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2D Necklace Flower Constellations

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Abstract

The 2D Necklace Flower Constellation theory is a new design framework based on the 2D Lattice Flower Constellations that allows to expand the possibilities of design while maintaining the number of satellites in the configuration. The methodology presented is a generalization of the 2D Lattice design, where the concept of necklace is introduced in the formulation. This allows to assess the problem of building a constellation in orbit, or the study of the reconfiguration possibilities in a constellation. Moreover, this work includes three counting theorems that allow to know beforehand the number of possible configurations that the theory can provide. This new formulation is especially suited for design and optimization techniques.

Keywords: `Space Mechanics`, `Satellite Constellation Design`, `Number Theory`, `Optimization Techniques`

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1. Introduction

The use of satellites provide countless possibilities including a great variety of missions such as Earth and space observation, telecommunications or global positioning systems. Moreover, many missions require multiple satellites working cooperatively to achieve a common mission, that is, a satellite constellation. In that sense, in the last years an increasing number of space missions have benefit from the advantages that satellite constellations provide, such as the improvement on the performance of the system, or the reduction of the costs associated with the mission. Examples of such missions are GPS, Galileo, Glonass, Iridium, A-train [1] or X-Tandem [2]. However, the simultaneous study of multiple satellites, and more importantly, the relations that appear in the internal structure of the constellation, increases the complexity of the problem to solve, but it also enhances the use of the available satellites, and the ability to expand the possibilities of design at our disposal.

Satellite constellation design has been since its beginning a process that required a high number of iterations due to the lack of established models for the generation and study of constellations. This situation resulted in the necessity of specific studies for each particular mission, being unable of extrapolate the results from one mission to another.

Fortunately, in the last decades, several satellite constellation design methodologies have appeared, such as Walker Constellations [3] for circular orbits or the design of Draim [4] for elliptic orbits. Later, in 2004, the Flower Constellation Theory [5, 6, 7] was presented, including in its formulation both circular and elliptic orbits, and containing the former designs of Walker and Draim. The theory was later improved by the 2D Lattice [8] and 3D Lattice [9] theories which simplified the formulation and made the configuration independent of any reference frame. Other more recent examples of satellite constellation design include the Ground-track Constellations [10, 11] for any kind of constellation configuration, the Helix constellation [2] for very safe formation flying, or polar constellations for discontinuous coverage [12].

Flower Constellations can be defined in any rotating frame of reference, although in general, the Earth Fix is considered due to its advantages for several missions. In these reference systems, the orbits acquire a shape that reminds the one of the petals of a flower, where these constellations take their name. The most important property of Flower Constellations however is that the distributions generated present a high number of symmetries, which makes this methodology of design very interesting for many applications, especially global coverage and global positioning.

In a Lattice Flower Constellation, the possible configurations that the theory provides is proportional to the number of satellites in the constellation, and thus, it imposes a great limitation in the design of small constellations. In order to solve this issue, the concept of necklace was introduced for the 2D Lattice formulation [13] where the condition for maintaining the uniformity and symmetries of the configurations was presented. However, necklaces were not included directly in the formulation of the constellation and its computation was difficult to handle in a computer. This resulted in the impossibility to automatize the computation of the different configurations and the requirement to calculate all the available positions instead of just the real locations of the satellites. Thus, a new design framework was required to solve these difficulties.

In this work we introduce the formulation of the 2D Necklace Flower Constellations. This design framework constitutes the generalization of the methodology presented in 2D Lattice Flower Constellations using necklaces [13, 14] and includes in its definition all the former 2D lattice configurations. This methodology of design allows also to study the sequence of launches for constellation building, as well as possible reconfigurations available in case of failure of some satellites of the distribution. In addition, three counting theorems are included, which allow to know beforehand the number of configurations obtained using this theory for the cases of fixed fictitious constellation, fixed symmetries of the configuration, and fixed number of satellites. This formulation is able to not only define the symmetries, but also to provide a methodology to easily define constellations, which will be used in future work for optimization, station-keeping [15], constellation reconfiguration [16] and satellite launching schedule studies.

2. Preliminaries

In this section we present the 2D Lattice Flower Constellation theory and its variant using necklaces. These are included as a background of the 2D Necklace Flower Constellation methodology that is introduced in this work. In addition, a brief description of the concepts of necklaces and Burnside's Lemma have been included, due to their use in several parts of this manuscript.

2.1. 2D Lattice Flower Constellations

A 2D Lattice Flower Constellation [8] (2D-LFC) is described by nine parameters: three integers and six continuous parameters. The first three parameters are the number of inertial orbits (N_o), the number of satellites per orbit (N_{so}) and the configuration number (N_c), which is a parameter that satisfies $N_c \in [0, N_o - 1]$ and governs the phasing of the constellation. In particular, the location of the satellites in a 2D-LFC corresponds to a lattice in the (Ω, M)-space [17], that is, a space generated in the orbital variables right ascension of the ascending node Ω and mean anomaly M of all the satellites of the constellation in a given instant. The (Ω, M)-space can be also regarded as a 2D torus (both axes, Ω and M, are modulo 2π) where the points represented coincide with the solutions of the following system of equations:

$$\begin{pmatrix} N_o & 0 \\ N_c & N_{so} \end{pmatrix} \begin{pmatrix} \Delta\Omega_{ij} \\ \Delta M_{ij} \end{pmatrix} = 2\pi \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \tag{1}$$

where $i = 1, \cdots, N_o$, $j = 1, \cdots, N_{so}$, and $N_c \in [0, N_o - 1]$, and $\Delta\Omega_{ij}$ and ΔM_{ij} represent the satellite distribution in the right ascension of the ascending node and the mean anomaly with respect to a reference satellite. Indexes (i, j) represent the j-th satellite on the i-th orbital plane. Note that this system of equations is derived from the Hermite Normal Form of the lattice, which is the minimum representation of a lattice in a 2D distribution [8].

On the other hand, the other six parameters are the semi-major axis (a), the eccentricity (e), the inclination (i) and the argument of perigee (ω) (which are the same for all the satellites of the constellation), and the longitude of the ascending node and the initial mean anomaly of the first satellite of the constellation, that is, Ω_{11} and M_{11}, which define a reference for the constellation.

2.2. 2D Lattice Flower Constellations using Necklaces

The theory of 2D Lattice Flower Constellations generates uniform and symmetric configurations. However, provided a set of satellites, the number of possible different configurations is limited by the combinations between the integer parameters that constitute the Hermite Normal Form (most importantly the number of orbital planes), and thus, bigger constellations generate a larger number of distributions. In order to solve this issue and allow more possible configurations in small constellations, the concept of necklaces was introduced for the case of 2D Lattice Flower Constellations [13].

2.2.1. Definition of a Necklace

A necklace is a subset of points selected from a set of n available positions that present modular arithmetic, that is, location 1 in the available positions is the same as location $n + 1$. They are represented by the subset $\mathcal{G} \subseteq \mathbb{Z}_n = \{1, ..., n\}$. As an example, if we have a configuration in which four positions are available, a necklace consisting in three points can be created as seen in Figure 1. In the figure, we have occupied three positions (the colored circles) form an available set of four positions, forming a necklace that is represented as $\mathcal{G} = \{1, 2, 4\} \subseteq \mathbb{Z}_4$.

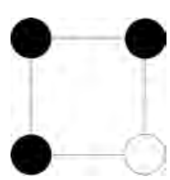

Figure 1: Example of necklace.

However, this is not the only representation that corresponds to this particular necklace. To be more precise, all the distributions that are obtained from a rotation of the whole configuration are considered identical. That is, two necklaces ($\mathcal{G}_1$ and $\mathcal{G}_2$) are considered to be identical, that is, an equivalence relation $\cong$, if they fulfill the following expression:

$$\mathcal{G}_1 \cong \mathcal{G}_2 \Longleftrightarrow \exists s : \mathcal{G}_1 = \mathcal{G}_2 + s \mod (n), \tag{2}$$

where s is an integer that belongs to the group $\mathbb{Z}_n$. Taking as an example the necklace from Figure 1 and varying the parameter s, all these configurations can be obtained:

$$\mathcal{G} = \{1,2,4\} \cong \{1,2,3\} \cong \{2,3,4\} \cong \{1,3,4\}\,; \tag{3}$$

which correspond to the graphical representation shown in Figure 2. As it can be seen, the difference between them is just a rotation in the circular loop, not changing the distribution in the process.

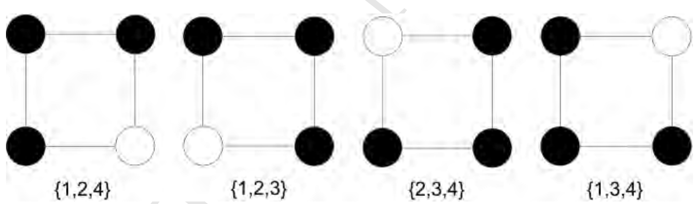

Figure 2: Identical necklaces.

2.2.2. Symmetry of a Necklace

The symmetry of a necklace is a parameter that provides information on how uniform the necklace distribution is [18]. This is done by counting the minimum number of times that the configuration can be rotated in the available positions in order to obtain the same necklace in the modular arithmetic.

Let $K(n)$ be the set of equivalence classes of necklaces modulo the relation defined by $\cong$:

$$K(n) = \{\text{necklaces} \subseteq \mathbb{Z}_n\}/\cong, \tag{4}$$

and let $\mathcal{G}$ be a necklace such that $\mathcal{G} \subseteq \mathbb{Z}_n$. The symmetry of a necklace ($Sym(\mathcal{G})$) is defined as the smallest value of $r \in \mathbb{Z}_n$ such that $\mathcal{G} + r = \mathcal{G}$ in $\mathbb{Z}_n$:

$$Sym(\mathcal{G}) = \min \{1 \leq r \leq n : \mathcal{G} + r = \mathcal{G} \quad \text{in} \quad \mathbb{Z}_n\}. \tag{5}$$

This means that r is the smallest value that the configuration has to be rotated in order to obtain the same initial configuration. In other words, if $\mathcal{G}_1 \cong \mathcal{G}_2$, then $Sym(\mathcal{G}_1) = Sym(\mathcal{G}_2)$ and thus, the symmetry can be defined over an equivalence class:

$$\begin{aligned} \overline{Sym}: \quad K(n) &\longrightarrow \mathbb{N} \\ \overline{\mathcal{G}} &\longmapsto \overline{Sym(\mathcal{G})}. \end{aligned} \tag{6}$$

Equivalent classes defined in this manner can be also regarded as the orbits that different symmetries of a necklace (seen as an action) generate in the group of possible combinations of elements taken from the available positions.

As an example of this concept, let assume that a configuration with six available positions is generated ($n = 6$), where a necklace $\mathcal{G} = \{1, 3, 5\} \subseteq \mathbb{Z}_6$ is defined. The representation of this example can be seen in Figure 3.

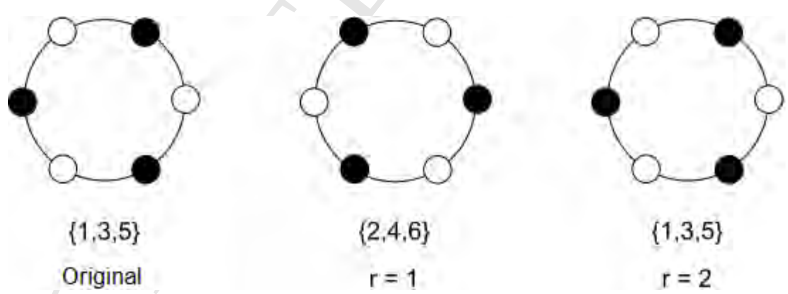

Figure 3: Symmetry of a necklace.

For this particular case, $Sym(\mathcal{G}) = 2$ because $\{1, 3, 5\} \equiv \{3, 5, 7\} \mod (6)$. Note that, in this example, although $\{2, 4, 6\}$ is an identical necklace with respect to $\mathcal{G}$, as defined in Equation (2), it does not fulfill the definition of symmetry of a necklace.

2.2.3. The Necklace problem

The necklace problem is a combinatorial problem that studies the number of different arrangements of n elements in a circular loop that can be generated assuming that each element comes in one of k different colors. In this definition, two arrangements are considered to be identical if they only differ by a rotation inside the loop (see Equation (2)). The number of different arrangements is given by the application of Burnside's counting theorem, which, applied to this particular case, can be summarized by the following formula[19]:

$$N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d}, \tag{7}$$

where the sum is taken over all the divisors d of n, and $\varphi(d)$ is called the Euler's totient function of d, an arithmetic function that counts the number of positive integers less than or equal to d that are coprime with d. It is important to note that the number of different arrangements of pearls provided by Equation (7) is also representing the number of equivalent classes (that is, orbits) defined by the group and actions considered.

The case of study is a simplification of the general necklace problem, since only two different states for each position are possible, the first one having the position occupied, and the second, the case in which it is not. Thus, for this particular case, the number of colors is $k = 2$.

However, the question of why using a representation in which the positions are distributed in a circular loop still remains. 2D Lattice Flower Constellations generate a distribution related to a reference satellite, which means that we are interested in the relative positions of the satellites ($\Delta\Omega_{ij}$ and ΔM_{ij}), and not the absolute positions. In fact, having two configurations with shifted positions in M only means that the same constellation is observed at a different time, while a shifting in Ω represents a rotation of the full constellation. Both shifting movements generate the same structure, and thus, there is no point in considering all combinations of parameters. Moreover, $\Delta\Omega_{ij}$ and ΔM_{ij} have modular arithmetic nature, which translates into the representation as a circular loop in the necklace.

2.2.4. Admissible pairs

We define the admissible pairs of a configuration to the set of constellation parameters that makes the distribution independent under a complete rotation in either the mean anomaly or the right ascension of the ascending node. This means that the constellation present the same relative geometry no matter the orbital plane or the moment in which it is observed.

Let $\mathcal{G} \subseteq \mathbb{Z}_{N_{so}}$ be a necklace defined in the variable mean anomaly. We know from Equation (1) that the values of the mean anomaly depend on the values of the longitude of the ascending node. Thus, we define $k \in \{1, \ldots, Sym(\mathcal{G})-1\}$ as the shifting parameter, which is a constant integer that represents the additional movement required by the necklace each time that we change the position in the variable $\Delta\Omega$. This parameter allows to obtain the symmetric configurations.

Expanding Equation (1) and computing the variation of the mean anomaly between two consecutive values of the right ascension of the ascending node, we obtain the ΔM-Shifting, defined as:

$$\Delta M = \frac{2\pi}{N_{so}} k - \frac{2\pi}{N_{so}} \frac{N_c}{N_o}, \tag{8}$$

where k is the shifting parameter. Moreover, imposing that the value of the mean anomaly is invariant under the addition of $N_o \Delta M$, we can obtain the relation that must be fulfilled by all admissible pairs:

$$Sym(\mathcal{G}) \mid kN_o - N_c, \tag{9}$$

which reads $Sym(\mathcal{G})$ divides $kN_o - N_c$. Equation (9) provides all possible admissible pairs given the values of the symmetry of the necklace $Sym(\mathcal{G})$, the number of orbits N_o and the configuration number N_c.

As an example, suppose that a constellation is distributed in six orbits ($N_o = 6$), where a necklace comprised by two satellites in four available positions ($\mathcal{G} = \{1, 2\} \subseteq \mathbb{Z}_4$, $N_{so} = 4$) is defined in each orbit, with a configuration number $N_c = 2$. From the definition of symmetry of the necklace (Equation (5)), we obtain $Sym(\mathcal{G}) = 4$. Now, we have to find the possible values of k that allow to obtain the same configuration when $\Delta\Omega = 2\pi$ following Equation (1).

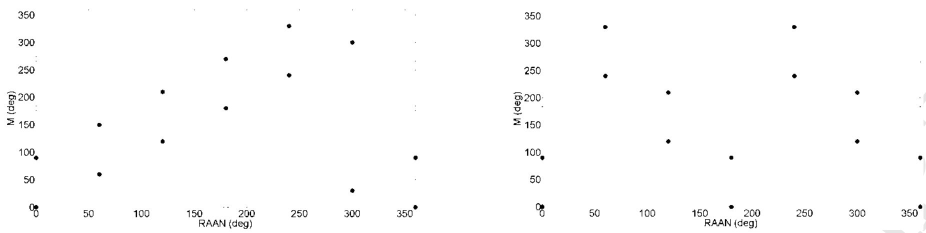

Figure 4: Satellite distribution for $k = 1$ (left) and $k = 3$ (right).

Using Equation (9) and applying it to the values of the example:

$$4 \mid 6k - 2, \tag{10}$$

where we can obtain the two values of the shifting parameter $k = 1, 3$ that fulfill that expression. The representation of both configurations can be seen in Figure 4. As it can be observed, both distributions are completely different and maintain the properties of symmetry that we were looking for, that is, the configuration is the same no matter the orbital plane observed..

2.3. Burnside's Lemma

In this work, we introduce three counting theorems that rely on Burnside's Lemma. Thus, and for the sake of completeness, a summary of the Lemma and the concepts that it introduces is presented in this section.

Let G be a group, and let $+$ be an action of this group over a set X, that is, an application defined as:

$$\begin{array}{rrcl} + : & G \times X & \longrightarrow & X \\ & (g, x) & \longmapsto & g + x, \end{array} \tag{11}$$

such that:

$$\left.\begin{array}{rcl} g_1 + (g_2 + x) & = & (g_1 + g_2) + x \\ 1_G + x & = & x \end{array}\right\} \quad \forall\, g_1, g_2 \in G,\ x \in X \tag{12}$$

In addition, let an orbit (orbit(x)) be the set of elements that can be obtained from x by the application of the action $(+)$, in other words:

$$\text{orbit}(x) = \{g + x \mid g \in G\} \subseteq X; \tag{13}$$

and let the fix of g (Fix(g)) be the elements of X that are invariant under the multiplication by g, that is:

$$\mathrm{Fix}(g) = \{x \in X \mid g + x = x\}. \tag{14}$$

The action partitions the set X into orbits, since if $y = g + x$, then orbit(y)=orbit(x). Thus, the number of orbits induced by the action $+$ is given by the Burnside's Lemma:

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|, \tag{15}$$

where we denote $|Y|$ to the number of elements of the set Y.

3. 2D Necklace Flower Constellation

We begin the Necklace Flower Constellation Theory with the case of a 2D Lattice. This is chosen in order to introduce in a clear way the new formulation that is carried out during the Necklace Flower Constellation Theory, as well as to serve as a common link between old and new formulations. In addition, this new formulation allows to have a better control in the design, since the necklace definition is performed directly in the formulation.

A 2D lattice can be generated in the same way as shown in Equation (1):

$$\begin{pmatrix} L_\Omega & 0 \\ L_{M\Omega} & L_M \end{pmatrix} \begin{pmatrix} \Delta\Omega_{ij} \\ \Delta M_{ij} \end{pmatrix} = 2\pi \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \tag{16}$$

where we denote L_Ω to the number of orbital planes, L_M to the number of satellites per orbit and $L_{M\Omega}$ to the combination number between the right ascension of the ascending node and the mean anomaly. Moreover, Equation (16) can be expanded in order to obtain the distribution as a function of the integers $i \in \{1, L_\Omega\}$ and $j \in \{1, L_M\}$:

$$\begin{aligned} \Delta\Omega_{ij} &= \frac{2\pi}{L_\Omega}(i-1), \\ \Delta M_{ij} &= \frac{2\pi}{L_M}(j-1) - \frac{2\pi}{L_M}\frac{L_{M\Omega}}{L_\Omega}(i-1), \end{aligned} \tag{17}$$

where this equation corresponds to a complete configuration. Now, instead of considering all the admissible locations, we select a set of satellites that maintain the properties of uniformity and symmetry of the former configuration, that is, the same distribution can be observed with independence on the orbital plane chosen. In order to do that, we define a necklace in the mean anomaly $\mathcal{G}_M$ as a subset of $\mathbb{Z}_{L_M}$ of cardinality N_M which contains the positions occupied by the necklace (and that also corresponds to the number of real satellites per orbit). A necklace is a subset $\mathcal{G}_M$ of the set of admissible locations:

$$\mathcal{G}_M \subseteq \{1, \dots, L_M\}, \tag{18}$$

such that $|\mathcal{G}_M| = N_M$ is the number of elements of the necklace $\mathcal{G}_M$. On the other hand, and in order to simplify the notation used, we assume that:

$$\mathcal{G}_M = \{\mathcal{G}_M(1), \dots, \mathcal{G}_M(j^*), \dots, \mathcal{G}_M(N_M)\}, \tag{19}$$

with

$$1 \leq \mathcal{G}_M(1) < \dots < \mathcal{G}_M(j^*) < \dots < \mathcal{G}_M(N_M) \leq L_M, \tag{20}$$

where the index j^* names each element of the necklace $\mathcal{G}_M$ and it is represented by an integer modulo N_M, that is, $j^* + N_M$ is the same index as j^*. This allows to interpret necklaces as injective functions:

$$\begin{array}{rccc} \mathcal{G}_M : & \mathbb{Z}_{N_M} & \longrightarrow & \mathbb{Z}_{L_M} \\ & j^* & \longmapsto & \mathcal{G}_M(j^*). \end{array} \tag{21}$$

For this reason, it makes sense to refer to $\mathcal{G}_M(j^*)$, where the integer parameter $j^* \in \{1, \dots, N_M\}$ represents the movement inside the necklace defined. In addition, and for simplicity of notation, we denote $\mathrm{mod}(a, b) = a \mod (b)$. Thus, due to the modular arithmetic inside the necklace:

$$\mathcal{G}_M(j^*) = \mathcal{G}_M(\mathrm{mod}(j^* + N_M, N_M)), \tag{22}$$

which corresponds to a complete loop in the available positions in the mean anomaly. It is important to note that this rotation is equivalent to a movement in the admissible locations defined by:

$$j = j + L_M \mod (L_M), \tag{23}$$

as both represent the same movement of the necklace, one using the parametrization of the necklace and the other using the parametrization of the fictitious constellation.

On the other hand, we require a parameter (the shifting parameter) that is able to modify the mean anomaly with respect to the change in the right ascension of the ascending node. Let $S_{M\Omega} \in \mathbb{Z}$ be that parameter. Thus, it is possible to define an application (T1) between the positions in the necklace necklace and the overall available positions:

$$\begin{aligned} \mathrm{T1}: \quad (\mathbb{Z}_{L_\Omega} \times \mathbb{Z}_{N_M}) &\longrightarrow (\mathbb{Z}_{L_\Omega} \times \mathbb{Z}_{L_M}) \\ (i, j^*) &\longmapsto (i, j), \end{aligned} \tag{24}$$

where the integer j is described as:

$$j = \mathcal{G}_M(j^*) + S_{M\Omega}(i-1). \tag{25}$$

In order to agree with the formulation introduced in Equation (16), one unit is subtracted from the previous expression leading to:

$$j - 1 = \mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1). \tag{26}$$

However, there is a modular behavior between the necklace and the available positions in the mean anomaly. Using the definition of symmetry of a necklace provided by Equation (5):

$$\mathcal{G}_M = \mathcal{G}_M + Sym(\mathcal{G}_M) \quad \text{in } \mathbb{Z}_{L_M}, \tag{27}$$

and thus, the movement in j is described as:

$$j - 1 = \mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1),\ Sym(\mathcal{G}_M)\right). \tag{28}$$

Introducing this expression in the original distribution shown in Equation (17), we obtain:

$$\begin{aligned}
\Delta\Omega_{ij^*} &= \frac{2\pi}{L_\Omega}\,(i-1)\,, \\
\Delta M_{ij^*} &= \frac{2\pi}{L_M}\left(\mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1),\ Sym(\mathcal{G}_M)\right)\right) - \\
&- \frac{2\pi}{L_M}\frac{L_{M\Omega}}{L_\Omega}\,(i-1)\,,
\end{aligned} \tag{29}$$

which describes all possible movements that the necklace $\mathcal{G}_M$ can perform in the space generated. Using this formulation, i represents the movement of the necklace in the right ascension of the ascending node while j^* defines the positions inside the necklace. One important thing to notice is that, although the shifting parameter $S_{M\Omega}$ can present any integer value, we only consider $S_{M\Omega} \in \{0, \ldots, Sym(\mathcal{G}_M) - 1\}$, since other values generate equivalent configurations due to the arithmetic nature of the problem in $Sym(\mathcal{G}_M)$.

Now, we impose the condition of symmetry, that is, a complete rotation in either variable, the right ascension of the ascending node or the mean anomaly, provides the same initial configuration. This definition is equivalent to:

$$\begin{aligned}
\text{Rotation in } M&: \left\{\begin{array}{ll} \Delta\Omega_{ij^*} & = \Delta\Omega_{i(j^*+N_M)}, \\ \Delta M_{ij^*} & = \Delta M_{i(j^*+N_M)}, \end{array}\right\} \\
\text{Rotation in } \Omega&: \left\{\begin{array}{ll} \Delta\Omega_{ij^*} & = \Delta\Omega_{(i+L_\Omega)j^*}, \\ \Delta M_{ij^*} & = \Delta M_{(i+L_\Omega)j^*}, \end{array}\right\}
\end{aligned} \tag{30}$$

where all relations must be fulfilled at the same time. From the first rotation in M there is no effect on the right ascension of the ascending node:

$$\frac{2\pi}{L_\Omega}\,(i-1) = \frac{2\pi}{L_\Omega}\,(i-1)\,, \tag{31}$$

while focusing on the mean anomaly, it must satisfy that:

$$\begin{aligned}
&\mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1),\ Sym(\mathcal{G}_M)\right) = \\
&= \mathrm{mod}\left(\mathcal{G}_M(\mathrm{mod}(j^* + N_M, N_M)) - 1 + S_{M\Omega}(i-1),\ Sym(\mathcal{G}_M)\right).
\end{aligned} \tag{32}$$

This relation is achieved without imposing further conditions since $\mathcal{G}_M(j^*) = \mathcal{G}_M(\mathrm{mod}(j^* + N_M, N_M))$ (see also Equation (22)). On the other hand, in the rotation of the right ascension of the ascending node, the first relation is automatically achieved:

$$\frac{2\pi}{L_\Omega}(i-1) = \frac{2\pi}{L_\Omega}(L_\Omega + i - 1) \mod (L_\Omega), \tag{33}$$

while the second relation does not. Imposing the condition:

$$\frac{L_M}{2\pi}\Delta M_{(i+L_\Omega)j^*} = \frac{L_M}{2\pi}\Delta M_{ij^*}, \tag{34}$$

provides the following expression:

$$\begin{aligned} &\mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(L_\Omega + i - 1),\ Sym(\mathcal{G}_M)\right) - \frac{L_{M\Omega}}{L_\Omega}(L_\Omega + i - 1) = \\ &= \mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1),\ Sym(\mathcal{G}_M)\right) - \frac{L_{M\Omega}}{L_\Omega}(i-1)\,. \end{aligned} \tag{35}$$

Then, by the properties of modular arithmetics, there exists $A \in \mathbb{Z}$ such that the former expression can be transformed into:

$$\begin{aligned} &\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(i-1) + ASym(\mathcal{G}_M) = \\ &= \mathcal{G}_M(j^*) - 1 + S_{M\Omega}(L_\Omega + i - 1) - L_{M\Omega}, \end{aligned} \tag{36}$$

Finally, the terms that are equal in both sides of the equation can be simplified, providing the expression:

$$ASym(\mathcal{G}_M) = S_{M\Omega}L_\Omega - L_{M\Omega}, \tag{37}$$

which relates the shifting parameter ($S_{M\Omega}$) with both the necklace ($\mathcal{G}_M$) and the fictitious orbit (L_Ω and $L_{M\Omega}$). Equation (37) can also be represented as:

$$Sym(\mathcal{G}_M) \mid S_{M\Omega}L_\Omega - L_{M\Omega}, \tag{38}$$

which reads, $Sym(\mathcal{G}_M)$ divides ($S_{M\Omega}L_\Omega - L_{M\Omega}$) and constitutes a Diophantine equation that is also subjected to modular arithmetic. It is important to note that Equation (38) is equivalent to Equation (9). However, the new formulation allows to show an alternative proof to the relation proposed in [13] as well as present a methodology that can be used for optimization since only the real positions of the satellites of the constellation have to be computed.

The combination of Equations (29) and (38) allows to compute all possible symmetric configurations for a particular necklace $\mathcal{G}_M$ and a fictitious expanded constellation. In the following sections, an example of application is shown and then, the number of possible configurations that the 2D Necklace Flower Constellations can provide is studied.

3.1. Example of application

As an example of a 2D Necklace Flower Constellation, we design a constellation made of 14 satellites in circular orbits $e = 0$, with semi-major axis $a = 14420\ km$ and inclination $i = 63.435^o$. The satellites are distributed in seven inertial orbits ($L_\Omega = 7$), which means that there are two satellites per orbit ($N_M = 2$). The number of possible configurations that we can obtain using the Lattice Flower Constellation Theory in this case is given by the possible values of the combination number $N_c = \{0, \dots, 6\}$, which is seven different distributions. However, the Necklace Flower Constellation Theory can be used to increase this number of possibilities.

Let $L_M = 20$ be the number of available positions in the mean anomaly that are defined in order to create a fictitious constellation composed by $L_\Omega L_M = 140$ satellites. In this fictitious constellation we look for the configurations with $N_M = 2$ that are symmetric in the sense of Equation (30). That way, we obtain 70 different distributions, ten times the former number of possible constellations.

In order to describe a simple example, we select only the distributions where $\mathcal{G}_M = \{1, 2\} \subseteq \mathbb{Z}_{20}$ and $L_{M\Omega} = 6$ from the set obtained. This implies that the symmetry of the necklace is $Sym(\mathcal{G}_M) = 20$, since $\{1, 2\} = \{1, 2\} + 20 \mod (20)$. Then, using Equation (38):

$$Sym(\mathcal{G}_M) \mid S_{M\Omega} L_\Omega - L_{M\Omega} \quad \Rightarrow \quad 20 \mid 7S_{M\Omega} - 6, \tag{39}$$

which leads to $S_{M\Omega} = 18$. Figure 5 shows the distribution of the constellation in the (Ω, M)-space, where, without losing generalization, we have chosen $\Omega_{11} = M_{11} = 0$ as the initial position of the reference satellite of the constellation. As it can be seen, the distribution when $\Omega = 0$ and when $\Omega = 2\pi$ is the same, and thus, the properties of symmetry of the constellation are maintained from the original lattice in L_Ω and L_M.

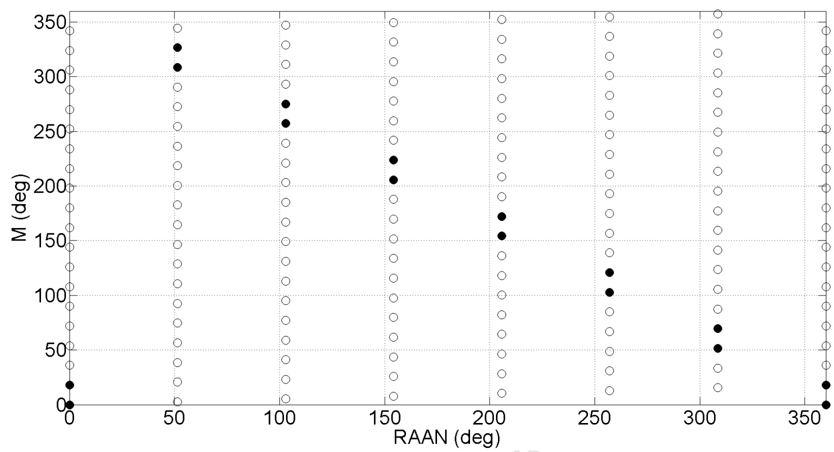

Figure 5: Representation of the initial positions of the satellites in the (Ω, M)-space.

On the other hand, in Figure 6, the (Ω, M)-torus representation of the constellation is shown. There, it can be observed clearer how the satellites are positioned following two closed lines (as $N_M = 2$) around the surface of the torus, not having any satellite outside this configuration.

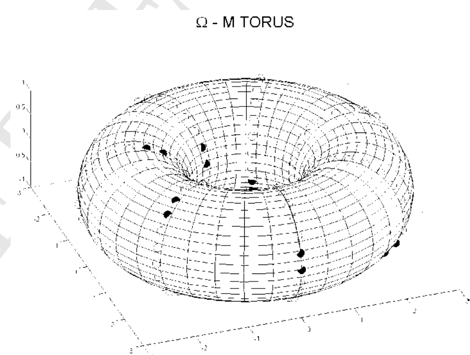

Figure 6: Representation of the initial positions of the satellites in the (Ω, M)-torus.

Finally, Figure 7 shows the inertial orbits of the constellation from an isometric view (left) and a polar view (right). This constellation presents two curious properties. First, all the satellites of the constellation are always positioned in an interval of Earth longitudes smaller than 90^o. This means, that they fly as a formation over the same regions of the Earth. Second, from the polar view, we observe that the constellation generates two heptagons of satellites that are bounded. In fact, during the motion of the constellation, these heptagons are maintained, from a polar perspective, creating a rigid structure that is rotating with no collisions between both structures.

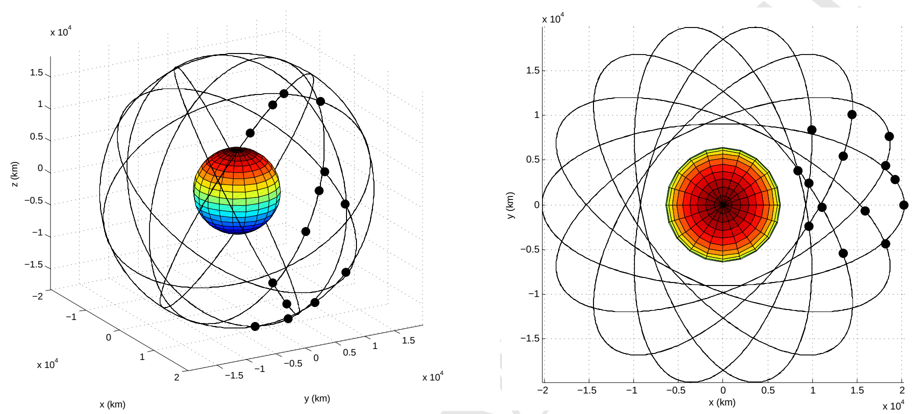

Figure 7: Initial distribution of the constellation in the ECI frame of reference.

As it can be seen, using this new formulation (see Equations (29) and (38)), we can expand the searching space as much as required without having to compute all available positions in the fictitious constellation generated. This allows to considerably reduce the amount of computations required, as only the real positions are calculated, a property that will be used in the future in optimization problems using this new design methodology.

4. Number of symmetric configurations in a 2D Necklace Flower Constellation

During this section, we deal with the computation of the number of configurations that the Necklace Flower Constellation Theory provides. In that respect, we consider three cases of interest which have different applications.

4.1. Fixing the necklace $\mathcal{G}_M$ and the Hermite Normal Form

In this case we focus on the study of the number of possibilities given a necklace $\mathcal{G}_M$ and the complete Hermite Normal Form for the fictitious constellation. By doing this, the available positions are fixed (they cannot shift), and thus, this methodology provides the number of symmetric configurations that follow a particular distribution given by the Hermite Normal Form. This is equivalent to compute the number of possible values that the shifting parameter $S_{M\Omega}$ can present in Equation (38).

Theorem 1. *Given a necklace in the mean anomaly $\mathcal{G}_M$ and a fixed Hermite Normal Form, there exists symmetric distributions in the constellation if and only if* $\gcd(Sym(\mathcal{G}_M), L_\Omega) \mid L_{M\Omega}$*, being the number of different configurations in that case:*

$$\gcd(Sym(\mathcal{G}_M), L_\Omega). \tag{40}$$

Proof. Equation (37) can be written as:

$$A Sym(\mathcal{G}_M) + L_\Omega S_{M\Omega} = L_{M\Omega}, \tag{41}$$

where A is a unknown integer. If we select A and $S_{M\Omega}$ as the variables of study, the expression becomes a linear Diophantine equation where, by the use of Bézout's identity, we can conclude that there exist solution if and only if:

$$\gcd(Sym(\mathcal{G}_M), L_\Omega) \mid L_{M\Omega}. \tag{42}$$

In the case the former expression is fulfilled, there are an infinite number of solutions of Equation (41) that have the form:

$$\begin{aligned} (S_{M\Omega})_\lambda &= (S_{M\Omega})_0 + \lambda \Delta l, \quad \text{with} \quad \Delta l = \frac{Sym(\mathcal{G}_M)}{\gcd(Sym(\mathcal{G}_M), L_\Omega)}, \\ (A)_\lambda &= (A)_0 - \lambda \frac{L_\Omega}{\gcd(Sym(\mathcal{G}_M), L_\Omega)}, \end{aligned} \tag{43}$$

where $(S_{M\Omega})_0$ and $(A)_0$ is a known pair of solutions, and λ is an integer number.

However, the variables and parameters from Equation (41) have some constraints due to the modular nature of the problem, in particular:

$$\begin{aligned} Sym(\mathcal{G}_M) &\in \{1, \ldots, L_M\}, \\ S_{M\Omega} &\in \{0, \ldots, Sym(\mathcal{G}_M) - 1\}, \\ L_{M\Omega} &\in \{0, \ldots, L_\Omega - 1\}, \end{aligned} \tag{44}$$

and thus, there are a finite number of different solutions to this problem. From the second boundary, we can derive that the difference between the maximum and the minimum value of $S_{M\Omega}$ is, at most, $\Delta S_{M\Omega} = (Sym(\mathcal{G}_M) - 1)$. Now, we are interested to know the number of different values of λ that allows Equation (43) to be inside this constraints. Thus, we first count the number of integer sections of length Δl that lay in the interval $\Delta S_{M\Omega}$, that is:

$$\begin{aligned} \left\lfloor \frac{\Delta S_{M\Omega}}{\Delta l} \right\rfloor &= \left\lfloor \frac{(Sym(\mathcal{G}_M) - 1)\gcd(Sym(\mathcal{G}_M), L_\Omega)}{Sym(\mathcal{G}_M)} \right\rfloor = \\ &= \left\lfloor \gcd(Sym(\mathcal{G}_M), L_\Omega) - \frac{\gcd(Sym(\mathcal{G}_M), L_\Omega)}{Sym(\mathcal{G}_M)} \right\rfloor, \end{aligned} \tag{45}$$

where $\lfloor x \rfloor$ is the round down integer of x.

It is elemental that $\gcd(Sym(\mathcal{G}_M), L_\Omega)$ is an integer, so Equation (45) can be expressed as:

$$\gcd(Sym(\mathcal{G}_M), L_\Omega) - \left\lceil \frac{\gcd(Sym(\mathcal{G}_M), L_\Omega)}{Sym(\mathcal{G}_M)} \right\rceil, \tag{46}$$

where $\lceil x \rceil$ is the round up integer of x. On the other hand, we know that $\gcd(Sym(\mathcal{G}_M), L_\Omega) \in [1, Sym(\mathcal{G}_M)]$ by the definition of greatest common divisor, thus:

$$\frac{\gcd(Sym(\mathcal{G}_M), L_\Omega)}{Sym(\mathcal{G}_M)} \in (0, 1]\,, \tag{47}$$

and applying this result we derive that the number of intervals is:

$$\gcd(Sym(\mathcal{G}_M), L_\Omega) - 1. \tag{48}$$

Finally, the number of intervals defines a set of different elements inside the interval $\Delta S_{M\Omega}$ equal to the number of intervals plus one. Consequently, the number of different values that $(S_{M\Omega})_\lambda$ can take is:

$$\gcd(Sym(\mathcal{G}_M), L_\Omega), \tag{49}$$

which is the number of solutions of Equation (41) provided that the number of orbital planes L_Ω, the combination number $L_{M\Omega}$, and the symmetry of the necklace $Sym(\mathcal{G}_M)$ are fixed. Note that this number of solutions only applies if the condition of existence of solution provided by Equation (42) is achieved.

□

4.2. Fixing the necklace $\mathcal{G}_M$, L_Ω and L_M

On the other hand, in this second case, we fix the necklace $\mathcal{G}_M$ and the size of the extended space, that is, the parameters L_Ω and L_M from the Hermite Normal Form. This provides the information of how many different distributions can be created with a given set of satellites (through the parameter $Sym(\mathcal{G}_M)$). This problem is equivalent to compute the amount of pairs $\{S_{M\Omega}, L_{M\Omega}\}$ that are solution of Equation (38).

Theorem 2. *Given a necklace in the mean anomaly $\mathcal{G}_M$ and a size of the fictitious constellation (L_Ω and L_M), the number of different symmetric constellation configurations is L_Ω.*

Proof. Equation (37) can be reordered as:

$$L_\Omega S_{M\Omega} - 1 L_{M\Omega} = ASym(\mathcal{G}_M), \tag{50}$$

where the parameters have the constraints shown in Equation (44). In this expression, we consider $S_{M\Omega}$ and $L_{M\Omega}$ the variables of the problem, and thus, the equation has solution only and only if:

$$\gcd(L_\Omega, 1) \mid ASym(\mathcal{G}_M), \tag{51}$$

which is always true as $\gcd(L_\Omega, 1) = 1$ and $ASym(\mathcal{G}_M)$ is an integer value. This provides an important result: given a symmetry of the necklace $Sym(\mathcal{G}_M)$, and a number of orbital planes L_Ω, there is always at least one solution to the equation. The objective now is to compute the number of solutions that this result represents.

Equation (50) is a linear Diophantine equation whose solutions are provided by the following relation:

$$\begin{aligned} (S_{M\Omega})_\lambda &= (S_{M\Omega})_0 + \lambda, \\ (L_{M\Omega})_\lambda &= (L_{M\Omega})_0 - \lambda L_\Omega, \end{aligned} \tag{52}$$

where $(S_{M\Omega})_0$ and $(L_{M\Omega})_0$ are a pair of possible solutions of Equation (50) and λ is an integer. From Equation (52), we can derive that there is only one solution for a fixed $ASym(\mathcal{G}_M)$, since $L_{M\Omega} \in \{0, \ldots, L_\Omega - 1\}$. Thus, the number of possible solutions is provided by the number of different equations in the form of Equation (52) (which is equivalent to the number of possible values of the integer A).

From Equation (50), the maximum and minimum values of $ASym(\mathcal{G}_M)$ can be obtained:

$$\begin{aligned} \min\left(ASym(\mathcal{G}_M)\right) &= -(L_\Omega - 1), \\ \max\left(ASym(\mathcal{G}_M)\right) &= (Sym(\mathcal{G}_M) - 1)L_\Omega. \end{aligned} \tag{53}$$

Then, we derive the maximum variation of the parameter $ASym(\mathcal{G}_M)$:

$$\begin{aligned} \Delta\left(ASym(\mathcal{G}_M)\right) &= \max\left(ASym(\mathcal{G}_M)\right) - \min\left(ASym(\mathcal{G}_M)\right) = \\ &= L_\Omega Sym(\mathcal{G}_M) - 1. \end{aligned} \tag{54}$$

Moreover, $Sym(\mathcal{G}_M)$ is constant in this variation, thus:

$$\Delta\left(ASym(\mathcal{G}_M)\right) = \Delta A Sym(\mathcal{G}_M), \tag{55}$$

where we can conclude that the admissible values of A lay in an interval of amplitude:

$$\Delta A = L_\Omega - \frac{1}{Sym(\mathcal{G}_M)}. \tag{56}$$

Now, we are interested in the number of complete intervals of amplitude 1 that are inside ΔA (remember that A is an integer number), since this number plus one defines the number of possible values of A. The number of complete intervals is:

$$\lfloor \Delta A \rfloor = \left\lfloor L_\Omega - \frac{1}{Sym(\mathcal{G}_M)} \right\rfloor = L_\Omega - \left\lceil \frac{1}{Sym(\mathcal{G}_M)} \right\rceil . \tag{57}$$

Moreover, since $\lceil Sym(\mathcal{G}_M)^{-1} \rceil \in (0,1]$, the number of complete intervals is $(L_\Omega - 1)$, which define L_Ω different values that the parameter A can take in Equation (52). The different values of A are providing the number of possible different equations that we can obtain from Equation (52). Furthermore, we already know that each equation has only one solution. Thus, the total number of solutions of Equation (52) is L_Ω.

□

One important thing to notice is that the number of solutions provided by Theorem 2 requires to set a particular symmetry of the necklace $Sym(\mathcal{G}_M)$. If the symmetry of the necklace is not fixed, and instead only the size of the fictitious constellation is fixed, that is, L_Ω and L_M, we have to use the Burnside's counting theorem applied to this particular case in addition to the methodology presented in this section. That way, the number of possible solutions that a fictitious constellation distributed in L_Ω orbital planes, with L_M available positions in each orbit is:

$$\frac{L_\Omega}{L_M} \sum_{d|L_M} \varphi(d) 2^{L_M/d}, \tag{58}$$

where the sum is taken over all the divisors d of L_M, and $\varphi(d)$ is the Euler's totient function of d. Equation (58) represents a combinatorial problem where the number of possible combinations of necklaces is given by Burnside's counting theorem while the number of pairs $\{L_{M\Omega}, S_{M\Omega}\}$ are given by Theorem 2. This combination can be freely performed since the number of pairs $\{L_{M\Omega}, S_{M\Omega}\}$ does not depend on the symmetry of the necklace (the only parameter that is changing in Burnside's counting theorem).

4.3. Fixing N_M, L_Ω and L_M

This case is an interesting variation of the previous counting methodology, where now, the real satellites per orbit, that is, $N_M = |\mathcal{G}_M|$, is fixed instead of the necklace. Thus, it provides information on the number of possibilities of design that are available with a set of satellites and a size of a fictitious constellation. It is important to note that, in this case, the sizes of both the real and the fictitious constellations are fixed.

Theorem 3. *Given a number of satellites per orbit N_M, and a size of fictitious constellation (L_Ω and L_M), the number of different symmetric constellation configurations is:*

$$\frac{L_\Omega}{L_M} \sum_{\substack{g=1 \\ g|L_M \\ \frac{L_M}{g}|N_M}}^{L_M} |Fix(g)|\,, \tag{59}$$

where $Fix(g)$ is the number of elements contained in the Fix of a given symmetry g, and can be computed using the following recursive function:

$$|Fix(g)| = \frac{L_M}{g}\left[\binom{g}{\frac{N_M}{L_M}g} - \sum_{\substack{g'=1 \\ g'|g \\ \frac{L_M}{g'}|N_M}}^{g-1} \frac{g'}{L_M}\,|Fix(g')|\right]. \tag{60}$$

Proof. The process followed in this case is based on applying Burnside's Lemma to count the number of different solutions. In order to use it, we require to set first a particular symmetry of a necklace and compute the Fix in the space of all possible configurations under that symmetry. Second, we remove the configurations that were considered in other symmetries before. Third, the number of orbits for a particular symmetry is computed using Burnside's Lemma. And finally, the total number of solutions is obtained as a sum of all the possible symmetries.

Let $+\mathbb{Z}_{L_M}$ be the possible actions that are considered in this problem, which correspond to the possible different rotations that a necklace $\mathcal{G}_M$ can perform in the modulo $\mathbb{Z}_{L_M}$. In addition, $G = \mathbb{Z}_{L_M}$ is the group of possible actions that can apply to any necklace defined in L_M available positions. That way, the map

ϕ can be defined as:

$$\begin{aligned} \phi: \quad G \times X &\longrightarrow X \\ (g, x) &\longmapsto x + g \mod (L_M). \end{aligned} \tag{61}$$

The objective is to apply the Burnside's Lemma to this application, and thus, we have to compute $|\mathrm{Fix}(g)|$ (see Equation (15)). The Fix of a given action is the set of elements that remain unaltered under the application of that action. In that respect, from the definition of symmetry of a necklace (see Equation (5)), we know that the only possible values of $g \in G$ that have elements in the $\mathrm{Fix}(X)$ are the ones that presents symmetries, that is, when an element fulfills $g = Sym(\mathcal{G}_M)$. This means that only the values such that $g|L_M$ and $\frac{L_M}{g}|N_M$ contribute to the elements of the Fix.

First, we focus in a particular value of symmetry of the necklace $g = Sym(\mathcal{G}_M)$ and its Fix ($\mathrm{Fix}(g)$). As there exists symmetry in the necklace, the configuration can be regarded as a pattern comprised of g available positions that is repeated L_M/g times in the L_M available positions. In this pattern, there must be $N_M g/L_M$ elements from the necklace since all the patterns must have the same number of elements. Thus, the number of possible combinations that exists in a pattern of size g ($PC(g)$) is:

$$PC(g) = \binom{g}{\frac{N_M}{L_M}g}. \tag{62}$$

On the other hand, each pattern can rotate L_M/g possible times in the L_M available positions while maintaining the same configuration (due to the symmetry that we are imposing). Thus, the number of combinations of N_M elements in L_M available positions that present a given symmetry g is:

$$\frac{L_M}{g} PC(g) = \frac{L_M}{g} \binom{g}{\frac{N_M}{L_M}g}. \tag{63}$$

However, this counting also includes some elements that belong to other symmetries, and thus, they must be removed from this set of combinations in order to

avoid duplicities in the counting process. For instance, if $L_M = 4$ and $N_M = 2$ and we consider $g = 4$ as the symmetry in study, the number of combinations
370 that we compute with Equation (62) include combinations of elements that also present symmetry of $g = 2$: $\{1, 3\}$ and $\{2, 4\}$; and thus, we could count them twice if we are not careful in the counting process. In order to avoid these cases, we only consider g as the smallest symmetry that a combination of elements can present.

From the definition of Fix, we know that the number of possible combinations of N_M elements with a particular symmetry g is the $|\mathrm{Fix}(g)|$ itself. In addition, the possible combinations of elements must have been generated based on patters of size g (as in Equation (62)). Thus, the number of different patterns that exist for a particular symmetry g' is:

$$PC(g') = \frac{g'}{L_M} \left|\mathrm{Fix}(g')\right| . \tag{64}$$

Then, we can remove from the counting process, of the different pattern generators with symmetry g, all the elements that belong to a different symmetry such that $g' < g$:

$$PC(g) = \binom{g}{\frac{N_M}{L_M} g} - \sum_{\substack{g'=1 \\ g' | g \\ \frac{L_M}{g'} | N_M}}^{g-1} \frac{g'}{L_M} \left|\mathrm{Fix}(g')\right| , \tag{65}$$

375 where the sum is performed in all the symmetries g' such that $g'|g$ and $\frac{L_M}{g'}|N_M$ since g' must also fulfill the conditions for symmetry.

Once the number of pattern combinations is computed, the $|\mathrm{Fix}(g)|$ can be obtained using Equation (65), leading to:

$$\left|\mathrm{Fix}(g)\right| = \frac{L_M}{g} \left[\binom{g}{\frac{N_M}{L_M} g} - \sum_{\substack{g'=1 \\ g' | g \\ \frac{L_M}{g'} | N_M}}^{g-1} \frac{g'}{L_M} \left|\mathrm{Fix}(g')\right| \right] , \tag{66}$$

which is a recursive function that can be easily computed. Equation (66) allows to obtain the number of different necklaces under a given symmetry g. This is done by the direct application of Burnside's Lemma (Equation (15)), where $G = \mathbb{Z}_{L_M}$ as pointed out before. That way, we can derive Corollary 1.

Corollary 1. *The number of different necklaces with a given symmetry g that can be obtained with N_M elements taken from L_M available positions is:*

$$\frac{1}{g}\left[\binom{g}{\frac{N_M}{L_M}g} - \sum_{\substack{g'=1 \\ g'|g \\ \frac{L_M}{g'}|N_M}}^{g-1} \frac{g'}{L_M}\left|Fix(g')\right|\right]. \tag{67}$$

where $|Fix(g')|$ is provided by Equation (66)*.*

In addition, if we fix the necklace, we obtain the same conditions as in Theorem 2, which implies that the number of possible different configurations that each necklace can provide is L_Ω. Thus, and for a given symmetry g, the number of possible configurations is:

$$\frac{L_\Omega}{g}\left[\binom{g}{\frac{N_M}{L_M}g} - \sum_{\substack{g'=1 \\ g'|g \\ \frac{L_M}{g'}|N_M}}^{g-1} \frac{g'}{L_M}\left|\mathrm{Fix}(g')\right|\right]. \tag{68}$$

Finally, since we already know the number of possible configurations that each symmetry can provide, we can sum all the contributions from the different symmetries to obtain the total number of configurations of a 2D Necklace Flower Constellation:

$$\sum_{\substack{g=1 \\ g|L_M \\ \frac{L_M}{g}|N_M}}^{L_M} \frac{L_\Omega}{g}\left[\binom{g}{\frac{N_M}{L_M}g} - \sum_{\substack{g'=1 \\ g'|g \\ \frac{L_M}{g'}|N_M}}^{g-1} \frac{g'}{L_M}\left|\mathrm{Fix}(g')\right|\right], \tag{69}$$

which can be rewritten as:

$$\frac{L_\Omega}{L_M} \sum_{\substack{g=1 \\ g|L_M \\ \frac{L_M}{g}|N_M}}^{L_M} |\mathrm{Fix}(g)|\,, \tag{70}$$

where $|\mathrm{Fix}(g)|$ is provided by Equation (66).

$\square$

The set of equations given by Theorem 3 are the general expressions to calculate the number of possible combinations that the 2D Necklace Flower Constellation methodology provides for a given number of satellites and a given size of the fictitious constellation. It also allows to fix the cost of the mission (the number of satellites and their general distribution), while providing information of the design possibilities available before starting the computation. That way, it is possible to decrease or increase the size of the fictitious constellation to adapt the number of possibilities to the memory and time available.

5. Generalizing into a double necklace

In Section 3 a necklace in the mean anomaly was introduced and then in Section 4 the number of possible configurations was assessed. In this section we introduce the formulation for a double necklace in the satellite distribution. This means that two necklaces are generated, one in the mean anomaly $\mathcal{G}_M$ and the other in the right ascension of the ascending node $\mathcal{G}_\Omega$.

Let N_Ω and L_Ω be the real and fictitious number of orbital planes in which the constellation is distributed. That way, the necklace in the right ascension of the ascending node can be defined as the subset:

$$\mathcal{G}_\Omega \subseteq \{1, \ldots, L_\Omega\}, \tag{71}$$

such that $|\mathcal{G}_\Omega| = N_\Omega$. In addition we define the index i^* as the parameter of distribution inside the necklace $\mathcal{G}_\Omega$. That way:

$$\mathcal{G}_\Omega(i^*) = \mathcal{G}_\Omega(\mathrm{mod}(i^* + N_\Omega, N_\Omega)), \tag{72}$$

which is equivalent to:

$$i = \mathrm{mod}(i + L_\Omega, L_\Omega). \tag{73}$$

Now, an application between i and i^* can be defined using the necklace $\mathcal{G}_\Omega$:

$$i = \mathcal{G}_\Omega(i^*), \tag{74}$$

and introducing this expression into Equation (29), we obtain:

$$\begin{aligned}
\Delta\Omega_{i^*j^*} &= \frac{2\pi}{L_\Omega}\left(\mathcal{G}_\Omega(i^*) - 1\right), \\
\Delta M_{i^*j^*} &= \frac{2\pi}{L_M}\left(\mathrm{mod}\left(\mathcal{G}_M(j^*) - 1 + S_{M\Omega}(\mathcal{G}_\Omega(i^*) - 1),\ Sym(\mathcal{G}_M)\right)\right) - \\
&- \frac{2\pi}{L_M}\frac{L_{M\Omega}}{L_\Omega}\left(\mathcal{G}_\Omega(i^*) - 1\right),
\end{aligned} \tag{75}$$

which is the general expression that allows to generate all the possible configurations when two necklaces are included. On the other hand the symmetric configurations of this formulation are still given by Equation (38) since the rotations in this new necklace does not modify the behavior of the system.

One important thing to notice is that this formulation represents the removal of complete orbital planes from the original configuration given by Equation (29). This means that, unless the necklace $\mathcal{G}_\Omega$ presents a symmetry, the configuration will lose the property of having an uniform distribution no matter the orbital plane observed. However, the resultant configuration still presents a structure related to the original distribution.

6. Generation of all the configurations

In this section, we present a general scheme in order to generate all the possible constellation configurations that the 2D Necklace theory can provide. In that respect, Figure 8 shows the summary of the process.

First, the general classic elements for the whole constellation are defined, namely, the semi-major axis a, the eccentricity e, the inclination i and the argument of perigee ω. Second, the sizes of the real and fictitious constellations are set (N_Ω, N_M for the real and L_Ω, L_M for the expanded distributions). Then,

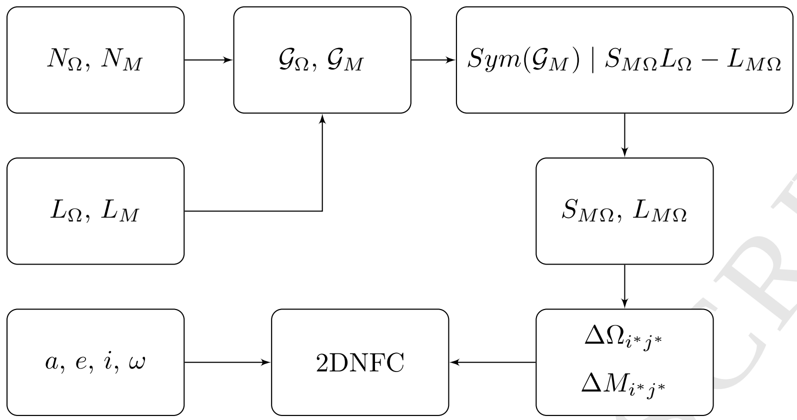

Figure 8: Flowchart of the 2D Necklace Flower Constellation generation process

using these sizes, all the possible necklaces are generated using a generation algorithm [20, 21]. With the results obtained, we apply Equation (38) to generate the shifting parameters $S_{M\Omega}$ and the configuration numbers $L_{M\Omega}$ that correspond to each combination of necklaces. Finally, the distribution in the right ascension of the ascending node ($\Delta\Omega_{i^*j^*}$) and in the mean anomaly ($\Delta M_{i^*j^*}$) is computed, and thus, in combination with the classical elements already defined, the configuration of the whole 2D Necklace Flower Constellation is defined.

This process can be parallelized in the generation of necklaces, the solution of the Diophantine equation and the generation of the distributions, allowing to generate and study a large number of configurations in a small amount of time. On the other hand, as the number of parameters required to define a constellation is very low, it is easy to store in memory all the possible combinations for later study in other applications.

7. Conclusion

This work presents a new methodology of satellite constellation design, the 2D Necklace Flower Constellations. This methodology allows to overcome the limitation on the number of possibilities of design that the original 2D Lattice Flower Constellations presented while maintaining the number of satellites of the configuration. This is achieved by an expansion of the configuration into a fictitious constellation in which a set of satellites that maintain the properties of uniformity and symmetry are selected. Other applications of this design framework are the definition of the sequence of launches for large constellations, the study of possible reconfiguration strategies of a given constellation with very little fuel consumption, or the assessment of the effect of failure in satellites of the configuration.

Compared to previous formulations, the main advantage of 2D Necklace Flower Constellations is that it introduces the concept of necklaces directly into its formulation, which allows to have closed expressions of the distributions that a constellation can present. This is especially interesting for design since it provides more control in the process, and for optimization techniques, since it is possible to generate any configuration that the theory can provide in a fast and easy procedure.

In addition, three counting theorems are presented, which allow to predict the number of possible combinations that the 2D Necklace Flower Constellations theory can provide. The first covers the number of constellation configurations where a particular distribution is fixed. The second theorem provides the information of the number of possibilities that a particular symmetry generates in the design methodology. On the other hand, the third theorem allows to compute the total number of configurations that a set of satellites can provide for a particular size of fictitious constellation.

Finally, it is important to notice that the number of possibilities obtained using this methodology depends on the size of the fictitious constellation, and thus, it can be increased as much as required. This property is very interesting from a design point of view, since it allows to optimize the methodology to the computational resources available.

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Highlights

- Necklace Flower Constellations is a new design framework to define constellations.
- It allows to expand the possibilities of design, maintaining the number of satellites.
- It contains, as a subset, Walker, Dufour, Draim and 2D Lattice Flower Constellations.
- The configurations obtained present stable structures with symmetric properties.
- The formulation is specially devised for optimization problems.