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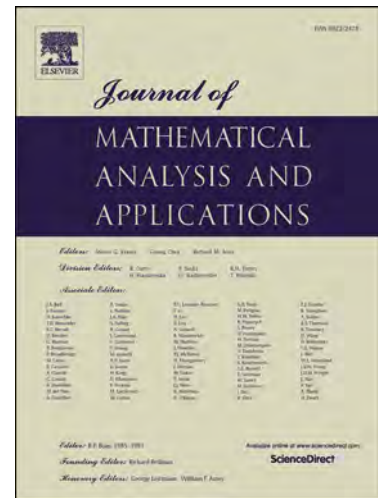
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NON-LOCAL FRACTIONAL DERIVATIVES. DISCRETE AND CONTINUOUS

LUCIANO ABADIAS, MARTA DE LEÓN, AND JOSÉ L. TORREA

ABSTRACT. We prove maximum and comparison principles for the discrete fractional derivatives in the integers. Regularity results when the space is a mesh of length h, and approximation theorems to the continuous fractional derivatives are shown. When the functions are good enough (Hölder continuous), these approximation procedures give a measure of the order of approximation. These results also allow us to prove the coincidence, for Hölder continuous functions, of the Marchaud and Grünwald-Letnikov derivatives in every point and the speed of convergence to the Grünwald-Letnikov derivative. The discrete fractional derivative will be also described as a Neumann-Dirichlet operator defined by a semi-discrete extension problem. Some operators related to the Harmonic Analysis associated to the discrete derivative will be also considered, in particular their behavior in the Lebesgue spaces $\ell^p(\mathbb{Z})$.

1. INTRODUCTION

Fractional derivatives on time have been used to propose nonlocal models to describe non-diffusive transport in magnetically confined plasmas, see [10]. They also appear in the study of parabolic problems in which it is natural to take into account the past, see[2]. Some porous medium flow with fractional time derivative have been considered recently, see [3]. In these cases, regarding the fractional derivative on time, several discretization techniques play a crucial role. In the literature, we can find different representations of the classical definition of the Riemann-Liouville fractional derivative as Caputo, Marchaud or Grünwald-Letnikov, see [18]. In general they do not coincide, for example they have different behavior respect to constant functions.

In this paper we study discrete fractional derivatives. We shall prove maximum and comparison principles as well as regularity results. Also approximation theorems to the continuous fractional derivatives will be shown. When the functions are (Hölder continuous), these approximation procedures give a measure of the order of approximation. These results also allows us to prove the coincidence, for (Hölder continuous), functions, of the Marchaud and Grünwald-Letnikov derivatives in every point and the speed of convergence to the Grünwald-Letnikov derivative. The discrete fractional derivatives will be also described as a Neumann-Dirichlet operator defined by a semi-discrete extension problem. Some operators related to the Harmonic Analysis associated to the discrete derivative will be also considered, in particular their behavior in the Lebesgue spaces $\ell^p(\mathbb{Z})$.

As far as we know S. Chapman in 1911, see [8], was the first author who consider "differences of fractional order". See also the paper of 1956 by B. Kuttner, [12]. For $s > 0$, given a sequence a_n they define

$$(1.1) \qquad \triangle^s a_n = \sum_{m=0}^{\infty} \binom{-s-1+m}{m} a_{n+m}.$$

Their motivation was to extend the obvious formula for differences of natural order. Also, by using Fourier transform in the integers it can be checked that the above definition produces $\widehat{(\triangle^s a_n)}(\theta) = (1 - e^{i\theta})^s \hat{a}_n(\theta)$ in coherence with the fact $\widehat{(a_n - a_{n+1})}(\theta) = (1 - e^{i\theta})\hat{a}_n(\theta)$.

Observe that Chapman's definition only cares about the future, but we shall also consider the discrete derivatives which cares about the past. For $f : \mathbb{Z} \to \mathbb{R}$, we define *"the discrete derivative from the right"* and *"the discrete derivative from the left"* as the operators given by the formulas

$$(1.2) \qquad \delta_{\text{right}} f(n) = f(n) - f(n+1) \ \text{ and } \ \delta_{\text{left}} f(n) = f(n) - f(n-1).$$

Given the function $G_t(n) = e^{-t}\frac{t^n}{n!}$, $n \in \mathbb{N}_0$, we define

$$(1.3) \qquad u(n,t) = \sum_{j=0}^{\infty} G_t(j) f(n+j).$$

It can be seen, see Section 2, that u satisfies the following semi-discrete transport equation

$$(1.4) \qquad \begin{cases} \partial_t u(n,t) + \delta_{\text{right}} u(n,t) = 0, & n \in \mathbb{Z}, \ t \geq 0, \\ u(n,0) = f(n), & n \in \mathbb{Z}. \end{cases}$$

The function $v(n,t) = \displaystyle\sum_{j=0}^{\infty} G_t(j) f(n-j)$ satisfies the analogous equation for δ_{left}. These equations will drive us to define the following nonlocal fractional operators

$$(\delta_{\text{right}})^\alpha f(n) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{u(n,t) - f(n)}{t^{1+\alpha}} dt, \ 0 < \alpha < 1,$$

and

$$(\delta_{\text{right}})^{-\alpha} f(n) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{u(n,t)}{t^{1-\alpha}} dt, \ 0 < \alpha < 1,$$

and the corresponding formula for $(\delta_{\text{left}})^\alpha$, $-1 < \alpha < 1$. We shall prove that this definition of $(\delta_{\text{right}})^\alpha$ coincides with the definition (1.1) given by Chapman in 1911, see Section 2.

Before the concrete presentation of our results, we observe that the definitions above can be given for a mesh with step length $h > 0$ instead of the integers mesh with step length 1. In other words, we can work in the field $\mathbb{Z}_h = \{jh : j \in \mathbb{Z}\}$. In this way we define

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h}, \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z},$$

and the associated $G_{t/h}(j)$

Now we state the main results of this paper.

Theorem 1.1. *Let $0 < \alpha < 1$ and $1 \leq p \leq \infty$.*

(i) *Let $u \in \ell^p(\mathbb{Z}_h)$ such that $u(j_0 h) = 0$ for some $j_0 \in \mathbb{Z}$, and $u(jh) \geq 0$ for all $j_0 \leq j$. Then $(\delta_{\text{right}})^\alpha u(j_0 h) \leq 0$. Moreover, $(\delta_{\text{right}})^\alpha u(j_0 h) = 0$ if and only if $u(jh) = 0$ for all $j_0 \leq j$.*

(ii) *Let $u, v \in \ell^p(\mathbb{Z}_h)$ such that $u(j_0 h) = v(j_0 h)$ for some $j_0 \in \mathbb{Z}$, and $u(jh) \geq v(jh)$ for all $j_0 \leq j$. Then $(\delta_{\text{right}})^\alpha u(j_0 h) \leq (\delta_{\text{right}})^\alpha v(j_0 h)$. Moreover, $(\delta_{\text{right}})^\alpha u(j_0 h) = (\delta_{\text{right}})^\alpha v(j_0 h)$ if and only if $u(jh) = v(jh)$ for all $j_0 \leq j$.*

The following result is a consequence of the above theorem.

Theorem 1.2. *Let $j_0 < j_1 \in \mathbb{Z}$ and $u, v \in \ell^p(\mathbb{Z}_h)$ with $1 \leq p \leq \infty$.*

(i) *Let u be a solution of*

$$\begin{cases} (\delta_{\text{right}})^\alpha u = f, & in \ [j_0, j_1) \\ \\ u = 0, & in \ [j_1, \infty). \end{cases}$$

If $f \geq 0$ in $[j_0, j_1)$ then $u \geq 0$ in $[j_0, \infty)$.

(ii) *If $(\delta_{\text{right}})^\alpha u \leq 0$ in $[j_0, j_1)$ and $u \leq 0$ in $[j_1, \infty)$, then*

$$\sup_{j \geq j_0} u(jh) = \sup_{j \geq j_1} u(jh).$$

(iii) *If $(\delta_{\text{right}})^\alpha u \geq 0$ in $[j_0, j_1)$ and $u \geq 0$ in $[j_1, \infty)$, then*

$$\inf_{j \geq j_0} u(jh) = \inf_{j \geq j_1} u(jh).$$

(iv) *If*

$$\begin{cases} (\delta_{\text{right}})^\alpha u \geq (\delta_{\text{right}})^\alpha v, & in \ [j_0, j_1) \\ \\ u \geq v, & in \ [j_1, \infty), \end{cases}$$

then $u \geq v$ in $[j_0, \infty)$. In particular, we have uniqueness of the Dirichlet problem

$$\begin{cases} (\delta_{\text{right}})^\alpha u = f, & in \ [j_0, j_1) \\ \\ u = g, & in \ [j_1, \infty). \end{cases}$$

Also we shall get regularity results for the discrete Hölder classes $C_h^{k,\beta}$, see Theorem 3.2. These results drive us to compare the discrete fractional derivatives with the discretization of the continuous fractional derivatives as defined in [2, 3, 5]. Given a function u defined on $\mathbb{R}$, we consider its restriction $r_h u$ (or discretization) to $\mathbb{Z}_h$, that is, $r_h u(j) = u(hj)$ for $j \in \mathbb{Z}$. We have the following theorems.

Theorem 1.3. *Let $0 < \beta \leq 1$ and $0 < \alpha < 1$.*

(i) *Let $u \in C^{0,\beta}(\mathbb{R})$ and $\alpha < \beta$. Then*

$$\|(\delta_{\text{right}})^\alpha(r_h u) - r_h((D_{right})^\alpha u)\|_{\ell^\infty} \leq C_\alpha [u]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

(ii) *Let $u \in C^{1,\beta}(\mathbb{R})$ and $\alpha < \beta$. Then*

$$\|-\delta_{\text{right}}(\delta_{\text{right}})^\alpha(r_h u) - r_h(\frac{d}{dx}(D_{right})^\alpha u)\|_{\ell^\infty} \leq C_\alpha [u]_{C^{1,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

Here, the operators $(D_{right/left})^\alpha$ are the Marchaud derivatives, see [18], that is,

$$(1.5) \qquad (D_{right/left})^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x \pm t) - f(x)}{t^{1+\alpha}} dt.$$

The classes $C^{k,\beta}(\mathbb{R})$, $k \in \mathbb{N}_0$, $\beta > 0$, are the usual Hölder classes on the real line (see Section 3).

There are analogous results when we substitute δ_{left} by δ_{right} and D_{left} by D_{right} respectively.

Consider the *fractional differences of order α*, with $\alpha > 0$

$$\Delta_{h,\pm}^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x \pm kh) = \sum_{k=0}^{\infty} \Lambda^{\alpha}(k) f(x \pm kh), \ x \in \mathbb{R}, h > 0.$$

The *Grünwald-Letnikov derivatives* of a function f are defined by

$$(1.6) \qquad\qquad f_{\pm}^{\alpha}(x) = \lim_{h \to +0} \frac{\Delta_{h,\pm}^{\alpha} f(x)}{h^{\alpha}},$$

see [18, pages 371–373].

The coincidence of the Marchaud and the Grünwald-Letnikov derivatives is known in almost everywhere sense or in $L^p(\mathbb{R}), 1 \leq p < \infty$, for $f \in L^r(\mathbb{R})$, with r and p independent, see [18, Theorems 20.2 ,20.4]. As a consequence of our Theorem 1.3 we shall prove that, for (Hölder continuous) functions, both derivatives coincide pointwise. Moreover we get the speed of convergence of the limit in (1.6), which is of order $h^{\beta-\alpha}$.

Theorem 1.4. *Let $0 < \alpha < \beta \leq 1$ and $f \in C^{0,\beta}(\mathbb{R})$. Then $f_{\pm}^{\alpha}(x) = (D_{right/left})^{\alpha} f(x)$ for every point $x \in \mathbb{R}$. Moreover, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\left| (D_{right/left})^{\alpha} f(x) - \frac{\Delta_{h,\pm}^{\alpha} f(x)}{h^{\alpha}} \right| \leq C_{\alpha,\beta} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}, \quad x \in \mathbb{R}.$$

The operators $(\delta_{\text{right/left}})^{\alpha}$ are non-local, but they can be understood as limits (when $t \to 0$) of a local extension problem on $\mathbb{R}_+ \times \mathbb{Z}$. In fact, by following the ideas of [7, 20, 11] we get the next result.

Theorem 1.5. *Let $f \in \ell^p(\mathbb{Z})$ and $0 < \gamma < 1$. Consider the equation*

$$(1.7) \qquad\qquad \partial_{zz}^2 U(z, \cdot) + \frac{1 - 2\gamma}{z} \partial_z U(z, \cdot) - \delta_{\text{right}} U(z, \cdot) = 0, \quad z \in S_{\pi/4},$$

where $S_{\pi/4} = \{ z \in \mathbb{C} \mid z \neq 0 \text{ and } |\arg z| < \pi/4 \}$. The formula

$$(1.8) \qquad U(z, \cdot) = \frac{z^{2\gamma}}{4^{\gamma} \Gamma(\gamma)} \int_0^{\infty} e^{-\frac{z^2}{4t}} u(\cdot, t) \frac{dt}{t^{1+\gamma}} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} e^{-\frac{z^2}{4t}} v(\cdot, t) \frac{dt}{t^{1-\gamma}}$$

solves (1.7) on $\ell^p(\mathbb{Z})$, where $u(\cdot, t)$ is defined in (1.3) and $v(\cdot, t)$ satisfies (1.4) with initial data $(\delta_{\text{right}})^{\gamma} f(\cdot)$. Moreover,

$$\lim_{z \to 0} U(z, \cdot) = f(\cdot) \quad and \quad \frac{1}{2\gamma} \lim_{z \to 0} z^{1-2\gamma} \partial_z U(z, \cdot) = \frac{\Gamma(-\gamma)}{4^{\gamma} \Gamma(\gamma)} (\delta_{\text{right}})^{\gamma} f(\cdot),$$

where both limits hold through proper subsectors of $S_{\pi/4}$ in the $\ell^p(\mathbb{Z})$ sense.
A parallel result can be stated for δ_{left}.

Remark 1.6. Extension problem for negative powers. *It is clear that if a function g is good enough, $(\delta_{\text{right}})^{-\gamma} g \in \ell^p(\mathbb{Z})$, and in (1.8) we substitute f by $(\delta_{\text{right}})^{-\gamma} g$ and $(\delta_{\text{right}})^{\gamma} f$ by g, U solves the same equation with the initial Dirichlet condition $(\delta_{\text{right}})^{-\gamma} g$ and the Neumann condition $\frac{\Gamma(-\gamma)}{4^{\gamma} \Gamma(\gamma)} g$. See [6, 11, 20].*

Remark 1.7. *We will also see that the formula (1.8) gives an explicit solution of (1.7) in the case $z \in (0, \infty)$, see Section 5. In fact the formula provides an expression of the generalized Poisson function associated to δ_{right}, see Section 6.*

Several versions of this theorem have been studied in [5].

The paper also contains results in the field of Harmonic Analysis related to the operators $\delta_{\text{right/left}}$. In particular, we analyze maximal operators and Littlewood-Paley square functions associated to the heat semigroups and generalized Poisson functions naturally linked to $\delta_{\text{right/left}}$. In the case of the heat semigroup both, the maximal function and the square function, in general are not bounded on $\ell^p(\mathbb{Z})$, see Section 6. However the maximal operator and the square function associated to the generalized Poisson functions have suitable boundedness properties in $\ell^p(\mathbb{Z})$.

We have the following result.

Theorem 1.8. *Let $1 < p < \infty$. Let S be either the maximal function or the square Littlewood-Paley function associated to the generalized Poisson functions defined in (5.1). Then S is bounded from $\ell^p(\mathbb{Z})$ into itself and from $\ell^1(\mathbb{Z})$ into weak-$\ell^1(\mathbb{Z})$ (For the maximal function p can be ∞).*

For the proof of this Theorem we shall use vector-valued Calderón-Zygmund theory. In short, we could say that the Calderón-Zygmund Theory relies in the proof of two facts. First the boundedness of the operator in a concrete Lebesgue space $\ell^p(\mathbb{Z})$ for some value of $p, 1 < p \leq \infty$. Secondly an estimate about the smoothness of the kernel. In our case both items are nontrivial. The key is the appropriate use of boundedness of the MacDonald function, with complex and real arguments, involving Cauchy integration. We think that these proofs will be of independent interest for the reader.

Throughout the paper, we use the variable constant convention, in which C denotes a constant which may not be the same from line to line. The constant is frequently written with subindexes to emphasize that it depends on some parameters.

2. The discrete fractional derivatives via a semigroup language

We shall use semigroup language as an alternative approach to these differences of fractional order. Given the function $G_t(n) = e^{-t}\frac{t^n}{n!}$, $n \in \mathbb{N}_0$, we define the operators

$$(2.1) \qquad T_{t,+}f(n) = \sum_{j=0}^{\infty} G_t(j)f(n+j), \text{ and } T_{t,-}f(n) = \sum_{j=0}^{\infty} G_t(j)f(n-j), \qquad t > 0, \, n \in \mathbb{Z}.$$

In this section we mainly prove that $T_{t,\pm}f(n)$ are markovian semigroups on $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, whose infinitesimal generators are $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$ and that the function $u(n,t) = T_{t,+}f(n)$ satisfies the Cauchy problem (1.4).

Lemma 2.1. *Let $f \in \ell^{\infty}(\mathbb{Z})$ and $\{T_{t,\pm}\}_{t \geq 0}$ be the families defined in (2.1), then*

$$\lim_{t \to 0} \frac{T_{t,+}f(n) - f(n)}{t} = -\delta_{\text{right}}f(n) \text{ and } \lim_{t \to 0} \frac{T_{t,-}f(n) - f(n)}{t} = -\delta_{\text{left}}f(n), \;\; n \in \mathbb{Z}.$$

Proof. Let $f \in \ell^{\infty}(\mathbb{Z})$ and $n \in \mathbb{Z}$. Observe that

$$\frac{T_{t,+}f(n) - f(n)}{t} = \frac{1}{t}e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}\left(f(n+j) - f(n)\right) = e^{-t}\sum_{j=1}^{\infty}\frac{t^{j-1}}{j!}\left(f(n+j) - f(n)\right)$$

$$= e^{-t}(f(n+1) - f(n)) + e^{-t}\sum_{j=2}^{\infty}\frac{t^{j-1}}{j!}\left(f(n+j) - f(n)\right).$$

Dominated Convergence gives the first identity in the statement. By a similar argument,

$$\lim_{t \to 0} \frac{T_{t,-}f(n) - f(n)}{t} = \lim_{t \to 0} e^{-t}(f(n-1) - f(n)) + \lim_{t \to 0} e^{-t} \sum_{j=2}^{\infty} \frac{t^{j-1}}{j!} \left(f(n-j) - f(n) \right) = -\delta_{\text{left}} f(n).$$

$\square$

The next Proposition shows that although the semigroups are not self adjoint, they satisfy the rest of the properties of the so called symmetric diffusion semigroups in the sense of E. M. Stein, see [19].

Proposition 2.2. *Let* $f \in \ell^p(\mathbb{Z})$ *with* $1 \le p \le \infty$. *The families of operators* $\{T_{t,\pm}\}_{t \ge 0}$ *satisfy*

(i) $\|T_{t,\pm}f\|_{\ell^p} \le \|f\|_{\ell^p}$ *and* $T_{0,\pm}f = f$.
(ii) $T_{t,\pm}T_{s,\pm}f = T_{t+s,\pm}f$.
(iii) $\lim_{t \to 0} T_{t,\pm}f = f$ *on* $\ell^p(\mathbb{Z})$.
(iv) $T_{t,\pm}f \ge 0$ *if* $f \ge 0$.
(v) $T_{t,\pm}1 = 1$.
(vi) $T_{t,\pm}^* = T_{t,\mp}$ *on* $\ell^2(\mathbb{Z})$.

Proof. $T_{0,\pm}f = f$ by definition. We prove the rest of the results for $T_{t,+}$ (the proof is analogous for $T_{t,-}$). Let $f \in \ell^p(\mathbb{Z})$ for $1 \le p < \infty$. By Minkowski's inequality

$$\|T_{t,+}f\|_{\ell^p} \le e^{-t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\sum_{n \in \mathbb{Z}} |f(n+j)|^p \right)^{\frac{1}{p}} = \|f\|_{\ell^p}.$$

For $p = \infty$, is analogous. In order to prove (ii) we use the Newton's binomial,

$$
\begin{aligned}
T_{t,+}T_{s,+}f(n) &= e^{-(t+s)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{h=0}^{\infty} \frac{s^h}{h!} f(n+j+h) = e^{-(t+s)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{u=j}^{\infty} \frac{s^{u-j}}{(u-j)!} f(n+u) \\
&= e^{-(t+s)} \sum_{u=0}^{\infty} \frac{f(n+u)}{u!} \sum_{j=0}^{u} \binom{u}{j} t^j s^{u-j} = T_{t+s,+}f(n).
\end{aligned}
$$

For (iii) we use that $f(n+j) - f(n) = 0$ for $j = 0$, and Minkowski's inequality to get

$$
\begin{aligned}
\|T_{t,\pm}f - f\|_{\ell^p} &= \left(\sum_{n \in \mathbb{Z}} \left| e^{-t} \sum_{j=1}^{\infty} \frac{t^j}{j!} (f(n+j) - f(n)) \right|^p \right)^{\frac{1}{p}} \\
&\le e^{-t} \sum_{j=1}^{\infty} \left(\sum_{n \in \mathbb{Z}} \left| \frac{t^j}{j!} (f(n+j) - f(n)) \right|^p \right)^{\frac{1}{p}} \\
&\le 2e^{-t} \|f\|_{\ell^p} \sum_{j=1}^{\infty} \frac{t^j}{j!} \to 0
\end{aligned}
$$

as $t \to 0$. For $p = \infty$ is similar. We leave the verification of (iv), (v) and (vi) to the reader. $\square$

Remark 2.3. *By Lemma 2.1 and Proposition 2.2 (i)-(iii), observe that the one-parameter operator families* $\{T_{t,\pm}\}_{t \ge 0}$ *are uniformly bounded* C_0-*semigroups on* $\ell^p(\mathbb{Z})$ *for* $1 \le p \le \infty$, *generated by* $-\delta_{\text{right/left}}$, *in the sense of the operator theory, see [4]. Furthermore, it is easy to see that the domain of* $\delta_{\text{right/left}}$ *on* $\ell^p(\mathbb{Z})$ *is the whole space.*

Now we shall see that the function $u(n,t) = T_{t,+}f(n) = \sum_{j \geq 0} G_t(j)f(n+j)$ is a solution of (1.4). Observe that

$$\frac{\partial}{\partial t}G_t(0) = -G_t(0), \quad \text{and} \quad \frac{\partial}{\partial t}(G_t(j)) = -G_t(j) + G_t(j-1), \quad j \geq 1.$$

On the other hand, for any $A > 0$, $\sum_j \frac{A^j}{j!} \leq C_A < \infty$, hence we can differentiate term by term the series and we have

$$\partial_t T_{t,+}f(n) = -G_t(0)f(n) + \sum_{j=1}^{\infty}(-G_t(j) + G_t(j-1))f(n+j)$$

$$= -\sum_{j \geq 0} G_t(j)f(n+j) + \sum_{j \geq 0} G_t(j)f(n+(j+1)) = -\delta_{\text{right}}T_t f(n).$$

The proof of the result for δ_{left} is analogous.

Once we have enclosed the fractional differences into the frame of semigroups, we take advantage of the method to highlight some properties and interesting results of these operators. We recall to the reader the following Gamma function formulas for an operator L.

$$(2.2) \qquad L^\alpha = \frac{1}{\Gamma(-\alpha)}\int_0^\infty (e^{-tL} - 1)\frac{dt}{t^{1+\alpha}} \quad \text{and} \quad L^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-tL}\frac{dt}{t^{1-\alpha}},$$

where e^{-tL} is the associated semigroup, see [5, 19, 20, 21]. In particular, for $f \in \ell^p, 1 \leq p \leq \infty$, we define

$$(\delta_{\text{right}})^\alpha f(n) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \frac{T_{t,+}f(n) - f(n)}{t^{1+\alpha}}dt, \ 0 < \alpha < 1,$$

and

$$(\delta_{\text{right}})^{-\alpha} f(n) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{T_{t,+}f(n)}{t^{1-\alpha}}dt, \ 0 < \alpha < 1,$$

and the corresponding formula for $(\delta_{\text{left}})^\alpha$, $-1 < \alpha < 1$. We shall prove that this definition of $(\delta_{\text{right}})^\alpha$ coincides with the definition (1.1) given by Chapman in 1911.

Along this paper, for $\alpha \in \mathbb{R}$, we denote $\Lambda^\alpha(m) = \binom{-\alpha-1+m}{m} = (-1)^m\binom{\alpha}{m}$, $m \in \mathbb{N}_0$. The sequences $\{\Lambda^\alpha(n)\}_{n \in \mathbb{N}_0}$ have been studied in a more general setting in [1, 22]. Here we highlight some properties of this kernel. For $0 < \alpha < 1$, we have $\sum_{n=0}^{\infty}\Lambda^\alpha(n) = 0$, $\sum_{n=1}^{\infty}\Lambda^\alpha(n) = -1$. In addition, we can observe that $\Lambda^\alpha(n) < 0$ for $n \in \mathbb{N}$, and

$$(2.3) \qquad \Lambda^\alpha(n) = \frac{1}{n^{\alpha+1}\Gamma(-\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right),$$

see [22, Vol.I, p.77, (1.18)].

Then

$$(\delta_{\text{right}})^\alpha f(n) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \frac{e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}(f(n+j) - f(n))}{t^{1+\alpha}}dt = \sum_{j=0}^{\infty}(f(n+j) - f(n))\int_0^\infty \frac{e^{-t}t^{j-\alpha}}{j!\Gamma(-\alpha)}\frac{dt}{t}$$

$$= \sum_{j=0}^{\infty}(f(n+j) - f(n))\frac{\Gamma(-\alpha+j)}{\Gamma(-\alpha)j!} = \sum_{j=0}^{\infty}\Lambda^\alpha(j)f(n+j),$$

where the interchange of the sum and the integral is justified because of the integral converges absolutely. In the last equality we have used that $\sum_{j=0}^{\infty} \Lambda^{\alpha}(j) = 0$. By a similar way we also get

$$(\delta_{\text{left}})^{\alpha} = \sum_{j=0}^{\infty} \Lambda^{\alpha}(j) f(n-j).$$

Remark 2.4. *The above expression of $(\delta_{\text{right/left}})^{\alpha}$ coincides with the formula of the fractional powers of $\delta_{\text{right/left}}$ as generators of uniformly bounded C_0-semigroups on $\ell^p(\mathbb{Z})$, in the sense of Balakrishnan, see* [21, Chapter IX, Section 11].

Remark 2.5. *(Probabilistic interpretation). Let u be a function defined on $\mathbb{Z}_h$ such that its progressive difference is zero, which is equivalent to write*

$$u(jh) = u((j+1)h), \quad j \in \mathbb{Z}.$$

This implies that the discrete function u is constant. We can interpret the above identity as the movement of a particle that compulsorily jumps to the adjacent right point on the mesh. If now we suppose that $(\delta_{\text{right}})^{\alpha} u = 0$, then

$$u(jh) = -\sum_{n=1}^{\infty} \Lambda^{\alpha}(n) u((j+n)h).$$

Since $-\sum_{n=1}^{\infty} \Lambda^{\alpha}(n) = 1$, the fractional identity can describe the movement of a particle which is able to jump to the right points $j+n$ with probability $-\Lambda^{\alpha}(n)$. It is easy to see that we recover the first situation as $\alpha \to 1^-$. If $\alpha \to 0^+$, the particle tends to be still.

3. Properties from a PDE point of view

In this section we study the discrete fractional derivatives, and their regularity on the discrete Hölder spaces.

In the previous section we have considered functions defined on $\mathbb{Z}$. Now we work on $\mathbb{Z}_h = h\mathbb{Z}$, for $h > 0$, since one of our main objectives will be to compare the discrete fractional derivatives on $\mathbb{Z}_h$ with the discretized continuous fractional derivatives as $h \to 0$. This will be done in the next section. Let $u : \mathbb{Z}_h \to \mathbb{R}$, the first order difference operators on $\mathbb{Z}_h$ are given by

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h}, \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z}.$$

Notice that $\{T_{\frac{t}{h}, \pm}\}_{t \geq 0}$ are the contraction semigroups on $\ell^p(\mathbb{Z}_h)$, $1 \leq p \leq \infty$, generated by $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$. Then, by the results of the last sections we can write

$$(\delta_{\text{right}})^{\alpha} u(nh) = \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (u((n+j)h) - u(nh)) \Lambda^{\alpha}(j).$$

and

$$(\delta_{\text{left}})^{\alpha} u(nh) = \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (u((n-j)h) - u(nh)) \Lambda^{\alpha}(j).$$

Now we shall prove the maximum and comparison principles for the fractional differences $(\delta_{\text{right}})^{\alpha}$ and the uniqueness of the corresponding Dirichlet problems stated in the Introduction as Theorem 1.1 and Theorem 1.2. Observe that the statements and proofs for δ_{left} are analogous.

Proof of Theorem 1.1: Part (ii) is a straightforward consequence of (i). For (i) we write
$(\delta_{\text{right}})^\alpha u(j_0 h) = \frac{1}{h^\alpha} \sum_{m \in \mathbb{N}_0} \Lambda^\alpha(m) u((j_0+m)h) = \frac{1}{h^\alpha} \sum_{m \in \mathbb{N}} \Lambda^\alpha(m) u((j_0+m)h) \leq 0$, since $\Lambda^\alpha(m) < 0$
for all $m \in \mathbb{N}$. Moreover, by the same argument, if $(\delta_{\text{right}})^\alpha u(j_0 h) = 0$ then $u((j_0 + m)h) = 0$ for
all $m \in \mathbb{N}$. □

Proof of Theorem 1.2: We prove part (i) by contradiction. We suppose that there exists
$m \in [j_0, j_1)$ such that $u(mh) < 0$ is the global minimum of u in $[j_0, \infty)$. So $u(jh) - u(mh) \geq 0$
for all $j \geq j_0$, then, by the maximum principle, $(\delta_{\text{right}})^\alpha(u - u(mh))(mh) = (\delta_{\text{right}})^\alpha u(mh) \leq 0$.
On the one hand, if $(\delta_{\text{right}})^\alpha u(mh) = 0$, then $u(jh) = u(mh) < 0$ for all $j \geq m$, which contradicts
that $u(jh) = 0$ for all $j \geq j_1$. On the other hand, if $(\delta_{\text{right}})^\alpha u(mh) < 0$, this contradicts the
hypothesis on f.

For part (ii), we use again an argument of contradiction. We suppose that $\sup_{j \geq j_0} u(jh)$ is not
attained in $[j_1, \infty)$. Then there exists $m \in [j_0, j_1)$ such that $u(mh)$ is the global maximum of u in
$[j_0, \infty)$. So $u(mh) - u(jh) \geq 0$ for all $j \geq j_0$, then, by the maximum principle, $(\delta_{\text{right}})^\alpha(u(mh) - u)(mh) = -(\delta_{\text{right}})^\alpha u(mh) \leq 0$, i.e, $(\delta_{\text{right}})^\alpha u(mh) \geq 0$. If $(\delta_{\text{right}})^\alpha u(mh) > 0$, it contradicts
that $(\delta_{\text{right}})^\alpha u \leq 0$ in $[j_0, j_1)$ and if $(\delta_{\text{right}})^\alpha u(mh) = 0$, then, by the maximum principle, $u(jh) = u(hm)$ for all $j \geq m$, so the $\sup_{j \geq j_0} u(jh)$ is attained in $[j_1, \infty)$.

Part (ii) implies part (iii) by taking $-u$. Finally, part (iv) is a consequence of (iii), and the
uniqueness is a straightforward consequence. □

The regularity results announced in the Introduction are based in the behavior of our fractional
powers when acting over some Hölder spaces that we define now. Following the notation in [9],
for $l, s \in \mathbb{N}_0$, we denote $\delta_{\text{right,left}}^{l,s} := (\delta_{\text{right}})^l (\delta_{\text{left}})^s$.

Definition 3.1. *([9, Definition 2.1]). Let $0 < \beta \leq 1$ and $k \in \mathbb{N}_0$. A function $u : \mathbb{Z}_h \to \mathbb{R}$ belongs
to the discrete Hölder space $C_h^{k,\beta}$ if*

$$[\delta_{\text{right,left}}^{l,s} u]_{C_h^{0,\beta}} = \sup_{m \neq j} \frac{|\delta_{\text{right,left}}^{l,s} u(jh) - \delta_{\text{right,left}}^{l,s} u(hm)|}{h^\beta |j - m|^\beta} < \infty$$

for each pair $l, s \in \mathbb{N}_0$ such that $l + s = k$. The norm in the spaces $C_h^{k,\beta}$ is given by

$$\|u\|_{C_h^{k,\beta}} = \max_{l+s \leq k} \sup_{m \in \mathbb{Z}} |\delta_{\text{right,left}}^{l,s} u(mh)| + \max_{l+s=k} [\delta_{\text{right,left}}^{l,s} u]_{C_h^{0,\beta}}.$$

For simplicity, we only write the following theorem for $(\delta_{\text{right}})^\alpha$ since it is analogous for $(\delta_{\text{left}})^\alpha$.

Theorem 3.2. *Let $0 < \beta \leq 1$ and $0 < \alpha < 1$.*

(i) *Let $u \in C_h^{0,\beta}$ and $\alpha < \beta$. Then $(\delta_{\text{right}})^\alpha u \in C_h^{0,\beta-\alpha}$ and*
$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{0,\beta-\alpha}} \leq C\|u\|_{C_h^{0,\beta}}.$$

(ii) *Let $u \in C_h^{1,\beta}$ and $\alpha < \beta$. Then $(\delta_{\text{right}})^\alpha u \in C_h^{1,\beta-\alpha}$ and*
$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{1,\beta-\alpha}} \leq C\|u\|_{C_h^{1,\beta}}.$$

(iii) *Let $u \in C_h^{1,\beta}$ and $\alpha > \beta$. Then $(\delta_{\text{right}})^\alpha u \in C_h^{0,\beta-\alpha+1}$ and*
$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{0,\beta-\alpha+1}} \leq C\|u\|_{C_h^{1,\beta}}.$$

(iv) *Let $u \in C_h^{k,\beta}$ and assume that $k + \beta - \alpha$ is not an integer, with $\alpha < k + \beta$. Then $(\delta_{\mathrm{right}})^\alpha u \in C_h^{l,s}$ where l is the integer part of $k + \beta - \alpha$ and $s = k + \beta - \alpha - l$.*

The positive constants C are independent of h and u.

Proof. Let $j, l \in \mathbb{Z}$. We write

$$|(\delta_{\mathrm{right}})^\alpha u(jh) - (\delta_{\mathrm{right}})^\alpha u(lh)| = \frac{1}{h^\alpha}|I_1 + I_2|,$$

with $I_1 = \displaystyle\sum_{1 \leq m \leq |j-l|} (u((j+m)h) - u(jh) - u((l+m)h) + u(lh))\Lambda^\alpha(m)$, and

$I_2 = \displaystyle\sum_{m > |j-l|} (u((j+m)h) - u(jh) - u((l+m)h) + u(lh))\Lambda^\alpha(m)$.

To prove (i), note that

$$|I_1| \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} \sum_{1 \leq m \leq |j-l|} \frac{m^\beta}{m^{\alpha+1}} \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j - l|^{\beta - \alpha},$$

where we have used (2.3). For I_2, observe that

$$|u(jh) - u(lh))|, |u((j+m)h) - u((l+m)h)| \leq h^\beta [u]_{C_h^{0,\beta}} |j - l|^\beta,$$

then

$$|I_2| \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j - l|^\beta \sum_{m > |j-l|} \frac{1}{m^{\alpha+1}} \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j - l|^{\beta - \alpha},$$

using again (2.3).

Part (ii) is a straightforward consequence of (i). By definition, if $u \in C_h^{1,\beta}$, then $\delta_{\mathrm{right}} u \in C_h^{0,\beta}$, and as $\delta_{\mathrm{right,left}}$ commutes with $(\delta_{\mathrm{right}})^\alpha$, by using (i) we get $\delta_{\mathrm{right/left}}(\delta_{\mathrm{right}})^\alpha u \in C_h^{0,\beta-\alpha}$.

For part (iii), suppose without loss of generality that $j > l$. We can write $u((j+m)h) - u(jh) = -h \displaystyle\sum_{p=0}^{m-1} \delta_{\mathrm{right}} u((j+p)h)$. Then

$$|I_1| \leq h \sum_{1 \leq m \leq |j-l|} \sum_{p=0}^{m-1} |\delta_{\mathrm{right}} u((j+p)h) - \delta_{\mathrm{right}} u((l+p)h)|\Lambda^\alpha(m)$$

$$\leq C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j - l|^\beta \sum_{1 \leq m \leq |j-l|} \frac{1}{m^\alpha} \leq C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j - l|^{\beta+1-\alpha}.$$

To bound I_2 we write $u((j+m)h) - u((l+m)h) = -h \displaystyle\sum_{p=0}^{|j-l|-1} \delta_{\mathrm{right}} u((l+m+p)h)$, then

$$|I_2| \leq h \sum_{m > |j-l|} \sum_{p=0}^{|j-l|-1} |\delta_{\mathrm{right}} u((l+m+p)h) - \delta_{\mathrm{right}} u((l+p)h)|\Lambda^\alpha(m)$$

$$\leq C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j - l| \sum_{m > |j-l|} \frac{m^\beta}{m^\alpha} \leq C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j - l|^{\beta+1-\alpha}.$$

Iterating parts (i), (ii) and (iii) we get (iv). $\qquad\qquad\square$

4. Approximation of fractional derivatives in the line by discrete fractional derivatives. Marchaud and Grünwald-Letnikov fractional derivatives

Now we compare the discrete fractional derivatives and the discretized continuous fractional derivatives on Hölder spaces, and we estimate the error of the approximation on $\ell^\infty(\mathbb{Z})$.

In [5], the fractional powers of the derivatives from the right and the left are considered, where

$$D_{\text{right}}f(x) = \lim_{t\to 0^+} \frac{f(x) - f(x+t)}{t} \ \text{ and } \ D_{\text{left}}f(x) = \lim_{t\to 0^+} \frac{f(x) - f(x-t)}{t}$$

are the continuous derivatives from the right and from the left, respectively. We recall that the fractional derivatives for $0 < \alpha < 1$ are given by

$$(D_{\text{right/left}})^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x\pm t) - f(x)}{t^{1+\alpha}}\,dt,$$

for sufficiently smooth functions f.

Also, recall that a continuous real function u belongs to the Hölder space $C^{k,\beta}(\mathbb{R})$ if $u \in C^k(\mathbb{R})$ and

$$[u^{(k)}]_{C^{0,\beta}(\mathbb{R})} = \sup_{x\neq y\in\mathbb{R}} \frac{|u^{(k)}(x) - u^{(k)}(y)|}{|x-y|^\beta} < \infty,$$

where $u^{(k)}$ denotes the k-th derivative of u. The norm in the spaces $C^{k,\beta}(\mathbb{R})$ is

$$\|u\|_{C^{k,\beta}(\mathbb{R})} = \sum_{l=0}^{k} \|u^{(l)}\|_{L^\infty(\mathbb{R})} + [u^{(k)}]_{C^{0,\beta}(\mathbb{R})}.$$

We only compare $(\delta_{\text{right}})^\alpha$ with $(D_{\text{right}})^\alpha$ but the result and the proof are analogous for $(\delta_{\text{left}})^\alpha$ and $(D_{\text{left}})^\alpha$. To prove Theorem 1.3 we need a previous lemma.

Lemma 4.1. *Let $0 < \alpha < 1$ and $j \in \mathbb{Z}$. We have*

$$(4.1) \qquad \left| \frac{1}{\Gamma(-\alpha)} \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} - \frac{\Lambda^\alpha(m)}{h^\alpha} \right| \leq \frac{C_\alpha}{h^\alpha m^{2+\alpha}}, \quad m \in \mathbb{N},$$

and

$$(4.2) \qquad \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} \leq \frac{C_\alpha}{h^\alpha m^{1+\alpha}}, \quad m \in \mathbb{N}.$$

Proof. By doing the change of variable $t - jh = zh$ we have

$$\left| \frac{1}{\Gamma(-\alpha)} \int_{(j+m)h}^{(j+m+1)h} \frac{1}{(t-jh)^{1+\alpha}}\,dt - \frac{\Lambda^\alpha(m)}{h^\alpha} \right| = \left| \frac{1}{h^\alpha\Gamma(-\alpha)} \int_m^{m+1} \frac{dz}{z^{1+\alpha}} - \frac{\Lambda^\alpha(m)}{h^\alpha} \right|$$

$$\leq \left| \frac{1}{h^\alpha\Gamma(-\alpha)} \int_m^{m+1} \left(\frac{1}{z^{1+\alpha}} - \frac{1}{m^{1+\alpha}} \right) dz \right| + \frac{1}{h^\alpha} \left| \frac{1}{\Gamma(-\alpha)m^{1+\alpha}} - \Lambda^\alpha(m) \right|.$$

The Mean Value Theorem implies

$$\left| \int_m^{m+1} \left(\frac{1}{z^{1+\alpha}} - \frac{1}{m^{1+\alpha}} \right) dz \right| \leq C_\alpha \left| \int_m^{m+1} \frac{1}{m^{2+\alpha}}\,dz \right| \leq \frac{C_\alpha}{m^{2+\alpha}},$$

and

$$\left| \frac{1}{\Gamma(-\alpha)m^{1+\alpha}} - \Lambda^\alpha(m) \right| \leq \frac{C_\alpha}{m^{2+\alpha}}$$

by (2.3). So, (4.1) is proved. For (4.2), we have

$$\int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} \leq \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(mh)^{1+\alpha}} \leq \frac{C_\alpha}{h^\alpha m^{1+\alpha}}.$$

$$\square$$

Proof of Theorem 1.3:
We suppose the hypothesis of part (i). Let $j \in \mathbb{Z}$, then

$$
\begin{aligned}
r_h((D_{\text{right}})^\alpha u)(j) &= \frac{1}{\Gamma(-\alpha)} \sum_{m \in \mathbb{N}_0} \int_{(j+m)h}^{(j+m+1)h} \frac{u(t) - u(jh)}{(t-jh)^{1+\alpha}} \, dt \\
&= \frac{1}{\Gamma(-\alpha)} \left(\sum_{m \in \mathbb{N}_0} \int_{(j+m)h}^{(j+m+1)h} \frac{u(t) - u((j+m)h)}{(t-jh)^{1+\alpha}} \, dt \right. \\
&\quad \left. + \sum_{m \in \mathbb{N}} \int_{(j+m)h}^{(j+m+1)h} \frac{u((j+m)h) - u(jh)}{(t-jh)^{1+\alpha}} \, dt \right) \\
&= \frac{1}{\Gamma(-\alpha)} (I_1 + I_2).
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
|I_1| &\leq C[u]_{C^{0,\beta}(\mathbb{R})} \sum_{m \in \mathbb{N}_0} \int_{(j+m)h}^{(j+m+1)h} \frac{|t - (j+m)h|^\beta}{(t-jh)^{1+\alpha}} \, dt \\
&\leq C_\alpha h^{\beta-\alpha} [u]_{C^{0,\beta}(\mathbb{R})} \left(1 + \sum_{m \in \mathbb{N}} \frac{1}{m^{1+\alpha}} \right) = C_\alpha h^{\beta-\alpha} [u]_{C^{0,\beta}(\mathbb{R})},
\end{aligned}
$$

where we have used (4.2). On the other hand we compare I_2 with $(\delta_{\text{right}})^\alpha(r_h u)(j)$. By (4.1),

$$
\begin{aligned}
&\left| \frac{I_2}{\Gamma(-\alpha)} - (\delta_{\text{right}})^\alpha(r_h u)(j) \right| \\
&\leq \sum_{m \in \mathbb{N}} |u((j+m)h) - u(jh)| \left| \frac{1}{\Gamma(-\alpha)} \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} - \frac{\Lambda^\alpha(m)}{h^\alpha} \right| \\
&\leq C_\alpha h^{\beta-\alpha} [u]_{C^{0,\beta}(\mathbb{R})} \sum_{m \in \mathbb{N}} \frac{m^\beta}{m^{2+\alpha}} \leq C_\alpha h^{\beta-\alpha} [u]_{C^{0,\beta}(\mathbb{R})}.
\end{aligned}
$$

For (ii), observe that δ_{right} commutes with $(\delta_{\text{right}})^\alpha$ and $\frac{d}{dx}$ with $(D_{\text{right}})^\alpha$. Then we write

$$
\begin{aligned}
\| -\delta_{\text{right}}(\delta_{\text{right}})^\alpha(r_h u) - r_h(\tfrac{d}{dx}(D_{\text{right}})^\alpha u) \|_{\ell^\infty} &\leq \| (\delta_{\text{right}})^\alpha(-\delta_{\text{right}})(r_h u) - (\delta_{\text{right}})^\alpha(r_h \tfrac{d}{dx} u) \|_{\ell^\infty} \\
&\quad + \| (\delta_{\text{right}})^\alpha(r_h \tfrac{d}{dx} u) - r_h((D_{\text{right}})^\alpha(\tfrac{d}{dx} u)) \|_{\ell^\infty}.
\end{aligned}
$$

We apply the part (i) to the second term. Let $j \in \mathbb{Z}$. For the first one, we apply the Mean Value Theorem and the fact that $\sum_{m=0}^{\infty} \Lambda^{\alpha}(m) = 0$,

$$
\left| (\delta_{\text{right}})^{\alpha}(-\delta_{\text{right}})(r_h u)(j) - (\delta_{\text{right}})^{\alpha}(r_h \frac{d}{dx} u)(j) \right| =
$$

$$
= \frac{1}{h^{\alpha}} \left| \sum_{m \in \mathbb{N}} \Lambda^{\alpha}(m) \left(\frac{u((j+m+1)h) - u((j+m)h)}{h} - u'((j+m)h) - \frac{u((j+1)h) - u(jh)}{h} + u'(jh) \right) \right|
$$

$$
= \frac{1}{h^{\alpha}} \left| \sum_{m \in \mathbb{N}} \Lambda^{\alpha}(m) \left(u'(\xi_{j+m}) - u'((j+m)h) - u'(\xi_j) + u'(jh) \right) \right|
$$

$$
\leq \frac{C}{h^{\alpha}} [u']_{C^{0,\beta}(\mathbb{R})} \sum_{m \in \mathbb{N}} |\Lambda^{\alpha}(m)| h^{\beta} \leq C_{\alpha} [u']_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha},
$$

where $\xi_j \in (jh, (j+1)h)$ and $\xi_{j+m} \in ((j+m)h, (j+m+1)h)$. $\square$

Proof of Theorem 1.4:

Given $x \in \mathbb{R}$ and $h > 0$, there exists a $j_0 \in \mathbb{Z}$ such that $j_0 h \leq x < j_0 h + h$. Then, we have

$$
\left| (D_{\text{right}})^{\alpha} f(x) - \frac{\Delta_{h,+}^{\alpha}}{h^{\alpha}} f(x) \right| \leq |(D_{\text{right}})^{\alpha} f(x) - (D_{\text{right}})^{\alpha} f(j_0 h)| + \left| (D_{\text{right}})^{\alpha} f(j_0 h) - \frac{\Delta_{h,+}^{\alpha}}{h^{\alpha}} f(j_0 h) \right|
$$

$$
+ \left| \frac{\Delta_{h,+}^{\alpha}}{h^{\alpha}} f(j_0 h) - \frac{\Delta_{h,+}^{\alpha}}{h^{\alpha}} f(x) \right| = I + II + III.
$$

On the one hand,

$$
I = \left| \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{f(x+t) - f(x)}{t^{1+\alpha}} dt - \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{f(j_0 h + t) - f(j_0 h)}{t^{1+\alpha}} dt \right|
$$

$$
\leq \left| \frac{1}{\Gamma(-\alpha)} \int_0^h \frac{f(x+t) - f(x)}{t^{1+\alpha}} dt \right| + \left| \frac{1}{\Gamma(-\alpha)} \int_0^h \frac{f(j_0 h + t) - f(j_0 h)}{t^{1+\alpha}} dt \right|
$$

$$
+ \left| \frac{1}{\Gamma(-\alpha)} \int_h^{\infty} \frac{f(x+t) - f(j_0 h + t)}{t^{1+\alpha}} dt \right| + \left| \frac{1}{\Gamma(-\alpha)} \int_h^{\infty} \frac{f(j_0 h) - f(x)}{t^{1+\alpha}} dt \right|
$$

$$
\leq C_{\alpha} [f]_{C^{0,\beta}(\mathbb{R})} \left(\int_0^h \frac{t^{\beta}}{t^{1+\alpha}} dt + \int_h^{\infty} \frac{h^{\beta}}{t^{1+\alpha}} dt \right) = C_{\alpha,\beta} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.
$$

By using Theorem 1.3 we obtain

$$
II \leq C_{\alpha} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.
$$

On the other hand, as $\sum_{k=0}^{\infty} \left| \binom{\alpha}{k} \right| \leq C_{\alpha}$, we obtain

$$
III \leq \left| \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} (f(j_0 h + kh) - f(x + kh)) \right| \leq \frac{C[f]_{C^{0,\beta}(\mathbb{R})}}{h^{\alpha}} \sum_{k=0}^{\infty} \left| \binom{\alpha}{k} \right| h^{\beta}
$$

$$
\leq C_{\alpha} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha},
$$

so the result follows.

The proof is analogous for $(D_{\text{left}})^{\alpha}$ and $\frac{\Delta_{h,-}^{\alpha}}{h^{\alpha}}$. $\square$

5. EXTENSION PROBLEM

Theorem 1.5 is a straightforward consequence of Remark 2.3 and [11, Theorem 2.1 and Remark 2.2, (ii)]. The formula (1.8) provides an explicit expression in our case, that is,

(5.1)

$$u(t,n) = P_{t,\pm}^{\gamma} f(n) = \frac{t^{2\gamma}}{4^{\gamma}\Gamma(\gamma)} \int_0^{\infty} e^{-\frac{t^2}{4s}} T_{s,\pm} f(n) \frac{ds}{s^{1+\gamma}} = \sum_{j=0}^{\infty} \frac{t^{2\gamma} f(n \pm j)}{4^{\gamma}\Gamma(\gamma)j!} \int_0^{\infty} e^{-s-t^2/4s} s^{j-\gamma} \frac{ds}{s}$$

$$= \sum_{j=0}^{\infty} \frac{t^{j+\gamma}}{2^{j+\gamma-1}\Gamma(\gamma)j!} K_{j-\gamma}(t) f(n \pm j)$$

where we have used the identity [15, 2.3.16.1, p. 344]. The function K_{ν} is the *Macdonald's function* (also called *modified Bessel function of the third type*) defined in [13, Section 5.7, p. 108]. By completeness we prove that the previous formula solves (1.7) pointwise for $t \in (0,\infty)$. We shall use the following identities (see [13, Section 5.7]):

$$K_{\nu+1}(t) = \frac{2\nu}{t} K_{\nu}(t) + K_{\nu-1}(t), \qquad \frac{d}{dt}(t^{\nu} K_{\nu}(t)) = -t^{\nu} K_{\nu-1}(t), \ \forall \nu \in \mathbb{R}.$$

We have

$$\partial_t P_{t,+}^{\gamma} f(n) = \sum_{j=0}^{\infty} \frac{f(n+j)}{2^{j+\gamma-1}\Gamma(\gamma)j!} (2\gamma t^{j+\gamma-1} K_{j-\gamma}(t) - t^{\gamma+j} K_{j-\gamma-1}(t)),$$

and

$$\partial_{tt} P_{t,+}^{\gamma} f(n) = \sum_{j=0}^{\infty} \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!} \left(\frac{2\gamma(2\gamma-1)}{t^2} K_{j-\gamma}(t) - \frac{4\gamma+1}{t} K_{j-\gamma-1}(t) + K_{j-\gamma-2}(t) \right)$$

$$= \sum_{j=0}^{\infty} \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!} \left(\left(\frac{2\gamma(2\gamma-1)}{t^2} + 1 \right) K_{j-\gamma}(t) - \frac{2\gamma+2j-1}{t} K_{j-\gamma-1}(t) \right).$$

Then, we obtain

$$(\partial_{tt}^2 + \frac{1-2\gamma}{t} \partial_t) P_{t,+}^{\gamma} f(n) = \sum_{j=0}^{\infty} \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!} (K_{j-\gamma} - \frac{2j}{t} K_{j-\gamma-1}(t))$$

$$= \delta_{\text{right}} P_{t,+}^{\gamma} f(n).$$

The analogous result for $\delta_{\text{left}} P_{t,-}^{\gamma}$ can be also proved by the same way.

Observe that, when $\gamma = 1/2$, $P_{t,\pm}^{1/2} f$ is precisely the Poisson semigroup associated to $\delta_{\text{right/left}}$.

6. MAXIMAL OPERATORS. LITTLEWOOD-PALEY FUNCTIONS

In this last section we shall consider the maximal functions and the Littlewood-Paley square functions associated to the heat semigroup and generalized Poisson function. Our first observation is that both functions have bad behavior in the case of the heat semigroups. In fact we have the following.

Claim 1 *The maximal functions of the heat semigroups defined in (2.1), that is,*

$$T_{\pm}^* f = \sup_{t \geq 0} |T_{t,\pm} f|,$$

are not bounded from $\ell^p(\mathbb{Z})$ into itself for $1 \leq p \leq 2$.

In fact, let $f(0) = 1$ and $f(j) = 0$ for $j \neq 0$, then $T_{t,-}f(n) = 0$ for $n < 0$ and $T_{t,-}f(n) = e^{-t}\frac{t^n}{n!}$ for $n \geq 0$. The maximum of the function $e^{-t}\frac{t^n}{n!}$ is $e^{-n}\frac{n^n}{n!}$, and, by Stirling's formula, it behaves asymptotically like $\frac{1}{\sqrt{n}}$ as $n \to \infty$.

Claim 2 *The Littlewood-Paley functions of the heat semigroups defined by*

$$\mathfrak{g}_{\pm}(f) = \left(\int_0^\infty |t\partial_t T_{t,\pm}f|^2 \frac{dt}{t} \right)^{1/2},$$

are not bounded from $\ell^2(\mathbb{Z})$ into itself.

We shall consider the Fourier transform $\hat{f}(\theta) = \sum_n f(n)e^{-in\theta}$. Observe that $\widehat{T_{t,+}f}(\theta) = e^{-t(1-e^{i\theta})}\hat{f}(\theta)$, and then it follows

$$\widehat{\partial_t T_{t,+}f}(\theta) = (e^{i\theta} - 1)e^{-t(1-e^{i\theta})}\hat{f}(\theta).$$

Let $f \in \ell^2(\mathbb{Z})$ such that $\int_0^{2\pi} \left(\frac{|\hat{f}(\theta)|}{\theta} \right)^2 d\theta = \infty$. Hence, by Plancherel's and Fubini's Theorems, we have

$$\|\mathfrak{g}_+(f)\|_{\ell^2}^2 = \sum_{n \in \mathbb{Z}} \int_0^\infty |t\partial_t T_{t,+}f(n)|^2 \frac{dt}{t} = \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{T}} |t\widehat{\partial_t T_{t,+}}f(\theta)|^2 d\theta \frac{dt}{t}$$

$$= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{T}} |t(e^{i\theta}-1)e^{-t(1-e^{i\theta})}|^2 |\hat{f}(\theta)|^2 d\theta \frac{dt}{t}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |\hat{f}(\theta)|^2 \int_0^\infty t^2 |e^{i\theta}-1|^2 e^{-2t\Re(1-e^{i\theta})} \frac{dt}{t} d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} |\hat{f}(\theta)|^2 \int_0^\infty t^2 (1-\cos\theta) e^{-2t(1-\cos\theta)} \frac{dt}{t} d\theta$$

$$\underbrace{=}_{2(1-\cos\theta)t=u} \frac{1}{4\pi} \int_0^{2\pi} \frac{|\hat{f}(\theta)|^2}{1-\cos\theta} \int_0^\infty u e^{-u} du\, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \frac{|\hat{f}(\theta)|^2}{1-\cos\theta} d\theta,$$

and this integral does not converge.

However, the behavior of the generalized Poisson function is suitable with the classical results in Harmonic analysis. Consider the maximal function associated to the generalized Poisson function

$$P_{\pm}^{\gamma,*} f = \sup_{t \geq 0} |P_{t,\pm}^\gamma f|, \quad 0 < \gamma < 1,$$

and the Littlewood-Paley square functions

$$g_{\pm}^\gamma(f) = \left(\int_0^\infty |t\partial_t P_{t,\pm}^\gamma f|^2 \frac{dt}{t} \right)^{1/2}, \quad 0 < \gamma < 1.$$

To prove Theorem 1.8 for these operators, the tool that we shall use is the vector valued Calderón-Zygmund Theory in spaces of homogeneous type, more specifically in the particular case of the integers with the natural distance $d(n,m) = |n-m|$ and measure $\mu(n) = 1$. Given a

Banach space E, we denote by $\ell_E^p(\mathbb{Z})$, $1 \leq p \leq \infty$, the space of E-valued functions f defined on $\mathbb{Z}$ such that $\|f\|_E$ belongs to $\ell^p(\mathbb{Z}, d, \mu)$.

Definition 6.1 (Vector-valued (convolution) Calderón-Zygmund operator on $(\mathbb{Z}, d, \mu)$). *We say that a linear operator $\mathfrak{T}$ on the space $(\mathbb{Z}, d, \mu)$ is a Calderón-Zygmund operator if it satisfies the following conditions.*

(I) *There exists $1 \leq p_0 \leq \infty$ such that $\mathfrak{T}$ is bounded from $\ell^{p_0}(\mathbb{Z})$ into $\ell_E^{p_0}(\mathbb{Z})$.*

(II) *For bounded functions f with compact support, $\mathfrak{T}f$ can be represented as*

$$\mathfrak{T}f(n) = \sum_{j \in \mathbb{Z}} K(j) f(j+n),$$

where $K(j) \in \mathcal{L}(\mathbb{R}, E)$ is the space of bounded linear operator from $\mathbb{R}$ to E, and satisfies

(II.1) $\|K(j)\|_{\mathcal{L}(\mathbb{R}, E)} \leq \dfrac{C}{|j|}$, *for every $j \neq 0$;*

(II.2) $\|K(j) - K(j_0)\|_{\mathcal{L}(\mathbb{R}, E)} \leq C \dfrac{|j - j_0|}{|j_0|^2}$, *whenever $|j_0| > 2|j - j_0|$, $\quad j_0 \neq 0$;*

for some constant $C > 0$.

The Calderón–Zygmund theorem says that if $\mathfrak{T}$ is a Calderón–Zygmund operator on $(\mathbb{Z}, d, \mu)$ as above then $\mathfrak{T}$ is bounded from $\ell^p(\mathbb{Z})$ into $\ell_E^p(\mathbb{Z})$, for any $1 < p \leq \infty$, and it is also of weak type $(1, 1)$. For full details see [14, 16, 17].

Now we ready to prove the Theorem 1.8. We shall proof only the cases associated to $P_{t,+}^\gamma$. For $P_{t,-}^\gamma$. the proof is analogous.

Proof of Theorem 1.8.

Case 1. The maximal function.

For convenience, we will write $P_{t,+}^\gamma f(n) = \sum_{j=0}^\infty P_t^\gamma(j) f(n+j)$, where

$$P_t^\gamma(j) = \frac{t^{2\gamma}}{4^\gamma \Gamma(\gamma) j!} \int_0^\infty e^{-s - t^2/4s} s^{j-\gamma} \frac{ds}{s} \, , \, j \in \mathbb{N}_0.$$

Consider the vector-valued operator

$$\mathfrak{T}f(n) = \left\{ \sum_{j \in \mathbb{Z}} P_t^\gamma(j) f(n+j) \right\}_{t \geq 0} = \sum_{j \in \mathbb{Z}} \left\{ P_t^\gamma(j) \right\}_{t \geq 0} f(n+j),$$

where we have assumed $P_t^\gamma(j) = 0$ for $j < 0$, $t \geq 0$. The operator $\mathfrak{T}$ satisfies $\mathfrak{T} : \ell^\infty(\mathbb{Z}) \longrightarrow \ell_{L^\infty(0,\infty)}^\infty(\mathbb{Z})$. In fact

$$\|\mathfrak{T}f\|_{\ell_{L^\infty}^\infty(\mathbb{Z})} = \sup_{n \in \mathbb{Z}} \sup_{t \geq 0} |P_{t,+}^\gamma f(n)| \leq C_\gamma \|f\|_\infty \sup_{t \geq 0} \sum_{j=0}^\infty \frac{1}{j!} \int_0^\infty e^{-u} e^{-\frac{t^2}{4u}} u^j \left(\frac{t^2}{u} \right)^\gamma \frac{du}{u}$$

$$= C_\gamma \|f\|_\infty \sup_{t \geq 0} \int_0^\infty e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma \frac{du}{u} \underbrace{=}_{\frac{t^2}{4u} = v} C_\gamma \|f\|_\infty \sup_{t \geq 0} \int_0^\infty e^{-v} (4v)^\gamma \frac{dv}{v} < \infty,$$

where we have applied Fubini's Theorem.

Moreover the kernel $\{P_t^\gamma(j)\}_{t \geq 0}$ satisfies

$$\|P_t^\gamma(j)\|_{L^\infty} = \sup_{t \geq 0} \frac{1}{4^\gamma \Gamma(\gamma) j!} \int_0^\infty e^{-u} e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma u^j \frac{du}{u} \leq \sup_{t \geq 0} \frac{C_\gamma}{j!} \int_0^\infty e^{-u} u^j \frac{du}{u} = C_\gamma \frac{1}{j},$$

for $j > 0$ and $\|P_t^\gamma(j)\|_{L^\infty} = 0 \leq \frac{C_\gamma}{|j|}$ for $j < 0$, where we have used that the function $g(u) = e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma$ reaches its maximum at $u = t^2/4\gamma$.

Regarding (II.2), it is easy to see that it is enough to prove that for each $j \in \mathbb{N}$,

$$\sup_{t \geq 0} |P_t^\gamma(j) - P_t^\gamma(j+1)| \leq \frac{C_\gamma}{j^2}.$$

If $j \geq 1$, then for all $t \geq 0$ we have

$$
\begin{aligned}
|P_t^\gamma(j) - P_t^\gamma(j+1)| &= \frac{1}{4^\gamma \Gamma(\gamma) j!} \left| \int_0^\infty e^{-u} e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma u^j \left(1 - \frac{u}{j+1} \right) \frac{du}{u} \right| \\
&= \frac{1}{4^\gamma \Gamma(\gamma)(j+1)!} \left| \int_0^\infty \partial_u (e^{-u} u^{j+1}) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma \frac{du}{u} \right| \\
&= \frac{1}{4^\gamma \Gamma(\gamma)(j+1)!} \left| \int_0^\infty e^{-u} u^{j+1} \partial_u \left(e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma \frac{1}{u} \right) du \right|,
\end{aligned}
$$

where we have used integration by parts. As $\left| \partial_u \left(e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u} \right)^\gamma \frac{1}{u} \right) \right| \leq \frac{C_\gamma}{u^2}$, then

$$|P_t^\gamma(j) - P_t^\gamma(j+1)| \leq \frac{C_\gamma}{(j+1)!} \int_0^\infty e^{-u} u^{j-1} du = \frac{C_\gamma \Gamma(j)}{(j+1)!} = \frac{C_\gamma}{j^2}.$$

Finally since $\|\mathfrak{T}f(n)\|_{L^\infty} = P_+^{\gamma,*} f(n)$ the result follows for the maximal operator, by choosing $p_0 = \infty$, and $E = L^\infty(\mathbb{R}_+)$.

Case 2. Littlewood-Paley functions Consider the vector-valued operator

$$\mathfrak{T}f(n) = \left\{ \sum_{j \in \mathbb{Z}} t \partial_t P_t^\gamma(j) f(n+j) \right\}_{t \geq 0} = \sum_{j \in \mathbb{Z}} \left\{ t \partial_t P_t^\gamma(j) \right\}_{t \geq 0} f(n+j).$$

In this case we write

$$P_t^\gamma(j) = \frac{t^{2\gamma}}{4^\gamma \Gamma(\gamma) j!} \int_0^\infty e^{-s - t^2/4s} s^{j-\gamma} \frac{ds}{s} = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-r} G_{t^2/4r}(j) \frac{dr}{r^{1-\gamma}}, \ j \in \mathbb{N}_0.$$

We have performed the change of variables $r = \frac{t^2}{4s}$ and the sequence G_t is defined in (2.1). Again we assume $P_t^\gamma(j) = 0$ for $j < 0$, $t \geq 0$. The operator $\mathfrak{T}$ is bounded from $\ell^2(\mathbb{Z})$ into $\ell^2_{L^2((0,\infty), \frac{dt}{t})}$. In fact

$$\|\mathfrak{T}f\|^2_{\ell^2_{L^2((0,\infty), \frac{dt}{t})}} = \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{T}} |t \widehat{\partial_t P_{t,+}^\gamma} f(\theta)|^2 d\theta \frac{dt}{t} = \frac{1}{2\pi} \int_{\mathbb{T}} \int_0^\infty |\widehat{t \partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} |\hat{f}(-\theta)|^2 d\theta.$$

We shall see that $\int_0^\infty |\widehat{t \partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} < C_\gamma$, where $C_\gamma > 0$ does not depend on θ. Observe that

$$\widehat{t \partial_t P_t^\gamma}(\theta) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-r} \frac{t^2}{2r} (e^{i\theta} - 1) e^{-\frac{t^2}{4r}(1 - e^{i\theta})} \frac{dr}{r^{1-\gamma}}.$$

Notice that if $\theta = 0, 2\pi$, the statement is trivial, so we have to consider three cases: $0 < \theta < \frac{\pi}{4}$, $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$ and $\frac{7\pi}{4} < \theta < 2\pi$.

If $0 < \theta < \frac{\pi}{4}$, we define $z_0 = t(1 - e^{i\theta})^{1/2}$, with $\varphi_0 = \arg z_0 \in (-\pi/4, \pi/4)$. Then,

$$\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} \;=\; C_\gamma \int_0^\infty \Big|\frac{z_0^2}{2} \int_0^\infty e^{-r} e^{-z_0^2/(4r)} \frac{dr}{r^{2-\gamma}}\Big|^2 \frac{dt}{t}.$$
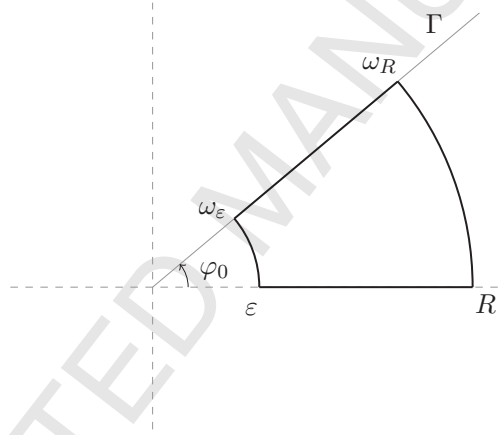
By applying Cauchy's Theorem, for $\varepsilon, R > 0$, we get

$$\int_\varepsilon^R e^{-r} e^{-z_0^2/(4r)} \frac{dr}{r^{2-\gamma}} = \Big(\int_{\Gamma_\varepsilon} - \int_{\Gamma_R} + \int_{\omega_\varepsilon}^{w_R}\Big) e^{-\omega} e^{-z_0^2/(4\omega)} \frac{d\omega}{\omega^{2-\gamma}},$$

where

$$\Gamma = \{\omega = sz_0 \mid 0 \le s \le \infty\}, \quad \Gamma_\varepsilon = \{\omega = \varepsilon e^{i\varphi} \mid \varphi \in [0, \varphi_0]\}, \quad \Gamma_R = \{\omega = Re^{i\varphi} \mid \varphi \in [0, \varphi_0]\},$$

and $\omega_\varepsilon = \varepsilon z_0$, $\omega_R = Rz_0$, see the next figure.

Notice that

$$\Big|\int_{\Gamma_\varepsilon} e^{-\omega} e^{-z_0^2/(4\omega)} \frac{d\omega}{\omega^{2-\gamma}}\Big| \le \int_{[0,\varphi_0]} |e^{-\varepsilon e^{i\varphi}} e^{-\frac{z_0^2}{4\varepsilon} e^{-i\varphi}}| \frac{d\varphi}{\varepsilon^{1-\gamma}} \le \frac{|\varphi_0| e^{-\frac{|z_0|^2 \cos(2\varphi_0)}{4\varepsilon}}}{\varepsilon^{1-\gamma}} \to 0, \text{ as } \varepsilon \to 0,$$

since $e^{-\varepsilon \cos\varphi} \le 1$ and it can be checked that $\Re(\frac{z_0^2}{4\varepsilon} e^{-i\varphi}) = \frac{|z_0|^2}{4\varepsilon} \cos(2\varphi_0 - \varphi) \ge \frac{|z_0|^2}{4\varepsilon} \cos(2\varphi_0)$.
Similarly, we get

$$\Big|\int_{\Gamma_R} e^{-\omega} e^{-z_0^2/(4\omega)} \frac{d\omega}{\omega^{2-\gamma}}\Big| \le \frac{|\varphi_0| e^{-\frac{|z_0| \cos(2\varphi_0)}{4R}}}{R^{1-\gamma}} \to 0, \text{ as } R \to \infty.$$

Hence

$$\frac{z_0^2}{2} \int_0^\infty e^{-r} e^{-z_0^2/(4r)} \frac{dr}{r^{2-\gamma}} = \frac{z_0^2}{2} \int_\Gamma e^{-\omega} e^{-z_0^2/(4\omega)} \frac{d\omega}{\omega^{2-\gamma}} = \frac{z_0^{1+\gamma}}{2} \int_0^\infty e^{-z_0 s} e^{-z_0/(4s)} \frac{ds}{s^{2-\gamma}}$$

and

$$
\begin{aligned}
\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} &= C_\gamma \int_0^\infty \left| \frac{z_0^{1+\gamma}}{2} \int_0^\infty e^{-z_0 s} e^{-z_0/(4s)} \frac{ds}{s^{2-\gamma}} \right|^2 \frac{dt}{t} \\
&\leq C_\gamma \int_0^\infty \left(t^{1+\gamma} \theta^{\frac{1+\gamma}{2}} \int_0^\infty e^{-ct\sqrt{\theta}\cos(\varphi_0)s} e^{-ct\sqrt{\theta}\cos(\varphi_0)/(4s)} \frac{ds}{s^{2-\gamma}} \right)^2 \frac{dt}{t} \\
&= C_\gamma \int_0^\infty \left(t^{1+\gamma} \theta^{\frac{1+\gamma}{2}} K_{\gamma-1}(ct\sqrt{\theta}\cos(\varphi_0)) \right)^2 \frac{dt}{t},
\end{aligned}
$$

where we have used that $|1 - e^{i\theta}| \sim \theta$, for θ being close to 0. In the last identity K_ν denotes the Mcdonald's function, see [15, 2.3.16.1, p. 344]. The following properties can be found in [13, Section 5.7 and Section 5.16]

$$
(6.1) \quad K_{-\nu}(z) = K_\nu(z), \ K_\nu(t) \sim \frac{2^{\nu-1}\Gamma(\nu)}{t^\nu} \text{ as } t \to 0, \text{ and } K_\nu(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \text{ as } t \to \infty, \ \nu > 0.
$$

Then

$$
\begin{aligned}
\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} &\leq C_\gamma \int_0^\infty \left(t^{1+\gamma} \theta^{\frac{1+\gamma}{2}} K_{1-\gamma}(ct\sqrt{\theta}\cos(\varphi_0)) \right)^2 \frac{dt}{t} \\
&\leq C_\gamma \int_0^{\frac{1}{c\sqrt{\theta}\cos\varphi_0}} \frac{t^{2+2\gamma}\theta^{1+\gamma}}{t^{2-2\gamma}\theta^{1-\gamma}(\cos\varphi_0)^{2-2\gamma}} \frac{dt}{t} + C_\gamma \int_{\frac{1}{c\sqrt{\theta}\cos\varphi_0}}^\infty \frac{t^{2+2\gamma}\theta^{1+\gamma}e^{-2ct\sqrt{\theta}\cos\varphi_0}}{t\sqrt{\theta}\cos\varphi_0} \frac{dt}{t} \\
&\underset{ct\sqrt{\theta}\cos\varphi_0 = u}{=} C_\gamma \int_0^1 \frac{du}{u^{1-4\gamma}} + C_\gamma \int_1^\infty u^{2\gamma+1} e^{-2u} \frac{du}{u} < \infty.
\end{aligned}
$$

Observe that in the last identity we have used that $\sqrt{2}/2 \leq \cos\varphi_0 \leq 1$.

The case $\frac{7\pi}{4} < \theta < 2\pi$ follows analogously, by using that $|1 - e^{i\theta}| = |1 - e^{i(\theta-2\pi)}| \sim |\theta - 2\pi|$, for θ being closed to 2π.

On the other hand, if $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$, then by using again [15, 2.3.16.1, p. 344] we get

$$
\begin{aligned}
|\widehat{t\partial_t P_t^\gamma}(\theta)| &\leq \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-r} \frac{t^2}{2r} |e^{i\theta} - 1| e^{-\frac{t^2}{4r}\Re(1-e^{i\theta})} \frac{dr}{r^{1-\gamma}} \leq C_\gamma \int_0^\infty e^{-r} \frac{t^2}{2r} e^{-\frac{t^2}{4r}(1-\cos\theta)} \frac{dr}{r^{1-\gamma}} \\
&\leq C_\gamma \int_0^\infty e^{-r} \frac{t^2}{2r} e^{-\frac{t^2}{4r}(1-\sqrt{2}/2)} \frac{dr}{r^{1-\gamma}} \leq C_\gamma \frac{t^2}{2} t^{\gamma-1} K_{1-\gamma}\left(t\sqrt{1-\sqrt{2}/2} \right) \\
&= C_\gamma t^{1+\gamma} K_{1-\gamma}(\tilde{c}t).
\end{aligned}
$$

Hence by estimates (6.1)

$$
\begin{aligned}
\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2 \frac{dt}{t} &= C_\gamma \int_0^{1/\tilde{c}} \frac{t^{2+2\gamma}}{(\tilde{c}t)^{2-2\gamma}} \frac{dt}{t} + C_\gamma \int_{1/\tilde{c}}^\infty \frac{t^{2+2\gamma}e^{-2\tilde{c}t}}{\tilde{c}t} \frac{dt}{t} \\
&\underset{\tilde{c}t=u}{=} C_\gamma \int_0^1 \frac{du}{u^{1-4\gamma}} + C_\gamma \int_1^\infty u^{2\gamma+1} e^{-2u} \frac{du}{u} < \infty.
\end{aligned}
$$

Therefore, we have proved that $\|\mathfrak{T}f\|_{\ell^2_{L^2((0,\infty),\frac{dt}{t})}}^2 \leq C_\gamma \|f\|_2$.

By the representation of the Poisson kernel, (5.1), we can write, for all $n \in \mathbb{Z}$, $t \geq 0$,

$$t\partial_t P_{t,\pm}^{\gamma} f(n) = \sum_{j=0}^{\infty} \frac{f(n \pm j)}{2\sqrt{\pi}j!} \int_0^{\infty} e^{-u} u^{j-\gamma} \left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} t^{2\gamma} \frac{du}{u} = \sum_{j=0}^{\infty} t\partial_t P_t^{\gamma}(j) f(n \pm j).$$

Then, by Minkowski's integral inequality, we get

$$\|t\partial_t P_t^{\gamma}(j)\|_{L^2((0,\infty),\frac{dt}{t})} = C_{\gamma} \left(\int_0^{\infty} \left|\int_0^{\infty} \left(2\gamma - \frac{t^2}{2u}\right) e^{-u} \frac{u^{j-\gamma}}{j!} e^{-\frac{t^2}{4u}} t^{2\gamma} \frac{du}{u}\right|^2 \frac{dt}{t}\right)^{1/2}$$

$$\leq C_{\gamma} \int_0^{\infty} \frac{u^j}{j!} e^{-u} \left(\int_0^{\infty} \left|2\gamma - \frac{t^2}{2u}\right|^2 e^{-\frac{t^2}{2u}} \left(\frac{t^2}{u}\right)^{2\gamma} \frac{dt}{t}\right)^{1/2} \frac{du}{u}$$

$$= C_{\gamma} \int_0^{\infty} \frac{u^j}{j!} e^{-u} I^{1/2} \frac{du}{u}, \quad \text{for } j \geq 1.$$

Observe that

$$I \leq C \int_0^{2\sqrt{\gamma u}} 4\gamma^2 e^{-\frac{t^2}{2u}} \left(\frac{t^2}{u}\right)^{2\gamma} \frac{dt}{t} + C \int_{2\sqrt{\gamma u}}^{\infty} \frac{t^4}{4u^2} e^{-\frac{t^2}{2u}} \left(\frac{t^2}{u}\right)^{2\gamma} \frac{dt}{t}$$

$$\underset{\frac{t}{2\sqrt{\gamma u}}=v}{\leq} C_{\gamma} \int_0^1 e^{-2\gamma v^2} v^{4\gamma} \frac{dv}{v} + C_{\gamma} \int_1^{\infty} v^{4+4\gamma} e^{-2\gamma v^2} \frac{dv}{v} \leq C_{\gamma}.$$

Hence, if $j \geq 1$,

$$\|t\partial_t P_t^{\gamma}(j)\|_{L^2((0,\infty),\frac{dt}{t})} \leq C_{\gamma} \int_0^{\infty} \frac{u^j}{j!} e^{-u} \frac{du}{u} = C_{\gamma} \frac{\Gamma(j)}{\Gamma(j+1)} \leq \frac{C_{\gamma}}{j},$$

and $\|t\partial_t P_t^{\gamma}(j)\|_{L^2((0,\infty),\frac{dt}{t})} = 0 \leq \frac{C_{\gamma}}{|j|}$ for $j < 0$.

Regarding (II.2), if $j \geq 1$,

$$\|t\partial_t P_t^{\gamma}(j) - t\partial_t P_t^{\gamma}(j+1)\|_{L^2((0,\infty),\frac{dt}{t})}$$

$$= C_{\gamma} \left(\int_0^{\infty} \left|\int_0^{\infty} \left(2\gamma - \frac{t^2}{2u}\right) \frac{e^{-u}}{j!} u^j \left(1 - \frac{u}{j+1}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^{\gamma} \frac{du}{u}\right|^2 \frac{dt}{t}\right)^{1/2}$$

$$= C_{\gamma} \left(\int_0^{\infty} \left|\int_0^{\infty} \frac{1}{j!} \left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^{\gamma} \frac{1}{u} \frac{\partial_u(e^{-u} u^{j+1})}{j+1} du\right|^2 \frac{dt}{t}\right)^{1/2}$$

$$= C_{\gamma} \left(\int_0^{\infty} \left|\int_0^{\infty} \frac{1}{(j+1)!} e^{-u} u^{j+1} \partial_u \left\{\left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^{\gamma} \frac{1}{u}\right\} du\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Note that

$$\left|\partial_u \left\{\left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^{\gamma} \frac{1}{u}\right\}\right| \leq C_{\gamma} \frac{e^{-\frac{t^2}{4u}}}{u^2} \left(\frac{t^2}{u}\right)^{\gamma} \left[1 + \left(\frac{t^2}{u}\right) + \left(\frac{t^2}{u}\right)^2\right].$$

Hence, by Minkowski's inequality,

$$\|t\partial_t P_t^\gamma(j) - t\partial_t P_t^\gamma(j+1)\|_{L^2((0,\infty),\frac{dt}{t})}$$
$$\leq \frac{C_\gamma}{(j+1)!} \int_0^\infty e^{-u} u^{j-1} \left(\int_0^\infty \left| e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \left[1 + \left(\frac{t^2}{u}\right) + \left(\frac{t^2}{u}\right)^2 \right] \right|^2 \frac{dt}{t} \right)^{1/2} du$$
$$\leq C_\gamma \frac{\Gamma(j)}{\Gamma(j+2)} \leq \frac{C_\gamma}{j^2}.$$

Finally since $\|\mathfrak{T}f(n)\|_{L^2((0,\infty),\frac{dt}{t})} = g_+^\gamma f(n)$, the result follows for the maximal operator, by choosing $p_0 = 2$, and $E = L^2((0,\infty),\frac{dt}{t})$.

□

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