Models, axiomatics and geometry of the hyperbolic plane

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28 de junio de 2018

Prologue

What made me choose

“Modelos, axiomática y geometría del plano hiperbólico”

as a topic of this work?

Many times I ask myself what the trajectory of the planes is and how is it under the geometrical concepts. With a lot of effort and with the great help of my tutor I have obtained a clear answer based on scientific concepts. However, I consider this work in dimension 2 as a first step towards understanding the geometry in the hyperbolic space.

First of all, we need to understand why there are other geometries such as Hyperbolic geometry besides the intuitive Euclidean geometry. Who discovered this geometry? Lobachevski, a young scientist who decided to leave the medical career to devote himself completely to the study of a geometry that he called “imaginary geometry”, is a founder of this geometry. He made progress not only in mathematics but also in physics, such as Einstein’s theory of relativity.

How did Lobachevski come up with this geometry? Before giving an answer to this question, let’s see what an axiomatic system is.

The axiomatic method allows us to show that a statement is correct based on a set of given “rules”. This method consists of using tools such as definitions, axioms, and postulates, that are trivial truths, to obtain a logical deduction, called **theorem**. The first axiomatic text that has been preserved appears in Euclid’s Elements around 300 BC in Greece. In this book, Euclid wrote definitions, axioms, and postulates which give the foundation of what we now call Euclidean geometry. The parallel postulate was for long suspected to be superfluous in Euclid’s axiomatic system and hence there were numerous attempts to deduce it from the other four postulates.

In the 19th century Lobachevsky thought that the fifth postulate was independent of the previous four and, denying the V postulate and using the previous four, he arrived at the construction of a new coherent logical model. Nowadays this axiomatic system is known as **hyperbolic geometry**.

Accepting that there exist at least two lines through any given point that do not intersect another given line, we can deduct that points, lines, and any figures in the new plane, the hyperbolic plane, will be defined differently than in the Euclidean plane. Another interesting difference with the Euclidean geometry is that the angles of a triangle can add up to *any positive real number less than* π*!*

There are a lot of models that allow us to visualize the hyperbolic plane. However, in this work we are going to describe two of the most important ones, i.e the Poincaré Disc and the Klein Model. In each of these models the objects such as points, lines, etc. are defined differently and each model has its own advantages. For instance, lines are easier to visualize and work with in the Klein model, meanwhile angles are easier to measure in the Poincaré model. There is a useful geometric conformal isomorphism between both models. This isomorfism will allow us to map points and lines from the Klein model into the Poincaré and hence help us measure angles in the Klein model.

In this work we will describe the basic ruler and compass constructions in the Poincaré disk such as drawing lines through two given points, inversion with respect to a circle, angles of parallelism (this is a consequence of the existence of more than one parallel line passing through a given point).

A second approach will be more analytical, such as the formula of Bolyai-Lobachevsky which is a fundamental theorem that relates the angle of parallelism to distance. This formula also provides a way to connect hyperbolic and circular functions.

Finally, we will prove some trigonometric identities that involve both circular and hyperbolic functions.

To write this thesis we have read and reworded ideas contained in the following interesting articles [1, 4, 6] and books [2, 3, 5].

0.1 Resumen

Vamos a ver primero qué es un sistema axiomático (notar que debe cumplir la consistencia y la independencia). Dicho método consiste en un conjunto de axiomas que se utilizan, mediante deducciones, para demostrar teoremas. Ejemplos de sistemas axiomcos deductivos son la geometría euclidiana compilada por Euclides en los Elementos.

Este libro empieza con sus cinco postulados, de los cuales el quinto: “Dada una recta l y un punto $P \notin l$, existe una única recta paralela a l y que pasa por el punto P.” Durante muchos anos se sospechaba que el quinto postulado se deducía de los cuatro anteriores. Sin embargo, no se llegó a ninguna demostración válida.

Otro ejemplo de sistema axiomático es la teoría de conjuntos en la que se emplean los axiomas de Zermelo-Fraenkel, complementados por el axioma de elección (axiom of choice).

Durante el siglo XIX algunos matemáticos trataron de llevar a cabo un proceso de formalización de la matemática a partir de la teoría de conjuntos. Gottlob Frege intentó culminar este proceso creando una axiomática de la teoría de conjuntos. Lamentablemente, Bertrand Russell descubrió en 1901 una contradicción, la llamada paradoja de Russell.

Poco más tarde, Lobachevsky y Bolai declararon que el quinto postulado es independiente de los anteriores, formando así sistema axiomático. Más aún, Lobachevsky declararó que:

Dada una recta l y un punto $P \notin l$ existen al menos dos rectas l_1, l_2 t.q. $l_1 \cap l = \emptyset$ $(l_1 \parallel l)$ y $l_2 \cap l = \emptyset$ $(l_2 \parallel l)$

(Notar que es la negación del 5 postulado). Ver el dibujo de las rectas paralelas (ver Figura 1)

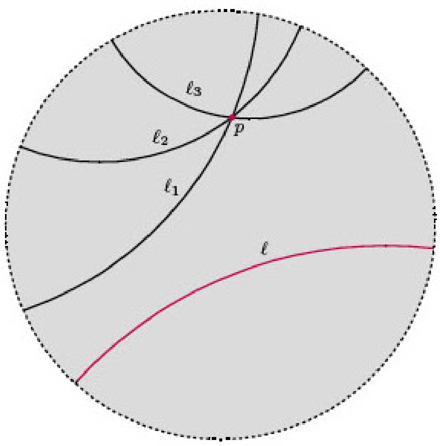

Figure 1: Rectas paralelas por el mismo punto.

Y así partiendo de la contradicción de dicho postulado, junto con los cuatro anteriores, llegó a la construcción de nueva geometría, llamada **Geometría Hiperbólica**.

De hecho, existen otras geometrías no euclideas,como por ejemplo, la esférica.

El objetivo de este trabajo es estudiar los modelos y la geometría del plano hiperbólico. Describiremos los modelos más conocidos, Modelo de Poincaré y Modelo de Klein, que nos permitirán visualizar el plano hiperbólico. Puesto que el Modelo de Poincaré tiene más ventajas, por ejemplo, es conforme (es decir, los ángulos se miden como en el plano euclídeo), nos vamos a centrar en estudiar dicho modelo. También mostraremos construcciones básicas con regla y compás de los objetos en el disco de Poincaré.

Ahora bien, uno se puede plantear la siguiente pregunta: ¿Y todo esto tiene algo en común con el plano euclídeo? La respuesta es la siguiente: cerca del origen, las geometría euclidea y la hiperbólica representan propiedades similares, razón por la cual, tanto las rectas, como los triángulos (de hecho, cualquier figura) se ven casi iguales. En cambio, conforme nos vayamos alejando del centro del disco unidad, dichas geometrías son muy distintas. Todo esto nos hace pensar que la distancia entre dos puntos en el plano hiperbóico no podrá ser igual a la distancia euclidea. Para ello hablarémos de un nuevo concepto, el *estar entre*, gracias a cual obtendrémos la distancia hiperbólica entre dos puntos en los distintos modelos. Observar que, dado que los puntos y rectas vienen definidas de distinas formas en el modelo de Poincaré y en el de Klein, las disntancias no serán exactamente igual.
Una manera de entender mejor la geometría hiperbólica es a través de la obra de Escher (ver anexo), en la que el disco de Poincaré es representado por animales teselados. Escher (1898-1972) fue un artista neerlandés conocido por sus grabados xilográficos, y sus dibujos que consisten en figuras imposibles, teselados y mundos imaginarios.

Verémos también el isomorfismo entre los modelos mencionados anteriormente, que nos ayudará a la hora de demostrar algunas identidades trigonométricas, así como medir los ángulos.
Para poder medir en la geometría hiperbólica se definen las funciones trigonométricas hiperbólicas a partir de cuales se establecen algunos resultados destacados de esta geometría. Darémos también una demostración de la fórmula de Bolyai-Lobachevky que relaciona ángulo de paparelismo (en el plano hiperbólico) y la distancia euclidea. Dicha fórmula nos permitirá hallar la distancia euclidea entre dos puntos, sabiendo la magnitud del ángulo de paralelismo.

Por último, daremos las demostraciones de identidades trigonométricas para un triángulo cualquiera en el plano hiperbólico.

Contents

Chapter 1

Introduction

1.1 Historical introduction

In the ancient world, geometry was used as a practical tool to solve problems in fields such as architecture and navigation. As fragmented knowledge grew, mathematicians felt the need to approach geometry in a more systematic fashion. This resulted in a breakthrough in Greece around 300 BC with the publication of Euclid's Elements, a mathematical treatise that was regarded as a paradigm of rigorous mathematical reasoning for the next two thousand years [Mueller, 1969]. In this work, Euclid wrote definitions, axioms and postulates which give the foundation of what we now call Euclidean geometry. **The five postulates in Elements** are interesting in particular, and **can be rephrased as follows** (compare with [Euclid, 1908, page 154-155]):

1. There is one and only one line segment between any two given points.
2. Any line segment can be extended continuously to a line.
3. There is one and only one circle with any given center and any given radius.
4. All right angles are congruent to one another.
5. If a line falling on two lines make the interior angles on the same side less than two right angles, then those two lines, if extended indefinitely, meet on the side on which the angles are less than two right angles.

The fifth postulate, which is seemingly the most complex one, is called **the parallel postulate**, as a pair of parallel lines is interpreted as two lines that do not intersect. Given the other four postulates, the postulate is equivalent to *Playfair's axiom*, which has a simpler formulation:

Axiom 1.1.1 (Euclidean Parallel Axiom)**.** *Given a line and a point not on the line, there is at most one line through the point that is parallel to the given line.*

The parallel postulate was for long suspected of being superfluous in Euclid's axiomatic system and hence there were numerous attempts to deduce it from the other four postulates. [Cannon et al., 1997] and other sources lists many mathematicians who attempted this, beginning as early as the fifth century.

By assuming that the postulate was false and looking for a contradiction, they discovered many interesting and counterintuitive results. The following is a brief discussion of the most well-known attempts. The Italian Gerolamo Saccheri (1667-1733) showed in 1733 that one of the following statements must be true for each geometry: either the angles of a triangle add up to a number less than, equal to or greater than π respectively.

Saccheri proved that the third statement leads to a contradiction under Euclid's first four postulates. However, his proof of the falseness of the first statement was flawed. The second statement can be shown to be equivalent to the parallel postulate, so if Saccheri's proof had been correct, he would have succeeded in his task of proving the parallel postulate.

It was not until the 19th century when mathematicians abandoned these efforts for reasons which will now be explained. Consider an axiomatic system that includes Euclid's first four postulates but replaces the fifth one with the following:

Axiom 1.1.2 (The Hyperbolic Parallel Axiom)**.** *Given any line ℓ and any point P not on ℓ, there exists more than one line m such that P is on m and m is parallel to ℓ. In other words, there exists a line ℓ such that for some point P not on ℓ at least two lines parallel to ℓ pass through P.*

A consistent model of this axiomatic system implies that the parallel postulate is logically independent of the first four postulates. Deep and independent investigation by **János Bolyai (1802-1860)** from Hungary and **Nikolai Lobachevsky (1793-1856)** from Russia led them conclude that this **axiomatic system**, which we today call **hyperbolic geometry**, was seemingly consistent, hence these two mathematicians have traditionally been given credit for showing the logical independence of the parallel postulate and for the discovery of hyperbolic geometry.

Hyperbolic geometry is an imaginative challenge that lacks important features of Euclidean geometry such as a natural coordinate system. Its discovery had implications that went against then-current views in theology and philosophy, with philosophers such as Immanuel Kant (1724-1804) having expressed the widely-accepted view at the time that our minds will impose a Euclidean structure on things a priori, meaning essentially that the existence of non-Euclidean geometry is impossible. Only with the work of later mathematicians, hyperbolic geometry found acceptance, which occurred after the death of both Bolyai and Lobachevsky.

One way to better understand the hyperbolic geometry is through Escher's artwork, in which the Poincaré disk is represented by tessellated animals. Escher (1898-1972) was a Dutch artist known for his xylographic prints, and his drawings consisting of impossible figures, tessellations and imaginary worlds (see 1.1).

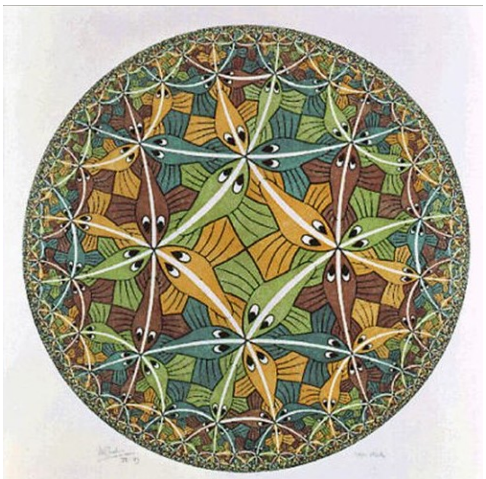

Figure 1.1: Hyperbolic tessellation by Escher.

Chapter 2

Hyperbolic Models

As we know Hyperbolic geometry is non-Euclidean geometry in which the parallel postulate from Euclidean geometry is replaced. As a result, in hyperbolic geometry, there is more than one line through a certain point that does not intersect another given line. The hyperbolic plane is a plane with constant negative curvature and this is that makes us see trigonometric figures (in said plane) in a different way than in the Euclidean plane. Here we have an example of different forms of triangles in the hyperbolic plane.

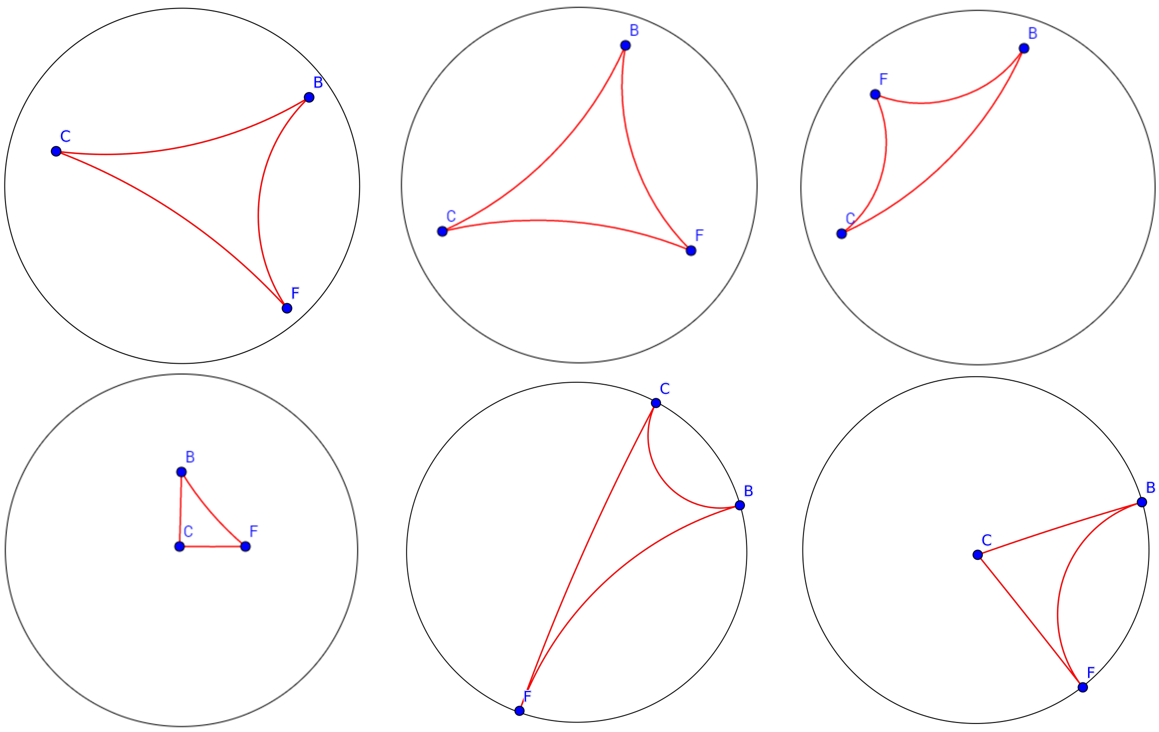

Figure 2.1: Examples of hyperbolic triangles in the Poincaré model

Euclidean plane geometry is usually represented by points and lines in $\mathbb{R} \times \mathbb{R}$, however, there are different models to represent the hyperbolic plane. The most commonly used are: the Klein Model, the Poincaré Disc, the upper half Poincaré plane, and Lorentz's model or hyperboloid. The general characteristics of these models are the following:

- **Klein's Model**, also known as projective disk and Beltrami-Klein's model, it represents the plane as the interior of a disk and its lines are the chords of the circle.

- **Poincaré Disc** also represents the plane as the interior $\mathbb{D}$ of the unit disk $\bar{\mathbb{D}}$, however lines are represented by either arcs of orthogonal circles to the circumference γ or diameters of γ.
- **The upper half Poincaré plane** is the interior of the upper half plane of the Euclidean plane. Its lines are either vertical lines or semi-circumferences centered at points on the x-axis.
- **Lorentz's model or hyperboloid**. In this case a sheet of a hyperboloid of revolution is used. The points are equivalence classes of vectors that satisfy a certain quadratic form and the lines result from the intersection of certain planes with the hyperboloid.

2.1 Poincaré Model

Now we are going to describe the Poincaré Model described above.

Consider a unit disk $\bar{\mathbb{D}}$ in the Euclidean plane. Now we are going to define points, lines and betweenness in hyperbolic plane.

Definition 2.1.1 (The Poincaré Model)**.**

1. **Points of the hyperbolic plane**: they are defined as the points in $\mathbb{D}$, the interior of the unit disk.
2. **Lines of the hyperbolic plane** are the diameters of the circumference γ and arcs of circumferences that are perpendicular to γ.
3. **Betweenness** Let A, B, C be three aligned points, that is to say on a hyperbolic line m. If m is a diameter we keep the Euclidean concept. In another case, m is an arc coming from an orthogonal circle δ with center in R. We define B **to be between** A and C if the Euclidean line AC separates R and B.

How can we know **which arcs of circles are hyperbolic lines**? In other words, how can we know if a circle cuts another orthogonally? Before we respond this question we are going to define the inverse of a point.

Definition 2.1.2 (The inverse of a point)**.** Let γ_r be a circumference of radius r with center O. For any point $P \neq O$ the inverse P' of P with respect to γ_r is the unique point P' on the ray $\overrightarrow{OP}$ such that $|OP| \cdot |OP'| = r^2$, where $|AB|$ is the Euclidean distance between A and B.

The map

Note that when we are working in the Poincaré Model, the radius r is always equal to 1. So in definition 2.1.2, two points P and P' are inverses if $|OP| \cdot |OP'| = 1$.

To *determine if an arc is orthogonal to* γ_r *or not* we are going to see the following:

Proposition 2.1.1. *Let P be any point that does not lie on γ_r and that does not coincide with the center O of γ_r, and let δ be a circumference through P. Then δ cuts γ_r orthogonally if and only if δ passes through the inverse point P' of P with respect to γ_r.*

Given a circumference γ_r. The map that sends a point to its inverse with respect to γ_r is called *inversion*. Inversions with respect to γ_r leave the circumference γ_r pointwise fixed and by Proposition 2.1.1 they leave any orthogonal circumference invariant. This proves the following.

Proposition 2.1.2. *Let γ and γ_r be orthogonal circumferences as above. Denote by δ the arc of γ_r that is inside the unit disk $\mathbb{D}^2$ and by R_1 and R_2 the two regions separated by δ in $\mathbb{D}^2$. Then the inversion with respect to γ_r interchanges R_1 and R_2 leaving $R_1 \cap R_2 = \delta$ fixed.*

In particular, given any point P in $\mathbb{D}^2$, there is an inversion that sends P to O and a given arc through P to a diameter.

Proof. The ray $\overrightarrow{OP}$ intersects γ_r at a point O' outside γ. Take the circumference δ centered at O' and orthogonal to γ. The inversion ϕ with respect to δ satisfies the following:

- $\phi(\gamma) = \gamma$ since δ is orthogonal to γ.
- $\phi|_\delta$ is the identity.
- $\phi(\overline{OP}) = \overline{OP}$ since $O' \in \overline{OP}$.
- $\phi(\gamma_r)$ is a line passing through $\phi(P)$, since $O' \in \gamma_r$.
- $\phi(P) = O$ by construction, since O is the center of γ an orthogonal circumference to δ and the tangencies to δ from O are aligned with P.

□

Note that betweenness is one of the several things that is defined the same in both the Euclidean and hyperbolic planes.

In the hyperbolic plane, distance from one point to another is different than what we call distance in the Euclidean plane. In order to determine the distance, we need to know what the cross-ratio is.

Definition 2.1.3 (Cross-ratio)**.** Let P, Q, A and B distinct points in $\mathbb{D}^2$, then their cross-ratio is $[P, Q, A, B] = \frac{|PB| \cdot |QA|}{|PA| \cdot |QB|}$ where $|PB|$, $|QA|$, $|PA|$ and $|QB|$ are the Euclidean lengths of those segments.

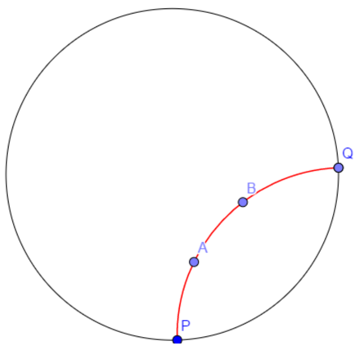

Figure 2.2: Four distinct points on a line $\ell_{A,B}$.

2.1.1 Poincaré distance

Definition 2.1.4 (Poincaré length)**.** Let A, B be two distinct points in $\mathbb{D}^2$. Consider $\ell_{A,B}$ the hyperbolic line passing through A and B. Take the points P and Q in $\ell_{A,B}$ on the circumference γ as in Figure 2.2. We define the *Poincaré length* $d(A, B)$ as $d(A, B) = |\ln([P, Q, A, B])|$.

One can prove that the Poincaré length in fact defines a distance in the Poincaré disk.

The hyperbolic distance is a distance

Let us prove that d defines a distance:

1. $d(A,B) \geq 0$ by definition. However, proving that $d(A,B) = 0$ if and only if $A = B$ is more subtle. Note that $d(A,B) = 0$ if and only if $|PB| \cdot |QA| = |PA| \cdot |QB|$, that is, $\frac{|PA|}{|QA|} = \frac{|PB|}{|QB|}$. If one defines the function $f : \ell \to \mathbb{R}$ from the hyperbolic arc between P and Q as $f(C) = \frac{|PC|}{|QC|}$, the result follows from the injectivity of f.

   Proof. To prove the injectivity of the function $f : \ell \to \mathbb{R}$ defined by $f(C) = \frac{|PC|}{|QC|}$ it is enough to see that its derivative is different from zero. Consider ℓ the hyperbolic line passing through A. Take the points $P = (1,0)$ and $Q = (x,y)$ in ℓ on the circumference γ. Then draw the tangent line to γ at P $T = \{x = 1\}$ and choose a point $\bar{Q} = (1,a)$ on T as a center of a δ. Note that the new cirumference has a radius a. How can we determine the coordinates of the point Q? Let calculate the Euclidean distance from Q to $\bar{Q} = (1,a)$.

   Since $Q \in \delta$ and δ has a radius a

   $$d(Q,\bar{Q})^2 = (x-1)^2 + (y-a)^2 = a^2 \tag{2.1}$$

   As $Q \in \gamma$, we have

   $$x^2 + y^2 = 1 \tag{2.2}$$

   Now, by (2.1), we obtain: $x^2 - 2x + 1 + y^2 - 2ay + a^2 = a^2$, so $2 = 2(x + ay)$. Hence $x = 1 - ay$ and by (2.2) we then have $(1-ay)^2 + y^2 = 1$, so $1 + a^2y^2 - 2ay + y^2 = 1$, then $y(-2a + y + ay^2) = 0$. Hence there are two solutions: $y = 0$, which is the point P and the other is $y = \frac{2a}{1+a^2}$. Therefore, $x = 1 - a(\frac{2a}{1+a^2}) = \frac{1-a^2}{1+a^2}$. So we already have the point $Q = (\frac{1-a^2}{1+a^2}, \frac{2a}{1+a^2})$.
   Note that the point $A \in \delta$, hence $A = (1 + a\cos t, a + a\sin t)$ and it depends on t.

   Now, since A is an interior point of γ, the Euclidean distance $|PA|$ is given by:

   $$d(P,A)^2 = |PA|^2 = a^2\cos^2 t + a^2(1+\sin t)^2 = 2a^2 + 2a^2\sin t = 2a^2(1+\sin t)$$

   The distance from Q to A is defined by:

   $$\begin{aligned} d(Q,A)^2 = |QA|^2 &= a^2(a\cos t - \frac{2a^2}{1+a^2})^2 + a^2(1+\sin t - \frac{2}{1+a})^2 = \\ &= a^2\Big[1 - 4\frac{a}{1+a^2}cost + \frac{4a^2}{(1+a^2)^2} + \frac{(a^2-1)^2}{(1+a^2)^2} + 2\frac{a^2-1}{1+a^2}\sin t\Big] \\ &= 2a^2\Big[1 + \frac{a^2-1}{1+a^2}\sin t - \frac{2a}{1+a^2}\cos t\Big] \end{aligned}$$

   Hence, $\tilde{f}(t) = f^2(A) = \frac{|QA|^2}{|PA|^2} = \frac{1+\frac{1}{1+a^2}((a^2-1)\sin t - 2a\cos t)}{(1+\sin t)}$

   and $\tilde{f}'(t) = \frac{\frac{1}{1+a^2}((a^2-1)\cos t + 2a\sin t)(1+\sin t)) - (1+\frac{1}{1+a^2}((a^2-1)\sin t - 2a\cos t))\cos t}{(1+\sin t)^2}$.
   Note that the denominator is always positive, so it is enough to check the sign of the numerator. Then we have: $\frac{1}{1+a^2}(2a + (a^2-1)\cos t + 2a\sin t) - \cos t = 2a + (a^2-1)\cos t + 2a\sin t - (a^2+1)\cos t = 2a - 2\cos t + 2a\sin t$.
   Then, let's see that $a - \cos t + a\sin t = a(1+\sin t) - \cos t$ is a possitive number.
   Since $\sin t \in [-1,0)$ and $a > 0$ because it is the radius of δ,
   we obtain $a(1+\sin t) - \cos t > 0$.
   Then $\tilde{f}'(t) > 0$. (Note that as A is closer to P, $\frac{|QA|}{|PA|}$ is getting bigger).

   □

2. the symmetric property $d(A,B) = d(B,A)$ is immediate since

$$\begin{aligned} d(A,B) &= |\ln([P,Q,A,B])| = \left|\ln\left(\tfrac{|PB|\cdot|QA|}{|PA|\cdot|QB|}\right)\right| \\ &= \left|-\ln\left(\tfrac{|PA|\cdot|QB|}{|PB|\cdot|QA|}\right)\right| = |\ln([P,Q,B,A])| \\ &= d(B,A) \end{aligned}$$

3. Finally, the triangular inequality

$$d(A,B) \leq d(A,C) + d(C,B)$$

for any three points in $\mathbb{D}^2$.

Distance in a special case

In the particular case where $A = O$ formulas become easier. In that case, the line $\ell_{A,B}$ is a Euclidean line (the diameter). For simplicity, one can assume that $O = (0,0)$ and $B = (x,0)$, then $P = (-1,0)$ and $Q = (1,0)$. We can see that the Euclidean length of PB is $(1+x)$ and that of QO is 1. Hence, $|PB|\cdot|QO| = (1+x)$. Now, since we know that the Euclidean distance of $|QB|$ is $(1-x)$, so $|PO|\cdot|QB| = (1-x)$. Therefore,

$$d(O,B) = \left|\ln\left(\frac{|PB|\cdot|QO|}{|PO|\cdot|QB|}\right)\right| = \left|\ln\left(\frac{1+x}{1-x}\right)\right|. \tag{2.3}$$

where x denotes the Euclidean distance $|OB|$.

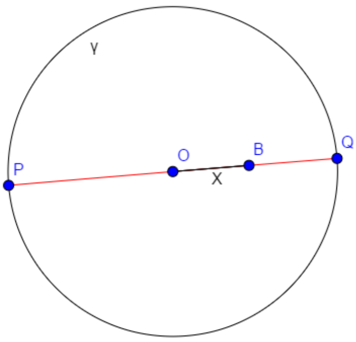

Figure 2.3: Poincaré distance from the origin.

Note that: the Euclidean distance $t = |OB|$ can never be equal to 1 because B is a point in the Poincaré model, and thus it is inside the unit circle γ. However, as t approaches 1, the Poincaré length from B to O is going to infinity.

By (2.3) we obtain the following theorem :

Theorem 2.1.1 (Poincaré length)**.** *If a point B inside the unit disc is at a Euclidean distance t from the origin O, then the Poincaré length from B to O is given by $d(O,B) = |\ln(\frac{1+t}{1-t})|$. Moreover,*

$$\sinh d = \tfrac{2t}{1-t^2}, \quad \cosh d = \tfrac{1+t^2}{1-t^2}, \quad \tanh d = \tfrac{2t}{1+t^2}.$$

2.1.2 Poincaré angles

Another similarity between the Euclidean and hyperbolic planes is angle congruence, which will be denoted by $\cong$. This has the same meaning in both planes. For the Poincaré model, since lines can be circular arcs, we need to define the measure of an angle.

In the hyperbolic plane, the way we find the degrees in an angle is conformal to the Euclidean plane. In the Poincaré model, we have three cases to consider, which are described in Figure 2.4:

- Case 1: where two circular arcs intersect.
- Case 2: where one circular arc intersects an ordinary ray.
- Case 3: where two ordinary rays intersect.

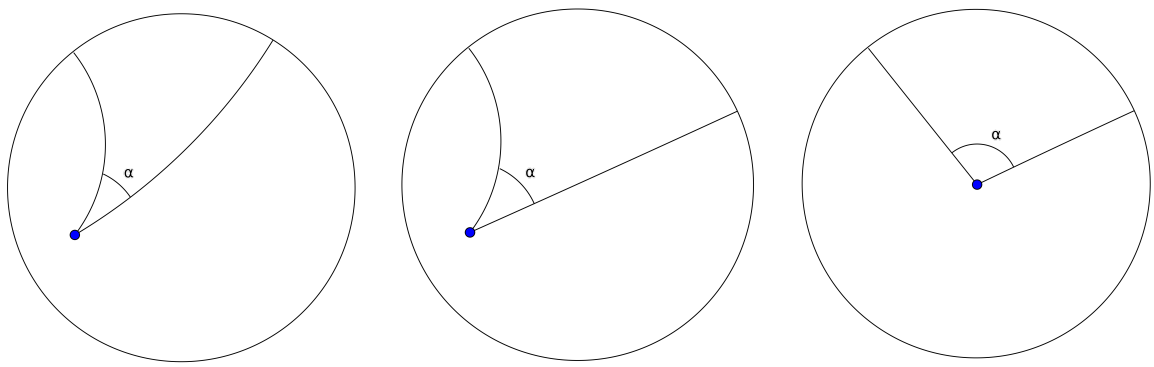

Figure 2.4: Examples of angles in the Poincaré model

Summarizing, in this model the circumference of the disc represents infinity. All points and lines exist only inside the disc. Lines in this plane are called geodesics and are defined as arcs of circles that meet the circumference orthogonally.

2.2 Klein Model

The Klein Model, also known as the Beltrami-Klein Model, is a disc model of hyperbolic geometry via projective geometry.

Let κ be a circle in the Euclidean plane with center O and radius OR. Let's see how are points and lines defined in the hyperbolic plane using this new model.

Definition 2.2.1 (The Klein Model)**.**

1. **Points in the hyperbolic plane** are all points X inside κ, i.e. such that $OX < OR$.
2. **Lines of the hyperbolic plane** are chords inside the circle κ excluding their endpoints.

Betweenness in the Klein model is defined similarly as betweenness in the Poincaré model.

To define the distance we need to use the cross-ratio again.

Definition 2.2.2 (Klein distance between two points). Let A and B be two points in the circle κ and P and Q be the endpoints of the chord $c_{A,B}$. Then, the Klein distance $d_\kappa(A,B)$ between the points A and B is defined as

$$d_\kappa(A,B) = \frac{|\ln([P,Q,B,A])|}{2}.$$

Now, we ask ourselves the following question: **How can we measure angles in the Klein model?** Remember that in the Poincaré model, we had to consider three cases, however, for the Klein model is much more complicated. The Klein model is only conformal at the origin. As a result, finding the measurement of angles at the origin is the same as finding them in the Euclidean plane. The difficulty begins when an angle is not at the origin. To give an answer to the previous question, first we need to introduce an isomorphism between the Klein and Poincaré models. This isomorphism will allow us to map lines and points from the Klein model into the Poincaré model. Hence, to ease the process of measuring an angle in the Klein model, we will map that angle into the Poincaré model and then measure it there. Before we describe this isomorphism in Theorem 3.2.1 we need to develop some tools.

Chapter 3

Basic geometric constructions

In this section we are going to see different geometric constructions in Poincaré's hyperbolic plane which will be very useful in the future.

3.1 Constructions of lines in the Poincaré model

Let's start with the lines in this plane. How do we draw them? A general way to draw a hyperbolic line in $\mathbb{D}^2$ can be described as follows:

1. Pick a point A on the circumference γ.
2. Construct t the line perpendicular to the ray $\overrightarrow{OA}$ at A, that is, the tangent Euclidean line to γ at A.
3. Choose a point P on t as the center of a circumference γ_r passing through A, where $r = |AP|$.
4. Let B denote the second point of intersection of γ_r with γ. Since $|PB| = |PA|$, the chord Theorem implies that $\overrightarrow{PB}$ is tangent to γ and hence γ and γ_r are orthogonal at B. Then the arc AB represents a hyperbolic line.

In fact, this way one can draw *almost* every hyperbolic line *passing* through A. Note that the only missing line through A is the diameter of $\mathbb{D}^2$ containing A.

With this basic construction we will address the problem of drawing the line through two given points A, B in $\mathbb{D}^2$. To do it we need to consider these three cases:

- Case 1: A and B belong to γ.
- Case 2: A belongs to γ and B lies inside γ.
- Case 3: A and B both lie inside γ.

Case 1: Let O be the center of γ. Now construct rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then draw the lines t_A (resp. t_B) perpendicular to $\overrightarrow{OA}$ (resp. $\overrightarrow{OB}$) at A (resp. B). Let Q be the point of intersection of those two lines. The circle γ_r centered at Q with radius $r = |QA| = |QB|$ intersects γ at A and B. The arc of γ_r that lies inside γ is the hyperbolic line between A and B. Note that this is the line we needed and that is orthogonal by construction (see Figure 3.1).

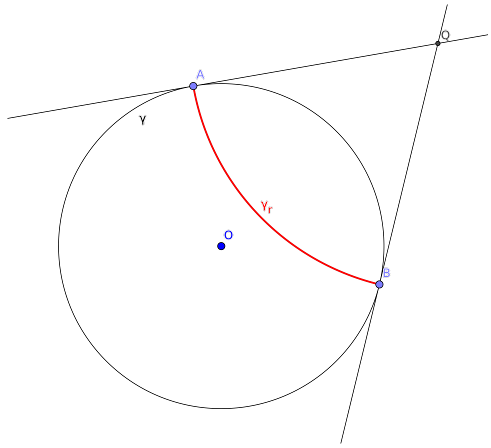

Figure 3.1: Hyperbolic line through $A, B \in \gamma$.

Case 2: Construct rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where A is a point on γ and B lies inside $\mathbb{D}^2$. Now construct t_A as above the line perpendicular to $\overrightarrow{OA}$ at A, that is, the tangent line to γ at A. Then draw the segment $\overline{AB}$ and construct its perpendicular bisector l.

Let Q be the point of intersection of t_A and l. Now consider the circle γ_r centered at Q with radius $r = |QA| = |QB|$. This circumference obviously contains A and B. Let's see that the arc of γ_r that lies inside $\mathbb{D}^2$ is the hyperbolic line containing A and B (see Figure 3.2).

By construction, this arc is orthogonal to γ at A. Now, we want to see that it is also orthogonal at the other point of intersection with γ. Let that point of intersection be D. Then, $D \in \gamma$, in other words, $|OA| = |OD|$ (i). Since D lies on γ_r it follows that $|QD| = |QA|$ (ii).

Therefore the triangles $\triangle OAQ$ and $\triangle ODQ$ are congruent (by (i), (ii) and because they share the side OQ). It means that $\angle ODQ = \angle OAQ$ which is a right angle.

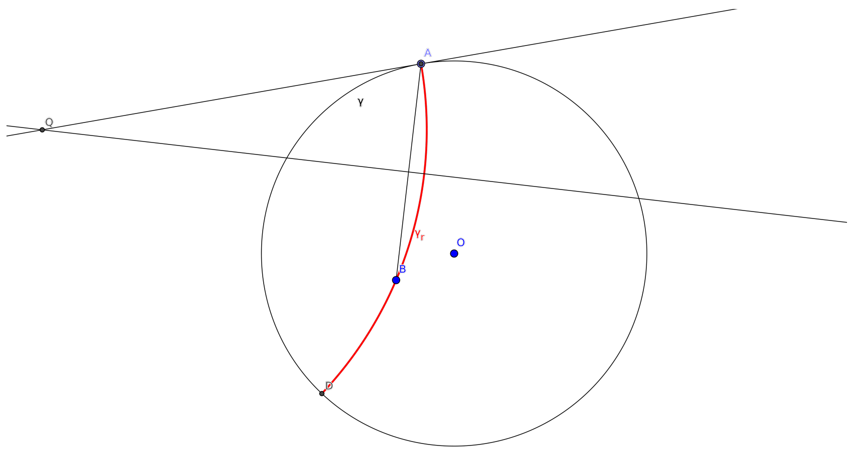

Figure 3.2: Hyperbolic line through $A \in \gamma$ and $B \in \mathbb{D}^2$.

Case 3: Construct the ray $\overrightarrow{OA}$ and then t_A the perpendicular line to $\overrightarrow{OA}$ at A. This intersects γ in points X and Y. Now construct the tangents to γ at X and Y. These tangents lines intersect at a point C. To determine the center of the new circle γ_r, draw the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$. They cut at Q which is the center of γ_r (iii). Note that this circumference passes through A, B, and C. The arc of γ_r that lies inside $\mathbb{D}^2$ is the hyperbolic line containing A and B.

We need to check that this arc is perpendicular to γ. In other words, let T and S be the points of this arc and lie on γ, we need to prove that the circles γ and γ_r are orthogonal at T and S. By construction we have that $\angle XOC = \angle XOA$, then the right triangles $\triangle OXC \cong \triangle OAX$.

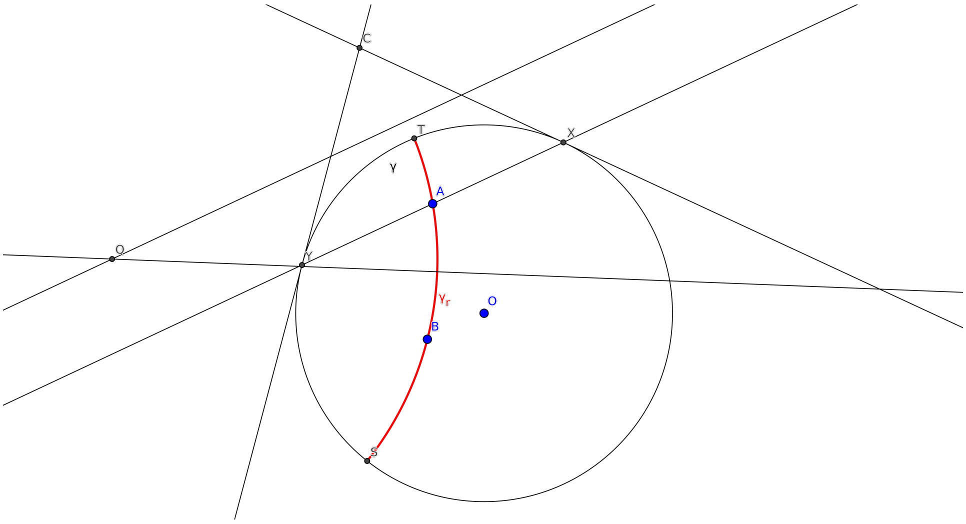

Figure 3.3: Hyperbolic line through $A, B \in \mathbb{D}^2$.

Hence

$$\frac{|OC|}{|OX|} = \frac{|XC|}{|AX|} = \frac{|OX|}{|OA|}$$

which implies

$$|OA| \cdot |OC| = |OX|^2 = r^2. \tag{3.1}$$

We know that Q lies on the perpendicular bisector of $\overline{AB}$ and $\overline{AC}$ (by construction (iii)). There is a point T on the circle γ_r so that the tangent line to γ_r at T passes through O.
Now, construct the line through Q and O which intersects γ_r in two points G_1 and G_2 so that G_1 lies between Q and O. Since $\triangle QOT$ is a right triangle, we obtain

$$|QO|^2 = |OT|^2 + |TQ|^2$$

that is,

$$\begin{aligned}
|OT|^2 &= |QO|^2 - |QT|^2 \\
&= (|QO| - |QT|) \cdot (|QO| + |QT|) \\
&= |OG_1| \cdot |OG_2| && \text{Because } |QT| = |QG_1| = |QG_2| = r \\
& && \text{and by construction } |QG_1| + |G_1O| = |QO| \\
& && \text{and } |G_2Q| + |QO| = |G_2O|). \\
&= |OA| \cdot |OC| && \text{By the chord Theorem.} \\
&= r^2 && \text{By (3.1)}
\end{aligned}$$

So $|OT| = r$. Therefore, T lies on γ, and γ and γ_r are orthogonal at T. If we apply a similar argument to the other point S we check that both circumferences are orthogonal at S.

3.2 Isomorphism from the Poincaré model to the Klein model

As mentioned in the previous section, there is a useful conformal geometric isomorphism between the Poincaré model and the Klein model, that preserves the incidence, betweenness, and congruence axioms. Let $\mathbb{D}^2$ be the unit with center O. Let $B = (t \cdot \cos\theta, t \cdot \sin\theta)$ be a point inside $\mathbb{D}^2$, that has polar coordinates $B_\rho = (t, \theta)$ for $t \in [0, 1]$ and $\theta \in [0, 2\pi)$. Now, we define the following map

$$\begin{array}{rccc} F: & I \times [0, 2\pi) & \to & \mathbb{R} \times [0, 2\pi) \\ & (t, \theta) & \to & (f(t), \theta) \end{array} \tag{3.2}$$

where $f(t) = \frac{2t}{1+t^2}$.

Now, let A be the point that bisects the chord $\overline{RS}$. We want to see that this isomorphism holds for any points along the chord. If $B_\rho = (t, \theta)$, so $A_\rho = (\frac{2t}{1+t^2}, \theta)$. The Euclidean distance from O to B is t, and we want to see that the hyperbolic distance from O to A is $f(t)$.

Recall the definition of the inverse of a point in the hyperbolic plane, 2.1.2. By Proposition 2.1.1, if a point in the Poincaré model lies on an orthogonal arc of γ_r to γ, then the corresponding inverse point lies also on γ_r, but outside of the Poincaré model. Let us denote the inverse of B by Q.

Let γ_r be a circumference that cuts γ orthogonally and that contains the point B and its inverse Q. Let P be the center of γ_r. Note that:

$$r = |PQ| = \frac{(\frac{1}{t} - t)}{2} = \frac{1 - t^2}{2t}$$

then obtain:

$$|OP| = |OB| + |BP| = t + \frac{1 - t^2}{2t} = \frac{t^2 + 1}{2t}.$$

Observe that we have similar triangles $\triangle OSP$ and $\triangle OAS$. Therefore

$$\frac{|OA|}{|OS|} = \frac{|OS|}{|OP|} \Rightarrow |OA| \cdot |OP| = |OS|^2 = 1$$

Then $|OA| = |OP|^{-1} = \frac{2t}{t^2+1}$. So we have already proved that the polar coordinates of A are $A_\rho = (f(t), \theta)$.

Now, consider the unit sphere $\mathbb{S} = \{x^2 + y^2 + z^2 = 1\}$ whose equator is contained in the plane $Z : z = 0$. Consider ϕ the stereographic projection of $\mathbb{S}$ onto Z as follows. Take a point B in Z and consider ℓ the line that passes through the north pole $N = (0, 0, 1)$ of $\mathbb{S}$ and the point B. The projection $\phi^{-1}(B)$ is geometrically defined as the remaining intersection of ℓ and $\mathbb{S}$ other than N. To calculate the coordinates of $\phi^{-1}(B)$ take $B = (x, y, 0)$ and let us calculate a parametric equation of the line ℓ, say

$$P_\lambda : (0, 0, 1) + (\lambda \cdot x, \lambda \cdot y, 1 - \lambda), \lambda \in \mathbb{R}$$

Now $P_\lambda \in S$ if and only if

$$\lambda^2 \cdot x^2 + \lambda^2 \cdot y^2 + (1 - \lambda)^2 = 1.$$

Note that

$$\lambda^2(x^2 + y^2 + 1) - 2 \cdot \lambda = \lambda(\lambda(x^2 + y^2 + 1) - 2) = 0$$

has two solutions: $\lambda = 0$, which is the north pole, and the other solution is $\lambda = \frac{2}{x^2+y^2+1}$.

When applied to the point $B = (t \cdot \cos\theta, t \cdot \sin\theta)$ with polar coordinates $B_\rho = (t, \theta)$, one obtains $\phi^{-1}(B) = (\frac{2t\cos\theta}{1+t^2}, \frac{2t\sin\theta}{1+t^2}, *)$. If we compose with the vertical projection π we obtain the map $\tilde{F} = \pi \circ \phi^{-1} : \mathbb{D}^2 \to \mathbb{D}^2$ defined as

$$\tilde{F}(t \cdot \cos\theta, t \cdot \sin\theta) = \left(\frac{2t}{1 + t^2} \cos\theta, \frac{2t}{1 + t^2} \sin\theta \right)$$

which in polar coordinates defines the map $F(t, \theta) = (\frac{2t}{1+t^2}, \theta)$.

Theorem 3.2.1. *The isomorphism between the Poincaré and Klein models is given geometrically as the composition of the vertical projection to the sphere with the stereographic projection back to the original plane described in* (3.2).

Chapter 4

Hyperbolic Trigonometry

As we know, trigonometry is the study of the relationships between the angles and the sides of a triangle. In the Euclidean plane, the idea of similar triangles was used to help define the sine, cosine, and tangent of an acute angle in a right triangle. From these definitions, we are able to extend the same ideas to find the cosecant, secant, and cotangent of such an angle.

For example, in the Euclidean plane, given a right triangle ABC where $\angle C$ is the right angle, we define $\cos A$ as the ratio of the adjacent side to the hypotenuse. In other words, $\cos A = \frac{b}{c}$.

However, the hyperbolic plane triangles come in different forms (see Figure 2.1). As a result, the Euclidean ratios no longer hold true in all cases so one has to define trigonometric functions differently in the hyperbolic plane. For the circular functions, their definitions are in terms of their Taylor series expansions:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{4.1}$$

and $\tan x = \frac{\sin x}{\cos x}$, etc. Note that trigonometry in the hyperbolic plane not only involves the circular functions but also the *hyperbolic functions* defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

and $\tanh x = \frac{\sinh x}{\cosh x}$, etc.

Similar to the circular functions, these hyperbolic functions can also be defined using their *Taylor series*:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{4.2}$$

Observe that the hyperbolic functions (4.2) are the circular functions (4.1) without the coefficients $(-1)^n$.

We ask ourselves the following question: *Where does the name hyperbolic functions come from?* It comes from the hyperbolic identity

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1$$

from which the parametric equations $x = \cosh\theta$ and $y = \sinh\theta$ give one part of the hyperbola $x^2 - y^2 = 1$ in Cartesian plane. It is important not to confuse θ in this sense with θ in the Euclidean plane.

4.1 Concepts of distances, angles, etc

In this section, we will see *the angle of parallelism*, which provides a direct link between the circular and the hyperbolic functions, and we will also present a theorem that allows us to directly solve for this angle. But, before that we will see *the relationship between the Euclidean distance of a point from the center of a circle and the hyperbolic.*

First, we introduce a metric by

$$dx = \frac{2dt}{1-t^2}$$

where x represents the hyperbolic distance and t is the Euclidean distance from the center of the circle. Note that $dx \to \infty$ as $t \to 1$. This means that lines are going to have infinite extent. Now, *the Euclidean distance of a point from the center of a circle and the hyperbolic distance are related by*:

$$x = \int_{u=0}^{t} \frac{2du}{1-u^2} =$$

Using $1-t^2 = (1-t)(1+t)$ and its decomposition in simple fractions $\frac{2}{1-u^2} = \frac{1}{1-u} + \frac{1}{1+u}$ one obtains the formula already given in Theorem 2.1.1

$$x = \ln\left(\frac{1+t}{1-t}\right) = 2\tanh^{-1}(t)$$

that is:

$$t = \tanh\left(\frac{x}{2}\right).$$

4.1.1 Angle of Parallelism and the Bolyai-Lobachevsky's formula

Recall that in the hyperbolic plane, the parallel postulate from the Euclidean plane 1.1.1 is replaced with the Hyperbolic Parallel Postulate 1.1.2

As a result of the existence of more than one parallel line in the hyperbolic plane, the following theorem holds.

Theorem 4.1.1 (Limiting parallel rays)**.** *Given any line ℓ and any point $P \notin \ell$, let Q denote the foot of the perpendicular from P to ℓ. Then there exist two hyperbolic rays $\overrightarrow{PY}$ and $\overrightarrow{PX}$ on opposite sides of $\overrightarrow{PQ}$ such that:*

1. *The rays $\overrightarrow{PY}$ and $\overrightarrow{PX}$ do not intersect ℓ.*
2. *A ray $\overrightarrow{PS}$ intersects the line ℓ if and only if $\overrightarrow{PS}$ is between $\overrightarrow{PY}$ and $\overrightarrow{PX}$.*
3. *$\angle YPQ \cong \angle XPQ$.*

Proof. A proof of this result can be seen in the Klein model for $P = O$, where X and Y are the intersections of ℓ and κ and the ray $\overrightarrow{PY}$ (resp. $\overrightarrow{PX}$) is the chord joining P and X (resp. Y). For the case where $P \neq O$ one can consider an inversion that sends P to O as in Proposition 2.1.2. □

The rays $\overrightarrow{PX}$ and $\overrightarrow{PY}$ in Theorem 4.1.1 are called *limiting parallel rays.* From the congruence relation (3) in Theorem 4.1.1, we conclude that either of these angles can be called *the angle of parallelism* of P with respect to ℓ. The important thing to remember about these two limiting parallel rays is that they *are situated such that they are symmetric with respect to the perpendicular line PQ to ℓ.*

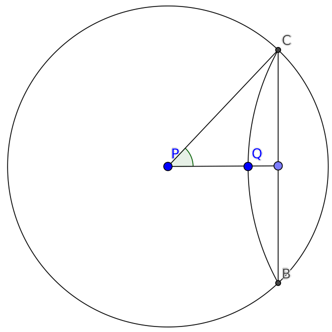

Figure 4.1: Angle of parallelism.

Theorem 4.1.2 (The formula of Bolyai-Lobachevsky)**.** *Let α be the angle of parallelism for P with respect to ℓ and let x be the hyperbolic distance from P to Q, where PQ is perpendicular to ℓ. We then we have, the following formula holds:*

$$\tan\left(\frac{\alpha}{2}\right) = e^{-x} \tag{4.3}$$

Proof. By Proposition 2.1.2 we can assume that $P = O$. By Theorem 3.2.1, if $Q = (t, 0)$, then $F(Q) = (\frac{2t}{1+t^2}, 0)$. By the tangent formula of the middle-angle we know that

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1-\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} = \left(\frac{1-t}{1+t}\right)^2.$$

Hence, $\tan\left(\frac{\alpha}{2}\right) = \frac{1-t}{1+t} = e^{-x}$ by Theorem 2.1.1. ☐

Remark 4.1.1. In formula (4.3) the angle of parallelism α depends on the hyperbolic distance x. If we look closer at formula (4.3), in particular as the hyperbolic distance x goes to 0, then we have

$$\lim_{x\to 0}\left(e^{-x}\right) = \lim_{x\to 0}\left(\tan\frac{\alpha_x}{2}\right) = 1$$

This implies that as $x \to 0$, $\alpha_x \to \pi/2$. In other words, as the distance x between points P and Q in Figure 4.1 goes to 0, *the angle of parallelism α is getting closer to $\pi/2 = 90^o$. Therefore, parallel lines in the hyperbolic plane are looking like parallel lines in the Euclidean plane as the points P and Q get closer.*

We can also transfer this idea to hyperbolic triangles and say that if the sides of the triangle are sufficiently small, then the triangle looks like a regular Euclidean triangle. See the 4^{th} circle of the Figure 2.1.

In contrast, if we look at formula (4.3) as x goes to ∞, then we see that $\alpha_x \to 0$. So, as the hyperbolic distance x between points P and Q in Figure 4.1 gets infinitely large, the limiting parallel ray $\overrightarrow{PX}$ essentially aligns with the line PQ.

The Bolyai-Lobachevsky formula is certainly *one of the most remarkable formulas in all of mathematics.* As we proved, this formula relates the angle of parallelism to distance. By simply rewriting the formula in a different way, we are able to also provide a link between hyperbolic and circular functions. To do it, we will consider the sine, cosine, and tangent of the angle of parallelism.
Note that Lobachevsky denoted α as $\prod(x)$. This notation is useful because makes it clear that the angle of parallelism relies on the hyperbolic distance x, so from now on, we will use it. Directly from equation (4.3), we obtain *the radiant measure of the angle of parallelism*:

$$\prod(x) = 2\arctan(e^{-x}) \tag{4.4}$$

Now, we are going to see some *alternative forms of the Bolyai-Lobachevsky Formula.*

Theorem 4.1.3 (Alternative forms of the Bolyai-Lobachevsky Formula). *Let $\prod(x)$ be the angle of parallelism and x be the hyperbolic distance from a point to a hyperbolic line. Then,*

$$\sin(\prod(x)) = \text{sech}(x) = \frac{1}{\cosh(x)} \tag{4.5}$$

$$\cos(\prod(x)) = \tanh(x) \tag{4.6}$$

$$\tan(\prod(x)) = \text{csch}(x) = \frac{1}{\sinh(x)} \tag{4.7}$$

Proof. If we write $y = \arctan(e^{-x})$, then $\tan(y) = e^{-x}$. Since $\tan^2(y) + 1 = \frac{\sin^2(y)}{\cos^2(y)} + 1 = \frac{\sin^2(y)+\cos^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}$, we have $\sec^2(y) = 1/\cos^2(y) = \tan^2(y) + 1$. Using the previous notation $\sec^2(y) = e^{-2x} + 1$, which implies

$$\cos(y) = \frac{1}{(e^{-2x} + 1)^{1/2}} \tag{4.8}$$

Similarly, we have:

$$\sin(y) = \tan(y)\cos(y) = \frac{e^{-x}}{(e^{-2x} + 1)^{1/2}} \tag{4.9}$$

By (4.4)

$$\prod(x) = 2\arctan(e^{-x}) = 2y \Rightarrow \sin(\prod(x)) = \sin(2y) = 2\sin(y)\cos(y)$$

Using (4.8) and (4.9) the previous formula becomes

$$\sin(\prod(x)) = 2\frac{e^{-x}}{(e^{-2x} + 1)^{1/2}}\frac{1}{(e^{-2x} + 1)^{1/2}} = 2\frac{e^{-x}}{e^{-2x} + 1} = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh(x)}$$

which proves equation (4.5).

Before proving the second formula, remember that the double-angle formula for cosine is: $\cos(2y) = \cos^2(y) - \sin^2(y)$. Then,

$$\begin{aligned} \cos(\prod(x)) &= \cos(2y) = \cos^2(y) - \sin^2(y) = \tfrac{1}{e^{-2x}+1} - \tfrac{e^{-2x}}{e^{-2x}+1} \\ &= \tfrac{e^x - e^{-x}}{e^x + e^{-x}} = \tfrac{e^x - e^{-x}}{2}\tfrac{2}{e^x + e^{-x}} = \tfrac{\sinh(x)}{\cosh(x)} = \tanh(x), \end{aligned}$$

which proves equation (4.6). The last equality is immediate. □

Therefore we conclude that the function $\prod$ provides a link between the hyperbolic and the circular functions.

4.2 Hyperbolic Identities

In the Euclidean plane, there are many trigonometric identities. These identities are equations that hold for all angles. In the hyperbolic plane, there are corresponding trigonometric identities that involve both circular and hyperbolic functions. In this subsection, we will see interesting hyperbolic identities.

4.2.1 Right Triangle Trigonometric Identities

As we know, in the Euclidean plane there are certain identities that can only be applied to a right triangle, such as Pythagoras' theorem. Similarly, in the hyperbolic plane, some identities only hold for right triangles.

Theorem 4.2.1 (Right Triangle Trigonometric Identities). *Given a right hyperbolic triangle $\triangle ABC$, with $\angle C$ being the right angle. Let a, b, and c denote the hyperbolic lengths of the corresponding sides. Then*

$$\sin A = \frac{\sinh a}{\sinh c} \qquad \text{and} \qquad \cos A = \frac{\tanh b}{\tanh c}. \tag{4.10}$$

$$\cosh c = \cosh a \cdot \cosh b = \cot A \cdot \cot B. \tag{4.11}$$

$$\cosh a = \frac{\cos A}{\sin B} \tag{4.12}$$

Proof. First, we see that formulas (4.11) and (4.12) follow from (4.10). Note that

$$1 = \sin^2 A + \cos^2 A \overset{(4.10)}{=} \frac{\sinh^2 a}{\sinh^2 c} + \frac{\tanh^2 b}{\tanh^2 c}.$$

Multiplying both sides by $\sinh^2 c$, we obtain

$$\sinh^2 c = \sinh^2 a + \cosh^2 c \cdot \frac{\sinh^2 b}{\cosh^2 b}$$

Adding 1 on both sides and using $1 + \sinh^2 x = \cosh^2 x$, this translates in

$$\cosh^2 c = \cosh^2 a + \cosh^2 c \cdot \tfrac{\sinh^2 b}{\cosh^2 b} \Rightarrow \cosh^2 c \cdot (\cosh^2 b - \sinh^2 b) = \cosh^2 a \cdot \cosh^2 b$$

$$\Rightarrow \cosh^2 c = \cosh^2 a \cdot \cosh^2 b \Rightarrow \cosh c = \cosh a \cdot \cosh b.$$

This gives the first equality of (4.11). Analogously one can check that (4.12) also follows from (4.10). First, one can apply (4.10) to B and obtain $\sin B = \frac{\sinh b}{\sinh c}$. Therefore,

$$\frac{\cos A}{\sin B} = \frac{\tanh b}{\tanh c} \cdot \frac{\sinh c}{\sinh b} = \frac{\cosh c}{\cosh b} \overset{(4.11)}{=} \cosh a.$$

To prove the second equality in (4.11), we will use the previous formula. Note that $\cosh b = \frac{\cos B}{\sin A}$ after applying (4.11) to $\cosh b$. Then we have

$$\cosh c = \cosh a \cdot \cosh b = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A} = \cot A \cdot \cot B.$$

Finally, let's see the proof of the first formula (4.10). The key point here is to assume that the vertex A of the right triangle coincides with the center O of the circle γ in the Poincaré model (see Proposition 2.1.2) as in Figure 4.2.

The points B' and C' are the images of B and C under the isomorphism F (3.2). Let B'' be the point of intersection between $\overrightarrow{OB}$ and the orthogonal circle δ that contains the Poincaré line BC. From the Euclidean triangle $\triangle(AB'C')$, we have

$$\cos A = \frac{|OC'|}{|OB'|}.$$

By Theorem 2.1.1, $\tanh c = f(|OB|) = |OB'|$. Hence,

$$\cos A = \frac{|OC'|}{|OB'|} = \frac{\tanh b}{\tanh c},$$

which is the second formula in (4.10). The first formula is analogous. □

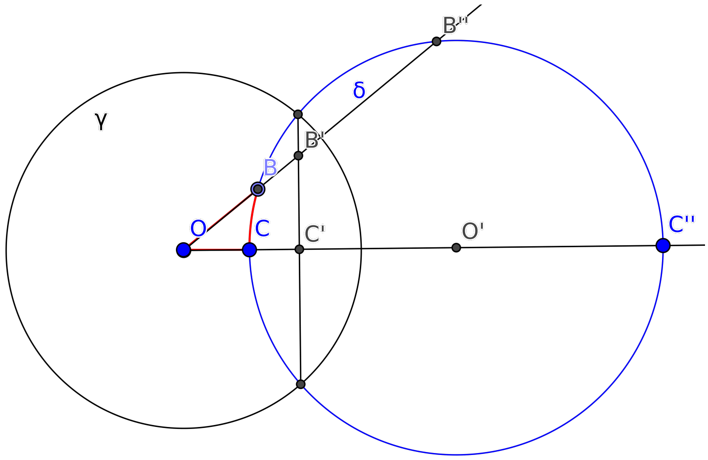

Figure 4.2: Calculation of $\sin A$.

4.2.2 Trigonometric Identities for any Triangle

What happen when we have a triangle which has not any right angle? We should apply the following theorem:

Theorem 4.2.2. *For any triangle $\triangle ABC$ in the hyperbolic plane,*

$$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C, \tag{4.13}$$

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}, \tag{4.14}$$

$$\cosh c = \frac{\cos A \cdot \cos B + \cos C}{\sin A \cdot \sin B}. \tag{4.15}$$

Proof. Recall that

$$\cos(x \pm y) = \cos x \cdot \cos y \pm \sin x \cdot \sin y. \tag{4.16}$$

$$\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y. \tag{4.17}$$

Let's start with the proof of equation (4.13). Given a hyperbolic triangle $\triangle ABC$, we will denote by B_0 the foot of the perpendicular from B to AC and analogously for the remaining feet. Let the length of $d(B, B_0) = b_0$ and $b = b_1 + b_2$, where $b_1 = d(B_0, A)$ and $b_2 = d(B_0, C)$.

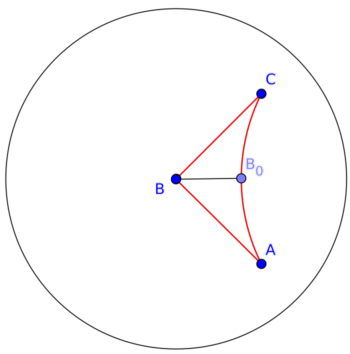

Figure 4.3: Foot of perpendicular in a hyperbolic triangle.

Now, using equations (4.10), (4.11), and (4.17), we have

$$\begin{aligned}
\cosh c &= \cosh b_1 \cdot \cosh b_0 \overset{(4.11)}{=} \cosh(b-b_2)\cdot \cosh b_0 \\
&\overset{(4.17)}{=} (\cosh b \cdot \cosh b_2 - \sinh b \cdot \sinh b_2) \cdot \cosh b_0 \\
&\overset{(4.11)}{=} \cosh b \cdot \cosh a - \sinh b \cdot \sinh a \cdot \tfrac{\cosh a \cdot \sinh b_2}{\cosh b_2 \cdot \sinh a} \\
&= \cosh b \cdot \cosh a - \sinh b \cdot \sinh a \cdot \tfrac{\tanh b_2}{\tanh a} \\
&\overset{(4.10)}{=} \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C.
\end{aligned}$$

This shows the first equation. To prove (4.14), note that

$$\frac{\sin A}{\sinh a} \overset{(4.10)}{=} \frac{1}{\sinh a} \cdot \frac{\sinh b_0}{\sinh c} = \frac{\sinh b_0}{\sinh a} \cdot \frac{1}{\sinh c} \overset{(4.10)}{=} \frac{\sin C}{\sinh c}.$$

One obtains the remaining equalities using the other perpendicular lines.

Finally, to prove (4.15) we will use a perpendicular from C to AB and call the length c_0. Also denote $c = c_1 + c_2$, where $c_1 = d(A, C_0)$, $c_2 = d(B, C_0)$. This perpendicular line divides $\angle C$ into two angles, namely $C_1 = \angle C_0CA$ and $C_2 = \angle C_0CB$. Using equations (4.10), (4.12), and (4.16), we have

$$\begin{aligned}
\cosh c &= \cosh(c_1+c_2) \overset{(4.17)}{=} \cosh c_1 \cdot \cosh c_2 + \sinh c_1 \cdot \sinh c_2 \\
&\overset{(4.12),(4.10)}{=} \tfrac{\cos C_1}{\sin A} \cdot \tfrac{\cos C_2}{\sin B} + \sinh b \cdot \sin C_1 \cdot \sinh a \cdot \sin C_2 \\
&= \tfrac{\cos C_1 \cdot \cos C_2 + \sin C_1 \cdot \sin C_2 \cdot \sinh^2 c_0}{\sin A \cdot \sin B} \\
&\overset{(4.17)}{=} \tfrac{\cos(C_1+C_2) + \sin C_1 \cdot \sin C_2 + (\sin C_1 \cdot \sin C_2 \cdot \sinh^2 b_0)}{\sin A \cdot \sin B} \\
&= \tfrac{\cos C + (\sin C_1 \cdot \cosh c_0)\cdot(\sin C_2 \cdot \cosh c_0)}{\sin A \cdot \sin B} \\
&= \tfrac{\cos C + \sin C_1 \cdot \sin C_2 (1+\sinh^2 c_0)}{\sin A \cdot \sin B} \overset{(4.12)}{=} \tfrac{\cos C + \cos A \cdot \cos B}{\sin A \cdot \sin B}.
\end{aligned}$$

It is important to note that for this proof we are working under the assumption that the dropped perpendiculars fall within the hyperbolic triangle $\triangle ABC$. Without this assumption, we could show in a proof that is generally the same as above that when the dropped perpendicular falls outside of the $\triangle ABC$ the equations (4.13), (4.14), and (4.15) still hold. □

4.2.3 Another Trigonometric Identities

Recall the formula for the circumference of a Euclidean circle is $C = 2\pi r$. Now, in the hyperbolic plane we have:

Theorem 4.2.3 (Gauss)**.** *In the hyperbolic plane, the circumference C of a circle of radius r is given by*

$$C = 2\pi \sinh r.$$

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