*Rev. Real Academia de Ciencias. Zaragoza.* **72:** 7–32, (2017).
ISSN: 0370-3207

# Distribution of mass in high-dimensional convex bodies

David Alonso-Gutiérrez

Departamento de Matemáticas - I.U.M.A.
Universidad de Zaragoza
Campus Plaza San Francisco
50009 Zaragoza, Spain
alonsod@unizar.es

*Premio a la Investigación de la Academia 2017. Sección de Exactas*

**Resumen**

En este artículo exploramos la interacción entre la geometría convexa y la probabilidad en el estudio de la distribución de volumen en cuerpos convexos de alta dimensión. Por una parte, un cuerpo convexo $K$ en $\mathbb{R}^n$ se puede entender como un espacio de probabilidad cuando se considera la medida de Lebesgue normalizada. Por lo tanto, las herramientas probabilísticas son muy útiles en el estudio del comportamiento de un vector aleatorio uniformemente distribuido en $K$. Esto nos lleva al entendimiento de cómo el volumen se distribuye en un cuerpo convexo y a la obtención de desigualdades geométricas. Por otra parte, cuando se consideran marginales de menor dimensión de la medida de probabilidad uniforme sobre $K$, se abandona la clase de las probabilidades uniformes sobre cuerpos convexos pero se permanece en la clase de las probabilidades log-cóncavas. Muchas desigualdades geométricas se pueden extender al contexto de las probabilidades log-cóncavas, llevándonos a desigualdades funcionales para funciones log-cóncavas.

**Abstract**

In this paper we will explore the interaction between convex geometry and probability in the study of the distribution of volume in high-dimensional convex bodies. On

the one hand, a convex body $K$ in $\mathbb{R}^n$ can be understood as a probability space when the normalized Lebesgue measure is considered. Thus, probabilistic tools become very handy in the study of the behavior of a random vector uniformly distributed in $K$. This leads to the understanding of how the volume is distributed in a convex body and the obtention of geometric inequalities. On the other hand, when considering lower-dimensional marginals of the uniform probability measure on $K$, we leave the class of uniform probabilities on convex bodies but remain in the class of log-concave probabilities. Many geometric inequalities can be extended to the context of log-concave probabilities, leading to functional inequalities for log-concave functions.

## 1 Introduction

A convex body $K \subseteq \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which is convex, compact, and has non-empty interior. For instance, the closed unit ball of any norm in $\mathbb{R}^n$ is a centrally symmetric convex body. Conversely, any centrally symmetric convex body is the closed unit ball of the norm defined by the Minkowski gauge

$$\|x\| := \inf\{\lambda \geq 0 : x \in \lambda K\}.$$

In this paper we will deal with questions that arise from the general problem of understanding how the mass is distributed in a high-dimensional convex body or, equivalently, how a random vector uniformly distributed in $K$ behaves. We say that a convex body $K$ is isotropic if it has volume (i.e., Lebesgue measure) $|K| = 1$, and satisfies the following two conditions

- $\displaystyle\int_K x dx = 0$
- $\displaystyle\int_K \langle x, \theta\rangle^2 dx = L_K^2$ for every $\theta \in S^{n-1}$.

Here $S^{n-1}$ denotes the Euclidean sphere in $\mathbb{R}^n$. These two conditions mean that if $X$ is a random vector uniformly distributed on $K$, then for every $\theta \in S^{n-1}$ the one-dimensional random variable $\langle X, \theta\rangle$ is centered and has variance $L_K^2$, independent of the direction $\theta$. This constant $L_K$ is called the isotropic constant of the convex body $K$.

It is clear from the definition that if $K$ is an isotropic convex body then, $\int_K |x|^2 dx = nL_K^2$ and that for any orthogonal map $U \in O(n)$, $U(K)$ is also isotropic and has the same isotropic constant. Besides, it is known that for any non-necessarily isotropic convex body $K$ there exists a unique (up to orthogonal transformations) non-degenerate affine map $T \in GL(n)$ such that $T(K)$ is isotropic. Consequently, the fact that a convex body is isotropic can be regarded as a normalization condition and for every convex body $K$ its isotropic constant can be defined as the isotropic constant of its isotropic image. Furthermore, this isotropic constant appears also as the solution to a minimization problem and can be defined as

$$nL_K^2 := \min\left\{ \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{a+TK} |x|^2 dx \,:\, a \in \mathbb{R}^n, T \in GL(n) \right\}. \tag{1}$$

It is widely-known (see e.g. [37]) that among all the $n$-dimensional convex bodies, the one with the smallest isotropic constant is the Euclidean ball $B_2^n$, whose value is greater than an absolute constant. i.e., a constant whose value does not depend neither on the dimension or any other parameter.

$$L_K \geq L_{B_2^n} = \frac{\Gamma\left(1+\frac{n}{2}\right)^{\frac{1}{n}}}{\sqrt{\pi(n+2)}} \geq c.$$

However, it is not known which convex body maximizes the value of the isotropic constant, neither whether its value is bounded from above by an absolute constant or not. This is the statement of the following conjecture, which was posed by Bourgain in [23].

**Conjecture 1.1** (Hyperplane conjecture)**.** *There exists an absolute constant $C$ such that for every $n \in \mathbb{N}$ and every $n$-dimensional convex body $K$*

$$L_K \leq C.$$

This conjecture is known as the hyperplane conjecture since it is equivalent to the following conjecture related to the maximal volume hyperplane section of a convex body.

**Conjecture 1.2** (Hyperplane conjecture)**.** *There exists an absolute constant $c$ such that for every $n \in \mathbb{N}$ and every $n$-dimensional convex body $K$ there exists a hyperplane $H$ such that*

$$|K \cap H| \geq c|K|^{\frac{n-1}{n}}.$$

The hyperplane conjecture has been proved to be true when we restrict ourselves to many families of convex bodies. However, the best general upper bound for the isotropic constant of $n$-dimensional convex bodies is due to Klartag [32] and gives an estimate depending on the dimension $L_K \leq Cn^{\frac{1}{4}}$. This estimate improves the one of the order $L_K \leq Cn^{\frac{1}{4}} \log n$, which had been given by Bourgain. Very recently, Lee and Vempala [35] gave a different proof of Klartag's estimate using techniques from stochastic differential equations following ideas of Eldan.

A random vector $X$ in $\mathbb{R}^n$ is said to be log-concave if it is distributed according to a log-concave probability measure, i.e., a measure $d\mu$ with a density with respect to the Lebesgue measure

$$d\mu(x) = e^{-u(x)}dx,$$

where $u : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. For instance, a random vector uniformly distributed on a convex body $K$ is log-concave, since it is distributed according to the probability measure $d\mu(x) = e^{-u(x)}dx$, where $u(x) = \log|K|$ if $x \in K$ and $u(x) = \infty$ if $x \notin K$. Furthermore, for any $k$-dimensional linear subspace $E \in G_{n,k}$, $P_E(X)$ is not uniformly distributed on $P_E(K)$, but it is a log-concave random vector. In fact, the class of log-concave random vectors is the smallest class, closed under limits, that contains the linear projections of random vectors uniformly distributed on convex bodies.

A log-concave random vector $X$ in $\mathbb{R}^n$ is called isotropic if

- $\mathbb{E}X = 0$
- $\mathbb{E}\langle X, \theta\rangle^2 = 1$ for every $\theta \in S^{n-1}$.

For instance, a random vector uniformly distributed on $L_K^{-1}K$, where $K$ is an isotropic convex body or a standard Gaussian random vector in $\mathbb{R}^n$ are isotropic. Conjecture 1.1 can be formulated in the more general setting of log-concave random vectors. This conjecture, as stated before in the context of convex bodies, where the normalization is different than in the setting of log-concave random vectors (notice the different value for $\mathbb{E}\langle X, \theta\rangle^2$ in both settings), is a conjecture regarding the behavior of the linear functional $f_\theta = \langle \cdot, \theta\rangle$, when it is applied to an isotropic log-concave random vector. More precisely, about its variance (under the normalization considered for convex bodies).

Another important conjecture in Asymptotic Geometric Analysis regards the behavior of the functional $f = |\cdot|^2$ applied to an isotropic log-concave random vector $X$. From the definition of isotropicity $\mathbb{E}|X|^2 = n$. The conjecture says that

**Conjecture 1.3** (Variance conjecture). *There exists an absolute constant $C$ such that for every $n \in \mathbb{N}$ and any isotropic log-concave random vector in $\mathbb{R}^n$*

$$Var\ |X|^2 \leq C\mathbb{E}|X|^2 = Cn.$$

This conjecture was considered by Bobkov and Koldobsky in the context of the Central Limit Problem for isotropic convex bodies, in which the question of finding directions $\theta \in S^{n-1}$ for which the random variable $\langle X, \theta \rangle$ is almost Gaussian, was considered (see [22]). The variance conjecture is a stronger conjecture than the hyperplane conjecture (see [27]) and the best general estimate of the constant in the inequality is due to Lee and Vempala [35] gives

$$\mathrm{Var}\ |X|^2 \leq Cn^{\frac{1}{2}}\mathbb{E}|X|^2$$

for every $n$-dimensional isotropic log-concave random vector. There is a third conjecture, regarding the behavior of $f(X)$ for any integrable locally Lipschitz function $f$ which states the following

**Conjecture 1.4** (Kannan-Lovász-Simonovits conjecture). *There exists an absolute constant $C$ such that for every $n \in \mathbb{N}$, any centered log-concave random vector in $\mathbb{R}^n$, and any integrable locally Lipschitz function $f$*

$$Var\ f(X) \leq C\lambda_X^2\mathbb{E}|\nabla f(X)|^2,$$

*where $\lambda_X^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}\langle X, \theta \rangle^2$.*

This conjecture was proposed by Kannan-Lovász and Simonovits in the context of Theoretical Computer Science in relation to the problem of finding an efficient algorithm to compute the volume of a convex body $K$. It is equivalent to the following conjecture, regarding the value of the constant in a Cheeger isoperimetric type inequality

**Conjecture 1.5** (Kannan-Lovász-Simonovits conjecture). *There exists an absolute constant $C$ such that for every $n \in \mathbb{N}$, any centered log-concave probability measure $\mu$ in $\mathbb{R}^n$, and any Borel set $A \subseteq \mathbb{R}^n$ with $\mu(A) \leq \frac{1}{2}$*

$$\mu^+(A) \geq \frac{C}{\lambda_\mu}\mu(A)$$

*where* $\lambda_\mu^2 = \sup_{\theta\in S^{n-1}} \mathbb{E}\langle X,\theta\rangle^2$ *with* $X$ *distributed according to* $\mu$ *and* $\mu^+(A)$ *is defined as*

$$\mu^+(A) = \liminf_{\varepsilon\to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

*being* $A^\varepsilon = \{x \in \mathbb{R}^n : d(x,A) \leq \varepsilon\}$.

The best known value of the constant in Conjecture 1.4 was proved by Lee and Vempala [35] and gives an inequality

$$\text{Var } f(X) \leq C n^{\frac{1}{2}} \lambda_X^2 \mathbb{E}|\nabla f(X)|^2$$

for every isotropic log-concave random vector $X$ and any integrable locally Lipschitz function $f$. Notice that since Conjecture 1.4 involves every integable locally Lipschitz function $f$, by changing variables it can be considered only for isotropic random vectors. Notice also that Conjecture 1.3 is the particular case of Conjecture 1.4 for $f(x) = |x|^2$ and $X$ isotropic. Thus, the Kannan-Lovász Simonovits (KLS) conjecture is stronger than the variance conjecture. In [26], Eldan proved that the variance conjecture implies the KLS conjecture up to a logarithmic factor in the value of the constant. Besides, in [21] the authors proved that if a particular family of log-concave random vectors verify the KLS conjecture, then the same family of log-concave vectors verify the hyperplane conjecture. We refer the reader to [5] for an overview on these conjectures and related topics.

In the three aforementioned conjectures, the distribution of mass on a convex body (or the distribution of a log-concave random vector $X$) is studied via studying the random variable $f(X)$, where $f$ is some function from $f : \mathbb{R}^n \to \mathbb{R}$. A second approach in order to understand the distribution of a log-concave random vector $X$ is the following: Take $X_1, \ldots, X_N$ independent copies of $X$, construct a convex body with these random vectors, and study the geometric properties of such random convex body. Typically, the random convex bodies that are considered are random polytopes, defined as

$$K_N := \text{conv}\{\pm X_1, \ldots, \pm X_N\}, \quad L_N := \text{conv}\{X_1, \ldots, X_N\}$$

where conv$A$ denotes the intersection of all the convex sets that contain $A$, and is the smallest convex set containing $A$.

A third approach in the study of the distribution of log-concave vectors is the study of their density functions, i.e., the study of log-concave functions and the inequalities that they verify. It is well-known that many geometric inequalities have their functional counterpart. For instance, consider the classical isoperimetric inequality, which states that for every bounded Borel set $A \subseteq \mathbb{R}^n$, if $|\partial A|$ denotes the surface measure of $A$, which can be defined as $m^+(A)$ being $m$ the Lebesgue measure, then

$$|\partial A| \geq n|B_2^n|^{\frac{1}{n}}|A|^{\frac{n-1}{n}}. \tag{2}$$

The isoperimetric inequality is equivalent to the following Sobolev inequality, which states that for every compactly supported smooth function $f$

$$\||\nabla f|\|_1 \geq n|B_2^n|^{\frac{1}{n}}\|f\|_{\frac{n}{n-1}}. \tag{3}$$

The set of log-concave integrable functions contains the set of convex bodies that contain the origin via one of the following injections

- $K \to \chi_K$
- $K \to e^{-\|x\|_K}$

and, as discussed before, it is the smallest class of functions, closed under limits, that contains the densities of the marginals of uniform measures on higher dimensional convex bodies. In view of such a close relation between convex bodies and log-concave functions, it is reasonable to ask whether some purely geometric inequalities and concepts could be extended to the more general framework of log-concave functions.

In this paper we will explore these three approaches and discuss some of the most recent results proved by the author. The paper is organised as follows: In Section 2 we will discuss our results related to the three aforementioned conjectures. Section 3 will be devoted to our results concerning the geometry of random convex bodies. This section will be divided in two subsections, being the first of them devoted to the study of the hyperplane conjecture on random polytopes, and the second one of them to the study of some geometric parameters of random convex bodies. Finally, in Section 4 we will study extensions of purely geometric concepts and inequalities to the framework of log-concave functions.

## 2 The hyperplane conjecture, the variance conjecture, and the KLS conjecture

In this section we are going to explore the first of the previously mentioned approaches for the study of the distribution of mass in an $n$-dimensional convex body, i.e., the study of the distribution of functionals applied to an isotropic random vector.

Let us recall that a polytope $P$ is the convex hull of a finite number of points $P = \text{conv}\{P_1, \ldots, P_N\}$ and that for any convex body $K \subseteq \mathbb{R}^n$ and any $\varepsilon > 0$, there exists a polytope $P_\varepsilon$ such that

$$P_\varepsilon \subseteq K \subseteq (1+\varepsilon)P_\varepsilon.$$

Since there exists an affine map $T$ such that $TP_\varepsilon$ is isotropic, taking into account (1), we have that

$$\begin{aligned} nL_K^2 &\leq \frac{1}{|TK|^{1+\frac{2}{n}}} \int_{TK} |x|^2 dx \leq \frac{1}{|TP_\varepsilon|^{1+\frac{2}{n}}} \int_{(1+\varepsilon)TP_\varepsilon} |x|^2 dx \\ &= \frac{(1+\varepsilon)^{n+2}}{|TP_\varepsilon|^{1+\frac{2}{n}}} \int_{TP_\varepsilon} |x|^2 dx = n(1+\varepsilon)^{n+2} L_{P_\varepsilon}^2. \end{aligned}$$

Choosing $\varepsilon = \frac{1}{n}$ we obtain the following

**Proposition 2.1.** *There exists an absolute constant $C$ such that for any convex body $K \subseteq \mathbb{R}^n$ there exists a polytope $P$ such that*

$$L_K \leq C L_P.$$

As a consequence of this proposition, if the family of polytopes verify Conjecture 1.1, then every convex body does. In [2], the following to estimates for the isotropic constant of a polytope were proved, extending known results for symmetric polytopes to the case of non-necessarily symmetric polytopes. The first one gives an estimate in terms of the number of vertices.

**Theorem 2.2.** *There exists an absolute constant $C$ such that for every polytope $P \subseteq \mathbb{R}^n$ with $N$ vertices we have*

$$L_P \leq C \log N.$$

The second one gives an estimate in terms of the number of facets ($(n-1)$-dimensional faces).

**Theorem 2.3.** *There exists an absolute constant $C$ such that for every polytope $P \subseteq \mathbb{R}^n$ with $N$ facets we have*

$$L_P \leq C\sqrt{\log \frac{N}{n}}.$$

In [8], we followed a different approach to give an upper bound for the isotropic constant of a polytope in terms of the number of vertices and obtained the following result, which proves that the class of polytopes with a number of vertices proportional to the dimension verify Conjecture 1.1.

**Theorem 2.4.** *There exists an absolute constant $C$ such that for every polytope $P \subseteq \mathbb{R}^n$ with $N$ vertices we have*

$$L_P \leq C\sqrt{\frac{N}{n}}.$$

The approach followed to prove Theorem 2.4 for symmetric polytopes is the following: For every symmetric polytope $P$ with $N$ vertices there exists an $n$-dimensional linear subspace $E$ of $\mathbb{R}^N$, $E \in G_{N,n}$ and an isomorphism $T : \mathbb{R}^n \to E$ such that $TP = P_E B_1^N$, the projection onto $E$ of $B_1^N = \{x \in \mathbb{R}^N : \|x\|_1 \leq 1\}$. Thus, in view of (1),

$$nL_P^2 \leq \frac{1}{|P_E B_1^N|^{\frac{2}{n}}} \frac{1}{|P_E B_1^N|} \int_{P_E B_1^N} |x|^2 dx.$$

In order to bound this quantity from above by a quantity independent of $E$, we bound the volume of any projection of $B_1^N$ form below using the fact that $B_1^N$ contains a Euclidean ball of radius $\frac{1}{\sqrt{N}}$ and then there is an absolute constant $c$ such that

$$|P_E B_1^N|^{\frac{2}{n}} \geq \frac{1}{N}|P_E B_2^N|^{\frac{2}{n}} = \frac{1}{N}|B_2^n|^{\frac{2}{n}} \geq \frac{c}{Nn},$$

and bound from above the average of $|x|^2$ over $P_E B_1^N$. To do that, we extend Cauchy's formula for projections onto hyperplanes and show that $P_E B_1^N$ can be decomposed as the disjoint union (up to measure 0 sets) of projections of some $n$-dimensional faces of $B_1^N$, where for each face $F$, $P_E : F \to P_E F$ is an affine isomorphism. This leads us to the existence of some numbers $\{c_i\}_{i=1}^l$ and some $n$-dimensional faces $\{F_i\}_{i=1}^l$ such that $\sum_{i=1}^l c_i = 1$ and

$$\frac{1}{|P_E B_1^N|} \int_{P_E B_1^N} |x|^2 dx = \sum_{i=1}^l c_i \frac{1}{|F_i|} \int_{F_i} |P_E x|^2 dx \leq \sum_{i=1}^l c_i \frac{1}{|F_i|} \int_{F_i} |x|^2 dx.$$

Since every $n$-dimensional face of $B_1^N$ is a regular simplex $\Delta_n$, for every $1 \leq i \leq l$

$$\frac{1}{|F_i|}\int_{F_i}|x|^2dx = \frac{1}{|\Delta_n|}\int_{\Delta_n}|x|^2dx$$

and then

$$\frac{1}{|P_E B_1^N|}\int_{P_E B_1^N}|x|^2dx \leq \frac{1}{|\Delta_n|}\int_{\Delta_n}|x|^2dx = \frac{2}{n+2},$$

which gives us the estimate. in the case of non-symmetric polytopes the same approach is taken but considering $n$-dimensional projections of the $N$-dimensional regular simplex.

The same approach as the one used in the proof of the latter theorem has been considered in order to study the variance conjecture (Conjecture 1.3) for projections of $B_1^n$ (and more generally of $B_p^n = \{x \in \mathbb{R}^n : |x|_P \leq 1\}$. In such case, projections of $B_p^n$ onto lower-dimensional subspaces are not isotropic in general. However, since Conjecture 1.3 is the particular case of Conjecture 1.4 for $f = |\cdot|^2$ and isotropic vectors, one can also consider the same particular case without the isotropicity condition, which leads us to

**Conjecture 2.5** (General variance conjecture)**.** *There exists an absolute constant $C$ such that for every $n \in \mathbb{N}$ and any log-concave random vector in $\mathbb{R}^n$*

$$Var\,|X|^2 \leq C\lambda_X^2 \mathbb{E}|X|^2,$$

*where $\lambda_X^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}\langle X, \theta\rangle^2$.*

This conjecture can be considered for random vectors uniformly distributed on projections of the $B_p^n$. In the case of projections onto hyperplanes, we prove the following

**Theorem 2.6.** *For every $1 \leq p \leq \infty$, there exists a constant $C_p$ such that for any hyperplane $H$, if $X$ is a random vector uniformly distributed on $P_H B_p^n$*

$$Var\,|X|^2 \leq C_p \lambda_X^2 \mathbb{E}|X|^2.$$

*Furthermore, $C_p \leq C\log(1+p)$ if $1 \leq p \leq n$ and $C_p \leq C$ if $p \geq n$, where $C$ is an absolute constant.*

Notice that even though the constant $C_p$ depends on the value of $p$, it does not depend on the value of $n$ or on the hyperplane $H$. The cases $p = 1, \infty$, were considered in [4]. In these cases we are considering families of polytopes and the same approach as in Theorem 2.4 could be used. The rest of the cases were considered in [6], where a similar approach was taken, relying heavily in a probabilistic representation of the cone measure on the boundary of $B_p^n$.

In the case $p = \infty$, this approach can be used for lower-dimensional projections. In [7] we have shown the following

**Theorem 2.7.** *There exists an absolute constant $C$ such that for any $1 \leq k \leq \sqrt{n}$ and any $E \in G_{n,n-k}$, if $X$ is a random vector uniformly distributed on $P_E B_p^n$ then*

$$Var\,|X|^2 \leq C\lambda_X^2 \mathbb{E}|X|^2.$$

## 3 The geometry of random convex sets

In this section we explore the second mentioned approach for the study of the distribution of log-concave random vectors. Given a log-concave random vector $X$, we will study the geometry of a random convex body $K_N$ generated by $N$ copies of $X$. Many geometric parameters can be considered, such as the volume, the surface area or the mean width and, since these geometric parameters of random convex bodies are random variables, their expectation, variance, or whether their value is of some order with high probability (tending to 1 as the dimension grows to infinity) is studied.

We treat separately the case of the isotropic constant of random polytopes. The purpose which started the study of this parameter was the search for counterexamples for the hyperplane conjecture, since even though an explicit construction would not be given, maybe one could show that with positive probability there is an $n$-dimensional random polytope $P_n \subseteq \mathbb{R}^n$ such that $L_{P_n} \geq C_n$, with $C_n$ tending to $\infty$. On the contrary, for the distributions considered up to now it turned out that the isotropic constant is bounded with high probability. This will be treated in the first subsection, while the second subsection will be devoted to the study of expectations of geometric parameters, focusing on the mean width.

### *3.1 The hyperplane conjecture on random polytopes*

Since the remarkable result of Gluskin on the diameter of the diameter of the Minkowski compactum, [29] random polytopes are known to provide many examples of convex bodies (and related normed spaces) with a pathologically bad behavior of various parameters of a linear and geometric nature (we refer to the survey [36] and references therein). For this reason, they were a natural candidate for a potential counterexample for the hyperplane conjecture and in [34] the authors studied the value of the isotropic constant of Gaussian random polytopes and showed that they did not provide a counterexample. On the contrary, they verified the hyperplane conjecture with high probability. More precisely, they showed

**Theorem 3.1** (Klartag-Kozma, (2006)). *Let $X_1, \dots X_N$ be independent copies of a standard Gaussian random vector $X$ in $\mathbb{R}^n$ ($N \geq n$) and let $K_N = conv\{\pm X_1, \dots, \pm X_N\}$. Then*

$$L_{K_N} \leq C$$

*with probability greater than $1 - c_1 e^{-c_2 n}$, where $C, c_1, c_2$ are absolute constants.*

In the same paper the authors also considered the non-symmetric case, as well as some other distributions for the random vector $X$. In all the cases considered in [34] the random vector $X$ had independent coordinates. Motivated by this result, other distributions for the random vector $X$, with non-necessarily independent coordinates, were considered. Following the approach initiated by Klartag and Kozma, we proved in [1] that random polytopes generated by random vectors uniformly distributed on the Euclidean sphere $S^{n-1}$ also verify the hyperplane conjecture with high probability. Namely, we proved the following

**Theorem 3.2.** *Let $X_1, \dots X_N$ be independent copies of a random vector $X$ uniformly distributed on $S^{n-1}$ ($N \geq n$) and let $K_N = conv\{\pm X_1, \dots, \pm X_N\}$. Then*

$$L_{K_N} \leq C$$

*with probability greater than $1 - c_1 e^{-c_2 n}$, where $C, c_1, c_2$ are absolute constants.*

In the case in which $X$ is an isotropic log-concave vector, the following estimate was proved in [14]

**Theorem 3.3.** *Let $X_1, \dots, X_N$ be independent copies of an isotropic log-concave random vector $X$ ($N \geq n$) and let $K_N = conv\{\pm X_1, \dots, \pm X_N\}$. Then*

$$L_{K_N} \leq C\sqrt{\log \frac{2N}{n}}$$

*with probability greater than* $1 - c_1 e^{-c_2\sqrt{n}}$, *where* $C, c_1, c_2$ *are absolute constants.*

This estimate was simultaneously proved in [28] and improves the ones known for general polytopes in the range $n \leq N \leq e^{\sqrt{n}}$. Besides, other distributions for $X$ have been considered, obtaining similar results. For instance, random vectors inside an unconditional convex body (invariant under reflections on the coordinate hyperplanes) (see [24]), or random vectors uniformly distributed on the boundary of $B_p^n$ distributed according to the cone measure, extending the result of Theorem 3.2 (see [30]).

### *3.2 Geometric parameters of random convex bodies*

In this section we will study the expectation of some random parameters associated to random convex bodies generated by log-concave random vectors. We will focus on the mean width. First of all, let us give some definitions.

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is the set $A + B := \{x + y : x \in A, y \in B\}$. If $A$ and $B$ are convex bodies then also $A+B$ is. Steiner's formula says that for any convex body $K$, the volume of $K + tB_2^n$ is a polynomial of degree $n$ in $t$.

$$|K + tB_2^n| = \sum_{k=0}^{n} \binom{n}{k} W_k(K) t^k.$$

The numbers $W_k(K)$ that appear as coefficients of this polynomial are called the quermaßintegrals of $K$ and can be interpreted geometrically in several ways. For instance, $W_0(K)$ is the volume of $K$, $nW_1(K)$ is the surface area of $K$ and $W_{n-1}(K)$ equals $2w(K)$, where $w(K)$ is the mean width of the convex body $K$, i.e., half the average over the sphere of the distance between the closest hyperplanes orthogonal to $\theta$ that have $K$ contained in between.

Many identities and inequalities are known to be satisfied between the quermaßintegrals of a convex body. In particular, calling $Q_k(K) = \left(\frac{W_{n-k}(K)}{|B_2^n|}\right)^{\frac{1}{k}}$, we have, by Kubota's formula, that

$$Q_k(K) = \left( \frac{1}{|B_2^k|} \int_{G_{n,k}} |P_E(K)| d\mu(E) \right)^{\frac{1}{k}},$$

where $d\mu$ is the Haar probability measure on the Grassmanian manifold. Besides, with this notation, we have that the sequence $Q_k(K)$ is decreasing in $k$. In [25], the authors proved the following

**Theorem 3.4** (Dafnis, Giannopoulos, Tsolomitis (2013)). *Let $X_1, \dots, X_N$ be independent copies of an isotropic log-concave random vector $X$ and let $K_N = conv\{\pm X_1, \dots, \pm X_N\}$. Then, if $n \le N \le e^{\sqrt{n}}$ we have that*

$$c_1 \sqrt{\log \frac{2N}{n}} \le \mathbb{E}Q_n(K_N) \le \cdots \le \mathbb{E}Q_1(K_N) \le c_2 \sqrt{\log N}.$$

In particular, if $n^2 \le N \le e^{\sqrt{n}}$ we have that for every $1 \le k \le n$ $\mathbb{E}Q_k(K_N) \sim \sqrt{\log N}$. However, this estimate is not sharp in the range $n \le N \le n^2$. In [15] and [16] we took care of the case $k = 1$ in this range of $N$. Notice that

$$Q_1(K_N) = w(K_N) = \int_{S^{n-1}} h_{K_N}(\theta) d\sigma(\theta),$$

where

$$h_{K_N}(\theta) = \max_{1 \le i \le N} |\langle X_i, \theta \rangle|$$

and $\sigma$ denotes the uniform probability measure on $S^{n-1}$. Thus, the expected value of the mean width of a random polytope is an average over the sphere of the expected value of the maximum of the random variables $|\langle X_i, \theta \rangle|$. Besides, for any vector $y \in \mathbb{R}^N$, the expected value of $\max_{1 \le i \le N} y_i |\langle X_i, \theta \rangle|$ is approximately equal (up to absolute constants) to the value of the Luxemburg norm

$$\|y\|_{M_\theta} := \inf \left\{ s > 0 \, : \, \sum_{i=1}^N M_\theta \left( \frac{|y_i|}{s} \right) \le 1 \right\},$$

where $M_\theta$ is an Orlicz function that depends on the distribution of $\langle X, \theta \rangle$. Thus,

$$\mathbb{E}h_{K_N}(\theta) \sim \|(1, \dots, 1)\|_{M_\theta}.$$

Using this representation for the expected value of the support function of a random polytope, we computed in [15] the expected value of the support function in the coordinate directions of a random polytope generated by random vectors uniformly distributed in $B_p^n$.

**Theorem 3.5.** *There exist absolute constants $c, c_1, c_2$ such that if $X_1, \dots X_N$ are independent copies of a random vector $X$ uniformly distributed on $|B_p^n|^{-\frac{1}{n}} B_p^n$ ($n \le N \le e^{cn}$) and*

$K_N = conv\{\pm X_1, \dots, \pm X_N\}$, *then*

$$c_1(\log N)^{\frac{1}{p}} \leq \mathbb{E}h_{K_N}(e_1) \leq c_2(\log N)^{\frac{1}{p}}.$$

In [16] we used the representation of the expected value of the support function of $K_N$ as an Orlicz norm, together with the Central Limit Theorem proved by Klartag [33], to show that the expected value of the mean width of a random polytope generated by isotropic log-concave random vectors is of order $\sqrt{\log N}$ also in the range $n \leq N \leq n^2$.

**Theorem 3.6.** *There exist absolute constants $c_1, c_2$ such that if $X_1, \dots X_N$ are independent copies of an isotropic log-concave random vector $X$ ($n \leq N \leq e^{\sqrt{n}}$) and $K_N = conv\{\pm X_1, \dots, \pm X_N\}$, then*

$$c_1\sqrt{\log N} \leq \mathbb{E}w(K_N) \leq c_2\sqrt{\log N}.$$

The aforementioned representation leads in a natural way to consider the Luxemburg norm of other vectors and not only of the vector $(1, \dots, 1)$. This corresponds geometrically with considering the expected value of a perturbed random polytope. If the random vector encoding the perturbation is chosen at random, several high probability results were given in [17] depending on the distribution of the perturbation vector. For instance, for Gaussian perturbation we have the following

**Theorem 3.7.** *Let $X_1, \dots, X_N$ be independent copies of an isotropic log-concave random vector in $\mathbb{R}^n$ ($n \leq N \leq e^{\sqrt{n}}$) and let $G$ be a Gaussian random vector in $\mathbb{R}^N$ independent of $X_1, \dots, X_N$. Denote by $K_{N,G} = conv\{\pm G_1X_1, \dots, \pm G_NX_N\}$. Then there exist absolute constants $c, c_1, c_2$ such that for every $t > 0$*

$$\mathbb{P}_G\left(c_1(1-t) \leq \frac{\mathbb{E}_{X_1,\dots,X_N}w(K_{N,G})}{\log N} \leq c_2(1+t)\right) \geq 1 - \frac{1}{N^{ct^2}}.$$

Also the case in which the perturbation vector is uniformly distributed on $S^{N-1}$ or in $B_p^N$ were considered.

Besides considering the random polytope generated by the random vectors, one can consider other convex bodies whose support function not only involves the maximum of the random variables $|\langle X_i, \theta\rangle|$, but other order statistics. More precisely, given $X_1, \dots X_N \in \mathbb{R}^n$, for any $1 \le \ell \le N$ and any $q \ge 1$, we consider $K_{N,\ell,q}$ the convex body whose support function is given by

$$h_{K_{N,\ell,q}}(\theta) := \left( \frac{1}{\ell} \sum_{k=1}^{\ell} \underset{1\le i\le N}{\text{k-max}} |\langle X_i, \theta\rangle|^q \right)^{1/q}.$$

If $\ell = 1$ this convex body is the polytope $\operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$. Taking $X_1, \dots, X_N$ random vectors, $K_{N,\ell,q}$ is a random convex body. In the following theorem, proved in [18], we extend the result of Theorem 3.6 to this larger family of random convex bodies

**Theorem 3.8.** *Let $n, N \in \mathbb{N}$ with $n \le N \le e^{\sqrt{n}}$ and let $X_1, \dots, X_N$ be independent copies of an isotropic log-concave random vector $X$ in $\mathbb{R}^n$. Then, for all $1 \le \ell \le N$ and any $q \ge 1$,*

$$c_1 \min\left\{\max\left\{\sqrt{q}, \sqrt{\log(N/\ell)}\right\}, \sqrt{\log N}\right\} \le \mathbb{E}\, w\big(K_{N,\ell,q}\big) \le c_2 \min\left\{\max\left\{\sqrt{q}, \sqrt{\log(N/\ell)}\right\}, \sqrt{\log N}\right\},$$

*where $c_1, c_2$ are absolute constants.*

In [9] we considered a different geometric parameter, the so called mean outer radii of a convex body. For a convex body $K \subseteq \mathbb{R}^n$, the $k$-th mean outer radius of $K$, $1 \le k \le n$, is defined as

$$\widetilde{R}_k(K) = \int_{G_{n,k}} R(P_E K)\, d\mu(E),$$

where $R(P_E K)$ denotes the circumradius of the projection of $K$ onto $E$, i.e., the radius of the smallest Euclidean ball containing $P_E K$. We showed the following

**Theorem 3.9.** *There exist absolute constants $c_1, c_2$ such that if $X_1, \dots X_N$ are independent copies of an isotropic log-concave random vector $X$ ($n \le N \le e^{\sqrt{n}}$) and $K_N = \operatorname{conv}\{\pm X_1, \dots, \pm X_N\}$, then for every $1 \le k \le n$*

$$c_1 \max\left\{\sqrt{k}, \sqrt{\log N}\right\} \le \mathbb{E}\widetilde{R}_k(K_N) \le c_2 \max\left\{\sqrt{k}, \sqrt{\log N}\right\}.$$

Furthermore, a high probability result was proved which gives the right order as long as $n^2 \le N \le e^{\sqrt{n}}$. A different proof of this result has later been shown in [28].

## 4 Geometric inequalities for log-concave functions

In this section we will explore the third approach mentioned in the introduction and see how some geometric concepts and inequalities are extended to the setting of log-concave functions, which are the densities of lower-dimensional projections of random vectors uniformly distributed on convex bodies. We will focus on the concept of volume ratio and on Rogers-Shephard type inequalities.

By John's theorem [31], for every convex body $K \subseteq \mathbb{R}^n$ there exists a unique ellipsoid $\mathcal{E}(K)$ that maximizes the volume among all the ellipsoids contained in $K$. The volume ratio of a convex body $K \subseteq \mathbb{R}^n$ is then defined as

$$\text{v.rat}(K) := \left( \frac{|K|}{|\mathcal{E}(K)|} \right)^{\frac{1}{n}}.$$

This is an affine invariant quantity, which can be used to measure how far is $K$ from an ellipsoid. If $\mathcal{E}(K) = B_2^n$ it is said that $K$ is in John's position and this position is characterized by the fact that the Euclidean ball is contained in $K$, together with the existence of some contact points $u_1, \dots u_m \in \partial K \cap S^{n-1}$ that give a decomposition of the identity. This characterization, together with Brascamp-Lieb inequality provide the convex bodies that maximize the volume ratio (see [20])

- $\text{v.rat}(K) \le \text{v.rat}(B_\infty^n)$ if $K$ is centrally symmetric,
- $\text{v.rat}(K) \le \text{v.rat}(\Delta^n)$ if $K$ is not necessarily centrally symmetric.

When passing from the context of convex bodies to the context of integrable log-concave functions via the injection $K \to \chi_K$, it is clear that the concept of volume is generalized by the integral. We generalize the concept of ellipsoid by ellipsoidal functions, which are functions proportional to the characteristic function of an ellipsoid $\mathcal{E}^a = a\chi_{\mathcal{E}}$. In [12] we proved that

**Theorem 4.1.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable log-concave function. There exists a unique ellipsoidal function $\mathcal{E}(f) = \mathcal{E}^{t_0\|f\|_\infty}$ for some $t_0 \in [e^{-n}, 1]$, such that*

- $\mathcal{E}(f) \le f$
- $\displaystyle\int_{\mathbb{R}^n} \mathcal{E}(f)(x)dx = \max\left\{ \int_{\mathbb{R}^n} \mathcal{E}^a(x)dx \,:\, \mathcal{E}^a \le f \right\}.$

This ellipsoidal function is called the John's ellipsoid of $f$ and is denoted by $\mathcal{E}(f)$. Thus, we can define the integral ratio of an integrable log-concave function in a similar way as the volume ratio like

$$I.rat(f) = \left( \frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} \mathcal{E}(f)(x)dx} \right)^{\frac{1}{n}}.$$

We also showed which log-concave functions maximize the integral ratio in the even and non-even case.

**Theorem 4.2.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable log-concave function. Then,*

$$I.rat(f) \leq I.rat(g_c),$$

*where $g_c(x) = e^{-\|x\|_{\Delta_n - c}}$ for any $c \in \Delta^n$. Furthermore, there is equality if and only if $\frac{f}{\|f\|_\infty} = g_c \circ T$ for some affine map $T$ and some $c \in \Delta^n$. If we assume $f$ to be even, then*

$$I.rat(f) \leq I.rat(g),$$

*where $g(x) = e^{-\|x\|_{B_\infty^n}}$, with equality if and only if $\frac{f}{\|f\|_\infty} = g \circ T$ for some linear map $T \in GL(n)$.*

In [3], the concept of volume ratio was used to give a stability version of the affine isoperimetric inequality. Before stating it let us introduce some concepts. Given a convex body $K \subseteq \mathbb{R}^n$, its polar projection body $\Pi^*(K)$ is the unit ball of the norm defined by

$$\|x\|_{\Pi^*(K)} = |x||P_{x^\perp} K|.$$

The quantity $|K|^{n-1}|\Pi^*(K)|$ is an affine invariant, i.e., $|TK|^{n-1}|\Pi^*(TK)| = |K|^{n-1}|\Pi^*(K)|$ for any non-degenerate affine map $T$. Petty's projection inequality [38] states that for any convex body $K \subseteq \mathbb{R}^n$

$$|K|^{n-1}|\Pi^*(K)| \leq |B_2^n|^{n-1}|\Pi^*(B_2^n)|, \tag{4}$$

with equality if and only if $K$ is an ellipsoid. This inequality is known as the affine isoperimetric inequality and it implies the isoperimetric inequality (2). In [3], we gave the following stability version of Petty's projection inequality

**Theorem 4.3.** *Let $K \subseteq$ be a convex body. Then*

$$|K|^{n-1}|\Pi^*(K)| \geq \frac{1}{v.rat(K)^n}|B_2^n|^{n-1}|\Pi^*(B_2^n)|.$$

In the same way as the isoperimetric inequality (2) is equivalent to Sobolev's inequality (3), the affine isoperimetric inequality has an equivalent functional form (see [42], which states that for any $f \in W^{1,1}(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,:\, \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n) \quad \forall i \right\}$,

$$\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}} \leq \frac{|B_2^n|}{2|B_2^{n-1}|}, \tag{5}$$

where $\Pi^*(f)$ is the unit ball of the norm

$$\|x\|_{\Pi^*(f)} = \int_{\mathbb{R}^n} |\langle \nabla f(y), x\rangle| dy.$$

In the case of log-concave functions, we proved in [12] a stability version to this affine Sobolev's inequality, involving the integral ratio in the same spirit as Theorem 4.3.

**Theorem 4.4.** *Let $f \in W^{1,1}(\mathbb{R}^n)$ be a log-concave function. Then*

$$\frac{\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)} \geq \frac{1}{e^{\frac{\int_{\mathbb{R}^n} f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\int_{\mathbb{R}^n} f(x)dx}}\|f\|_\infty^{\frac{1}{n}}\left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}} I.rat(f)}.$$

Petty's projection inequality (4) has a reverse form, Zhang's inequality [41], which states that the affine invariant quantity $|K|^{n-1}|\Pi^*(K)|$ is minimized when $K$ is a simplex.

$$|K|^{n-1}|\Pi^*(K)| \geq |\Delta^n|^{n-1}|\Pi^*(\Delta^n)|. \tag{6}$$

In [13] we studied the properties of the $\theta$-convolution bodies of two convex bodies $K, L$, which are defined as

$$K +_\theta L = \{x \in K + L \,:\, |K \cap (x - L)| \geq \theta \max_{z\in\mathbb{R}^n} |K \cap (z - L)|\}$$

and gave a different proof and an extension of inequality (6) as well as recovered the classical Rogers-Shephard inequality (see [39] and [40]), which states that for any convex body $K \subseteq \mathbb{R}^n$

$$|K + (-K)| \leq \binom{2n}{n} |K| \tag{7}$$

and, more generally, for any two convex bodies $K, L \subseteq \mathbb{R}^n$

$$\max_{x_0 \in \mathbb{R}^n} |K \cap (x_0 - L)||K + L| \leq \binom{2n}{n} |K||L|. \tag{8}$$

In [10] we studied the properties of a more generally defined convolution bodies and proved the following extension of Rogers-Shephard inequality (7), which involves the surface area measure of convex bodies.

**Theorem 4.5.** *Let $K, L \subseteq \mathbb{R}^n$ be two convex bodies. Then*

$$|K + L| \leq \binom{2n}{n} \frac{|K||\partial L| + |L||\partial K|}{2 \max_{x_0 \in \mathbb{R}^n} |\partial (K \cap (x_0 - L))|}.$$

Rogers and Shephard also proved in [40] the following volume inequality for the volume of the convex hull of two convex bodies

$$|K \cap L||\text{conv}\{K, -L\}| \leq 2^n |K||L| \tag{9}$$

They proved that there is equality when $L = K$ in this inequality if and only if $K$ is a simplex with 0 as a vertex and they suggested that it is likely that equality is attained if and only if $K = L$ is a simplex and 0 is one of its vertices, but they did not prove that equality can only be attained when $L = K$.

In [11] we proved the following three theorems, which give functional extensions to (8), Theorem 4.5, and (9). The first theorem is the following, which extends (8). In the statement of the theorem $f * g$ is the convolution of $f$ and $g$, which is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(z) g(x - z) dz$$

and it represents the functional extension of $|K \cap (x - L)|$ for any $x \in \mathbb{R}^n$. $f \star g$ is the

Asplund product of two log-concave functions is defined by

$$f \star g(x) = \sup_{z \in \mathbb{R}^n} f(z)g(x-z)$$

and it represents the functional version of the sum in the context of log-concave functions, as $-\log f \star g$ is a convex function whose epigraph is the Minkowski sum of the epigraphs of $-\log f$ and $-\log g$.

**Theorem 4.6.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable log-concave functions with full-dimensional support such that $f$ and $g$ are continuous when restricted to their supports. Then*

$$\|f * g\|_\infty \int_{\mathbb{R}^n} f \star g(x)dx \leq \binom{2n}{n} \|f\|_\infty \|g\|_\infty \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(x)dx.$$

*Furthermore, this inequality becomes an equality if and only if $\frac{f(x)}{\|f\|_\infty} = \frac{g(-x)}{\|g\|_\infty}$ is the characteristic function of an $n$-dimensional simplex.*

The second theorem is the following, extending Theorem 4.5. In the notation used in the theorem, the quermaßintegral $W_1$ (surface area) of a log-concave function is defined by integrating the corresponding quermaßintegral on the level sets

$$W_1(f) := \int_0^\infty W_1(\{x \in \mathbb{R}^n \, : \, f(x) \geq t\})dt.$$

**Theorem 4.7.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable log-concave functions with full-dimensional support and continuous when restricted to their supports. Then*

$$\int_{\mathbb{R}^n} f \star g(x)dx \leq \binom{2n}{n} \|f\|_\infty \|g\|_\infty \frac{W_1(g) \int_{\mathbb{R}^n} f(x)dx + W_1(f) \int_{\mathbb{R}^n} g(x)dx}{2 \max_{x_0 \in \mathbb{R}^n} W_1(f(\cdot)g(x_0 - \cdot))}.$$

*Furthermore, when $n \geq 3$ this inequality becomes an equality if and only if $\frac{f(x)}{\|f\|_\infty} = \frac{g(-x)}{\|g\|_\infty}$ is the characteristic function of an $n$-dimensional simplex.*

The third theorem not only generalizes inequality (9) but strengthens it. It states the following

**Theorem 4.8.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two integrable log-concave functions with full-*

*dimensional supports and continuous when restricted to them. Then*

$$\int_{\mathbb{R}^n} \sqrt{f(x)g(-x)}dx \int_{\mathbb{R}^n} f \star g(2x)dx \leq 2^n \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(x)dx.$$

*Equality holds if and only if the following two conditions are satisfied:*

- *supp* $f = -$*supp* $g$ *is a translation of a cone* $C$ *with vertex at 0 with simplicial section, and*
- $f(x) = c_1 e^{-\langle a,x\rangle}$ *on supp* $f$ *and* $g(x) = c_2 e^{-\langle b,x\rangle}$ *on supp* $g$ *for some* $c_1, c_2 > 0$ *and some* $a, b \in \mathbb{R}^n$ *such that* $\langle a,x\rangle \geq 0 \geq \langle b,x\rangle$ *for every* $x \in C$.

A more general result, without characterizing the equality cases has been proved in [19]. As a consequence one obtains the following geometric inequality, which improves (9)

$$|K \cap L||\text{conv}\{K, -L\}| \leq \left|\left(\frac{K^\circ - L^\circ}{2}\right)^\circ\right| |\text{conv}\{K, -L\}| \leq 2^n |K||L|,$$

and allows the characterization of the equality cases in (9) that were conjectured by Rogers and Shephard. Here $K^\circ$ denotes the polar body of a convex body that contains the origin is defined as

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \, \forall y \in K\}.$$

When $K$ is centrally symmetric it is the unit ball of the dual norm associated to $K$.

**Acknowledgments.** I would like to thank all the members of 'Real Academia de Ciencias Exactas, Físicas, Químicas y Naturales de Zaragoza', specially the members of the 'Sección de Exactas' for nominating me for the Academy award. I would also like to thank Prof. Jesús Bastero and Prof. Julio Bernués for many useful and helpful discussions and advices along my whole researcher career as well as the rest of the members of the 'Departamento de Matemáticas' (in special the 'Análisis Matemático' section) at 'Universidad de Zaragoza', for creating such a good working environment. I wish to thank Professors Apostolos Giannopoulos, Franck Barthe, Keith Ball, and Alain Pajor, who took me as an early stage researcher in their institutions during the time I was struggling to obtain my PhD, as well as Professors Nicole Tomczak-Jaegermann and Alexander Litvak, who took me as a postdoctoral fellow the years after obtaining my PhD, and Prof. María Ángeles Hernández-Cifre, who supervised and collaborated with my work during my period as a

'Juan de la Cierva' researcher. Finally, I would also like to thank the rest of my collaborators, Nikos Dafnis, Bernardo González, Carlos Hugo Jiménez, Joscha Prochno, Christoph Thäle, Pawel Wolff, and Rafael Villa, for many useful discussions. This work is partially supported by MINECO/FEDER projects MTM2016-77710-P.

## References

[1] Alonso-Gutiérrez D. *On the isotropy constant of random convex sets.* Proceedings of the AMS. **136** (9), (2008), pp. 3293–3300.

[2] Alonso-Gutiérrez D. *A remark on the isotropy constant of polytopes.* Proceedings of the AMS **139** (7), (2011),pp. 2565–2569.

[3] Alonso-Gutiérrez D. *On a reverse Petty projection inequality for projections of convex bodies.* Advances in Geometry **14**, (2014) pp. 215–223.

[4] Alonso-Gutiérrez D., Bastero J. *The variance conjecture on some polytopes.* In Asymptotic Geometric Analysis, Proceedings of the Fall 2010, Fields Institute Thematic Program, pp. 1–20, (Springer, 2013)

[5] Alonso-Gutiérrez D., Bastero J. *Approaching the Kannan-Lovász-Simonovits and variance conjectures.* Lecture Notes in Mathematics, Springer. Monograph. **2131**, (2015).

[6] Alonso-Gutiérrez D., Bastero J. *The variance conjecture on hyperplane projections of the $\ell_p^n$ balls.* To appear in Revista Matemática Iberoamericana.

[7] Alonso-Gutiérrez D., Bernués J. *The variance conjecture on hyperplane projections of the cube.* Preprint.

[8] Alonso-Gutiérrez D. Bastero J. Bernués J., Wolff P. *On the isotropy constant of projections of polytopes.* J. Functional Analysis **258**, (2010) pp. 1452–1465.

[9] Alonso-Gutiérrez D. Dafnis N., Hernández-Cifre M.A. Prochno J. *On mean outer radii of random polytopes.* Indiana University Mathematics Journal, **63** (2), (2014), pp. 579–595.

[10] Alonso-Gutiérrez D., González B., Jiménez C.H. *Volume inequalities for the i-th convolution bodies.* J. Math. Anal. and App. **424** (1), (2015) ,pp. 385–401.

[11] Alonso-Gutiérrez D., González B., Jiménez C.H. Villa R. *Rogers-Shephard inequality for log-concave functions.* J. Functional Analysis **271** (11), (2016), pp. 3269–3299.

[12] Alonso-Gutiérrez D., González B., Jiménez C.H. Villa R. *John's ellipsoid and the integral ratio of a log-concave function* To appear in J. Geom. Anal. DOI 10.1007/s12220-017-9858-4.

[13] Alonso-Gutiérrez D., Jiménez C.H., Villa R. *Brunn-Minkowski and Zhang inequalities for convolution bodies.* Adv. in Math. **238** (2013), pp. 50–69.

[14] Alonso-Gutiérrez D., Litvak A.E., Tomczak-Jaegermann N. *On the Isotropic Constant of Random Polytopes.* J. Geom. Anal. **26** (1), (2016), pp 645–662.

[15] Alonso-Gutiérrez D., Prochno J. *Estimating support functions of random polytopes via Orlicz norms.* Discrete Comput. Geom. **49** (2013),pp. 558–588.

[16] Alonso-Gutiérrez D., Prochno J. *On the Gaussian behavior of marginals and the mean width of random polytopes.* Proc. Amer. Math. Soc. **143** (2), (2015),pp. 821–832.

[17] Alonso-Gutiérrez D., Prochno J. *Mean width of random perturbations of random polytopes.* Advances in Geometry. **17** (1), (2017),pp. 75–90.

[18] Alonso-Gutiérrez D., Prochno J. *On the geometry of random convex sets between polytopes and zonotopes.* J. Math. Anal. and App. **450** (1), (2017), pp. 670–690.

[19] Arstein S., Einhorn K., Florentin D.I., Ostrover Y. *On Godbersen's conjecture.* Geometriae Dedicata,**178** (1), (2015), pp 337–350.

[20] Ball K. *Volume ratios and a reverse isoperimetric inequality.* J. london Mathematical Society **44** (2), (1991), pp. 351–359.

[21] K. Ball, V. H. Nguyen. *Entropy jumps for random vectors with log-concave density and spectral gap.* Studia Mathematica **213** (1), (2012), pp. 81–96.

[22] Bobkov S., Koldobsky A. *On the central limit property of convex bodies.* Geometric aspects of functional analysis. Lecture Notes in Math. **1807**, pp. 44–52. (Springer, Berlin, 2003)

[23] Bourgain J. *On high-dimensional maximal functions associated to convex bodies.* Amer. J. Math. **108** (6), (1986),pp. 1467–1476.

[24] Dafnis N., Giannopoulos A., Guédon O. *On the isotropic constant of random polytopes.* Advances in Geometry **10** (2010), pp. 311–321.

[25] Dafnis N., Giannopoulos A., Tsolomitis A. *Quermaßintegrals and asymptotic shape of random polytopes in an isotropic convex body.* Michigan Math. J. **62** (1), (2013), pp. 59–79.

[26] Eldan R. *Thin shell implies spectral gap up to polylog via stochastic localization scheme*, Geom. Funct. Anal. **23** (2013), pp. 532–569.

[27] R. Eldan, B. Klartag, *Approximately gaussian marginals and the hyperplane conjecture*, Contemporary Math. **545**, (2011), pp.55–68.

[28] Giannopoulos A., Hioni L., Tsolomitis A. *Asymptotic shape of the convex hull of isotropic log-concave random vectors.* Advances in Applied Mathematics **75**, (2016), pp. 116–143.

[29] Gluskin E.D. *The diameter of Minkowski compactum roughly equals to $n$.* Funktsional. Anal. i Prilozhen. **15** (1), (1981), pp. 72–73. English translation: Functional Anal. Appl. **15** (1), (1981), pp. 72–73.

[30] Hörmann J. Prochno J., Thäle C. *On the Isotropic Constant of Random Polytopes with Vertices on an $\ell_p$-Sphere.* To appear in J. Geom. Anal. arXiv:1605.09160v3

[31] John F. *Extremum problems with inequalities as subsidiary conditions.* Studies and Essays Presented to R. Courant on his 60th Birthday . Interscience (New York, 1948). pp. 187–204.

[32] Klartag B. *On convex perturbations with a bounded isotropic constant.* Geom. and Funct. Anal. (GAFA), **16** (6), (2006), pp. 1274–1290.

[33] Klartag B., *P*ower-law estimates for the central limit theorem for convex sets., Journal of Functional Analysis **245**, (2007), pp. 284–310.

[34] Klartag B., Kozma G. *On the hyperplane conjecture for random convex sets.* Israel J. Math. **170** (1), (2009), pp. 1–33.

[35] Lee Y.T., Vempala S.. *Eldan's stochastic localization and the KLS Hyperplane conjecture: An improved lower bound for expansion.* arXiv:1612.01507v2

[36] Mankiewicz P., Tomczak-Jaegermann N. *Q*uotients of finite-dimensional Banach spaces; random phenomena. In:"Handbook in the geometry of Banach spaces, Vol 2", ed. W. B. Johnson, J. Lindenstrauss, Elsevier, 1201–1246.

[37] MILMAN V.D., PAJOR A. *Isotropic positions and inertia ellipsoids and zonoids of the unit ball of a normed $n$-dimensional space.* GAFA Seminar 87-89, Springer Lecture Notes in Math. **1376** (1989), 64–104.

[38] PETTY C.M.*Projection bodies* Proc. Colloquium on Convexity (Copenhagen, 1965), pp. 234–241, Kobenhavns Univ. Mat. Inst., Copenhagen 1967.

[39] ROGERS C. A., SHEPHARD G. C. *The difference body of a convex body.* Arch. Math. **8** (1957), pp. 220–233.

[40] ROGERS C. A., SHEPHARD G. C. *Convex bodies associated with a given convex body.* J. Lond. Math. Soc. **33** (1958), pp. 270–281.

[41] ZHANG G. *Restricted chord projection and affine inequalities.* Geom. Dedicata **39**, (1991), pp. 213–222.

[42] ZHANG G. *The affine Sobolev inequality.* J. Differential geometry **53** (1999), pp. 183–202.