is exactly what we get applying ψ to the right hand side. □

Lemma 2.38 and Lemma 2.39 show part of the following result.

Proposition 2.40. *The maps $\varphi : G \to \mathcal{A}$ and $\psi : \mathcal{A} \to G$ are well-defined group isomorphisms.*

Proof. We have already seen that both maps are group homomorphisms. We claim that they are inverses of each other. This will prove the result. Let $s \in V(\Gamma)$, $s \neq z$. We have $(\varphi \circ \psi)(s) = \varphi(s) = s$ and $(\psi \circ \varphi)(s) = \psi(s) = s$. Also $(\varphi \circ \psi)(z) = \varphi(b_1) = z$ and $(\psi \circ \varphi)(b_1) = \psi(z) = b_1$. Moreover, Lemma 2.36 implies that b_1 and the Artin generators of $\mathcal{A}_1$ generate the whole group G, so $\psi \circ \varphi$ is the identity map of G. Similarly, $\varphi \circ \psi$ is the identity map of $\mathcal{A}$. □

Now, we obtain immediately our main result.

Theorem 2.41. *Every even Artin group of FC type is poly-free.*

Proof. By Proposition 2.40, $\mathcal{A}_\Gamma \simeq G = F \rtimes \mathcal{A}_1$. By induction we may assume that $\mathcal{A}_1$ is poly-free, thus $\mathcal{A}$ is also poly-free. □

Chapter 3

Poly-freeness: large even Artin groups

The main objective of this chapter is to prove that large even Artin groups are poly-free.

Recall that an Artin group is said to be **large** if $m_e \geq 3$ for every $e \in E$. This is a family of Artin groups which has been frequently studied. One of the most interesting results proved for them is a solution to the word problem. Holt and Rees described a set of normal forms for the elements of large Artin groups [56], these normal forms will play a key role in this chapter.

In section 3.1 we review some results by Holt and Rees about normal forms in large Artin groups. Section 3.2 is rather technical: we use the normal forms of section 3.1 together with other results from [56] to gain information about geodesic words in large even Artin groups.

In section 3.3 we will see how to split any large even Artin group as semidirect product of a parabolic subgroup and certain normal subgroup. Later on we will show that this normal subgroup is free and this semidirect product decomposition will be crutial to argue by induction and deduce our main result. Finally, in section 3.4 we will prove the main result and in section 3.5 we use a similar strategy to prove that any Artin group based on an even triangle graph is poly-free.

3.1 Normal forms in Artin groups

In this section we will recall some definitions and results about normal forms which can be found in [56], [15], [57] applying them to the particular case of even Artin groups.

Definition 3.1. *We call* ***alphabet*** *to a finite set L. An element $a \in L$ is* *called a **letter**. A **word** over L is a finite sequence of letters.*

*Formally, a word can be defined as a map $w : \{1, ..., n\} \to L$ where $w(i)$ is the i-th letter of the word. The **length** of a word w is the integer n and it is denoted by $|w|$.*

When $n = 0$, we say that w is the **empty word** over L, and it is denoted by ϵ. We denote by L^* the set of all words over the alphabet L. Given a word $w = abc$, with possibly empty $a, b, c \in L^*$, the word a is said to be a **prefix** of w, c is a **suffix** of w and b is a **subword** of w. Given a word w we denote by $f[w]$ and $l[w]$ the first and last letter of w respectively. So, if $|w| = n$, $f[w] = w(1)$ and $l[w] = w(n)$.

From now on we fix $L = S \cup S^{-1}$, where S is a generating set of the group G. A letter $a \in L$ is **positive** if $a \in S$ and is **negative** otherwise. The **name** of a letter is its positive form.

If two words w, v represent the same element in a group G, we will write $w =_G v$.

Definition 3.2. *We say that a word $w \in L^*$ is **positive** if all its letters are positive, **negative** if all its letters are negative and **unsigned** otherwise.*

Definition 3.3. *A word $w \in L^*$ is **freely reduced** if it does not admit any subword of the form aa^{-1} or $a^{-1}a$ for any letter a. We say that a not freely reduced word **admits a free reduction**.*

*A word $w \in L^*$ is **geodesic** if for any other word v such that $w =_G v$, we have that $|w| \leq |v|$.*

Definition 3.4. *Let L be an alphabet. Given $<_{lex}$ an arbitrary lexicographic ordering on L (which induces a lexicographic order, that we denote $<_{lex}$, in L^*), the **shortlex** ordering $<_{slex}$ on L^* is defined by*

$$w <_{slex} v \text{ if and only if } |w| < |v| \text{ or } |w| = |v| \text{ and } w <_{lex} v.$$

Definition 3.5. *A word w is said to be a **shortlex minimal representative** if for every word $v \neq w$ such that $w =_G v$, $w <_{slex} v$.*

Remark 3.6. *Every shortlex minimal representative is geodesic.*

3.1.1 Dihedral Artin groups

Definition 3.7. *The **dihedral Artin group** $A_2(m)$, $m \in \mathbb{Z}^+ \cup \{+\infty\}$ is the Artin group based on the graph consisting of two vertices joined by an edge labelled with m or two disconnected vertices if $m = \infty$. If $m < \infty$ this is the group with presentation $\langle a, b \mid {}_m(a, b) = {}_m(b, a)\rangle$.*

We want to study how to obtain a shortlex representative of a given word in $A_2(m)$ for m arbitrary. If $m = \infty$, $A_2(\infty)$ is the free group on two variables and it is easy to see that every freely reduced word $w \in A_2(\infty)$ is shortlex minimal. So we only need to consider the cases where $m < \infty$.

We define ${}_m(a, b)$ as the alternating product of a and b of length m beginning with a. Similarly, we define $(a, b)_m$ as the alternating procuct of a and b of length m ending with b. Notice that if m is even, we have ${}_m(a, b) = (a, b)_m$ and we can use any of the notations indistinctly.

Definition 3.8. *Let w be a freely reduced word in $A_2(m)$ over the alphabet $L = \{a, a^{-1}, b, b^{-1}\}$. Consider the integers:*

$$r_1 = \max\{r \mid {}_r(a, b) \textit{ or } {}_r(b, a) \textit{ is a subword of } w\},$$

$$r_2 = \max\{r \mid {}_r(a^{-1}, b^{-1}) \textit{ or } {}_r(b^{-1}, a^{-1}) \textit{ is a subword of } w\},$$

$p(w) = \min\{r_1, m\}$ *and* $n(w) = \min\{r_2, m\}$.

Geodesic words w in $A_2(m)$ are characterized by the values $p(w)$ and $n(w)$.

Proposition 3.9. *[65] Let $g \in A_2(m)$ and let $w \in L^*$ be a freely reduced word representing g.*

1. *If $p(w)+n(w) < m$, then w is the unique geodesic representative for g.*
2. *If $p(w) + n(w) = m$, then w is one of the geodesic representatives for g.*
3. *If $p(w) + n(w) > m$, then w is not geodesic. Furthermore, w has a subword w' such that $p(w') + n(w') = m$.*

Let w be a shortlex minimal word representing g. Then since w is geodesic, $p(w) + n(w) \leq m$. Moreover, if $p(w) + n(w) < m$, then w is the unique geodesic word for g and such word must be the shortlex minimal representative of g.

Definition 3.10. *Let w be a freely reduced word in $A_2(m)$. Let $\{x, y\} = \{z, t\} = \{a, b\}$ and put $p = p(w)$ and $n = n(w)$. The word w is called a* ***critical word*** *if $p + n = m$ and it has one of the following forms. In these forms, ξ_+ represents some positive word in L^*, ξ_- some negative word in L^* and η some word in L^*.*

If w is a positive word, then

$$w = \xi_+(x,y)_m \text{ or } w = {}_m(x,y)\xi_+,$$

where w has exactly one alternating positive subword of length m.

If w is a negative word, then

$$w = \xi_-(x^{-1},y^{-1})_m \text{ or } w = {}_m(x^{-1},y^{-1})\xi_-,$$

where w has exactly one alternating negative subword of length m.

If w is an unsigned word, then

$$w = {}_p(x,y)\eta(z^{-1},t^{-1})_n \text{ or } w = {}_n(x^{-1},y^{-1})\eta(z,t)_p.$$

We denote by T the set of all critical words.

Let us consider the Garside element $\Delta := (a,b)_m$. Notice that Δ is central if m is even but $a^\Delta = b$ if m is odd. We define the automorphism ν of L^* such that $\nu(w) = w^\Delta$. Notice that for the case of even Artin groups $\nu = Id_{L^*}$.

Definition 3.11. *Define a map τ on the critical words as follows:*

$$\begin{aligned}
(x,y)_m &\mapsto (y,x)_m, \\
\xi_+(x,y)_m &\mapsto {}_m(t,z)\nu(\xi_+), \text{ where } z = f[\xi_+], \\
{}_m(x,y)\xi_+ &\mapsto \nu(\xi_+)(z,t)_m, \text{ where } z = l[\xi_+], \\
(x^{-1},y^{-1})_m &\mapsto (y^{-1},x^{-1})_m, \\
\xi_-(x^{-1},y^{-1})_m &\mapsto {}_m(t^{-1},z^{-1})\nu(\xi_-), \text{ where } z = f[\xi_-]^{-1}, \\
{}_m(x^{-1},y^{-1})\xi_- &\mapsto \nu(\xi_-)(z^{-1},t^{-1})_m, \text{ where } z = l[\xi_-]^{-1}, \\
{}_p(x,y)\eta(z^{-1},t^{-1})_n &\mapsto {}_n(y^{-1},x^{-1})\nu(\eta)(t,z)_p, \text{ where } p+n=m, \ \ n,p \neq 0, \\
{}_n(x^{-1},y^{-1})\eta(z,t)_p &\mapsto {}_p(y,x)\nu(\eta)(t^{-1},z^{-1})_n, \text{ where } p+n=m, \ \ n,p \neq 0.
\end{aligned}$$

Here $\{x,y\} = \{z,t\}$, ξ_+ is a non-empty positive word, ξ_- a non-empty negative word and η can be empty.

These are called τ-moves.

Proposition 3.12. *[56][15] For any critical word w:*

1. *$\tau(w)$ is also critical, $\tau(w) =_G w$ and $\tau(\tau(w)) = w$.*

2. *$p(\tau(w)) = p(w)$ and $n(\tau(w)) = n(w)$.*

3. *$f[w]$ and $f[\tau(w)]$ have different names, the same is true for $l[w]$ and $l[\tau(w)]$.*

4. $f[w]$ and $f[\tau(w)]$ have the same sign if w is positive or negative, but different signs if w is unsigned; the same is true for $l[w]$ and $l[\tau(w)]$.

Definition 3.13. *Let w be a word that admits a factorization $w = w_1w_2w_3$ where w_2 is a critical word. If $w_1\tau(w_2)w_3$ admits a free reduction or if $w_1\tau(w_2)w_3 <_{lex} w_1w_2w_3$, we say that w admits a* ***critical reduction****.*

We say that $w_1\tau(w_2)w_3$ is a ***right length reduction*** *(or that w admits a right lengt reduction) if $l[\tau(w_2)] = f[w_3]^{-1}$.*

We say that $w_1\tau(w_2)w_3$ is a ***left lex reduction*** *(or that w admits a left lex reduction) if $f[\tau(w_2)] <_{lex} f[w_2]$ and $w_1\tau(w_2)w_3$ does not admit a free reduction.*

Theorem 3.14. *[56][15] Let W be the set of all words that do not admit any right length reduction or left lex reduction. Then W is the set of shortlex minimal representatives of elements of $A_2(m)$.*

Remark 3.15. *Since W is the set of shortlex minimal representatives, it is clear that if $w \in W$, then for every prefix u of w, also $u \in W$.*

3.1.2 Large Artin groups

To find normal forms for arbitrary large Artin groups we will use the results we had for dihedral groups. Let A_Γ be a large even Artin group with n generators $a_1, ..., a_n$, let $m_{i,j}$ be the label of the edge between a_i and a_j, $L = \{a_1^{\pm 1}, ..., a_n^{\pm 1}\}$. To find normal forms for arbitrary large Artin groups we will use the results we had for dihedral groups. We fix a lexicographic order on L. Given an arbitrary word $w \in L^*$, we can consider all the subwords of w involving only 2 generators, say a_i and a_j, as words in the dihedral group $A_2(m_{i,j})$. For such a subword u, we define $p(u)$, $n(u)$ and τ as before. We will denote by $T_{i,j}$ the set of critical words in the subgroup $A_2(m_{i,j})$.

Definition 3.16. *Let w be a freely reduced word over A_Γ. Suppose that w has a factorization $\alpha w_1 u w_2 \beta$, where $u \in T_{i,j}$.*

If w_1 is a 2-generator subword on a_{i_1}, a_{j_1} such that $|\{i,j\} \cap \{i_1, j_1\}| = 1$, the name of $l[w_1]$ is not in $\{a_i, a_j\}$ and $w_1 f[\tau(u)] \in T_{i_1,j_1}$, then we say that we have a ***critical left overlap*** *in $w_1 f[\tau(u)]$.*

Similarly, if w_2 is a 2-generator subword on a_{i_2}, a_{j_2} such that $|\{i,j\} \cap \{i_2, j_2\}| = 1$, the name of $f[w_2]$ is not in $\{a_i, a_j\}$ and $l[\tau(u)]w_2 \in T_{i_2,j_2}$, then we say that we have a ***critical right overlap*** *in $l[\tau(u)]w_2$.*

Definition 3.17. *Let $\alpha_1 u_1 \beta_1$ be a freely reduced word with u_1 a critical subword. Consider a sequence:*

$$\alpha_1 u_1 \beta_1, \quad \alpha_1 \tau(u_1)\beta_1 = \alpha_2 u_2 \beta_2 \quad \ldots \quad \alpha_k \tau(u_k)\beta_k.$$

where all the u_i's are critical subwords, $u_i = w_i f[\tau(u_{i-1})]^*$ for w_i a suffix of α_{i-1} (in this case we have a critical left overlap) or $u_i = l[\tau(u_{i-1})]^* w_i$ for w_i a prefix of β_{i-1} (in this case we have a critical right overlap), $i = 2, \ldots k$, where $*$ represents some positive power ≥ 1.

If at each step we have a left (resp. right) critical overlap, we say that this is a ***leftward*** *(resp.* ***rightward****)* ***critical sequence****.*

If a critical sequence is such that $\alpha_k \tau(u_k)\beta_k$ is not freely reduced, the sequence is called a ***length reducing sequence****. In this situation, we say that β_k is the* ***tail*** *of the sequence. If it is reduced and $\alpha_k \tau(u_k)\beta_k <_{lex} \alpha_1 u_1 \beta 1$, then it is a* ***lex reducing sequence****.*

Now, we are going to see some examples to illustrate how critical sequences work. For both examples we are going to consider the large even Artin group based on a triangle with labels $(4, 4, 4)$,

$$A_\Gamma = \langle a, b, c \mid abab = baba, acac = caca, bcbc = cbcb \rangle.$$

Example 3.18. Let us consider the word $w = cbcabacbcb$ and the lexicographic order $a < a^{-1} < b < b^{-1} < c < c^{-1}$.

$$w = cbcaba\overbrace{cbcb}^{u_1} \xrightarrow[\tau(u_1)]{} w_1 = cbc\rlap{$\overbrace{}^{u_2}$}aba\underbrace{bcbc}_{\tau(u_1)} \xrightarrow[\tau(u_2)]{} w_2 = \rlap{$\overbrace{}^{u_3}$}cbc\underbrace{baba}_{\tau(u_2)}cbc$$

$$\xrightarrow[\tau(u_3)]{} w_3 = \underbrace{bcbc}_{\tau(u_3)}abacbc.$$

And since $w_3 <_{lex} w$, this is a leftward lex reducing sequence.

Example 3.19. *Let us consider the word $w = a^{-1}b^3abc^{-1}a^2cb^{-1}aba$ and the lexicographic order $a < a^{-1} < b < b^{-1} < c < c^{-1}$.*

$$w = \overbrace{a^{-1}b^3ab}^{u_1}c^{-1}a^2cb^{-1}aba \xrightarrow[\tau(u_1)]{} w_1 = \underbrace{bab^3\rlap{$\overbrace{\phantom{a^{-1}c^{-1}a^2c}}^{u_2}$}a^{-1}}_{\tau(u_1)}c^{-1}a^2cb^{-1}aba$$

$$\xrightarrow[\tau(u_2)]{} w_2 = bab^3\underbrace{ca^2c^{-1}\rlap{$\overbrace{\phantom{a^{-1}b^{-1}ab}}^{u_3}$}a^{-1}}_{\tau(u_2)}b^{-1}aba \mathrel{=}\!\!\xrightarrow[\tau(u_3)]{} w_3 = bab^3ca^2c^{-1}\underbrace{bab^{-1}\rlap{$\overbrace{\phantom{a^{-1}a}}^{Free\ red}$}a^{-1}}_{\tau(u_1)}a$$

$$\xrightarrow[FreeRed]{} w' = bab^3ca^2c^{-1}bab^{-1}.$$

Since $|w'| < |w|$, this is a rightward length reducing sequence.

Theorem 3.20. *[56] Let G be a large Artin group. Let W be the set of all freely reduced words w that admit no rightward length reducing sequence or leftward lex reducing sequence of any length $k \geq 1$. Then W is the set of shortlex representatives.*

Proposition 3.21. *[56] Suppose that $w \in W$ and $a \in L$ is such that wa is freely reduced but $wa \notin W$. Then applying a single rightward length reducing or leftward lex reducing sequence followed by a free reduction in the rightward case (where the letter a will be the tail of the sequence) to wa we get an element of W.*

Remark 3.22. *Given $w \in L^*$ denote by $sl(w)$ the shortlex minimal representative of w. We use the same notation for $g \in G$ and put $sl(g)$ for its shortlex minimal representative.*

Remark 3.23. *Again, since W is the set of shortlex minimal representatives, it is clear that if $w \in W$, then $u \in W$, for every subword u of w.*

3.2 Technical results about geodesic words in large Artin groups

In this section we will prove several results concerning geodesic words in large Artin groups that we will later use to prove that groups in this family are polyfree.

Definition 3.24. *Given a word w of length n, we will denote by $pr(w, k)$, $k = 1, ..., n$ the prefix of length k of w and by $w(k)$ the k-th letter of w.*

Our first technical lemma is equivalent to Proposition 4.5 (1) in [56]. However, we will incude a proof to introduce some of the techniques that we will use in the rest of the chapter.

Lemma 3.25. *Let G be a large Artin group, $a \in L$. There are no two geodesic words $aw_1, a^{-1}w_2$ such that $aw_1 =_G a^{-1}w_2$.*

Proof. Let us suppose that both words are geodesic and represent the same element of the large Artin group G. We may assume that w_1 is a shortlex representative. Without loss of generality we will suppose that $a < a^{-1}$ are the first letters in the lexicographic order. Let us call $\hat{w} = a^{-1}w_2$, then we can suppose that $sl(\hat{w}) = aw_1$ (it must have this form since a is the first letter of the lexicographic order and the word is geodesic). Let $n = |\hat{w}|$.

It is clear that since $\hat{w}$ is geodesic, $pr(\hat{w}, k)$ is also geodesic for every $k = 1, ..., n$. Obviously, we have $pr(\hat{w}, k) = a^{-1}\alpha_k$, $k = 1, ..., n$, where $\alpha_1 = \epsilon$.

Since in the lexicographic order before a^{-1} there is only a and $sl(pr(\hat{w}, n)) = sl(\hat{w}) = aw_1$, there exists $m \in \{2, ..., n\}$ such that $f[sl(pr(\hat{w}, m))] = a$ and $f[sl(pr(\hat{w}, m-1))] = a^{-1}$. But since $sl(pr(\hat{w}, m-1)) \in W$, there is a leftward lex reducing sequence that transforms $sl(pr(\hat{w}, m-1))\hat{w}(m)$ into $sl(pr(\hat{w}, m))$. Let us call l to the length of that sequence, and let u_l be the last critical word of the sequence.

Since that secuence must change the first letter of the word, we have that $\tau(u_l)$ must begin by a and u_l must begin by a^{-1}. But, by Proposition 3.12, $f[\tau(u_l)]$ and $f[u_l]$ must have different names, getting a contradiction. □

Lemma 3.26. *Consider a large Artin group G and let $b \in L$. Let $t > 0$, w a word. Then:*

- *A word $b^t w$ is geodesic if and only if bw is geodesic.*
- *A word $b^{-t} w$ is geodesic if and only if $b^{-1} w$ is geodesic.*

Proof. We are going to prove only the case when the word is $b^t w$, the case of $b^{-t} w$ is completely analogous. The fact that if $b^t w$ is geodesic then bw is geodesic is obvious by Remark 3.23. Let us suppose then that bw is geodesic but $b^t w$ is not geodesic. Thus, there must exist $k > 1$ such that $b^k w$ is not geodesic but $b^{k-1} w$ is geodesic. Therefore, $w^{-1} b^{-(k-1)}$ is geodesic, but $w^{-1} b^{-k}$ is not geodesic. Hence, $sl(w^{-1} b^{-(k-1)}) b^{-1}$ is not geodesic and by Theorem 3.21 there exists a rightward length reducing sequence in which b^{-1} is going to be the tail that will be eliminated. Let us call l to the length of that sequence, and let u_l be the last critical word of the sequence. Therefore, $sl(w^{-1} b^{-(k-1)}) =_G \hat{w}\tau(u_l)$ is geodesic and $\tau(u_l)$ ends by b. But then, $sl(w^{-1} b^{-(k-1)})$ has a geodesic representative ending by b, so $b^{k-1} w$ has a geodesic representative beginning with b^{-1}, which is impossible by Lemma 3.25. □

Lemma 3.27. *Consider a large Artin group G and let $b, h \in L$, $w' \in L^*$. If $w = b^t w'$ (resp. $w = b^{-t} w'$) is geodesic but wh doesn't have a geodesic representative beginning with b^t (resp. b^{-t}), then $w'h$ has a geodesic representative beginning with b^{-1} (resp. b).*

Proof. We are going to prove the result only in the case $w = b^t w'$ with $t > 0$, the other case is completely analogous.

The hypothesis implies that $b^t w' h$ is not geodesic, thus by Lemma 3.26, $bw'h$ is not geodesic. Let us consider the word $w_1 = bw'$ which is geodesic, then $w_1 h = bw'h$ doesn't have a geodesic representative beginning by b (otherwise, wh would have a geodesic representative begining by b^t contradicting the hypothesis). We have

$$(w_1 h)^{-1} = h^{-1} w_1^{-1} = h^{-1} w'^{-1} b^{-1} =_G sl(h^{-1} w'^{-1}) b^{-1}.$$

But since w_1h doesn't have a geodesic representative beginning by b, then $sl(h^{-1}w'^{-1})b^{-1}$ is not geodesic and by Theorem 3.21 there is a rightward length reducing sequence for $sl(h^{-1}w'^{-1})b^{-1}$ such that b^{-1} is the tail and $sl(h^{-1}w'^{-1}) =_G \hat{w}\tau(u_l)$ geodesic with $l[\tau(u_l)] = b$.

Therefore, there exists a geodesic representative of $h^{-1}w'^{-1}$ ending by b, and then $w'h$ has a geodesic representative beginning by b^{-1}. □

In the next technical results we want to understand when we can have elements admiting geodesic representatives of the form $a^s w_1 =_G b^t w_2$.

Example 3.28. Consider the group $A_\Gamma = \langle a, b \mid (ab)^2 = (ba)^2 \rangle$. Note that the relation $abab = baba$ implies $[a, bab] = [b, aba] = 1$. From this we get:

$$b^2(aba)^2 = abababab = a(baba)bab = a^2babbab = a^2(bab)^2.$$

The motivation of the following technical results is precisely to understand better this kind of situation. From now on, let A_Γ be a large Artin group and let W be the set of words that admit neither rightward length reducing sequence nor leftward lex reducing sequence. Recall that then W is the set of shorlex representatives by Theorem 3.20 (in particular, words in W are geodesic).

Notice that the set W of shortlex representatives depends on the chosen order in the generating set. In the following technical results we only want to prove certain restrictions on the form of the geodesic representatives and to do so, we use appropiate lexicographic orders in each case. This has the effect that we will have to check consistence of the chosen order when applying these results.

Lemma 3.29. *Let $G = A_\Gamma$ be a large Artin group. Suppose that a (resp. a^{-1}) is the first letter of the lexicographic order and that there exist u geodesic word and a letter l such that*

- *ul is geodesic,*
- *$sl(ul)$ begins with a^s, $s > 1$, (resp. $s < -1$),*
- *$sl(u)$ doesn't begin with a^s.*

Then $sl(u)$ must begin by a^i with $i = s - 1$ (resp. $i = s + 1$).

Proof. Without loss of generality we may suppose that u is a shorlex representative. We are going to argue by induction on $|ul| = k$. We will only prove the result for $s > 1$, the case $s < -1$ is analogous.

If $|ul| = 2$, then $ul = a^2$ and therefore the only option is $u = l = a$, so $s = 2$ and $i = s - 1 = 1$.

Let us suppose that the result is true for length smaller than k. Recall that $sl(ul) = a^s\hat{u}$ and that $sl(u) = a^i u'$ for some $i \in \mathbb{Z}$, some $\hat{u}$ and some u'. We may also assume that $\hat{u}$ and u' don't begin by a.

Note that as $sl(u)$ is shorltex and since a is the first letter of the lexicographic order (thus $a < a^{-1}$), u cannot have any geodesic representative begining by a^s. We want to prove that $i = s - 1$ if $s > 1$. Note that u' is also shortlex and does not begin with $a^{\pm 1}$ so it doesn't have any geodesic representative beginnig with a. Moreover, as u is geodesic, u' cannot have any geodesic representative begining with a^{-1}. Finally notice that by Lemma 3.25 i and s must have the same sign.

We know by Proposition 3.21 that there exists a leftward length reducing sequence that transforms $sl(u)l = a^i u'l$ into $sl(ul) = a^s\hat{u}$.

$$... \to \alpha_r u_r \beta_r \to \alpha_r \tau(u_r)\beta_r = a^s\hat{u}$$

Note that α_r is a prefix of both $sl(u)l = a^i u'l$ and $sl(ul) = a^s\hat{u}$, so the number of $a's$ at the beginning of α_r cannot be bigger than i. Since $\alpha_r\tau(u_r)\beta_r$ begins by a^s, $\alpha_r = a^p$, $p \leq i$, and as $f(u_r)$ and $f(\tau(u_r))$ must have different names, we conclude that $\alpha_r = a^i$, which implies $a^s\hat{u} = a^i a^{s-i}\hat{u}$ and $sl(u'l) = a^{s-i}\hat{u}$. But now, if $s-i \neq 1$, using the inductive hypothesis, we would have that $sl(u')$ begins with a which is impossible. So, $s - i = 1$ and therefore $i = s - 1$. $\square$

Corollary 3.30. *Suppose that a is the first letter of the lexicographic order and let $w = b^{\pm 1}w'$ be geodesic such that $sl(w) = a^s\tilde{w}$, $s > 0$, with $a \neq b$ a letter. Then there are prefixes $pr(w, l_1), ..., pr(w, l_s)$, $l_1 < l_2 < ... < l_s$, such that $sl(pr(w, l_i))$ begins by a^i, but not by a^{i+1}.*

Proof. We will argue by induction over s. If $s = 1$ there is nothing to prove. Let us suppose that it is true for $s - 1$. It is clear that $pr(w, j) = pr(w, j-1)w(j)$. Let r be the greatest integer such that $sl(pr(w, r-1)w(r)) = sl(pr(w, r)) = a^s\gamma$ and $sl(pr(w, r - 1))$ does not begin with a^s. Then, by Lemma 3.29, $sl(pr(w, r - 1)) = a^{s-1}w'$ and we set $l_s = r$. Now, we can assume that the result is true for $pr(w, r - 1)$ by induction hypothesis and we conclude the proof. $\square$

Remark 3.31. *Notice that with an analogous strategy we can prove the following statement. Let $w = b^{\pm 1}w'$ be geodesic such that $sl(w) = a^s\tilde{w}$, $s < 0$. There exist prefixes $pr(w, l_1), ..., pr(w, l_s)$, $l_1 < l_2 < ... < l_s$, such that $sl(pr(w, l_i))$ begins by a^i, but $sl(pr(w, l_i - 1))$ begins by a^{i+1} but not by a^i, $i = -1, ..., s$.*

Lemma 3.32. *In a large even Artin group, let $s \geq t \geq 1$ (resp. $s \leq t \leq -1$) and*

$$w = b^t w', \hat{w} = a^s \hat{w}' \text{ geodesic words such that } w =_G \hat{w}. \tag{3.1}$$

Let $2m$ be the label between a and b. The minimal length of such a word w is $|w| = |s| + |t|(2m-1)$. Moreover, the only word w of this length satisfying (3.1) is:

$$w = b^t[_{2m-1}(a,b)]^t a^{s-t}, \hat{w} = a^s[_{2m-1}(b,a)]^t$$

for $s, t \geq 1$ or

$$w = b^t[_{2m-1}(a^{-1}, b^{-1})]^{|t|} a^{-|s-t|}, \hat{w} = a^s[_{2m-1}(b^{-1}, a^{-1})]^{|t|}$$

for $s, t \leq -1$.

Proof. We will only prove the case $s, t \geq 1$, the case $s, t \leq -1$ follows by symmetry (it is enough to consider a^{-1}, b^{-1} as generators instead of a, b).

Without loss of generality, we fix a lexicographic order whose first letters are $a < b < a^{-1} < b^{-1}$. Then, as we may assume $\hat{w}' \in W$ we have $sl(w) = \hat{w} = a^s \hat{w}'$.

By Corollary 3.30 there exists a prefix $pr(w, l_1)$ such that $sl(pr(w, l_1)) = a\alpha'$ is a geodesic word such that α' doesn't have a geodesic representative beginning with a and $sl(pr(w, l_1 - 1))$ doesn't begin with a. Therefore, since the order is $a < b < a^{-1} < b^{-1}$ and w begins with b, $sl(pr(w, l_1 - 1))$ must also begin with b.

We claim that $sl(pr(w, l_1 - 1))$ must begin by b^t. Suppose that this does not happen, then there exists $l_2 < l_1$ such that $sl(w, l_2 - 1)$ begins with b^t but $sl(w, l_2)$ begins with b^r with $r < t$,where b^r be the biggest prefix of $sl(pr(w, l_2))$ that is a power of b. Observe that by construction $pr(w, l_2 - 1)$ begins by b^t. The fact that this element has a geodesic representative beginning by b^t and that our order is $a < b < a^{-1}$ implies that $b^r a$ must be a prefix of $sl(pr(w, l_2))$. Then there is a leftward lex reducing sequence

$$sl(pr(w, l_2 - 1))w(l_2) \to sl(pr(w, l_2)) = b^r a\gamma.$$

So if $u_n \to \tau(u_n)$ is the last τ-move in the leftward lex reducing sequence, we have that $f(\tau(u_n)) = a$. So, we know by Proposition 3.12 that u_n must be a positive critical word and it must have one of the following forms:

$$u_n = \begin{cases} \text{either} & b^{t-r}\xi(t,z)_{2m} \quad \text{if } t - r \geq 1, \{z, t\} = \{a, b\} \\ \text{or} & (b,a)_{2m}\, \xi \quad\quad\;\; \text{if } t - r = 1 \end{cases}$$

In both cases $u = b^r u_n$ is a prefix of a word representive of $pr(w, l_2)$. If we are in the first case, then $u = b^r u_n = b^t \xi(t, z)_{2m}$ is also a critical word and applying τ, we obtain a geodesic representative of $pr(w, l_2)$ that begins with a, getting a contradiction.

If we are in the second case, $u = b^r u_n = b^{t-1}(b,a)_{2m}\xi$, applying τ we get $\tau(u_n) = \xi(t,z)_{2m}$ with $\{z,t\} = \{a,b\}$. Therefore, $u =_G b^{t-1}\xi(t,z)_{2m}w'$. But notice that $b^{t-1}\xi(t,z)_{2m}$ is a critical word and if we apply τ we get a geodesic representative of $pr(w,l_2)$ that begins by a which is again impossible. So the claim follows.

Now, by Theorem 3.21 we know that there is a leftward lex reducing sequence that transforms $sl(pr(w,l_1{-}1))w(l_1)$ into $sl(pr(w,l_1))$. As $sl(pr(w,l_1-1))w(l_1)$ begins with b^t and $sl(pr(w,l_1))$ begins with a, the sequence must involve the first letter of these words and if we let r be the length of that sequence, $\tau(u_r) = {}_{2m-1}(a,b)b^t\xi$. Therefore,

$$w =_G [{}_{2m-1}(a,b)]b^t\tilde{w} = {}_{2m}(a,b)b^{t-1}\tilde{w}. \tag{3.2}$$

Besides, this (together with the lexicographic order) implies that ${}_{2m-1}(a,b)b^t = {}_{2m}(a,b)b^{t-1}$ is a prefix of $sl(pr(w,l_1))$.

Now we will argue by induction on t. If $t = 1$, notice that ${}_{2m-1}(a,b)b^t$ has length $2m$ and since by Corollary 3.30 we will need at least s different prefixes, the length of a word satisfying this must be at least $2m + s - 1$, that is exactly the length of ${}_{2m-1}(b,a)a^s =_G a^s\ {}_{2m-1}(b,a)$. Obviously, there are no other words of this length satisfying the property.

Let us suppose now that the result is true for $t < k$ and consider $t = k$. Since $t > 1$, we have that also $s > 1$. We already know that $w =_G {}_{2m-1}(a,b)b^t\tilde{w} = {}_{2m}(a,b)b^{t-1}\tilde{w}$. Now we claim that $sl(b^{t-1}\tilde{w})$ begins by a^{s-1}.

Assume that $sl(b^{t-1}\tilde{w})$ does not begin by a^{s-1}, then $sl(b^{t-1}\tilde{w}) = a^p\tilde{w}'$ with $0 \le p < s-1$ where $\tilde{w}'$ has no geodesic representative that begins with a. Besides, we know that

$$w =_G {}_{2m-1}(a,b)bb^{t-1}\tilde{w} =_G a_{2m-2}(b,a)ba^p\tilde{w}' =_G a^{p+1}\underbrace{{}_{2m-1}(b,a)\tilde{w}'}_{\alpha} =_G a^s w'.$$

Let us call $\alpha =_{2m-1} (b,a)\tilde{w}'$. As $p+1 < s$, $sl(\alpha)$ must begin by a, and so there must be a prefix $pr(\alpha, l_1')$ such that $sl(pr(\alpha,l_1'))$ begins by a but $sl(pr(\alpha, l_1'-1))$ begins by b because our lexicographic order is $a < b < a^{-1}$.

Therefore, there exists a leftward lex reducing sequence

$$\underbrace{sl(pr(\alpha,l_1'-1))}_{b\ldots}\alpha(l_1') \longrightarrow \underbrace{sl(pr(\alpha,l_1'))}_{a\ldots}.$$

Let us denote by $u_1, \ldots, u_r$ the critical words of the sequence. As $sl(pr(\alpha, l_1' - 1))\alpha(l_1')$ begins with b and $sl(pr(\alpha,l_1'))$ begins with a, the end of the sequence must involve the first letter of the words. Since $\tilde{w}'$ does not have any geodesic representative beginning by a, the last critical word of the sequence, u_r, must be of the form $\xi(b,a)_{2m}$ with $\xi = {}_{2m-1}(b,a)\xi'$ (ξ' possibly empty) and it must be a positive critical word. Notice that if $\xi' \neq \epsilon$, $f[\xi'] = b$ because by the form of the critical word it must begin by b or a. The latter case is impossible since $\tilde{w}'$ doesn't have a geodesic representative begining by a. So we are left with two cases

- either $\xi' = \epsilon$ and $f[u_r'] = b$
- or $\xi' \neq \epsilon$.

Then in both cases $u_r' = \xi'(b,a)_{2m}$ is also a critical word, and $\tau(u_r')$ begins by a. Thus the sequence $u_1, ..., u_{r-1}, u_r'$ is a reducing sequence for $\tilde{w}'$. After applying τ we get a geodesic representative beginning with a, yielding a contradiction. Thus, $sl(b^{t-1}\tilde{w})$ begins by a^{s-1}.

Therefore, we have the words

$$w =_G [{}_{2m}(a,b)]b^{t-1}\tilde{w} =_G [{}_{2m}(a,b)]a^{s-1}\tilde{w}' =_G a^s[{}_{2m-1}(b,a)]\tilde{w}'.$$

So, the minimal length is $2m + \mathrm{ML}(s-1, t-1)$ where $\mathrm{ML}(s-1,t-1)$ is the minimal length of the case with parameters $t-1$, $s-1$, and by induction hypothesis we know that $\mathrm{ML}(s-1,t-1) = s-1+(t-1)(2m-1)$ and the only words satisfying this are $b^{t-1}[{}_{2l-1}(a^{-1},b^{-1})]^{|t-1|}a^{s-t} =_G a^{s-1}[{}_{2l-1}(b^{-1},a^{-1})]^{|t-1|}$.

Thus, for parameters t, s, the minimal length is $2m+s-1+(t-1)(2m-1) = s+t(2m-1)$ and the only words of this length are $b^t[{}_{2m-1}(a,b)]^{|t-1|}a^{s-t} =_G a^s[{}_{2m-1}(b,a)]^{|t-1|}$ as we wanted. □

Lemma 3.33. *In a large even Artin group we cannot have two geodesic words $a^s w_1 =_G b^{-t}w_2$, $s,t \geq 2$.*

Proof. We may assume $s = t = 2$. We argue by contradiction. Let T be the set of elements of G that have geodesic representatives $w' = a^2w_1 =_G b^{-2}w_2 = w$ and assume $T \neq \emptyset$. Let $g \in T$ be an element of minimal geodesic length represented by $w =_G w'$ as before.

Without loss of generality we consider the lexicographic order $a < b < b^{-1} < a^{-1} < ....$ By the choice of the order we can suppose that $w' = sl(w)$. Besides, also by the choice of the order and the fact that w is geodesic together with Lemma 3.25 we deduce that if $sl(pr(w,i))$, $1 \leq i \leq |g|$, doesn't begin with b^{-1} it must begin by a.

Let $n = |g|$, we have $sl(pr(w,n)) = w'$. Since g is minimal we know that $sl(pr(w,n))$ is the first prefix in the series beginning by a^2 (in other case there would be a prefix α of w shorter than w and such that $\alpha \in$

T). Let $sl(pr(w,j))$ be the first prefix which doesn't begin by b^{-2}. Then, $sl(pr(w,j-1))$ begins by b^{-2} and $sl(pr(w,j))$ may begin by a or by $b^{-1}a$ (by the choice of the lexicographic order). Let us see that in both cases, we have that $sl(pr(w,n-1))$ must begin by ab.

By Theorem 3.21 there is a leftward lex reducing sequence that transforms $sl(pr(w,j-1))w(j)$ into $sl(pr(w,j))$.

1. Let us supppose that $sl(pr(w,j))$ begins by a, then we know that the last critical word of the reducing sequence must involve the first letter. Let r be the length of the reducing sequence, then $sl(pr(w,j))$ begins with $\tau(u_r)$. Then u_r is a critical word in a,b beginning with b^{-2} and such that $\tau(u_r)$ begins with a. Let $2m$ be the label between a and b, therefore: $u_r = b^{-1}b^{-1}\xi'(t,z)_{2m-1}$ where $\{t,z\} = \{a,b\}$. Thus, $\tau(u_r) = {}_{2m-1}(a,b)b^{-1}\xi' t^{-1}$ where $t \in \{a,b\}$ so $sl(pr(w,j))$ begins by ab.

   Therefore, since by minimality $sl(pr(w,n-1))$ cannot begin by a^2, it begins by ab (since $sl(pr(w,j))$ begins by ab and b is the second letter in the lexicographic order).

2. Assume now that $sl(pr(w,j))$ begins by $b^{-1}a$. Recall that $sl(pr(w,j-1))$ begins with b^{-2}. Then we know that there exists a leftward lex reducing sequence transforming $sl(pr(w,j-1))w(j)$ into $sl(pr(w,j))$, and that the last critical word of the reducing sequence must involve the second letter of the word and not the first one (i.e. every letter of the word except the fist one belongs to the reducing sequence). Let r be the length of the reducing sequence, then $sl(w,j)$ begins with $b^{-1}\tau(u_r)$. Then u_r is a critical word in a,b beginning with b^{-1} and such that $\tau(u_r)$ begins with a, therefore: $u_r = {}_n(b^{-1},a^{-1})\xi(t,z)_p$ where $\{t,z\} = \{a,b\}$, $p+n = 2m$. Thus, $\tau(u_r) = {}_p(a,b)\xi(t^{-1},z^{-1})_n$.

   By Corollary 3.30 there exists k with $j < k < n$ such that $sl(pr(w,k))$ begins with a and $sl(pr(w,k-1))$ begins with $b^{-1}a$ (by the chosen lexicographic order and Lemma 3.25). We know by Theorem 3.21 that there exists a leftward lex reducing sequence that transforms $sl(pr(w,k-1))w(k)$ into $sl(pr(w,k))$, let d be the length of this sequence. Then, since the first letter is changed, $sl(pr(w,k))$ must begin with $\tau(u_d)$, besides recall that $sl(pr(w,k-1))$ begins with $b^{-1}a$. Therefore, u_d has one of the following forms:

$$u_d = \begin{cases} \text{either} & b^{-1}(a\xi')\,{}_{2m-1}(t,z), \quad \{t,z\} = \{a,b\} \\ \text{or} & b^{-1}\,{}_{2m-1}(a,b) \end{cases}$$

So, then:

$$\tau(u_d) = \begin{cases} \text{either} & {}_{2m-1}(a,b)(a\xi')t^{-1}, \quad t \in \{a,b\} \\ \text{or} & {}_{2m-1}(a,b)b^{-1} \end{cases}$$

Therefore, $sl(pr(w,k))$ begins by ab. Moreover $sl(pr(w,n))$ begins with a^2 and $sl(pr(w,n-1))$ doesn't by minimality. Therefore, $sl(pr(w,n-1))$ must also begin by ab (since b is the second letter in the lexicographic order).

So we have that $sl(pr(w,n))$ begins by a^2 and $sl(pr(w,n-1))$ begins by ab. By Theorem 3.21 we know that there is a leftward lex reducing sequence that transforms $sl(pr(w,n-1))w(n)$ into $sl(pr(w,n))$, and if we call r' to the length of that sequence $sl(pr(w,n)) = a\tau(u_{r'})$. Therefore, by Proposition 3.12 $u_{r'}$ is a positive critical word beginning with b. Then

$$\tau(u_{r'}) = \begin{cases} \text{either} & a\xi'\,{}_{2m}(t,z), \quad \{z,t\} = \{a,b\} \\ \text{or} & {}_{2m}(a,b)\xi \end{cases}$$

Thus:

$$u_{r'} = \begin{cases} \text{either} & {}_{2m}(b,a)a\xi' \\ \text{or} & \xi\,{}_{2m}(t,z), \quad \{z,t\} = \{a,b\}, \end{cases}$$

In both cases, $au_{r'}$ is also a positive critical word beginning with a, therefore appliying τ to this critical word instead of applying it to $u_{r'}$ in the last step, we would obtain a geodesic representative of w beginning with b (by Proposition 3.12). But w begins by b^{-1} and this is impossible by Lemma 3.25. □

Lemma 3.34. *In a large even Artin group, let $w = b^t w'$ (resp. $w = a^{-t}\hat{w}'$), $\hat{w} = a^{-1}\hat{w}'$ (resp. $\hat{w} = bw'$), $t \geq 1$ be geodesic words such that $w =_G \hat{w}$. Let $2m$ be the label between a and b. The minimal length of such a word w is $|w| = t + (2m-1)$. Moreover, the only words w of this length satisfying that are: $w = b^t\,{}_{2m-1}(a^{-1}, b^{-1}) =_G {}_{2m-1}(a^{-1}, b^{-1})b^t$ (resp. $w = a^{-t}\,{}_{2m-1}(b,a) =_G {}_{2m-1}(b,a)a^{-t}$).*

Proof. Obviously, a word w like that must begin by b^t (resp. a^{-t}) and since it can be transformed, must contain a critical word, so it must have at least length $t + 2m - 1$.

The only way to get a word like that of this length is considering the shortest possible critical word, i.e. $w = b^t\,{}_{2m-1}(a^{-1}, b^{-1})$ (resp. $a^{-t}\,{}_{2m-1}(b,a)$). □

Finally, we will need the following result which is equivalent to Proposition 4.5 (3) in [56].

Given a set A, we will use $\#A$ to denote the cardinal of the set.

Lemma 3.35. *Let us consider the large even Artin group A_Γ. Given an element $g \in A_\Gamma$*

$$\#\{v \in V(\Gamma)^{\pm} \mid \text{exists } w \text{ geodesic representative of } g \text{ such that } f[w] = v\} \leq 2.$$

3.3 Technical key result

In this section we will find a presentation for the kernel of the map $\psi : A_\Gamma \to A_{\Gamma \setminus \{r\}}$ induced by $r \mapsto 1$ and $v \mapsto v$ if $v \neq r$. Notice that this is well defined because our group is an even Artin group.

Notation 3.36. *Let Γ be a simple labelled graph with even labels. Let $r \in V(\Gamma)$. We will consider the Artin groups $G = A_\Gamma$ and $G_1 = A_{\Gamma \setminus \{r\}}$. We will denote by $b_1, ..., b_n$ the vertices connected to r whith label $2k_j$, $k_j > 1$, $1 \leq j \leq n$ and by $c_1, ..., c_k$ the vertices non-connected to r.*

Definition 3.37. *For $i = 1, ..., n$ we define the following integer numbers:*

$$\begin{aligned}
p_i^+ &= \left\lfloor \frac{k_i}{2} \right\rfloor + 1 \\
n_i^- &= -\left(\left\lfloor \frac{k_i - 1}{2} \right\rfloor + 1 \right) \\
p_i^- &= \left\lfloor \frac{k_i}{2} \right\rfloor + 1 - k_i = p_i^+ - k_i \\
n_i^+ &= k_i - \left(\left\lfloor \frac{k_i - 1}{2} \right\rfloor + 1 \right) = k_i + n_i^-
\end{aligned}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Remark 3.38. *The following properties are satisfied for large even Artin groups (every $k_i \geq 2$):*

1. $p_i^+ \geq 2$, $n_i^+ > 0$, $n_i^- < 0, p_i^- \leq 0$.
2. $p_i^+ > |p_i^-|$
3. $|n_i^-| \geq n_i^+$ *and the equality holds if and only if k_i is even.*
4. $p_i^+ \geq |n_i^-|$ *and the equality holds if and only if k_i is odd.*

Lemma 3.39. *We have:*

1. $p_i^- - 1 = n_i^-$
2. $n_i^+ < p_i^+$

Proof.

1. $p_i^- - 1 = \left(\left\lfloor \frac{k_i}{2} \right\rfloor + 1 - k_i\right) - 1 = \left\lfloor \frac{k_i}{2} \right\rfloor - k_i = -\left(\left\lfloor \frac{k_i - 1}{2} \right\rfloor + 1\right) = n_i^-$.
2. $n_i^+ = k_i - \left(\left\lfloor \frac{k_i - 1}{2} \right\rfloor + 1\right) = k_i - \left\lfloor \frac{k_i - 1}{2} \right\rfloor - 1 < \left\lfloor \frac{k_i}{2} \right\rfloor + 1 = p_i^+$

□

Definition 3.40. *We will consider the following sets of elements of G:*

$$\Omega_i^+ = \left\{ g \in G \mid g \textit{ has a geodesic representative beginning with } b_i^{p_i^+} \right\}$$
$$\Omega_i^- = \left\{ g \in G \mid g \textit{ has a geodesic representative beginning with } b_i^{n_i^-} \right\}$$

for $i = 1, ..., n$.

Remark 3.41. *We have $\Omega_i^+ \cap \Omega_i^- = \emptyset$ by Lemma 3.25.*

Consider an Artin relation of even type of the form $(rb_i)^{k_i} = (b_i r)^{k_i}$. We can rewrite it as follows:

$$r^{b_i^{k_i}} = r^{b_i^{k_i - 1}} ... r^{b_i} r \left(r^{b_i}\right)^{-1} ... \left(r^{b_i^{k_i - 1}}\right)^{-1}$$

$$r^{-b_i} = r^{-1} \left(r^{b_i}\right)^{-1} ... \left(r^{b_i^{k_i - 2}}\right)^{-1} r^{b_i^{k_1 - 1}} r^{b_i^{k_i - 2}} ... r^{b_i} r$$

From here we can easily see that these expressions are equivalent to:

$$r^{b_i^{p_i^+}} = r^{b_i^{p_i^+ - 1}} ... r^{b_i^{p_i^-}} ... \left(r^{b_i^{p_i^+ - 1}}\right)^{-1}$$

$$r^{b_i^{n_i^-}} = \left(r^{b_i^{n_i^- + 1}}\right)^{-1} ... r^{b_i^{n_i^+}} ... r^{b_i^{n_i^- + 1}}$$

Taking these relations as inspiration, we define the following sets of relations:

Definition 3.42.

$$\hat{R}^+ = \left\{ r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+ - 1} g)} ... r^{sl(b_i^{p_i^-} g)} ... \left(r^{sl(b_i^{pi - 1} g)}\right)^{-1} ; b_i^{p_i^+} g \in \Omega_i^+ \textit{ for some } 1 \leq i \leq n \right\}$$
$$\hat{R}^- = \left\{ r^{sl(b_i^{n_i^-} g)} = \left(r^{sl(b_i^{n_i^- + 1} g)}\right)^{-1} ... r^{sl(b_i^{n_i^+} g)} ... r^{sl(b_i^{n_i^- + 1} g)} ; b_i^{n_i^-} g \in \Omega_i^- \textit{ for some } 1 \leq i \leq n \right\}$$
$$\hat{R} = \hat{R}^+ \cup \hat{R}^-$$

Remark 3.43. *Let $h = b_i^{p_i^+} \in \Omega_i^+$. The relation of $\hat{R}^+$ associated to h is $r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+-1} g)} \ldots r^{sl(b_i^{p_i^-} g)} \ldots \left(r^{sl(b_i^{pi-1} g)}\right)^{-1}$ and will be denoted by $R(h)$.*

Analogously, let $h = b_i^{n_i^-} g \in \Omega_i^-$. The relation of $\hat{R}^-$ associated to h is $r^{sl(b_i^{n_i^-} g)} = \left(r^{sl(b_i^{n_i^-+1} g)}\right)^{-1} \ldots r^{sl(b_i^{n_i^+} g)} \ldots r^{sl(b_i^{n_i^-+1} g)}$ will and will be denoted by $R(h)$.

Now, let us consider the map

$$\psi : A_\Gamma \to A_{\Gamma_{\setminus\{r\}}}$$

induced by $r \mapsto 1$ and $v \mapsto v$ if $v \neq r$. We want to obtain a presentation for the kernel of this map. To do so, we will observe that $A_\Gamma \simeq A_{\Gamma_{\setminus\{r\}}} \ltimes \ker(\psi)$ and use a standard argument that can be found in Appendix A of [66], to obtain a presentation for the semidirect product.

Let $K = \langle Y \mid C \rangle$ and $G = \langle Z \mid T \rangle$ be groups. Let G act on Y by permutations. Notice that $C \leq F(Y)$, the free group generated by Y, and observe that G also acts on $F(Y)$. We assume that this action preserves C. Let Y_0 be a set of representatives for the G-orbits in Y and C_0 be a set of representatives for the G-orbits in C. We observe that $C_0 \leq \langle t(a_0) \mid a_0 \in Y_0, t \in G \rangle$ that is, we may express elements of C_0 as products of elements in the G-orbit of Y_0. We then set $\hat{C}_0 \subset \langle t^{-1}a_0t \mid a_0 \in Y_0, t \in G \rangle$ to be the set of fixed expressions for the elements of C_0 where we have replaced the action of G on Y_0 by the conjugation of elements. The set $\hat{C}_0$ is thus a set of formal expressions which will be used later to express relations in groups.

Lemma 3.44. *[66] With the notation above, we have*

$$G \ltimes K = \langle Y_0, Z \mid \hat{C}_0, T, [Stab_G(y), y], y \in Y_0 \rangle,$$

where the semidirect product is given by the action of G on K.

Lemma 3.45. *Let G be a large even Artin group, r a generator and $G_1 = A_{\Gamma\setminus\{r\}}$ as before. If $g \in G_1$ satisfies $g^{-1}rg = r$, then $g = \epsilon$.*

Proof. Let w be a shortlex representative for g. As $r =_G w^{-1}rw$, the word $w^{-1}rw$ is obviously not shortlex, so it is clear that there exists a prefix α of $w^{-1}rw$ such that α contains r and admits a rightward length reducing sequence. But that means that at some moment of the sequence we will have a critical word u_i in two letters, one of them being r. Since G is large even, in every critical word the name of each of the letters appears at least twice, so one of the occurences of r must occur in w or w^{-1}. But since $g \in G_1$, it is impossible that neither r nor r^{-1} appear in w or w^{-1}. So it is impossible to have such a critical word u_i and hence $g = \epsilon$. □

Lemma 3.46. $\ker(\psi)$ *is isomorphic to:*

$$K := \left\langle r^{sl(g)}; g \in A_{\Gamma_1} \mid \hat{R} \right\rangle,$$

where $\hat{R}$ is the set of relations defined in Definition 3.42.

Proof. Note that $\ker(\psi)$ is the normal subgroup of A_Γ generated by r. We define an action of $G_1 = A_{\Gamma\setminus\{r\}} = \langle S_1 \mid C \rangle$ (where S_1 is the set of generators of G_1 and C is the set of Artin relations of G_1) on the abstract group K via

$$h^{-1}(r^{sl(g)})h = r^{sl(gh)}, \quad h \in G_1.$$

Let us see that this action preserves the relators of K. To prove that, we may suppose that h is a generator, i.e. that $h \in S_1$. Consider an $\hat{R}^+$ relation

$$r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+-1} g)} ... r^{sl(b_i^{p_i^-} g)} ... \left(r^{sl(b_i^{pi-1} g)}\right)^{-1},$$

where g has a representative w with $b_i^{p_i^+} w$ geodesic. If h acts on both sides of this relation we obtain:

$$r^{sl(b_i^{p_i^+} gh)} \text{ and } r^{sl(b_i^{p_i^+-1} gh)} ... r^{sl(b_i^{p_i^-} gh)} ... \left(r^{sl(b_i^{pi-1} gh)}\right)^{-1},$$

we want to see that these two elements of K are equal.

Assume first that the element gh has some geodesic representative u with $b_i^{p_i^+} u$ also geodesic. Then there is a $\hat{R}^+$ relation of the form

$$r^{sl(b_i^{p_i^+} gh)} = r^{sl(b_i^{p_i^+-1} gh)} ... r^{sl(b_i^{p_i^-} gh)} ... \left(r^{sl(b_i^{pi-1} gh)}\right)^{-1},$$

which is precisely the image under h of the previous relation.

Now, we are left with the case when g has a representative w with $b_i^{p_i^+} w$ geodesic but $b_i^{p_i^+} wh$ has no geodesic representative beginning by $b_i^{p_i^+}$. Then, by Lemma 3.27 gh has a geodesic representative that begins by b_i^{-1}, say $b_i^{-1}\alpha$. Thus, $b_i^{p_i^-} gh$ has a geodesic representative that begins by $b_i^{p_i^- -1}$. But, by Lemma 3.39 we have that $p_i^- - 1 = n_i^-$. So, it begins by $b_i^{n_i^-}$ and rearranging, we obtain:

$$r^{sl(b_i^{p_i^-} gh)} = r^{sl(b_i^{n_i^-} \alpha)} = \left(r^{sl(b_i^{n_i^-+1} \alpha)}\right)^{-1} ... r^{sl(b_i^{n_i^+} \alpha)} ... r^{sl(b_i^{n_i^-+1} \alpha)}$$

which is a relation of $\hat{R}^-$.

Following the same strategy, we can prove that the same happens with the $\hat{R}^-$ relations.

Now, applying Lemma 3.44 to our case, we obtain that

$$G_1 \ltimes K = \langle Y_0, Z \mid \hat{C}_0, T, [Stab_{G_1}(y), y], y \in Y_0 \rangle,$$

where $Y_0 = r$, $Z = V(G_1)$, $\hat{C}_0 = \hat{C}^+ \cup \hat{C}^-$, $T = C$ with

$$\begin{aligned}
\hat{C}^+ = \{ & sl(b_i^{p_i^+} g)^{-1} r sl(b_i^{p_i^+} g) = sl(b_i^{p_i^+ -1} g)^{-1} r sl(b_i^{p_i^+ -1} g) ... sl(b_i^{p_i^-} g)^{-1} r sl(b_i^{p_i} g) ... \\
& (sl(b_i^{pi-1} g)^{-1} r sl(b_i^{pi-1} g))^{-1}; b_i^{p_i^+} g \in \Omega_i^+ \text{ for some } 1 \leq i \leq n \} \\
\hat{C}^- = \{ & sl(b_i^{n_i^-} g)^{-1} r sl(b_i^{n_i^-} g) = (sl(b_i^{n_i^- +1} g)^{-1} r sl(b_i^{n_i^- +1} g))^{-1} ... sl(b_i^{n_i^+} g)^{-1} r sl(b_i^{n_i^+} g) ... \\
& sl(b_i^{n_i^- +1} g)^{-1} r sl(b_i^{n_i^- +1} g); b_i^{n_i^-} g \in \Omega_i^- \text{ for some } 1 \leq i \leq n \}
\end{aligned}$$

Therefore,

$$G_1 \ltimes K = \langle r, V(G_1) \mid \hat{C}_0, C, [Stab_{G_1}(r), r] \rangle.$$

Now, if $g \in Stab_{G_1}(r)$, we have that $g^{-1} r g = r^{sl(g)} = r$ in K because of the form of the action. Hence $sl(g) = \epsilon$ by Lemma 3.45 and thus we don't have any relation of this type. Thus:

$$G \ltimes K = \langle r, V(G_1) \mid \hat{C}_0, C \rangle.$$

But since in our presentation we have C, the set of relations of G_1, we can rewrite the relations of $\hat{C}_0$ in the following way:

$$\begin{aligned}
\hat{C}_1^+ = \{ & (b_i^{p_i^+} g)^{-1} r (b_i^{p_i^+} g) = (b_i^{p_i^+ -1} g)^{-1} r (b_i^{p_i^+ -1} g) ... \left(b_i^{p_i^-} g \right)^{-1} r (b_i^{p_i^-} g) ... \\
& ((b_i^{pi-1} g)^{-1} r (b_i^{pi-1} g))^{-1}; g \in G_1, b_i^{p_i^+} g \text{ geodesic}, 1 \leq i \leq n \} \\
\hat{C}_1^- = \{ & (b_i^{n_i^-} g)^{-1} r (b_i^{n_i^-} g) = ((b_i^{n_i^- +1} g)^{-1} r (b_i^{n_i^- +1} g))^{-1} ... \left(b_i^{n_i^+} g \right)^{-1} r (b_i^{n_i^+} g) ... \\
& (b_i^{n_i^- +1} g)^{-1} r (b_i^{n_i^- +1} g); g \in G_1, b_i^{n_i^-} g \text{ geodesic}, 1 \leq i \leq n \} \\
\hat{C}_1 = {} & \hat{C}^+ \cup \hat{C}^-
\end{aligned}$$

So we get the presentation:

$$G_1 \ltimes K = \langle V(G_1), r \mid \hat{C}_1, C \rangle.$$

We define:

$$C_1' = \{ (b^{p_i^+})^{-1} r (b^{p_i^+}) = (b^{p_i^+ -1})^{-1} r (b^{p_i^+ -1}) ... r ... ((b^{p_i^+ -1})^{-1} r (b^{p_i^+ -1}))^{-1}, i = 1, ..., n \}$$

So in fact, the relations in $\hat{C}_1 \setminus C_1'$ are obtained from the ones of C_1' by conjugation. So we can eliminate them from the presentation using Tietze transformations. Thus, we have:

$$G_1 \ltimes K = \langle V(G_1), r \mid C_1', C \rangle \simeq A_\Gamma$$

and the isomorphism maps K onto $\ker(\psi)$. Therefore, $K := \langle r^{sl(g)}; g \in G_1 \mid \hat{R} \rangle = \langle\langle r \rangle\rangle = \ker(\psi)$. □

3.4 Poly-freeness for large even Artin groups

In this section we are going to prove our main result, i.e. that every large even Artin group is polyfree.

Let Γ be a labelled graph with even labels ≥ 4 and A_Γ the associated Artin group. Let r be a vertex of the graph and $G_1 = A_{\Gamma \setminus \{r\}}$

By Lemma 3.46 we know that:

$$K := \langle\langle r \rangle\rangle = \langle r^{sl(g)}; g \in G_1 \mid \hat{R} \rangle$$

with

$$\hat{R}^+ = \left\{ r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+ -1} g)} \dots r^{sl(b_i^{p_i^-} g)} \dots \left(r^{sl(b_i^{pi-1} g)} \right)^{-1}; b_i^{p_i^+} g \in \Omega_i^+ \text{ for some } 1 \leq i \leq n \right\}$$

$$\hat{R}^- = \left\{ r^{sl(b_i^{n_i^-} g)} = \left(r^{sl(b_i^{n_i^- +1} g)} \right)^{-1} \dots r^{sl(b_i^{n_i^+} g)} \dots r^{sl(b_i^{n_i^- +1} g)}; b_i^{n_i^-} g \in \Omega_i^- \text{ for some } 1 \leq i \leq n \right\}$$

$$\hat{R} = R^+ \cup R^-$$

where $b_1, ..., b_n$ are the vertices connected to r, $2k_1, ..., 2k_n$ the labels of the connecting edges and

$$\Omega_i^+ = \left\{ h \in G \mid h \text{ has a geodesic representative beginning with } b_i^{p_i^+} \right\}$$

$$\Omega_i^- = \left\{ h \in G \mid h \text{ has a geodesic representative beginning with } b_i^{n_i^-} \right\}$$

for $i = 1, 2, ..., n$.

Recall that in Definition 3.37 we have defined the numbers $p_i^+, p_i^-, n_i^-, n_i^+$ and that by Remark 3.41 we already know that $\Omega_i^+ \cap \Omega_i^- = \emptyset$.

Lemma 3.47. *An element $g \in G$ can belong at most to two different sets $\Omega_i^\pm$, $i = 1, ..., n$.*

Proof. It is easily deduced from Lemma 3.35. □

Lemma 3.48. *If $i \neq j$, $\Omega_i^+ \cap \Omega_j^- \neq \emptyset$ if and only if $n_j^- = -1$, i.e. $k_j = 2$.*

Proof. Recall that $p_i^+ \geq 2$. By Lemma 3.33, given a geodesic word $w = a^s w'$, $s \geq 2$, there cannot be another geodesic representative of the same element begining by b^{-t}, $t > 1$. So, if $\Omega_i^+ \cap \Omega_j^- \neq \emptyset$, then $n_j^- = -1$. The fact that if $n_j^- = -1$ then $\Omega_i^+ \cap \Omega_j^- \neq \emptyset$ is clear by Lemma 3.34. □

Now, notice that by the form of the relations in $\hat{R}$, we see that each $r^{sl(b_i^{p_i^+} g)}$ (resp. $r^{sl(b_i^{n_i^-} g)}$) is conjugate in K to $r^{sl(b_i^{p_i^-} g)}$ (resp. $r^{sl(b_i^{n_i^+} g)}$).

Let us define the following maps:

$$\begin{aligned} \rho_i^\varepsilon : \Omega_i^\varepsilon &\longrightarrow G_1 \\ h &\mapsto b_i^{-\varepsilon k_i} \hat{g} \end{aligned}$$

for $i = 1, ..., n$ and $\varepsilon = \pm$.

Notice that if $\varepsilon = +$, then h has a geodesic representative of the form $b_i^{p_i^+} \bar{h}$ where $\bar{h}$ doesn't have a geodesic representative begining with b_i^{-1}. Thus, $\rho_i^+(h) = b_i^{p_i^-} \bar{h}$. Similarly, if $\varepsilon = -$, then h has a geodesic representative of the form $b_i^{n_i^-} \bar{h}$ where $\bar{h}$ doesn't have a geodesic representative begining with b_i. Thus, $\rho_i^-(h) = b_i^{n_i^+} \bar{h}$.

Let $\Omega = \cup_{i=1}^n (\Omega_i^+ \cup \Omega_i^-)$. We also define $\Lambda = G_1 \setminus \Omega$.

Let $\mathcal{P}$ be the set of subsets of G_1. We set:

$$\begin{aligned} \rho : \mathcal{P} &\longrightarrow \mathcal{P} \\ A &\mapsto \cup_{i=1}^n (\rho_i^+(A \cap \Omega_i^+)) \cup (\rho_i^-(A \cap \Omega_i^-)) \cup (A \cap \Lambda). \end{aligned} \tag{3.3}$$

When we consider the image under ρ of a one element subset $\{g\} \subset \mathcal{P}(W)$, we will write $\rho(g)$ instead of $\rho(\{g\})$.

Lemma 3.49. *Let $g \in \Omega$.*

(i) If g lies in only one of the sets $\Omega_i^\pm$, say $g \in \Omega_i^\epsilon$, then

$$\rho(g) = \{\rho_i^\epsilon(g)\}.$$

(ii) If g lies in two of the sets, say $g \in \Omega_i^\epsilon \cap \Omega_j^\delta$, then

$$\rho(g) = \{\rho_i^\epsilon(g), \rho_j^\delta(g)\}.$$

Note that by Lemma 3.47 there are no other possibilities.

Proof. The result is just an inmediate consequence of the definition of ρ. $\square$

Lemma 3.50. *Let $g \in \Omega_i^+$, $g_1 \in \Omega_i^-$. We have*

(i) $|g| > |\rho_i^+(g)|$

(ii) $|g_1| \geq |\rho_i^-(g_1)|$ *with equality if and only if $w_1 = b_i^{n_i^-} w_2$ is a geodesic representative of g_1, k_i is an even number and w_2 doesn't have a geodesic representative beginning by b_i^{-1} (and therefore, it cannot begin by $b_i^{\pm 1}$).*

Proof. The result follows from Remark 3.38. $\square$

Lemma 3.51. *Let $g \in \cup_{q=1}^n \Omega_q^-$ such that $|g| = |\rho_i^-(g)|$. Then $\rho_i^-(g)$ doesn't belong to the intersection of two of the sets $\Omega_j^\pm$, $j = 1, 2, ..., n$.*

Proof. The hypothesis and Lemma 3.50 imply that g has a geodesic representative of the form $w = b_i^{n_i^-} w'$ and that k_i is an even number.

Therefore, the element $\rho_i^-(g)$ has a geodesic representative of the form $b_i^{k_i+n_i^-} w'$ (because of the hypothesis on its length), where no geodesic representative of w' begins by b_i.

Assume that there exists $b_i^{p_i^+} u$ geodesic such that $b_i^{k_i+n_i^-} w' =_G b_i^{p_i^+} u$. This is impossible because $k_i + n_i^- < p_i^+$ by Lemma 3.39 and w' doesn't have any geodesic representative beginning with b_i. So $\rho_i^-(g) \notin \Omega_i^+$.

Analogously, we see that $\rho_i^-(g)$ cannot be in Ω_i^- (because $|n_i^+| > 0$ and and w' doesn't have any geodesic representative beginning with b_i^{-1}) and by Lemma 3.35 there is only other possible letter such that the word can begin with it so it can belong at most to one $\Omega_j^\pm$. $\square$

Lemma 3.52. *Assume that we choose a lexicographic order such that for every j, $v_j < v_j^{-1}$, then for every g in the suitable set $sl(g) >_{slex} sl(\rho_i^\pm(g))$.*

Proof. By Lemma 3.50 we only have to consider the case when g has a geodesic representative $w = b^{n_i^-} w'$, k_i is an even number and w' doesn't have any geodesic representative beginning with $b_i^{\pm 1}$.

In this case, by definition of ρ_i^- we know that $\rho_i^-(g) = b^{n_i^+} g'$ where g' is the element represented by the word w' and $|n_i^-| = |n_i^+|$. Notice that $sl(b_i^{n_i^-} w') >_{slex} sl(b_i^{n_i^+} w')$ if and only if $sl(b_i^{n_i^-} w'w'^{-1}) = b_i^{n_i^-} >_{slex} sl(b_i^{n_i^+} w'w'^{-1}) = b_i^{n_i *}$. But n_i^- is a negative number and n_i^+ is positive and since in the lexicographic order we have $b < b^{-1}$, the result follows. $\square$

Lemma 3.53. *Given $g \in \Omega$ there exists $l \in \mathbb{Z}^+$ such that if $h \in \rho^l(g) = \underbrace{\rho(\rho...\rho}_{l}(g))$ and $h \in \Omega$ then $|h| < |g|$.*

Proof. Let $g \in \Omega$, we know that $\#\rho(g) = 1$ or 2 (recall that we use $\#A$ to denote the cardinal of the set A.). We define $\beta = \{h \in \rho(w) \mid |h| = |g|\}$. If β is empty, the result follows for $l = 1$ by Lemma 3.50. So we can suppose β is not empty.

By Lemma 3.50 g must have a geodesic representative $w = b_i^{n_i^-} w'$ for some i such that k_i is even and w' doesn't have any geodesic representative beginning with $b_i^{\pm 1}$. Considering if necessary all the elements of β instead of g and using Lemma 3.51 we may assume that $\#\rho^l(g) = 1$ for all $l \geq 1$. It is enough to prove that there exists $l \in \mathbb{Z}^+$ such that $|\rho^l(g)| < |g|$.

We are going to argue by induction over m, the number of negative letters in w. Notice that since we are working on an even Artin group, the number of negative letters in a word is constant for any of its geodesic representatives (see Definition 3.11).

As $w = b_i^{n_i^-} w'$ for k_i even, $|n_i^-| \leq m$. Let $k = \min\{|n_j^-| \mid k_j \text{ is even}\}$. If $m = k$, then w' must be positive. But then, $sl(\rho(g)) = sl(b_i^{n_i^+} w')$ is positive, and therefore if $\rho^2(g) \in \Omega$, $|\rho^2(g)| < |\rho(g)| = |g|$.

In the general case, $sl(\rho(g)) = sl(b_i^{n_i^+} g')$ has less negative letters than w, and we distinguish three cases: if $\rho(g) \notin \Omega$ or $|\rho(\rho(g))| < |\rho(g)|$ we are done. Otherwise, we may apply the induction hypothesis and we obtain that there exists an $l-1 \in \mathbb{Z}^+$ such that either $\rho^{l-1}(\rho(g)) \notin \Omega$ or $|\rho^{l-1}(\rho(g))| < |\rho(g)| = |g|$. That is, either $\rho^l(g) \notin \Omega$ or $|\rho^l(g)| < |g|$. □

Corollary 3.54. *Given $g \in \Omega$, there exists $l \in \mathbb{Z}^+$ such that $\rho^l(g) \subset \Lambda$.*

Proof. We will argue by induction over the geodesic length of g. If $|w| = 0$ the result is obvious.

Let $\gamma = |g|$, and suppose that it is true for length less than γ. By Lemma 3.53 there exists $l_1 \in \mathbb{Z}$ such that for every $g' \in \rho^{l_1}(g)$ either $g' \in \Lambda$ or $|g'| < |g| = \gamma$. Let $\beta = \{g' \in \rho^{l_1}(g) \mid g' \notin \Lambda\} = \{g'_1, ..., g'_\mu\}$, by induction hypothesis there is $l'_i \in \mathbb{Z}$ such that $\rho^{l'_i}(g'_i) \subset \Lambda$ for $i = 1, ..., \mu$. Let $l_2 = \max\{l'_i\}$. Thus, for $l = l_1 + l_2$ we have that $\rho^l(g) \subset \Lambda$. □

Definition 3.55. *Given $g \in \Omega$ we define $\alpha(g)$ as the smallest positive integer such that $\rho^{\alpha(g)}(g) \subset \Lambda$. This way, we define the map:*

$$\begin{aligned} \delta : \Omega &\longrightarrow \mathcal{P}(\Lambda) \\ g &\mapsto \delta(g) := \rho^{\alpha(g)}(g) \end{aligned}$$

We denote $H = \langle b_1^{k_1}, b_2^{k_2}, ..., b_n^{k_n} \rangle$.

Remark 3.56. *Notice that the elements in $\delta(g)$ lie in the intersection of the coset Hg with Λ.*

Lemma 3.57. *H is freely generated by $b_1^{k_1}, ..., b_n^{k_n}$.*

Proof. Observe first that any freely reduced word in the alphabet $\{b_1^{\pm k_1}, b_2^{\pm k_2}, ..., b_n^{\pm k_n}\}$ can also be seen as a freely reduced word in the alphabet $\{b_1^{\pm 1}, b_2^{\pm 1}, ..., b_n^{\pm 1}\}$. Now, assume that w, w' are freely reduced words in $\{b_1^{\pm k_1}, b_2^{\pm k_2}, ..., b_n^{\pm 1}\}$ which represent the same element in G. We may assume that w' is shortlex and w is not.

Therefore, by Theorem 3.20 w should admit a critical reducing sequence, so it must have a critical subword. But by Definition 3.10 it is impossible to have a critical subword on $b_i^{\pm k_i}, b_j^{\pm k_j}$ if $k_i, k_j \geq 2$ and $m_{b_i,b_j} \geq 4$. Thus w doesn't admit a critical reducing sequence, and w must be shortlex. Therefore, $w = w'$. ☐

Lemma 3.58. *Assume $g \in \Omega_i^{\pm} \cap \Omega_j^{\pm}$ and $\hat{w}_i \in \delta(\rho_i^{\pm}(g))$ and $\hat{w}_j \in \delta(\rho_j^{\pm}(g))$, then $\hat{w}_i \neq_G \hat{w}_j$.*

Proof. Note that $\hat{w}_i, \hat{w}_j$ both lie in the coset Hg. Notice as well that $\delta(\rho_i^{\pm}(g))$, $\delta(\rho_j^{\pm}(g)) \subset \delta(g)$. Since $\hat{w}_i \in \delta(g)$, then $\hat{w}_i =_G h_i g_i$ ($h_i \in H$ beginning with $b_i^{\pm}$). Analogously, $\hat{w}_j =_G h_j g_j$ ($h_j \in H$ beginning with $b_j^{\pm}$). Then, $\hat{w}_i =_G \hat{w}_j$ implies that $h_i =_G h_j$, but this is impossible since H is free by Lemma 3.57 and h_i, h_j begin by different letters. ☐

Lemma 3.59. *The minimal geodesic length of an element $g \in \Omega_i^{\pm} \cap \Omega_j^{\pm}$, $i \neq j$ is bounded below by $|n_j^-| + |n_i^-|(2m-1)$ where $|n_j^-| \geq |n_i^-|$ and $m = m_{b_i,b_j}$. Moreover, there is always an element g of that geodesic length in the set $\Omega_i^- \cap \Omega_j^-$.*

Proof. Assume first that $g \in \Omega_i^+ \cap \Omega_j^-$. Then, by Lemma 3.41, $n_j^- = -1$ thus also $n_i^- = -1$. Using Lemma 3.34 we have that the geodesic length of g is at least

$$|p_i^+| + (2m-1) > 1 + (2m-1) = |n_j^-| + |n_i^-|(2m-1) \tag{3.4}$$

where the inequality is strict since $p_j^+ \geq 2$ by Remark 3.38 (1).

Now assume that $g \in \Omega_i^- \cap \Omega_j^+$, by Lemma 3.33 and Remark 3.38 (1) $n_i^- = -1$. By Lemma 3.34 the minimal geodesic length of such an element is

$$|p_j^+| + (2m-1) \geq |n_j^-| + (2m-1) = |n_j^-| + |n_i^-|(2m-1) \tag{3.5}$$

where by Lemma 3.39 the equality holds if and only if k_j is odd.

If $g \in \Omega_i^+ \cap \Omega_j^+$, by Lemma 3.32 the minimal geodesic length of g is

$$|p_j^+| + |p_i^+|(2m-1) \geq |n_j^-| + |n_i^-|(2m-1) \tag{3.6}$$

where by Lemma 3.39 the equality holds if and only if k_i, k_j are odd.

Finally, if $g \in \Omega_i^- \cap \Omega_j^-$, by Lemma 3.32 the minimal geodesic length of g is

$$|n_j^-| + |n_i^-|(2m-1)$$

and also by Lemma 3.32 we know that there exists an element of this length satisfying $g \in \Omega_i^- \cap \Omega_j^-$. $\square$

Remark 3.60. *Let $t = n_i^-$, $s = n_j^-$, $b = v_i, a = v_j$. If k_j is even, in Lemma 3.59 the inequalities (3.4), (3.5) and (3.6) are strict and therefore there is only one element $g \in \Omega_i^\pm \cap \Omega_j^\pm$, $i \neq j$ with geodesic length $|n_j^-| + |n_i^-|(2m-1)$. This element is:*

$$b^t[{}_{2m-1}(a^{-1}, b^{-1})]^{|t|} a^{-|s-t|} =_G a^s[{}_{2m-1}(b^{-1}, a^{-1})]^{|t|}$$

If k_j is odd and k_i is even, then $k_i \geq 2$ and therefore by Lemma 3.48 $\Omega_i^+ \cap \Omega_j^- = \Omega_i^- \cap \Omega_j^+ = \emptyset$. The inequality (3.6) is strict. Therefore there is again only one element $g \in \Omega_i^\pm \cap \Omega_j^\pm$ of geodesic length $|n_j^-| + |n_i^-|(2m-1)$. This element is

$$b^t[{}_{2m-1}(a^{-1}, b^{-1})]^{|t|} a^{-|s-t|} =_G a^s[{}_{2m-1}(b^{-1}, a^{-1})]^{|t|}$$

If k_i, k_j are both odd numbers, applying Lemmas 3.32, 3.33 and 3.34, we obtain that there are two elements of that geodesic length in $\Omega_i^\pm \cap \Omega_j^\pm$:

$$b^t[{}_{2m-1}(a^{-1}, b^{-1})]^{|t|} a^{-|s-t|} =_G a^s[{}_{2m-1}(b^{-1}, a^{-1})]^{|t|},$$

$$b^{-t}[{}_{2m-1}(a, b)]^{|t|} a^{|s-t|} =_G a^{-s}[{}_{2m-1}(b, a)]^{|t|}.$$

In this way, if we consider the order $a < a^{-1} < b < b^{-1} < \ldots$ we have that the shortlex representative in $\Omega_i^\pm \cap \Omega_j^\pm$ is :

- *If k_i, k_j are both odd numbers,*

  $$b^{-t}[{}_{2m-1}(a, b)]^{|t|} a^{|s-t|} =_G a^{-s}[{}_{2m-1}(b, a)]^{|t|}$$

  (which is a positive word).

- *Otherwise,*

  $$b^t[{}_{2m-1}(a^{-1}, b^{-1})]^{|t|} a^{-|s-t|} =_G a^s[{}_{2m-1}(b^{-1}, a^{-1})]^{|t|}$$

  (which is a negative word).

Recall that we are considering the group:

$$K := \langle\langle r \rangle\rangle = \langle r^{sl(g)}; g \in G_1 \mid \hat{R} \rangle$$

with

$$\begin{aligned}
&\hat{R}^+ = \{r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+-1} g)} ... r^{sl(b_i^{p_i^-} g)} ... (r^{sl(b_i^{pi-1} g)})^{-1}; b_i^{p_i^+} g \in \Omega_i^+ \text{ for some } 1 \leq i \leq n\} \\
&\hat{R}^- = \{r^{sl(b_i^{n_i^-} g)} = (r^{sl(b_i^{n_i^-+1} g)})^{-1} ... r^{sl(b_i^{n_i^+} g)} ... r^{sl(b_i^{n_i^-+1} g)}; b_i^{n_i^-} g \in \Omega_i^- \text{ for some } 1 \leq i \leq n\} \\
&\hat{R} = \hat{R}^+ \cup \hat{R}^-
\end{aligned}$$

Notice that the relations of $\hat{R}^+$ and $\hat{R}^-$ have the following form:

$$\begin{aligned}
&\hat{R}^+ = \{r^{sl(h)} = \alpha r^{sl(\rho_i^+(h))} \alpha^{-1}; h \in \Omega_i^+ \text{ for some } 1 \leq i \leq n\} \\
&\hat{R}^- = \{r^{sl(h)} = \alpha r^{sl(\rho_i^-(h))} \alpha^{-1}; h \in \Omega_i^- \text{ for some } 1 \leq i \leq n\} \\
&\hat{R} = \hat{R}^+ \cup \hat{R}^-
\end{aligned}$$

Proposition 3.61. *K is a free group.*

Proof. We are going to prove it using Tietze transformations. We order the vertices $b_1, ..., b_n$ linked to r, in such way that $k_1 \leq k_2 \leq ... \leq k_n$ and we consider the lexicographic order

$$b_1 < b_1^{-1} < b_2 < b_2^{-1} < ... < b_n < b_n^{-1} < c_1 < c_1^{-1} < ... < c_k < c_k^{-1}.$$

Notice that the choice of this order is consistent with the results of Section 3.2, since there we only used the chosen orders as a technical tool to prove the possible existence or not of determinate kinds of geodesic representatives, but the results themselves didn't depend on the chosen order.

Firstly, let us consider those generators of the form $r^{sl(g)}$, such that $g \in \Omega$. For such an element there is at least one relation in R which is of the form $r^{sl(g)} = \alpha r^{sl(\rho_i^{\pm}(h))} \alpha^{-1}$ (see Remark 3.43), it will belong to R^+ or R^- depending on whether g has a geodesic representative beginning with $b_i^{p_i^+}$ or $b_i^{n_i^-}$ respectively ($i = 1, 2, ..., n$). Thus, using Tietze transformations we can erase that relation and the generator $r^{sl(g)}$. This can be done to erase every generator $r^{sl(g)}$ with $g \in \Omega$, but maybe not every relation of R. Because if g lies in two of the sets $\Omega_i^{\pm}$, $\Omega_j^{\pm}$ (remind that by Lemma 3.47, any element can belong to at most two sets $\Omega_i^{\pm}$), then after using Tietze transformations we are left with a relation of the following form:

$$\alpha^{-1} r^{sl(\rho_i^s(w_1))} \alpha = \beta^{-1} r^{sl(\rho_j^t(w_2))} \beta. \tag{3.7}$$

We define $\tilde{R}$ as the set of remaining relations after this proccess. Recall that $\Lambda = G_1 \setminus \Omega$.

After this process we get a presentation:

$$K = \langle r^{sl(g)}; g \in G_1, g \in \Lambda \mid \tilde{R} \rangle.$$

Now we will see that we can also get rid of the relations in $\tilde{R}$. Notice that

$$\Omega := \bigcup \left\{ \Omega_i^{\pm} \cap \Omega_j^{\pm} \mid i, j = 1, ..., n, i \neq j \right\} \subset \{ g \in G_1 \mid \#(Hg \cap \Lambda) > 1 \}$$

since the elements in Ω are of the form $b_{j_1}^{\epsilon_1 k_{j_1}} ... b_{j_n}^{\epsilon_n k_{j_n}} g$ and if two expressions represent the same group element, then by Lemma 3.58 the $b'_{j_i} s$ must also be equal. Notice that there is a bijection between the relations in $\hat{R}$ and the elements of Ω since each relation appears when we have an element in Ω. Also recall that for any $g \in \Omega$ we have $\#\rho(sl(g)) = 2$ by Lemma 3.49 (ii).

Let us explain a little the strategy that we are going to follow. At this point we have a presentation of the group K with:

1. Set of generators $\{r^{sl(g)}, g \in \Lambda\}$, bijective to Λ.

2. Set of relators $\tilde{R} = \{r(\alpha) \mid \alpha \in \Omega\}$, bijective to Ω.

We want to show that it is possible to remove all the relators and some generators using Tietze transformations. To do that we proceed inductively. More precisely, we first order the elements of Ω using the shortlex order:

$$\alpha_1 < \alpha_2 < ... < \alpha_i < ...$$

To each element α_i we associate a $u_i \in \Lambda$ such that the relator $r(\alpha_i)$ can be written as:

$$r^{u_i} = \gamma r^{w_i} \gamma^{-1}$$

where γ is a word in the alphabet $\{r^{sl(g)}, g \in \Lambda - \{u_1, ..., u_i\}\}$.

Notice that by the order that we have stablish in the vertices, we need to take $\alpha_1 \in \Omega_1^{\pm} \cap \Omega_2^{\pm}$. By Remark 3.60 for the element α_1 we have the following possibilities:

- If k_1, k_2 are both odd numbers, then α_1 admits the following two geodesic representatives:

  $w_1 = b_1^{p_1^+}{}_{2m-1}(b_2, b_1)^{p_2^+} b_2^{p_2^+ - p_1^+}$, $w_2 = b_2^{p_2^+}{}_{2m-1}(b_1, b_2)^{p_2^+}$.

- Otherwise, α_1 admits the following two geodesics representatives: $w_1 = b_1^{n_1^-}{}_{2m-1}(b_2^{-1}, b_1^{-1})^{|n_2^-|} b_2^{n_2^- - n_1^-}$, $w_2 = b_2^{n_2^-}{}_{2m-1}(b_1^{-1}, b_2^{-1})^{|n_2^-|}$.

Where in both cases $m = m_{12}$.

We know that $|\rho_s^-(w_s)| = |w_s|$ if and only if k_s is even and that $|\rho_s^+(w_s)| < |w_s|$. Then, we can distinguish the following cases:

1. If k_1, k_2 are odd numbers, we have that for $s = 1, 2$, $|\rho_s^+(w_s)| < |w_s|$, which implies by minimality that $\rho_s^+(w_s) \notin \Omega$. Thus $\#\delta(\rho_s^+(w_s)) = 1$, so $\delta(sl(\alpha_1)) = \{\delta(\rho_1^+(w_1)), \delta(\rho_2^+(w_2))\}$ has only two elements.

2. If k_2 is even and k_1 is odd (or the other way around), then, $|\rho_1^-(w_1)| < |w_1|$, so $\rho_1^-(w_1) \notin \Omega$ and $\#\delta(\rho_i^-(w_i)) = 1$. And $|\rho_2^-(w_2)| = |w_2|$ and by Lemma 3.51 $\rho_2^-(w_2) \notin \Omega$ and since by Lemma 3.52 $sl(\rho_2^-(w_2)) <_{slex} sl(w_2)$ and $w_1 =_G w_2$ are the shorltex minimal words in Ω, we have that $\#\delta(\rho_2^-(w_2)) = 1$. So, $\delta(sl(\alpha_1)) = \{\delta(\rho_1^-(w_1)), \delta(\rho_2^-(w_2))\}$ has only two elements.

3. If k_1, k_2 are both even, then for $s = 1, 2$ $|\rho_s^-(w_s)| = |w_s|$ and by Lemma 3.51 $\rho_s^-(w_s) \notin \Omega$ and again since by Lemma 3.52 $sl(\rho_s^-(w_s)) <_{slex} sl(w_s)$ and $w_1 =_G w_2$ are the shorltex minimal words in Ω, we have that $\#\delta(\rho_s^-(w_s)) = 1$. So, $\delta(sl(\alpha_1)) = \{\delta(\rho_1^-(w_1)), \delta(\rho_2^-(w_2))\}$ has only two elements.

So, in every possible case $\delta(\alpha_1) = \{\hat{w}_1, \hat{w}_2\} \subset \Lambda$ and both lie in the coset $H\alpha_1$ in Λ. Besides, $\hat{w}_1 \neq_G \hat{w}_2$ by Lemma 3.58.

Therefore, we may assume $sl(\hat{w}_2) >_{slex} sl(\hat{w}_1)$ and using a Tietze transformation we can erase the relation and the generator $r^{sl(\hat{w}_2)}$.

We define the set $\Lambda_1 = \Lambda \setminus \{sl(\hat{w}_2)\}$. Now, we can define the natural projection $\pi_1 : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda_1)$ given by $\pi_1(A) = A \cap \Lambda_1$. After this, we define $\delta_1 = \pi_1 \circ \delta : \Omega \longrightarrow \mathcal{P}(\Lambda_1)$.

We are going to prove that we can construct a family of subsets $\Lambda = \Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset ...$ such that for each element g of G_1 if $sl(g) <_{slex} sl(\alpha_k)$, then $\#(\Lambda_l \cap \delta(g)) = 1$ for every $l \geq k-1$. Once we have constructed these sets, we can define the applications $\pi_k : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda_k)$ and $\delta_k = \pi_k \circ \delta : \Omega \longrightarrow \mathcal{P}(\Lambda_k)$.

1. For $k = 1$, take Λ_1.

2. Assume $\Lambda_1, ..., \Lambda_{k-1}$ have been constructed. For k we know that $\#(\Lambda_{k-1} \cap \delta(g)) = 1$ for every g such that $sl(g) <_{slex} sl(\alpha_k)$. Also $\alpha_k \in \Omega_i^\pm \cap \Omega_j^\pm$ for some $i, j \in \{1, ..., n\}$ with $i \neq j$. At a first step we have two geodesic representatives of α_k: w_i begining with $b_i^{p_i^+}$ or $b_i^{n_i^-}$ and w_j begining with $b_j^{p_j^+}$ or $b_j^{n_j^-}$.

   By Lemma 3.53 for each $s = i, j$ there exist $l_s \in \mathbb{Z}$ such that $|\rho^{l_s}(w_s)| < |w_s|$ (and $|\rho^{d_s}(w_s)| = |w_s|$ for $d_s < l_s$). Note that by Lemma 3.51 each of the elements represented by w_s, $\rho(w_s)$,..., $\rho^{l_s-1}(w_s)$ lies at most on one of the sets $\Omega_r^{\pm}$, $r \in \{1, ..., n\}$.

Therefore, $sl(\rho^{l_s}(w_s)) <_{slex} sl(\alpha_k)$ and by construction of Λ_{k-1}, $\Lambda_{k-1} \cap (\rho^{l_i}(w_s)) = \tilde{w}_s$. Therefore $\delta_{k-1}(\alpha) \cap \Lambda_{k-1} = \{\tilde{w}_i, \tilde{w}_j\}$ (and both must be different by Lemma 3.58).

Now, we may assume $sl(\tilde{w}_j) >_{slex} sl(\tilde{w}_i)$, and we define $\Lambda_k = \Lambda_{k-1} \setminus \{\tilde{w}_j\}$.

Now, at each step of the induction, when we obtain $\delta_{k-1}(\alpha) \cap \Lambda_{k-1} = \{\tilde{w}_i, \tilde{w}_j\}$, we know that the relation $R(\alpha_k)$ can be written as $r^{sl(\tilde{w}_j)}$ equals to a conjugate of $r^{sl(\tilde{w}_i)}$. And using a Tietze transformation we may eliminate the relation and the generator $r^{sl(\tilde{w}_j)}$.

Notice that by the construction of the family $\{\Lambda_k\}$, the word $\hat{w}_1$ obtained in the construction of Λ_1 verifies that $\hat{w}_1 \in \Lambda_k$ for any $k \in \mathbb{N}$. Therefore, notice that $\cap_{k\in\mathbb{Z}}\Lambda_k \neq \emptyset$. In this way, after each inductive step we have a presentation of the group K with:

1. Set of generators $\{r^{sl(g)}, g \in \Lambda_k\}$, bijective to Λ_k.
2. Set of relators $\tilde{R}_k = \{r(\alpha_i) \mid i = k+1, ...\}$, bijective to $\Omega \setminus \{\alpha_1, ..., \alpha_k\}$.

Thus, eventually we can remove every relation and we conseve a non-empty set of generators $\cap_{k\in\mathbb{Z}}\Lambda_k$. Therefore, the group is free. □

And now, as an immediate consequence we obtain our main theorem:

Theorem 3.62. *Any even Artin group based on a large graph is poly-free.*

Proof. It follows immediately from Lemma 3.46 and Proposition 3.61. □

3.5 Poly-freeness for Artin groups based on even triangle graphs

Consider the case when our graph is a triangle. We can distinguish four different types of triangles according to the number of edges with label 2:

(i) $(2, 2, 2)$,

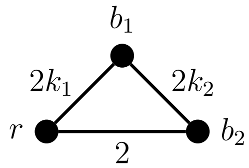
(ii) $(2k_1, 2, 2)$,

(iii) $(2k_1, 2k_2, 2)$,

(iv) $(2k_1, 2k_2, 2k_3)$.

with $k_i \geq 2$.

The Artin group associated to a triangle of type (i) is $\mathbb{Z}^3$, so it is poly-free. Artin groups associated to type (ii) triangles are even of FC type and so we know that it is also poly-free by [11] (In fact, these groups are of the form $A_2(2k_1) \times \mathbb{Z}$ so they are obviously poly-free). And Artin groups associated to triangles of type (iv) are large even Artin groups and thus they are also poly-free by Theorem 3.62. The only remaining case are Artin groups associated to triangles of type (iii).



The problem for this case is that as far as we know there are not known normal forms for the associated group.

However, notice that in every proof along the chapter we have only used normal forms in the small subgroup $A_{\Gamma\setminus\{r\}}$, never in the big Artin group A_Γ. Thus, almost the exact same proof that we have used to prove poly-freeness for large even Artin groups works also for any even Artin group Γ satisfying that there exists a vertex $r \in V(\Gamma)$ such that $A_{\Gamma\setminus\{r\}}$ is a large even Artin group.

The only place where we need to change a bit our proof is in Lemma 3.46 since Lemma 3.45 is not true for this Artin group. In this way, in $G_1 \ltimes K$ we could not have discarded the relations $[Stab(r), r]$. We are going to give a different proof for Lemma 3.46 in this situation.

Lemma 3.63. *Let* $A_\Gamma = \langle r, b_1, b_2 \mid (rb_1)^{k_1} = (b_1r)^{k1}, rb_2 = b_2r, (b_1b_2)^{k_2} = (b_2b_1)^{k_2}\rangle$ *and* $A_{\Gamma_1} = A_{\Gamma\setminus\{r\}} = \langle b_1, b_2 \mid (b_1b_2)^{k_2} = (b_2b_1)^{k_2}\rangle$*. Let us consider the map:*

$$\psi : A_\Gamma \longrightarrow A_{\Gamma_1},$$

induced by $r \mapsto 1$ *and* $b_i \mapsto b_i$ *for* $i = 1, 2$.

$\ker(\psi)$ *is isomorphic to:*

$$K := \langle r^{sl(g)}; g \in A_{\Gamma_1} \mid \hat{R}\rangle,$$

where $\hat{R}$ *is the set of relations defined in Definition 3.42.*

Proof. Note that $\ker(\psi)$ is the normal subgroup of A_Γ generated by r.

Notice that in this case we have $p_1^+ = 2, p_1^- = 0, n_1^- = -1, n_1^+ = 1, p_2^+ = 1, p_2^- = 0, n_2^- = -1, n_2^+ = 0$. Hence, we have $\hat{R} = \hat{R}^+ \cup \hat{R}^-$ with:

$$R^+ = \{r^{sl(b_1^2 g)} = r^{sl(b_1 g)} r^{sl(g)} \left(r^{sl(b_1^{-1}g)}\right)^{-1}; b_1^2 g \in \Omega_1^+\} \cup \{r^{sl(b_2 g)} = r^{sl(g)}; b_2 g \in \Omega_2^+\}$$

$$R^- = \{r^{sl(b_1^{-1}g)} = \left(r^{sl(g)}\right)^{-1} r^{sl(b_1 g)} r^{sl(g)}; b_1^{-1} g \in \Omega_1^-\} \cup \{r^{sl(b_2^{-1}g)} = r^{sl(g)}; b_2^{-1} g \in \Omega_2^-\}$$

As before, we define an action of $G_1 = A_{\Gamma_1} = A_{\Gamma\setminus\{r\}} = \langle b_1, b_2 \mid (b_1 b_2)^{k_2} = (b_2 b_1)^{k_2}\rangle$ on the abstract group K via

$$h^{-1}(r^{sl(g)})h = r^{sl(gh)}, \quad h \in G_1.$$

The proof is exactly the same as in Lemma 3.46 until we obtain the presentation of the semidirect product applying Lemma 3.44. In our case, we obtain

$$G_1 \ltimes K = \langle r, b_1, b_2 \mid \hat{C}_0, T, [Stab_{G_1}(r), r]\rangle.$$

where $\hat{C}_0 = \hat{C}^+ \cup \hat{C}^-$, $T = \{(b_1 b2)^{k_2} = (b_2 b_1^{k_2})\}$ with

$$\begin{aligned}\hat{C}^+ = \{&sl(b_1^2 g)^{-1} r sl(b_1^2 g) = sl(b_1 g)^{-1} r sl(b_1 g) sl(g)^{-1} r sl(g) (sl(b_1 g)^{-1} r sl(b_1 g))^{-1};\\ &b_1^2 g \in \Omega_1^+\} \cup \{sl(b_2 g)^{-1} r sl(b_2 g) = sl(g)^{-1} r sl(g); b_2 g \in \Omega_2^+\}\\ \hat{C}^- = \{&sl(b_1^{-1} g)^{-1} r sl(b_1^{-1} g) = (sl(g)^{-1} r sl(g))^{-1} sl(b_1 g)^{-1} r sl(b_1 g) sl(g)^{-1} r sl(g);\\ &b_1^{-1} g \in \Omega_1^-\} \cup \{sl(b_2^{-1} g)^{-1} r sl(b_2^{-1} g) = sl(g)^{-1} r sl(g); b_2^{-1} g \in \Omega_2^-\}\end{aligned}$$

Since in our presentation we have $T = \{(b_1 b_2)^{k_2} = (b_2 b_1^{k_2})\}$, the only relation of G_1, we can rewrite the relations of $\hat{C}_0$ in the following way:

$$\begin{aligned}\hat{C}_1^+ = \{&(b_1^2 g)^{-1} r (b_1^2 g) = (b_1 g)^{-1} r (b_1 g)(g)^{-1} r(g)((b_1 g)^{-1} r (b_1 g))^{-1}; b_1^2 g \text{ geodesic}\} \cup\\ \{&(b_2 g)^{-1} r (b_2 g) = (g)^{-1} r (g); b_2 g \text{ geodesic}\}\\ \hat{C}_1^- = \{&(b_1^{-1} g)^{-1} r (b_1^{-1} g) = ((g)^{-1} r(g))^{-1} (b_1 g)^{-1} r (b_1 g)(g)^{-1} r(g); b_1^{-1} g \text{ geodesic}\} \cup\\ \{&(b_2^{-1} g)^{-1} r (b_2^{-1} g) = (g)^{-1} r(g); b_2^{-1} g \text{ geodesic}\}\\ \hat{C}_1 = \hat{C}&^+ \cup \hat{C}^-\end{aligned}$$

So we get the presentation:

$$G_1 \ltimes K = \langle b_1, b_2, r \mid \hat{C}_1, T, [Stab_{G_1}(r), r]\rangle.$$

We define:

$$C_1' = \{b_1^{-2} r b_1^2 = b_1^{-1} r b_1 r (b_1^{-1} r b_1)^{-1}, b_2^{-1} r b_2 = r\}.$$

So in fact, the relations in $\hat{C}_1 \setminus C'_1$ are obtained from the ones of C'_1 by conjugation. So we can eliminate them from the presentation using Tietze transformations. Thus, we have:

$$G_1 \ltimes K = \langle b_1, b_2, r \mid C'_1, T, [Stab_{G_1}(r), r] \rangle.$$

Note that $\{C'_1, T\}$ is in fact the set of relations in A_Γ. Therefore, we have an epimorphism

$$A_\Gamma \twoheadrightarrow G_1 \ltimes K.$$

To end the proof it is enough to see that the relations $[Stab_{G_1}(r), r]$ are also satisfied in A_Γ.

Let us consider $g \in Stab_{G_1}(r)$, therefore by the definition of our action $r^{sl(g)} =_K r$. Now, take into account the following property of the set of relations $\hat{R}$. By construction, each time that we have a relation $\hat{R}^+$:

$$r^{sl(b_i^{p_i^+} g)} = r^{sl(b_i^{p_i^+ -1} g)} \dots r^{sl(b_i^{p_i^-} g)} \dots \left(r^{sl(b_i^{pi-1} g)} \right)^{-1},$$

then in the original A_Γ it is satisfied:

$$r^{b_i^{p_i^+} g} =_{A_\Gamma} r^{b_i^{p_i^+ -1} g} \dots r^{b_i^{p_i^-} g} \dots \left(r^{b_i^{pi-1} g} \right)^{-1}.$$

Analogously, we have a similar situation for the relations of $\hat{R}^-$.

Therefore, $r^{sl(g)} =_K r$ implies that $r^g =_{A_\Gamma} r$, i.e. $g^{-1} r g =_{A_\Gamma} r$. Thus, $g^{-1} r g =_{A_\Gamma} r$ for every $g \in Stab_{G_1}(r)$.

But, $[Stab_{G_1}(r), r] = \{g^{-1} r g = r \mid g \in Stab_{G_1}(r)\}$, so the relations $[Stab_{G_1}(r), r]$ are also satisfied in A_Γ.

Thus, we have:

$$G_1 \ltimes K = \langle b_1, b_2, r \mid C'_1, T \rangle \simeq A_\Gamma$$

and the isomorphism maps K onto $\ker(\psi)$. Therefore, $K := \langle r^{sl(g)}; g \in G_1 \mid \hat{R} \rangle = \langle\langle r \rangle\rangle = \ker(\psi)$. □

Remark 3.64. *Notice that this proof could have also been applied to the case of large Artin groups proved before. But we have preferred to present the proofs in this way because we think that Lemma 3.45 has importance by itself.*

As we have commented before, the rest of the proof for polyfreeness works in almost the same way as for large Artin groups. The proof of Proposition 3.61 can be used also in this case just taking into account the following remarks:

- The proof of Lemma 3.48 cannot be used in this case, but in our particular case we have $n_1^- = n_2^- = -1$, so we don't need to apply it.
- It is not necessary to use Remark 3.60 in order to check different cases. Since we are considering a particular case, we know that for this group $\alpha_1 = a^{-1}b^{-1}a^{-1}b^{-1} =_G b^{-1}a^{-1}b^{-1}a^{-1}$.

Therefore:

Corollary 3.65. *The Artin group based on the triangle graph* $(2k_1, 2k_2, 2)$ *is poly-free.*

Corollary 3.66. *Any even Artin group based on a triangle graph is poly-free.*

Chapter 4

Residually finiteness

In this chapter we will study the notion of residually finiteness for some families of even Artin groups (and also for a family of general Artin groups). In Section 4.1 we will define residually finiteness and we will see some background about this property. Later on, in Section 4.2 we will prove that even Artin groups of FC type are residually finite, this result also appears in [11]. Finally, in section 4.3 we will prove residually finiteness for other families of Artin groups like Artin groups based on forest graphs. These results have been published in [10] and this last section is based on that paper.

4.1 Definitions and background

Definition 4.1. *A group G is said to be* residually finite *if, for every $g \in G \setminus \{1\}$, there is a normal subgroup of finite index in G not containing g. Equivalently, G is residually finite if for every $1 \neq g \in G$ there exists a homomorphism α from G to a finite group such that $\alpha(g) \neq 1$.*

It is well-known that being residually finite is not closed under short exact sequences, in the sense that, if N is a normal subgroup of G and both N and G/N are residually finite, then one cannot deduce the same for G itself. However, the situation changes if we work under some extra hypothesis. For example, a direct product of residually finite groups is residually finite. This can be generalized to the following result of Boler–Evans [12] that will be crucial in our argument.

Theorem 4.2 (Boler–Evans [12]). *Let G_1, G_2 be residually finite groups, and let $K \leq G_1, G_2$ such that both split as $G_i = H_i \rtimes K$. Then the amalgamated free product $G = G_1 *_K G_2$ is residually finite.*

For the sake of completeness we will reproduce a proof of this theorem.

Proposition 4.3. *Let G be a group fitting in a split short exact sequence:*

$$1 \longrightarrow K \longrightarrow G \overset{\pi}{\rightleftarrows} H \longrightarrow 1$$

i.e. $G = K \rtimes H$.

Let us suppose that K, H are residually finite and K is finitely generated. Then G is residually finite.

Proof. Let us consider $g \in G$, $g \neq 1$. If $h = \pi(g) \neq 1$,then as H is residually finite, there exists $\varphi : H \longrightarrow F$ with F finite such that $\varphi(h) \neq 1$. So for $\psi = \varphi \circ \pi : G \longrightarrow F$, $\psi(g) = \varphi(h) \neq 1$.

On the other hand, if $\pi(g) = 1$, $g \in K$ and as $g \neq 1$, there exists a finite index subgroup L_1 such that $g \notin L_1$. Put $m = [K : L_1]$, since K is finitely generated there are only finitely many subgroups: $L_1, L_2, ..., L_l$ of index m ([52]).

We define $M = \cap_{i=1}^{l} L_i$. Then M is of finite index (it is intersection of finite index subgroups), $g \notin M$ and M is a characteristic subgroup.

Therefore, we get:

$$H \xrightarrow{f_1} Aut(K) \xrightarrow{f_2} Aut(K/M)$$

where f_1 is defined by the semidirect product and f_2 from M being characteristic.

We define $\varphi_1 = f_2 \circ f_1 : H \longrightarrow Aut(K/M)$ and $\pi_1 : K \longrightarrow K/M$ the projection.

Let us consider:

$$\psi : K \rtimes H \longrightarrow K/M \rtimes Aut(K/M)$$
$$(k, h) \mapsto (\pi_1(k), \varphi_1(h))$$

which clearly is a homomorphism and verifies that $\psi(g) = (\pi_1(g), 1) \neq 1$ (since $g \notin M$ so $\pi_1(g) \neq 1$).

As $K/M \rtimes Aut(K/M)$ is finite, we deduce that G is residually finite. □

Lemma 4.4. *Let us consider*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1.$$

If K, H are residually finite, K is finitely generated and $Z(K) = \{1\}$, G is residually finite.

Proof. Let us consider $g \in G$, $g \neq 1$. If $h = \pi(g) \neq 1$, then as H is residually finite there exists $\varphi : H \longrightarrow F$ with F finite such that $\varphi(h) \neq 1$. So for $\psi = \varphi \circ \pi : G \longrightarrow F$, $\psi(g) = \varphi(h) \neq 1$.

On the other hand, if $\pi(g) = 1$, $g \neq 1$, we choose $g_1 \in K$ such that $gg_1 \neq g_1 g$, and we consider $h = g_1 g g_1^{-1} g^{-1} \in K \setminus \{1\}$. By the same argument as in the proof of Proposition 4.3, there exists a characteristic subgroup of finite index $N \lhd_f K$ such that $h \notin N$.

Therefore, we get:

$$G \xrightarrow{f_1} Aut(K) \xrightarrow{f_2} Aut(K/M)$$

where f_1 is defined by the conjugation and f_2 from M being characteristic.

We define $\varphi = f_2 \circ f_1 : G \longrightarrow Aut(K/M)$ and $\pi : K \longrightarrow K/M$ the projection. We will denote $\bar{k} = \pi(k)$.

Let us see that $\varphi(g) \neq Id$.

$$\varphi(g)(\bar{g}_1)\bar{g}_1^{-1} = \bar{g}\bar{g}_1\bar{g}^{-1}\bar{g}_1^{-1} \neq 1.$$

Therefore, $\varphi(g)(\bar{g}_1) \neq \bar{g}_1$ so $\varphi(g) \neq Id$. Thus, G is residually finite. □

Our following lemma will be an inmediate consequence of this theorem due to Gruenberg:

Theorem 4.5. *[50] If $\mathcal{P}$ is a root property, then every free product of residually $\mathcal{P}$ groups is itself residually $\mathcal{P}$ if and only if every free group is residually $\mathcal{P}$.*

Just applying it to the case where $\mathcal{P}$ is the property of being finite, we obtain:

Lemma 4.6. *If G_1 and G_2 are residually finite, then $G = G_1 * G_2$ is also residually finite.*

So now we can prove Theorem 4.2:

Theorem 4.2. We consider:

$$\iota_i : K \hookrightarrow G_i$$

$$\pi_i : G_i \to K$$

such that $\pi_i \circ \iota_i = Id_K$. Then $H_i = Ker(\pi_i)$.

We have the sequence:

$$1 \longrightarrow H_i \longrightarrow G_i \overset{\pi_i}{\rightleftarrows} K \longrightarrow 1.$$

Then, $G_i = H_i \rtimes K$. We have homomorphisms:

$$\varphi_i : K \longrightarrow Aut(H_i)$$

Let us consider $H = H_1 * H_2$, we define:

$$\varphi : K \longrightarrow Aut(H)$$

such that $\varphi(k)|_{H_i} = \varphi_i(k)$ (i.e. $\varphi = \varphi_1 * \varphi_2$).

Thus we have $G = H \rtimes_\varphi K$. We take $g \in G$. Let us consider $\pi : G \to K$, $H = Ker\pi$. If $k = \pi(g) \neq 1$, there exists $\alpha : K \longrightarrow F$ with F finite such that $\varphi(k) \neq 1$ (since K is residually finite). So for $\psi = \alpha \circ \pi : G \longrightarrow F$, $\psi(g) = \alpha(k) \neq 1$.

If $\pi(g) = 1$, $g \in H$, we write $g = g_1 g_2 ... g_l$ where:

$$\begin{gathered} g_i \in H_1 \setminus \{1\} \text{ or } g_i \in H_2 \setminus \{1\} \\ \text{and} \\ g_i \in H_1 \setminus \{1\} \implies g_{i+1} \in H_2 \setminus \{1\} \\ g_i \in H_2 \setminus \{1\} \implies g_{i+1} \in H_1 \setminus \{1\} \end{gathered}$$

We can suppose without loss of generality that $g_{2i+1} \in H_1 \setminus \{1\}$ and $g_{2i} \in H_2 \setminus \{1\}$ for all i.

There exist N_1, N_2 such that:

$$N_1 \lhd_f G_1, \quad g_1, g_3, ... \notin N_1$$

$$N_2 \lhd_f G_2, \quad g_2, g_4, ... \notin N_2$$

We will denote $\bar{N}_j = N_j \cap H_j$, then:

$$\bar{N}_1 \lhd_f H_1, \quad g_1, g_3, ... \notin \bar{N}_1$$

$$\bar{N}_2 \lhd_f H_2, \quad g_2, g_4, ... \notin \bar{N}_2$$

Consider $X = \langle\langle \bar{N}_1 \cup \bar{N}_2 \rangle\rangle$. By definition $X \lhd H$. Take $k \in K \subset G_j$ then $kN_jk^{-1} = N_j$ since N_j is normal. It is also true that $kH_jk^{-1} = H_j$ (since it is the kernel). Therefore, $k\bar{N}_jk^{-1} = \bar{N}_j$. Moreover, $kXk^{-1} = X$ since it is invariant by the generators). Besides:

$$H/X \simeq (H_1/N_1) * (H_2/N_2)$$

which is finitely generated since it is a free product of finite groups.

So $G/X = H/X \rtimes K$ with the induced action. By Lemma 4.6 H/X is residually finite and therefore by Proposition 4.3 so is G/X.

Now, let us consider $\pi : G \longrightarrow G/X$. By construction $\pi(g) \neq 1$. And since G/X is residually finite, there exists a homomorphism α from G/X to a finite group such that $\alpha\pi(g) \neq 1$. Hence we have that also G is residually finite. $\square$

4.2 Residually finiteness on even Artin groups of FC type

In this section we will show that even Artin groups of FC type are residually finite.

Theorem 4.7. *Every even Artin group of FC type is residually finite.*

Proof. We argue by induction on the number of vertices of Γ. Assume first that Γ is complete. Then, by definition, $\mathcal{A}_\Gamma$ is of spherical type, hence, by Cohen–Wales [31] and Digne [38], $\mathcal{A}_\Gamma$ is linear, and therefore $\mathcal{A}_\Gamma$ is residually finite.

Assume that Γ is not complete. Then we can choose two distinct vertices $s, t \in S = V(\Gamma)$ such that $m_{s,t} = \infty$. Set $X = S \setminus \{s\}$, $Y = S \setminus \{t\}$ and $Z = S \setminus \{s, t\}$. From the presentation of $\mathcal{A}$ follows that $\mathcal{A} = \mathcal{A}_X *_{\mathcal{A}_Z} \mathcal{A}_Y$. Moreover, since A is even, the inclusion map $\mathcal{A}_Z \hookrightarrow \mathcal{A}_X$ has a retraction $\pi_{X,Z} : \mathcal{A}_X \to \mathcal{A}_Z$ which sends r to r for all $r \in Z$ and sends t to 1, hence $\mathcal{A}_X = \mathrm{Ker}(\pi_{X,Z}) \rtimes \mathcal{A}_Z$. Similarly, the inclusion map $\mathcal{A}_Z \hookrightarrow \mathcal{A}_Y$ has a retraction $\pi_{Y,Z} : \mathcal{A}_Y \to \mathcal{A}_Z$, hence $\mathcal{A}_Y = \mathrm{Ker}(\pi_{Y,Z}) \rtimes \mathcal{A}_Z$. By the inductive hypothesis, $\mathcal{A}_X$ and $\mathcal{A}_Y$ are residually finite, hence, by Theorem 4.2, $\mathcal{A}$ is also residually finite. $\square$

4.3 Residually finiteness on Artin groups based on forest graphs

In this section we will see that the families of triangle free even Artin groups and general Artin groups based on forest are also residually finite.

Definition 4.8. *A graph Γ is* ***triangle free*** *if no full subgraph of Γ is a triangle.*

In fact, notice that every triangle free even Artin groups is an even Artin group of FC type, so the fact that it is residually finite is already proved by the previous result.

Moreover, it was also already known that Artin groups based on forest are residually finite since when Γ is a forest, the Artin group $\mathcal{A}_\Gamma$ is the fundamental group of a graph manifold by Gordon [49], and is thus virtually special by Przytycki-Wise [72], hence is linear and residually finite.

Nevertheless, our proofs are quite different to the previously known ones, and we include it here because of their interest.

Definition 4.9. *Given the set $S = V(\Gamma)$.A partition of S is a set $\mathcal{P}$ of pairwise disjoint subsets of S satisfying $\cup_{X\in\mathcal{P}} X = S$.*

We say that a partition is ***admissable*** *if, for all $X, Y \in \mathcal{P}$, $X \neq Y$, there is at most one edge in Γ connecting an element of X with an element of Y. In particular, if $s \in X$ and $t \in Y$ are connected in Γ by an edge and $s' \in X$, $s' \neq s$, then s' is not connected in Γ by any edge to any vertex of Y.*

An admissible partition $\mathcal{P}$ determines a new graph $\Gamma/\mathcal{P}$ defined as follows. The set of vertices of $\Gamma/\mathcal{P}$ is $\mathcal{P}$. Two distinct elements $X, Y \in \mathcal{P}$ are connected by an edge labelled by m if there exists $s \in X$ and $t \in Y$ such that $m_{s,t} = m(\neq \infty)$.

Lemma 4.10. *If a graph Γ has one or two vertices then $\mathcal{A}_\Gamma$ is residually finite.*

Proof. If Γ has only one vertex, then $\mathcal{A}_\Gamma \simeq \mathbb{Z}$ which is residually finite. Suppose that Γ has two vertices s, t. If $m_{s,t} = \infty$, then $\mathcal{A}_\Gamma$ is a free group of rank 2 which is residually finite. If $m_{s,t} \neq \infty$, then Γ is of spherical type, hence by Digne [38] and Cohen-Wales [31], $\mathcal{A}_\Gamma$ is linear, and therefore is residually finite. $\square$

Lemma 4.11. *Let Γ be a graph and let $\mathcal{A} = \mathcal{A}_\Gamma$. Let $s \in S$. We set $Y = S \setminus \{s\}$, we denote $\Gamma_1, ..., \Gamma_l$ the connected components of Γ_Y, and for $i \in \{1, ..., l\}$, we denote by Y_i the set of vertices of Γ_i. If $\mathcal{A}_{Y_i\cup\{s\}}$ is residually finite for all $i \in \{1, ..., l\}$, then A is residually finite.*

Proof. We argue by induction on l. If $l = 1$; then $Y \cup \{s\} = S$ and $\mathcal{A}_{Y\cup\{s\}} = A$, so A is obviously residually finite. Suppose that $l \geq 2$ and that the inductive hypothesis holds. We set $X_1 = Y_1 \cup ... \cup Y_{l-1} \cup \{s\}$, $X_2 = Y_l \cup \{s\}$ and $X_0 = \{s\}$. Let $G_1 = \mathcal{A}_{X_1}$, $G_2 = \mathcal{A}_{X_2}$, and $L = \mathcal{A}_{X_0} \simeq \mathbb{Z}$. The groups G_1 and G_2 are residually finite by induction. It is easily seen in the presentation of A that $A = G_1 *_L G_2$. Furthermore, the homomorphism $\rho_1 : G_1 \to L$ which sends t to s for all $t \in X_1$ is a retraction of the inclusion map $L \hookrightarrow G_1$, hence G_1 splits as a semi-direct product $G_1 = H_1 \rtimes L$. Similarly, G_2 splits as a semi-direct product $G_2 = H_2 \rtimes L$. We conclude by Theorem 4.2 that $\mathcal{A}$ is residually finite. $\square$

Lemma 4.12. *Let Γ be a graph, let $\mathcal{A} = \mathcal{A}_\Gamma$, and let $\mathcal{P}$ be an admissible partition of S such that*

a the group $\mathcal{A}_X$ is residually finite for all $X \in \mathcal{P}$,

b the graph $\Gamma/\mathcal{P}$ has at most two vertices.

Then $\mathcal{A}$ is residually finite.

Proof. If $|\mathcal{P}| = 1$ there is nothing to prove. Suppose that $|\mathcal{P}| = 2$ and one of the elements of $\mathcal{P}$ is a singleton. We set $\mathcal{P} = \{X, Y\}$ where $X = S \setminus \{t\}$ and $Y = \{t\}$ for some $t \in S$. If there is no edge in Γ connecting t to an element of X, then $\mathcal{A} = \mathcal{A}_X * \mathcal{A}_Y$, hence $\mathcal{A}$ is residually finite. So, we can assume that there is an edge connecting t to an element $s \in X$. Notice that this element is unique by definition of admissibility. We denote by $\Gamma_1, ..., \Gamma_l$ the connected components of $\Gamma_{X\setminus\{s\}}$ and, for $i \in \{1, ..., l\}$, we denote X_i the set of vertices of Γ_i. For all $i \in \{1, ..., l\}$ the group $\mathcal{A}_{X_i\cup\{s\}}$ is residually finite since $\mathcal{A}_{X_i\cup\{s\}} \subset \mathcal{A}_X$. On the other hand, $\mathcal{A}_{\{s,t\}}$ is residually finite by Lemma 4.10. Noticing that the connected components of $\Gamma_{S\setminus\{s\}}$ are precisely $\Gamma_1, ..., \Gamma_l$ and $\{t\}$, we deduce from Lemma 4.11 that A is residually finite.

Now assume that $|\mathcal{P}| = 2$ and both elements of $\mathcal{P}$ are of cardinality ≥ 2. Set $\mathcal{P} = \{X, Y\}$. If there is no edge in Γ connecting an element of X with an element of Y, then $\mathcal{A} = \mathcal{A}_X * \mathcal{A}_Y$, hence $\mathcal{A}$ is residually finite. So, we can assume that there is an edge connecting an element $s \in X$ to an element $t \in Y$. Again, this edge is unique. Let $\Omega_1, ..., \Omega_p$ be the connected components of $\Gamma_{X\setminus\{s\}}$ and let $\Gamma_1, ..., \Gamma_q$ be the connected components of Γ_Y. We denote by X_i the set of vertices of Ω_i for all $i \in \{1, ..., p\}$ and by Y_j the set of vertices of Γ_j for all $j \in \{1, ..., q\}$. The group $\mathcal{A}_{X_i\cup\{s\}}$ is residually finite since $X_i \cup \{s\} \subset X$ for all $i \in \{1, ..., p\}$, and, by the above, the group $\mathcal{A}_{y_j\cup\{s\}}$ is residually finite for all $j \in \{1, ..., q\}$. It follows by Lemma 4.11 that $\mathcal{A}$ is residually finite. $\square$

Remark 4.13. *Alternative arguments from Pride [71] and/or from Burillo-Martino [24] can be used to prove partially or completely Lemma 4.12.*

Therefore, we can obtain our main results:

Theorem 4.14. *Let Γ be a graph, let $\mathcal{A} = \mathcal{A}_\Gamma$, and let $\mathcal{P}$ be an admissible partition of S such that*

a) the group $\mathcal{A}_X$ is residually finite for all $X \in \mathcal{P}$,

b) the graph $\Gamma/\mathcal{P}$ is even and triangle free.

Then $\mathcal{A}$ is residually finite.

Proof. We argue by induction on the cardinality $|\mathcal{P}|$ of $\mathcal{P}$. The case $|\mathcal{P}| \leq 2$ is covered by Lemma 4.12. So, we can suppose that $|\mathcal{P}| \geq 3$ and that the inductive hypothesis holds. Since $\Gamma/\mathcal{P}$ is triangle free, there exists $X, Y \in \mathcal{P}$ such that none of the elements of X is connected to an element of Y. We set

$U_1 = S \setminus X$, $U_2 = S \setminus Y$, and $U_0 = S \setminus (X \cup Y)$. We have $\mathcal{A} = \mathcal{A}_{U_1} *_{\mathcal{A}_{U_0}} \mathcal{A}_{U_2}$ and, bu inductive hypothesis, $\mathcal{A}_{U_1}$ and $\mathcal{A}_{U_2}$ are residually finite. Since $\Gamma/\mathcal{P}$ is even, the inclusion map $\mathcal{A}_{U_0} \hookrightarrow \mathcal{A}_{U_1}$ admits a retraction $\rho_1 : \mathcal{A}_{U_1} \to \mathcal{A}_{U_0}$ which send t to 1 if $t \in T$ and sends t to t if $t \in U_0$. Similarly, the inclusion map $\mathcal{A}_{U_0} \hookrightarrow \mathcal{A}_{U_2}$ admits a retraction $\rho_2 : \mathcal{A}_{U_2} \to \mathcal{A}_{U_0}$. By Theorem 4.2 it follows that $\mathcal{A}$ is residually finite. □

Theorem 4.15. *Let Γ be a graph, let $\mathcal{A} = \mathcal{A}_\Gamma$, and let $\mathcal{P}$ be an admissible partition of S such that*

a) the group $\mathcal{A}_X$ is residually finite for all $X \in \mathcal{P}$,

b) the graph $\Gamma/\mathcal{P}$ is a forest.

Then $\mathcal{A}$ is residually finite.

Proof. We argue by induction on the cardinality $|\mathcal{P}|$ of $\mathcal{P}$. The case $|\mathcal{P}| \le 2$ is covered by Lemma 4.12. So, we can assume that $|\mathcal{P}| \ge 3$ and that the inductive hypothesis holds. Set $\Omega = \Gamma/\mathcal{P}$. Let $\Omega_1, ..., \Omega_l$ be the connected components of Ω. For $i \in \{1, ..., l\}$ we denote by $\mathcal{P}_i$ the set of vertices of Ω_i and we set $Y_i = \cup_{X \in \mathcal{P}_i} X$ and $\Gamma_i = \Gamma_{Y_i}$. The set $\mathcal{P}_i$ is an admissible partition of Y_i and $\Gamma_i/\mathcal{P}_i = \Omega_i$ is a tree for all $i \in \{1, ..., l\}$. Moreover, we have $\mathcal{A} = \mathcal{A}_{Y_1} * ... * \mathcal{A}_{Y_l}$. hence A is residually finite if and only if $\mathcal{A}_{Y_i}$ is residually finite for all $i \in \{1, ..., l\}$. So, we can assume that $\Omega = \Gamma/\mathcal{P}$ is a tree.

Since $|\mathcal{P}| \ge 3$, Ω has a vertex X of valence ≥ 2. Choose $Y \in \mathcal{P}$ connected to X by an edge of Ω. Let $s \in X$ and $t \in Y$ such that s and t are connected by an edge of Γ. Recall that by definition s and t are unique. Let $\mathcal{Q}'$ be the connected component of $\Omega_{\mathcal{P} \setminus \{X\}}$ containing Y, let $\mathcal{P}_{\mathcal{Q}'}$ be the set of vertices of $\mathcal{Q}'$, let $U' = \cup_{Z \in \mathcal{P}_{\mathcal{Q}'}} Z$, let $U = U' \cup \{s\}$, and let $\mathcal{P}_\mathcal{Q} = \mathcal{P}_{\mathcal{Q}'} \cup \{\{s\}\}$. Observe that $\mathcal{P}_\mathcal{Q}$ is an admissible partition of U, that $\mathcal{A}_Z$ is residually finite for all $Z \in \mathcal{P}_\mathcal{Q}$, that $\Gamma_U/\mathcal{P}_\mathcal{Q}$ is a tree, and that $|\mathcal{P}_\mathcal{Q}| < |\mathcal{P}|$. By the inductive hypothesis it follows that $\mathcal{A}_U$ is residually finite. Let R be connected component of $\Omega_{\mathcal{P} \setminus \{Y\}}$ contatining X, let $\mathcal{P}_R$ be the set of vertices of R, and let $V = \cup_{Z \in \mathcal{P}_R} Z$. Observe that $\mathcal{P}_R$ is an admissible partition of V, that $\mathcal{A}_Z$ is residually finite for all $Z \in \mathcal{P}_R$, that $\Gamma_V/\mathcal{P}_R$ is a tree, and that $|\mathcal{P}_R| < |\mathcal{P}|$. By the inductive hypothesis it follows that $\mathcal{A}_V$ is residually finite. Let $\Delta_1, ..., \Delta_q$ be the connected components of $\Gamma_{S \setminus \{s\}}$. Let $i \in \{1, ..., q\}$. Let Z_i be the set of vertices of Δ_i. It is easily seen that either $Z_i \cup \{s\} \subset U$, or $Z_i \cup \{s\} \subset V$, hence by the above, $\mathcal{A}_{Z_i \cup \{s\}}$ is residually finite. We conclude by Lemma 4.11 that $\mathcal{A}$ is residually finite. □

And, from Lemma 4.14 and Lemma 4.15 we obtain:

Corollary 4.16. *Let Γ be a graph, let $\mathcal{A} = \mathcal{A}_\Gamma$.*

(1) If Γ is even and triangle free, then $\mathcal{A}$ is residually finite.

(2) If Γ is a forest, then $\mathcal{A}$ is residually finite.

Chapter 5

Quasi-projectivity

The question of classification of quasi-projective groups, which today is referred to as Serre's question, has been frequently alluded to since Zariski [76] and Van Kampen [59] proposed it for complements of curves in the projective plane. The search for properties of such groups goes back to Enriques [46] and O.Zariski [77, Chapter VIII]. This has developed in the search for obstructions for a group to be quasi-projective (resp. quasi-Kähler) starting with Morgan [67], Kapovich-Millson [60], Arapura [3, 4], Libgober [62], Dimca [39], Dimca-Papadima-Suciu [40], and Artal-Cogolludo-Matei [5, 6].

In this chapter we concentrate in the possible characterization of quasi-projective Artin groups, as stated in [40, p. 451]. These results have been published in [9] in which this chapter is mainly based. Any proof of such results requires the use of obstructions to disregard the negative cases as well as the constructive part of finding realizations for the positive cases.

A first approach to this problem is given in [40, Thm. 11.7] where quasi-projective right-angled Artin groups are characterized by complete multipartite graphs corresponding to direct products of free groups. In the more general case of even Artin groups, that is, Artin groups associated with even-labelled graphs, the label plays an important role and not all multipartite graphs produce quasi-projective Artin groups.

In order to describe such graphs we define the concept of QP-*irreducible graph*. In this context, *graph* means for us simple graph. Let us denote by $\mathcal{G}_{\text{QP}}$ the family of labelled graphs whose associated Artin groups are quasi-projective. Given two labelled graphs $\Gamma_1 = (V_1, E_1, m_1)$, $\Gamma_2 = (V_2, E_2, m_2)$, we define their 2-join $\Gamma_1 *_2 \Gamma_2 = (V, E, m)$ as the labelled graph given by the join of Γ_1 and Γ_2 whose connecting edges have all label 2, that is

$$m(e) = \begin{cases} m_i(e) & \text{if } e \in E_i \\ 2 & \text{if } e \in E \setminus (E_1 \cup E_2). \end{cases}$$

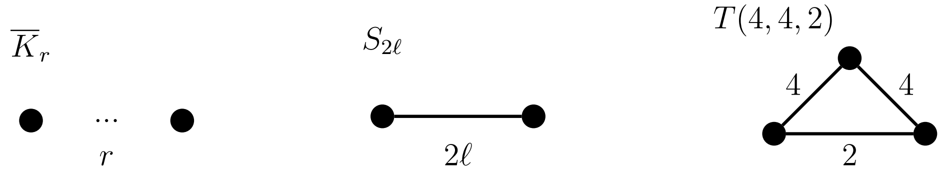


Figure 5.1: QP-irreducible graphs of type $\overline{K}_r$, $S_{2\ell}$, and $T(4,4,2)$.

We say $\Gamma \in \mathcal{G}_{\text{QP}}$ is a QP-irreducible graph if Γ is not a 2-join of two graphs in $\mathcal{G}_{\text{QP}}$.

Denote by $\overline{K}_r$ a disjoint graph with r vertices and no edges. Also denote by S_m the graph given by two vertices joined by an edge with label m. Finally, denote by $T(4,4,2)$ the triangle as shown in Figure 5.1. It will be shown that these are the only QP-irreducible even graphs. In other words, the main result of this chapter is the following.

Theorem 5.1. *Let $\Gamma = (V, E, 2\ell)$ be an even-labelled graph and $\mathcal{A}_\Gamma$ its associated even Artin group. Then the following are equivalent:*

1. *$\mathcal{A}_\Gamma$ is quasi-projective, that is, $\Gamma \in \mathcal{G}_{\text{QP}}$.*
2. *Γ is the 2-join of finitely many copies of $\overline{K}_r$, $S_{2\ell}$, and T.*

Moreover, if $\Gamma \in \mathcal{G}_{\text{QP}}$, then $\mathcal{A}_\Gamma = \pi_1(X)$ where $X = \mathbb{P}^2 \setminus \mathcal{C}$ is a curve complement.

As mentioned in the Section 1.2, the $K(\pi,1)$ conjecture referred to an Artin group $\mathcal{A}_\Gamma$ claims that a certain space, that appears as a quotient of the complement of the Coxeter arrangement by the action of the Coxeter group associated to Γ is an Eilenberg-MacLane space whose fundamental group is $\mathcal{A}_\Gamma$ – or a $K(\mathcal{A}_\Gamma, 1)$ space – see for instance [70] for a detailed explanation of this conjecture. In the context of quasi-projective groups, we can also ask ourselves whether or not a quasi-projective Artin group is realizable by an Eilenberg-MacLane space.

Conjecture 5.2 (Quasi-projective $K(\pi,1)$ conjecture)**.** *Any quasi-projective Artin group $\mathcal{A}_\Gamma$ can be realized as $\mathcal{A}_\Gamma = \pi_1(X)$ for a smooth, connected, quasi-projective Eilenberg-MacLane space X.*

The other main result of this chapter is a positive answer to the quasi-projective $K(\pi,1)$ conjecture for even Artin groups.

Theorem 5.3. *Quasi-projective even Artin groups satisfy the quasi-projective $K(\pi, 1)$ conjecture.*

This chapter is organized as follows: in section 5.1 the general definition of quasi-projective groups will be given as well as the notion of characteristic varieties as an invariant of a group. Section 5.2 will be devoted to studying kernels of cyclic quotients of Artin groups, called co-cyclic subgroups. Section 5.3 focuses on the problem of finding QP-irreducible graphs. The main theorems will be proved in section 5.4.

5.1 Settings and definitions

5.1.1 Quasi-projective groups

The main focus of this section is the study of those groups that can appear as fundamental groups in an algebraic geometry context, in particular as fundamental groups of smooth connected quasi-projective varieties. Recall that a *quasi-projective* variety is the complement of a hypersurface in a projective variety defined simply as the zero locus of a finite number of homogeneous polynomials in $\mathbb{C}[x_0, ..., x_n]$.

Definition 5.4. *A group G is* quasi-projective *if $G = \pi_1(X)$ for a smooth connected quasi-projective variety X.*

Example 5.5. *Since the fundamental group of the complement of a smooth plane curve of degree d in $\mathbb{P}^2$ is the cyclic group $\mathbb{Z}_d$ and the complement of two lines in $\mathbb{P}^2$ has the homotopy type of $\mathbb{C}^*$, all cyclic groups are quasi-projective. Moreover, since the complement of $r + 1$ irreducible smooth curves $C_0, ..., C_r$ of degrees $d_i = \deg C_i$ intersecting transversally has fundamental group*

$$\pi_1(\mathbb{P}^2 \setminus C_0 \cup ... \cup C_r) = \mathbb{Z}^r \oplus \mathbb{Z}_d,$$

where $d = \gcd(d_0, ..., d_r)$, one immediately obtains that all abelian groups are quasi-projective.

This example points out that the quasi-projective variety whose fundamental group realizes a quasi-projective group is clearly not unique in any geometrical sense, since the torsion part d can be attained in many different ways.

Example 5.6. *On the other end of abelianization properties, the free group of rank r is also quasi-projective since it can be realized as the fundamental group of the complement of $r + 1$ points in the complex projective line $\mathbb{P}^1$.*

The following important properties of quasi-projective groups are well known.

Proposition 5.7.

1. *If G is a quasi-projective group and $K \subset G$ is a finite index subgroup of G, then K is also a quasi-projective group.*

2. *If G_1, G_2 are quasi-projective groups, then $G_1 \times G_2$ is also a quasi-projective group.*

We will denote the set of all quasi-projective varieties by Σ_Q.
We can consider the following family:

$\Sigma_H := \{\mathbb{P}^n \setminus V \mid V \text{ is a projective hypersurface } \} \subset \Sigma_Q.$

For now, we will focus on this subset. We can also consider the family:

$$\Sigma_C := \{\mathbb{P}^2 \setminus C \mid C \text{ is a projective curve}\} \subset \Sigma_H.$$

One interesting result involving these families is that:

Proposition 5.8. $\{\pi_1(F) \mid F \in \Sigma_C\} = \{\pi_1(F) \mid F \in \Sigma_H\}$

We get this result using a weak version of Lefschetz's hyperplane theorem.

Theorem 5.9. *(Lefschetz's hyperplane theorem)*
Let X be an n-dimensional complex projective variety in $\mathbb{P}^n$, and let Y be a generic hyperplane section of X such that $X \setminus Y$ is smooth. Then, the natural map:

$$\pi_k(Y) \longrightarrow \pi_k(X)$$

is an isomorphism for $k < n-1$ and is surjective for $k = n-1$.

A sketch of the proof of Proposition 5.8 would be to consider a hypersurface V and intersect it with a generic plane section H.

The inclusion map $i : H \cap (\mathbb{P}^n \setminus V) \longrightarrow \mathbb{P}^n \setminus V$ defines a morphism of fundamental groups:

$$I : \pi_1(H \cap (\mathbb{P}^n \setminus V)) \longrightarrow \pi_1(\mathbb{P}^n \setminus V)$$

which by Lefschetz theorem (Theorem 5.9) is an isomorphism if $n > 2$.

Applying the theorem repeatedly we get to $H \setminus (V \cap H)$ on the left side where $H = \mathbb{P}^2$ and $V \cap H$ is a curve.

We will study $\{\pi_1(\mathbb{P}^n \setminus V) \mid V \subset \mathbb{P}^n \text{ hypersurface}\}$ and for that it will be enough to compute $\{\pi_1(\mathbb{P}^2 \setminus C) \mid C \subset \mathbb{P}^2 \text{ curve}\}$.

We will focus now on computing $\pi_1(\mathbb{P}^2 \setminus C)$.

5.1.2 Computing $\pi_1(\mathbb{P}^2 \setminus C)$ using braid monodromy

The objective of this section is to compute $\pi_1(\mathbb{P}^2 \setminus C)$. Before starting, we are going to give some basic notions about the action of braids in free groups which will be useful in the process.

Action of a braid on a free group

Let $\alpha \in \mathbb{B}r_d(\mathbb{C})$ (group of braids) and $\mathbb{F}_d$ the free group of rank d.

There exists a right action (the action which permutes the d meridians we have talked earlier):

$$\nabla : \mathbb{B}r_d(\mathbb{C}) \times \mathbb{F}_d \longrightarrow \mathbb{F}_d$$

such that

$$\nabla(\sigma_i, \mu_j) = \mu_j^{\sigma_i} = \begin{cases} \mu_j & \text{if } j < i \text{ or } j > i+1 \\ \mu_{i+1} & \text{if } j = i \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} & \text{if } j = i+1 \end{cases}$$

Besides, $\nabla(\alpha\beta, \mu) = \mu^{\alpha\beta} = \nabla(\beta, \nabla(\alpha, \mu))$.

Thus, it is straight forward to check that if $d = 2$ ($\mathbb{F}_d = \langle a, b \rangle$), $a^{\sigma^n} = a$ is the relation corresponding to two joined vertices with label n in an Artin group, i.e. if $n = 2k$ we get $(ab)^k = (ba)^k$.

Computation of $\pi_1(\mathbb{P}^2 \setminus C)$

We choose adequate coordinates such that $P = [0 : 1 : 0] \notin C$ and $\{z = 0\} \not\subset C$.

Since $\mathbb{P}^2 \setminus \{z = 0\} = \mathbb{C}^2$, we have $\pi_1(\mathbb{P}^2 \setminus (C \cup \{z = 0\})\} = \pi_1(\mathbb{C}^2 \setminus C)$. Our curve C will be the set of zeros of a homogeneous polynomial $f(x, y, z)$ of degree d, i.e. $C = \{f(x, y, z) = 0\}$. So:

Proposition 5.10. *$P \notin C \iff f(0, 1, 0) \neq 0$, which happens if and only if $f(x, y, z)$ has degree d in the variable y.*

Besides:

$$\mathbb{P}^2 \setminus \{z = 0\} = \{[x : y : z] \mid z \neq 0\} = \{[\tfrac{x}{z} : \tfrac{y}{z} : 1]\} \leftrightarrow \{(x, y) \in \mathbb{C}^2\}.$$

Considering $[u : v : 1] \mapsto (u, v)$ and $(x, y) \mapsto [x : y : 1]$, we have an isomorphism between the last two sets. The equation of $C \cap \mathbb{C}^2$ is $\{f(x, y, 1) = 0\}$ (From now on we will consider $f(x, y) = f(x, y, 1)$) .

Remark 5.11. *To recover the original equation of the curve from this one, we only have to homogenize the polynomial, i.e. we have to divide x and y by z and then multiply by z^d the whole equation.*

Now, if we consider the projection $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that $(x, y) \mapsto x$. Given a generic point $x_0 \in \mathbb{C}$, we have that the fiber in x_0, $F_{x_0} = \{(x_0, y) \in \mathbb{C}^2 \mid f(x_0, y) = 0\}$ is generically the union of d different points of $\mathbb{C} = \pi^{-1}(x_0)$ (since the polynomial $f(x, y)$ has degree d over the variable y).

Remark 5.12. *In some points of $\mathbb{C}$ this is going to fail due to the existence of singularities in the curve or tangencies with respect to a line $x = k$ for some constant k.*

More precisely, there exists a finite set $\Delta \subset \mathbb{C}$ such that $\#F_{x_0} = d$ if and only if $x_0 \in \mathbb{C} \setminus \Delta$. (In fact, $\Delta = \{x_0 \in \mathbb{C} \mid \frac{\partial f}{\partial y}(x_0, y) = f(x_0, y) = 0\}$).

Definition 5.13. We say that the fiber bundle is **locally trivial** if for each point of the base x, the fiber (i.e., the preimage of a neighbourhood of the point, U^x) is diffeomorphic to the product of the preimage of the point and the neighbourhood U^x. In other words, if for each $x \in B$, $\exists U^x$ neighbourhood such that $\pi^{-1}(U^x) \simeq \pi^{-1}(x) \times U^x$.

So, we have the following result:

Theorem 5.14. $\pi : (\mathbb{C}^2 \setminus \pi^{-1}(\Delta), C \setminus \pi^{-1}(\Delta)) \longrightarrow (\mathbb{C} \setminus \Delta)$ *is a locally trivial fiber bundle, i.e., for each $x_0 \in \mathbb{C} \setminus \Delta$, $\exists U^{x_0} \subset \mathbb{C} \setminus \Delta$ neighbourhood such that:*

$$(\pi^{-1}(U^{x_0}), C) \simeq (\pi^{-1}(x_0) \times U^{x_0}, F_{x_0} \times U^{x_0}).$$

This proccess is known as **trivialization**.

Definition 5.15. A sequence of groups and homomorphisms:

$$G_0 \to G_1 \to G_2 \to ...$$

is called **exact** if the image of each homomorphism is equal to the kernel of the next.

We can trivialize through a path joining open sets. Thus, the fibers at two different points are diffeomorphic, $F_{x_0} \simeq F_{x_1}$. So we can speak about the fiber (F) of a locally trivial fiber bundle. This defines an exact sequence (denoting E the total space and B the fiber bundle base):

$$\ldots \longrightarrow \pi_2(B) \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \longrightarrow 0$$

In our case, $F = \mathbb{C} \setminus \{$ d points$\}$ and $\pi_1(F) = \mathbb{F}_d$ (free group of rank d). Besides, $B = \mathbb{C} \setminus \Delta$, so $\pi_2(\mathbb{C} \setminus \Delta) = 0$ and $\pi_1(\mathbb{C} \setminus \Delta) = \mathbb{F}_r$ (free group of rank r) where $r = \#\Delta$. And $E = \mathbb{C}^2 \setminus (C \cup L_\Delta)$, where $L_\Delta = \cup_{x_1 \in \Delta} \pi_1^{-1}(x_1)$. Therefore, we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{F}_d \longrightarrow \pi_1(\mathbb{C}^2 \setminus (C \cup L_\Delta)) \longrightarrow \mathbb{F}_r \longrightarrow 0$$

The generators of $\mathbb{F}_d$ are meridians (loops around the d points), similarly the generators of $\mathbb{F}_r$ are loops around our r points. We consider that the paths are taken in the front side. When we trivialize and we go along a closed path in the base, the d points are the same but they don't necessarily maintain their order, so a braid is formed. When we end, we have got the same points but not in the same order. Thus, we have:

$$\pi_1(\mathbb{C}^2 \setminus (C \cup L_\Delta)) = \langle \mu_1, \ldots, \mu_d, \gamma_1, \ldots, \gamma_r : \mu_j^{\gamma_i} = \mu_j \rangle.$$

We have that $\mu_j^{\gamma_i} \in \mathbb{F}_d$. In order to compute $\pi_1(\mathbb{C}^2 \setminus C)$ and be capable of eliminating L_Δ, we have to factor out by the normal subgroup generate by the γ_i.

Theorem 5.16. *(Zariski-Van-Kampen)*

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle \mu_1, \ldots, \mu_d : \mu_j^{\gamma_i} = \mu_j \rangle.$$

5.1.3 When is $\pi_1(\mathbb{P}^2 \setminus C)$ an Artin group?

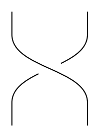
Local Braids

We have studied before that we can encounter different kinds of non-regular values of the projection of the curve, these may occur due to the existence of tangencies or singularities in the curve. Our objective now is to study what kind of braid is formed from each of these situations.

1. Tangencies

   If we have a special fiber of the type $y^2 = x$, we take a circunfernece around 0, $x(t) = e^{2\pi i t}$. Then $y = \pm e^{\pi i t}$.

   Thus, we obtain the braid σ



and the action obtained is:

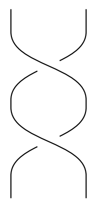
$$\mu_1^\sigma = \mu_2 = \mu_1$$
$$\mu_2^\sigma = \mu_2^{-1}\mu_1\mu_2 = \mu_1.$$

So the action is equivalent to identifying both generators.

In general, for tangencies we obtain σ or a conjugate of it (when the base point is far from the local point).

2. If we have a singularity of the kind $y^2 = x^2$, we take a circunfernece around 0, $x(t) = e^{2\pi it}$. Then $y = \pm e^{2\pi it}$.

   Then, we get the braid σ^2

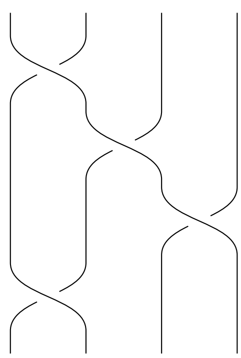
   and the relation obtained is $\mu_1\mu_2 = \mu_2\mu_1$ (Artin relation for label 2).

3. In general, if we have a singularity of the type $y^2 = x^k$ and we take a circunfernece around 0, $x(t) = e^{2\pi it}$ and so $y = \pm e^{k\pi it}$. Then, we get the braid σ^k and the relation obtained is the Artin relation for label k.

4. There also exits other kind of singularities, but they don't produce Artin relations.

   For example, if we have a singularity of the kind $y^d = x^d$ and we take a circunference around 0, $x(t) = e^{2\pi it}$. Then $y = \eta_d^i x$, $i = 0, ..., d-1$, where η_d^i are unity roots.

   Then we obtain the braid: $(\sigma_1\sigma_2...\sigma_{d-1})^d$.

   ....

Remark 5.17. *To get even Artin relations, the singularities must locally be of the form* $y^2 = x^{2k}$ *(eventually, there may be some tangencies, but they are not shown in the group).*

Algebraic Curves

Theorem 5.18. *(Bezout's Theorem)*

If C_1, C_2 *are plane curves of degrees* d_1, d_2 *respectively (without commmon components) then:*

$$\textstyle\sum_{p\in C_1\cap C_2} \nu_p(C_1, C_2) = d_1 d_2.$$

If we can parametrize C_1 near p:

$$C_1 = \psi(t) = (\psi_1(t), \psi_2(t)).$$

We take $C_2 = \{f_2(x, y) = 0\}$, and then $\nu_p(C_1, C_2) = ord_t(f_2 \circ \psi(t))$.

Example 5.19. If $C_1 = \{y = x^k\}$ and $C_2 = \{y = -x^k\}$, we want to compute $\nu_0(C_1, C_2)$ ($C_1 \cap C_2 = \{0\}$).

A parametrization of C_1 near 0 is $\psi(t) = (t, t^k)$. And $C_2 = \{f_2(x, y) = y + x^k = 0\}$.

Thus:

$$f_2 \circ \psi(t) = (y + x^k) \circ \psi(t) = t^k + t^k.$$

Then, $\nu_0(C_1, C_2) = ord_t(f_2 \circ \psi(t)) = k$.

Therefore $(y - x^k)(y + x^k)$ are two branches which intersect with intersection multiplicity k.

Coming back to our case of even Artin groups, we had seen that the singularities must locally have the form $y^2 = x^{2k}$. But this is equivalent to $(y - x^k)(y + x^k) = 0$, so locally there are two branches which intersect in a point.

5.1.4 Serre's question for Artin groups

The question about deciding whether a certain group is quasi-projective is known as *Serre's question.* This question is solved for right-angled Artin groups, but almost nothing is known for more general Artin groups.

Theorem 5.20 ([40, Theorem 11.7])**.** *The right-angled Artin group* $\mathcal{A}_\Gamma$ *is quasi-projective if and only if* $\mathcal{A}_\Gamma$ *is a product of finitely generated free groups, i.e.* $\mathcal{A}_\Gamma = \mathbb{F}_{n_1} \times ... \times \mathbb{F}_{n_r}$.

The direct implication is proved by exploiting the obstructions of resonance varieties of quasi-projective groups. The converse is achieved by realizing such groups as fundamental groups of quasi-projective varieties, built as products of complements of points in $\mathbb{C}$.

In fact, this result can be interpreted in terms of the graphs via the 2-join construction as follows.

Definition 5.21. *Consider Γ_1 and Γ_2 two labelled graphs. The* 2-join *of Γ_1 and Γ_2, denoted by $\Gamma = \Gamma_1 *_2 \Gamma_2$ is the labelled graph Γ defined as the join of the graphs and whose label is defined as*

$$m(e) = \begin{cases} m_i(e) & \text{if } e \in E(\Gamma_i) \\ 2 & \text{otherwise.} \end{cases}$$

The Artin group of a 2-join is the product of the Artin groups, that is,

$$\mathcal{A}_{\Gamma_1 *_2 \Gamma_2} = \mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2}. \tag{5.1}$$

From Example 5.5 note that a free abelian group of rank r is an Artin group corresponding to a complete right-angled Artin group of r vertices, or 2-joins of r points. From Example 5.6 note that a free group of rank r is also an Artin group corresponding to a totally disconnected graph of r vertices. Using (5.1), Theorem 5.20 can be rewritten as follows.

Theorem 5.22. *Let Γ be a right-angled graph, then $\mathcal{A}_\Gamma$ is quasi-projective if and only if Γ is the 2-join of finitely many totally disconnected graphs.*

For triangle Artin groups and general type Artin groups, partial results on their quasi-projectivity are given in [5], among those we describe the following associated with Artin groups of type $\mathcal{A}_{S_{2\ell}}$ and $\mathcal{A}_T$ in Figure 5.1.

Theorem 5.23 ([5, Chapter 5])**.** *The Artin groups $\mathcal{A}_{S_{2\ell}} = \langle a, b \mid (ab)^\ell = (ba)^\ell \rangle$ and $\mathcal{A}_T = \langle a, b, c \mid abab = baba, acac = caca, bc = cb \rangle$ are quasi-projective.*

Proof.
- First let us prove that $\mathcal{A}_{S_{2l}}$ is quasi-projective. Let's consider $C = \{(y - x^l)y = 0\}$ an affine curve (Figure 5.2). ($C^2 - C = \mathbb{P}^2 - (z = 0 \cup \bar{C})$ where $\bar{C} = z(yz^{l-1} - x^l) = 0$).

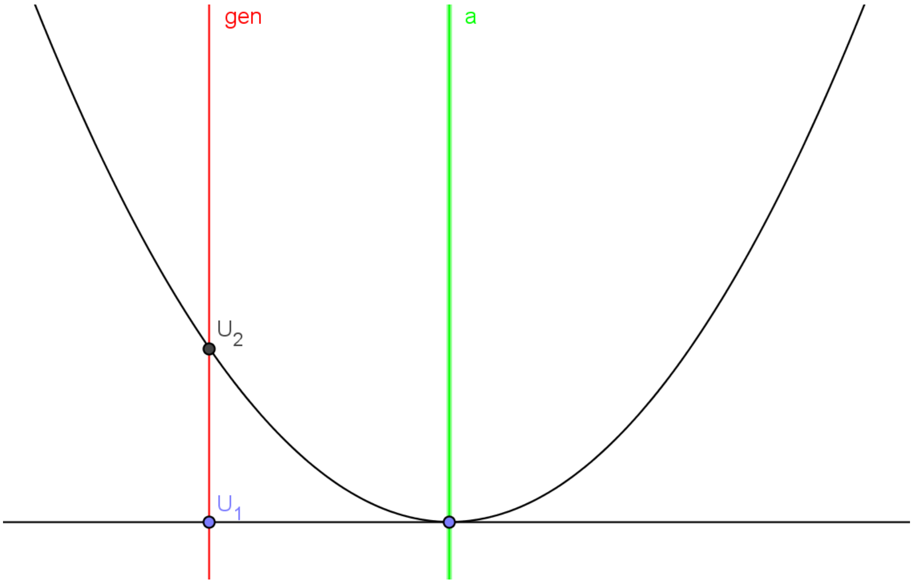


Figure 5.2: Curve C for $\mathcal{A}_{S_{2l}}$

There is only one special fiber (where there are less than two points of intersection, coloured in green). We take a generic fiber (coloured in red). Studying that special fiber (fiber a), we get the braid σ^{2l} (Figure 5.3):

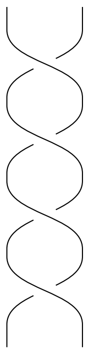


Figure 5.3: Braid σ^{2l}

which generates the relation:

$$(\mu_1\mu_2)^l = (\mu_2\mu_1)^l.$$

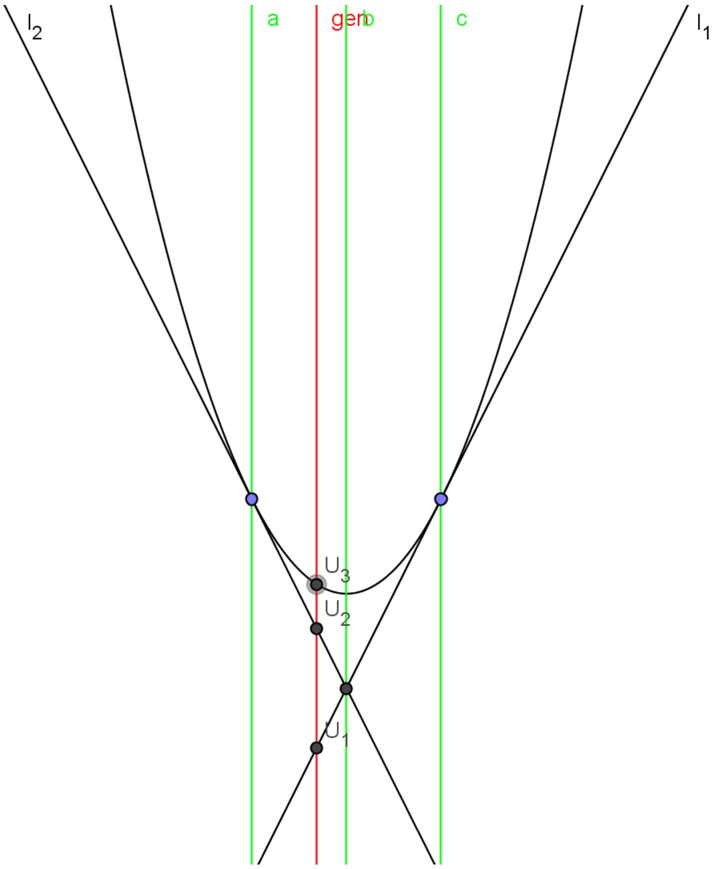


Figure 5.4: Curve C for $\mathcal{A}_T$

Thus, we get the even Artin group of two vertices and label $2l$:

$$\mathcal{A}_{S_{2l}} = \langle a, b \mid (ab)^l = (ba)^l \rangle.$$

- Now let us see that $\mathcal{A}_T$ is quasi-projective. Now, let's consider $l_1 = \{2x - y = 1\}$, $l_2 = \{2x + y = -1\}$ and $C = \{(y - x^2)l_1 l_2 = 0\}$ (Figure 5.4). ($C^2 - C = \mathbb{P}^2 - (z = 0 \cup \bar{C})$ where $\bar{C} = z(yz - x^2)(2x - y - z)(2x + y + z) = 0$). There are three special fibers (coloured in green, where there are less than three points of intersection). We take a generic fiber (gen, coloured in red). Studying the first special fiber (fiber a), we get the braid σ_2^4 (Figure 5.5) which generates the relations:

$$\mu_1 = \mu_1$$
$$(\mu_2\mu_3)^2 = (\mu_3\mu_2)^2.$$

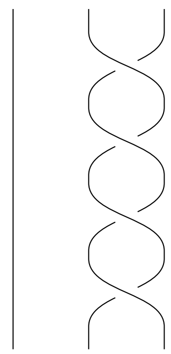


Figure 5.5: Braid σ_2^4

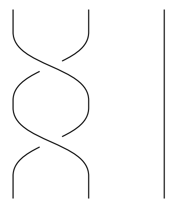


Figure 5.6: Braid σ_1^2

So, from the first braid we obtain the relation:

$$(\mu_2\mu_3)^2 = (\mu_3\mu_2)^2. \tag{5.2}$$

Now in the second one (b), we have the braid σ_1^2 (Figure 5.6), so we get the relations:

$$\mu_1\mu_2 = \mu_2\mu_1$$
$$\mu_3 = \mu_3.$$

Therefore, from the second braid we obtain the relation:

$$\mu_1\mu_2 = \mu_2\mu_1. \tag{5.3}$$

Studying the fiber c, we have the braid $\sigma_1\sigma_2^4\sigma_1^{-1}$ (Figure 5.7) and we get the relations:

$$\mu_1 = (\mu_1)^{\sigma_1\sigma_2^4\sigma_1^{-1}} \Rightarrow \mu_1 = \mu_3\mu_1\mu_3\mu_1\mu_3^{-1}\mu_1^{-1}\mu_3^{-1} \Rightarrow (\mu_3\mu_1)^2 = (\mu_1\mu_3)^2 \tag{5.4}$$

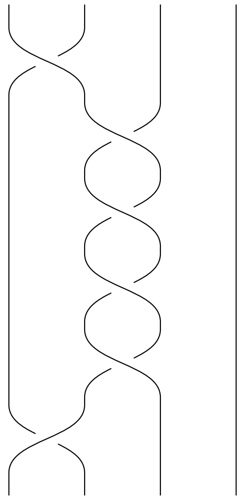


Figure 5.7: Braid $\sigma_1\sigma_2^4\sigma_1^{-1}$

$$\mu_2 = (\mu_2)^{\sigma_1\sigma_2^4\sigma_1^{-1}} \Rightarrow \mu_2 = \mu_3\mu_1\mu_3\mu_1\mu_3^{-1}\mu_1^{-1}\mu_3^{-1}\mu_1^{-1}\mu_2\mu_3\mu_1\mu_3\mu_1\mu_3^{-1}\mu_1^{-1}\mu_3^{-1}\mu_1^{-1}. \tag{5.5}$$

And using relation (5.4) in (5.5), we get that the second relation (5.5) is equivalent to $\mu_2 = \mu_2$. So, from this braid we obtain the relation:

$$(\mu_3\mu_1)^2 = (\mu_1\mu_3)^2. \tag{5.6}$$

Thus, we get the even Artin group of three vertices and three edges, two of them with label 4 and the other label 2:

$$\mathcal{A}_T = \langle a, b, c \mid ab = ba, (ac)^2 = (ca)^2, (bc)^2 = (cb)^2 \rangle.$$

□

Our objective in this chapter is to give a similar characterization to Theorem 5.22 for even Artin groups.

5.1.5 Characteristic Varieties

Characteristic varieties are a sequence of invariants of a group. They were introduced by Hillman in [54] for links using Alexander modules. Properties of characteristic varieties have been studied extensively: by Cohen-Suciu [33], Libgober [61, 62] Arapura in [3, 4], Dimca [39] and together with Papadima and Suciu [40, 41], Artal-Cogolludo-Matei [5, 6], Liu-Maxim [63], Budur-Wang [23, 22]and Budur-Liu-Wang [21]. It should also mentioned that the connection between Alexander modules and cohomology of local systems was first proved by Hironaka [55].

For expository reasons we will mainly follow [61] and we will only provide specific references for the more specialized results. Let X be a finite CW-complex and $G = \pi_1(X)$ its fundamental group. For the sake of simplicity, we assume that the abelianization $H_1(G) = G/G'$ of G is torsion-free, say $H_1(G) = \mathbb{Z}^r$. Consider the universal abelian cover $\tilde{X} \xrightarrow{\phi} X$, where $\mathrm{Deck}(\phi) = \mathbb{Z}^r$ is generated by $t_1, \ldots, t_r \in \mathrm{Deck}(\phi)$. Since $\mathrm{Deck}(\phi)$ acts on $H_1(\tilde{X})$, the group $H_1(\tilde{X})$ inherits a module structure over the ring $\Lambda = \mathbb{Z}[\mathrm{Deck}(\phi)] = \mathbb{Z}[\mathbb{Z}^r]$. This module $M_X = H_1(\tilde{X})$ is called the Alexander module of X. As any Λ-module, M_X has a sequence of invariants given by the Fitting ideals or analogously by the sequence of annihilators of its exterior powers as follows:

$$I_k = \mathrm{Ann}_\Lambda \left(\bigwedge^k M_X \right) \subset \Lambda,$$

where $\mathrm{Ann}_R(A) = \{r \in R \mid ra = 0 \ \forall a \in A\} \subset R$ is by definition the annihilator ideal of an R-module A. After tensoring Λ by $\mathbb{C}$, a new ring $\Lambda^{\mathbb{C}}$ is obtained over which one can take an algebraic geometrical point of view and consider the zero locus of $I_r \otimes \Lambda^{\mathbb{C}}$ inside the torus $\operatorname{Spec} \Lambda^{\mathbb{C}} = (\mathbb{C}^*)^r$.

Definition 5.24. *We define the sequence of* characteristic varieties *of X as:*

$$V_1(X) := Z(I_1) \supset \ldots \supset V_k(X) := Z(I_k) \supset \ldots$$

where $Z(I_k) \subset (\mathbb{C}^*)^r$ is the zero locus of I_k.

There is an alternative way to define characteristic varieties using Fitting ideals.

Definition 5.25. *Let $\varphi : A_2 \to A_1$ be a map of free modules over a ring R. We define the ideal $\tilde{F}_k(\varphi) \subset R$ as the image of the canonical map:*

$$\bigwedge^k A_2 \otimes \bigwedge^k A_1^* \to R$$

induced by φ.

Definition 5.26. *Let M be a finitely presented module over R and consider a free resolution*

$$\varphi : A_2 \to A_1 \to M \to 0$$

of M such that A_1 (resp. A_2) is a finitely generated R-module of rank r (resp. s). For every integer $k \geq 0$ we define the k-th Fitting ideal of M to be

$$F_k(M) := \tilde{F}_{r-k}(\varphi).$$

Proposition 5.27. *Under the above conditions, the sequence of characteristic varieties $V_k(X)$ coincides with the zero locus of the Fitting ideals of its Alexander module $F_k(M_X)$.*

Proof. This is an immediate consequence of [20, Cor. 1.3]. □

Characteristic varieties of quasi-projective spaces satisfy the following result:

Proposition 5.28. *[3, 41] The irreducible components of the characteristic varieties associated to a quasi-projective group G are algebraic translated tori by torsion points, that is, they are intersection of zero-sets of polynomials of the form*

$$P(t_1, ..., t_r) = \prod_i (t_1^{n_1} ... t_r^{n_r} - \nu_i),$$

where ν_i is a root of unity.

Moreover, the intersection of two such irreducible components is a finite union of torsion points.

From the computational point of view, a third way to calculate the sequence of characteristic varieties from a finite presentation of a group

$$G = \pi_1(X) = \langle a_1, ..., a_n : R_1 = ... = R_m = 1 \rangle \tag{5.7}$$

is provided via Fox calculus (see [47]).

Formally, one associates a matrix

$$A = \left(\frac{\partial R_i}{\partial a_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n},$$

to the presentation (5.7), where the derivative of a word in the letters $a_1, \ldots, a_n$ is obtained by extending the following defining properties by linearity:

$$\frac{\partial uv}{\partial a_j} = \frac{\partial u}{\partial a_j} + \phi(u) \frac{\partial v}{\partial a_j}, \qquad \frac{\partial 1}{\partial a_j} = 0, \qquad \text{and} \qquad \frac{\partial a_i}{\partial a_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is called the *Alexander matrix* associated with (5.7) and it turns out to be the matrix of the free resolution of a module which is not the Alexander module, but the *Alexander invariant* $\tilde{M}_X = H_1(\tilde{X}, \phi^{-1}(p))$, which is the relative homology of the universal abelian cover of X relative to the preimage of a point as a Λ-module exactly as was done for the Alexander module M_X. As in knot theory, both invariants are related (see for instance [30, Ch. 1]).

Proposition 5.29. *The sequence of characteristic varieties of X can be calculated via Fox calculus as*

$$V_k(X) \setminus \bar{\mathbf{1}} = Z(F_{k+1}(M_X)) \setminus \bar{\mathbf{1}} = Z(F_{k+1}(\tilde{M}_X)) \setminus \bar{\mathbf{1}}.$$

The computational advantage of $F_{k+1}(\tilde{M}_X)$ is that it can be computed from the Alexander matrix A of a free resolution of $\tilde{M}_X$ as follows:

$$F_{k+1}(\tilde{M}_X) = \begin{cases} \Lambda & \text{if } k > n \\ 0 & \text{if } k \leq \max\{0, n-m\} \\ (\text{minors of order } n-k \text{ of } A) & \text{otherwise.} \end{cases}$$

5.2 Preliminaries

Characteristic varieties of even Artin groups are too similar to those of quasi-projective groups and hence they cannot be used to tell them apart. However, some of their finite index subgroups can be detected as not quasi-projective. This is why we present a study of a certain type of subgroups of even Artin groups that will be key in the discussion on quasi-projectivity.

5.2.1 Co-cyclic subgroups of even Artin groups

Let us consider the even Artin group associated with $\Gamma = (V, E, 2\ell)$.

The Artin group associated with Γ has a presentation $\mathcal{A}_\Gamma = \langle v; v \in V \mid A_{\ell_e}(e); e \in E\rangle$, where $A_{\ell_e}(e)$ denotes the relation $(uv)^{\ell_e} = (vu)^{\ell_e}$ with $e = \{u, v\}$. Let us fix a vertex, say $u \in V$ and an integer $k > 1$; our purpose is to give a presentation of the index k subgroup $\mathcal{A}_{\Gamma,u,k}$ of $\mathcal{A}_\Gamma$ defined as the kernel of the following morphism:

$$\begin{array}{rcl} \alpha_{u,k} : \mathcal{A}_\Gamma & \longrightarrow & \mathbb{Z}_k \\ v & \mapsto & \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

One can think of these as finite index normal subgroups of a group that appear as the kernel of a surjection onto a finite cyclic group, and refer to them as *co-cyclic subgroups.*

Note that, for any $v \in V$ the conjugation of v by u^i is in $\mathcal{A}_{\Gamma,u,k}$, $v_i := u^i v u^{-i}$. Also, $\bar{u} := u^k$ will be in the kernel of $\alpha_{u,k}$. In order to write a presentation for $\mathcal{A}_{\Gamma,u,k}$ we need some notation. Let us denote by $\langle x, y\rangle^l_{i,\varepsilon}$ a formal word in the letters $\{x_0, ..., x_{k-1}, y\}$ as follows

$$\langle x, y\rangle^l_{i,\varepsilon} = (x_i \cdots x_{k-1} y x_0 \cdots x_{i-1})^c \underbrace{x_i \cdots}_{r_e+\varepsilon},$$

where $l = ck + r$, $i \in \mathbb{Z}_k$ and $\varepsilon = 0, 1$. Note that $\langle x, y\rangle^l_{i,\varepsilon}$ can be thought of as a cyclic product of the letters $x_0, \ldots, x_{k-1}$, and y starting at x_i and with length $c(k + 1) + r + \varepsilon$.

Also, let us consider the set of vertices in V adjacent to u with label 2:

$$V_{2,u} = \{v \in V \mid e = \{u, v\} \in E, m_e = 2\ell_e = 2\}.$$

The remaining vertices will be denoted by $W = V \setminus (\{u\} \cup V_{2,u})$.

One obtains the following presentation for $\mathcal{A}_{\Gamma,u,k}$.

Theorem 5.30. *The co-cyclic subgroup $\mathcal{A}_{\Gamma,u,k}$ is generated by*

$$\{\bar{u}\} \cup V_{2,u} \cup \bigcup_{w \in W} \{w_0, ..., w_{k-1}\}$$

and the following is a complete set of relations:

(R) 1. *$A_1(v, \bar{u})$, for $v \in V_{2,u}$,*
2. *$A_{\ell_e}(v, v')$, for $v, v' \in V_{2,u}$, $e = \{v, v'\} \in E$,*
3. *$A_{\ell_e}(v, w_i)$, for $v \in V_{2,u}$, $w \in W$, $i \in \mathbb{Z}_k$, $e = \{v, w\} \in E$,*
4. *$A_{\ell_e}(w_i, w'_i)$, for $w, w' \in W$, $i \in \mathbb{Z}_k$, $e = \{w, w'\} \in E$.*

(RB) $B^i_{\ell_e,k}(w, \bar{u})$, for $w \in W \cap \mathrm{lk}(u)$, $i \in \mathbb{Z}_k$, $e = \{u, w\} \in E$,

where $\ell_e = c_e k + r_e$ and $B^i_{\ell_e}(w, \bar{u})$ is the relation

$$\langle w, \bar{u}\rangle^{\ell_e}_{i,\varepsilon} = \langle w, \bar{u}\rangle^{\ell_e}_{i+1,\varepsilon} \qquad \text{and} \quad \varepsilon = \begin{cases} 0 & \text{if } 0 \le i < k - r_e \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The proof is a direct application of Reidemeister-Schreier's theorem (c.f. [48]) to obtain a presentation of $\mathcal{A}_{\Gamma,u,k}$ as the kernel of $\alpha_{u,k}$

$$\mathcal{A}_{\Gamma,u,k} \overset{j}{\hookrightarrow} \mathcal{A}_\Gamma \overset{\alpha_{u,k}}{\to} \mathbb{Z}_k.$$

Consider the Reidemeister's section $s : \mathbb{Z}_k \to \mathcal{A}_\Gamma$ of the map $\alpha_{u,k}$ given as $s(i) := u^i$. Then $\mathcal{A}_{\Gamma,u,k}$ admits a presentation generated by the letters

$$\{\bar{u}\} \cup \bigcup_{v \in V} \{v_0, ..., v_{k-1}\},$$

where $j(\bar{u}) = u^k$ and $j(v_i) = u^i v u^{-i}$ whose relations are:

1. $A_{\ell_e}(v_i, w_i)$, for $v, w \in V$, $i \in \mathbb{Z}_k$, if $e = \{v, w\} \in E$,

2. $B^i_{\ell_e,k}(w,\bar{u})$, for $i \in \mathbb{Z}_k$, if $v \in \mathrm{lk}(u)$.

However, note that if $v \in V_{2,u}$, then $u^i v u^{-i} = v_i = v_j = u^j v u^{-j}$ which implies a reduction in the set of generators, which now becomes as stated:

$$\{\bar{u}\} \cup V_{2,u} \cup \bigcup_{w \in W} \{w_0, ..., w_{k-1}\}.$$

Finally, note that the only relations affected by this elimination of generators are those of type $A_{\ell_e}(v_i, w_i)$ for $v \in V_{2,u}$, which now become $A_{\ell_e}(v, w_i)$, and those of type $B^i_{1,k}(v,\bar{u})$ for $v \in V_{2,u}$, which now be reduced to $A_1(v,\bar{u})$, as stated. □

Remark 5.31. *Our purpose will be to study the characteristic varieties of the co-cyclic subgroups. As presented in section 5.1.5 these are subvarieties of* $\operatorname{Spec} \mathbb{C}[G/G']$*, for $G = \mathcal{A}_{\Gamma,u,k}$. First we will describe the abelianization of $\mathcal{A}_{\Gamma,u,k}$. Since G is finitely presented consider $\mathbb{F} \to G$ the map from the free group $\mathbb{F}$ in the generators of G. The kernel K of this homomorphism is a free subgroup generated by the set of relations in G. Consider $G \overset{\Phi_G}{\to} G/G'$, $g \to t_g$ the abelianization map (with a multiplicative structure). According to Theorem 5.30 the abelianization $G/G' = \Phi_{\mathbb{F}}(\mathbb{F})/\Phi_{\mathbb{F}}(K)$ is generated by*

$$\{t_{\bar{u}}\} \cup \{t_v\}_{v \in V_{2,u}} \cup \bigcup_{w \in W} \{t_{w,0}, ..., t_{w,k-1}\},$$

where for convenience $t_{w,i}$ is used to denote t_{w_i}. Note that (R).(1)-(4) *considered as words in the free group $\mathbb{F}$ belong in fact to $\mathbb{F}'$ and hence their image by the abelianitation map $\Phi_{\mathbb{F}}$ is trivial. On the other hand, the words $B^i_{\ell_e,k}(w,\bar{u})$, $w \in W \cap \mathrm{lk}(u)$ produce the following relations in homology:*

$$t_{w,i} = t_{w,i+d_e} = ... = t_{w,i+nd_e} \quad \text{if } e = \{u,w\}, d_e = \gcd(\ell_e, k), \tag{5.8}$$

Definition 5.32. *The presentation described in Theorem 5.30 will be referred to as the* standard presentation *of $\mathcal{A}_{\Gamma,u,k}$.*

5.2.2 Fox calculus on the co-cyclic subgroups $\mathcal{A}_{\Gamma,u,k}$

Fox derivatives of a standard presentation

We want to describe the Fox derivatives of the relations of a standard presentation of the subgroup $\mathcal{A}_{\Gamma,u,k}$.

The first set of relations of type (R) in Theorem 5.30 are classical Artin relations. In order to describe their Fox derivatives we introduce the polynomial $p_l(t) = \frac{t^l-1}{t-1}$ and as above, we denote by t_g the homology class of an element g. In the following results we present the Fox derivatives of certain relations of type $W_1 = W_2$, by this we mean the derivative of the abstract word $W_1 W_2^{-1}$.

Lemma 5.33. *Under the above conditions*

$$\frac{\partial A_{\ell_e}(a,b)}{\partial g} = \begin{cases} -(t_b-1)p_{\ell_e}(t_a t_b) & \textit{if } g = a \\ (t_a - 1)p_{\ell_e}(t_a t_b) & \textit{if } g = b \\ 0 & \textit{otherwise.} \end{cases}$$

In order to describe the derivatives of relations of type (RB), let us use some conventions:

$$\bar{t}_{w,i,j} = \begin{cases} t_{w,i} \cdots t_{w,j-1} & \text{if } 0 \le i < j \le k \\ 1 & \text{if } i = j \\ \bar{t}_w t_0 / \bar{t}_{w,j,i} & \text{if } 0 \le j < i \le k \end{cases}$$

where $t_{w,i} = t_{w_i}$ with $w_i = u^i w u^{-i}$, $t_0 = t_{\bar{u}}$, and $\bar{t}_w = \bar{t}_{w,0,k}$. Let us define the following set that will be useful for the statement of the following lemma:

$$\Delta_{i,j} = \begin{cases} \{i < k < j\} & \text{if } i < j \\ \{k > i\} \cup \{k < j\} & \text{if } j < i \end{cases}$$

We define also the number

$$\alpha_e = \begin{cases} 0 & \text{if } r_e = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.34. *Under the above conditions,*

$$\frac{\partial B^i_{\ell_e,k}(w,\bar{u})}{\partial g} = \begin{cases} \bar{t}_{w,i,k}(1 - t^{-1}_{w,i+r_e})p_{c_e+\varepsilon}(t_0\bar{t}_w) & \textit{if } g = \bar{u} \\ \left(1 - \dfrac{t_0\bar{t}_w}{t_{w,i+r_e}}\right) p_{c_e}(t_0\bar{t}_w) + \alpha_e (t_0\bar{t}_w)^{c_e} & \textit{if } g = w_i \\ \bar{t}_{w,i,j}(1 - t^{-1}_{w,i+r_e})p_{c_e+1}(t_0\bar{t}_w) & \textit{if } g = w_j, j \in \Delta_{i,i+r_e} \\ \bar{t}_{w,i,j}(1 - t^{-1}_{w,i+r_e})p_{c_e}(t_0\bar{t}_w) - \alpha_e \dfrac{\bar{t}_{w,i,i+r_e}}{t_{w,i+r_e}}(t_0\bar{t}_w)^{c_e} & \textit{if } g = w_{i+r_e} \\ \bar{t}_{w,i,j}(1 - t^{-1}_{w,i+r_e})p_{c_e}(t_0\bar{t}_w) & \textit{otherwise.} \end{cases}$$

Proof. The proof is straightforward. First notice that:

$$\frac{\partial B^i_{\ell_e,k}(w,\bar{u})}{\partial g} = \frac{\partial \langle w,\bar{u}\rangle^{\ell_e}_{i,\varepsilon}(\langle w,\bar{u}\rangle^{\ell_e}_{i+1,\varepsilon})^{-1}}{\partial g}.$$

Let us suppose first that $\varepsilon = 0$, we will calculate $\dfrac{\langle w, \bar{u}\rangle_{i,0}^{\ell_e}}{\partial g}$. It is straightforward that

$$\frac{\partial\langle w, \bar{u}\rangle_{i,0}^{\ell_e}}{\partial g} = \begin{cases} \bar{t}_{w,i,k} p_{c_e}(t_0\bar{t}_w) & \text{if } g = \bar{u} \\ \bar{t}_{w,i,j} p_{c_e}(t_0\bar{t}_w) & \text{if } g = w_j, j < i \\ \bar{t}_{w,i,j} p_{c_e+\alpha_e}(t_0\bar{t}_w) & \text{if } g = w_i \\ \bar{t}_{w,i,j} p_{c_e+1}(t_0\bar{t}_w) & \text{if } g = w_j, i < j < i + r_e \\ \bar{t}_{w,i,j} p_{c_e}(t_0\bar{t}_w) & \text{if } g = w_j, j \geq i + r_e. \end{cases}$$

Now, using the multiplication rule and $0 = \frac{\partial uu^{-1}}{\partial v} = \frac{\partial u}{\partial v} + t_u \frac{\partial u^{-1}}{\partial v}$ one obtains:

$$\frac{\partial(\langle w, \bar{u}\rangle_{i+1,0}^{\ell_e})^{-1}}{\partial g} = -\frac{1}{(t_0\bar{t}_w)^{c_e}\bar{t}_{i+1,i+1+r_e}} \frac{\partial\langle w, \bar{u}\rangle_{i+1,0}^{\ell_e}}{\partial g}$$

Therefore:

$$\frac{\partial B_{\ell_e,k}^i(w,\bar{u})}{\partial g} = \frac{\partial\langle w, \bar{u}\rangle_{i,0}^{\ell_e}}{\partial g} + (t_0\bar{t}_w)^{c_e}\bar{t}_{w,i,i+r_e} \frac{\partial(\langle w, \bar{u}\rangle_{i+1,0}^{\ell_e})^{-1}}{\partial g}$$

And so, doing the computations we obtain:

$$\frac{\partial B_{\ell_e,k}^i(w,\bar{u})}{\partial g} = \begin{cases} \bar{t}_{w,i,k}(1 - t_{w,i+r_e}^{-1}) p_{c_e}(t_0\bar{t}_w) & \text{if } g = \bar{u} \\ \bar{t}_{w,i,j}(1 - t_{w,i+r_e}^{-1}) p_{c_e}(t_0\bar{t}_w) & \text{if } g = w_j, j < i \\ \left(1 - \dfrac{t_0\bar{t}_w}{t_{w,i+r_e}}\right) p_{c_e}(t_0\bar{t}_w) + \alpha_e(t_0\bar{t}_w)^{c_e} & \text{if } g = w_i \\ \bar{t}_{w,i,j}(1 - t_{w,i+r_e}^{-1}) p_{c_e+1}(t_0\bar{t}_w) & \text{if } g = w_j, i < j < i + r_e \\ \bar{t}_{w,i,j}(1 - t_{w,i+r_e}^{-1}) p_{c_e}(t_0\bar{t}_w) - \alpha_e \dfrac{\bar{t}_{w,i,i+r_e}}{t_{w,i+r_e}}(t_0\bar{t}_w)^{c_e} & \text{if } g = w_{i+r_e} \\ \bar{t}_{w,i,j}(1 - t_{w,i+r_e}^{-1}) p_{c_e}(t_0\bar{t}_w) & \text{if } g = w_j, j < i + r_e. \end{cases}$$

We will made the computations in detail for a sample case. Assume $g = w_i$, then:

$$\frac{\partial B_{\ell_e,k}^i(w,\bar{u})}{\partial w_i} = p_{c_e+\alpha_e}(t_0\bar{t}_w) + (t_0\bar{t}_w)^{c_e}\bar{t}_{w,i,i+r_e} \frac{(\langle w, \bar{u}\rangle_{i+1,0}^{\ell_e})^{-1}}{\partial w_i} =$$

$$= p_{c_e}(t_0\bar{t}_w) + \alpha_e(t_0\bar{t}_w)^{c_e} - (t_0\bar{t}_w)^{c_e}\bar{t}_{w,i,i+r_e} \frac{(\bar{t}_{w,i+1,i}) p_{c_e}(t_0\bar{t}_w)}{(t_0\bar{t}_w)^{c_e}\bar{t}_{w,i+1,i+1+r_e}} =$$

$$= p_{c_e}(t_0\bar{t}_w) + \alpha_e(t_0\bar{t}_w)^{c_e} - \bar{t}_{w,i,i+r_e} \frac{(t_0\bar{t}_w)}{\bar{t}_{w,i,i+1+r_e}} p_{c_e} = \left(1 - \frac{t_0\bar{t}_w}{t_{w,i+r_e}}\right) p_{c_e}(t_0\bar{t}_w) + \alpha_e(t_0\bar{t}_w)^{c_e}.$$

Similarly, we will suppose now that $\varepsilon = 1$. Again we will calculate $\dfrac{\langle w,\bar u\rangle_{i,1}^{\ell_e}}{\partial g}$. It is straight-forward that

$$\frac{\partial\langle w,\bar u\rangle_{i,1}^{\ell_e}}{\partial g} = \begin{cases} \bar t_{w,i,k} p_{c_e+1}(t_0\bar t_w) & \text{if } g=\bar u \\ \bar t_{w,i,j} p_{c_e+1}(t_0\bar t_w) & \text{if } g=w_j, j<i+r_e \\ \bar t_{w,i,j} p_{c_e}(t_0\bar t_w) & \text{if } g=w_j, i+r_e\le j<i \\ \bar t_{w,i,j} p_{c_e+\alpha_e}(t_0\bar t_w) & \text{if } g=w_i \\ \bar t_{w,i,j} p_{c_e+1}(t_0\bar t_w) & \text{if } g=w_j, j>i. \end{cases}$$

We use again the multiplication rule as in the previous case and we obtain:

$$\frac{\partial(\langle w,\bar u\rangle_{i+1,1}^{\ell_e})^{-1}}{\partial g} = -\frac{1}{(t_0\bar t_w)^{c_e}\bar t_{i+1,i+1+r_e}}\frac{\partial\langle w,\bar u\rangle_{i+1,1}^{\ell_e}}{\partial g}$$

Therefore:

$$\frac{\partial B^i_{\ell_e,k}(w,\bar u)}{\partial g} = \frac{\partial\langle w,\bar u\rangle_{i,1}^{\ell_e}}{\partial g} + (t_0\bar t_w)^{c_e}\bar t_{w,i,i+r_e}\frac{\partial(\langle w,\bar u\rangle_{i+1,1}^{\ell_e})^{-1}}{\partial g}$$

And so, doing the computations we obtain that $\dfrac{\partial B^i_{\ell_e,k}(w,\bar u)}{\partial g}$ is equal to:

$$\begin{cases} \bar t_{w,i,k}(1-t^{-1}_{w,i+r_e})p_{c_e+1}(t_0\bar t_w) & \text{if } g=\bar u \\ \bar t_{w,i,j}(1-t^{-1}_{w,i+r_e})p_{c_e+1}(t_0\bar t_w) & \text{if } g=w_j, j<i+r_e \\ \bar t_{w,i,j}(1-t^{-1}_{w,i+r_e})p_{c_e}(t_0\bar t_w) - \alpha_e\dfrac{\bar t_{w,i,i+r_e}}{t_{w,i+r_e}}(t_0\bar t_w)^{c_e} & \text{if } g=w_{i+r_e} \\ \bar t_{w,i,j}(1-t^{-1}_{w,i+r_e})p_{c_e}(t_0\bar t_w) & \text{if } g=w_j, i+r_e<j<i \\ \left(1-\dfrac{t_0\bar t_w}{t_{w,i+r_e}}\right)p_{c_e}(t_0\bar t_w)+\alpha_e(t_0\bar t_w)^{c_e} & \text{if } g=w_i \\ \bar t_{w,i,j}(1-t^{-1}_{w,i+r_e})p_{c_e+1}(t_0\bar t_w) & \text{if } g=w_j, j<i+r_e. \end{cases}$$

□

Alexander matrices for co-cyclic subgroups of even Artin groups

Given $\Gamma=(V,E,2\ell)$ an even labelled graph. Let us fix $u\in V$ and an integer $k>1$. We will denote by M_Γ (resp. $M_{\Gamma,u,k}$) the Alexander matrix associated with the Artin presentation of $\mathcal{A}_\Gamma$, (resp. the standard presentation of $\mathcal{A}_{\Gamma,u,k}$ given in §5.2.1). The purpose of this section is to describe some relevant properties of both M_Γ and $M_{\Gamma,u,k}$.

Among these properties, the most relevant for our purposes refer to their rank. Note that, since these matrices have coefficients in a ring of Laurent polynomials $R = \mathbb{C}[\mathbb{Z}^m]$, a matrix $A \in \operatorname{Mat}(R)$ has rank at least r if and only if there is a value $p = (t_1, \dots, t_m) \in \mathbb{C}^m$ such that $A \otimes R/\mathfrak{m}_p \in \operatorname{Mat}(\mathbb{C})$ has an $r \times r$ non-zero minor, where $\mathfrak{m}_p$ denotes the maximal ideal at p. This operation will be called *evaluating* and will be used oftentimes to simplify notation.

Lemma 5.35. *The rank of the Alexander matrix M_Γ defined above is exactly $|V| - 1$.*

Proof. Consider M_T the row submatrix of M_Γ given by the $|V|-1$ relations determined by the edges of a maximal tree T in Γ. Since M_T clearly has rank $|V|-1$, the matrix M_Γ has rank at least $|V|-1$.

To see the equality, consider $\bar{\Gamma} = (V, \bar{E}, 2\bar{\ell})$ the completion of the graph Γ obtained from Γ by adding an edge of label 2 for every pair of disconnected vertices. The matrix $M_{\bar{\Gamma}}$ associated with this graph contains M_Γ as a submatrix. Choose any vertex $v \in V$, we will show that the $|V|-1$ rows associated to the relations involving v generate the remaining rows.

Consider $e = \{w, w'\} \in \bar{E}$, using Lemma 5.33, the row f_e associated with the classical Artin relation $A_{\bar{\ell}_e}(w, w')$ has the form:

$$p_{\bar{\ell}_e}(t_w t_{w'}) \begin{pmatrix} 0 & \dots & 0 & (1-t_{w'}) & 0 & \dots & 0 & (t_w - 1) & 0 & \dots & 0 \end{pmatrix} \tag{5.9}$$

where the non-zero elements are at the columns corresponding to the vertices w and w' respectively.

Note that, since $\bar{\Gamma}$ is a complete labelled even graph, the three vertices $v, w, w' \in \bar{V} = V$ form a triangle, that is, $e = \{w, w'\}$, $e_1 = \{v, w\}$, $e_2 = \{v, w'\}$. Moreover, the rows f_e, f_{e_1}, and f_{e_2} satisfy the following linear combination:

$$\frac{(t_v - 1)}{p_{\bar{\ell}_e}(t_w t_{w'})} f_e + \frac{(t_{w'} - 1)}{p_{\bar{\ell}_{e_1}}(t_v t_w)} f_{e_1} + \frac{(t_w - 1)}{p_{\bar{\ell}_{e_2}}(t_v t_{w'})} f_{e_2} = 0.$$

Thus, $M_{\bar{\Gamma}}$ has rank less than or equal to $|V|-1$. Since M_Γ is a submatrix of $M_{\bar{\Gamma}}$ the result follows. $\square$

Notation 5.36. *Recall from Theorem 5.30 that the generators of a standard presentation of $\mathcal{A}_{\Gamma,u,k}$ can be distinguished in three type groups $\{\bar{u}\} \cup V_{2,u} \cup W_{k,u}$, where*

$$V_{2,u} = \{v \in V \mid e = \{u, v\} \in V, \ell_e = 1\}$$

and

$$W_{k,u} = \{w_{i,j} \mid w_i \in W = V \setminus (\{\bar{u}\} \cup V_{2,u}), j \in \mathbb{Z}_k\}.$$

In the sequel, the elements in $V_{2,u}$ will be denoted by $v_1, ..., v_m$, where m is the number of vertices adjacent to u with label 2. Analogously, the elements of $W_{k,u}$ will be denoted by $w_{i,j}$, for $w_i \in W$, where $1 \leq i \leq n = |V| - m - 1$ and $j \in \mathbb{Z}_k$.

From the results of the two previous sections, we immediately obtain the following description of the Alexander matrix $M_{\Gamma,u,k}$.

Lemma 5.37. *The Alexander matrix $M_{\Gamma,u,k}$ of $\mathcal{A}_{\Gamma,u,k}$ associated with its standard presentation has the following form:*

$$
\begin{array}{c}
\begin{array}{cccccccc}
w_{*,0} & w_{*,1} & \dots & w_{*,k-1} & v_1 & \dots & v_m & \bar{u}
\end{array} \\
\left(\begin{array}{c|c|c|c|ccc|c}
0 & 0 & \dots & 0 & \multicolumn{3}{c|}{A_k} & 0 \\ \hline
A'_0 & 0 & \dots & 0 & \multicolumn{3}{c|}{A_0} & 0 \\ \hline
0 & A'_1 & \dots & 0 & \multicolumn{3}{c|}{A_1} & 0 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
0 & 0 & \dots & A'_{k-1} & \multicolumn{3}{c|}{A_{k-1}} & 0 \\ \hline
 & & & & t_{\bar{u}} - 1 & & & 1 - t_{v_1} \\
0 & 0 & 0 & \dots & & \ddots & & \vdots \\
 & & & & & & t_{\bar{u}} - 1 & 1 - t_{v_m} \\ \hline
\multicolumn{8}{c}{} \\
\multicolumn{8}{c}{M_B} \\
\multicolumn{8}{c}{}
\end{array}\right)
\end{array}
$$

where:

1. *$w_{*,j}$ denotes the set of columns associated with all the generators of type $w_{i,j} \in W_{k,u}$ for a fixed $j \in \mathbb{Z}_k$, with $w_i \in W$, as in Notation 5.36.*

2. *A_k is the Alexander matrix corresponding with relations of type $R(2)$ in Theorem 5.30 with respect to the generators $\{v_1, \ldots, v_m\}$.*

3. *the submatrices A'_j and A_j are so that the matrix $(A'_j \mid A_j)$ is the Alexander matrix of the relations of type $R(3)$ in Theorem 5.30 with respect to the generators $\{w_{*,j}, v_1, \ldots, v_m\}$, i.e. their rows are of the form:*

$$f_{a,b} \equiv p_{c_{ab}}(t_a t_b) \; (0 \;\; \dots \;\; 0 \;\; (1 - t_b) \;\; 0 \;\; \dots \;\; 0 \;\; (t_a - 1) \;\; 0 \;\; \dots \;\; 0)$$

   for $a = v_l \in V_{2,u}$ and $b = w_{i,j} \in W_{k,u}$.

4. *The submatrix M_B is the Alexander matrix associated with the relations of type (RB) in Theorem 5.30. Note that this is a block matrix whose blocks are the submatrices $M_{B(w,u)}$ associated with the relations of type $B^i_{\ell_e,k}(w, \bar{u})$, for $i \in \mathbb{Z}_k$ and $\{u, w\} \in E$.*

Lemma 5.38. *The submatrix $M_{B(w,u)}$ has maximal rank.*

Proof. As was mentioned above, we are assuming $\{u, w\} \in E$. Let us distinguish two cases depending on whether or not ℓ_e is a multiple of k.

1. Assume $\ell_e \equiv 0 \mod k$. Using Lemma 5.34 and evaluating $t_{w,0} = t_{w,1} = \dots = t_{w,k-2} = 1$ in $M_{B(w,u)}$ the following upper triangular matrix is obtained:

$$M = \begin{array}{c} \begin{array}{cccccc} w_0 & w_1 & \dots & w_{k-2} & w_{k-1} & \bar{u} \end{array} \\ \left(\begin{array}{ccccc|c} t_{\bar{u}}-1 & t_{\bar{u}}-1 & \dots & t_{\bar{u}}-1 & t_{\bar{u}}-1 & 1-t_{w,k-1} \\ 1-t_{w,k-1}t_{\bar{u}} & 0 & \dots & 0 & 0 & 0 \\ 0 & 1-t_{w,k-1}t_{\bar{u}} & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1-t_{w,k-1}t_{\bar{u}} & 0 & 0 \end{array}\right) \end{array}$$

which has maximal rank.

2. Assume $\ell_e \not\equiv 0 \mod k$. Write $\ell_e = c_e k + r_e$, with $0 < r_e < k$. Analogously to the previous case, using Lemma 5.34 and evaluating now at $t_{w,0} = t_{w,1} = \dots = t_{w,k-2} = t_{w,k-1} = 1$, the following matrix is obtained:

$$M = \begin{array}{c} \begin{array}{cccccccccc} w_0 & \dots & w_{r-1} & w_r & \dots & w_{k-r-1} & w_{k-r} & \dots & w_{k-1} & \bar{u} \end{array} \\ \left(\begin{array}{ccccccccc|c} 1 & & & -t_{\bar{u}}^{c} & & & & & & 0 \\ & \ddots & & & \ddots & & & & & \vdots \\ & & \ddots & & & \ddots & & & & \vdots \\ & & & \ddots & & & \ddots & & & \vdots \\ & & & & \ddots & & & \ddots & & \vdots \\ & & & & & 1 & & & -t_{\bar{u}}^{c} & \\ \hline -t_{\bar{u}}^{c+1} & & & & & & 1 & & & \vdots \\ & \ddots & & & & & & \ddots & & \\ & & -t_{\bar{u}}^{c+1} & & & & & & 1 & 0 \end{array}\right) \end{array}$$

Formally, $t_{\bar{u}} = 0$ produces a matrix of maximal rank and hence the result follows using small enough values of $t_{\bar{u}}$.

□

Remark 5.39. *Note that, in the previous Lemma, the submatrix of $M_{B(w,\bar{u})}$ resulting from deleting the column $\bar{u}$ has a maximal rank. Therefore, in order* *to study the rank of* $M_{\Gamma,u,k}$*, and after row operations, one can assume that* $M_{B(w,\bar{u})}$ *is equivalent to:*

$$\begin{array}{cccccc} w_0 & w_1 & \dots & w_{k-2} & w_{k-1} & \bar{u} \end{array}$$
$$\left(\begin{array}{ccccc|c} * & * & \dots & * & * & * \\ 0 & * & \dots & * & * & * \\ 0 & \ddots & \ddots & * & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & * & * \end{array}\right)$$

Recall that the corank of a matrix M, is defined as

$$\operatorname{corank}(M) = \#\text{columns}(M) - \operatorname{rank}(M).$$

Then one has the following result on the corank of $M_{\Gamma,u,k}$.

Lemma 5.40. *Under the conditions above* $\operatorname{corank}(M_{\Gamma,u,k}) \leq 1$.

Proof. Let us consider $\Gamma_u = \Gamma \setminus \{u\}$. We will first assume that Γ_u is connected. Following the notation above, recall that $V_{2,u} = \{v_1, \dots, v_m\}$ denotes the set of vertices adjacent to u with label 2 and $W = \{w_1, \dots, w_n\}$ denotes the set of remaining vertices of Γ_u. We will consider the matrix M obtained eliminating the column corresponding to u from the Alexander matrix $M_{\Gamma,u,k}$ (which has $(nk + m + 1) - 1 = nk + m$ columns). We will prove the result showing that M has maximal rank:

1. If $n = 0$, the matrix M becomes:

$$M = \begin{array}{c} \begin{array}{ccc} v_1 & \dots & v_m \end{array} \\ \begin{pmatrix} 1 - t_{\bar{u}} & & \\ & \ddots & \\ & & 1 - t_{\bar{u}} \\ * & * & * \\ \dots & \dots & \dots \\ * & * & * \end{pmatrix} \end{array}$$

   which has maximal rank.

2. If $n \neq 0$, consider T a spanning tree on Γ_u.

(a) Assume $m \neq 0$. In this case we will describe certain submatrices of M_Γ which will appear as blocks in $M_{\Gamma,u,k}$ of the appropriate rank.

In order to do this, note that T will contain at least n edges $e_1, .., e_n$ satisfying that each e_i involves at least one vertex in W and $W \subset V(\{e_1, ..., e_n\})$. Let us denote by $S \subset T$ the forest containing the edges $e_1, .., e_n$. Note that S defines a submatrix M_0 of M_{Γ_u}. We will show that columns and rows can be ordered in such a way that M_0 is upper triangular, every diagonal element is non-zero, and the columns associated with the vertices W come first.

This can be easily seen by induction. In case Γ_u has only two vertices, say v and w (this is by hypothesis the minimum number of vertices), and only one edge, the matrix M_0 is a row matrix of type (5.9) whose columns can be reordered as wanted. Now, suppose the result is true for $\lambda - 1$ vertices and consider now the case when Γ_u has exactly λ vertices. Choose a vertex w' in $V(S)$ of degree 1. Note that, by definition, S must contain at least one such vertex in W, so one can assume $w' \in W$. Then $S \setminus \{w'\}$ verifies the result. The matrix M_0 results from the latter after adding one column (associated with w') and one row f (associated with the edge containing w'). Note that placing w' as the first column and f as the first row concludes the proof.

Also note that the submatrix M_n of M_0 resulting from keeping only the columns associated with the vertices in W appears as is in k blocks in $M_{\Gamma,u,k}$ corresponding to the copies of the vertices in W and the relations associated with the edges of S. This produces a square submatrix M_k of $M_{\Gamma,u,k}$ of size kn and non-zero determinant. Finally, let us add to M_k the columns associated with all vertices in $V_{2,u}$ placed at the end. Since every $v_i \in V_{2,u}$ is adjacent to u with label 2 the relations associated with these edges result in rows producing an upper triangular square submatrix M

of size $kn+m$ whose determinant is non-zero as below.

$$\begin{array}{rc} & \begin{array}{cc} \qquad W \qquad\qquad & \qquad V \qquad \end{array} \\ M = & \left(\begin{array}{ccc|ccc} M_n & & & & & \\ 0 & \ddots & & & & \\ \vdots & 0 & M_n & & & \\ \hline \vdots & \vdots & \ddots & 1-t_{\bar{u}} & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 1-t_{\bar{u}} \end{array}\right) \end{array}$$

This ends this case.

(b) If $m=0$, then the spanning tree T consists of n vertices and $n-1$ edges. Let us consider w_n a vertex in Γ adjacent to u (there must be at least one since Γ is connected). Consider the Alexander submatrix $M_{B(w_n,\bar{u})}$ associated with relations of type (RB), which by Remark 5.39, is equivalent to:

$$\begin{array}{c} \begin{array}{ccccc} w_{n,0} & w_{n,1} & \dots & w_{n,k-2} & w_{n,k-1} \end{array} \\ \left(\begin{array}{ccccc} * & * & \dots & * & * \\ 0 & * & \dots & * & * \\ 0 & \ddots & \ddots & * & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & * \end{array}\right) \end{array}$$

Let $f_1, ..., f_k$ denote the k rows of this matrix.

On the other hand, let M_T be the $(n-1)\times n$ submatrix M_Γ associated with T. Let us order the relations in such a way that M_T is upper triangular with non-zero diagonal elements and whose last column corresponds to w_n – in other words, the vertex associated to the first column must have degree 1.

For each group of copies of the vertices $w_{j,p}$, there is a copy of the tree T with an analogous matrix $M_{T,p}$. Now, one can write the Alexander matrix M_Γ in the following way: the first rows correspond to the matrix M_T, then the row f_1 completed with zeroes where necessary, then the rows corresponding to the matrix $M_{T,1}$, then the row f_2. Finally the matrix $M_{T,k-1}$ and the row f_k. This matrix is clearly upper triangular and it has maximal rank $(kn = kn+m)$.

Summarizing, if Γ_u is connected, then $\operatorname{rank}(M_{\Gamma,u,k}) \geq nk+m$, and hence $\operatorname{corank}(M_{\Gamma,u,k}) \leq 1$.

Assume now that Γ_u is not connected, and denote by $\Gamma_1, ..., \Gamma_s$ its connected components.

Then, the Alexander matrix $M_{\Gamma,u,k}$ after removing the column $\bar{u}$ is of the form:

$$
\begin{array}{cccc}
C_1 & C_2 & ... & C_s \\
\end{array}
$$
$$
\begin{pmatrix}
M_{C_1} & 0 & \dots & 0 \\
0 & M_{C_2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \dots & 0 & M_{C_s}
\end{pmatrix}
$$

where M_{C_i} corresponds to a connected graph. The result follows from the connected case since the matrix is block-diagonal. □

Lemma 5.41. *Assume $e = \{w, u\} \in E$ such that $\ell_e \equiv 0 \mod k$, then the matrix $M_{B(w,\bar{u})}$ has rank 1 over $\Lambda/\mathfrak{p}$, where $\mathfrak{p}$ is the ideal generated by $1 - t_{\bar{u}} t_{w,0} \cdots t_{w,k-1}$.*

Proof. By Lemma 5.34 we know that $M_{B(w,\bar{u})}$ is a multiple by $p_{c_e}(t_{\bar{u}}\bar{t}_w)$ of the following matrix

$$
M = \begin{pmatrix}
1 - \bar{t}_{w,0,k} & t_{\bar{u}} - 1 & ... & \bar{t}_{w,0,k-1}(t_{\bar{u}} - 1) \\
\bar{t}_{w,1,k}(t_{w,0} - 1) & 1 - \bar{t}_{w,1,0} & ... & \bar{t}_{w,1,k-1}(t_{w,0} - 1) \\
... & ... & ... & ... \\
\bar{t}_{w,k-1,k}(t_{w,k-2} - 1) & \bar{t}_{w,k-1,0}(t_{w,k-2} - 1) & ... & t_{w,k-2} - 1
\end{pmatrix}
$$

Note that, mod $\mathfrak{p}$, M can be written as

$$
\begin{pmatrix}
t_{\bar{u}}^{-1}(t_0 - 1) & t_{\bar{u}} - 1 & ... & \bar{t}_{w,0,k-1}(t_{\bar{u}} - 1) \\
t_{\bar{u}}^{-1} t_{w,0}^{-1}(t_{w,0} - 1) & t_{w,0}^{-1}(t_{w,0} - 1) & ... & \bar{t}_{w,1,k-1}(t_{w,0} - 1) \\
... & ... & ... & ... \\
t_{\bar{u}}^{-1} t_{w,0}^{-1} ... t_{w,k-2}^{-1}(t_{w,k-2} - 1) & t_{w,0}^{-1} ... t_{w,k-2}^{-1}(t_{w,k-2} - 1) & ... & t_{w,k-2} - 1
\end{pmatrix}
$$

If f_j denotes the j-th row of M note that

$$
(t_{\bar{u}} - 1) t_{w,0} \cdots t_{w,j-2} f_j = (t_{w,j-2} - 1) f_1
$$

for any $j = 2, ..., k-1$ and thus the result follows. □

5.3 QP-Irreducible graphs

As mentioned in the Introduction, a graph is called *quasi-projective* – or QP-graph – if its associated Artin group is in $\mathcal{G}_{\text{QP}}$. The purpose of this section is to describe the simplest QP-graphs, referred to as QP-*irreducible* graphs for even Artin groups.

Definition 5.42. *We call Γ a* QP-***irreducible graph*** *if $\mathcal{A}_\Gamma$ is quasi-projective and it cannot be obtained as a 2-join of two quasi-projective graphs.*

By Proposition 5.7(2), the 2-join of QP-graphs must be a QP-graph. However, in general, properties on Artin groups are not easily read from subgraphs. This result allows one to read an obstruction to quasi-projectivity from certain subgraphs to graphs.

Definition 5.43. *We say that Γ_1 is a* v***-subgraph*** *of Γ if Γ_1 is obtained from Γ by deleting some vertices. We will denote it $\Gamma_1 \subset_v \Gamma$. In this situation Γ is called a* v***-supergraph*** *of Γ_1.*

Lemma 5.44. *Let $\mathcal{A}_{\Gamma_1}$ be the Artin group of $\Gamma_1 = (V_1, E_1, m_1)$. Assume that for certain $k \in \mathbb{Z}_{\geq 2}$, and $u \in \Gamma_1$, the subgroup $\hat{G}_k := \mathcal{A}_{\Gamma_1,u,k} \subset \mathcal{A}_{\Gamma_1}$ satisfies that there exist two ideals $\hat{I}_1, \hat{I}_2 \subset \hat{\Lambda}_k := \mathbb{C}[H_1(\hat{G}_k)]$ such that:*

(C1) *$Z(\hat{I}_i) \subset V_{r_i}(\hat{G}_k)$, $r_i \geq 1$ for $i = 1, 2$,*

(C2) *$\dim(Z(\hat{I}_1 + \hat{I}_2)) \geq 1$, and*

(C3) (a) *either $Z(\hat{I}_1 + \hat{I}_2) \subset V_r(\hat{G}_k)$ for $r > \max\{r_1, r_2\}$,*
(b) *or $\hat{I}_1, \hat{I}_2$ are prime ideals of $\hat{\Lambda}_k$.*

Then $\mathcal{A}_{\Gamma_1}$ is not quasi-projective.

Moreover, if $\Gamma = (V, E, m)$ is any v-supergraph of Γ_1 such that m_e is even for any $e = \{v, w\} \in E$, $v \in V_1$, $w \in V \setminus V_1$, then $\mathcal{A}_\Gamma$ is not quasi-projective.

Proof. Let us prove the first part by contradiction. Assuming that $\mathcal{A}_{\Gamma_1}$ is quasi-projective would imply that the co-cyclic group $\mathcal{A}_{\Gamma_1,u,k}$ is also quasi-projective by Proposition 5.7(1). The strategy of this proof is to reach a contradiction on the quasi-projectivity of $\mathcal{A}_{\Gamma_1,u,k}$ by finding two irreducible components of its characteristic variety intersecting in a positive dimensional component and thus contradicting Proposition 5.28. Let us assume that $r_1 \geq r_2$. Note that the set of zeroes $Z(\hat{I}_i)$ may be non-irreducible, but, using condition (C3)(a) in the statement, there exists an irreducible component, say H_1 (resp. H_2) in $Z(\hat{I}_1)$ (resp. $Z(\hat{I}_2)$) which is not contained in $Z(\hat{I}_2)$

(resp. in $Z(\hat{I}_1)$). By condition (C2) their intersection $H_1 \cap H_2$ has dimension greater or equal to 1.

To prove the *moreover* part, we will show that $\mathcal{A}_\Gamma$ also satisfies the hypotheses of the first part, that is, that there exist two ideals $I_1, I_2 \subset \Lambda_k := \mathbb{C}[H_1(G_k)]$ satisfying conditions (C1)-(C2) and either (C3)(a) or (C3)(b) for the subgroup $G_k := \mathcal{A}_{\Gamma,u,k} \subset \mathcal{A}_\Gamma$. Note that, the condition on the parity of the labels joining vertices from V_1 and $V \setminus V_1$ ensures the existence of a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & G_k & \to & G_\Gamma & \to & \mathbb{Z} & \to & 1 \\ & & & & \downarrow & & || & & \\ 1 & \to & \hat{G}_k & \to & G_{\Gamma_1} & \to & \mathbb{Z} & \to & 1 \end{array}$$

which allows for the existence of a morphism $H_1(G_k) \to H_1(\hat{G}_k)$ extending to $\Lambda_k \to \hat{\Lambda}_k$. Moreover, $\hat{\Lambda}_k = \Lambda_k / I$ for a certain ideal. In order to describe it let us decompose V as a disjoint union $V = V_1 \cup \tilde{V}_{2,u} \cup W$, where $\tilde{V}_{2,u} = \{v \in V \setminus V_1 \mid e = \{u,v\} \in E, m_e = 2\}$. Then

$$I = \text{Ideal}\left(\{t_v - 1 \mid v \in V \setminus \tilde{V}_{2,u}\} \cup \{t_{w,j} - 1 \mid w \in W, j \in \mathbb{Z}_k\}\right).$$

Since the tensor product is right exact, the matrix $\hat{M}_{\Gamma,u,k} = M_{\Gamma,u,k} \otimes \Lambda/I$ determines the Alexander $\hat{\Lambda}_k$-module of $\hat{G}_k$. We claim that

$$\hat{M}_{\Gamma,u,k} = \left(\begin{array}{c|c} M_{\Gamma_1,u,k} & 0 \\ \hline 0 & A' \end{array}\right). \tag{5.10}$$

In order to check this, first note that the submatrix of $\hat{M}_{\Gamma,u,k}$ whose rows are associated to the edges of Γ_1 has the form

$$\left(\begin{array}{c|c} M_{\Gamma_1,u,k} & 0 \end{array}\right).$$

The claim will follow if we prove that the remaining rows, associated with the edges in $E \setminus E_1$, satisfy that any entry in a column in V_1 is in $\hat{I}$. The latter is a consequence of (5.9) and Lemma 5.34.

Finally, note that if condition (C3)(a) (resp. (C3)(b)) is satisfied for $\hat{I}_i$, then also condition (C3)(a) (resp. (C3)(b)) is satisfied for $I_i = I + \hat{I}_i$ using (5.10) (resp. using that $Z(I_i) = Z(\hat{I}_i) \times \{1\}$ is irreducible). Therefore the ideals $I_1, I_2 \subset \Lambda_k$ also satisfy the conditions of the statement for $\mathcal{A}_\Gamma$ and the result follows. $\square$

Remark 5.45. *By Theorem 5.22, the only* QP*-irreducible right-angled graphs are sets of r disconnected vertices, $\overline{K}_r$. On the other hand, we have established by Theorem 5.23 that both the segment $S_{2\ell}$ with label 2ℓ ($\ell > 1$) and the triangle $T(4,4,2)$ are also* QP*-irreducible graphs.*

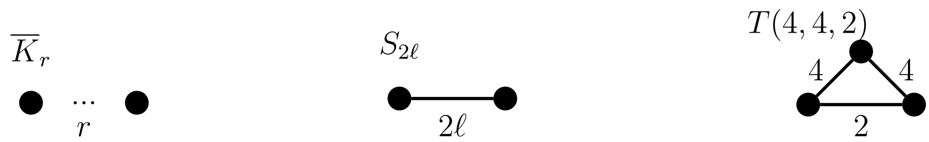


Figure 5.8: QP-irreducible graphs of type $\overline{K}_r$, $S_{2\ell}$, and $T(4,4,2)$.

The purpose of this section is to show that the only QP-irreducible graphs are $\overline{K}_r$, $S_{2\ell}$ ($\ell > 1$), and $T(4,4,2)$.

First we can assume that our graph has at least three vertices, otherwise it is QP-irreducible if and only if it is disconnected $\overline{K}_r$ ($r = 1, 2$) or a segment $S_{2\ell}$ ($\ell > 1$). The second reduction is given in [5], for *strictly even* graphs, that is, even graphs that are not right-angled. We recall it here.

Theorem 5.46 ([5, Thm. 5.26])**.** *If* Γ *is a strictly even, non-complete graph with at least three vertices, then* $\mathcal{A}_\Gamma$ *is not quasi-projective.*

This result is shown by proving that the characteristic varieties of the Artin groups of non-complete strictly even graphs contain two irreducible components having a positive dimensional intersection, which contradicts Proposition 5.28.

For the sake of completeness, we will include a proof of this theorem which will be developed in the next three lemmas. Notice that to prove Theorem 5.46 is enough to discard the following three cases:

Lemma 5.47. *If* $\hat{\Gamma}$ *is an even graph which contains a v-subgraph* $\Gamma \subset_v \hat{\Gamma}$ *as in Figure 5.9 with* $k, r \geq 2$*, then* $\mathcal{A}_{\hat{\Gamma}}$ *is not quasiprojective.*

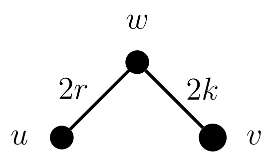


Figure 5.9

Proof. We will consider the subgroup $\mathcal{A}_{\Gamma,u,r} \subset \mathcal{A}_\Gamma$ of index r. The Alexander matrix, $M_{\Gamma,u,r}$, of the associated group is:

$$
\begin{array}{c}
\begin{array}{ccccccccc}
\bar{u} & v_0 & w_0 & v_1 & w_2 & \dots & v_{r-1} & w_{r-1}
\end{array}\\
\begin{pmatrix}
0 & p_{2k}(t_{v,0}t_{w,0})(1-t_{w,0}) & p_{2k}(t_{v,0}t_{w,0})(t_{v,0}-1) & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & p_{2k}(t_{v,1}t_{w,1})(1-t_{w,1}) & p_{2k}(t_{v,1}t_{w,1})(t_{v,1}-1) & \dots & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & 0 & 0 & \dots & p_{2k}(t_{v,r-1}t_{w,r-1})(1-t_{w,r-1}) & p_{2k}(t_{v,r-1}t_{w,r-1})(t_{v,r-1}-1)\\
1-\bar{t}_{v,0,r} & t_{\bar{u}}-1 & 0 & \bar{t}_{v,0,1}(t_{\bar{u}}-1) & 0 & \dots & \bar{t}_{v,0,r-1}(t_{\bar{u}}-1) & 0\\
\bar{t}_{v,1,r}(t_{v,0}-1) & 1-\bar{t}_{v,1,0} & 0 & t_{v,1}-1 & 0 & \dots & \bar{t}_{v,1,r-1}(t_{v,1}-1) & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\bar{t}_{v,r-1,r}(t_{v,r-2}-1) & \bar{t}_{v,r-1,0}(t_{v,r-2}-1) & 0 & \bar{t}_{v,r-1,1}(t_{v,r-2}-1) & 0 & \dots & t_{v,r-2}-1 & 0
\end{pmatrix}
\end{array}
$$

By Lemma 5.40, $M_{\Gamma,u,r}$ has rank $\geq 2r$. Since $M_{\Gamma,u,r}$ has exactly $2r$ rows, its rank must be $2r$, so it has corank 1. We now consider two ideals:

$$
\begin{aligned}
I_1 &= (t_{v,0}-1, t_{v,1}-1)\\
I_2 &= (t_{\bar{u}}-1, t_{w,1}-1) + \sum_{i\neq 1}(t_{v,i}-1).
\end{aligned}
$$

If we study the corank of $M_{\Gamma,u,r}|_{I_1}$ (resp. $M_{\Gamma,u,r}|_{I_2}$), it is easily seen that:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1} = \operatorname{corank} M_{\Gamma,u,r}|_{I_2} = 2 > \operatorname{corank} M_{\Gamma,u,r} = 1.$$

Now, $I_1 + I_2$ annuls the last r rows corresponding to the Pseudo Artin submatrix as well as the second row. Therefore, there are only $r-1$ non-zero rows and:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1+I_2} \geq r+2 > \max\{\operatorname{corank} M_{\Gamma,u,r}|_{I_1}, \operatorname{corank} M_{\Gamma,u,r}|_{I_2}\} = 2.$$

Moreover, note that, $Z(I_1+I_2)$ has dimension ≥ 1 since the variable $t_{w,0}$ is free. Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective. $\square$

Lemma 5.48. *If $\hat{\Gamma}$ is an even graph which contains a v-subgraph $\Gamma \subset_v \hat{\Gamma}$ as in Figure 5.10 with $r \geq 2$. Then $\mathcal{A}_{\hat{\Gamma}}$ is not a quasiprojective group.*

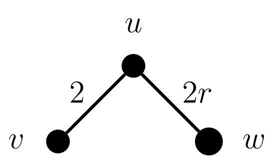


Figure 5.10

Proof. We will consider the subgroup $\mathcal{A}_{\Gamma,u,r} \subset \mathcal{A}_{\Gamma}$ of index r. The Alexander matrix of the associated group is:

$$M_{\Gamma,u,r} = \begin{array}{c} \begin{array}{cccccc} u & v & w_0 & w_1 & \dots & w_{r-1} \end{array} \\ \begin{pmatrix} 1-t_v & t_{\bar{u}}-1 & 0 & 0 & \dots & 0 \\ 1-\bar{t}_{w,0,r} & 0 & t_{\bar{u}}-1 & \bar{t}_{w,0,1}(t_{\bar{u}}-1) & \dots & \bar{t}_{w,0,r-1}(t_{\bar{u}}-1) \\ \bar{t}_{w,1,r}(t_{w,0}-1) & 0 & 1-\bar{t}_{w,1,0} & t_{w,0}-1 & \dots & \bar{t}_{w,1,r-1}(t_{w,0}-1) \\ \bar{t}_{w,2,r}(t_{w,1}-1) & 0 & \bar{t}_{w,2,0}(t_{w,1}-1) & 1-\bar{t}_{w,2,1} & \dots & \bar{t}_{w,2,r-1}(t_{w,1}-1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \bar{t}_{w,r-1,r}(t_{w,r-2}-1) & 0 & \bar{t}_{w,r-1,0}(t_{w,r-2}-1) & \bar{t}_{w,r-1,1}(t_{w,r-2}-1) & \dots & t_{w,r-2}-1 \end{pmatrix} \end{array}$$

By Lemma 5.38,the submatrix obtained by eliminating the first row has rank r, and the first row is independent with respect to the others (note the entries in column v), so $M_{\Gamma,u,r}$ has rank $r+1$ (so corank 1). We now consider two ideals:

$$I_1 = (t_{\bar{u}} - 1) + \sum_{i \neq 0} (t_{w,i} - 1),$$
$$I_2 = (t_{\bar{u}} - 1) + \sum_{j \neq 1} (t_{w,j} - 1).$$

Note that I_1 (resp. I_2) makes the columns v and w_0 (resp. w_1) vanish. Therefore:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1} = \operatorname{corank} M_{\Gamma,u,r}|_{I_2} = 2 > \operatorname{corank} M_{\Gamma,u,r} = 1.$$

Now, clearly:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1+I_2} \geq 3 > \max\{\operatorname{corank} M_{\Gamma,u,r}|_{I_1}, \operatorname{corank} M_{\Gamma,u,r}|_{I_2}\} = 2.$$

Moreover, $Z(I_1+I_2)$ has dimension ≥ 1 since the variable t_v is free. Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective. □

Lemma 5.49. *Assume $\hat{\Gamma}$ is an even graph which contains a v-subgraph $\Gamma \subset_v \hat{\Gamma}$ as in Figure 5.11 with $r \geq 1$. Then $\mathcal{A}_{\hat{\Gamma}}$ is not a quasiprojective group.*

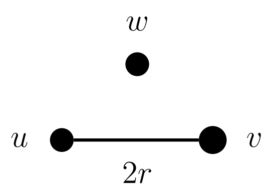


Figure 5.11

Proof. If $r = 1$ we have a right-angled graph, so we already know that it is not quasiprojective (by Theorem 5.22), so it is not a simple block. Therefore, we can supposse $r > 1$. Consider the subgroup $\mathcal{A}_{\Gamma,u,r} \subset \mathcal{A}_\Gamma$ of order r. Note that the Alexander matrix of the associated group is:

$$\begin{array}{cc}
 & \begin{array}{ccccccc} u & v_0 & \dots & v_{r-1} & w_0 & \dots & w_{r-1} \end{array} \\
M_{\Gamma,u,r} = & \left(\begin{array}{cccc|c}
1-\bar{t}_{v,0,r} & t_{\bar{u}}-1 & \dots & \bar{t}_{v,0,r-1}(t_{\bar{u}}-1) & \\
\bar{t}_{v,1,r}(t_{v,0}-1) & 1-\bar{t}_{v,1,0} & \dots & \bar{t}_{v,1,r-1}(t_{v,0}-1) & 0 \\
\dots & \dots & \dots & \dots & \\
\bar{t}_{v,r-1,r}(t_{v,r-2}-1) & \bar{t}_{v,r-1,0}(t_{v,r-2}-1) & \dots & t_{v,r-2}-1 &
\end{array}\right)
\end{array}$$

By Lemma 5.38, this matrix has rank r (i.e. maximal rank since it is a $(2r+1) \times r$ matrix), so it has corank $r+1$. Consider the following ideals:

$$I_1 = (t_{\bar{u}} - 1) + \sum_{i \neq 1} (t_{w,i} - 1),$$
$$I_2 = (t_{\bar{u}} - 1) + \sum_{j \neq 0} (t_{w,j} - 1).$$

Note that I_1 (resp. I_2) makes the column v_1 (resp. v_0) vanish. Therefore:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1} = \operatorname{corank} M_{\Gamma,u,r}|_{I_2} = r+2 > \operatorname{corank} M_{\Gamma,u,r} = r+1.$$

Now, clearly:

$$\operatorname{corank} M_{\Gamma,u,r}|_{I_1+I_2} = 2r+1 > \max\{\operatorname{corank} M_{\Gamma,u,r}|_{I_1}, \operatorname{corank} M_{\Gamma,u,r}|_{I_2}\} = r+2.$$

Moreover, $Z(I_1 + I_2)$ has dimension ≥ 1 since the variables $t_{w,p}$ are free. Therefore, by Lemma 5.44, the group $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective. □

Note that QP-irreducible even graphs – other than a point – are necessarily strictly even. Hence, the purpose of the rest of the section is to study complete QP-irreducible graphs of three or more vertices.

5.3.1 Complete QP-irreducible graphs with 3 vertices

Theorem 5.50. *Assume Γ is an even v-supergraph of $T(2r, 2k, 2\ell)$ with $r \geq 3$ and $k \geq 2$ – see Figure 5.12. Then $\mathcal{A}_\Gamma$ is not quasi-projective.*

Proof. Without loss of generality, one can assume $r \geq k \geq \ell$. Four separate cases will be considered.

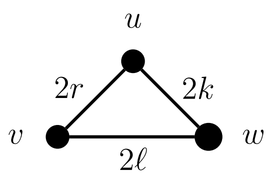


Figure 5.12

1. In case $r \geq 4$, $k \geq 2$, one can consider the index r subgroup $\mathcal{A}_{T,u,r} \subset \mathcal{A}_T$, where $T = T(2r, 2k, 2\ell)$. According to Lemma 5.37 $M_{T,u,r}$ has two B-Artin submatrices $M_{B(v,\bar{u})}$ and $M_{B(w,\bar{u})}$ of r rows each and an Artin submatrix of r rows (corresponding to the r relations $A_\ell(v_i, w_i)$ for $i \in \mathbb{Z}_r$. Hence $M_{T,u,r}$ is a $3r \times (2r+1)$ matrix whose corank is ≤ 1 by Lemma 5.40. Let us define $p = 1 - t_{\bar{u}}\bar{t}_v$ and consider the following ideals

$$I_1 = (p, t_{v,0} - 1, t_{v,1} - 1, t_{w,0} - 1, t_{w,1} - 1)$$
$$I_2 = (p, t_{v,0} - 1, t_{v,2} - 1, t_{w,0} - 1, t_{w,2} - 1).$$

Note that $\operatorname{rank}(M_{B(v,\bar{u})}|_{I_i}) = 1$ by Lemma 5.41. In addition, note that the first two rows of $M_{T,u,r}|_{I_1}$ are zero and also the first and third rows of $M_{T,u,r}|_{I_2}$. Summarizing, $M_{T,u,r}|_{I_i}$ contains three submatrices $M_{i,A}$, $M_{i,B(v,\bar{u})}$, and $M_{i,B(w,\bar{u})}$, where $\operatorname{rank}(M_{i,A}) \leq r-2$, $\operatorname{rank}(M_{i,B(v,\bar{u})}) = 1$, and $\operatorname{rank}(M_{i,B(w,\bar{u})}) \leq r$. Therefore $\operatorname{rank}(M_{T,u,r}|_{I_i}) \leq 2r-1$. Since $M_{T,u,r}$ has $2r+1$ columns, one has

$$\operatorname{corank}(M_{T,u,r}|_{I_i}) \geq 2 > \operatorname{corank} M_{T,u,r}.$$

Also

$$\operatorname{corank}(M_{T,u,r}|_{I_1+I_2}) < \max\{\operatorname{corank} M_{T,u,r}|_{I_1}, \operatorname{corank} M_{T,u,r}|_{I_2}\},$$

since I_1+I_2 makes one extra row vanish, which is originally independent from the others. Finally, $Z(I_1+I_2)$ has dimension ≥ 1 since the variable $t_{\bar{u}}$ is free (since $r \geq 4$, P gives a relation between $t_{v,3}$ and $t_{\bar{u}}$ but does not fix any of them). Therefore, by Lemma 5.44, $\mathcal{A}_\Gamma$ is not quasi-projective.

According to Remark 5.31, if $\gcd(k, r) = 1$ (resp. $\gcd(k, r) = 2$), then $t_{w,0} = t_{w,1} = t_{w,2}$ (resp. $t_{w,0} = t_{w,2}$). However, the variables $t_{v,i}$ are all different due to the choice of r, the label of the edge $\{u, v\}$, as the index of the finite subgroup. Hence ideals I_1 and I_2 satisfy the properties of Lemma 5.44 anyway.

2. Case $r = 3$ and $k = 3$ can be treated by considering $\mathcal{A}_{T,u,3} \subset \mathcal{A}_T$ the subgroup of index 3.

   The associated Alexander matrix, $M_{T,u,3}$, is:

$$
\begin{array}{c}
\begin{matrix} u & v_0 & w_0 & v_1 & w_1 & v_2 & w_2 \end{matrix} \\
\begin{pmatrix}
0 & p_\ell(t_{v,0}t_{w,0})(1-t_{w,0}) & p_\ell(t_{v,0}t_{w,0})(t_{v,0}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_\ell(t_{v,1}t_{w,1})(1-t_{w,1}) & p_\ell(t_{v,1}t_{w,1})(t_{v,1}-1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_\ell(t_{v,2}t_{w,2})(1-t_{w,2}) & p_\ell(t_{v,2}t_{w,2})(t_{w,2}-1) \\
1-t_{v,0}t_{v,1}t_{v,2} & t_{\bar{u}}-1 & 0 & t_{v,0}(t_{\bar{u}}-1) & 0 & t_{v,0}t_{v,1}(t_{\bar{u}}-1) & 0 \\
t_{v,1}t_{v,2}(t_{v,0}-1) & 1-t_{v,1}t_{v,2}t_{\bar{u}} & 0 & t_{v,0}-1 & 0 & t_{v,1}(t_{v,0}-1) & 0 \\
t_{v,2}(t_{v,1}-1) & t_{v,2}t_{\bar{u}}(t_{v,1}-1) & 0 & 1-t_{v,2}t_{\bar{u}}t_{v,0} & 0 & t_{v,1}-1 & 0 \\
1-t_{w,0}t_{w,1}t_{w,2} & 0 & t_{\bar{u}}-1 & 0 & t_{w,0}(t_{\bar{u}}-1) & 0 & t_{w,0}t_{w,1}(t_{\bar{u}}-1) \\
t_{w,1}t_{w,2}(t_{w,0}-1) & 0 & 1-t_{w,1}t_{w,2}t_{\bar{u}} & 0 & t_{w,0}-1 & 0 & t_{w,1}(t_{w,0}-1) \\
t_{w,2}(t_{w,1}-1) & 0 & t_{w,2}t_{\bar{u}}(t_{w,1}-1) & 0 & 1-t_{w,2}t_{\bar{u}}t_{w,0} & 0 & t_{w,1}-1
\end{pmatrix}
\end{array}
$$

   By Lemma 5.40, it has corank ≤ 1. We now consider the ideals:

$$
\begin{aligned}
I_1 &= (t_{\bar{u}}-1, t_{v,1}-1, t_{v,2}-1, t_{w,0}-1, t_{w,1}-1) \\
I_2 &= (t_{\bar{u}}-1, t_{v,1}-1, t_{v,2}-1, t_{w,0}-1, t_{w,2}-1).
\end{aligned}
$$

   Now,

$$
M_{\Gamma,w_0,3}|_{I_1} =
\begin{array}{c}
\begin{matrix} u & v_0 & w_0 & v_1 & w_1 & v_2 & w_2 \end{matrix} \\
\begin{pmatrix}
0 & 0 & p_\ell(t_{v,0}t_{w,0})(t_{v,0}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_\ell(t_{v,2}t_{w,2})(1-t_{w,2}) & 0 \\
1-t_{v,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
(t_{v,0}-1) & 0 & 0 & t_{v,0}-1 & 0 & (t_{v,0}-1) & 0 \\
0 & 0 & 0 & 1-t_{v,0} & 0 & 0 & 0 \\
1-t_{w,2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{w,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-t_{w,2} & 0 & 0
\end{pmatrix}
\end{array}
$$

   has corank 2, and:

$$
M_{\Gamma,w_0,3}|_{I_2} =
\begin{array}{c}
\begin{matrix} u & v_0 & w_0 & v_1 & w_1 & v_2 & w_2 \end{matrix} \\
\begin{pmatrix}
0 & 0 & p_\ell(t_{v,0}t_{w,0})(t_{v,0}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_\ell(t_{v,1}t_{w,1})(1-t_{w,1}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-t_{v,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
(t_{v,0}-1) & 0 & 0 & t_{v,0}-1 & 0 & (t_{v,0}-1) & 0 \\
0 & 0 & 0 & 1-t_{v,0} & 0 & 0 & 0 \\
1-t_{w,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{w,1} & 0 & 0 & 0 & 0 \\
(t_{w,1}-1) & 0 & (t_{w,1}-1) & 0 & 0 & 0 & t_{w,1}-1
\end{pmatrix}
\end{array}
$$

   has also clearly corank 2. Therefore

$$
2 = \max(\operatorname{corank} M_{T,u,3}|_{I_1}, \operatorname{corank} M_{T,u,3}|_{I_2}) > 1 \geq \operatorname{corank} M_{T,u,3}
$$

Also

$$
M_{T,u,3}|_{I_1+I_2} = \begin{array}{c} \begin{array}{ccccccc} u & v_0 & w_0 & v_1 & w_1 & v_2 & w_2 \end{array} \\ \begin{pmatrix}
0 & 0 & p_\ell(t_{v,0}t_{w,0})(t_{v,0}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-t_{v,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
(t_{v,0}-1) & 0 & 0 & t_{v,0}-1 & 0 & (t_{v,0}-1) & 0 \\
0 & 0 & 0 & 1-t_{v,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

has corank 3. Therefore

$$\operatorname{corank} M_{T,u,3}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{T,u,3}|_{I_1}, \operatorname{corank} M_{T,u,3}|_{I_2}\} = 2.$$

Moreover, $Z(I_1+I_2)$ has dimension ≥ 1 since the variable $t_{v,0}$ is free. Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective.

3. Case $r=3$, $k=\ell=2$ follows after considering the subgroup $\mathcal{A}_{T,u,2} \subset \mathcal{A}_T$ of index 2. The associated Alexander matrix is

$$
M_{T,u,2} = \begin{array}{c} \begin{array}{ccccc} \bar{u} & v_0 & w_0 & v_1 & w_1 \end{array} \\ \begin{pmatrix}
0 & p_0(1-t_{w,0}) & p_0(t_{v,0}-1) & 0 & 0 \\
0 & 0 & 0 & p_1(1-t_{w,1}) & p_1(t_{v,1}-1) \\
1-t_{v,0}t_{v,1} & t_{\bar{u}}-1 & 0 & t_{v,0}(t_{\bar{u}}-1) & 0 \\
t_{v,1}(t_{v,0}-1) & 1-t_{v,1}t_{\bar{u}} & 0 & t_{v,0}-1 & 0 \\
1-t_{w,0}t_{w,1} & 0 & t_{\bar{u}}-1 & 0 & t_{w,0}(t_{\bar{u}}-1) \\
t_{w,1}(t_{w,0}-1) & 0 & 1-t_{w,1}t_{\bar{u}} & 0 & t_{w,0}-1
\end{pmatrix} \end{array}
$$

with $p_i = 1 + t_{w,i}t_{v,i} + t_{w,i}^2 t_{v,i}^2$, $i = 0,1$.

By Lemma 5.40, it has corank ≤ 1. We now consider the ideals

$$
\begin{aligned}
I_1 &= (1 - t_{\bar{u}}\bar{t}_w, 1 - t_{\bar{u}}\bar{t}_v, p_0) \\
I_2 &= (1 - t_{\bar{u}}\bar{t}_w, 1 - t_{\bar{u}}\bar{t}_v, p_1),
\end{aligned}
$$

By Lemma 5.41, it is clearly seen that $M_{T,u,2}|I_i$ has corank 2. Therefore:

$$2 = \max(\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}) > 1 \geq \operatorname{corank} M_{T,u,2}.$$

It is also easy to see that $M_{T,u,2}|_{I_1+I_2}$ has corank 3. So, clearly:

$$\operatorname{corank} M_{T,u,2}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}\} = 2.$$

Moreover, $Z(I_1 + I_2)$ has dimension ≥ 1, since $I_1 + I_2$ is generated by four polynomials in five variables. Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective.

4. Finally, if $r = 3$, $k = 2$, and $\ell = 1$, the result follows after considering $\mathcal{A}_{T,v,3} \subset \mathcal{A}_T$ the subgroup of index 3 which has associated Alexander matrix:

$$M_{\Gamma,v,3} = \begin{array}{c} \begin{array}{ccccc} \bar{v} & w & u_0 & u_1 & u_2 \end{array} \\ \begin{pmatrix} 1 - t_w & t_{\bar{v}} - 1 & 0 & 0 & 0 \\ 0 & p_0(1 - t_{u,0}) & p_0(t_w - 1) & 0 & 0 \\ 0 & p_1(1 - t_{u,1}) & 0 & p_1(t_w - 1) & 0 \\ 0 & p_2(1 - t_{u,2}) & 0 & 0 & p_2(t_w - 1) \\ 1 - t_{u,0}t_{u,1}t_{u,2} & 0 & t_{\bar{v}} - 1 & t_{u,0}(t_{\bar{v}} - 1) & t_{u,0}t_{u,1}(t_{\bar{v}} - 1) \\ t_{u,1}t_{u,2}(t_{u,0} - 1) & 0 & 1 - t_{u,1}t_{u,2}t_{\bar{v}} & t_{u,0} - 1 & t_{u,1}(t_{u,0} - 1) \\ t_{u,2}(t_{u,1} - 1) & 0 & t_{u,2}t_{\bar{v}}(t_{u,1} - 1) & 1 - t_{\bar{v}}t_{u,0}t_{u,2} & t_{u,1} - 1 \end{pmatrix} \end{array}$$

where $p_i = 1 + t_w t_{u,i}$, $i = 0, 1, 2$. By Lemma 5.40, it has corank ≤ 1. We now consider the ideals:

$$I_1 = (p_0, p_1, 1 - t_{\bar{v}}\bar{t}_u)$$
$$I_2 = (p_0, p_2, 1 - t_{\bar{v}}\bar{t}_u),$$

By Lemma 5.41, it is clearly seen that $M_{\Gamma,v,3}|I_i$ has corank 2. Therefore

$$2 = \max(\operatorname{corank} M_{\Gamma,v,3}|_{I_1}, \operatorname{corank} M_{\Gamma,v,3}|_{I_2}) > 1 \geq \operatorname{corank} M_{\Gamma,v,3}.$$

It is also easy to see that $M_{\Gamma,v,3}|_{I_1+I_2}$ has corank 3. So, clearly

$$\operatorname{corank} M_{\Gamma,v,3}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{\Gamma,v,3}|_{I_1}, \operatorname{corank} M_{\Gamma,v,3}|_{I_2}\} = 2.$$

Moreover, $Z(I_1 + I_2)$ has dimension ≥ 1 since $I_1 + I_2$ is generated by four polynomials in five variables. Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective.

□

Theorem 5.51. *Assume Γ is an even v-supergraph of $T(4,4,4)$. Then, $\mathcal{A}_\Gamma$ is not quasi-projective.*

Proof. Consider $T = T(4,4,4)$ with vertices $V = \{u, v, w\}$ and the index 2 co-cyclic subgroup $\mathcal{A}_{T,u,2} \subset \mathcal{A}_T$. According to Lemma 5.37 the Alexander matrix of the associated group is

$$
\begin{array}{cccccc}
 & v_0 & w_0 & v_1 & w_1 & \bar{u} \\
M_{T,u,2} = & \multicolumn{5}{c}{\begin{pmatrix}
p_0(1-t_{w,0}) & p_0(t_{v,0}-1) & 0 & 0 & 0 \\
0 & 0 & p_1(1-t_{w,1}) & p_1(t_{v,1}-1) & 0 \\
t_{\bar{u}}-1 & 0 & t_{v,0}(t_{\bar{u}}-1) & 0 & 1-t_{v,0}t_{v,1} \\
1-t_{v,1}t_{\bar{u}} & 0 & t_{v,0}-1 & 0 & t_{v,1}(t_{v,0}-1) \\
0 & t_{\bar{u}}-1 & 0 & t_{w,0}(t_{\bar{u}}-1) & 1-t_{w,0}t_{w,1} \\
0 & 1-t_{w,1}t_{\bar{u}} & 0 & t_{w,0}-1 & t_{w,1}(t_{w,0}-1)
\end{pmatrix}}
\end{array}
$$

with $p_i = 1 + t_{v,i}t_{w,i}$, $i = 0, 1$. By Lemma 5.40, $M_{T,u,2}$ has corank ≤ 1. Consider the ideals

$$
\begin{aligned}
I_1 &= (1 - t_{\bar{u}}\bar{t}_v, 1 - t_{\bar{u}}\bar{t}_w, p_0) \\
I_2 &= (1 - t_{\bar{u}}\bar{t}_v, 1 - t_{\bar{u}}\bar{t}_w, p_1).
\end{aligned}
$$

By Lemma 5.41, it is clearly seen that $M_{T,u,2}|I_i$ has corank 2. Therefore

$$2 = \max\{\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}\} > 1 \geq \operatorname{corank} M_{T,u,2}$$

It is also easy to see that $M_{T,u,2}|_{I_1+I_2}$ has corank 3, which implies

$$\operatorname{corank} M_{T,u,2}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{T,u,2}|_{I_1}, \operatorname{corank} M_{T,u,2}|_{I_2}\} = 2.$$

Moreover, $Z(I_1 + I_2)$ has dimension ≥ 1 since $I_1 + I_2$ is generated by four polynomials in five variables. Therefore, by Lemma 5.44, $\mathcal{A}_\Gamma$ is not quasi-projective. □

The previous results combined prove the following.

Corollary 5.52. *The only strictly even complete* QP*-graphs with three vertices are $T(2\ell, 2, 2)$ with $\ell \geq 2$ and $T(4, 4, 2)$. Moreover, the latter is the only* QP*-irreducible even graph with three vertices.*

5.3.2 QP-irreducible even graphs with 4 vertices

As an immediate consequence of Theorems 5.50, 5.51, and 5.46, the only candidates to QP-irreducible even graphs with 4 vertices must be complete even v-supergraphs of either $T(2\ell, 2, 2)$ or $T(4, 4, 2)$. Figure 5.13 shows all possible such graphs.

This list can easily be obtained using the following observation.

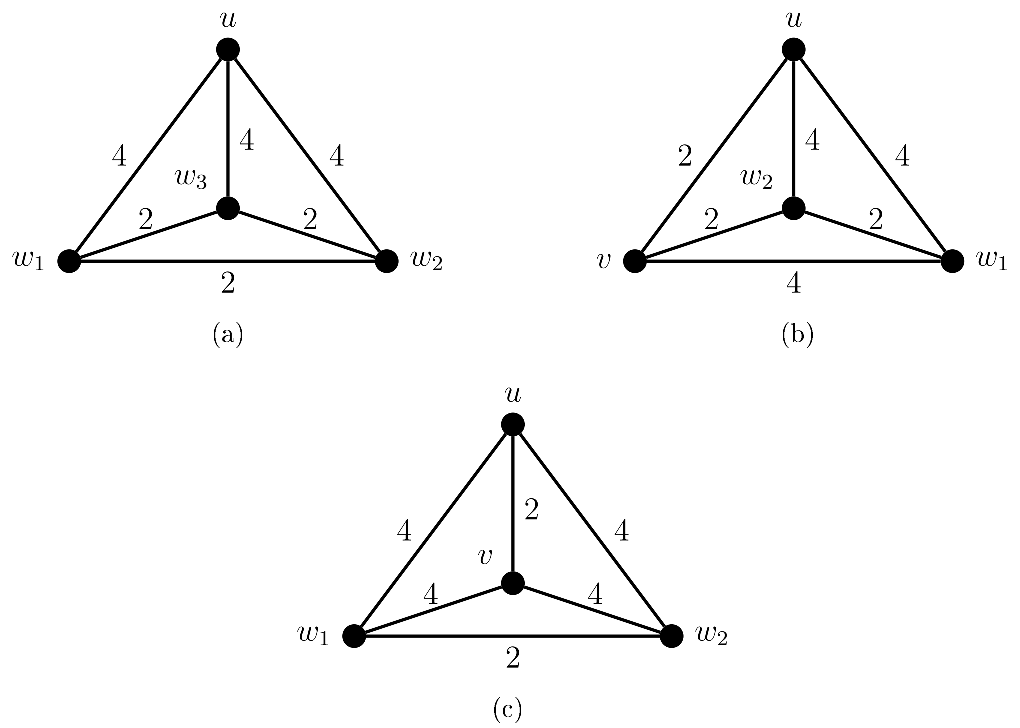


Figure 5.13

Lemma 5.53. *Any* QP*-irreducible even graph with at least 3 vertices has labels no larger than* 4.

Proof. By Theorem 5.46 one can assume the graph Γ is complete. Assume $m_e \geq 6$ for some edge $e \in E$ of Γ. By Theorem 5.50 all edges adjacent to e must have a label 2. Since Γ is complete $\Gamma = \{e\} *_2 \Gamma'$, where Γ' is the resulting v-subgraph after deleting the vertices of e. $\square$

Note that the 4-graph in Figure 5.13(a) is the only candidate containing $T(2,2,2)$, Figure 5.13(b) is the only candidate containing $T(4,4,2)$, but no $T(2,2,2)$, and Figure 5.13(c) is the only candidate containing $T(4,4,2)$ but no $T(2\ell,2,2)$.

We are going to see that these three candidates cannot be QP-irreducible graphs.

Theorem 5.54. *There are no* QP*-irreducible even graphs of four vertices.*

Moreover, an even graph containing any of the graphs in Figure 5.13 as a v-subgraph is not quasi-projective.

Proof. As discussed above, one only needs to rule out the list of graphs shown in figure 5.13. We will do this separately and using similar arguments.

- For graph 5.13(a) let us consider the index 2 subgroup $\mathcal{A}_{\Gamma,u,2} \subset \mathcal{A}_\Gamma$. Its Alexander matrix is given as follows:

$$M_{\Gamma,u,2} = \begin{array}{c} \begin{array}{ccccccc} w_{1,0} & w_{2,0} & w_{3,0} & w_{1,1} & w_{2,1} & w_{3,1} & \bar{u} \end{array} \\ \begin{pmatrix} 1-t_{2,0} & t_{1,0}-1 & 0 & 0 & 0 & 0 & 0 \\ 1-t_{3,0} & 0 & t_{1,0}-1 & 0 & 0 & 0 & 0 \\ 0 & 1-t_{3,0} & t_{2,0}-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-t_{2,1} & t_{1,1}-1 & 0 & 0 \\ 0 & 0 & 0 & 1-t_{3,1} & 0 & t_{1,1}-1 & 0 \\ 0 & 0 & 0 & 0 & 1-t_{3,1} & t_{2,1}-1 & 0 \\ t_{\bar{u}}-1 & 0 & 0 & t_{1,0}(t_{\bar{u}}-1) & 0 & 0 & 1-t_{1,0}t_{1,1} \\ 1-t_{1,1}t_{\bar{u}} & 0 & 0 & 1-t_{1,0} & 0 & 0 & t_{1,1}(t_{1,0}-1) \\ 0 & t_{\bar{u}}-1 & 0 & 0 & t_{2,0}(t_{\bar{u}}-1) & 0 & 1-t_{2,0}t_{2,1} \\ 0 & 1-t_{2,1}t_{\bar{u}} & 0 & 0 & t_{2,0}-1 & 0 & t_{2,1}(t_{2,0}-1) \\ 0 & 0 & t_{\bar{u}}-1 & 0 & 0 & t_{3,0}(t_{\bar{u}}-1) & 1-t_{3,0}t_{3,1} \\ 0 & 0 & 1-t_{3,1}t_{\bar{u}} & 0 & 0 & t_{3,0}-1 & t_{3,1}(t_{3,0}-1) \end{pmatrix} \end{array}$$

  where $t_{i,j}$ denotes $t_{w_{i,j}}$. We now consider the following prime ideals:

$$I_1 = (t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1, t_{2,1}-1, t_{3,0}-1)$$
$$I_2 = (t_{\bar{u}}-1, t_{1,0}-1, t_{1,1}-1, t_{2,1}-1, t_{3,0}-1).$$

  Note that $\operatorname{corank}(M_{\Gamma,u,2}|_{I_1}) = \operatorname{corank}(M_{\Gamma,u,2}|_{I_2}) = 2$, $\operatorname{corank}(M_{\Gamma,u,2}|_{I_1+I_2}) = 4$ and $Z(I_1+I_2)$ is equal to:

$$\{(t_{\bar{u}}, t_{1,0}, t_{1,1}, t_{2,0}, t_{2,1}, t_{3,0}, t_{3,1}) = (1,1,1,1,1,1,\lambda) \mid \lambda \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^7.$$

  The result follows from Lemma 5.44 and the fact that $\dim Z(I_1+I_2) = 1$.

- For graph 5.13(b) consider $\mathcal{A}_{\Gamma,u,2} \subset \mathcal{A}_\Gamma$. Computing the Alexander polynomials using SAGE we obtain that two of the prime ideals associated to the irreducible components are:

$$I_1 = (t_v-1, t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1), \quad I_2 = (t_{\bar{u}}-1, t_{1,1}-1, t_{2,0}-1, 1+t_{1,0}t_v).$$

  And it's easy to check that $Z(I_1+I_2)$ has dimension 1 instead of 0 ($\{(1,1,-1,1,1,\lambda) \in (\mathbb{C}^*)^5 \mid \lambda \in \mathbb{C}^*\} \subset V(I)$) and hence contradicts Theorem 5.28 .

  Therefore, by Theorem 5.28 and Proposition 5.7(1) $\mathcal{A}_\Gamma$ is not quasi-projective.

- For graph 5.13(c) the result follows considering the subgroup $\mathcal{A}_{\Gamma,u,2} \subset \mathcal{A}_\Gamma$. The Alexander matrix associated to the group is:

$$
M_{\Gamma,u,2} = \begin{array}{c}
\begin{array}{cccccc} \bar{u} & v & w_{1,0} & w_{2,0} & w_{1,1} & w_{2,1} \end{array} \\
\begin{pmatrix}
1-t_v & t_{\bar{u}}-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{2,0} & t_{1,0}-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-t_{2,1} & t_{1,1}-1 \\
0 & p_{1,0}(1-t_{1,0}) & p_1(t_v-1) & 0 & 0 & 0 \\
0 & p_{2,0}(1-t_{2,0}) & 0 & p_2(t_v-1) & 0 & 0 \\
0 & p_{1,1}(1-t_{1,1}) & 0 & 0 & p_{1,1}(t_v-1) & 0 \\
0 & p_{2,1}(1-t_{2,1}) & 0 & 0 & 0 & p_{2,1}(t_v-1) \\
1-t_{1,0}t_{1,1} & 0 & t_{\bar{u}}-1 & 0 & t_{1,0}(t_{\bar{u}}-1) & 0 \\
t_{1,1}(t_{1,0}-1) & 0 & 1-t_{\bar{u}}t_{1,1} & 0 & t_{1,0}-1 & 0 \\
1-t_{2,0}t_{2,1} & 0 & 0 & t_{\bar{u}}-1 & 0 & t_{2,0}(t_{\bar{u}}-1) \\
t_{2,1}(1-t_{2,0}) & 0 & 0 & 1-t_{\bar{u}}t_{2,1} & 0 & t_{2,0}-1
\end{pmatrix}
\end{array}
$$

where $p_{1,i} = 1 + t_{1,i}t_v$, and $p_{2,i} = 1 + t_{2,i}t_v$.

By Lemma 5.40, $M_{\Gamma,u,2}$ has corank ≤ 1. We consider the prime ideals

$$
I_1 = (t_{\bar{u}} - 1, t_{1,0} - 1, t_{2,1} - 1, p_{1,1}), \quad I_2 = (t_{\bar{u}} - 1, t_{1,0} - 1, t_{2,1} - 1, p_{2,0}).
$$

Now,

$$
M_{\Gamma,w_0,2}|_{I_1} = \begin{array}{c}
\begin{array}{cccccc} \bar{u} & v & w_{1,0} & w_{2,0} & w_{1,1} & w_{2,1} \end{array} \\
\begin{pmatrix}
1-t_v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{2,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{1,1}-1 \\
0 & 0 & p_{1,0}(s_1-1) & 0 & 0 & 0 \\
0 & p_{2,0}(1-t_{2,0}) & 0 & p_{2,0}(t_v-1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{2,1}(t_v-1) \\
1-t_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{1,1} & 0 & 0 & 0 \\
1-t_{2,0} & 0 & 0 & 0 & 0 & 0 \\
(1-t_{2,0}) & 0 & 0 & 0 & 0 & t_{2,0}-1
\end{pmatrix}
\end{array}
$$

has clearly corank 2 and

$$
M_{\Gamma,u,2}|_{I_2} = \begin{array}{cccccc}
\bar{u} & v & w_{1,0} & w_{2,0} & w_{1,1} & w_{2,1} \\
\end{array}
\begin{pmatrix}
1-t_v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{2,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{1,1}-1 \\
0 & 0 & p_{1,0}(t_v-1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{1,1}(1-t_{1,1}) & 0 & 0 & p_{1,1}(t_v-1) & 0 \\
0 & 0 & 0 & 0 & 0 & p_{2,1}(t_v-1) \\
1-t_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{1,1} & 0 & 0 & 0 \\
1-t_{2,1} & 0 & 0 & 0 & 0 & 0 \\
(1-t_{2,0}) & 0 & 0 & 0 & 0 & t_{2,0}-1
\end{pmatrix}
$$

has also corank 2.

Therefore:

$$2 = \max(\operatorname{corank} M_{\Gamma,u,2}|_{I_1}, \operatorname{corank} M_{\Gamma,u,2}|_{I_2}) > 1 \geq \operatorname{corank} M_{\Gamma,u,2}$$

Besides:

$$
M_{\Gamma,u,2}|_{I_1+I_2} = \begin{array}{cccccc}
\bar{u} & v & w_{1,0} & w_{2,0} & w_{1,1} & w_{2,1} \\
\end{array}
\begin{pmatrix}
1-t_v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{2,0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{1,1}-1 \\
0 & 0 & p_{1,0}(t_v-1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{2,1}(t_v-1) \\
1-t_{1,1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-t_{1,1} & 0 & 0 & 0 \\
1-t_{2,0} & 0 & 0 & 0 & 0 & 0 \\
(1-t_{2,0}) & 0 & 0 & 0 & 0 & t_{2,0}-1
\end{pmatrix}
$$

has corank 3. So, clearly:

$$\operatorname{corank} M_{\Gamma,u,2}|_{I_1+I_2} = 3 > \max\{\operatorname{corank} M_{\Gamma,u,2}|_{I_1}, \operatorname{corank} M_{\Gamma,u,2}|_{I_2}\} = 2$$

Besides, $Z(I_1 + I_2)$ has dimension ≥ 1 since we have six different variables but only 5 polynomials.

Therefore, by Lemma 5.44, $\mathcal{A}_{\hat{\Gamma}}$ is not quasiprojective.

$\square$

5.3.3 QP-irreducible even graphs with more than 4 vertices

As a consequence of the results obtained in the previous sections, no quasi-projective even Artin group can contain any of the following subgraphs:

1. A vertex with two edges with labels $2r$, $r \geq 3$ and $2k$, $k \geq 2$ – see Theorems 5.50 and 5.46.

2. A triangle $T(4,4,4)$ – see Theorem 5.51.

3. A three-edge tree of labels $4,4,4$ – see Theorems 5.46, 5.51, and 5.54.

Theorem 5.55. *There are no* QP*-irreducible even graphs with more than* 3 *vertices.*

Proof. The result follows for graphs with four vertices by the previous section.

For any QP-irreducible even graph Γ with more than four vertices note the following:

- Γ must be complete by Theorem 5.46.

- If Γ contains an edge e with label $m_e = 2r$, $r \geq 3$, then by (1) above, $\Gamma = \{e\} *_2 \hat{\Gamma}$ and hence Γ is not QP-irreducible.

- If Γ contains an edge e with label $m_e = 4$, then either $\Gamma = \{e\} *_2 \hat{\Gamma}$ (see (2) above) or $\Gamma = T(4,4,2) *_2 \hat{\Gamma}$ (see (3)).

□

5.4 Proofs of Main Theorems

5.4.1 Proof of Theorem 5.1

As a consequence of Theorems 5.22 and 5.23 graphs of type $\overline{K}_r$, $S_{2\ell}$, and $T(4,4,2)$ as in Figure 5.8 are QP-irreducible. Moreover, by Corollary 5.52 and Theorem 5.55 these are the only ones. Using equiation (5.1) and Proposition 5.7(2) any 2-join QP-irreducible graphs is quasi-projective. This completes the first part of the proof.

For the *moreover* part it is enough to check that the product of two fundamental groups of curve complements is also the fundamental group of a curve complement. This is a consequence of the following result due to Oka and Sakamoto.

Theorem 5.56. *[69] Let C_1 and C_2 be plane algebraic curves in $\mathbb{C}^2$. Assume that the intersection $C_1 \cap C_2$ consist of distinct $d_1 d_2$ points where d_i $(i = 1, 2)$ are respective degrees of C_1 and C_2. Then the fundamental group $\pi_1(\mathbb{C}^2 \setminus (C_1 \cup C_2))$ is isomorphic to the product of $\pi_1(\mathbb{C}^2 \setminus C_1)$ and $\pi_1(\mathbb{C}^2 \setminus C_2)$.*

5.4.2 Proof of Theorem 5.3

Since the product of two $K(\pi, 1)$ spaces is also a $K(\pi, 1)$ space it is enough to prove the result for the QP-irreducible even graphs $\overline{K}_r$, $S_{2\ell}$, and $T(4, 4, 2)$. The graph $\overline{K}_r$ is associated with the free group $\mathbb{F}_r$ of rank r, which can be realized as the fundamental group of the complement to r points in $\mathbb{C}$, which is an Eilenberg-MacLane space.

The group $\mathcal{A}_{S_{2\ell}}$ associated with the segment graph $S_{2\ell}$ is the fundamental group of the complement X to the affine curve $\{y - x^\ell\} \cup \{y = 0\}$. In projective coordinates X can be seen as the complement to the projective curve $\mathcal{C} = \{yz(yz^{\ell-1} - x^\ell) = 0\} \subset \mathbb{P}^2$, that is, $X = \mathbb{P}^2 \setminus \mathcal{C}$. Consider the projection $\pi : \mathbb{P}^2 \setminus \{[1 : 0 : 0]\} \to \mathbb{P}^1$, defined by $[x : y : z] \mapsto [y : z]$. Note that $\pi|_X :\to \mathbb{P}^1 \setminus \{[0 : 1], [1 : 0]\}$ is well defined, locally trivial fibration and moreover, the fiber at each point $[y : z]$ is homeomorphic to $\mathbb{C} \setminus \{\ell \text{ points}\}$. Thus X is also an Eilenberg-MacLane space.

Finally, the triangle Artin group $\mathcal{A}_T$ associated with the triangle $T = T(4, 4, 2)$ is the fundamental group of the complement X to the affine curve $\{y - x^2\} \cup \{2x - y - 1 = 0\} \cup \{2x + y + 1 = 0\}$ [5, Example 5.11]. Using the identification $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{z = 0\}$ we can think of X as the complement of a smooth conic and three tangent lines in the complex projective plane. After an appropriate change of coordinates, X can be given as $\mathbb{P}^2 \setminus \mathcal{C}$, where $\mathcal{C} = \{F(x, y, x) = xyz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0\} \subset \mathbb{P}^2$.

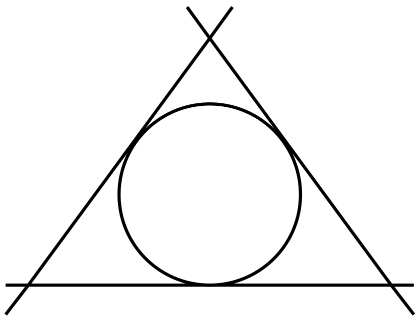


Figure 5.14: Projective curve $\mathcal{C} = \{F = 0\}$

We will consider a 4-fold cover X_4 of X. Since the higher homotopy groups of X and X_4 are isomorphic, it is enough to show that X_4 is an

Eilenberg-MacLane space. Consider the Kummer morphism $\kappa : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $\kappa([x:y:z]) = [x^2 : y^2 : z^2]$. Note that κ is a 4:1 ramified cover and its ramification locus is $R = \{xyz = 0\}$. Since $R \subset \mathcal{C}$ κ defines an unramified cover on $X_4 = \kappa^{-1}(X)$. Moreover, the preimage of $\mathcal{C}$ by κ is a product of 7 lines, three of which are the axis $xyz = 0$ and four of them are the preimage of the conic $(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0$. In particular

$$\begin{aligned} \mathcal{C}_2 = \kappa^{-1}(\mathcal{C}) = & \ \{F(x^2, y^2, z^2) = 0\} \\ = & \ \{xyz(x+y+z)(x+y-z)(x-y+z)(x-y-z) = 0\}. \end{aligned}$$

Geometrically this corresponds to a Ceva arrangement – formed by the six lines of a generic pencil of conics – with an extra line passing through two out of the three double points. In our equations, the pencil of conics can be defined as $F_{[\alpha:\beta]} = \alpha((x+z)^2 - y^2) - \beta((x-z)^2 - y^2)$. Note that for $\alpha = \beta = 1$ one obtains $F_{[1:1]} = 4xz$. The rational map $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by the pencil, where $\pi^{-1}[\alpha : \beta] = \{F_{[\alpha:\beta]} = 0\}$, that is, $\pi([x : y : z]) = [(x-z)^2 - y^2 : (x+z)^2 - y^2]$ is not defined at the base points of the pencil. Since the curve $\mathcal{C}_2$ contains these base points, one obtains that $\pi|_{X_4}$ is well defined, where $X_4 = \kappa^{-1}(X) = \mathbb{P}^2 \setminus \mathcal{C}_2$.

After our discussion above, recall that the special fibers of π are the six lines $\{xz(x+y+z)(x+y-z)(x-y+z)(x-y-z) = 0\}$. Finally, note that the line $y = 0$ is a multisection since $\pi|_{y=0}$ is defined by $\pi([x : 0 : z]) = [(x-z)^2 : (x+z)^2]$ which is 2:1 and ramifies only at $[0 : 1]$ and $[1 : 0]$, therefore the map

$$\begin{aligned} \pi|_{X_4} : X_4 & \to \mathbb{P}^1 \setminus \{[0:1], [1:0], [1:1]\} \\ [x:y:z] & \mapsto [(x-z)^2 - y^2 : (x+z)^2 - y^2] \end{aligned}$$

is a well-defined locally trivial fibration whose generic fiber is the smooth conic of the pencil with six points removed (the four base points and the two points of intersection with the multisection $\{y = 0\}$). Therefore X_4 is an Eilenberg-MacLane space.

5.5 An example

To end this chapter we take a closer look into the triangle Artin group $\mathcal{A}_T$, $T = T(4,4,2)$ given by geometrical methods coming from its quasi-projectivity property.

First we will show that $\mathcal{A}_T$ is not an extension of free groups. To do so we first study the surjections of $\mathcal{A}_T$ onto a free group $\mathbb{F}_r$ of rank r. Any surjection of groups $G_1 \twoheadrightarrow G_2$ induces an injection of characteristic varieties $V_i(G_2) \hookrightarrow V_i(G_1)$ via the change of base $* \otimes_{\mathbb{C}[G_2/G_2']} \mathbb{C}[G_1/G_1']$ that turns an ideal in $\mathbb{C}[G_2/G_2']$ to an ideal in $\mathbb{C}[G_1/G_1']$ (see [61]). An Alexander matrix of $\mathcal{A}_T$ can be obtained immediately from Lemma 5.33 and (5.9) as

$$M_{\mathcal{A}_T} = \begin{pmatrix} -(t_0t_1+1)(t_1-1) & (t_0t_1+1)(t_0-1) & 0 \\ -(t_0t_2+1)(t_2-1) & 0 & (t_0t_2+1)(t_0-1) \\ 0 & -t_2+1 & t_1-1 \end{pmatrix}$$

and thus its characteristic variety $V_1(\mathcal{A}_T) = \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3$ is the zero set of the Fitting ideal generated by the 2×2-minors of $M_{\mathcal{A}_T}$, where

$$\begin{aligned} \mathbb{T}_1 &= \{(-t^{-1}, t, 1) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3, \\ \mathbb{T}_2 &= \{(-t^{-1}, 1, t) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3, \\ \mathbb{T}_3 &= \{(-t^{-1}, t, t) \mid t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^3 \end{aligned}$$

are three one-dimensional complex tori in $(\mathbb{C}^*)^3$. Since the characteristic variety of the free group $\mathbb{F}_r$ has dimension r, this implies that the only possible surjection $\mathcal{A}_T \twoheadrightarrow \mathbb{F}_r$ is restricted to $r = 1$.

Note that any short exact sequence

$$1 \to \mathbb{F}_s \to \mathcal{A}_T \to \mathbb{Z} \to 0$$

splits and the action of $\mathbb{Z}$ on $\mathcal{A}_T$ is trivial in homology. Therefore $\mathcal{A}_T = \mathbb{F}_s \rtimes \mathbb{Z}$ is called an *IA-product of free groups* and by [32, Corollary 3.4] the Poincaré polynomial $P_{\mathcal{A}_T}(t)$ of $\mathcal{A}_T$ should factor as a product of linear terms in $\mathbb{Z}[t]$. However, since the complement $X = \mathbb{P}^2 \setminus \mathcal{C}$ of the conic an three tangent lines shown in Figure 5.14 is a $K(\mathcal{A}_T, 1)$-space it is enough to calculate $P_X(t)$. One can easily check that $h_0(X) = 1$ and $h_1(X) = 3$. Moreover, using the additivity of the Euler characteristic

$$\begin{aligned} \chi(X) &= \chi(\mathbb{P}^2) - \sum \chi(\mathcal{C}_i) + \#\,\mathrm{Sing}(\mathcal{C}) = 3 - 4\chi(\mathbb{P}^1) + 3 = 1 \\ &= h_0(X) - h_1(X) + h_2(X) = -2 + h_2(X), \end{aligned}$$

where $\mathcal{C}_i$ are the irreducible components of $\mathcal{C}$ and $\chi(\mathcal{C}_i) = \chi(\mathbb{P}^1) = 2$ since they are all rational curves. Therefore $h_2(X) = 3$ and thus

$$P_{\mathcal{A}_T}(t) = P_X(t) = 3t^2 + 3t + 1$$

which is not a product of linear factors in $\mathbb{Z}[t]$.

However, as shown in the proof of Theorem 5.3, – see section 5.4.2 – its 4-fold cover X_4 is the complement of a line arrangement of fibered type whose fundamental group $\pi_1(X_4)$ is a finite index normal subgroup of $\mathcal{A}_T$ which is an IA-free product of free groups $\mathbb{F}_3 \rtimes \mathbb{F}_3$.

Conclusiones y trabajo futuro

En conclusión, durante mis años de doctorado he estado trabajando en la extensión de tres resultados y propiedades conocidas para los grupos de Artin de ángulo recto a otras familias más grandes de grupos de Artin (pares):

- En relación con la propiedad de la poli-libertad, hemos sido capaces de probar que los grupos de Artin pares de tipo FC y de tipo large son poli-libres. Por el momento, no podemos extenderlo más debido a lo siguiente: por un lado, el tipo de técnicas que usamos se basan en el hecho de que nuestras relaciones son de tipo par. Por otro lado, una extensión a familias más generales de grupos de Artin pares parece difícil ya que necesitaríamos formas normales y no hay solución conocida del problema de la palabra para grupos de Artin pares en general.

  Durante una estancia de investigación que hice en Newcastle, empecé a trabajar con Sarah Rees en el problema de extender su algoritmo para formas normales de grupos de Artin de tipo large a familias más grandes de grupos de Artin. Este es actualmente un proyecto abierto pero creemos que seremos capaces de extender el algortimo un poco más.

  Si tenemos éxito con este problema, un proyecto futuro sería aplicar el algoritmo obtenido para tratar de extender nuestras demostraciones de la poli-libertad a la nueva familia de grupos de Artin pares para la que tendríamos formas normales.

- Respecto a la propiedad de ser residualmente finitos, hemos demostrado esta propiedad para los grupos de Artin pares de tipo FC y para los grupos de Artin basados en grafos que son bosques. Por el momento no somos capaces de extender más esta propiedad, pero otro proyecto interesante para el futuro sería tratar de extenderla a familias más grandes.

- Además, hemos sido capaces de caracterizar en términos del grafo cuándo un grupo de Artin par es cuasiproyectivo. Ahora estamos estudiando este problema omitiendo la condición de que las relaciones sean pares y tratando de resolver el problema para grupos de Artin generales. Si consiguiésemos esa tarea, también querríamos comprobar si se verifica la conjetura $K(\pi, 1)$ cuasiproyectiva que hemos establecido.

Además, actualmente estoy trabajando en otros problemas interesantes. Uno de ellos es el problema de la rigidez en los grupos de Artin pares, esto es, de saber si dados dos grupos de Artin $A_\Gamma \simeq A_\Omega$ implica o no que $\Gamma \simeq \Omega$, en el cual esto trabajando con Luis Paris.

Finalmente, junto con mis directores de tesis Conchita Martínez-Pérez y José Ignacio Cogolludo-Agustín, estoy estudiando los grupos de homología de los núcleos de Artin, una estructura relacionada con los grupos de Artin.

Conclusions and future work

Summing up, during these years of my PhD we have been working on the extension to more general families of (even) Artin groups of known results and properties of three types for RAAGs:

- In relation to the property of poly-freeness we have been able to prove that even Artin groups of FC type and large even Artin groups are poly-free. For the moment we can't extend it anymore because of the following: on the one hand, the type of techniques that we use rely on the fact that our relators are of even type. On the other hand, an extension to more general families of even Artin groups seems difficult since we would need normal forms and there is no known solution for the word problem for general even Artin groups.

  During a research stay in Newcastle I began to work with Sarah Rees in the problem of extending their algorithm for normal forms of large Artin groups to bigger families of Artin groups. This is nowadays work in progress but we think we will be able to extend the algorithm a bit more.

  If we succeed with this problem, a future project would be to apply the obtained algorithm to try to extend also our proofs of poly-freeness to the new family of even Artin groups for which we would have normal forms.

- With respect to residually finiteness, we have established this property for the family of even Artin groups of FC type and for Artin groups based on forests. For the moment we haven't been able of extending more this property, but another interesting future project would be to try to extend it to bigger families.

- Besides, we have been able of characterizing in terms of the graph when an even Artin group is quasi-projective. Now we are studying this problem without the even condition and trying to solve the problem for general Artin groups. Moreover, if we succeed in this task we would also like to check the Quasi-projective $K(\pi, 1)$ conjecture that we have stated.

Moreover, nowadays I am working on other interesting problems. One of them is the problem of the rigidity of even Artin groups, i.e. knowing if $\mathcal{A}_\Gamma \simeq \mathcal{A}_\Omega$ implies or not that $\Gamma \simeq \Omega$, in which I am working on together with Luis Paris .

Finally, together with my PhD advisors Conchita Martínez-Pérez and José Ignacio Cogolludo-Agustín, I am studying the homology groups of Artin kernels, a structure related with Artin groups.

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