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Comparing pivoting strategies for almost strictly sign regular matrices

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Abstract

In this paper some properties of two-determinant pivoting for Neville elimination are presented. In particular, we consider a zero-increasing property and we show an optimal normwise growth factor. Comparisons with other pivoting strategies for Neville elimination and with Gaussian elimination with partial pivoting of almost strictly sign regular matrices are performed. Numerical examples are included.

Keywords: pivoting strategies, Neville elimination, almost strictly sign regular matrices
2000 MSC: 65F05, 65F15, 65F35

1. Introduction

Numerical methods adapted to structured classes of matrices have been studied recently. A very important class of structured matrices due to its applications is the class of *sign regular* (SR) matrices. A matrix is SR if all its minors of the same order have the same sign. The importance of nonsingular SR matrices comes from their characterization as variation diminishing transformations. This property has played a crucial role in the applications to Statistics, Economy or Computer-Aided Geometric Design (see [1, 2, 3]).

A relevant subclass of SR matrices is formed by *totally positive* (TP) matrices, that is, matrices such that all their minors are nonnegative. The study of TP matrices began in 1930 with the work of Schoenberg (see [4]).

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The exhaustive research carried out in the field of TP matrices is reflected in books written several decades ago, such as Gantmacher and Krein, whose original was published in Russian in 1941 and has an English version of 2002 (see [5]), or Karlin's 1968 book (see [6]). There are also more recent texts such as that edited by Gasca and Michelli in 1996 (see [7]) and more recently Pinkus' book (see [8]) and Fallat and Johnson's book (see [9]). In contrast, the knowledge about the class of SR matrices is much smaller. This is due, above all, to the much greater difficulties that arise from their study.

The goal of this work is the study of another subclass of SR matrices: the *almost strictly sign regular* (ASSR) matrices (see [10]). These matrices have all their nontrivial minors of the same order with the same strict sign and this subclass contains the nonsingular *almost strictly totally positive* (ASTP) matrices, introduced by Gasca, Miccheli and Peña (see [11]). A nonsingular matrix is ASTP if a minor with consecutive rows and columns is positive if and only if it has positive diagonal entries. Hurwitz matrices and B-splines collocation matrices are examples of ASTP matrices. In general, problems with ASSR matrices are much more difficult to deal with than the corresponding problems with ASTP matrices.

This work analyzes several aspects about the application of some pivoting strategies to ASSR matrices using Neville elimination (NE) or Gaussian elimination (GE). NE is an elimination procedure alternative to GE, very useful when dealing with SR matrices and their subclasses.

In [12] the scaled partial pivoting with respect to the l_∞-norm and Euclidean norm are studied for GE and NE applied to totally positive linear systems. It is proved that in exact arithmetic row exchanges are not necessary. In [13] a backward error analysis of Neville procedure is presented (without pivoting strategy). In the case of TP matrices, the error bounds are similar to those obtained previously by other authors for GE. In 2007 a pivoting strategy called two-determinant pivoting (see [14]) is analyzed. The application of NE with two-determinant pivoting strategy to ASSR matrices is studied in [15]. It is shown that this procedure preserves the almost strict sign regularity and that the associated Wilkinson-type growth factor is optimal. In [16] some componentwise growth factors, taking into account several pivoting strategies, are analyzed.

In this paper, we continue the study of the properties of NE with two-determinant pivoting strategy. In this work, we consider a normwise growth factor, in contrast to the Wilkinson-type growth factor analyzed in [15]. We prove that this pivoting strategy has an optimal normwise growth factor.

A zero-increasing property is also introduced and satisfied by this pivoting strategy. In addition, we show that GE with partial pivoting and other pivoting pivoting strategies for NE do not satisfy the interesting properties proved for NE with two-determinant pivoting.

This work is organized as follows. Section 2 is dedicated to preliminary definitions and notations related to ASSR matrices. In Section 3 we present several pivoting strategies for NE. In Section 4 we compare these pivoting strategies and GE with partial pivoting, using several illustrative examples. Moreover, we present the better properties of the two-determinant pivoting strategies under several points of view.

2. Almost strictly sign regular matrices

For $k, n \in \mathbb{N}$, with $1 \leq k \leq n$, $Q_{k,n}$ denotes the set of all increasing sequences of k natural numbers not greater than n. For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in Q_{k,n}$ and A an $n \times n$ real matrix, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A containing rows $\alpha_1, \ldots, \alpha_k$ and columns $\beta_1, \ldots, \beta_k$ of A. If $\alpha = \beta$, we denote by $A[\alpha] := A[\alpha|\alpha]$ the corresponding principal submatrix.

ASSR matrices present zero entries in certain positions, and can be classified in two classes that are defined below, type-I and type-II staircase.

A matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is called type-I staircase if it satisfies simultaneously the following conditions

- $a_{11} \neq 0, \; a_{22} \neq 0, \; \ldots, a_{nn} \neq 0$;
- $a_{ij} = 0, \, i > j \Rightarrow a_{kl} = 0, \, \forall l \leq j, \; i \leq k$;
- $a_{ij} = 0, \, i < j \Rightarrow a_{kl} = 0, \, \forall k \leq i, \; j \leq l$.

As usual, hereinafter we denote by P_n the backward identity matrix $n \times n$, whose element (i, j) is defined as

$$\begin{cases} 1, & \text{if } i + j = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

So, A is a type-II staircase matrix if it satisfies that $P_n A$ is a type-I staircase matrix.

Next, we present some definitions and basic results.

Definition 1. *For a real matrix $A = (a_{ij})_{1\le i,j\le n}$ type-I (type-II) staircase, a submatrix $A[\alpha|\beta]$, with $\alpha, \beta \in Q_{m,n}$, is nontrivial if all its main diagonal (secondary diagonal) entries are nonzero.*

The minor associated to a nontrivial submatrix ($A[\alpha|\beta]$) is called a nontrivial minor ($\det A[\alpha|\beta]$).

Definition 2. *A vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^n$ is a signature sequence, or simply, a signature, if $|\varepsilon_i| = 1$, $\forall i \in \mathbb{N}$, $i \le n$.*

Definition 3. *A $n \times n$ real matrix A is said to be ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ if it is either type-I or type-II staircase and all its nontrivial minors $\det A[\alpha|\beta]$ satisfy that*

$$\varepsilon_m \det A[\alpha|\beta] > 0, \quad \alpha, \beta \in Q_{m,n}, \qquad m \le n. \tag{1}$$

Some results regarding with ASSR matrices can be seen in [10, 15, 17, 18].

3. Some pivoting strategies for Neville elimination

In this section we briefly present the NE and several row pivoting strategy associated to this method.

NE is a very convenient procedure when working with ASSR matrices and other related types of matrices. If A is a nonsingular $n \times n$ matrix, NE consists of at most $n - 1$ successive major steps, resulting in a sequence of matrices as follows:

$$A = \widetilde{A}^{(1)} \to A^{(1)} \to \cdots \to \widetilde{A}^{(n)} = A^{(n)} = U \tag{2}$$

where U is an upper triangular matrix.

For each t, $1 \le t \le n$, $A^{(t)} = \left(a_{ij}^{(t)}\right)_{1\le i,j\le n}$ has zeros in the positions $a_{ij}^{(t)}$, for $1 \le j \le t$, $j \le i \le n$. Besides it holds that

$$a_{it}^{(t)} = 0, \; i \ge t \Rightarrow a_{ht}^{(t)} = 0, \; \forall h \ge i. \tag{3}$$

The matrix $A^{(t)}$ is obtained from $\widetilde{A}^{(t)}$ reordering rows t, $t+1$, $\ldots$, n according to a row pivoting strategy that satisfies (3).

To obtain $\widetilde{A}^{(t+1)}$ from $A^{(t)}$ we produce zeros in the column t below the main diagonal by subtracting a multiple of the ith row from the $(i+1)$th, for $i = n-1, n-2, \ldots, t$, according to the following formula:

$$\tilde{a}_{ij}^{(t+1)} = \begin{cases} a_{ij}^{(t)}, & 1 \leq i \leq t, \\ a_{ij}^{(t)} - \frac{a_{it}^{(t)}}{a_{i-1,t}^{(t)}} a_{i-1,j}^{(t)}, \quad \text{if } a_{i-1,t}^{(t)} \neq 0, & t+1 \leq i \leq n, \\ a_{ij}^{(t)}, \quad \text{if } a_{i-1,t}^{(t)} = 0, & t+1 \leq i \leq n, \end{cases} \tag{4}$$

for all $j = 1, 2, \ldots, n$.

The element

$$p_{ij} = a_{ij}^{(j)}, \quad 1 \leq j \leq i \leq n \tag{5}$$

is called the (i, j) *pivot* of NE of A and the number

$$m_{ij} = \begin{cases} \dfrac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} \left(= \dfrac{p_{ij}}{p_{i-1,j}} \right), & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0, \end{cases} \tag{6}$$

the (i, j) *multiplier*. Note that $m_{ij} = 0$ if and only if $p_{ij} = 0$ and, by (3),

$$m_{ij} = 0 \implies m_{hj} = 0, \ \forall h > i. \tag{7}$$

Now, we present some pivoting strategies for NE, using row exchanges with similar purposes to those of the pivoting strategies for GE. Recall that $A^{(t)}$ is obtained by reordering the rows of matrix $\widetilde{A}^{(t)}$ by an adequate pivoting strategy with a criterion for the choice of the pivots p_{ij}. GE with partial pivoting chooses the pivots so that all multipliers have absolute value not greater than 1. With a similar purpose, one can define *NE with partial pivoting*. For it, we interchange the rows of $\widetilde{A}^{(t)}$ so that $A^{(t)}$ satisfies

$$|a_{tt}^{(t)}| \geq |a_{t+1,t}^{(t)}| \geq \cdots \geq |a_{nt}^{(t)}|$$

and from $A^{(t)}$ we construct $\widetilde{A}^{(t+1)}$ as in (4).

In [14] a row pivoting strategy associated to NE for nonsingular SR matrices is introduced. It will be called two-determinant pivoting strategy due to the special role played by some 2×2 determinants of some matrices appearing along the Neville procedure.

The criterion of the *two-determinant pivoting* strategy to obtain $A^{(t)}[t, \ldots, n]$ from a reordering of the rows of $\widetilde{A}^{(t)}[t, \ldots, n]$ is the following:

- If $\widetilde{a}_{tt}^{(t)} = 0$: then we reverse the ordering of the rows, that is, $A^{(t)}[t,\ldots,n] := P_{n-t+1}\widetilde{A}^{(t)}[t,\ldots,n]$.
- If $\widetilde{a}_{nt}^{(t)} = 0$: then we do not perform rows exchanges, that is, $A^{(t)} := \widetilde{A}^{(t)}$.
- If $\widetilde{a}_{tt}^{(t)} \neq 0$ and $\widetilde{a}_{nt}^{(t)} \neq 0$, then we compute the determinant $d_1 = \det \widetilde{A}^{(t)}[t,t+1]$.
  - If $d_1 > 0$ then $A^{(t)} := \widetilde{A}^{(t)}$.
  - If $d_1 < 0$ then $A^{(t)}[t,\ldots,n] := P_{n-t+1}\widetilde{A}^{(t)}[t,\ldots,n]$.
  - If $d_1 = 0$ then compute the determinant $d_2 = \det \widetilde{A}^{(t)}[n-1,n|t,t+1]$.
    * If $d_2 > 0$ then $A^{(t)} := \widetilde{A}^{(t)}$.
    * If $d_2 < 0$ then $A^{(t)}[t,\ldots,n] := P_{n-t+1}\widetilde{A}^{(t)}[t,\ldots,n]$.

In Section 3 of [14] it is shown that this pivoting strategy is well defined for nonsingular SR matrices. The computational cost of the NE without row exchanges for an $n \times n$ matrix coincides with the cost of GE without row exchanges. So it has a cost of $\frac{4n^3+3n^2-7n}{6} \simeq \frac{2n^3}{3}$ flops (floating-point operations). Using the two-determinant pivoting strategy, this cost is increased with at most $2n-2$ subtractions and $4n-4$ multiplications. Besides, by Theorem 4.1 of [14], for a nonsingular SR matrix, the two-determinant pivoting strategy for NE is a scaled partial pivoting strategy for any monotone vector norm.

Other pivoting strategies used in both GE and NE are called pairwise pivoting (see [19, 20]). These are suitable for implementations on parallel computers and reduce the communication cost considerably.

In contrast to row pivoting strategies described above (associated with (2)) pairwise pivoting strategies interchange consecutive rows in each step and then produce a zero. In fact, *pairwise pivoting for NE* is defined as follows: to produce a zero at position $(n,1)$ in $\widetilde{A}^{(1)}$, one compares the element $\widetilde{a}_{n1}^{(1)}$ with $\widetilde{a}_{n-1,1}^{(1)}$. If $|\widetilde{a}_{n1}^{(1)}| > |\widetilde{a}_{n-1,1}^{(1)}|$ the corresponding rows are exchanged, in order that $|\widetilde{a}_{n1}^{(1)}/\widetilde{a}_{n-1,1}^{(1)}| \leq 1$. Then the elements of row n are updated producing a zero in the $(n,1)$ entry. We continue with the first column until producing a zero in the $(2,1)$ entry, and then we would continue with the second column and later columns, analogously to NE, until obtaining an upper triangular matrix U. For a nonsingular matrix A, NE with pairwise pivoting consists of at most $n(n-1)/2$ successive steps, resulting in a sequence of matrices as follows:

$$A = \widetilde{A}^{(1)} \to A^{(1)} \to \cdots \to \widetilde{A}^{\left(\frac{n(n-1)}{2}\right)} = A^{\left(\frac{n(n-1)}{2}\right)} = U, \tag{8}$$

where U is an upper triangular matrix.

Remark 1. *Note that the process defined in (8) can be expressed as*

$$E_n\left(-\frac{a_{n,n-1}^{(\frac{n(n-1)}{2}-1)}}{a_{n-1,n-1}^{(\frac{n(n-1)}{2}-1)}}\right)Q_{\frac{n(n-1)}{2}-1}\cdots E_{n-1}\left(-\frac{a_{n-1,1}^{(2)}}{a_{n-2,1}^{(2)}}\right)Q_2E_n\left(-\frac{a_{n,1}^{(1)}}{a_{n-1,1}^{(1)}}\right)Q_1A = U, \tag{9}$$

where, for all $t = 1, 2, \dots, n(n-1)/2-1$, Q_t is the permutation matrix of the corresponding step of (8) and $E_i(\alpha)$ the bidiagonal lower triangular matrix whose (r,s) entry $(1 \le r, s \le n)$ is given by

$$\begin{cases} 1, & \text{if } r = s, \\ \alpha, & \text{if } (r,s) = (i, i-1), \\ 0, & \text{elsewhere.} \end{cases}$$

4. Comparison of the pivoting strategies

In this section, some aspects regarding the application of pivoting strategies defined in the previous section to ASSR matrices are studied. We compare those pivoting strategies under different points of view.

Let A be an $n \times n$ matrix. We shall define several normwise growth factors.

If NE with partial pivoting (g_{NE}^{pp}) (with two-determinant pivoting (g_{NE}^{td}), respectively) associated with (2) is applied, we consider the value

$$g_{NE}^{pp}(A) = \frac{\| Q_1|L_1|Q_2|L_2|\cdots Q_{n-1}|L_{n-1}||U| \|_2}{\|A\|_2} \ (= g_{NE}^{td}(A),\ \textit{respectively}), \tag{10}$$

where $\| \ \|_2$ denotes the Euclidean norm, U is upper triangular, and for all $t = 1, 2, \dots, n-1$, Q_t is the permutation matrix of the corresponding step of (2), $L_t = E_n(m_{nt})E_{n-1}(m_{n-1,t}) \dots\ E_{t+1}(m_{t+1,t})$ and $E_i(\alpha)$ the bidiagonal lower triangular matrix given in Remark 1.

On the other hand, if NE with pairwise pivoting (g_{NE}^{pw}) is applied, then

$$g_{NE}^{pw}(A) = \frac{\| Q_1|L_1|Q_2|L_2|\cdots Q_{(n(n-1)/2-1)}|L_{(n(n-1)/2-1)}||U| \|_2}{\|A\|_2}, \tag{11}$$

where, for all $t = 1, 2, \ldots, n(n-1)/2-1$, Q_t is the permutation matrix of the corresponding step of (8) and L_t is the corresponding $E_i(\alpha)$ obtained from (9).

Note that if NE without pivoting is considered then the growth factor is denoted by $g_{NE}(A)$ and can be defined as in (10) but taking into account that $Q_1 = Q_2 = \cdots = Q_{n-1} = I$.

When GE with partial pivoting (g_{GE}^{pp}) is considered, then

$$g_{GE}^{pp}(A) = \frac{\||L||U|\|_2}{\|PA\|_2}, \tag{12}$$

where P denote the permutation matrix so that $PA = LU$ (a similar value for GE without pivoting can be found in Section 9.3 of [21]).

Next, we present three nice properties of NE with two-determinant pivoting when it is applied to ASSR matrices.

The first property guarantees that almost strict sign regularity is inherited during the elimination process, and it is proved in Theorem 2 of [15]:

Theorem 1. *Let $A = (a_{ij})_{1\le i,j,\le n}$ be an ASSR matrix, and let us apply NE with two-determinant pivoting strategy. Then, for all $t \in \{1, \ldots, n\}$, all matrices $\widetilde{A}^{(t)}[t, \ldots, n]$ are ASSR and $\varepsilon_1(A) = \varepsilon_1(\widetilde{A}^{(t)})$.*

A second property corresponds to the optimal growth factor defined in (10).

Theorem 2. *Let $A = (a_{ij})_{1\le i,j\le n}$ be an $n \times n$ ASSR matrix. Then the growth factor (10) corresponding to NE with two-determinant pivoting is optimal:*

$$g_{NE}^{td}(A) = 1. \tag{13}$$

Proof. By Theorem 1 all multipliers (6) are nonnegative and so all matrices L_i, with $i = 1, \ldots, n-1$, are nonnegative. Besides, the entries of U have the constant sign of the entries of A ($\varepsilon_1(U) = \varepsilon_1(\widetilde{A}^{(n)}) = \varepsilon_1(A)$). Then

$$Q_1|L_1|Q_2|L_2|\cdots Q_{n-1}|L_{n-1}||U| = |Q_1L_1Q_2L_2\cdots Q_{n-1}L_{n-1}U| = |A|,$$

and so

$$g_{NE}^{td}(A) = \frac{\|Q_1|L_1|Q_2|L_2|\cdots Q_{n-1}|L_{n-1}||U|\|_2}{\|A\|_2} = \frac{\|A\|_2}{\|A\|_2} = 1.$$

This growth factor is optimal because, given a general $n \times n$ matrix $A = (a_{ij})_{1\le i,j\le n}$, one has that

$$Q_1|L_1|Q_2|L_2|\cdots Q_{n-1}|L_{n-1}||U| \ge |Q_1L_1Q_2L_2\cdots Q_{n-1}L_{n-1}U| = |A|,$$

and so

$$g_{NE}^{td}(A) \ge 1.$$

□

Remark 2. *Let us observe that by (26) of [17] the computed matrices $\widehat{L}_i$, for $i = 1,\ldots,n$, $\widehat{U}$, obtained with two-determinant pivoting, satisfy that*

$$\widehat{L}_i \to L_i,\ \widehat{U} \to U \quad \text{as} \quad u \to 0.$$

The last property shows that the strategy is always zero-increasing for ASSR matrices. Let us define this property.

Definition 4. *A pivoting strategy for NE is called zero-increasing if $A^{(t-1)} \neq \widetilde{A}^{(t)}$ implies that $\widetilde{A}^{(t)}$ has more zeros than $\widetilde{A}^{(t-1)}$ and the zero entries of $A^{(t-1)}$ are again zero entries of $\widetilde{A}^{(t)}$, for $t = 2,\ldots,n$.*

Clearly, the zero-increasing property has computational advantages because it guarantees that zeros are preserved during the elimination process.

Theorem 3. *The applications of NE with two-determinant pivoting to ASSR matrices is zero-increasing.*

Proof. By Remark 3 of [15] we have that $A^{(1)}[2,\ldots,n]$ and $\widetilde{A}^{(2)}[2,\ldots,n]$ have the zero entries exactly in the same positions, and $\widetilde{A}^{(2)}$ can have mores zeros due to the elimination process in the first column. So, the number of zeros of $\widetilde{A}^{(2)}$ is greater than or equal to the number of zeros of $A^{(1)}$, which coincides with the number of zeros of $A = \widetilde{A}^{(1)}$.

By Theorem 1, $\widetilde{A}^{(2)}[2,\ldots,n]$ is also an ASSR matrix and performing an step of NE with two-determinant pivoting we obtain $\widetilde{A}^{(3)}[2,\ldots,n]$. Applying again Remark 3 of [15], we also conclude that the zeros of $A^{(2)}[2,\ldots,n]$ are preserved in $\tilde{A}^{(3)}[2,\ldots,n]$ and that the number of zeros of $\widetilde{A}^{(3)}[2,\ldots,n]$ is greater than or equal to the number of zeros of $\widetilde{A}^{(2)}[2,\ldots,n]$. Then, the zero-increasing property also holds for the second step.

Continuing analogously, the result holds. □

Finally, several numerical results are presented.

Given an ASSR matrix A we can observe that NE with partial pivoting, GE with partial pivoting and pairwise pivoting for NE do not preserve the structure of this matrix. The following example illustrates this situation.

Let A be an ASSR matrix with signature $\varepsilon = (-1, 1, -1, 1, -1, -1)$

$$A = \widetilde{A}^{(1)} = \begin{pmatrix} -1 & -4 & 0 & 0 & 0 & 0 \\ -2 & -10 & -10 & -16 & -2 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix}.$$

Taking into account that $|a_{21}| = 2 > |a_{11}| = 1$, the application of NE with partial pivoting (which coincides with GE with partial pivoting in the first step) implies that the rows 1 and 2 must be exchanged, so

$$A^{(1)} = \begin{pmatrix} -2 & -10 & -10 & -16 & -2 & 0 \\ -1 & -4 & 0 & 0 & 0 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix},$$

$$\widetilde{A}^{(2)} = \begin{pmatrix} -2 & -10 & -10 & -16 & -2 & 0 \\ 0 & 1 & 5 & 8 & 1 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix}$$

and $\widetilde{A}^{(2)}[2, \ldots, 6]$ is not an ASSR matrix. Note that $\widetilde{A}^{(2)}$ has less zeros than A.

If pairwise pivoting with NE is considered, then

$$A = \widetilde{A}^{(1)} = A^{(1)} = \widetilde{A}^{(2)} = A^{(2)} = \widetilde{A}^{(3)} = A^{(3)} = \widetilde{A}^{(4)}$$

and the matrices $A^{(4)}$ and $\widetilde{A}^{(5)}$ can be expressed as

$$A^{(4)} = \begin{pmatrix} -2 & -10 & -10 & -16 & -2 & 0 \\ -1 & -4 & 0 & 0 & 0 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix},$$

$$\widetilde{A}^{(5)} = \begin{pmatrix} -2 & -10 & -10 & -16 & -2 & 0 \\ 0 & 1 & 5 & 8 & 1 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix}.$$

Observe that the property of almost strict sign regularity is not inherited by $\widetilde{A}^{(5)}[2, \ldots, 6]$. Note also that $\widetilde{A}^{(5)}$ has less zeros than A.

On the other hand, the value of the norm growth factors previously defined are $g_{NE}^{pw}(A) = g_{GE}^{pp}(A) = 1.01641131$ and $g_{NE}^{pp}(A) = 1.20884472$.

As for two-determinant pivoting strategy, we get the following matrices $A = \widetilde{A}^{(1)} = A^{(1)}$ and

$$\widetilde{A}^{(2)} = \begin{pmatrix} -1 & -4 & 0 & 0 & 0 & 0 \\ 0 & -2 & -10 & -16 & -2 & 0 \\ 0 & -6 & -33 & -60 & -21 & 0 \\ 0 & -8 & -46 & -92 & -70 & -36 \\ 0 & 0 & -9 & -60 & -242 & -316 \\ 0 & 0 & -6 & -60 & -443 & -2823 \end{pmatrix}.$$

In this case $\widetilde{A}^{(2)}[2, \ldots, 6]$ is ASSR and $\widetilde{A}^{(2)}$ has more zeros than A. These properties also hold by matrices $\widetilde{A}^{(t)}[t, \ldots, 6]$, for all $t = 3, \ldots, 6$, when we apply NE with two-determinant pivoting and the zero-increasing property holds. Besides, $g_{NE}^{td}(A) = 1$ and in this case the stability of the strategy is assured.

Focusing in the norm growth factors and considering the following ASSR matrices

$$A_1 = \begin{pmatrix} -10^{-5} & -1 & -1 \\ -2 & -5 & -2 \\ -3 & -1 & 0 \end{pmatrix}, \quad \varepsilon = (-1, -1, 1),$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1-10^{-7} & 10^{-7} \end{pmatrix}, \quad \varepsilon = (1, -1),$$

$$A_3 = \begin{pmatrix} -260 & -100 & -71 & 0 \\ -179 & -70 & -51 & -10 \\ -10 & -4 & -3 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \quad \varepsilon = (-1, 1, 1, -1),$$

$$A_4 = (a_{ij})_{1\le i,j\le 10},\ a_{ij} = j^{10-i}, \quad \varepsilon = (1, -1, -1, 1, 1, -1, -1, 1, 1, -1).$$

The results obtained are collected in Table 1.

	g_{NE}	g_{NE}^{pp}	g_{NE}^{pw}	g_{NE}^{td}	g_{GE}^{pp}
A_1	1.63926169e+05	1	1	1	1
A_2	1.61803385	1.61803385	1.61803385	1	1.61803385
A_3	1.00001235	1.00001138	1.00001138	1	1.00000683
A_4	11.01193352	11.01193352	11.01193352	1	6.22301661

Table 1: Numerical results: norm growth factors

With matrices A_1 and A_3 all strategies exchange some rows, up to NE without pivoting. With matrix A_2, only two-determinant pivoting implies row exchanges.

Let us observe that A_4 is a Vandermonde matrix with reversed rows and it is an almost strictly sign regular matrix, where the sign of a $k \times k$ minor is $\varepsilon_k = (-1)^{k(k-1)/2}$, with $k = 1, 2, \ldots, 10$. This matrix appears, for instance, in moment problems, when determining the weights for quadrature rules.

Table 1 allows us to observe, with regard to normwise growth factor, that the best value for all cases is obtained with NE with two-determinant pivoting; with this strategy $g_{NE}^{td}(A) = 1$ and optimal stability is assured, confirming our theoretical result.

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