

2D Well-balanced Augmented ADER schemes for the Shallow Water Equations with bed elevation and extension to the rotating frame

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Abstract

In this work, an arbitrary order augmented WENO-ADER scheme for the resolution of the 2D Shallow Water Equations (SWE) with geometric source term is presented and its application to other shallow water models involving non-geometric sources is explored. This scheme is based in the 1D Augmented Roe Linearized-ADER (ARL-ADER) scheme, presented by the authors in a previous work and motivated by a suitable compromise between accuracy and computational cost. It can be regarded as an arbitrary order version of the Augmented Roe solver, which accounts for the contribution of continuous and discontinuous geometric source terms at cell interfaces in the resolution of the Derivative Riemann Problem (DRP). The main novelty of this work is the extension of the ARL-ADER scheme to 2 dimensions, which involves the design of a particular procedure for the integration of the source term with arbitrary order that ensures an exact balance between flux fluctuations and sources. This procedure makes the scheme preserve equilibrium solutions with machine precision and capture the transient waves accurately. The scheme is applied to the SWE with bed variation and is extended to handle non-geometric source terms such as the Coriolis source term. When considering the SWE with bed variation and Coriolis, the most relevant equilibrium states are the still water at rest and the geostrophic equilibrium. The traditional well-balanced property is extended to satisfy the geostrophic equilibrium. This is achieved by means of a geometric reinterpretation of the Coriolis source term. By doing this, the formulation of the source terms is unified leading to a single geometric source regarded as an apparent topography. The numerical scheme is tested for a broad variety of situations, including some cases where the first order scheme ruins the solution.

Keywords: ADER, Shallow water, Source terms, Coriolis, Well-balanced, High order accuracy.

1. Introduction

Finite Volume (FV) numerical schemes are a very common choice for the resolution of complex flows of hyperbolic nature. Such schemes have experienced a great improvement in the past decades thanks to the development of sophisticated schemes that ensure a high order of accuracy both in space and time when computing the solution. In the framework of FV, the introduction of the ENO and, specially, the WENO reconstruction techniques [4, 5] supposed a major step when seeking arbitrary order of accuracy in space. On the other hand, the preservation of high order in time was generally done by means of a Runge-Kutta time integrators, which sometimes proved to be inefficient due to Butcher's barrier [6]. This issue was circumvented by the ADER approach [7, 8], which provides a fully discrete scheme of arbitrary order. The original ADER idea can be regarded as a high order generalization of Godunov's method by means of a Taylor power series expansion in time, where time derivatives are computed from the reconstructed space derivatives by means of the Cauchy-Kowalewski (CK) procedure. ADER schemes consist of two steps: first, a high-order spatial reconstruction procedure and secondly, the resolution of a high order extension of the Riemann Problem (RP), called Derivative Riemann Problem (DRP) [9]. The initial condition for the DRP consists of piecewise polynomial

16 data with K nontrivial derivatives, usually constructed by means of non-oscillatory and arbitrary-
17 order reconstruction procedures, such as the aforementioned ENO or WENO methods. The ADER
18 approach successfully allows the construction of arbitrary order schemes for systems of hyperbolic
19 conservation laws [9, 10, 11, 6, 12]. It is of particular interest for this work to consider the application
20 of ADER schemes for the resolution of geophysical problems [13, 14, 15], specially for the resolution of
21 the Shallow Water Equations (SWE) [16, 15, 17, 18, 19], which will be the focus of this work.

22 An approach for the resolution of the DRP_K was first presented in [9] and the proposed solver
23 was called Toro–Titarev (TT) solver. This solver allows to reduce the DRP_K to a series of classical
24 Riemann problems where classic Riemann solvers are of application. The DRP_K consists of one RP for
25 the leading term, referred to as DRP_0 , plus K additional RPs for the derivatives. Up to date, a broad
26 variety of Derivative Riemann solvers have been designed with the aim of providing accurate and fast
27 solutions. Apart from the TT solver, the most common solvers we can find in the literature are the
28 HEOC solver [16, 4] and the Castro–Toro (CT) solver [16]. A semi-implicit version of those schemes
29 was proposed by Montecinos [20], allowing to deal with more stiff source terms. It is worth pointing
30 out that the traditional ADER approach using the CK procedure may become rather cumbersome
31 when dealing with complex systems of equations and may not provide the expected performance when
32 dealing with very stiff source terms. In such a case, a successful solution would be to replace the CK
33 procedure by a local spacetime Galerkin method, as done by the DET solver, proposed by Dumbser in
34 [21].

35 The TT, CT and HEOC solvers assume that there is a single solution, referred to as the star solution
36 [22], with independence of the presence of source terms. However, when dealing with geometric source
37 terms it was shown that the consideration of the source term in the resolution of the DRP is more
38 convenient in order to ensure certain properties of the numerical solution [23, 24]. Those solvers that
39 account for the source term in the resolution of the (D)RP are called augmented solvers [23] and allow
40 to exactly preserve the equilibrium states [25, 26, 27].

41 In this work, we will focus on the use of augmented solvers for the resolution of the DRP in order to
42 construct arbitrary order schemes for the 2D SWE with source term. To account for the contribution
43 of the source term in the DRP, the FS and LFS augmented solvers were introduced in [28, 29]. They
44 can be combined with the Augmented Roe (ARoe) solver [24] and HLLS solvers [26], which are the
45 augmented versions of the Roe [30] and HLL solvers [31, 32]. Such solvers, called AR-ADER, ARL-
46 ADER, HLLS-ADER and HLLSL-ADER schemes, provide an arbitrary order approximate solution to
47 the DRP. Here, the LFS solver in combination with the ARoe solver will be used. Such method showed
48 a good performance when solving the SWE with geometric source term and is computationally more
49 efficient than the FS solver [29].

50 This work has two main aims: (a) to extend the aforementioned methods to 2D to solve the 2D
51 SWE with geometric source term and (b) to explore the resolution of the SWE with source terms
52 of non-geometric nature by doing a geometric reinterpretation of those sources, which allows the
53 preservation of equilibrium states. We first develop the 2D extension of the 1D ARL-ADER scheme
54 in [29] for hyperbolic systems of equations with geometric source terms. This is achieved by means
55 of two techniques: (a) the use of the augmented LFS solver and (b) a particular procedure for the
56 integration of the source term inside cells that ensures an exact balance between flux fluctuations and
57 source terms. This procedure preserves the equilibrium states with machine precision and guarantees
58 an arbitrary order of accuracy thanks to the use of Romberg integration method [51], provided an
59 optimal derivation of time derivatives. Then, the aforestated methods are extended for the numerical
60 treatment of other types of source terms by expressing them in geometric form

61 When considering the numerical resolution of the SWE with source terms, it must be borne in
62 mind that a suitable discretization of such terms must be considered in order to provide physically
63 feasible solutions. If considering first a quiescent equilibrium, where water is at rest with no rotation,
64 the very first property that the numerical scheme must satisfy is the preservation of such quiescent

65 equilibrium in the discrete level. Numerical schemes able to preserve still water at rest are called
66 well-balanced methods [34, 35]. There is a broad variety of well-balanced methods based on Riemann
67 solver techniques [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. For a comparison of different
68 well-balanced methods in the framework of DG schemes, see [50]. The well-balanced property can still
69 be enhanced by considering energy conservation criteria in the numerical scheme [51, 52, 53, 54, 55,
70 56, 57, 58, 28, 29, 55].

71 The SWE in the rotating frame represents a good model for large scale phenomena in geophysical
72 flows [60, 59]. For such model, the most relevant equilibrium solution is the so-called geostrophic
73 equilibrium state, often referred to as *jet in the rotating frame* [59]. In the last decade, a great effort
74 has been put on the design of FV well-balanced numerical schemes capable to maintain the geostrophic
75 equilibrium by following different approaches [61, 62, 63, 64, 65, 66, 60, 59].

76 For the particular case of the SWE with bed elevation and Coriolis, the well-balanced property that
77 allowed to correctly simulate the still water at rest is extended to preserve the geostrophic equilibrium.
78 We define two directional primitive variables for the x and y contributions of the Coriolis force, as done
79 in [65, 60, 59], which allows to express again the Coriolis source term as a geometric source term. By
80 doing this, both source terms can be merged into a single geometric source where the scalar variable
81 can be regarded as an apparent topography [65, 59]. The numerical techniques designed here for the
82 discretization of the bed slope source term can be extended to the Coriolis source term while retaining
83 the well-balanced property, even in cases with discontinuous bed topography. However, the high order
84 of accuracy provided by the scheme will help to accurately converge to such solution. It is worth
85 pointing out that the preservation of the geostrophic equilibrium is done at the cost of loosing the
86 optimal accuracy of the integration in time, as the CK procedure has to be carried out in a particular
87 way that combines the use of conserved and primitive source term variables.

88 The outline of the paper is next presented. In section 2, an introduction to nonlinear systems
89 of conservation laws with geometric source terms is provided. In Section 3, we present the general
90 formulation of 2D ADER schemes in Cartesian grids and provide the details for the construction of the
91 ARL-ADER scheme. Chapter 4 is devoted to the application of the scheme to the SWE and describes
92 a well-balanced integration method for both the bed elevation and Coriolis source term. Chapter 5
93 includes a variety of test cases for the SWE with bed variation, rotation and with both features at the
94 same time. Finally, in Chapter 6 we present a summary of the work and the concluding remarks.

95 2. Nonlinear systems of equations with source term

96 The basic ideas underlying this work can be illustrated by examining hyperbolic nonlinear systems
97 of equations with source terms in 2D, that can be expressed in differential formulation as

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E}(\mathbf{U}) = \mathbf{S}, \quad \forall x, y \in \Omega \subseteq \mathbb{R}^2 \quad (1)$$

98 where $\mathbf{U} = \mathbf{U}(\mathbf{x}, t) \in \mathcal{C} \subset \mathbb{R}^{N_\lambda}$ is the vector of conserved quantities that takes values on \mathcal{C} , with N_λ
99 the number of equations. $\mathbf{E}(\mathbf{U}) : \mathcal{C} \rightarrow \mathbb{R}^{N_\lambda \times 2}$ is the matrix of fluxes, that will hereafter referred to as
100 $\mathbf{E} = (\mathbf{F}, \mathbf{G})$, where $\mathbf{F} = \mathbf{F}(\mathbf{U}) : \mathcal{C} \rightarrow \mathbb{R}^{N_\lambda}$ and $\mathbf{G} = \mathbf{G}(\mathbf{U}) : \mathcal{C} \rightarrow \mathbb{R}^{N_\lambda}$ are the physical fluxes on the
101 coordinate directions x and y . Note that $\mathbf{x} = (x, y)$.

102 It is possible to define two Jacobian matrices for the fluxes $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ as

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}, \quad \mathbf{B}(\mathbf{U}) = \frac{\partial \mathbf{G}(\mathbf{U})}{\partial \mathbf{U}}, \quad (2)$$

103 that allow to rewrite (1) as

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial y} = \mathbf{S}, \quad \forall x, y \in \Omega \subseteq \mathbb{R}^2 \quad (3)$$

104 and provide sufficient information for the hyperbolicity of the system. The system in (1) is said to be
 105 *hyperbolic* if the matrix $\mathcal{J}(\mathbf{U}) \in \mathbb{R}^{N_\lambda \times N_\lambda}$ defined as

$$\mathcal{J}(\mathbf{U}) = k_1 \mathbf{A}(\mathbf{U}) + k_2 \mathbf{B}(\mathbf{U}), \quad (4)$$

106 is diagonalizable with real eigenvalues for all $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$ and for all $\mathbf{U} \in C$ with $C \subseteq \mathbb{R}^{N_\lambda}$ the
 107 subset of physically relevant values of \mathbf{U} . If the N_λ eigenvalues are distinct, then the system is said to
 108 be *strictly hyperbolic* [67].

109 Assuming that the system is hyperbolic with N_λ real eigenvalues $\lambda^m(\mathbf{U})$ and N_λ linearly indepen-
 110 dent right eigenvectors $\mathbf{e}^m(\mathbf{U})$, it is possible define two matrices $\mathbf{P}(\mathbf{U}) = (\mathbf{e}^1(\mathbf{U}), \mathbf{e}^2(\mathbf{U}), \dots, \mathbf{e}^{N_\lambda}(\mathbf{U}))$
 111 and $\mathbf{P}^{-1}(\mathbf{U})$ with the property that they diagonalize the Jacobian as

$$\mathcal{J}(\mathbf{U}) = \mathbf{P}(\mathbf{U}) \mathbf{\Lambda}(\mathbf{U}) \mathbf{P}^{-1}(\mathbf{U}) \quad (5)$$

112 with $\mathbf{\Lambda}(\mathbf{U}) = \text{diag}(\lambda^1(\mathbf{U}), \dots, \lambda^{N_\lambda}(\mathbf{U}))$ a diagonal matrix composed by the eigenvalues of the Jacobian.

113 In this work, we put an special emphasis on the so-called *geometric source terms* for the momentum
 114 equation, whose vector components are generally expressed as $S(\mathbf{U}, x, y) = S_s(\mathbf{U}) \partial_\varphi \phi(x, y)$, where φ
 115 can either be x or y (depending on the vector component of the momentum which the source is applied
 116 to). $S_s(\mathbf{U})$ is a function of the conserved quantities and $\phi(x, y)$ is the geometric function, normally a
 117 potential that depends upon the position x, y and that can be discontinuous.

118 3. General formulation of 2D ADER schemes in Cartesian grid

119 Let us consider again the system of conservation laws in (1) to compose the following Initial
 120 Boundary Value Problem (IBVP)

$$\left\{ \begin{array}{l} \text{PDEs: } \frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{E}(\mathbf{U}) = \mathbf{S} \\ \text{IC: } \mathbf{U}(\mathbf{x}, 0) = \mathring{\mathbf{U}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \\ \text{BC: } \mathbf{U}(\mathbf{x}, t) = \mathbf{U}_{\partial\Omega}(\mathbf{x}, t) \quad \forall \mathbf{x} \in \partial\Omega \end{array} \right. \quad (6)$$

121 defined in the domain $\Omega \times [0, T]$, where $\Omega = [a, b] \times [c, d]$ is the spatial domain. Note that the initial
 122 condition is given by $\mathring{\mathbf{U}}(\mathbf{x})$ and the boundary condition by $\mathbf{U}_{\partial\Omega}(\mathbf{x}, t)$. The spatial domain is discretized
 123 in $N_x \times N_y$ volume cells, defined as $\Omega_{ij} \subseteq \Omega$, such that $\Omega = \bigcup_{i,j=1}^N \Omega_{ij}$, with cell edges at

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N_x - \frac{1}{2}} < x_{N_x + \frac{1}{2}} = b, \quad (7)$$

124 and

$$c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \dots < y_{N_y - \frac{1}{2}} < y_{N_y + \frac{1}{2}} = d, \quad (8)$$

125 Cells and cell are sizes defined as

$$\Omega_{ij} = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right], \quad i = 1, \dots, N_x, j = 1, \dots, N_y \quad (9)$$

126 and

$$\vartheta_{ij} = (x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) \cdot (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}), \quad i = 1, \dots, N_x, j = 1, \dots, N_y, \quad (10)$$

127 respectively, and in the case of regular grid we have $\vartheta_{ij} = \Delta x^2$.

128 Inside each cell at time $t^n = \sum_{l=1}^n \Delta t_l$, with Δt_l the time step dynamically calculated, the conserved
 129 quantities are defined as cell averages as

$$\bar{\mathbf{U}}_{ij}^n = \frac{1}{\vartheta_{ij}} \int_{\Omega_{ij}} \mathbf{U}(\mathbf{x}, t^n) dA \quad i = 1, \dots, N_x, j = 1, \dots, N_y. \quad (11)$$

130 where $dA = dx dy$ is the differential element of surface. Let us consider again the system in (1) and
 131 integrate it over the discrete domain $\Omega_{ij} \times \Delta t$, where $\Delta t = t^{n+1} - t^n$. Application of the Gauss-
 132 Ostrogradsky theorem yields

$$\bar{\mathbf{U}}_{ij}^{n+1} = \bar{\mathbf{U}}_{ij}^n - \frac{1}{\vartheta_{ij}} \int_0^{\Delta t} \int_{\partial\Omega_{ij}} \mathbf{E} \cdot \hat{\mathbf{n}} dl dt + \frac{1}{\vartheta_{ij}} \int_0^{\Delta t} \int_{\Omega_{ij}} \mathbf{S} dAdt, \quad (12)$$

133 where dl is the differential length. If considering a regular Cartesian grid, all cells have a constant cell
 134 area Δx^2 and we obtain the following fully-discrete updating formula

$$\bar{\mathbf{U}}_{ij}^{n+1} = \bar{\mathbf{U}}_{ij}^n - \frac{\Delta t}{\Delta x^2} \left(\sum_{r=1}^4 \mathcal{F}_r^- - \bar{\mathbf{S}}_{ij} \right), \quad (13)$$

135 where

$$\bar{\mathbf{S}}_{ij} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i+1/2}}^{x_{i-1/2}} \int_{y_{i+1/2}}^{y_{i-1/2}} \mathbf{S} dy dx d\tau. \quad (14)$$

136 is the approximation of the space-time integral of the source term inside the cell and \mathcal{F}_r^- is the space-
 137 time integral of the numerical fluxes over the r -th cell edges. To construct a numerical scheme of order
 138 $K + 1$ -th, it is sufficient to approximate the integral of the flux, \mathcal{F}_r^- , using a $K + 1$ -th order Gaussian
 139 quadrature rule as

$$\mathcal{F}_r^- = \frac{\Delta x}{2} \sum_{q=1}^k w_q \mathcal{F}_{r,q}^-, \quad (15)$$

140 where w_l are the Gaussian weights inside the interval $[-1, 1]$ at the $q = 1, \dots, k$ quadrature points
 141 along the cell edge and $\mathcal{F}_{r,q}^-$ the numerical fluxes at each of these points, computed by means of the
 142 resolution of a 1D approximation to the Cauchy problem with at least K non-trivial derivatives, to
 143 ensure high order not only in space but also in time. It is worth recalling that the use of k quadrature
 144 points for the Gaussian integration allows to construct a $2k - 1$ -th order approximation of the integral.

145 It is worth noting that the notation $\mathcal{F}_{r,q}^-$ for the numerical fluxes has been adopted in order to to
 146 express them as cell-leaving fluxes and provide a more compact formulation of the scheme. Generally,
 147 the flux \mathcal{F} stands for $\mathbf{F} \cdot \mathbf{n}_x + \mathbf{G} \cdot \mathbf{n}_y$, but it can only be either $\mathcal{F} = \pm \mathbf{F}$ or $\mathcal{F} = \pm \mathbf{G}$ when considering a
 148 Cartesian mesh. In this particular case, the numerical fluxes $\mathcal{F}_{r,q}^-$ can be expressed as $\mathcal{F}_{N,q}^- = \mathbf{G}_{i,j+1/2,q}^-$,
 149 $\mathcal{F}_{S,q}^- = -\mathbf{G}_{i,j-1/2,q}^+$, $\mathcal{F}_{E,q}^- = \mathbf{F}_{i+1/2,j,q}^-$ or $\mathcal{F}_{W,q}^- = -\mathbf{F}_{i-1/2,j,q}^+$. Note that the subscripts for the r loop
 150 have been labeled as 3=N, 1=S, 2=E and 4=W, which stand for north, south, east and west.

151 The numerical fluxes are computed solving an arbitrary order approximation of the Cauchy problem
 152 at cell interfaces. This is given by a 1D DRP, which is defined in the x direction for the numerical
 153 fluxes on the east and west interfaces and in the y direction for those fluxes on the north and south
 154 interfaces. It is worth noting that the source term is included in the definition of the DRP, according
 155 to [29]. The DRP $_K$ defined in the x direction, at the interface $i + 1/2$ and quadrature point q , reads
 156 [29]

$$\begin{cases} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S} \\ \mathbf{U}(x, t = 0) = \begin{cases} \mathbf{U}_{ij}(x, y_{ij,q}) & x < 0 \\ \mathbf{U}_{i+1j}(x, y_{i+1j,q}) & x > 0 \end{cases} \end{cases} \quad (16)$$

157 where $\mathbf{U}_{ij}(x, y_{i+1/2,j,q})$ and $\mathbf{U}_{i+1j}(x, y_{i+1/2,j,q})$ are smooth functions of the position, defined using
 158 suitable reconstruction procedures, such as the WENO method. Such functions are evaluated at the
 159 particular $y = y_{i+1/2,j,q}$ location where the 1D x -oriented DRP is defined.

160 The solution for the DRP in (16) can be constructed using the flux expansion approach as

$$\begin{aligned}\mathbf{F}_{i+1/2,j,q}^- &= \mathbf{F}_{i+1/2,j,q}^{-,(0)} + \sum_{k=1}^K \mathbf{F}_{i+1/2,j,q}^{-,(k)} \frac{\Delta t^k}{(k+1)!}, \\ \mathbf{F}_{i+1/2,j,q}^+ &= \mathbf{F}_{i+1/2,j,q}^{+,(0)} + \sum_{k=1}^K \mathbf{F}_{i+1/2,j,q}^{+,(k)} \frac{\Delta t^k}{(k+1)!},\end{aligned}\tag{17}$$

161 where $\mathbf{F}_{i+1/2,j,q}^{-,(0)}$, $\mathbf{F}_{i+1/2,j,q}^{-,(k)}$, $\mathbf{F}_{i+1/2,j,q}^{+,(0)}$ and $\mathbf{F}_{i+1/2,j,q}^{+,(k)}$ are computed by solving the DRP $_K$. When using
 162 the LFS-ARoe solver, the coefficients of (17) read

$$\begin{aligned}\mathbf{F}_{i+1/2}^{-,(k)} &= \mathbf{F}_{iE}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^- \alpha^{(k)} - \beta^{-,(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K \\ \mathbf{F}_{i+1/2}^{+,(k)} &= \mathbf{F}_{(i+1)W}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^+ \alpha^{(k)} - \beta^{+,(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K\end{aligned}\tag{18}$$

163 where $\mathbf{F}_{iE}^{(k)}$ and $\mathbf{F}_{(i+1)W}^{(k)}$ are the left and right-hand limits to the cell edge of the physical flux ($k = 0$)
 164 and their k -th time derivatives, $\alpha^{(k)}$ are the wave strengths, $\beta^{-,(k)}$ the source strenghts and $\tilde{\lambda}^\pm$ and
 165 $\tilde{\mathbf{e}}_{i+1/2}^m$ the approximate wave celerities and eigenvectors defined using Roe's averages. The computation
 166 of the aforementioned quantities is detailed in Appendix A

167 Analogously, the DRP in (16) can be defined in the y direction and the numerical fluxes on the
 168 north and south interfaces, $\mathbf{G}_{i,j+1/2,q}^-$ and $\mathbf{G}_{i,j-1/2,q}^+$, can be computed as follows

169 4. Application to the Shallow Water Equations

170 The SWE with bottom topography in a rotating frame can be expressed in matrix form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}, \quad \mathbf{S} = \mathbf{S}_b + \mathbf{S}_c\tag{19}$$

171 where

$$\mathbf{U} = (h, hu, hv)^T, \quad \mathbf{F} = \left(hu, hu^2 + \frac{1}{2}gh^2, huv \right)^T, \quad \mathbf{G} = \left(hv, huv, hv^2 + \frac{1}{2}gh^2 \right)^T,\tag{20}$$

172 are the vectors of conserved quantities and physical fluxes in the x and y directions and

$$\mathbf{S}_b = \left(0, -gh \frac{dz}{dx}, -gh \frac{dz}{dy} \right)^T, \quad \mathbf{S}_c = (0, fhv, -fhu)^T,\tag{21}$$

173 are the vectors of sources due to bed variation and Coriolis force, respectively, where h is the water
 174 depth, u and v are the x and y velocities respectively, z is the bed elevation and f the Coriolis
 175 coefficient. Equations (19)–(21) represent a good model for both small and large scale phenomena in
 176 geophysical flows, the latter being dominated by the Coriolis source term.

177 In the following subsections, a detailed derivation of the approximation of the source term for the
 178 SWE with bottom elevation and Coriolis, to construct a well-balanced scheme in Cartesian grids, is
 179 presented. Details on the calculation of the Roe's averages and source strengths for the construction
 180 of the approximate solution using the LFS solver are presented in Appendix B.

181 *4.1. Formulation of a well-balanced scheme for the SWE with bottom topography*

182 For now, we will consider the SWE in (19)–(21) with $\mathbf{S}_c = 0$, that is, in a fixed frame. The
 183 well-balanced formulation can be regarded as a weaker exact conservation property than the energy
 184 balanced formulation. The energy balanced formulation ensures the exact conservation property for
 185 both quiescent and moving equilibrium cases by considering the mechanical energy, $e = |\mathbf{v}|^2/2g + h + z$.
 186 On the other hand, the well-balanced formulation only satisfies the exact conservation property when
 187 $\mathbf{v} = 0$, that is, for still water at rest.

188 Under steady conditions, when the velocity vanishes, $\mathbf{v} = 0$, the equation for the conservation of
 189 energy and momentum yield the same result

$$\nabla(h + z) = 0 \quad (22)$$

190 which is known in the literature as *lake at rest* condition. At the discrete level and considering a
 191 Cartesian grid, between two points \mathbf{x}_1 and \mathbf{x}_2 , Equation (22) can be decomposed into the Cartesian
 192 directions as

$$\delta(h + z)_{x_2, x_1} = 0, \quad \delta(h + z)_{y_2, y_1} = 0 \quad (23)$$

193 To construct a well-balanced scheme, the previous discrete conditions in (23) must be satisfied.
 194 This can only be achieved if the WENO reconstruction method is applied to $\eta = h + z$ and z first, and
 195 h is computed from the difference of these reconstructions as $h_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)} - z_{(\cdot)}^{(0)}$, where $\eta_{(\cdot)}^{(0)}$ and $z_{(\cdot)}^{(0)}$ are
 196 the reconstructed water surface elevation and bottom elevation and $h_{(\cdot)}^{(0)}$ the computed water depth.
 197 The discharges hu and hv are also reconstructed using the WENO method.

198 **Theorem 1.** *Let us consider that the initial piecewise constant data for the problem satisfy the quiescent*
 199 *equilibrium condition in (23), that variables $\eta_{(\cdot)}^{(0)}$, $hu_{(\cdot)}^{(0)}$, $hv_{(\cdot)}^{(0)}$ and $z_{(\cdot)}^{(0)}$ are reconstructed using the*
 200 *WENO method and that $h_{(\cdot)}^{(0)}$ is reconstructed as $h_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)} - z_{(\cdot)}^{(0)}$. Then, the reconstructed variables*
 201 *satisfy the quiescent equilibrium condition in (23).*

202 *Proof.* We will assume that the initial data is in equilibrium state. Therefore, the cell averages
 203 $\overline{hu}_{ij} = \overline{hv}_{ij} = 0$ and $\overline{\eta}_{ij} = \overline{h}_{ij} + \overline{z}_{ij} \equiv \text{const}$. Then, the WENO reconstruction of such quantities
 204 will lead to $hu_{(\cdot)}^{(0)} = hv_{(\cdot)}^{(0)} = 0$ and $\eta_{(\cdot)}^{(0)} \equiv \text{const}$ according to the properties of the WENO reconstruc-
 205 tion. The water depth, reconstructed as $h_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)} - z_{(\cdot)}^{(0)}$, will satisfy (23), as $\delta(h^{(0)} + z^{(0)})_{(\cdot, \cdot)} =$
 206 $\delta(\eta^{(0)} - z^{(0)} + z^{(0)})_{(\cdot, \cdot)}$, which yields $\delta(\eta^{(0)})_{(\cdot, \cdot)} = 0$ \square

207 The keystone for the design of a well balanced scheme is to ensure that the cell fluctuations, which
 208 are discrete differences between flux variations and source contributions between two different points of
 209 the grid, are canceled out. In other words, the discrete flux variation between two points must coincide
 210 with the approximation of the integral of the source term along the path defined by those points. To
 211 this end, a particular discretization of the integral of the source term that fulfills the aforementioned
 212 requirement must be found. Such discretization will be derived by decomposing cell fluctuations in the
 213 Cartesian directions and locally imposing the equilibrium condition to each of them.

214 To design a well-balanced scheme, the fluctuation form of the scheme in (13) is more suitable. This
 215 formulation reads

$$\overline{\mathbf{U}}_{ij}^{n+1} = \overline{\mathbf{U}}_{ij}^n - \frac{\Delta t}{\Delta x^2} \left(\sum_{r=1}^4 \delta \mathbf{M}_r^- + \delta \mathbf{M}_{ij} \right), \quad (24)$$

216 where $\delta \mathbf{M}_r^-$ are the contribution of the incoming waves at cell interfaces and $\delta \mathbf{M}_{ij}$ is the centered fluc-
 217 tuation, that accounts for the variation of physical fluxes and source terms inside the cell. To construct

218 a well-balanced scheme, it is required that all fluctuations become nil under quiescent conditions, that
 219 is $\delta\mathbf{M}_r^- = \delta\mathbf{M}_{ij} = 0$.

220 Centered fluctuations read

$$\delta\mathbf{M}_{ij} = \sum_{r=1}^4 \frac{\Delta x}{2} \sum_{q=1}^k w_q \mathcal{F}_{r,q} - \bar{\mathbf{S}}_{ij} \quad (25)$$

221 where

$$\bar{\mathbf{S}}_{ij} \approx \begin{pmatrix} 0 \\ \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i+1/2}}^{x_{i-1/2}} \int_{y_{i+1/2}}^{y_{i-1/2}} S_{ij}^x(x, y, \tau) dy dx d\tau \\ \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i+1/2}}^{x_{i-1/2}} \int_{y_{i+1/2}}^{y_{i-1/2}} S_{ij}^y(x, y, \tau) dy dx d\tau \end{pmatrix} \quad (26)$$

222 is the approximation of the integral of the source term inside the cell. Note that the q loop is done
 223 over quadrature points. In this case, a suitable approximation of the integral in (26) is required to
 224 ensure $\delta\mathbf{M}_{ij} = 0$.

225 On the other hand, upwind fluctuations are given by

$$\delta\mathbf{M}_r^- = \sum_{r=1}^4 \frac{\Delta x}{2} \sum_{q=1}^k w_q (\mathcal{F}_{r,q}^- - \mathcal{F}_{r,q}) \quad (27)$$

226 where $\mathcal{F}_{r,q} = \mathbf{F}_{i_\eta,j,q} \cdot n_x + \mathbf{G}_{i,j_\eta,q} \cdot n_y$ in Cartesian grid, where η can be either L or R . For instance
 227 $\mathcal{F}_{E,q} = \mathbf{F}_{i_E,j,q}$, $\mathcal{F}_{W,q} = -\mathbf{F}_{i_W,j,q}$, $\mathcal{F}_{N,q} = \mathbf{G}_{i,j_N,q}$ and $\mathcal{F}_{S,q} = -\mathbf{G}_{i,j_S,q}$. For (27), the equilibrium
 228 condition is $\mathcal{F}_{r,q}^- = \mathcal{F}_{r,q}$. Remark that q will be hereafter denoted by α in the x direction and by β in
 229 the y direction.

230 Let us consider first the equilibrium condition for centered fluctuations, $\delta\mathbf{M}_{ij} = 0$. First of all, we
 231 propose to decompose (26) in each coordinate direction by performing a Gaussian integration along the
 232 transverse direction with respect to the direction of the variation of the geometric term while seeking
 233 a suitable discretization of the source term for the integral along the former direction. This means

$$\bar{\mathbf{S}}_{ij} = \begin{pmatrix} 0 \\ \frac{\Delta x}{2} \sum_{\beta=1}^k w_\beta \bar{S}_{ij,\beta}^x \\ \frac{\Delta x}{2} \sum_{\alpha=1}^k w_\alpha \bar{S}_{ij,\alpha}^y \end{pmatrix} \quad (28)$$

234 where

$$\bar{S}_{ij,\beta}^x \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{x_{i-1/2}}^{x_{i+1/2}} S_{ij}^x(x, y_\beta, \tau) dx d\tau$$

$$\bar{S}_{ij,\alpha}^y \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{y_{i-1/2}}^{y_{i+1/2}} S_{ij}^y(x_\alpha, y, \tau) dy d\tau \quad (29)$$

235 are the sought approximations. Equation (25) can also be decomposed in each of the Cartesian direc-
 236 tions as follows

$$\begin{aligned} \delta \mathbf{M}_{ij} = & \frac{\Delta x}{2} \sum_{\beta=1}^k w_{\beta} \left(\mathbf{F}_{i_E,j,\beta} - \mathbf{F}_{i_W,j,\beta} - \begin{pmatrix} 0 \\ \bar{S}_{ij,\beta}^x \\ 0 \end{pmatrix} \right) + \\ & \frac{\Delta x}{2} \sum_{\alpha=1}^k w_{\alpha} \left(\mathbf{G}_{i,j_N,\alpha} - \mathbf{G}_{i,j_S,\alpha} - \begin{pmatrix} 0 \\ 0 \\ \bar{S}_{ij,\alpha}^y \end{pmatrix} \right) \end{aligned} \quad (30)$$

237 where the source term has been decomposed in each of the coordinate directions. It is worth recalling
238 that the source term is constructed as a power series expansion in time as follows

$$\mathbf{S}_{ij}(x, y, \tau) = \mathbf{S}_{ij}(x, y, 0) + \sum_{k=1}^K \left[\frac{\partial^k \mathbf{S}_{ij}}{\partial t^k} \right]_{x,y,t=0} \frac{\tau^k}{k!}, \quad (31)$$

239 and so are the physical fluxes \mathbf{F} and \mathbf{G} . We can then rewrite the 1D discretizations of the source term
240 as

$$\begin{aligned} \bar{S}_{ij,\beta}^x & \approx \int_{x_{i-1/2}}^{x_{i+1/2}} \left(S_{ij}^x(x, y_{\beta}, 0) + \sum_{k=1}^K \left[\frac{\partial^k S^x}{\partial t^k} \right]_{x,y_{\beta},t=0} \frac{\Delta t^k}{(k+1)!} \right) dx \\ \bar{S}_{ij,\alpha}^y & \approx \int_{y_{i-1/2}}^{y_{i+1/2}} \left(S_{ij}^y(x_{\alpha}, y, 0) + \sum_{k=1}^K \left[\frac{\partial^k S^y}{\partial t^k} \right]_{x_{\alpha},y,t=0} \frac{\Delta t^k}{(k+1)!} \right) dy \end{aligned} \quad (32)$$

241 or in its compact form

$$\bar{S}_{ij,\beta}^x = \bar{S}_{ij,\beta}^{x,(0)} + \sum_{k=1}^K \bar{S}_{ij,\beta}^{x,(k)} \frac{\Delta t^k}{(k+1)!}, \quad \bar{S}_{ij,\alpha}^y = \bar{S}_{ij,\alpha}^{y,(0)} + \sum_{k=1}^K \bar{S}_{ij,\alpha}^{y,(k)} \frac{\Delta t^k}{(k+1)!} \quad (33)$$

242 To derive the well-balanced formulation, steady conditions are considered. Hence, for the derivation
243 of the suitable approximation of the 1D integrals of the source term in (33), we only consider the leading
244 terms of the equations, as time derivatives vanish in the steady state. Equation (30) is rewritten for
245 the leading term only

$$\begin{aligned} \delta \mathbf{M}_{ij}^{(0)} = & \frac{\Delta x}{2} \sum_{\beta=1}^k w_{\beta} \left(\mathbf{F}_{i_E,j,\beta}^{(0)} - \mathbf{F}_{i_W,j,\beta}^{(0)} - \begin{pmatrix} 0 \\ \bar{S}_{ij,\beta}^{x,(0)} \\ 0 \end{pmatrix} \right) + \\ & \frac{\Delta x}{2} \sum_{\alpha=1}^k w_{\alpha} \left(\mathbf{G}_{i,j_N,\alpha}^{(0)} - \mathbf{G}_{i,j_S,\alpha}^{(0)} - \begin{pmatrix} 0 \\ 0 \\ \bar{S}_{ij,\alpha}^{y,(0)} \end{pmatrix} \right) \end{aligned} \quad (34)$$

246 from which we notice the one-dimensional conditions

$$\mathbf{F}_{i_E,j,\beta}^{(0)} - \mathbf{F}_{i_W,j,\beta}^{(0)} - \begin{pmatrix} 0 \\ \bar{S}_{ij,\beta}^{x,(0)} \\ 0 \end{pmatrix} = 0 \quad \forall \beta = 1, \dots, k \quad (35)$$

$$\mathbf{G}_{i,j_N,\alpha}^{(0)} - \mathbf{G}_{i,j_S,\alpha}^{(0)} - \begin{pmatrix} 0 \\ 0 \\ \bar{S}_{ij,\alpha}^{y,(0)} \end{pmatrix} = 0 \quad \forall \alpha = 1, \dots, k \quad (36)$$

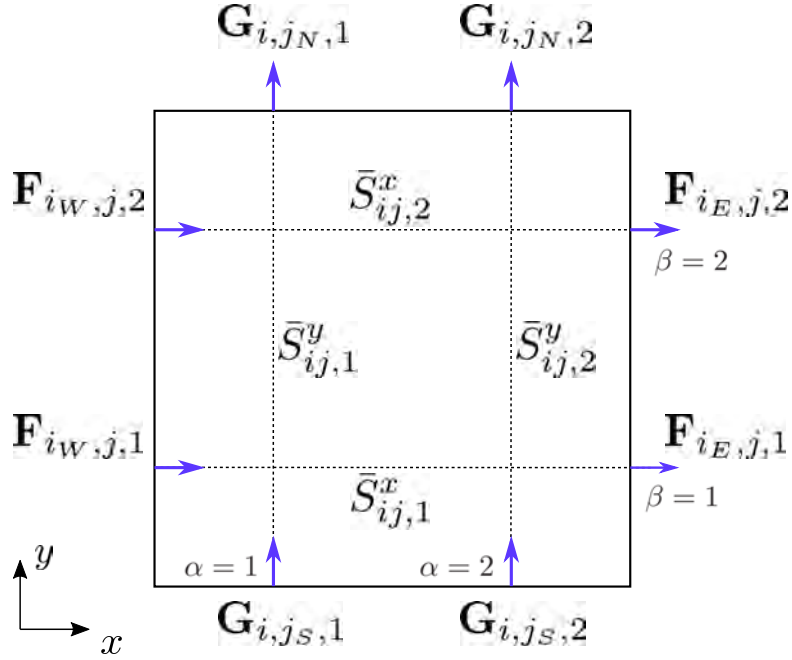


Figure 1: Representation of the Gaussian integration points, the 1D sub-cell integration lines (dashed line) in the Cartesian directions and the the physical fluxes at cell interfaces (blue vectors) for the case of $k = 2$. Integration lines are identified using the notation for the quadrature points α and β .

247 The notation for the 1D balancing of fluxes and sources in Equation (35) and (36) is depicted in
 248 Figure 1 for the case of $k = 2$. In this figure, the 1D integration lines in the Cartesian directions are
 249 depicted using dashed lines and the physical fluxes at both sides of the integration line, as well as the
 250 integral of the source term along it, are also represented. It is shown that such 1D integration lines are
 251 defined by the quadrature points at cell interfaces. Let us consider now the condition in the x -direction
 252 in Equation (35) (the condition in y is the same due to the rotational invariance). In what follows,
 253 subscripts j and β are omitted for the sake of clarity, assuming a pure 1D problem. Under quiescent
 254 steady conditions, we have that $\mathbf{F}(\mathbf{U}) = (0, \frac{1}{2}gh^2, 0)^T$ yielding

$$(g\bar{h}\delta h)_{i_E, i_W}^{(0)} - \bar{S}_i^{x, (0)} = 0 \quad (37)$$

255 If approaching the integral of the source term as

$$\bar{S}_i^{x, (0)} = -g\bar{h}\delta z_{i_W, i_E}^{(0)}, \quad (38)$$

256 it yields $g\bar{h}\delta(h+z)_{i_W, i_E}^{(0)} = 0$, where one notices that $(h+z)_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)}$ is the equilibrium reconstruction
 257 variable and therefore $g\bar{h}\delta\eta_{i_W, i_E}^{(0)} = 0$ is satisfied with machine precision.

258 The discretization of the source term used above, though well-balanced, is only 2-nd order accurate
 259 in space. To obtain a $K + 1$ -th order scheme, it is necessary to extend this integration technique to
 260 arbitrary order in space. To this end, we can use Romberg integration, which is a result that can be
 261 obtained from Richardson's extrapolation. The main role of Romberg integration is to extend the well-
 262 balanced integration in Equation (38) to arbitrary order, without losing the properties of Equation
 263 (38) for reproducing the discrete equilibrium. Note that traditional quadrature rules are not able to
 264 maintain the discrete equilibrium.

265 An arbitrary order integral of the source term is denoted as $\left\{ \bar{S}_i^{x, (0)} \right\}_m^n$, where m is the number of
 266 subdivisions of the initial interval $\Delta x = [x_{i-1/2}, x_{i+1/2}]$, with step size $h_m = \frac{\Delta x}{2^m}$ and n the number

267 of Romberg iterations, having a magnitude of the residual of $\mathcal{O}(\Delta x^{2(n+1)})$. To construct a $K + 1$ -th
 268 order ADER scheme, both n and m will take values up to $\lceil \frac{K-1}{2} \rceil$ and the order of accuracy of the
 269 method will be either $\mathcal{O}(\Delta x^{K+2})$ if K is even or $\mathcal{O}(\Delta x^{K+1})$ if K is odd. The expression for $\left\{ \bar{S}_i^{x,(0)} \right\}_m^n$
 270 is computed recursively departing from the trapezoid integrals, that is $n = 0$ and $m = 0, \dots, \lceil \frac{K-1}{2} \rceil$,
 271 and computing the following levels $n = 1, \dots, \lceil \frac{K-1}{2} \rceil$ as

$$\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^{n+1} = \frac{4^{n+1} \left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^n - \left\{ \bar{S}_i^{x,(0)} \right\}_m^n}{4^{n+1} - 1}. \quad (39)$$

272 For $n = 0$, the integrals are given by $\left\{ \bar{S}_i^{x,(0)} \right\}_0^0 = (-g\bar{h}\delta z)_{i_E, i_W}^{(0)}$ and $\left\{ \bar{S}_i^{x,(0)} \right\}_1^0 = (-g\bar{h}\delta z)_{i_E, i}^{(0)} +$
 273 $(-g\bar{h}\delta z)_{i, i_W}^{(0)}$, for $m = 0$ and $m = 1$ respectively.

274 **Theorem 2.** *The extrapolating function $\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^{n+1}$ in Equation (39) posses the following property:*
 275 *if the two arguments of this function are equal, the function yields the argument of the function, as*
 276 *follows*

$$\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^{n+1} \left(\left\{ \bar{S}_i^{x,(0)} \right\}_m^n, \left\{ \bar{S}_i^{x,(0)} \right\}_m^n \right) = \left\{ \bar{S}_i^{x,(0)} \right\}_m^n \quad (40)$$

277 *Proof.* Let us consider that $\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^n = \left\{ \bar{S}_i^{x,(0)} \right\}_m^n$, then (39) becomes

$$\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^{n+1} = \frac{(4^{n+1} - 1) \left\{ \bar{S}_i^{x,(0)} \right\}_m^n}{4^{n+1} - 1} \quad (41)$$

278 which yields $\left\{ \bar{S}_i^{x,(0)} \right\}_{m+1}^{n+1} = \left\{ \bar{S}_i^{x,(0)} \right\}_m^n$.

279 □

280 **Theorem 3.** *Let us consider that the initial integrals for the iterative extrapolation in Equation (39)*
 281 *are equal and read $\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = \left\{ \bar{S}_i^{x,(0)} \right\}_0^0 \forall m = 1, \lceil \frac{K-1}{2} \rceil$. Then, all the higher order iterations*
 282 *$n = 1, \dots, \lceil \frac{K-1}{2} \rceil$ will yield $\left\{ \bar{S}_i^{x,(0)} \right\}_m^n = \left\{ \bar{S}_i^{x,(0)} \right\}_0^0$.*

283 *Proof.* The integrals are computed iteratively from a set of initial integrals $\left\{ \bar{S}_i^{x,(0)} \right\}_m^0$. If all the initial
 284 integrals (at $n = 0$) are equal, all the approximations for $m = 1, \dots, \lceil \frac{K-1}{2} \rceil$ at $n = 1$ will be equal
 285 according to the property in Theorem 2, and so on for $n = 1, \dots, \lceil \frac{K-1}{2} \rceil$.

286 □

287 **Theorem 4.** *Let us consider that the initial piecewise constant data for the problem satisfy the quiescent*
 288 *equilibrium condition in (23) and that the integral in Equation (38) is used in combination with the*
 289 *reconstruction procedure in Theorem 1. Then, all the initial approximations of the integrals $\left\{ \bar{S}_{ij,\beta}^{x,(0)} \right\}_m^0$*
 290 *are equal to $\left\{ \bar{S}_{ij,\beta}^{x,(0)} \right\}_0^0 = -g\bar{h}\delta z_{i_W, i_E}^{(0)}$*

291 *Proof.* Let us consider that the interval $\Delta x = [x_{i_W}, x_{i_E}]$ where the integral is approached is subdivided
 292 m times, leading to an step size of $h_m = \Delta x / 2^m$. Inside that interval, the initial integral is approached
 293 as $\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = \sum_{p=1}^{2^m} -g\bar{h}\delta z_{p-1,p}^{(0)}$. Notice that $p = 0$ and $p = 2^m$ are equivalent to i_W and i_E , as they
 294 are the extrema of the interval.

295 As outlined by Theorem 1, considering that the reconstructed $\eta_{(\cdot)}^{(0)} = h_{(\cdot)}^{(0)} + z_{(\cdot)}^{(0)}$ preserves the
 296 discrete equilibrium, we can substitute $\delta z_{p-1,p}^{(0)}$ by $-\delta h_{p-1,p}^{(0)}$, yielding

$$\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = \sum_{p=1}^{2^m} g \bar{h} \delta h_{p-1,p}^{(0)} \quad (42)$$

297 which can be rewritten as

$$\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = \frac{g}{2} \left(\sum_{p=1}^{2^m} \left(h_p^{(0)} \right)^2 - \sum_{p=1}^{2^m} \left(h_{p-1}^{(0)} \right)^2 \right) \quad (43)$$

298 where $2^m - 1$ terms are canceled and yields

$$\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = \frac{g}{2} \left(\left(h_p^{(0)} \right)^2 - \left(h_{p-1}^{(0)} \right)^2 \right) \equiv g \bar{h} \delta h_{i_W, i_E}^{(0)} \quad (44)$$

299 It is straightforward to notice that $\delta z_{i_W, i_E}^{(0)} = -\delta h_{i_W, i_E}^{(0)}$, therefore $\left\{ \bar{S}_i^{x,(0)} \right\}_m^0 = -g \bar{h} \delta z_{i_W, i_E}^{(0)}$. This
 300 completes the proof of the theorem.

301 □

302 Concerning the derivative terms, there is no need of a particular discretization technique of the
 303 source term to ensure the well-balanced property as time derivatives vanish under steady state. Here,
 304 we use a 2D Gaussian integration

$$\bar{S}_{ij}^{x,(k)} = \sum_{\beta=1}^k w_\beta \sum_{\alpha=1}^k w_\alpha \left(-g h^{(k)} \partial_x z \right)_{\alpha, \beta} \quad (45)$$

$$\bar{S}_{ij}^{y,(k)} = \sum_{\beta=1}^k w_\beta \sum_{\alpha=1}^k w_\alpha \left(-g h^{(k)} \partial_y z \right)_{\alpha, \beta} \quad (46)$$

305 where $h_{\alpha, \beta}^{(k)}$ is the k -th time derivative of h at the quadrature point. The complete integral of the source
 306 term will be computed as

$$\bar{\mathbf{S}}_{ij} = \frac{\Delta x}{2} \left(\begin{array}{c} 0 \\ \sum_{\beta=1}^k w_\beta \left\{ \bar{S}_{ij, \beta}^{x,(0)} \right\}_m^n + \sum_{k=1}^K \bar{S}_{ij}^{x,(k)} \frac{\Delta t^k}{(k+1)!} \\ \sum_{\alpha=1}^k w_\alpha \left\{ \bar{S}_{ij, \alpha}^{y,(0)} \right\}_m^n + \sum_{k=1}^K \bar{S}_{ij}^{y,(k)} \frac{\Delta t^k}{(k+1)!} \end{array} \right) \quad (47)$$

307 where $m = n = \lceil \frac{K-1}{2} \rceil$.

308 **Theorem 5.** *Let us consider that the initial piecewise constant data for the problem satisfy the quiescent*
 309 *equilibrium condition in (23). Then, the source term discretization in (38)–(47) in combination with*
 310 *the reconstruction procedure in Theorem 1 satisfies the discrete equilibrium for the centered fluctuations,*
 311 *that is $\delta \mathbf{M}_{ij} = 0$.*

312 *Proof.* Our goal is to prove that (30) equals zero for any formal order of accuracy of (47). For the sake
 313 of brevity, the proof will be done for the second component of (30), since the third component will be
 314 analogous and the first component is already zero.

315 We will assume the result from Theorem 1, that is, the reconstructed $\eta_{(0)}^{(\cdot)}$ is constant and $hu_{(\cdot)}^{(0)} =$
316 $hv_{(\cdot)}^{(0)} = 0$. We will also assume that the particular derivation of time derivatives using the CK procedure
317 yields nil derivatives when departing from equilibrium data (see [28] for more details). With the latter
318 assumption, we can consider all time derivatives nil, hence the higher order terms of the power series
319 expansion of the fluxes are zero, those of (47) are also zero and the equilibrium condition can be simply
320 stated for (34), instead of (30). Considering the x -component of (34) (second component) and omitting
321 subscripts j and β , the equilibrium condition at any arbitrary formal order of accuracy yields

$$\mathbf{F}_{i_E}^{(0)} - \mathbf{F}_{i_W}^{(0)} - \begin{pmatrix} 0 \\ \left\{ \bar{S}_i^{x,(0)} \right\}_m^n \\ 0 \end{pmatrix} = 0 \quad (48)$$

322 where $\left\{ \bar{S}_i^{x,(0)} \right\}_m^n$ is computed using (38) and (39), and $\mathbf{F}(\mathbf{U}) = (0, \frac{1}{2}gh^2, 0)^T$ under steady conditions.
323 If considering $m = n = 0$, we have that

$$\left(\frac{1}{2}gh^2 \right)_{i_E}^{(0)} - \left(\frac{1}{2}gh^2 \right)_{i_W}^{(0)} - \left\{ \bar{S}_i^{x,(0)} \right\}_0^0 = 0 \quad (49)$$

324 When choosing $\left\{ \bar{S}_i^{x,(0)} \right\}_0^0 = -g\bar{h}\delta z_{i_W, i_E}^{(0)}$, as given in (38), Equation (49) yields $g\bar{h}\delta(h+z)_{i_W, i_E}^{(0)} =$
325 0 , where one notices that $(h+z)_{(\cdot)}^{(0)} = \eta_{(\cdot)}^{(0)}$ is the equilibrium reconstruction variable and therefore
326 $g\bar{h}\delta\eta_{i_W, i_E}^{(0)} = 0$. Hence, Equation (49) is satisfied.

327 When considering a higher order extrapolation of the integral ($m, n > 0$), it is sufficient to show
328 that $\left\{ \bar{S}_{i_j, \beta}^{x,(0)} \right\}_m^n = \left\{ \bar{S}_{i_j, \beta}^{x,(0)} \right\}_0^0 = -g\bar{h}\delta z_{i_W, i_E}^{(0)}$, for any m and n , to confirm that (48) is satisfied. The
329 proof for the aforementioned statement is given by Theorems 2, 3 and 4. This completes the proof of
330 this theorem. □

332 On the other hand, when considering the upwind fluctuations in (27), it is straightforward to prove
333 that the following approximation for the leading term of the integral of the source term in (A.12)

$$\bar{S}_{i+1/2}^{x,(0)} = (-g\bar{h}\delta z)_{i+1/2}^{(0)}, \quad (50)$$

$$\bar{S}_{i+1/2}^{y,(0)} = 0, \quad (51)$$

334 satisfies the steady state equilibrium condition $\left(\tilde{\lambda}^- \alpha^{(0)} - \beta^{-(0)} \right)_{i+1/2}^m = 0$ in Equation (A.7), yielding
335 $\mathbf{F}_{i+1/2, j, \beta}^- = \mathbf{F}_{i_E, j, \beta}$. Higher order terms are computed as

$$\bar{\mathbf{S}}_{i+1/2}^{x,(k)} = \left(-g\bar{h}^{(k)} \delta z^{(0)} \right)_{i+1/2}. \quad (52)$$

336 Note that the approximation of the integral of the source term for the y -oriented DRPs is analogous
337 and the same properties are satisfied.

338 **Theorem 6.** *Let us consider that the initial piecewise constant data for the problem satisfy the quiescent*
339 *equilibrium condition in (23). Then, the source term discretization in (50)–(52) in combination with*
340 *the reconstruction procedure in Theorem 1 satisfies the discrete equilibrium for the upwind fluctuations,*
341 *that is $\delta \mathbf{M}_r^- = 0$.*

342 *Proof.* To prove this theorem, it is sufficient to show that at cell interfaces, the numerical fluxes coincide
 343 with the physical fluxes evaluated at such point. According to Equation (27), when the aforementioned
 344 hypothesis is satisfied, the equality $\delta\mathbf{M}_r^- = 0$ holds.

345 For instance, if considering the x flux on the right hand side wall, the equality $\mathbf{F}_{i+1/2,j,\beta}^- = \mathbf{F}_{i_E,j,\beta}$ is
 346 sought. In the equilibrium state, the derivative terms for the flux $\mathbf{F}_{i+1/2,j,\beta}^-$ in (17) vanish. Therefore,
 347 it is sufficient to prove that $\mathbf{F}_{i+1/2,j,\beta}^{-(0)} = \mathbf{F}_{i_E,j,\beta}$ holds. According to the properties of the ARoe solver,
 348 detailed in [33], it is shown that if using the source term discretization in (50), the numerical fluxes
 349 evaluated at the equilibrium state yield $\mathbf{F}_{i+1/2}^{-(0)}(\mathbf{U}_{i_E}^{(0)}, \mathbf{U}_{(i+1)W}^{(0)}, \bar{S}_{i+1/2}^{x,(0)}) = \mathbf{F}(\mathbf{U}_{i_E}^{(0)})$. The same result can
 350 be derived for the numerical fluxes at the other cell walls.

351 □

352 4.2. Formulation of a well-balanced scheme for the SWE in the rotating frame

353 We now consider the SWE with bed elevation in a rotating frame by means of including both the
 354 bed slope and the Coriolis source terms, \mathbf{S}_b and \mathbf{S}_c respectively. Equations (19)–(21) represent a good
 355 model for large scale phenomena in geophysical flows, in which oceanic and atmospheric circulations
 356 are often perturbations of the so-called geostrophic equilibrium [59].

357 When designing a numerical scheme for the resolution of a particular system of equations, it is
 358 of importance to design the scheme in such a way that allows the preservation of the steady-state
 359 equilibrium solutions, since many phenomena of interest are often perturbations of those equilibrium
 360 states. It is worth recalling that when the Coriolis effect is neglected, the SWE in (19)–(21) satisfies
 361 the quiescent equilibrium steady state in (22). Numerical schemes satisfying (22) in the discrete form
 362 are called well-balanced schemes.

363 When considering the Coriolis source term, equilibrium states become more complex as they now
 364 include the circulation of the flow in particular directions. For the system in (19)–(21), the most
 365 relevant equilibrium solution to be considered in the design of the numerical scheme is the so-called
 366 geostrophic equilibrium state, which arises from the balance of the Coriolis force with the hydrostatic
 367 pressure change due to the surface elevation gradient. This steady state is often referred to as *jet in*
 368 *the rotating frame* [59]. According to [59], the geostrophic equilibrium satisfies

$$369 \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad g \frac{\partial}{\partial x}(h+z) = fv, \quad g \frac{\partial}{\partial y}(h+z) = -fu, \quad (53)$$

369 which can be rewritten as

$$370 \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad g \frac{\partial}{\partial x}(h+z-V) = 0, \quad g \frac{\partial}{\partial y}(h+z+U) = 0, \quad (54)$$

370 with $\partial_x V = fv/g$ and $\partial_y U = fu/g$ the primitive functions of the Coriolis force, also called *apparent*
 371 *topography* [62, 59]. For the sake of simplicity, we will define the potentials

$$372 \quad L = h + z - V, \quad K = h + z + U, \quad (55)$$

372 which are functionals that are conserved in the geostrophic equilibrium. We can identify two particular
 373 jets in the rotating frame [59], which satisfy

$$374 \quad u = 0 \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial h}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad L \equiv constant, \quad (56)$$

374 and

$$375 \quad v = 0 \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial h}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = 0, \quad K \equiv constant, \quad (57)$$

375 In this section, a well-balanced WENO-ADER scheme using the Augmented Roe solver, which
 376 preserves the jets in (56) and (57), is proposed. The keystone of this scheme is the treatment of
 377 the Coriolis source terms as geometric sources in order to discretize them in the same way than the
 378 bed elevation source term. As done for the SWE with variable bed elevation and no Coriolis terms,
 379 it is necessary to identify first which quantities the reconstruction procedure will be applied to and
 380 which other quantities will be computed from the reconstructed data in order to satisfy the discrete
 381 equilibrium. When Coriolis was not considered, the reconstruction technique was applied to hu , hv , z
 382 and $h+z$ (the equilibrium variable) and then, the water depth h was computed from the reconstructions.
 383 When Coriolis forces are present, the equilibrium variable is not anymore $h+z$ and instead, the exact
 384 conservation must be ensured for both K and L . In order to satisfy this, the WENO reconstruction
 385 will be carried out for hu , hv , h , z , L and K first, and then, V and U will be computed from the
 386 reconstructed data as

$$V_{(\cdot)}^{(0)} = h_{(\cdot)}^{(0)} + z_{(\cdot)}^{(0)} - L_{(\cdot)}^{(0)}, \quad U_{(\cdot)}^{(0)} = K_{(\cdot)}^{(0)} - h_{(\cdot)}^{(0)} - z_{(\cdot)}^{(0)} \quad (58)$$

387 where $h_{(\cdot)}^{(0)}$, $z_{(\cdot)}^{(0)}$, $L_{(\cdot)}^{(0)}$ and $K_{(\cdot)}^{(0)}$ are the reconstructed water depth, bottom elevation, L potential and
 388 K potential, and $V_{(\cdot)}^{(0)}$ and $U_{(\cdot)}^{(0)}$ the computed Coriolis primitive variables. Note that the proposed
 389 approach requires to perform $2\times$ more reconstructions than a traditional SWE model. With WENO
 390 limiting, this may be a large additional expense which is worth accepting as the well-balanced property
 391 is totally required to reproduce most practical cases.

392 When using WENO-ADER schemes, the problem data is discretized in the form of cell averages,
 393 which are required in the first step of the scheme to carry out the WENO reconstructions. The
 394 discretization of h , z , hu and hv as cell averages is straightforward, however, for V and U is not that
 395 simple and has to be done in a more elaborate way. According to the definitions of V and U , we can
 396 express them as the following integrals

$$V(x, y) = \int_0^x \frac{fv(\chi, y)}{g} d\chi + V(0), \quad U(x, y) = \int_0^y \frac{fu(x, \chi)}{g} d\chi + U(0) \quad (59)$$

397 where $V(0) = U(0) = 0$. When considering piecewise constant data in a Cartesian grid with cell size
 398 Δx , we can compute V and U at cell interfaces as

$$V_{i+1/2, j} = \sum_{t=0}^i \left(\frac{f\bar{v}}{g} \right)_{tj} \Delta x, \quad U_{i, j+1/2} = \sum_{t=0}^j \left(\frac{f\bar{u}}{g} \right)_{it} \Delta x \quad (60)$$

399 and then calculate the cell averages as

$$\bar{V}_{ij} = \frac{1}{2} (V_{i+1/2, j} + V_{i-1/2, j}), \quad \bar{U}_{ij} = \frac{1}{2} (U_{i, j+1/2} + U_{i, j-1/2}) \quad (61)$$

400 After computing the cell averages for V and U , those for K and L can be computed and the
 401 reconstruction procedure for all variables can be carried out. It is worth noting that the use of the
 402 global integrals U and V enforces global communication in a parallelized code, which increases the
 403 computational expense.

404 **Theorem 7.** *Let us consider that the initial piecewise constant data for the problem satisfy the*
 405 *geostrophic equilibrium in (56) and (57), that variables $L_{(\cdot)}^{(0)}$, $K_{(\cdot)}^{(0)}$, $h_{(\cdot)}^{(0)}$, $hu_{(\cdot)}^{(0)}$, $hv_{(\cdot)}^{(0)}$ and $z_{(\cdot)}^{(0)}$ are*
 406 *reconstructed using the WENO method and that $V_{(\cdot)}^{(0)}$ and $U_{(\cdot)}^{(0)}$ are reconstructed using Equation (58).*
 407 *Then, the reconstructed variables satisfy the quiescent equilibrium condition in (23).*

408 *Proof.* We will assume that the initial data is in equilibrium state. Therefore, the cell averages $\overline{hu}_{ij} =$
409 $\overline{hu}_{ij} = 0$, $\overline{L}_{ij} = \overline{h}_{ij} + \overline{z}_{ij} - \overline{V}_{ij} \equiv \text{const}$ and $\overline{K}_{ij} = \overline{h}_{ij} + \overline{z}_{ij} + \overline{U}_{ij} \equiv \text{const}$, where \overline{U}_{ij} and \overline{V}_{ij}
410 are computed using (61). Then, the WENO reconstruction of such quantities will lead to $hu_{(\cdot)}^{(0)} =$
411 $hv_{(\cdot)}^{(0)} = 0$, $L_{(\cdot)}^{(0)} \equiv \text{const}$ and $K_{(\cdot)}^{(0)} \equiv \text{const}$ according to the properties of the WENO reconstruction.
412 The Coriolis primitives V and U , reconstructed using Equation (58), will satisfy (56) and (57) as
413 $\delta(h^{(0)} + z^{(0)} - V^{(0)})_{(\cdot,\cdot)} = \delta(L^{(0)})_{(\cdot,\cdot)}$ and $\delta(h^{(0)} + z^{(0)} + U^{(0)})_{(\cdot,\cdot)} = \delta(K^{(0)})_{(\cdot,\cdot)}$, and from the WENO
414 reconstruction we have that $\delta(L^{(0)})_{(\cdot,\cdot)} = \delta(K^{(0)})_{(\cdot,\cdot)} = 0$.

415 □

416 Concerning the integration of the source terms, \mathbf{S}_b and \mathbf{S}_c , to satisfy the well-balance property, we
417 must follow the same approach of the previous section for the SWE with variable bed. The numerical
418 scheme must be written in fluctuation form (24) and a particular discretization of the source term must
419 be sought in such a way that all fluctuations, those at cell interfaces and those inside the cell, must
420 become nil under geostrophic conditions, satisfying (56) and (57). As done in the previous section,
421 the scheme is designed to satisfy the discrete equilibrium in the Cartesian directions and therefore the
422 source term integration is reduced to a one-dimensional formulation. For the x -geostrophic balance in
423 (56), the 0-th discretization of the source term reads

$$\bar{S}_i^{x,(0)} = -(g\bar{h}\delta z)_{i_W,i_E}^{(0)} + (g\bar{h}\delta V)_{i_W,i_E}^{(0)} \quad (62)$$

424 yielding $g\bar{h}\delta(h+z-V)_{i_W,i_E}^{(0)} = 0$ where $(h+z-V)_{(\cdot)}^{(0)} = L_{(\cdot)}^{(0)}$ is the equilibrium reconstruction variable
425 and therefore $g\bar{h}\delta L_{i_W,i_E}^{(0)} = 0$ is satisfied with machine precision. To extend the 2-nd order integral
426 in (62) to higher order of accuracy, we use the Romberg integration method detailed in the previous
427 section. Concerning the derivative terms, there is no need of a particular discretization technique of
428 the source term to ensure the well-balanced property, as outlined in the previous section. Here, we use
429 a 2D Gaussian integration

$$\bar{S}_{ij}^{x,(k)} = \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} \left(-gh^{(k)} \partial_x z + f(hv)^{(k)} \right)_{\alpha,\beta} \quad (63)$$

$$\bar{S}_{ij}^{y,(k)} = \sum_{\alpha=1}^k w_{\alpha} \sum_{\beta=1}^k w_{\beta} \left(-gh^{(k)} \partial_y z - f(hu)^{(k)} \right)_{\alpha,\beta} \quad (64)$$

430 where $h_{\alpha,\beta}^{(k)}$ and $hu_{\alpha,\beta}^{(k)}$ are the k -th time derivative of h and hu at the quadrature point.

431 All time derivatives should vanish under steady regime. However, this is only achieved when using
432 a particular expression of the CK procedure for the computation of time derivatives in terms of space
433 derivatives. The keystone to ensure that all time derivatives vanish in the geostrophic equilibrium is
434 to use spatial derivatives of V and U when constructing the first derivative using the CK procedure,
435 instead of directly using fhv and fhu . For instance, the first derivative for the discharge in x yields

$$\partial_t(hu) = -\partial_x \left(hu^2 + \frac{1}{2}gh^2 \right) - \partial_y(huv) - gh\partial_x z + gh\partial_x V \quad (65)$$

436 noticing that under the x -geostrophic equilibrium in (56), it becomes

$$\partial_t(hu) = -\partial_x \left(\frac{1}{2}gh^2 \right) - gh\partial_x z + gh\partial_x V \quad (66)$$

437 If substituting $L = h + z - V$ in Equation (66), it yields $\partial_t(hu) = -gh\partial_x L = 0$, hence the equilibrium
 438 is satisfied. Second and higher order derivatives cannot be computed using V and U because there is
 439 not evolution equation for such variables.

440 Let us consider now the upwind fluctuations in (27). As done in the previous section, the ap-
 441 proximate integrals of the source term at cell interfaces are constructed considering variations in the
 442 direction of definition of the DRP. For an x -oriented DRP we have

$$\bar{S}_{i+1/2}^{x,(0)} = (-g\bar{h}\delta z)_{i+1/2}^{(0)} + (g\bar{h}\delta V)_{i+1/2}^{(0)} \quad (67)$$

$$\bar{S}_{i+1/2}^{y,(0)} = 0 \quad (68)$$

443 satisfying the steady state equilibrium condition. Higher order terms are computed as in (52), neglect-
 444 ing the contribution of the derivatives of the Coriolis term due to its non-geometric nature. When
 445 considering an y -oriented DRP, we have

$$\bar{S}_{j+1/2}^{x,(0)} = 0 \quad (69)$$

$$\bar{S}_{j+1/2}^{y,(0)} = (-g\bar{h}\delta z)_{j+1/2}^{(0)} - (g\bar{h}\delta U)_{j+1/2}^{(0)} \quad (70)$$

446 which satisfies the steady state equilibrium condition.

447 **Theorem 8.** *Let us consider that the initial piecewise constant data for the problem satisfy the*
 448 *geostrophic equilibrium condition in equilibrium in (56) and (57). Then, the source term discretiza-*
 449 *tions proposed in Section 4.2 in combination with the reconstruction procedure in Theorem 7 satisfy*
 450 *the discrete equilibrium for both cell centered and upwind fluctuations, that is $\delta\mathbf{M}_{ij} = 0$ and $\delta\mathbf{M}_{\bar{r}} = 0$.*

451 *Proof.* The proof for this theorem is similar to that for Theorems 5 and 6, if considering that the
 452 variable z in the aforementioned theorems is substituted by an apparent topography, $z_{ap}^x = z - V$ and
 453 $z_{ap}^y = z + U$, in the x and y directions respectively.

454 □

455 5. Numerical results

456 5.1. Convergence test for the SWE with bed elevation

457 In this section, a convergence rate test for the ARL-ADER well-balanced scheme is presented. The
 458 following initial condition is imposed

$$z(x, y) = 0.1 \exp\left(-\frac{(x-50)^2 + (y-50)^2}{80}\right), \quad \forall(x, y) \in \Omega \quad (71)$$

459 $h(x, y, 0) = 1$ m and $hu(x, y, 0) = hv(x, y, 0) = 0$ m²/s, $\forall(x, y) \in \Omega$, with x and y given in m. The
 460 computational domain is $\Omega = [0, 100] \times [0, 100]$ m and the solution is computed at $t = 5$ s setting
 461 $CFL = 0.2$ using the 1-st and 3-rd order scheme.

462 Numerical errors and convergence rates for h and hu computed in four different grids composed
 463 of 50×50 , 100×100 , 200×200 and 400×400 cells are presented in Tables 1 and 2, respectively.
 464 Numerical errors have been computed using a reference solution computed by the 3-rd order scheme in
 465 a 2000×2000 grid and are measured using the L_1 , L_2 and L_∞ error norms. In Figure 2, a logarithmic
 466 plot of the numerical errors for the water depth and discharge provided by the 1-st and 3-rd order
 467 schemes are represented against the computational time. It can be observed that the theoretical
 468 convergence rates are achieved.

Scheme	N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
1st	50	1.13E-03		2.16E-05		8.65E-03	
	100	6.43E-04	0.81	1.25E-05	0.79	4.65E-03	0.89
	200	3.46E-04	0.90	6.79E-06	0.88	2.44E-03	0.93
	400	1.80E-04	0.94	3.55E-06	0.94	1.26E-03	0.96
3rd	50	6.45E-05		1.24E-06		4.09E-04	
	100	8.69E-06	2.89	1.71E-07	2.86	5.83E-05	2.81
	200	1.10E-06	2.99	2.17E-08	2.98	7.39E-06	2.98
	400	1.36E-07	3.01	2.71E-09	3.00	9.21E-07	3.00

Table 1: Section 5.1. Convergence rate test for h using L_1 and L_2 and L_∞ error norms for the 1-st and 3-rd order ARL-ADER schemes. CFL=0.2.

Scheme	N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
1st	50	2.17E-03		4.52E-05		1.86E-02	
	100	1.25E-03	0.80	2.67E-05	0.76	1.12E-02	0.73
	200	6.74E-04	0.89	1.47E-05	0.87	6.22E-03	0.85
	400	3.51E-04	0.94	7.70E-06	0.93	3.29E-03	0.92
3rd	50	1.37E-04		3.02E-06		1.30E-03	
	100	1.82E-05	2.91	4.12E-07	2.87	1.80E-04	2.85
	200	2.30E-06	2.98	5.25E-08	2.97	2.35E-05	2.94
	400	2.86E-07	3.01	6.56E-09	3.00	3.00E-06	2.97

Table 2: Section 5.1. Convergence rate test for hu using L_1 and L_2 and L_∞ error norms for the 1-st and 3-rd order ARL-ADER schemes. CFL=0.2.

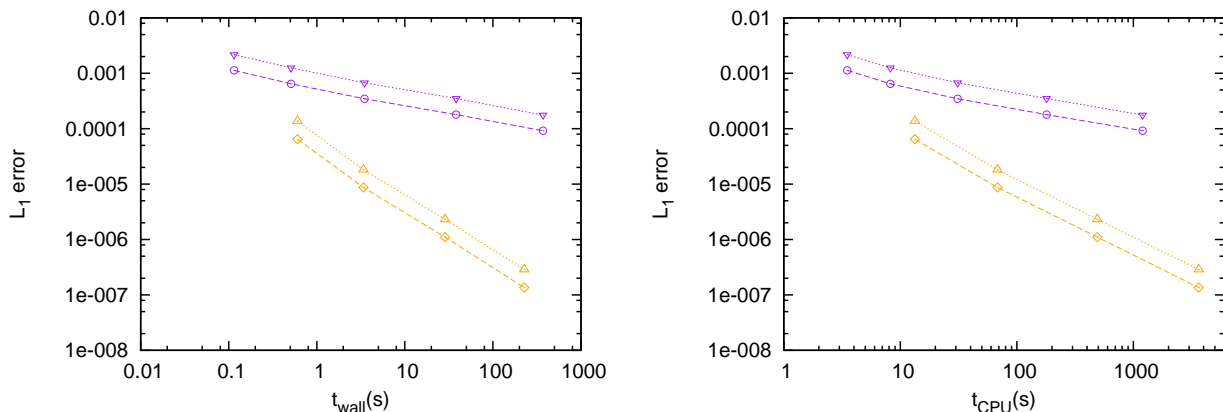


Figure 2: Section 5.1. Convergence rate test: logarithmic plot of the L_1 error against the wall-clock time (bottom-left) and CPU time (bottom-right). Solution computed using a 1-st (purple) and 3-rd (orange) order schemes.

469 5.2. Steady supercritical flow through a cross-section discontinuity

470 The *Mach 3 wind tunnel with a step* test problem, introduced [70], has proven to be a useful test for
471 a large number of methods and a large number of years in the framework of Euler equations. Here, we
472 propose the analogous problem for the SWE with variable bed elevation and use it to test the numerical
473 performance of the ADER schemes when solving complex 2D shock structures. The configuration of
474 the case is detailed below.

475 We consider a computational domain given by $\Omega = [0, 70] \times [0, 25] \setminus \Pi_s$ where $\Pi_s = [15, 70] \times [0, 5]$
476 (units in m) is a solid obstacle which mimics the cross-section contraction and is not included in
477 the computational domain. The boundary condition at the inlet ($x = 0$) is defined as supercritical
478 flow setting $h = 1$ m and $hu = 10.5$ m²/s, while at the outlet ($x = 75$ m), free flow condition is
479 imposed. On the remaining boundaries, we impose solid wall conditions. The initial condition is given

480 by $h(x, y, 0) = 1$ m, $hu(x, y, 0) = 10.5$ m²/s and $hv(x, y, 0) = 0$ m²/s $\forall(x, y) \in \Omega$.

481 Bed elevation is not constant in this case and is given by the following piecewise function

$$z(x, y) = \begin{cases} 0 & \text{if } x \leq 5 \\ 0.3x^{0.3} - 0.486 & \text{if } x > 5 \end{cases} \quad (72)$$

482 Two different grids are used in this test: a coarse grid, composed of 100×280 cells and a fine
 483 grid, composed of 400×1120 cells. The numerical solution for the water surface elevation at $t = 100$
 484 s computed by the 1-st and 3-rd order well-balanced ARL-ADER scheme in each grid is presented in
 485 Figure 3. We have used $CFL = 0.3$.

486 In order to give a closer insight to the structure and features of the solution, a representation of
 487 the numerical solution of $|\nabla(h + z)|$ (top), Froude number (middle) and velocity magnitude (bottom)
 488 is presented in Figure 4. The solution is represented at the time when the Kelvin-Helmholtz instability
 489 is better observed. From such figures we can see that the initial effect of the presence of the obstacle
 490 in the supercritical flow is the formation of a bow shock (hydraulic jump) that is reflected on the
 491 top solid wall and eventually forms a Mach stem that is joined to the incident wave (bow shock) and
 492 reflected wave at the so-called triple point. It is also worth mentioning that at the triple point, a
 493 shear layer appears and a Kelvin-Helmholtz instability develops, triggered by the numerical shockwave
 494 instabilities at the Mach stem and amplified by the physical instability at the slip line.

495 In Figure 3, it is observed that the position of the triple point does depend on the grid and the
 496 accuracy of the numerical scheme. The coarser the grid and the lower the accuracy is, the higher the
 497 triple point is located. Moreover, we observe that the solution provided by the 3-rd order scheme in the
 498 coarse grid is of about the same accuracy than the solution provided by the 1-st order scheme in the
 499 fine grid, which has 16 times more cells. The position of the triple point/upper Mach stem proved to
 500 be a good estimation for the accuracy of the scheme and can be used to assess the convergence [71]. In
 501 Figure 5, the x position of the Mach stem at $y = 24.5$ m computed by the 1-st and 3-rd order schemes
 502 is plotted against the number of cells in the x -direction, denoted by N_x , chosen in the discretization.
 503 It is observed that for the 3-rd order scheme, when discretizing the domain with more than 1000 cells
 504 in the x -direction, the variation in the position of the Mach stem is of the order of 10^{-2} m, while it is
 505 more than an order of magnitude higher for the 1-st order scheme.

506 5.3. 2D Riemann problem

507 Compared with the relatively simple 1D RPs, the solution of 2D RPs include complex geometric
 508 wave patterns that pose a computational challenge for high-resolution numerical schemes. In this
 509 section, we solve a 2D RP whose initial condition is given by piecewise constant data in each of the
 510 four quadrants of the Cartesian plane. Hereafter, such quadrants will be denoted by Q_1, Q_2, Q_3, Q_4 .
 511 The problem is solved using the 1-st and 3-rd order ADER scheme in a grid composed of 800×800
 512 cells using $CFL = 0.4$ in the domain $\Omega = [0, 100] \times [0, 100]$ m, at a time $t = 3$ s.

513 The initial condition is given by

$$h(x, y) = \begin{cases} 10 & \text{if } (x, y) \in Q_1 \\ 3 & \text{if } (x, y) \in Q_2 \\ 0.2 & \text{if } (x, y) \in Q_3 \\ 3 & \text{if } (x, y) \in Q_4 \end{cases}, z(x, y) = \begin{cases} 1 & \text{if } (x, y) \in Q_1 \\ 0 & \text{if } (x, y) \in Q_2 \\ -0.5 & \text{if } (x, y) \in Q_3 \\ 0 & \text{if } (x, y) \in Q_4 \end{cases} \quad (73)$$

$$u(x, y) = \begin{cases} 0 & \text{if } (x, y) \in Q_1 \\ 3 & \text{if } (x, y) \in Q_2 \\ 3 & \text{if } (x, y) \in Q_3 \\ 0 & \text{if } (x, y) \in Q_4 \end{cases}, v(x, y) = \begin{cases} 0 & \text{if } (x, y) \in Q_1 \\ 0 & \text{if } (x, y) \in Q_2 \\ 3 & \text{if } (x, y) \in Q_3 \\ 3 & \text{if } (x, y) \in Q_4 \end{cases} \quad (74)$$

514 with h and z in m and u and v in m/s.

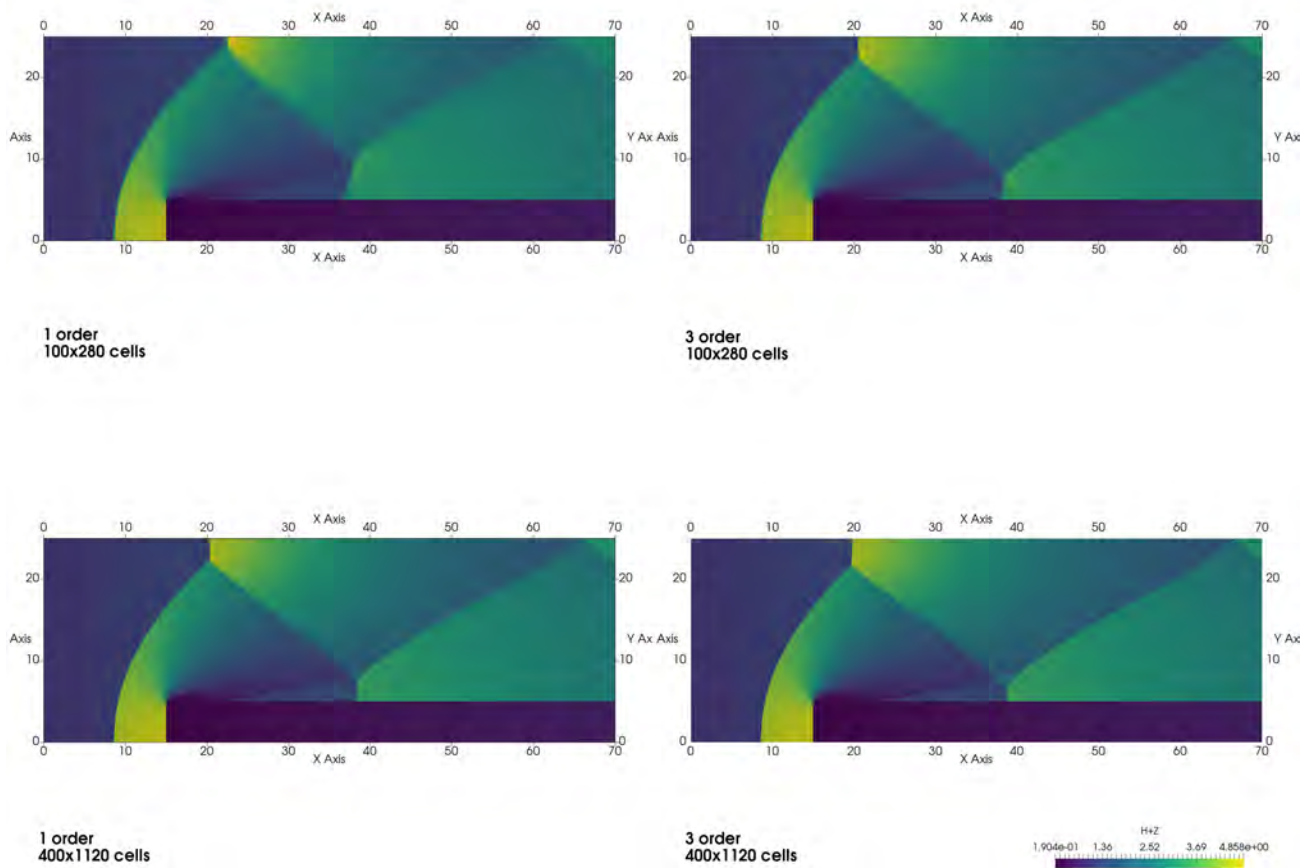


Figure 3: Section 5.2. Numerical solution for $h + z$ (m) at $t = 100$ s provided by the 1-st order (left) and 3-rd order scheme (right) using the 100×280 grid (top) and the 400×1120 grid (bottom).

515 The structure of the solution is composed by a complex variety of waves: at the four edges of
 516 the quadrants limiting with the axis, there are four contact discontinuities due to the bed steps. In
 517 the first quadrant, there are two interacting rarefaction waves moving in the x and y directions. In
 518 the second and fourth quadrants, there are two rarefaction waves moving in the y and x direction
 519 respectively, which interact with a transverse shock wave. Finally, the most complex wave pattern
 520 can be found in the third quadrant. Here, there are two pairs of shocks in each coordinate direction,
 521 moving perpendicularly with respect to each other, and from their interaction a jet stream pointing
 522 to the bottom-left corner of the domain is generated. Such jet is bounded by multiple shocks and a
 523 strong recirculation is observed at both sides of the jet. It is worth pointing out that this RP is a
 524 resonant problem as the number of waves is greater than the number of eigenvalues.

525 A shadowgraph for the numerical $h + z$ and the velocity vector field has been represented in Figure
 526 6. A cross-sectional representation of the numerical $h + z$ and q computed using a 1-st and 3-rd order
 527 scheme in a 200×200 and 800×800 grid is also presented in Figure 7 and compared with a reference
 528 solution that has been computed using the 3-rd order scheme in a very fine mesh.

529 It is observed that only when using the 3-rd order scheme, the finest details such as the normal
 530 shock produced by the deceleration of the reverse flow in the jet in the third quadrant, can be captured.
 531 Moreover, the velocity peak in the jet is underestimated when using the 1-st order scheme. Important

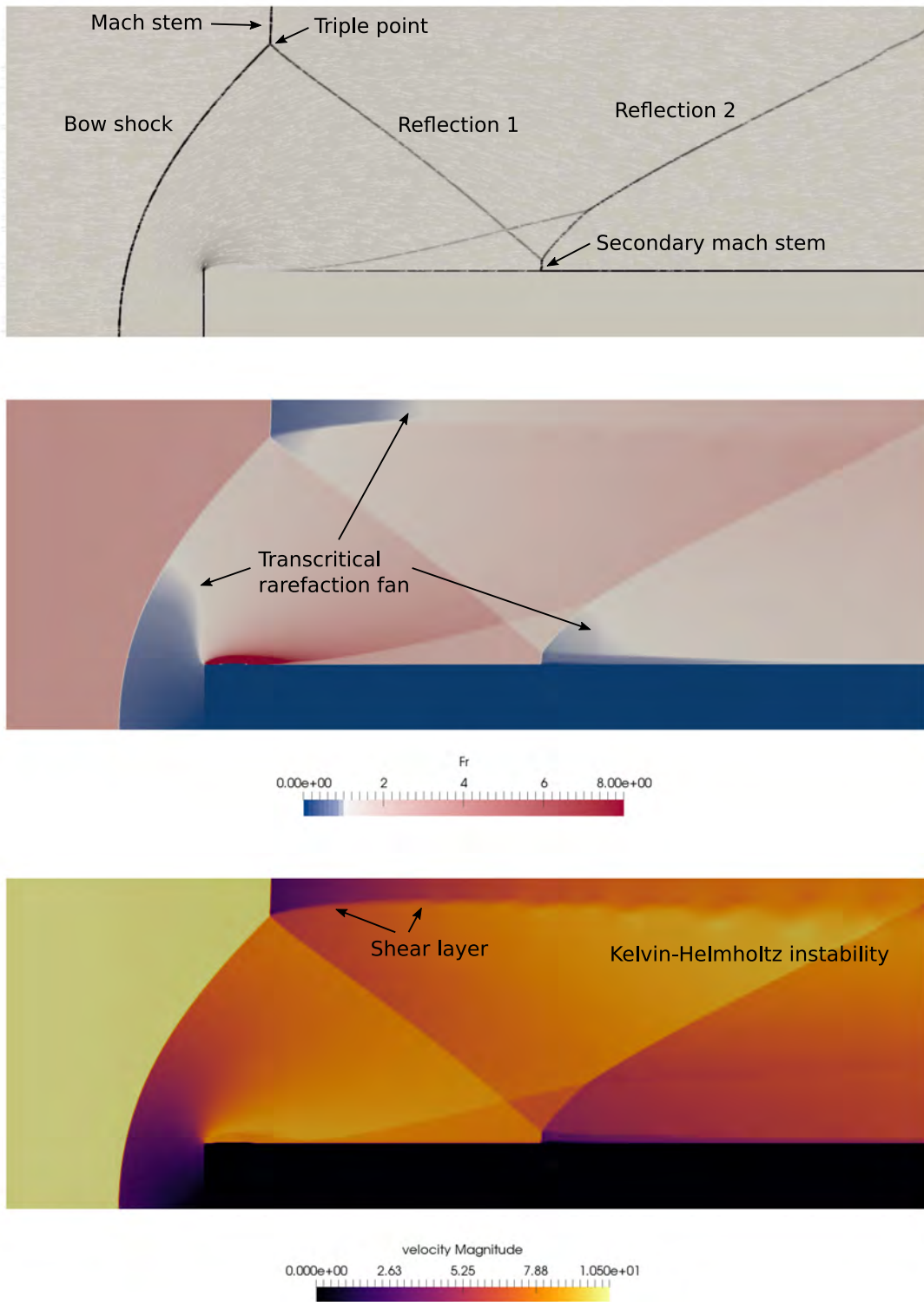


Figure 4: Section 5.2. Numerical solution for the water surface elevation gradient (top), Froude number (middle) and velocity magnitude (m²/s) (bottom), including the relevant features of this particular flow.

532 differences can also be found in the first quadrant, where the solution provided by the 1-st order scheme
 533 shows larger diffusion and does not capture the correct shape of the interaction of the rarefaction fans.

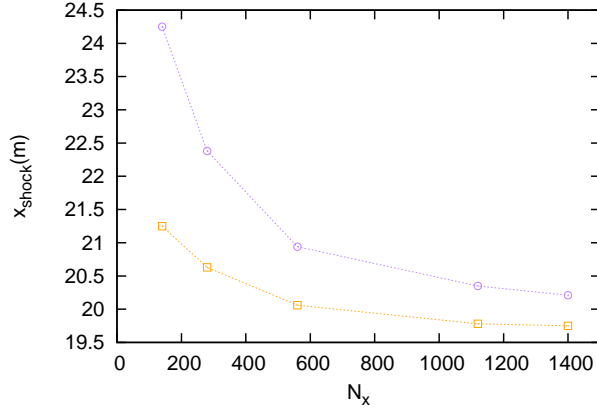


Figure 5: Section 5.2. Representation of the x position of the Mach stem at $y = 24.5$ m computed by the 1-st (purple) and 3-rd order (orange) ARL-ADER schemes against the number of cells in the x -direction

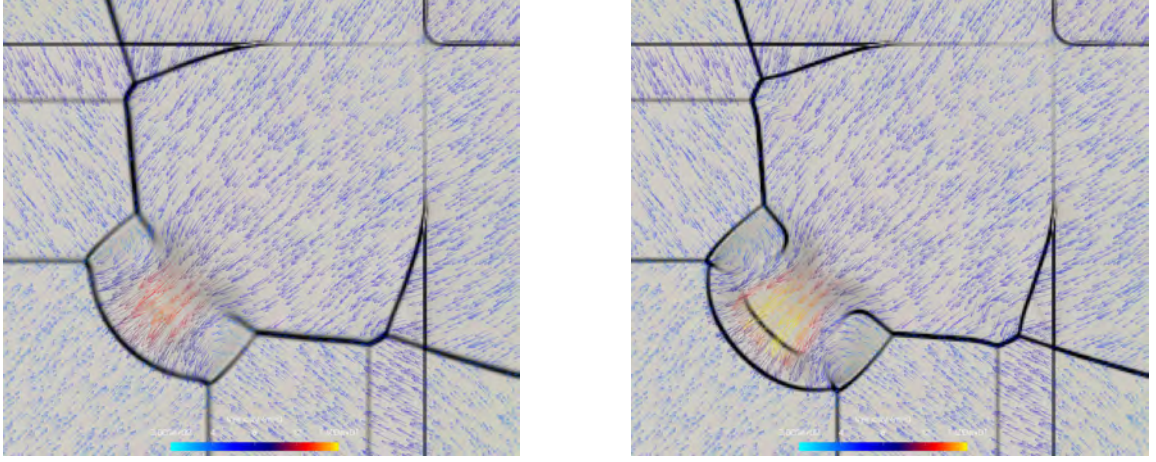


Figure 6: Section 5.3. Water surface elevation gradient and velocity field (m/s) provided by the 1-st order scheme (left) and 3-rd order scheme (right) in a 200×200 grid.

534 5.4. 1D geostrophic equilibrium with bed variation

535 This test case consist of a one-dimensional flow in the y -direction that is initially at geostrophic
 536 equilibrium [59]. The computational domain is $y \in [-5, 5]$ m and the initial condition is given by

$$K(x, y) = 2, \quad u(x, y) = \frac{2g}{f}y \exp(-y^2), \quad v(x, y) = 0, \quad (75)$$

537 where K is given in m, u and v in m^2/s , $f = 1 \text{ s}^{-1}$ and $g = 9.8 \text{ ms}^{-2}$. The bottom elevation is given
 538 by

$$z(x, y) = 0.1 \sin(0.2\pi y). \quad (76)$$

539 The numerical solution for $h + z$, hu , hv and K at time $t = 10$ s provided by the 3-rd order
 540 ARL-ADER scheme is presented in Figure 8 and is compared with the exact solution. The numerical
 541 solution provided by a non well-balanced version of the ARL-ADER scheme is also presented in Figure
 542 8. This non well-balanced scheme is constructed using the WENO reconstruction over $h + z$, hu , hv and
 543 z , the traditional Roe solver for the fluxes, the ADER approach for the time stepping and a traditional
 544 Gaussian integration for the source term. Primitive variables are not used and the source term is only
 545 accounted for inside cells and not at cell interfaces.

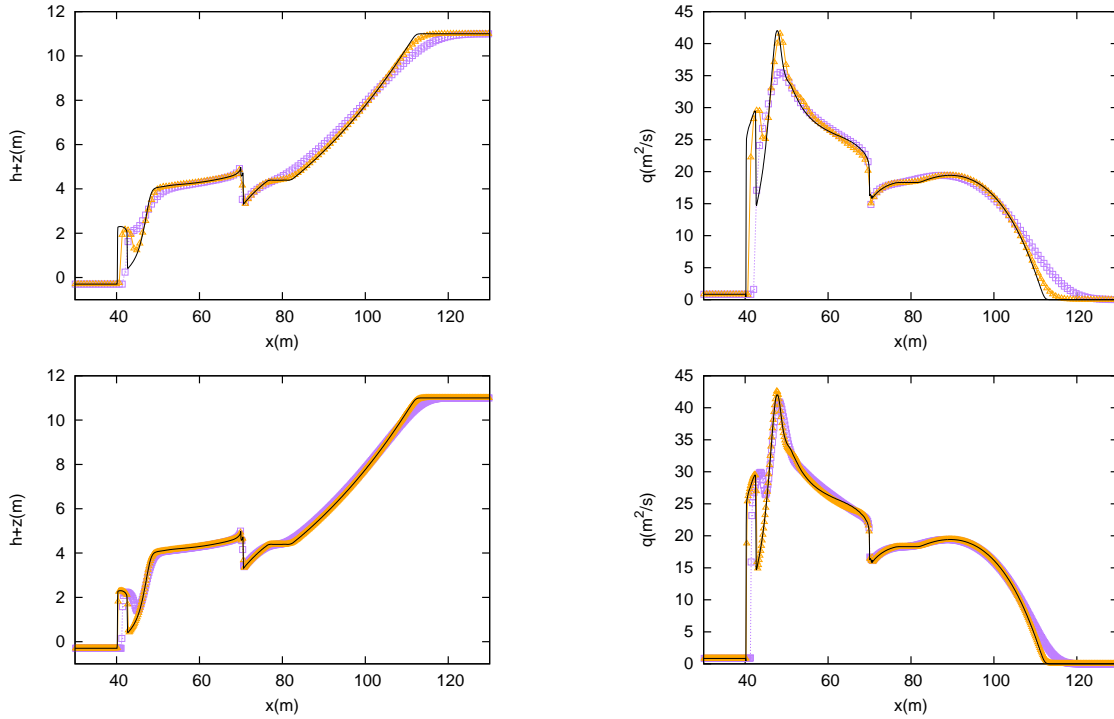


Figure 7: Section 5.3. Cross-sectional representation of the numerical $h + z$ (m) (left) and q (m^2/s) (right) computed using a 1-st ($-\square-$) and 3-rd order ($-\triangle-$) scheme in a 200×200 (top) and 800×800 (bottom) grid.

	1-st order	3-rd order
$t(\text{s})$	L_∞ error	L_∞ error
5	4.44088E-016	4.44089E-016
10	4.44088E-016	4.44089E-016

Table 3: Section 5.4. Numerical errors for K provided by the 1-st and 3-rd order ARL-ADER scheme, measured with L_∞ error norm at $t = 5$ and $t = 10$ s. Double precision is used.

546 According to the numerical results, the proposed ARL-ADER scheme preserves the geostrophic
547 equilibrium up to machine precision, with round-off errors for K of magnitude 10^{-16} m, as shown in
548 Table 3. Hence, the scheme satisfies the well-balanced property. On the other hand, the non well-
549 balanced ADER scheme leads to the appearance of artificial waves and is not able to preserve the
550 geostrophic equilibrium.

551 In order to study the convergence properties of the non well-balanced ADER scheme, the evolution
552 in time of the L_1 error norm for K computed by a 1-st and 3-rd order scheme in two different grids
553 with $\Delta x = 0.2$ m and $\Delta x = 0.1$, is depicted in Figure 9. It is observed that the numerical solution
554 converges to the exact geostrophic state as the grid is refined and the order is increased, however, it
555 can never achieve the level of accuracy of the well-balanced scheme.

556 5.5. Convergence test for the SWE with bed elevation and Coriolis

557 A convergence rate test for the ARL-ADER well-balanced scheme with bed elevation and Coriolis
558 is presented. The following initial condition is imposed

$$z(x, y) = 0.1 \exp\left(-\frac{(x-50)^2 + (y-50)^2}{80}\right), \quad \forall(x, y) \in \Omega \quad (77)$$

559 $h(x, y, 0) = 0.5$ m and $hu(x, y, 0) = hv(x, y, 0) = 0$ m^2/s $\forall(x, y) \in \Omega$, $g = 1$ m/s^2 and $f = 0.05$ s^{-1} .

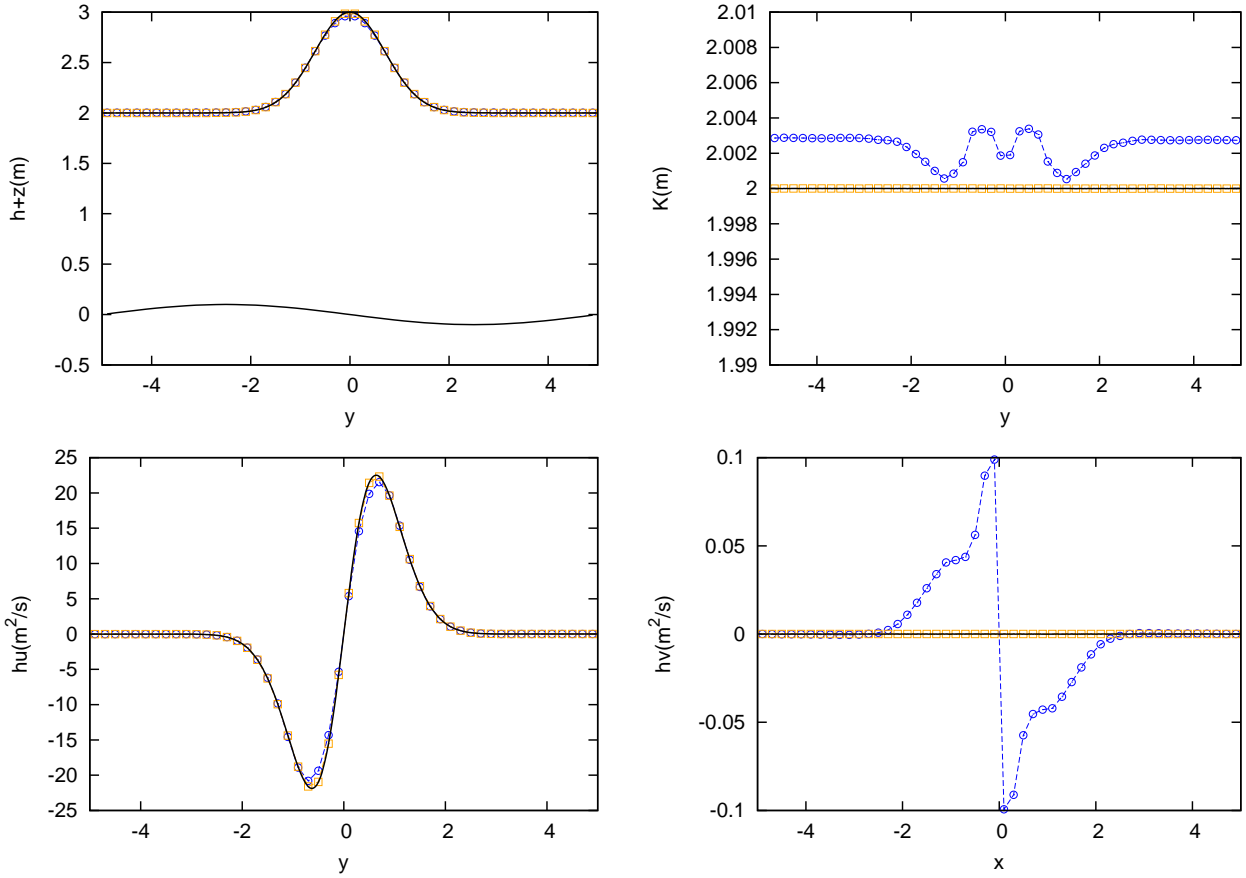


Figure 8: Section 5.4. Numerical solution for $h + z$, K , hu and hv at $t = 10$ s provided by the 3-rd order well-balanced ARL-ADER scheme ($-\square-$) and by its non well-balanced version ($-\circ-$), using $\Delta x = 0.2$ m.

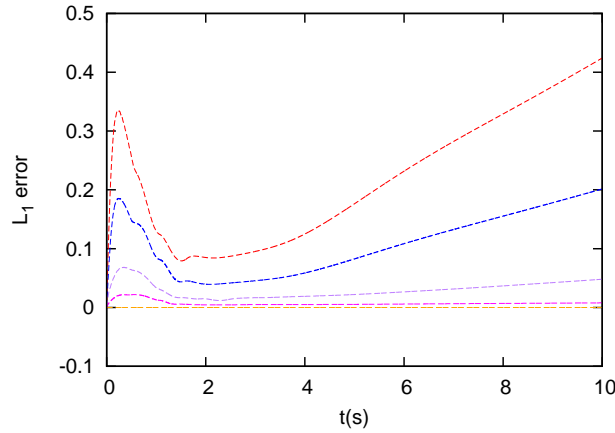


Figure 9: Section 5.4. Evolution in time of L_1 error norm for K computed by the well-balanced ARL-ADER scheme ($-$) and by its non well-balanced version using a 1-st order scheme and $\Delta x = 0.2$ m ($-$), a 1-st order scheme and $\Delta x = 0.1$ ($-$), 3-rd order scheme and $\Delta x = 0.2$ ($-$) and 3-rd order scheme and $\Delta x = 0.1$ ($-$).

560 The computational domain is $\Omega = [0, 100] \times [0, 100]$ m and the solution is computed at $t = 3$ s setting
 561 $CFL = 0.1$ using the 3-rd order scheme.

562 Numerical errors and convergence rates for h and hu computed in four different grids composed of
 563 50×50 , 100×100 , 200×200 and 400×400 cells are presented in Table 4. Numerical errors have been

Variable	N	L_1 error	Order	L_2 error	Order	L_∞ error	Order
h	50	5.39E-06		2.06E-07		2.50E-04	
	100	7.33E-07	2.88	2.89E-08	2.83	3.82E-05	2.71
	200	9.51E-08	2.95	3.82E-09	2.92	5.27E-06	2.86
	400	1.21E-08	2.97	4.97E-10	2.94	7.18E-07	2.88
hu	50	2.69E-06		1.07E-07		9.74E-05	
	100	3.25E-07	3.05	1.30E-08	3.05	1.23E-05	2.98
	200	3.84E-08	3.08	1.52E-09	3.09	1.42E-06	3.11
	400	4.57E-09	3.07	1.76E-10	3.11	1.68E-07	3.08

Table 4: Section 5.5. Convergence rate test for h and hu using L_1 and L_2 and L_∞ error norms for the 3-rd order ARL-ADER scheme. CFL=0.1.

Scheme	N	$t = 0.3$ s				$t = 3$ s			
		L_1 error	Order	L_∞ error	Order	L_1 error	Order	L_∞ error	Order
3-rd	25	4.26E-06		2.23E-04		3.73E-06		5.90E-05	
	50	6.50E-07	2.71	3.50E-05	2.67	5.74E-07	2.70	1.04E-05	2.50
	100	8.53E-08	2.93	4.90E-06	2.84	1.41E-07	2.03	2.47E-06	2.07
5-th	25	1.21E-06		6.23E-05		3.11E-06		1.25E-04	
	50	5.78E-08	4.39	3.13E-06	4.31	1.68E-07	4.21	7.10E-06	4.14
	100	2.08E-09	4.80	1.20E-07	4.71	2.20E-08	2.93	1.10E-06	2.69

Table 5: Section 5.5. Convergence rate test for h using L_1 and L_∞ error norms for the 3-rd and 5-th order ARL-ADER scheme. The solution is computed at $t = 0.3$ and $t = 3$ s using CFL=0.02.

564 computed for h and hu using a reference solution computed by the 3-rd order scheme in a 2000×2000
565 grid and are measured using the L_1 , L_2 and L_∞ error norms. Numerical errors for hu are not presented
566 due to the symmetry of the case. The theoretical convergence rate is achieved.

567 5.5.1. Computation of stiff non-geometric source terms

568 The numerical method proposed here considers a geometric reinterpretation of the Coriolis source
569 term that allows to apply the well-balancing techniques derived for the SWE with bed elevation to
570 the SWE in the rotating frame. To ensure the discrete equilibrium, first order time derivatives are
571 derived expressing the source term in terms of the primitive variables. By contrast, second and higher
572 order time derivatives are derived considering the original source terms fhu and fhv . This yields to a
573 suboptimal integration in time that can be evidenced when computing very stiff source terms.

574 If the test case in the previous section is repeated, setting $f = 2 \text{ s}^{-1}$, a suboptimal convergence
575 rate is observed in the solution. Table 5 shows the numerical errors provided by a 3-rd and 5-th order
576 ARL-ADER scheme at $t = 0.3$ and $t = 3$ s, using CFL=0.02. It is observed that the convergence rates
577 at the shorter time, $t = 0.3$ s, are optimal. However, as the solution advances in time, a degradation
578 of accuracy is observed. This is because at shorter times, the spatial error is much higher than
579 the temporal error, hence the overall convergence is maintained. When moving to longer times, the
580 temporal error, which has a suboptimal behavior, grows over the spatial error and hence the overall
581 convergence is reduced.

582 It is worth recalling that this test case considers a ratio between the Coriolis parameter and the
583 gravity constant of $f/g = 2$ while in the atmospheric and oceanic circulation we usually find that
584 $f/g = 0.00001$. Therefore, the present test case is beyond the range of application of the model,
585 however the authors consider of importance to show the actual limitations of the scheme.

586 5.6. 2D geostrophic adjustment

587 Here we consider the test case proposed in [61] (see also [59, 64]), which consists of a initial asymmet-
588 rical column of water that falls under a strong rotation that leads to a 2D geostrophic adjustment the

589 numerical scheme must be able to reproduce. The computational domain is $\Omega = [-10, 10] \times [-10, 10]$
 590 m, the bottom topography is flat and the initial condition is given by

$$h(x, y) = 1 + 0.5A_0 \left(1 - \tanh \left(\frac{\sqrt{(\sqrt{\lambda}x)^2 + (y/\sqrt{\lambda})^2} - R_i}{R_E} \right) \right), \quad (78)$$

$$u(x, y) = v(x, y) = 0, \quad (79)$$

591 where $A_0 = 0.5$, $\lambda = 2.5$, $R_E = 0.1$, $R_i = 1$ and the gravity and Coriolis parameters are set to $g = 1$
 592 m/s^2 and $f = 1 \text{ s}^{-1}$. The numerical solution is computed using the 3-rd order ARL-ADER scheme
 593 using a grid of 400×400 cells and setting CFL=0.4. Initially, the elliptical column of water is not
 594 at equilibrium and evolves in a nonaxisymmetric way due to the rotation effect. Two successive shock
 595 (gravity) waves are generated and leave behind a small smooth hump that is slowly spinning clockwise.
 596 The numerical result for the water surface elevation at times $t = 0$, $t = 4$, $t = 8$, $t = 12$, $t = 16$ and
 597 $t = 20$ s are presented in Figure 10. It is observed that the expected behavior of the evolution of the
 598 solution is reproduced by the numerical scheme and that the numerical results are qualitatively very
 599 similar to those presented in [59, 64].

600 A cross sectional representation of the solution for $h + z$ and L at $y = 10$ m and $t = 4$ s, provided
 601 by a 1-st and 3-rd order ARL-ADER scheme in two grids composed of 101×101 and 401×401 cells
 602 is depicted in Figure 11.

603 5.7. 2D propagation of Rossby waves on the equatorial β -plane

604 This test case considers the propagation of a Rossby soliton on the equatorial beta-plane, for which
 605 an asymptotic solution exists to the inviscid SWE. Theoretically, the soliton should propagate to the
 606 west at fixed phase speed, without change of shape. Since the uniform propagation and shape preserva-
 607 tion of the soliton are achieved through a fine balance between linear wave dynamics and nonlinearity,
 608 this is a good context in which to look for erroneous wave dispersion and/or numerical damping and
 609 has proven to be a good benchmark for atmosphere and ocean models ([http://marine.rutgers.edu/po](http://marine.rutgers.edu/po/index.php?model=test-problems)
 610 [/index.php?model=test-problems](http://index.php?model=test-problems)) [72]. The interest in this test problem is to assess the spurious
 611 dispersion and dissipation effects of the numerical scheme, and how they relate to the choice of grid
 612 resolution and the accuracy of the scheme.

613 Long, weakly nonlinear, equatorial Rossby waves are governed by either Korteweg–de Vries (KDV)
 614 or the modified Korteweg–de Vries (MKDV) equation [72, 73]. Here, a *zero*-th order asymptotic
 615 solution to the SWE is used [73]. The initial condition for a dipole at $(x, y) = (0, 0)$ m can be found in
 616 [72, 73]. Here, the dipole is translated to $(x, y) = (72, 6)$ m by means of evaluating the initial condition
 617 at $x' = x - 72$ m and $y' = y - 6$ m. The gravity constant is set to $g = 1 \text{ m/s}^2$ and the Coriolis
 618 parameter is calculated using the β -plane approximation $f(y) = f_0 + \beta y$ with $f_0 = 0 \text{ s}^{-1}$ and $\beta = 1$
 619 $\text{m}^{-1}\text{s}^{-1}$.

620 The case considered here consists of the *zero*-th order soliton described above over a flat bed, that
 621 is $z(x, y) = 0$ m, computed inside the domain $\Omega = [0, 96] \times [0, 12]$, units in m, at time $t = 120$ s. The
 622 numerical solution provided by the 1-st order (12) and 3-rd order ARL-ADER scheme (13) at times
 623 $t = 0$, $t = 30$, $t = 60$, $t = 90$ and $t = 120$ s using two grid sizes of $\Delta x = 0.2$ and $\Delta x = 0.1$ m, are
 624 presented and compared with the exact solution at $t = 120$ s. CFL is set to 0.4. It is observed that
 625 the 1-st order scheme in the coarsest grid does not perform well as it generates spurious waves. When
 626 moving to the finest grid size, the performance of the scheme is improved though it is still very diffusive
 627 and dispersive. The 3-rd order scheme does provide an accurate solution with both grids and ensures
 628 a much lower dispersion and diffusion of the soliton.

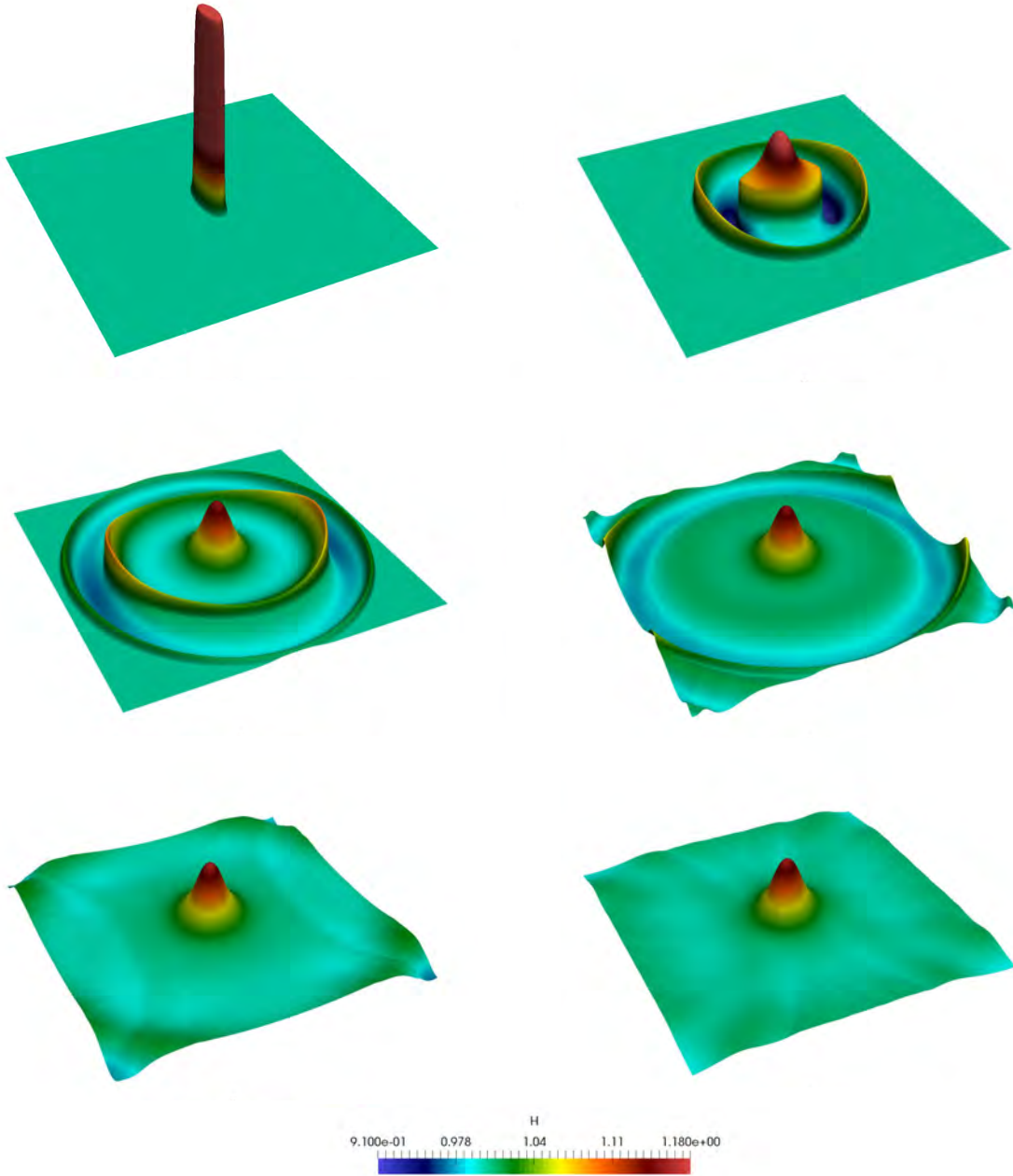


Figure 10: Section 5.6. Numerical $h+z$ (m) at $t = 0$ (top-left), $t = 4$ (top-right), $t = 8$ (middle-left), $t = 12$ (middle-right), $t = 16$ (bottom-left) and $t = 20$ s (bottom-right) provided by the 3-rd order ARL-ADER scheme in a 400×400 grid.

629 To assess the performance of the numerical schemes, we have used the following metrics: the
 630 damping factor, ν , which accounts for the numerical damping of the solution and the relative speed,
 631 c_r , which accounts for the numerical dispersion of the solution, defined in [72].

632 It is worth noting that all maximum and minimum water depth values are cell-averaged values and
 633 no interpolation is used. Numerical values for the metrics described above and other related data is
 634 presented in Table 6 using the results provided by the 1-st and 3-rd AR-ADER scheme in two grids
 635 of $\Delta x = 0.2$ and $\Delta x = 0.1$ m. It is evidenced that the 3-rd order scheme outperforms the 1-st order
 636 scheme in terms of numerical dispersion and damping, as it was expected. If comparing with the

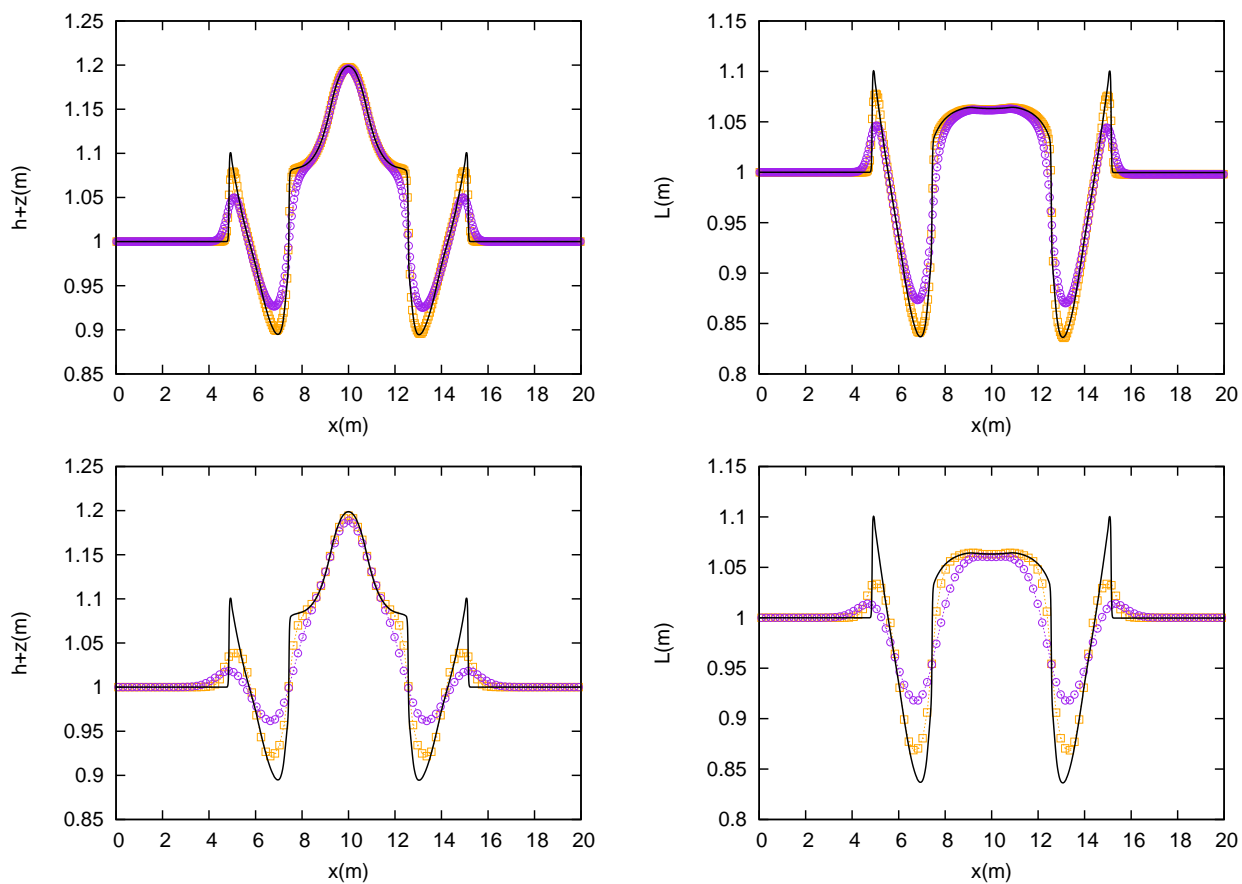


Figure 11: Section 5.6. Cross sectional representation of the solution for $h + z$ (m) and L (m) at $y = 10$ (m) and $t = 4$ s provided by a 1-st and 3-rd order ARL-ADER scheme in a 101×101 and 401×401 grid.

637 numerical results in [72] it is observed that the measures for dispersion and damping are of the same
 638 magnitude.

639 Figure 14 shows a comparison between the numerical solution at $t = 30$ s provided by the 3-rd
 640 order ARL-ADER scheme and the non well-balanced version of this scheme, in a $\Delta x = 0.1$ m grid.
 641 It is observed that the non well-balanced scheme produces spurious waves arising from the imbalance
 642 of the Coriolis terms in the scheme. This result motivates the necessity of well-balanced schemes,
 643 specially for the simulation of such kind of geophysical events that consist of the evolution in time of
 644 perturbations of a certain equilibrium state. Only when the equilibrium state is accurately preserved,
 645 those perturbations will be properly captured.

646 5.8. Kelvin front generation on the equatorial β -plane

647 In this section, the generation of nonlinear planetary (Rossby) and Kelvin waves at Earth's equa-
 648 torial line is reproduced by the ARL-ADER numerical scheme. When the equatorial area is perturbed
 649 (by changing winds, for instance), its adjustment to the new equilibrium state is done by means of
 650 wave propagation. Such perturbations are usually of a very low frequency and therefore gravity waves
 651 are not excited, instead, only certain type of waves such as Kelvin waves, mixed waves and planetary
 652 waves (Rossby waves) appear. The short wavelength Kelvin waves carry energy eastward direction,
 653 whereas the long wavelength planetary waves carry energy to westward direction.

654 An additional phenomena is considered in this test case. It has been recognized for some time that
 655 nonlinear equatorial Kelvin waves can steepen and break, forming a broken wave, or front, propagating

	3-rd order		1-st order	
	$\Delta x = 0.2$	$\Delta x = 0.1$	$\Delta x = 0.2$	$\Delta x = 0.1$
$h_{max,t=0}(m)$	1.16948	1.17032	1.16948	1.17032
$h_{min,t=0}(m)$	1.00000	1.00000	1.00000	1.00000
$h_{max,t=120}(m)$	1.14019	1.15332	1.05989	1.08135
$h_{min,t=120}(m)$	0.99509	0.99263	≈ 0	0.99835
ν	0.975	0.985	0.906	0.924
$x_{end}(m)$	24.9	24.84	26.9	26.2
c_r	0.981	0.983	0.940	0.954
Time _{CPU} (s)	212.8	1777.5	7.2	39.5
Time _{wall-clock} (s)	12.55	100.75	0.95	6.7

Table 6: Section 5.7. Numerical values of the relevant metrics for the assessment of numerical dispersion and damping of the scheme.

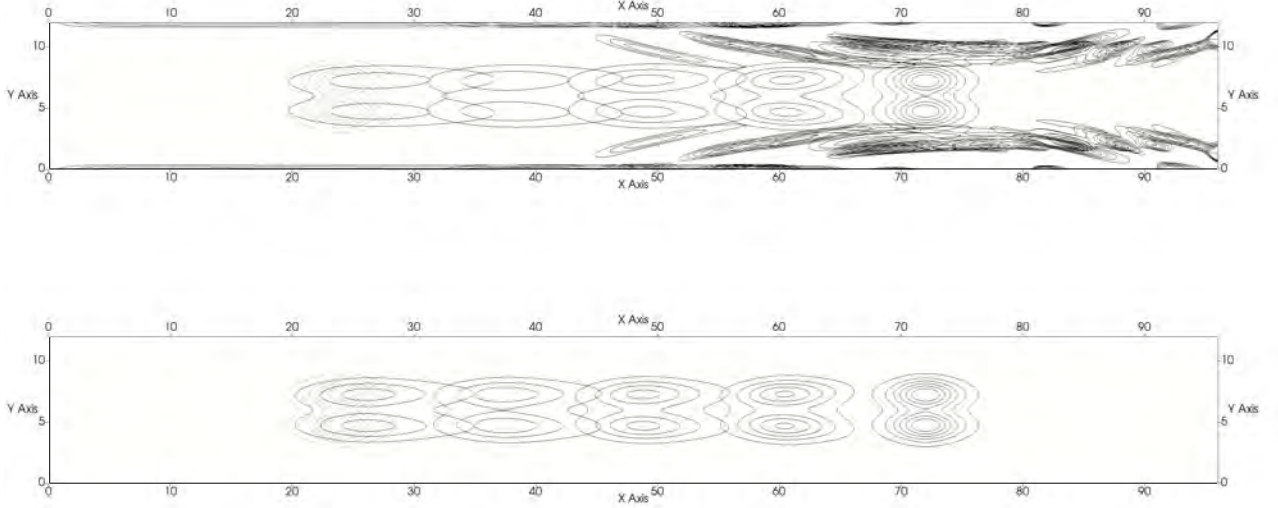


Figure 12: Section 5.7. Numerical solution for h (m) provided by the 1-st order scheme at times $t = 0$, $t = 30$, $t = 60$, $t = 90$ and $t = 120$ s, using two different grids with $\Delta x = 0.2$ (top) and $\Delta x = 0.1$ m (bottom). The contour plot has been generated using 6 intervals from 1.02 to 1.14 m.

656 eastward [74]. This leads to the generation of equatorially trapped inertial-gravity (or Poincare) waves,
657 which is an analogous mechanism than for nonlinear coastal Kelvin waves.

658 In this test case, we aim to show that the proposed numerical scheme is able to reproduce the
659 formation and propagation of both Rossby and Kelvin waves over a non-flat bed elevation and eventu-
660 ally the generation of the Kelvin front and resonant formation of Poincare waves. The computa-
661 tional domain for this test case is $\Omega = [0, 70] \times [0, 12]$, units in m. The initial perturbation is given by a
662 Gaussian water surface elevation anomaly, which reads

$$h(x, y) = h_0 + 0.8 \exp\left(-\frac{(x - 30)^2 + (y - 6)^2}{3}\right) \quad (80)$$

663 where $h_0 = 2$ m. The bed elevation is given by

$$z(x, y) = \begin{cases} 0 & \text{if } x \geq 40 \\ 0.025x - 1 & \text{if } x < 40 \end{cases} \quad (81)$$

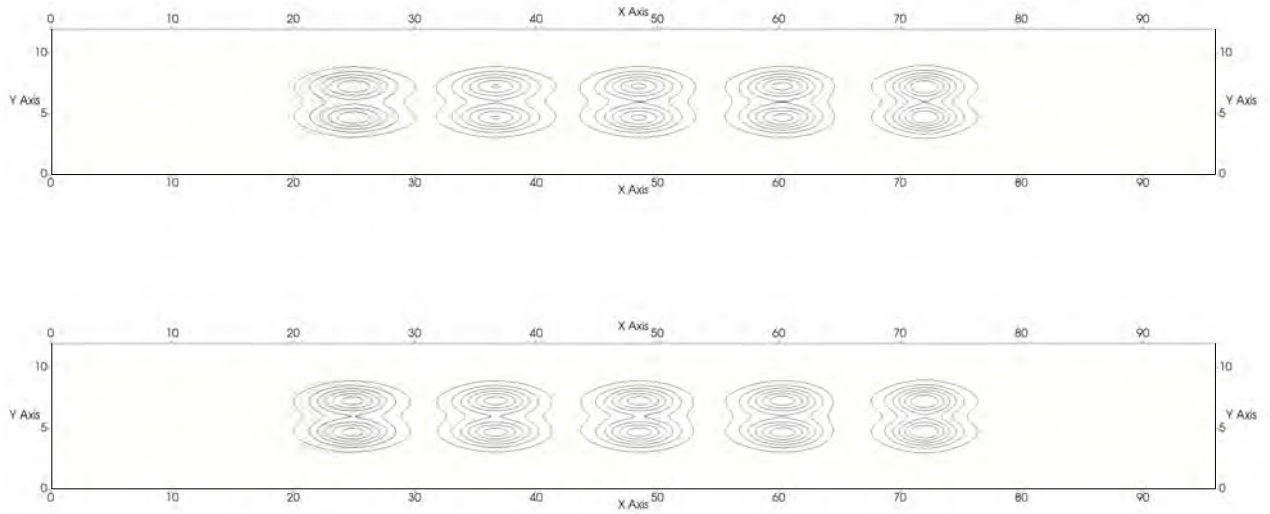


Figure 13: Section 5.7. Numerical solution for h (m) provided by the 3-rd order scheme at times $t = 0$, $t = 30$, $t = 60$, $t = 90$ and $t = 120$ s, using two different grids with $\Delta x = 0.2$ (top) and $\Delta x = 0.1$ m (bottom). The contour plot has been generated using 6 intervals from 1.02 to 1.14 m.

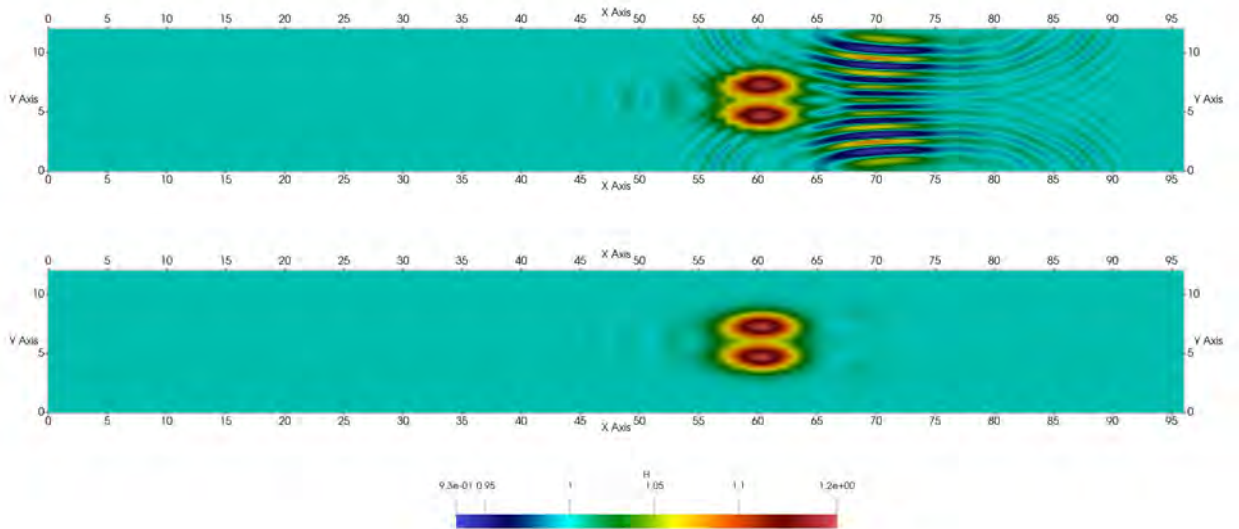


Figure 14: Section 5.7. Numerical solution for h (m) provided by a well-balanced (bottom) and non well-balanced (top) 3-rd order ADER scheme at time $t = 30$ s using $\Delta x = 0.1$ m.

664 The numerical solution is computed using the 1-st and 3-rd order ARL-ADER scheme in a 1400×240
 665 grid, using $CFL = 0.4$. The solution for $h + z$ at $t = 40$ s is depicted in Figure 15 using a contour
 666 plot with 20 intervals from 1.94 to 2.36 m. It is observed that only when using the 3-rd order scheme,
 667 an accurate resolution of the Kelvin front formation is possible and Poincare waves are captured.
 668 Regarding the planetary waves moving westward, it is worth mentioning that both schemes are able
 669 to reproduce the expected physical behavior, though the first order scheme is more diffusive and

670 dispersive.

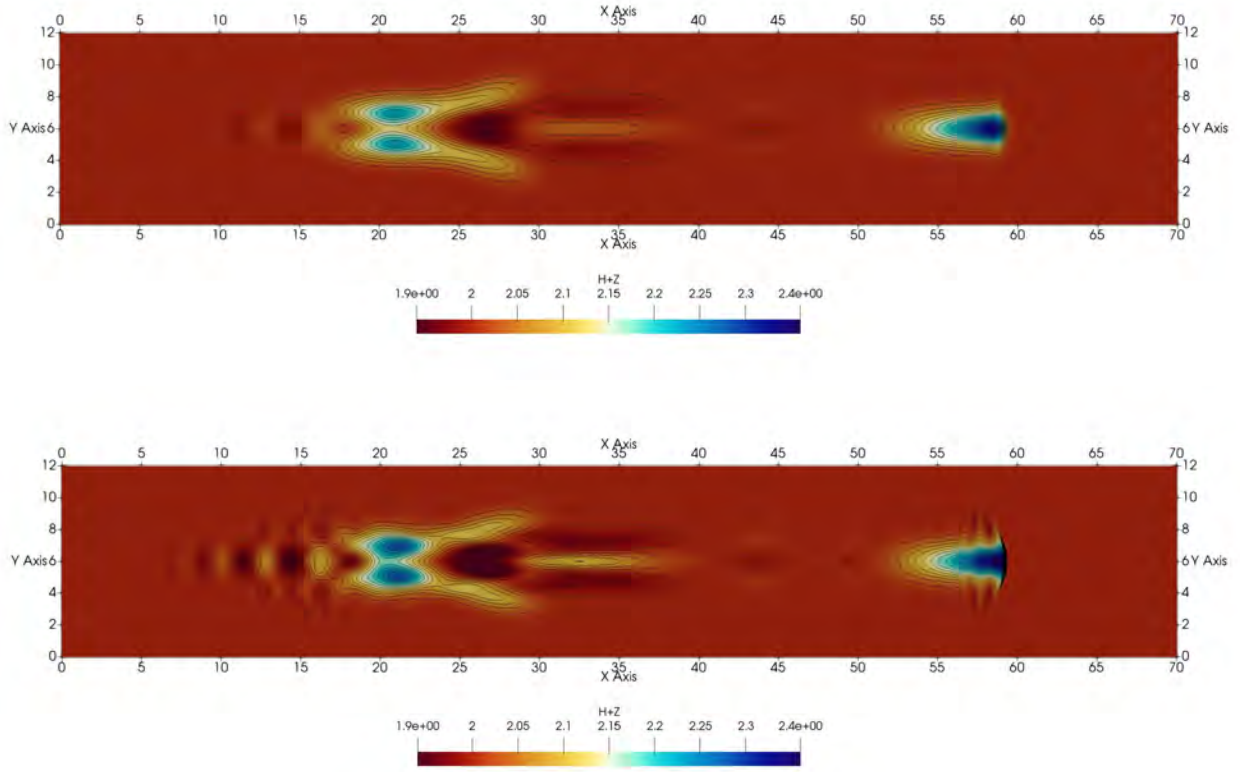


Figure 15: Section 5.8. Numerical solution for $h + z$ (m) provided by the 1-st (top) and 3-rd order (bottom) ARL-ADER scheme in the fine mesh at $t = 40$ s using $CFL = 0.4$. The contour plot has been generated using 20 intervals from 1.94 to 2.36 m.

671 *5.9. Anticyclonic propagation in the β -plane*

672 The proposed scheme is applied here to a more realistic case from [75] that consists of a initially
 673 symmetric vortex propagating westward due to the effect of the variation of the Coriolis coefficient
 674 in the y -direction. The domain extent is an idealized 2000 km \times 1200 km rectangular basin and the
 675 initial condition is given by a Gaussian distribution of the free surface centered at the origin of the
 676 domain, prescribed together with a velocity field which is in geostrophic balance. The water depth at
 677 the initial time is given by

$$h(x, y) = h_0 + \zeta(x, y), \quad (82)$$

678 where $h_0 = 1.631$ m is the water depth reference level and $\zeta(x, y)$ is the surface height anomaly given
 679 by

$$\zeta(x, y) = Ae^{-(x^2+y^2)/B^2} \quad (83)$$

680 with $A = 0.95$ m, $B = 130$ km and x and y given in km. The initial velocity field is given by

$$u_1(x, y) = 2A \frac{g}{f(y)} \frac{y}{B^2} e^{-(x^2+y^2)/B^2}, \quad u_2(x, y) = -2A \frac{g}{f(y)} \frac{x}{B^2} e^{-(x^2+y^2)/B^2}, \quad (84)$$

681 with $f(y)$ the Coriolis parameter, evaluated at 25° N using the β -plane approximation with $f_0 =$
 682 $6.1635 \times 10^{-5} \text{ s}^{-1}$ and $\beta = 2.0746 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$. The gravity constant is set to $g = 9.81 \text{ m/s}^2$.

683 The solution is computed using the 1-st and 3-rd order ARL-ADER schemes at $t = 8$ weeks. Figure
 684 16 shows the numerical solution for the 1-st (left) and 3-rd order scheme (right) using a 100×60 grid
 685 (top) and a 200×120 grid (bottom). As reported in [64], the first order scheme is not able to reproduce
 686 the physical behavior of the solution and is much more mesh dependent than the 3-rd order scheme,
 687 which provides a rather accurate solution even for the coarsest grid.

688 To compare the performance of the designed well-balanced 3-rd order scheme to that of a non
 689 well-balanced scheme, the solution has been computed using the 3-rd order non well-balanced scheme
 690 used in Section 5.4. The numerical solution at $t = 8$ weeks is presented in Figure 17 for a 200×120
 691 grid (left) and a 400×240 grid (right). It is evidenced that the non well-balanced scheme provides a
 692 poorer resolution of the moving eddy and would require a very fine mesh to capture the solution with
 693 a similar accuracy than the well-balanced scheme. Note that a level of refinement higher than those
 694 of Figure 16 has been used, as the solution for the 100×60 grid was very inaccurate.

695 In Figure 18, the trajectory of the center of the moving vortex computed by the 1-st and 3-rd order
 696 well-balanced schemes is plotted in the $x - y$ plane. The trajectories are computed using three different
 697 grids composed of 100×60 cells, 200×120 cells and 400×240 cells. It is observed that the 3-rd order
 698 scheme provides an accurate prediction of the trajectory of the westward moving eddy, even for the
 699 coarsest grid, while the 1-st order scheme has a much lower convergence rate and requires more than
 700 400×240 cells to predict the trajectory within an acceptable level of accuracy.

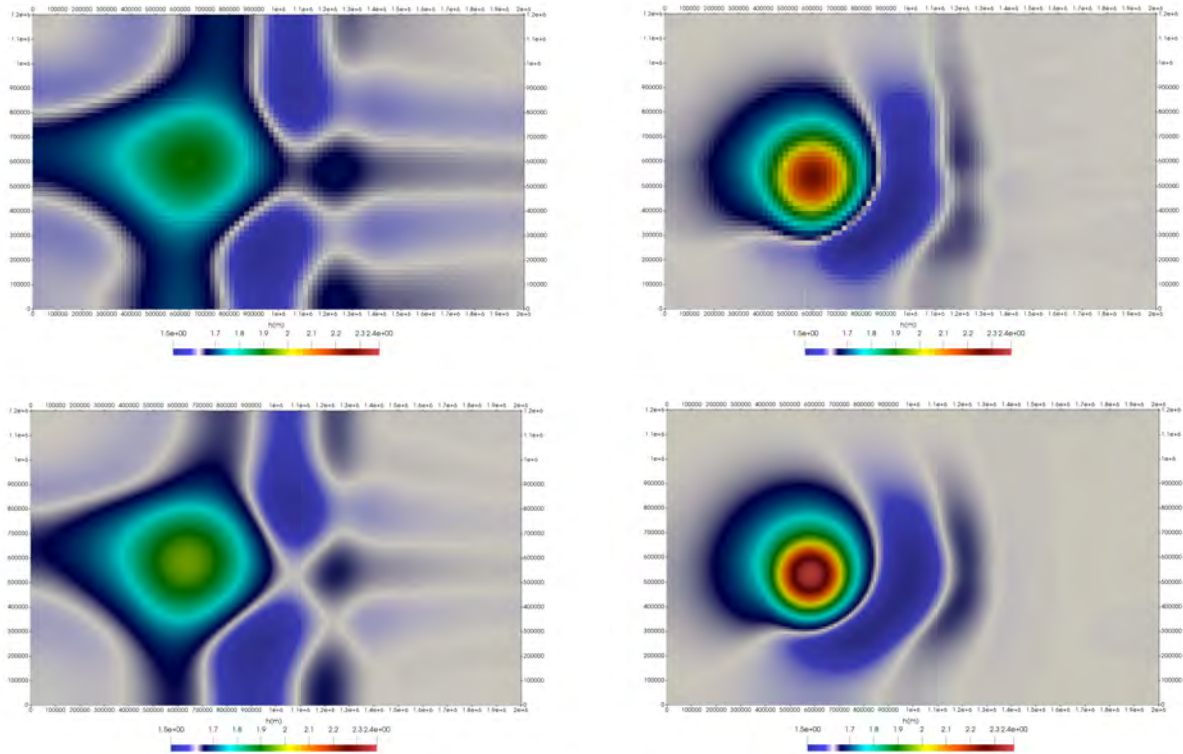


Figure 16: Section 5.9. Numerical solution for $h + z$ (m) computed by the 1-st (left) and 3-rd order scheme (right) using a 100×60 grid (top) and a 200×120 grid.

701 6. Summary and concluding remarks

702 In this work, an arbitrary order augmented WENO-ADER scheme for the resolution of the 2D

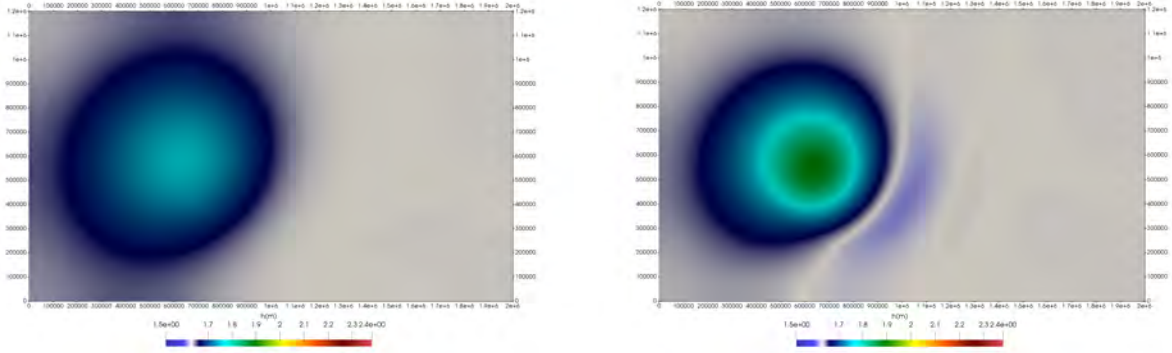


Figure 17: Section 5.9. Numerical solution for $h + z$ (m) computed by the non well-balanced 3-rd order scheme using a 200×120 grid (left) and a 400×240 grid (right).

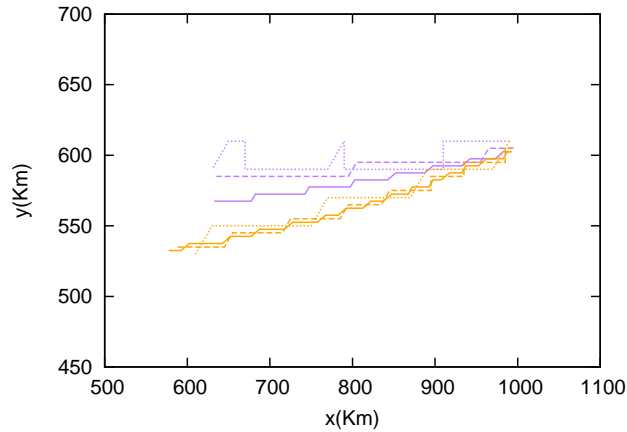


Figure 18: Section 5.9. Trajectory of the center of the moving vortex in the $x - y$ plane computed by the 1-st (purple) and 3-rd order scheme (orange) in a 100×60 cell grid (dotted line), 200×120 cell grid (dashed line) and 400×240 cell grid (solid line).

703 SWE with geometric source term has been presented. This scheme is the 2D extension of the ARL-
 704 ADER scheme in [29] which is based on the LFS Derivative Riemann solver. The solution of the
 705 DRP is computed using an Augmented solver, namely the ARoe solver. Compared to other tradi-
 706 tional homogeneous solvers, the ARoe solver enhances the performance of the numerical scheme as
 707 it accounts for the contribution of the source term at cell interfaces. This allows to provide more
 708 accurate numerical approximations when dealing with geometric source terms, as well as to ensure the
 709 exact balance between fluxes and sources without requiring any additional correction. The extension
 710 of the aforementioned methods to the resolution of problems non-geometric source terms has also been
 711 explored in this work. This is the case of the SWE in the rotating frame.

712 The main novelty presented in this work is the 2D extension of the ARL-ADER scheme by using
 713 a particular arbitrary order discretization of the source term inside cells that ensures the exact preser-
 714 vation of the equilibrium states of relevance. This is achieved by reducing the 2D problem to two 1D
 715 problems in the Cartesian directions. This allows to derive a 2D arbitrary order approximation of the
 716 integral of the source term by means of the combination of 1D arbitrary order integrals using Gaussian
 717 quadrature and Romberg integration. The keystone for the preservation of equilibrium with very high
 718 order is the use of the Romberg integration method, which allows to extend the well-balanced dis-
 719 cretization of the source term to arbitrary order. As a result, the proposed scheme is able to preserve
 720 the steady states of relevance while retaining a high order of accuracy for the resolution of transient

721 wave propagation.

722 The ARL-ADER scheme has been applied to the resolution of the SWE with bed variation and
723 rotation, for which the relevant equilibrium states are the quiescent equilibrium and the geostrophic
724 equilibrium. When considering rotation, the Coriolis source term has been reinterpreted as a geometric
725 source term, allowing to apply the well-balanced algorithms applied for general geometric source terms.
726 To ensure an exact equilibrium in the discrete level, special attention must be paid to the formulation
727 of the CK procedure for the reconstruction of time derivatives.

728 The 2D ARL-ADER scheme has been assessed in a broad variety of cases involving bed variation
729 and rotation. Such cases comprise smooth and discontinuous transient events, steady flows in complex
730 geometries and steady and transient problems in the rotating frame including realistic scenarios with
731 Rossby and Kelvin wave propagation. Most of the test cases propose the exercise of the scheme in
732 situations where source terms are dominant. The numerical results show that the proposed scheme
733 performs well in all cases, ensuring convergence to the reference solutions. The effectiveness of the well-
734 balanced property is made visible in most of the presented tests. The capabilities of the augmented
735 solver are evidenced in the test number 3, where the scheme is able to reproduce the full wave structure
736 of the solution of a 2D resonant RP. It is also remarkable to mention the good performance of the
737 proposed scheme for the resolution of the SWE in the rotating frame, compared to other non well-
738 balanced schemes. In test case number 9 a realistic scenario involving the propagation of an anticyclonic
739 eddy in the northern hemisphere is considered. Here, only when using the high order version of the
740 well-balanced scheme, the physical solution is reproduced.

741 The convergence rate of the solution has been experimentally assessed. It is observed that the
742 scheme achieves the expected accuracy when considering smooth cases with bed variation and Coriolis
743 forces. It has also been shown that when considering a stiff Coriolis source term, the convergence
744 rate may be reduced. The numerical results evidence that such loss of accuracy has to do with the
745 integration in time, which is not optimally done due to the CK procedure. This underscores the
746 importance of an optimal derivation of time derivatives and shows that when considering complex
747 source terms, such as the geometric reinterpretation of the Coriolis source term, the CK procedure
748 may become rather cumbersome and other techniques may be worth being used [21].

749 Concerning the overall computational expenses of the scheme, it is worth noting that the achieve-
750 ment of the well-balanced property while ensuring high order in space and time is done at the cost
751 of using complex procedures. Such procedures increase the computational cost when compared to a
752 typical non well-balanced method. The increased expense is mainly due to three aspects: the Romberg
753 integration, doubling the number of spatial WENO reconstructions and the calculation of the Coriolis
754 primitive variables. Though the latter does not involve a great computational cost, it enforces global
755 communication in a parallelized code, which increases the computational expense.

756 The numerical results evidence that high order schemes are not only recommended, due to the
757 improved efficiency of the methods, but sometimes necessary when the first order schemes are not able
758 to reproduce the physical solution. The simulation tool proposed in this work provides an appropriate
759 balance between computational efficiency and complexity of implementation. It is able to reproduce a
760 broad variety of flows dominated by source terms by using a unified strategy based on the consideration
761 of geometric source terms that allow to satisfy certain equilibrium conditions. The scheme uses a
762 Cartesian mesh, but no mesh-dependency of the solution is observed due to the high order of accuracy.
763 The scheme can be easily extended to other systems of equations with source terms of a broader variety.

764 **Acknowledgment**

765 This work has been funded by the Spanish Ministry of Science and Innovation under the research
766 project CGL2015-66114-R.

Decoration	Meaning
$(\cdot)_{i\pm 1/2,j}$	Relative to cell's right and left interfaces
$(\cdot)_{i,j\pm 1/2}$	Relative to cell's top and bottom interfaces
$(\cdot)_{i\pm 1/2,j,q}$	Relative to cell's right and left interfaces, at Gaussian point q
$(\cdot)_{i,j\pm 1/2,q}$	Relative to cell's top and bottom interfaces, at Gaussian point q
$(\cdot)_{i_r,j,q}$	Left ($r = W$) and right ($r = E$) limit to the cell's interface, at Gaussian point q
$(\cdot)_{i_s,j,q}$	Upper ($r = N$) and lower ($r = S$) limit to the cell's interface, at Gaussian point q
$(\cdot)_{r,q}$	Relative to the location at the interface $r = N, S, E, W$ and Gaussian point q
$(\cdot)_{ij}^n$	Cell average at time t^n
$(\cdot)_{j\pm 1/2}$	Relative to cell's top and bottom interfaces
$(\cdot)_{\pm}$	Relative to the numerical solution at left ($-$) and right ($+$) limits to the RP interface
$(\cdot)_{i+1/2}$	Arithmetic mean at $i + 1/2$, that is, using cell averages at i and $i + 1$
$(\cdot)_{i+1/2}$	Roe's average at $i + 1/2$, that is, using cell averages at i and $i + 1$
$\delta(\cdot)_{i+1/2}$	Discrete difference at $i + 1/2$, that is, $\delta(\cdot)_{i+1/2} = (\cdot)_{i+1} - (\cdot)_i$
$(\cdot)^{(k)}$	k -th time derivative
$\partial_{\phi}^{(k)}$	k -th derivative with respect to variable ϕ
$\{(\cdot)\}_m^n$	High order extension of quadrature rule (\cdot) using Romberg integration
(\cdot)	Evaluation of a variable at the initial time $t = 0$
$(\cdot)_{\partial\Omega}$	Evaluation of a variable at the boundary of the domain Ω

Table 7: Dictionary of decorations

Variable	Meaning
x, y	Spatial coordinates
Δx	Cell size
t	Time
Δt	Time step
\mathbf{U}	Vector of conserved quantities
\mathbf{E}	Matrix of physical fluxes
\mathbf{S}	Vector of source terms
\mathbf{F}, \mathbf{G}	Vectors of physical fluxes in x and y directions.
$\delta\mathbf{M}$	Flux fluctuation
\mathbf{A}, \mathbf{B}	Jacobian matrices in x and y directions.
\mathbf{J}	General notation for the Jacobian matrix
\mathcal{J}	Jacobian matrix composed of \mathbf{A} and \mathbf{B}
\mathbf{e}^m	m -th eigenvector of the Jacobian
\mathbf{P}	Matrix containing all eigenvectors \mathbf{e}^m in columns
$\mathbf{\Lambda}$	Diagonal matrix of the Jacobian
λ^m	m -th eigenvalue
α^m	m -th wave strength
β^m	m -th source strength
Ω	Spatial domain
N_x	Number of cells in the x direction
N_y	Number of cells in the y direction
N^λ	Number of equations/eigenvalues
\mathcal{F}	Normal flux to a cell interface
h	Water depth
u, v	Velocities in x and y directions
η	Water surface elevation $h + z$
z	Bed elevation
g	Gravity
f	Coriolis coefficient
V, U	Primitive variables of the x and x Coriolis forces
L, K	Equilibrium variables in x and y directions in presence of Coriolis forces
ω	Gaussian quadrature weight

Table 8: Dictionary of variables

767 **Appendix A. Resolution of the 1D DRP_K using the LFS solver**

768 The coefficients of the polynomial time series expansion of the numerical fluxes in (17) are computed
 769 by solving the DRP_K. The keystone for the resolution of the DRP_K is that it can be decomposed
 770 in $K + 1$ conventional RPs, one for the evolution of the conserved quantities and K more RPs for
 771 the evolution of their derivatives. The former, hereafter referred to as DRP₀, allows to compute the
 772 leading terms of the series, $\mathbf{F}_{i+1/2}^{-,(0)}$ and $\mathbf{F}_{i+1/2}^{+,(0)}$, whereas the latter allow to compute the higher order
 773 terms, $\mathbf{F}_{i+1/2}^{-,(k)}$ and $\mathbf{F}_{i+1/2}^{+,(k)}$. Note that the subscripts j and q have been omitted for the sake of clarity
 774 and will not be displayed throughout this section. It is worth recalling that the DRP_K includes the
 775 contribution of the source term at the origin and a suitable solver, namely the FS or LFS solver in [29],
 776 is required to compute the solution. The details for the decomposition of the DRP_K in conventional
 777 RPs using the LFS are presented below.

778 The DRP₀ corresponds to the following RP

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \\ \mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_{i_E}^{(0)} & \text{if } x < 0 \\ \mathbf{U}_{(i+1)_W}^{(0)} & \text{if } x > 0 \end{cases} \end{array} \right. \quad (\text{A.1})$$

779 with $\mathbf{U}_{i_E}^{(0)} = \lim_{x \rightarrow 0^-} \mathbf{U}_i(x)$ and $\mathbf{U}_{(i+1)_W}^{(0)} = \lim_{x \rightarrow 0^+} \mathbf{U}_{i+1}(x)$ the initial data for the conserved quanti-
 780 ties at cell interfaces provided by an adequate reconstruction procedure, such as the WENO method.

781 The K RPs associated to the high order terms of the DRP_K are composed of the linearized evolution
 782 equations for time derivatives of the conserved variables, leading to the following RP

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \left(\partial_t^{(k)} \mathbf{U} \right) + \tilde{\mathbf{J}}_{i+1/2} \frac{\partial}{\partial x} \left(\partial_t^{(k)} \mathbf{U} \right) = \partial_t^{(k)} \mathbf{S} \\ \partial_t^{(k)} \mathbf{U}(x, 0) = \begin{cases} \mathbf{D}_{i_E}^{(k)} & \text{if } x < 0 \\ \mathbf{D}_{(i+1)_W}^{(k)} & \text{if } x > 0 \end{cases} \end{array} \right. \quad (\text{A.2})$$

783 where $\tilde{\mathbf{J}}_{i+1/2} = \tilde{\mathbf{J}}_{i+1/2}(\mathbf{U}_{i_E}^{(0)}, \mathbf{U}_{(i+1)_W}^{(0)})$ is an approximation of the Jacobian matrix of the flux in the x
 784 direction, $\mathbf{A}(\mathbf{U})$, at each interface $x_{i+1/2}$, and satisfies

$$\delta \mathbf{F}_{i+1/2}^{(0)} = \tilde{\mathbf{J}}_{i+1/2} \delta \mathbf{U}_{i+1/2}^{(0)}, \quad (\text{A.3})$$

785 by using the Roe averages [30].

786 The Jacobian in (A.3) is used here to approximate time derivatives of the flux as

$$\partial_t^{(k)} \mathbf{F}(\mathbf{U}) = \tilde{\mathbf{J}}_{i+1/2} \partial_t^{(k)} \mathbf{U}. \quad (\text{A.4})$$

787 with $\partial_t^{(k)} \mathbf{U}$ the time derivatives of the conserved quantities, which will be denoted as $\mathbf{D}^{(k)}$.

788 The CK procedure is used to express time derivatives of the variables in terms of their spatial deriva-
 789 tives, which are computed using a suitable derivative reconstruction procedure [68]. This procedure is
 790 based on the recursive computation of time derivatives of \mathbf{U} using (1), as follows

$$\partial_t^{(k)} \mathbf{U} = \partial_t^{(k-1)} (-\partial_x \mathbf{F}(\mathbf{U}) - \partial_y \mathbf{G}(\mathbf{U}) + \mathbf{S}(\mathbf{U})). \quad (\text{A.5})$$

791 For the sake of simplicity, the CK procedure is carried out using absolute coordinates and is
 792 expressed only in terms of space derivatives of the conserved quantities, as \mathbf{F} , \mathbf{G} and \mathbf{S} depend upon
 793 \mathbf{U} . Time derivatives of the source term will be denoted by $\mathbf{Q}^{(k)} = \partial_t^{(k)} \mathbf{S}$.

794 Due to the presence of the source term in the DRP, an augmented solver must be used to solve
 795 (A.1) and (A.2). Following [29], the ARoe solver is chosen to compute $\mathbf{F}_{i+1/2}^{-,(0)}$, $\mathbf{F}_{i+1/2}^{-,(k)}$, $\mathbf{F}_{i+1/2}^{+,(0)}$ and
 796 $\mathbf{F}_{i+1/2}^{+,(k)}$. The details for this procedure are summarized below.

797 In what follows, $\delta(\cdot)_{i+1/2}$ operator will represent the difference between the right and left state
 798 of the DRP centered in $i + 1/2$ for a given variable, as $\delta(\cdot)_{i+1/2} = (\cdot)_{(i+1)W} - (\cdot)_{iE}$ and $\delta(\cdot)_{i-1/2} =$
 799 $(\cdot)_{iW} - (\cdot)_{(i-1)E}$.

800 Considering that (A.1) is hyperbolic, a set of approximated eigenvalues $\tilde{\lambda}_{i+1/2}^m$ and eigenvectors
 801 $\tilde{\mathbf{e}}_{i+1/2}^m = (e_1^m, \dots, e_{N_\lambda}^m)^T$ can be calculated. The approximate Jacobian $\tilde{\mathbf{J}}_{i+1/2}$ can be expressed as

$$\tilde{\mathbf{J}}_{i+1/2} = \tilde{\mathbf{P}}_{i+1/2} \tilde{\mathbf{\Lambda}}_{i+1/2} \tilde{\mathbf{P}}_{i+1/2}^{-1}, \quad (\text{A.6})$$

802 with $\tilde{\mathbf{P}}_{i+1/2} = (\tilde{\mathbf{e}}^1, \dots, \tilde{\mathbf{e}}^{N_\lambda})_{i+1/2}$ an invertible matrix composed by the eigenvectors of $\tilde{\mathbf{J}}_{i+1/2}$ and $\tilde{\mathbf{\Lambda}}_{i+1/2}$
 803 the diagonal matrix composed by the eigenvalues of $\tilde{\mathbf{J}}_{i+1/2}$.

804 When using the ARoe solver, the coefficients of (17) read

$$\begin{aligned} \mathbf{F}_{i+1/2}^{-,(k)} &= \mathbf{F}_{iE}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^- \alpha^{(k)} - \beta^{-(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K \\ \mathbf{F}_{i+1/2}^{+,(k)} &= \mathbf{F}_{(i+1)W}^{(k)} + \sum_{m=1}^{N_\lambda} \left(\tilde{\lambda}^+ \alpha^{(k)} - \beta^{+(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m, \quad k = 0, K \end{aligned} \quad (\text{A.7})$$

805 with

$$\left(\tilde{\lambda}^\pm \right)_{i+1/2}^m = \left(\frac{\tilde{\lambda}^\pm |\tilde{\lambda}|}{2} \right)_{i+1/2}^m, \quad \left(\beta^{\pm,(0)} \right)_{i+1/2}^m = \left(\frac{\tilde{\lambda}^\pm}{\tilde{\lambda}} \beta^{(0)} \right)_{i+1/2}^m, \quad (\text{A.8})$$

806 $\mathbf{A}_{i+1/2}^{(k)} = (\alpha^{(k),1}, \dots, \alpha^{(k),N_\lambda})^T$ the wave strengths and $\mathbf{B}_{i+1/2}^{(k)} = (\beta^{(k),1}, \dots, \beta^{(k),N_\lambda})^T$ the source
 807 strengths.

808 It is worth showing that the term $\sum_{m=1}^{N_\lambda} \left(\tilde{\lambda} \alpha^{(k)} - \beta^{(k)} \right)_{i+1/2}^m \tilde{\mathbf{e}}_{i+1/2}^m$ stands for the flux fluctuation

$$\delta \mathbf{M}_{i+1/2} = \delta \mathbf{F}_{i+1/2} - \bar{\mathbf{S}}_{i+1/2} \quad (\text{A.9})$$

809 The fluctuation in (A.9) can be rewritten as

$$\delta \mathbf{F}_{i+1/2} - \bar{\mathbf{S}}_{i+1/2} = \mathbf{P} (\mathbf{\Lambda} \mathbf{P}^{-1} \delta \mathbf{U} - \mathbf{P}^{-1} \bar{\mathbf{S}})_{i+1/2} \quad (\text{A.10})$$

810 which helps understanding how the source term is introduced in the formulation of the numerical
 811 fluxes. This is the basis of the augmented approach.

812 Physical fluxes as well as wave and source strengths are computed differently for (A.1) and (A.2).
 813 Following [29] they read

$$\mathbf{F}_{iE}^{(k)} = \begin{cases} \mathbf{F} \left(\mathbf{U}_{iE}^{(0)} \right) & \text{if } k = 0 \\ \tilde{\mathbf{J}}_{i+1/2} \mathbf{D}_{iE}^{(k)} & \text{if } k > 0 \end{cases}, \quad \mathbf{A}_{i+1/2}^{(k)} = \begin{cases} \tilde{\mathbf{P}}_{i+1/2}^{-1} \delta \mathbf{U}_{i+1/2}^{(0)} & \text{if } k = 0 \\ \tilde{\mathbf{P}}_{i+1/2}^{-1} \delta \mathbf{D}_{i+1/2}^{(k)} & \text{if } k > 0 \end{cases} \quad (\text{A.11})$$

814 and $\mathbf{B}_{i+1/2}^{(k)} = \tilde{\mathbf{P}}_{i+1/2}^{-1} \bar{\mathbf{S}}_{i+1/2}^{(k)}$. A suitable approximation of the integral of the source term across the
 815 interface

$$\bar{\mathbf{S}}_{i+1/2}^{(0)} \approx \int_{x_{i+1/2}^-}^{x_{i+1/2}^+} \mathbf{S}(x, y, 0) dx, \quad \bar{\mathbf{S}}_{i+1/2}^{(k)} \approx \int_{x_{i+1/2}^-}^{x_{i+1/2}^+} \mathbf{Q}^{(k)} dx, \quad (\text{A.12})$$

816 must be found in order to satisfy an exact equilibrium of fluxes and sources at the interface.

817 **Appendix B. Approximate solution using the linearized ARoe (ARL) solver**

818 The approximate Jacobian $\tilde{\mathbf{J}}_{\xi+1/2}(\mathbf{U}_{\xi_R}^{(0)}, \mathbf{U}_{(\xi+1)_L}^{(0)})$ for the DRP_K in (A.3) is defined at the interface
819 $x_{\xi+1/2}$, with $\xi = i, j$, using the left and right states of the reconstructed variables and reads

$$\tilde{\mathbf{J}}_{\xi+1/2} = \begin{pmatrix} 0 & n_x & n_y \\ \tilde{c}^2 n_x - \tilde{u}(\tilde{\mathbf{v}} \cdot \hat{\mathbf{n}}) & \tilde{\mathbf{v}} \cdot \hat{\mathbf{n}} + \tilde{u} n_x & \tilde{u} n_y \\ \tilde{c}^2 n_y - \tilde{v}(\tilde{\mathbf{v}} \cdot \hat{\mathbf{n}}) & \tilde{v} n_x & \tilde{\mathbf{v}} \cdot \hat{\mathbf{n}} + \tilde{v} n_y \end{pmatrix}, \quad (\text{B.1})$$

820 It is constructed using the Roe averages [30] and is diagonalized by the eigenvectors $\tilde{\mathbf{e}}_{\xi+1/2}^1, \tilde{\mathbf{e}}_{\xi+1/2}^2, \tilde{\mathbf{e}}_{\xi+1/2}^3$,
821 leading to the diagonal matrix $\tilde{\mathbf{\Lambda}}_{\xi+1/2} = \text{diag}(\tilde{\lambda}_{\xi+1/2}^1, \tilde{\lambda}_{\xi+1/2}^2, \tilde{\lambda}_{\xi+1/2}^3)$.

822 Regarding the source strengths for the leading term, it is worth recalling that only when considering
823 geometric source terms the source strengths are not nil. Otherwise, the source term is not included
824 in the definition of the DRP. When dealing with geometric source terms, we only consider non-zero
825 source components for the momentum equations as follows

$$\bar{\mathbf{S}}_{\xi+1/2}^{(0)} = \left(0, \bar{S}_{\xi+1/2}^{x,(0)}, \bar{S}_{\xi+1/2}^{y,(0)} \right)^T, \quad (\text{B.2})$$

826 with $\bar{\mathbf{S}}_{\xi+1/2}^{(0)}$ defined in (A.12). Then, we can define the following vector $\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \in \mathbb{R}^2$, where M stands
827 for *momentum*, as

$$\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} = \left(\bar{S}_{\xi+1/2}^{x,(0)}, \bar{S}_{\xi+1/2}^{y,(0)} \right)^T, \quad (\text{B.3})$$

828 and use it for the definition of the source strengths as

$$\beta_{\xi+1/2}^{(0),1} = -\frac{1}{2\tilde{c}} \bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}, \quad \beta_{\xi+1/2}^{(0),2} = -\frac{1}{\tilde{c}} \bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}_{\perp}, \quad \beta_{\xi+1/2}^{(0),3} = -\beta_{\xi+1/2}^{(0),1} \quad (\text{B.4})$$

829 where $\hat{\mathbf{n}}_{\perp} = \mathbf{R}_{\pi/2} \hat{\mathbf{n}}$ is the unitary vector parallel to the cell interface and $\mathbf{R}_{\pi/2}$ a $\pi/2$ rad rotation
830 matrix. The expression of the source strengths for the higher order terms can be derived analogously.

831 If considering a geometric source term, the projection $\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}$ must be an approximation of

$$\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{\check{x}_{i+1/2}^-}^{\check{x}_{i+1/2}^+} S_s(\mathbf{U}) \nabla \phi \cdot \hat{\mathbf{n}} d\check{x}, \quad (\text{B.5})$$

832 and using the relations $dx = d\check{x}n_x$ and $dy = d\check{x}n_y$, it can be approached by

$$\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}} = S_s(\bar{\mathbf{U}}_{\xi+1/2}^{(0)}) \int_{\mathbf{x}_{i+1/2}^-}^{\mathbf{x}_{i+1/2}^+} \nabla \phi d\mathbf{x} = S_s(\bar{\mathbf{U}}_{v+1/2}^{(0)}) \delta(\phi)_{\xi+1/2}^{(0)}, \quad (\text{B.6})$$

833 On the other hand, the projection $\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}_{\perp}$ must be an approximation of

$$\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}_{\perp} \approx \frac{1}{\Delta t} \int_0^{\Delta t} \int_{\check{x}_{\xi+1/2}^-}^{\check{x}_{\xi+1/2}^+} S_s(\mathbf{U}) \nabla \phi \cdot \hat{\mathbf{n}}_{\perp} d\check{x}, \quad (\text{B.7})$$

834 where $\nabla \phi \cdot \hat{\mathbf{n}}_{\perp}$ is the directional derivative of ϕ in the direction parallel to the cell interface. According
835 to the definition of the DRP_K , we only consider variations of the variables in the normal direction to
836 the cell interface, hence $\bar{\mathbf{S}}_{\xi+1/2}^{M,(0)} \cdot \hat{\mathbf{n}}_{\perp} = 0$ and $\beta_{\xi+1/2}^{(0),2} = 0$.

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