Structure of polarimetric purity of three-dimensional polarization states

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It has recently been demonstrated that a general three-dimensional (3D) polarization state cannot be considered an incoherent superposition of (1) a pure state, (2) a two-dimensional unpolarized state, and (3) a 3D unpolarized state [J. J. Gil, Phys. Rev. A **90**, 043858 (2014)]. This fact is intimately linked to the existence of 3D polarization states with fluctuating directions of propagation, but whose associated polarization matrices $\mathbf{R}$ satisfy rank $\mathbf{R} = 2$. In this work, such peculiar states are analyzed and characterized, leading to a meaningful general classification and interpretation of 3D polarization states. Within this theoretical framework, the interrelations among the more significant polarization descriptors presented in the literature, as well as their respective physical interpretations, are studied and illustrated with examples, providing a better understanding of the structure of polarimetric purity of any kind of polarization state.

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I. INTRODUCTION

Regardless of the obvious fact that any state of polarization is realized in the physical three-dimensional space, two-dimensional (2D) polarization states are commonly defined as those whose electric field vector $\mathbf{E}$, at the considered point $\mathbf{r}$ in the space, evolves in a fixed plane during the measurement time; otherwise, the polarization state is a genuine three-dimensional (3D) state. In general, 3D polarization states have an intricate structure whose physical interpretation requires appropriate analyses beyond the common, but not general, case of 2D states. The main aim of this work is to characterize and classify the different types of 3D polarization states by means of their constructive synthesis and the detailed study of their structure of polarimetric purity. In particular, we emphasize that not all 3D states are superpositions of a fully polarized state, a 2D unpolarized state, and a 3D unpolarized state.

It is well known that a complete characterization of the second-order polarization properties of an electromagnetic field at a given point $\mathbf{r}$ in space is provided by the corresponding *3D polarization matrix* $\mathbf{R}$ (also called the *3D coherency matrix*) defined as

$$\mathbf{R} = \langle \boldsymbol{\varepsilon}(t) \otimes \boldsymbol{\varepsilon}^{\dagger}(t)\rangle = \begin{pmatrix} \langle \varepsilon_1(t)\varepsilon_1^*(t)\rangle & \langle \varepsilon_1(t)\varepsilon_2^*(t)\rangle & \langle \varepsilon_1(t)\varepsilon_3^*(t)\rangle \\ \langle \varepsilon_2(t)\varepsilon_1^*(t)\rangle & \langle \varepsilon_2(t)\varepsilon_2^*(t)\rangle & \langle \varepsilon_2(t)\varepsilon_3^*(t)\rangle \\ \langle \varepsilon_3(t)\varepsilon_1^*(t)\rangle & \langle \varepsilon_3(t)\varepsilon_2^*(t)\rangle & \langle \varepsilon_3(t)\varepsilon_3^*(t)\rangle \end{pmatrix}, \quad (1)$$

where $\boldsymbol{\varepsilon}(t)$ is the *3D instantaneous Jones vector* [1] at point $\mathbf{r}$ and at time t. Its components $\varepsilon_i(t)$ (with $i = 1,2,3$) are the analytic signals of the electric field components with respect to the given reference frame *XYZ*, the dagger indicates conjugate transpose, $\otimes$ stands for the Kronecker product, and $\langle \cdots \rangle$ represents time averaging over the measurement time. Thus the elements of $\mathbf{R}$ are the second-order moments of the (zero-mean) field variables $\varepsilon_i(t)$, and therefore $\mathbf{R}$ has the mathematical structure of a covariance matrix; that is, $\mathbf{R}$ is a positive semidefinite Hermitian matrix. Note that the subscripts 1, 2, 3 are used instead of *x, y, z* in order to simplify later mathematical expressions. The quantity $\mathrm{tr}\,\mathbf{R}$ represents the intensity of the state $\mathbf{R}$,

$$I = \mathrm{tr}\,\mathbf{R} = \langle \varepsilon_1^2(t)\rangle + \langle \varepsilon_2^2(t)\rangle + \langle \varepsilon_3^2(t)\rangle; \quad (2)$$

i.e., it is proportional to the sum of the electric energy densities of the field components along the axes *X, Y,* and *Z*.

We first consider the representation of an arbitrary 3D polarization matrix as an incoherent sum of matrices that correspond to pure states, meaning that their respective electric fields lie in a fixed plane and describe a well-defined polarization ellipse. Since the polarization matrix is Hermitian, it can be diagonalized and we may write

$$\mathbf{R} = \mathbf{U}\,\mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3)\mathbf{U}^{\dagger} = (\mathrm{tr}\,\mathbf{R})\mathbf{U}\,\mathrm{diag}\,(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3)\mathbf{U}^{\dagger},$$
$$[\hat{\lambda}_i \equiv \lambda_i/(\mathrm{tr}\,\mathbf{R}),\ \ (i = 1,2,3)], \quad (3)$$

where $\mathbf{U}$ is the unitary matrix whose columns are the eigenvectors of $\mathbf{R}$, and $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ represents the diagonal matrix composed of the ordered non-negative eigenvalues $(0 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1)$. It follows that $\mathbf{R}$ always can be expressed as the following convex expansion in terms of a set of r (where $r \equiv \mathrm{rank}\,\mathbf{R}$) arbitrary but independent complex unit vectors $\mathbf{w}_i$ belonging to the subspace generated by the eigenvectors of $\mathbf{R}$ with nonzero eigenvalues [2],

$$\mathbf{R} = (\mathrm{tr}\,\mathbf{R})\sum_{i=1}^{r} p_i \hat{\mathbf{R}}_{pi}, \quad \hat{\mathbf{R}}_{pi} \equiv \mathbf{w}_i \otimes \mathbf{w}_i^{\dagger},$$
$$p_i = \frac{1}{\sum_{j=1}^{r} \frac{1}{\lambda_j}\left|(\mathbf{U}^{\dagger}\mathbf{w}_i)_j\right|^2}, \quad (4)$$

where

$$\sum_{j=1}^{r} p_i = 1, \quad (5)$$

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or equivalently,

$$\mathbf{R} = \sum_{i=1}^{r} p_i \mathbf{R}_{pi}, \quad \mathbf{R}_{pi} \equiv (\mathrm{tr}\,\mathbf{R})(\mathbf{w}_i \otimes \mathbf{w}_i^\dagger). \tag{6}$$

We refer to Eq. (4) as *arbitrary decomposition* [1]. The term arbitrary is used here in the sense that any set of r independent unit vectors $\mathbf{w}_i$ allows generating r mutually incoherent pure components $\mathbf{R}_{pi} \equiv (\mathrm{tr}\,\mathbf{R})(\mathbf{w}_i \otimes \mathbf{w}_i^\dagger)$ of $\mathbf{R}$, with respective positive coefficients given by p_i.

Hereafter, when appropriate to emphasize that a given state $\mathbf{R}$ is pure, its polarization matrix will be denoted as $\mathbf{R}_p$. For the sake of clarity and completeness, Appendix A includes a proof of Eq. (4) as an adaptation of that corresponding to the arbitrary decomposition of covariance matrices associated with Mueller matrices, developed in [2].

The arbitrary decomposition provides theoretical support towards the analysis of the main object of this work, namely, the study of the structure of regular and nonregular states of polarization, defined in Sec. II.

An important special case of Eq. (4) is encountered when $\mathbf{w}_i$ are the eigenvectors $\mathbf{u}_i$ $(i = 1,2,3)$ of $\mathbf{R}$ (which constitute the respective columns of $\mathbf{U}$), whereby Eq. (4) becomes the *spectral decomposition*,

$$\mathbf{R} = (\mathrm{tr}\,\mathbf{R}) \sum_{i=1}^{r} \hat{\lambda}_i (\mathbf{u}_i \otimes \mathbf{u}_i^\dagger). \tag{7}$$

Equivalently,

$$\mathbf{R} = \hat{\lambda}_1[(\mathrm{tr}\,\mathbf{R})\mathbf{U}\,\mathrm{diag}\,(1,0,0)\mathbf{U}^\dagger] + \hat{\lambda}_2[(\mathrm{tr}\,\mathbf{R})\mathbf{U}\,\mathrm{diag}\,(0,1,0)\mathbf{U}^\dagger] + \hat{\lambda}_3[(\mathrm{tr}\,\mathbf{R})\mathbf{U}\,\mathrm{diag}\,(0,0,1)\mathbf{U}^\dagger], \tag{8}$$

where the terms (in square brackets) in the sum satisfy the following properties: (1) they have the same intensity $(\mathrm{tr}\,\mathbf{R})$ as $\mathbf{R}$ itself, (2) they are synthesized from the corresponding eigenvector $\mathbf{u}_i$ of $\mathbf{R}$, and (3) their weight in the convex sum is equal to the corresponding normalized eigenvalue $\hat{\lambda}_i$. Note that the term *spectral* is used here with reference to the eigenvalue spectrum of $\mathbf{R}$ and without any link to the frequency spectrum of the state represented by $\mathbf{R}$.

From a mathematical point of view, the arbitrary decomposition expresses the fact that any polarization matrix can be represented with respect to different matrix bases constituted by sets of three independent positive-definite Hermitian matrices of rank 1, which are not necessarily trace orthogonal as in the particular case of the spectral decomposition.

While 2D states are characterized by the conventional Stokes parameters and, thus, their arbitrary decompositions can be easily represented in the Poincaré sphere [1,3], 3D states are characterized by a set of nine generalized (or 3D) Stokes parameters [4–7], whose geometric representation is given by a feasible convex region of eight dimensions determined by the eight normalized 3D Stokes parameters (the parameter representing the intensity is taken as unity and excluded in the coordinate reference Stokes system, as occurs in the Poincaré sphere representation for 2D states).

The constraining inequalities that characterize the physically admissible sets of 3D Stokes parameters have an intricate mathematical structure that cannot be reduced to the eight-dimensional (8D) sphere, but such inequalities define the above-mentioned 8D convex region. Obviously, for 2D states, the unnecessary extra Stokes dimensions can be disregarded and the 8D convex region reduces to the Poincaré sphere.

The physical meaning of the arbitrary decomposition is that any polarization state can be interpreted as the incoherent superposition of a number r (with $r = \mathrm{rank}\,\mathbf{R}$) of different pure states, and provides all the (infinite) possible ways to do it. Equivalently, the arbitrary decomposition can be considered a universal procedure for the synthesis of any 3D state by means of the incoherent superposition (at the considered point in the space) of pure states. This synthesis can be performed experimentally by generating three fully polarized electromagnetic waves (for instance, collimated beams passing through respective polarizers), in general with different directions of propagation but intersecting at the considered point. Measurement of 3D polarization matrices (or, equivalently, 3D Stokes parameters [7]) requires unconventional experimental setups like the one suggested in [8].

Since the notion of orthogonality of states is widely used along this work, let us recall that two pure states $\mathbf{R}_{p1} \equiv (\mathrm{tr}\,\mathbf{R})\,(\mathbf{w}_1 \otimes \mathbf{w}_1^\dagger)$ and $\mathbf{R}_{p2} \equiv (\mathrm{tr}\,\mathbf{R})\,(\mathbf{w}_2 \otimes \mathbf{w}_2^\dagger)$ are said to be mutually orthogonal when their respective generating unit vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ satisfy $\mathbf{w}_1^\dagger \mathbf{w}_2 = \delta_{12}$, where δ_{12} stands for the Kronecker delta. Unlike the spectral components, in general the arbitrary components of $\mathbf{R}$ are not mutually orthogonal.

Next we bring up various parameters that provide information on the polarimetric purity of a given state $\mathbf{R}$. Polarimetric purity refers to the closeness of a polarization state to a pure state and is characterized by the *degree of polarimetric purity* of $\mathbf{R}$ [9] (also called the *degree of polarization* of three-component fields [4,11]):

$$P_{3D} = \sqrt{\frac{1}{2}\left[\frac{3\,\mathrm{tr}\,(\mathbf{R}^2)}{(\mathrm{tr}\,\mathbf{R})^2} - 1\right]} = \sqrt{\frac{1}{2}\left(3\sum_{i=1}^{3}\hat{\lambda}_i^2 - 1\right)}. \tag{9}$$

Conversely, polarimetric randomness refers to the closeness of a polarization state to a fully random 3D polarization state (or, equivalently, to the distance of a polarization state from a pure state) and is globally measured by the *degree of polarimetric randomness* that we define as

$$D_{3D} = \sqrt{2 - \frac{3}{2}\frac{\mathrm{tr}\,(\mathbf{R}^2)}{(\mathrm{tr}\,\mathbf{R})^2}} = \sqrt{\frac{3}{2}\left(1 - \sum_{i=1}^{3}\hat{\lambda}_i^2\right)} = \sqrt{1 - P_{3D}^2}. \tag{10}$$

States which from a polarimetric point of view are fully random, also called 3D unpolarized states, represented by polarization matrices proportional to the 3×3 identity matrix $\mathbf{I}$ and denoted as $\mathbf{R}_{u-3D} = \mathbf{I}/3$, are characterized by $D_{3D} = 1$ $(P_{3D} = 0)$, while pure states are characterized by $D_{3D} = 0$ $(P_{3D} = 1)$.

Since $\mathbf{R}$ has the structure of a covariance matrix of the zero-mean variables $\varepsilon_i(t)$, the detailed quantification of polarimetric randomness (or conversely, of polarimetric purity) is condensed in the eigenvalues λ_i of $\mathbf{R}$. Their sum, $I = \lambda_1 + \lambda_2 + \lambda_3$, representing the intensity of the state, acts as a common factor in the arbitrary decomposition of Eq. (4). Therefore, beyond the neutral information provided by I, and beyond the overall knowledge provided by P_{3D} (or analogously, by D_{3D}), the structure of polarimetric purity (randomness) of $\mathbf{R}$ is given by a pair of parameters. An important criterion for the choice of a best-suited set of such parameters is that they provide separate physical information in an optimally compressed and structured way. As seen in previous works [1,10], a privileged view of the structure of polarimetric purity is given by the *characteristic* or *trivial* decomposition of $\mathbf{R}$,

$$\begin{aligned} \mathbf{R} &= I[P_1\hat{\mathbf{R}}_p + (P_2 - P_1)\hat{\mathbf{R}}_m + (1 - P_2)\hat{\mathbf{R}}_{u-3D}], \\ \hat{\mathbf{R}}_p &\equiv \mathbf{U}\,\mathrm{diag}\,(1,0,0)\mathbf{U}^\dagger, \\ \hat{\mathbf{R}}_m &\equiv \tfrac{1}{2}\mathbf{U}\,\mathrm{diag}\,(1,1,0)\mathbf{U}^\dagger, \quad \hat{\mathbf{R}}_{u-3D} \equiv \tfrac{1}{3}\mathbf{I}, \end{aligned} \tag{11}$$

where (P_1,P_2) are the so-called *indices of polarimetric purity* (hereafter IPP) [9,12], defined as

$$P_1 = \hat{\lambda}_1 - \hat{\lambda}_2, \quad P_2 = \hat{\lambda}_1 + \hat{\lambda}_2 - 2\hat{\lambda}_3. \tag{12}$$

By virtue of the equality $\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 = 1$, P_2 can also be expressed as

$$P_2 = 1 - 3\hat{\lambda}_3. \tag{13}$$

The IPP have the properties

$$0 \leqslant P_1 \leqslant P_2 \leqslant 1, \quad P_{3D} = \sqrt{\frac{3P_1^2 + P_2^2}{4}}, \tag{14}$$

and give complete information about the polarimetric purity (randomness) of $\mathbf{R}$. The expression of the characteristic decomposition in terms of the eigenvalues of 3×3 polarization matrices was examined for the first time by Samson [4] through considering their expansion in a set of nondisjoint idempotent matrices, and later it was analyzed under the scope of radar polarimetry by Holm and Barnes [13] and by Cloude and Pottier [14], while Ellis, Dogariu, Brosseau, and co-workers [3,15,16] dealt with it for electromagnetic fields.

It is worth recalling now that a simplified and significant view of the features of a 3D polarization state is obtained through its corresponding *intrinsic polarization matrix* $\mathbf{R}_O$, defined as [17–19]

$$\mathbf{R}_O \equiv \mathbf{Q}_O\mathbf{R}\mathbf{Q}_O^T \equiv \begin{pmatrix} a_1 & -in_{O3} & in_{O2} \\ in_{O3} & a_2 & -in_{O1} \\ -in_{O2} & in_{O1} & a_3 \end{pmatrix}, \tag{15}$$

where $\mathbf{Q}_O$ is the orthogonal matrix that diagonalizes the (symmetric) real part of $\mathbf{R}$ and T denotes transpose. Thus, $\mathbf{R}_O$ represents the same state of polarization as $\mathbf{R}$, but referred with respect to the corresponding *intrinsic reference frame* (hereafter denoted as $X_OY_OZ_O$). The non-negative real parameters a_i, the eigenvalues of $\mathrm{Re}(\mathbf{R})$, are called the *principal intensities* ($\mathbf{Q}_O$ is defined so as to satisfy $a_1 \geqslant a_2 \geqslant a_3$), and n_{Oi} are the components of the intrinsic angular momentum vector $\mathbf{n}_O \equiv (n_{O1},n_{O2},n_{O3})^T$ of the state with respect to $X_OY_OZ_O$. Note that a_i satisfy the property that, by considering all possible laboratory reference frames *XYZ* (mutually linked by rotations determined by respective orthogonal transformations), a_1 is the maximal value achievable for a diagonal element of the corresponding representative polarization matrix, while a_3 is the minimal one. Moreover, when considering unitary similarity transformations of $\mathbf{R}$ (which include the above-mentioned orthogonal transformations as a subclass), the maximal and minimal diagonal elements are just λ_1 and λ_3, respectively [20], and therefore $\lambda_1 \geqslant a_1$ and $\lambda_3 \leqslant a_3$. As we will see in Sec. II D, these inequalities have important implications for distinguishing between different kinds of polarization states.

To complete this summary of notions that are required for the analyses performed in Secs. II and III, we recall that the information contained in $\mathbf{R}_O$ is constituted by $\mathbf{n}_O$ together with the intensity

$$I = \mathrm{tr}\,\mathbf{R}_O = \mathrm{tr}\,\mathbf{R} = a_1 + a_2 + a_3, \tag{16}$$

the *degree of linear polarization* (a measure of the maximum portion of the power that can be allocated to a linearly polarized incoherent component of $\mathbf{R}$) [18,21],

$$P_l \equiv \hat{a}_1 - \hat{a}_2, \quad (\hat{a}_i \equiv a_i/I, \ \ i = 1,2,3), \tag{17}$$

and the *degree of directionality* (a measure of how close the state represented by $\mathbf{R}$ is to a 2D state) [18,21],

$$P_d \equiv \hat{a}_1 + \hat{a}_2 - 2\hat{a}_3, \tag{18}$$

which, by virtue of the equality $\hat{a}_1 + \hat{a}_2 + \hat{a}_3 = 1$, can also be expressed as

$$P_d = 1 - 3\hat{a}_3. \tag{19}$$

The *degree of circular polarization* (a measure of the maximum portion of the power that can be allocated to a circularly polarized incoherent component of $\mathbf{R}$) is given by the following parameter derived from $\mathbf{n}_O$ [21]:

$$P_c \equiv 2\hat{n}_O, \ \left(\hat{n}_O \equiv \sqrt{n_{O1}^2 + n_{O2}^2 + n_{O3}^2}/I\right). \tag{20}$$

Let us finally recall that the set of parameters (P_d,P_l,P_c) constitutes the so-called set of *components of purity* (hereafter CP) of the state of polarization considered and satisfies the following expressions [21]:

$$0 \leqslant P_l \leqslant P_d \leqslant 1, \ \ P_{3D} = \sqrt{\frac{3\left(P_l^2 + P_c^2\right) + P_d^2}{4}}. \tag{21}$$

II. INTERPRETATION OF THE CHARACTERISTIC DECOMPOSITION OF THE 3D POLARIZATION MATRIX

At first sight, and as pointed out by several authors [1,3,10,15,16,22], the characteristic decomposition in Eq. (11) suggests that the state of polarization $\mathbf{R}$ can always be considered as representing a *regular state*, namely, an incoherent composition of a pure state $\mathbf{R}_p$, a 2D unpolarized state $\mathbf{R}_{u-2D}$, and a 3D unpolarized state $\mathbf{R}_{u-3D}$. As required for pure states (or totally polarized states), the electric field of the state $\mathbf{R}_p$ lies in a fixed plane and describes a well-defined polarization ellipse, while $\mathbf{R}_{u-3D}$ can always be interpreted as an equiprobable, fully random, mixture of three linearly polarized pure states whose respective electric fields lie in mutually orthogonal directions and thus can properly be called a 3D unpolarized state. Nevertheless, it has been proven that the component $\mathbf{R}_m$ of the characteristic decomposition in Eq. (11) does not necessarily represent a 2D unpolarized state [18]; that is, $\mathbf{R}_m$ cannot always be considered an equiprobable fully random composition of two mutually orthogonal pure states whose electric fields lie in a common fixed plane.

As pointed out in previous works [18,19], the only unitary transformations $\mathbf{U}^{\dagger}\mathbf{R}\mathbf{U}$ that can be physically realized as rotations of the laboratory reference frame XYZ are those where there exists an orthogonal matrix $\mathbf{Q}$ (i.e., $\mathbf{Q}$ satisfying $\mathbf{Q}^T = \mathbf{Q}^{-1}$ and $\det \mathbf{Q} = +1$) that satisfies $\mathbf{U}^{\dagger}\mathbf{R}\mathbf{U} = \mathbf{Q}^T\mathbf{R}\mathbf{Q}$. Therefore, given a polarization matrix $\mathbf{R}$ satisfying rank $\mathbf{R} = 2$ there is not always an orthogonal transformation $\mathbf{Q}^T\mathbf{R}\mathbf{Q}$ such that one of its diagonal elements is zero. In other words, all a_i are nonzero in all frames and the electric vector is not in a plane. To clarify this fact, and due to the key role played by $\mathbf{R}_m$ in the physical interpretation of $\mathbf{R}$, let us next analyze the different types of matrices $\mathbf{R}_m$ on the basis of its spectral decomposition.

A. Composition of two linearly polarized fields propagating in different directions

Regardless of the respective directions of propagation, the incoherent superposition, at a given point $\mathbf{r}$ in the space, of two linearly polarized states is a mixed state whose electric field lies in the plane defined by the pair of respective polarization axes. The polarization matrix of the combined state (assumed that the directions in which the respective fluctuating electric fields of the components lie are different, but not necessarily mutually orthogonal) can always be expressed as

$$\mathbf{R}_{ll} = \mathbf{Q}\mathbf{R}_O\mathbf{Q}^T,$$
$$\mathbf{R}_O \equiv \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = a_2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (a_1 - a_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (0 \leqslant a_2 \leqslant a_1), \tag{22}$$

where $\mathbf{Q}$ is an orthogonal matrix, while the real and nonnegative parameters a_1 and a_2 are the respective intensities of the linear spectral components.

In the case of an equiprobable mixture of two mutually orthogonal linear states (i.e., $a_1 = a_2 \equiv I/2$), the polarization matrix of the combined state has the form

$$\mathbf{R}_{u-2\mathrm{D}} = (I/2)\mathbf{Q}\,\mathrm{diag}\,(1,1,0)\mathbf{Q}^T, \tag{23}$$

which corresponds to a 2D unpolarized state whose electric field lies in the plane X_OY_O of the intrinsic reference frame. Consequently, this state is characterized by

$$P_l(\mathbf{R}_{u-2\mathrm{D}}) = 0, \quad P_c(\mathbf{R}_{u-2\mathrm{D}}) = 0, \quad P_d(\mathbf{R}_{u-2\mathrm{D}}) = 1,$$
$$P_1(\mathbf{R}_{u-2\mathrm{D}}) = 0, \quad P_2(\mathbf{R}_{u-2\mathrm{D}}) = 1, \quad P_{3\mathrm{D}}(\mathbf{R}_{u-2\mathrm{D}}) = 1/2, \tag{24}$$

and therefore, regardless of the respective directions of propagation of the linear components, the polarization state, at the point $\mathbf{r}$ considered (hence disregarding the coherence-polarization properties of the whole beam given by the two-point coherence matrix [7,23]), is indistinguishable from a 2D unpolarized state propagating along the direction determined by Z_O.

Note that the particular case (discarded from the above analysis) of an incoherent superposition of two linearly polarized states whose electric fields lie in the same direction results in a whole linearly polarized state (hence pure) that defines the direction X_O, but does not determine the plane X_OY_O nor the direction of propagation Z_O [18].

B. Equiprobable incoherent mixture of two mutually orthogonal pure states with a common direction of propagation

It is well known that the incoherent composition, at a given point $\mathbf{r}$ in the space, of two mutually orthogonal pure states with equal intensities $I/2$ and whose polarization ellipses lie in the same plane X_OY_O is represented by a polarization matrix that necessarily has the form of a 2D unpolarized state. In the three-dimensional formulation, the polarization matrix of such a field adopts the form

$$\mathbf{R}_{u-2\mathrm{D}} = (I/2)\mathbf{Q}\,\mathrm{diag}\,(1,1,0)\mathbf{Q}^T, \tag{25}$$

where the proper orthogonal transformation by $\mathbf{Q}$ represents a rotation from the intrinsic reference frame $X_OY_OZ_O$ to the particular reference frame XYZ considered. Note that, as shown in Sec. II A, the same result is obtained for an equiprobable incoherent mixture of two mutually orthogonal linearly polarized states with different directions of propagation. The IPP and CP of the present $\mathbf{R}_{u-2\mathrm{D}}$ have the values indicated in Eq. (24).

C. Equiprobable incoherent mixture of two mutually orthogonal pure states with different directions of propagation

Let us assume that at least one of the combined states has nonzero ellipticity (the case of linear components has been considered in Sec. II A). It is straightforward to prove that the polarization matrix associated with this kind of mixed state cannot be reduced to the form $\mathbf{R}_{u-2\mathrm{D}}$. That is, although in this case the polarization matrix still has the form

$$\mathbf{R}_m = (I/2)\mathbf{U}\,\mathrm{diag}\,(1,1,0)\mathbf{U}^{\dagger}, \tag{26}$$

it is not a real matrix. This occurs when there is no orthogonal matrix $\mathbf{Q}$ satisfying $\mathbf{Q}\,\mathrm{diag}\,(1,1,0)\mathbf{Q}^T = \mathbf{U}\,\mathrm{diag}\,(1,1,0)\mathbf{U}^{\dagger}$.

Therefore, when $\mathbf{R}_m$ is not a real matrix, the unitary similarity transformation in Eq. (26) cannot be considered a rotation of the reference frame. In other words, necessarily $a_3(\mathbf{R}_m) > 0$, which means that the fluctuating electric field of the state has three nonzero components with respect to any laboratory reference frame and, consequently, despite the fact that rank $\mathbf{R}_m = 2$, $\mathbf{R}_m$ represents a 3D state. In fact, this kind of mixed state is characterized by the following values of the components and indices of polarimetric purity:

$$P_l(\mathbf{R}_m) \geqslant 0, \quad P_c(\mathbf{R}_m) > 0, \quad P_d(\mathbf{R}_m) < 1,$$
$$P_1(\mathbf{R}_m) = 0, \quad P_2(\mathbf{R}_m) = 1. \tag{27}$$

It is remarkable that since, in this case, $\mathbf{R}_m$ is not a real matrix, some nonzero off-diagonal elements of $\mathbf{R}_m$ are not real and therefore the degree of circular polarization of $\mathbf{R}_m$ is necessarily nonzero.

To illustrate this peculiar type of polarization state, it is worth considering the case of an equiprobable composition of a right-circularly polarized state $\mathbf{R}_c$ propagating along the X axis and a linearly polarized state $\mathbf{R}_l$ whose electric field vibrates in the direction X (i.e., the direction of propagation of $\mathbf{R}_l$ is orthogonal to X). The representations of these states in the Cartesian reference frame XYZ are the following:

$$\mathbf{R}_c = (I/2)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & i/2 & 1/2 \end{pmatrix}, \quad \mathbf{R}_l = (I/2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Therefore, the polarization matrix of the incoherently combined state is given by

$$\mathbf{R}_{\text{cl}} = I\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & -i/4 \\ 0 & i/4 & 1/4 \end{pmatrix}, \tag{29}$$

which can be expressed as

$$\mathbf{R}_{\text{cl}} = (I/2)\mathbf{U}\,\text{diag}\,(1,1,0)\mathbf{U}^{\dagger}, \quad \mathbf{U} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}, \tag{30}$$

and satisfies $P_2(\mathbf{R}_{\text{cl}}) = 1$, while $\text{Re}(\mathbf{R}_{\text{cl}}) = \text{diag}(1/2,1/4,1/4)$, and thus $\text{rank}\,[\text{Re}(\mathbf{R}_{\text{cl}})] = 3$. Obviously, while this matrix has the form $\mathbf{R}_{\text{cl}} = (I/2)\,\mathbf{U}\,\text{diag}(1,1,0)\mathbf{U}^{\dagger}$, it is different from $\mathbf{R}_{u-2\text{D}}$ and therefore, there is no way to identify it with a 2D unpolarized state. In fact, besides its degree of directional purity given by $P_d(\mathbf{R}_{\text{cl}}) = 1/4$, $\mathbf{R}_{\text{cl}}$ exhibits a certain amount of linear and circular polarization given by $P_l(\mathbf{R}_{\text{cl}}) = 1/4$ and $P_c(\mathbf{R}_{\text{cl}}) = 1/2$.

D. Regular and nonregular states of polarization

Let us consider again a general polarization matrix $\mathbf{R}$ and its characteristic decomposition

$$\mathbf{R} = P_1\mathbf{R}_p + (P_2 - P_1)\mathbf{R}_m + (1 - P_2)\mathbf{R}_{u-3\text{D}}, \tag{31}$$

and recall that $\mathbf{R}$ is said to represent a regular state when $\mathbf{R}_m = \mathbf{R}_{u-2\text{D}}$. From the analysis performed in the previous sections it follows that this condition occurs if and only if $\mathbf{R}_m$ is a 2D state, that is, if and only if the following equivalent conditions are satisfied: (1) $\mathbf{R}_m$ is a real matrix, (2) $P_c(\mathbf{R}_m) = 0$, (3) $P_d(\mathbf{R}_m) = 1$, (4) $a_3(\mathbf{R}_m) = 0$, and (5) $m < 3$ (where $m \equiv \text{rank}\,[\text{Re}(\mathbf{R}_m)]$).

In the special cases where $P_1 = P_2$ (which will be analyzed in Sec. III), the component $\mathbf{R}_m$ is not present in the characteristic decomposition of $\mathbf{R}$, so that $\mathbf{R}$ can be considered as an incoherent composition of a pure state $\mathbf{R}_p$ and a 3D unpolarized state $\mathbf{R}_{u-3\text{D}}$ and therefore condition $P_1 = P_2$ implies that $\mathbf{R}$ is a particular type of regular state. Leaving aside states with $P_1 = P_2$, regular states (whose polarization matrix is hereafter denoted as $\mathbf{R}_r$ wherever appropriate) are characterized by $m < 3$, while nonregular states (hereafter denoted as $\mathbf{R}_n$) are characterized by $m = 3$. States $\mathbf{R}_n$ constitute a peculiar class of 3D polarization states in which the middle component $\mathbf{R}_m$ of the characteristic decomposition is not real and can be interpreted as an equiprobable mixture $\mathbf{R}_m = (I/2)(\mathbf{u}_1 \otimes \mathbf{u}_1^{\dagger}) + (I/2)(\mathbf{u}_2 \otimes \mathbf{u}_2^{\dagger})$ of two mutually orthogonal pure states with different directions of propagation (i.e., whose respective electric fields evolve in different planes and at least one of them has nonzero ellipticity). Such compositions were considered in Sec. II C above.

In summary, apart from states with $P_1 = P_2$ (which constitute a particular type of regular state), a given polarization matrix $\mathbf{R}$ represents a regular state if and only if its corresponding component $\mathbf{R}_m$, by itself, represents a regular state, in this case, a mere unpolarized 2D field. Equivalently, $\mathbf{R}$ represents a nonregular state if and only if its corresponding component $\mathbf{R}_m$, by itself, represents a nonregular state ($m = 3$), i.e., a true 3D field.

Let us now inspect other interesting features that distinguish regular and nonregular states. The inequality $\hat{\lambda}_3 \leqslant \hat{a}_3$ (see Sec. I), together with Eqs. (13) and (19), leads to

$$P_d \leqslant P_2. \tag{32}$$

Furthermore, as indicated above, $\mathbf{R}_m$ differs from $\mathbf{R}_{u-2\text{D}}$ if and only if $P_c(\mathbf{R}_m) \neq 0$ (i.e., $\mathbf{R}_m$ exhibits a nonzero degree of circular polarization). That is, $\mathbf{R}$ represents a regular state if and only if $P_c(\mathbf{R}_m) = 0$, which, in its turn is satisfied if and only if

$$P_l^2 + P_c^2 = P_1^2; \tag{33}$$

i.e., regular states are characterized by the fact that all the degree of circular polarization of $\mathbf{R}$ is allocated to the characteristic component $\mathbf{R}_p$ of its characteristic decomposition. Observe that, from Eqs. (14) and (21), it follows that

$$2P_{3\text{D}}^2 = 3P_1^2 + P_2^2 = 3\left(P_l^2 + P_c^2\right) + P_d^2, \tag{34}$$

and therefore condition (32) is entirely equivalent to $P_l^2 + P_c^2 \geqslant P_1^2$, where the equality is satisfied if and only if $P_d = P_2$, which constitutes a genuine property of regular states. Due to its particular physical significance, which will be analyzed in Sec. III, it is worth defining the *degree of elliptical purity* (that is, the combined contributions to purity due to circular and linear degrees of polarization, or equivalently, a measure of all the contributions to polarimetric purity except that due to the degree of directionality) as

$$P_e \equiv \sqrt{P_l^2 + P_c^2}. \tag{35}$$

Table I summarizes the main results obtained from the analysis of polarization matrices of the form $\mathbf{R}_m = (I/2)\,\mathbf{U}\,\text{diag}\,(1,1,0)\mathbf{U}^{\dagger}$.

A comparison of the main properties of regular and nonregular states is shown in Table II.

To complete the discussion on regular and nonregular states in terms of incoherent components, it is worth recalling that, by

TABLE I. Summary of the properties of $\mathbf{R}_m$ in terms of the value of the integer descriptor m.

$m \equiv \text{rank}\,[\text{Re}(\mathbf{R}_m)]$	$P_d(\mathbf{R}_m)$	$P_c(\mathbf{R}_m)$	$P_e(\mathbf{R}_m)$	$\mathbf{R}_m$	2D-3D	Regularity
$m = 2$	$P_d = P_2 = 1$	$P_c = 0$	$P_e = P_1 = 0$	$\mathbf{R}_m \in \mathbb{R}$	2D	Regular
$m = 3$	$P_d < P_2 = 1$	$P_c > 0$	$P_e > P_1 = 0$	$\mathbf{R}_m \notin \mathbb{R}$	3D	Nonregular

TABLE II. Summary of the properties of a polarization matrix **R** in terms of the value of the integer descriptor m.

$m \equiv \text{rank}\,[\text{Re}(\mathbf{R}_m)]$	$P_d(\mathbf{R})$	$P_c(\mathbf{R})$	$P_e(\mathbf{R})$	Regularity
$m = 2$	$P_d = P_2$	$P_c \geqslant 0$	$P_e = P_1$	Regular
$m = 3$	$P_d < P_2$	$P_c > 0$	$P_e > P_1$	Nonregular

combining the spectral and the characteristic decompositions of a given polarization state **R**, it can always be considered an incoherent composition of the following form [18]:

$$\mathbf{R} = I\left[\frac{P_2 + P_1}{2}\hat{\mathbf{R}}_{p1} + \frac{P_2 - P_1}{2}\hat{\mathbf{R}}_{p2} + (1 - P_2)\hat{\mathbf{R}}_{u-3\text{D}}\right], \quad (36)$$

where $\hat{\mathbf{R}}_{p1} = \mathbf{u}_1 \otimes \mathbf{u}_1^{\dagger}$ and $\hat{\mathbf{R}}_{p2} = \mathbf{u}_2 \otimes \mathbf{u}_2^{\dagger}$ represent the first two spectral components (hence mutually orthogonal). When the polarization ellipses of the respective polarization matrices $\hat{\mathbf{R}}_{p1}$ and $\hat{\mathbf{R}}_{p2}$ lie in a common plane, then **R** represents a regular state; otherwise (i.e., when $\hat{\mathbf{R}}_{p1}$ and $\hat{\mathbf{R}}_{p2}$ have different directions of propagation at the point **r** considered and at least one of them has nonzero ellipticity) **R** represents a nonregular state.

Once the properties of the component $\mathbf{R}_m$ and its implications on the interpretation of **R** have been analyzed, it is worth taking them to complete, in Table III, the general classification of the possible types of states of polarization performed in [7,18] in terms of the values of the integer descriptors,

$$r = \text{rank}\,\mathbf{R}, \quad t = \text{rank}[\text{Re}(\mathbf{R})], \quad m = \text{rank}[\text{Re}(\mathbf{R}_m)], \quad (37)$$

together with the corresponding values of the IPP and CP. Note that the following conditions satisfied by the integer descriptors have been considered in the case analysis shown in Table III [18]:

$$\begin{aligned} r = 1 &\Rightarrow t < 3, \\ r = 2 &\Rightarrow 1 < t \leqslant 3, \\ (r = 2,\ t = 2) &\Rightarrow m = 2, \\ (r = 2,\ t = 3) &\Rightarrow m = 3, \\ r = 3 &\Rightarrow t = 3, \\ r = 3 &\Rightarrow 1 < m \leqslant 3. \end{aligned} \quad (38)$$

III. DISCUSSION

The description, characterization, and classification of 3D polarization states by means of appropriate quantities are not only necessary from a very theoretical point of view, but they are also justified on the increasing practical interest in nanotechnologies, optical near-field phenomena, and other areas [5,8–34]. However, we note that evanescent (i.e., inhomogeneous) electromagnetic plane waves have complex-valued wave vectors and thereby the directionality of such fields is physically more involved than that of the usual propagating plane waves.

From the sole point of view of the structure of polarimetric purity of a given 3D state **R**, such structure is fully characterized by the corresponding indices of polarimetric purity, P_1 and P_2, which are insensitive to specific polarization features of **R**, which in turn are determined by the components of purity P_d, P_l, and P_c. The degree of polarimetric purity P_{3D} constitutes an overall measure of the closeness of **R** to a polarimetrically pure state and can be calculated either from the IPP or from the CP through the following expressions [19]:

$$P_{3D}^2 = \tfrac{3}{4}P_1^2 + \tfrac{1}{4}P_2^2 = \tfrac{3}{4}P_e^2 + \tfrac{1}{4}P_d^2 \quad (39)$$

(recall that, in general, $P_d \neq P_2$, where the equality holds exclusively for regular states).

Equation (39) provides two complementary ways to interpret the structure of polarimetric purity of a polarization state. On the one hand, it can be interpreted as shared among P_1 and P_2, where P_2 is the relative power of the incoherent component that is not 3D unpolarized, while P_1 is the relative power of the totally polarized component. On the other hand, it can be interpreted as shared among P_d and P_e where P_d is the contribution to polarimetric purity allocated to the degree of directionality (i.e., to the closeness to a 2D state) while P_e is the contribution to polarimetric purity allocated to the

TABLE III. General classification of states of polarization.

Integer descriptors	P_2, P_d	P_1, P_e	P_l	P_c	$P_{3\text{D}}$	2D-3D	Regularity	Characteristic components
$r = 1$ $t = 1$	$P_d = P_2 = 1$	$P_e = P_1 = 1$	$P_l = 1$	$(P_c = 0)$	$P_{3\text{D}} = 1$	2D	R	$\mathbf{R}_p$ linear
$r = 1$ $t = 2$	$P_d = P_2 = 1$	$P_e = P_1 = 1$	$0 \leqslant P_l < 1$	$0 < P_c \leqslant 1$	$P_{3\text{D}} = 1$	2D	R	$\mathbf{R}_p$ elliptic
$r = 2$ $t = 2$ $(m = 2)$	$P_d = P_2 = 1$	$0 \leqslant P_e = P_1 < 1$	$0 \leqslant P_l < 1$	$0 \leqslant P_c < 1$	$1/2 \leqslant P_{3\text{D}} < 1$	2D	R	$\mathbf{R}_p$ $\mathbf{R}_{u-2\text{D}}$
$r = 2$ $t = 3$ $(m = 3)$	$0 \leqslant P_d < P_2 = 1$	$0 \leqslant P_1 < P_e < 1$	$0 \leqslant P_l < 1$	$0 < P_c < 1$	$1/2 \leqslant P_{3\text{D}} < 1$	3D	NR	$\mathbf{R}_p$ $\mathbf{R}_m$
$r = 3$ $m = 2$	$0 \leqslant P_d = P_2 < 1$	$0 \leqslant P_e = P_1 < 1$	$0 \leqslant P_l < 1$	$0 \leqslant P_c < 1$	$0 \leqslant P_{3\text{D}} < 1$	3D	R	$\mathbf{R}_p$ $\mathbf{R}_{u-2\text{D}}$ $\mathbf{R}_{u-3\text{D}}$
$r = 3$ $m = 3$	$0 \leqslant P_d < P_2 < 1$	$0 \leqslant P_1 < P_e < 1$	$0 \leqslant P_l < 1$	$0 < P_c < 1$	$0 \leqslant P_{3\text{D}} < 1$	3D	NR	$\mathbf{R}_p$ $\mathbf{R}_m$ $\mathbf{R}_{u-3\text{D}}$

TABLE IV. Classification of states of polarization with $P_2 = 1$ and $0 < P_1 < 1$.

Integer descriptors	P_2, P_d	P_1, P_e	P_l	P_c	P_{3D}	2D-3D	Regularity	Characteristic components
$r = 2$ $t = 2$ $(m = 2)$	$P_d = P_2 = 1$	$0 < P_e = P_1 < 1$	$0 \leqslant P_l < 1$	$0 \leqslant P_c < 1$	$1/2 < P_{3D} < 1$	2D	R	$\mathbf{R}_p$ $\mathbf{R}_{u-2D}$
$r = 2$ $t = 3$ $(m = 3)$	$0 < P_d < P_2 = 1$	$0 < P_1 < P_e < 1$	$0 \leqslant P_l < 1$	$0 < P_c < 1$	$1/2 < P_{3D} < 1$	3D	NR	$\mathbf{R}_p$ $\mathbf{R}_m$

quadratic average of linear and circular degrees of polarization. This justifies the introduction in Sec. II of the name *degree of elliptical purity* for P_e.

The characteristic decomposition, whose coefficients are determined by the IPP, provides an insightful view of the polarimetric purity and randomness of a polarization state. As demonstrated in previous sections (see Table III), nonregular states are characterized by the inequality $P_e > P_1$, which means that, besides the amount of purity associated with the weight P_1 of the pure component, some amount of polarimetric purity is allocated to the circular and linear polarization of the middle component $\mathbf{R}_m$ of $\mathbf{R}$. Only when P_1 and P_2 are equal (and hence they both equal P_{3D}) the middle component of the characteristic decomposition in Eq. (11) vanishes and all three quantities describe the ratio of the intensity of the pure (polarized) part to the total intensity (see also case B below).

To go deeper into the physical interpretation of 3D states of polarization and of the role played by the IPP, the CP, and P_{3D}, let us analyze the particular cases where some component is not present in the characteristic decomposition of $\mathbf{R}$. Observe that the cases where the characteristic decomposition contains a unique component, namely, $\mathbf{R}_p$ ($P_1 = 1$), or $\mathbf{R}_m$ ($P_1 = 0$, $P_2 = 1$), or $\mathbf{R}_{u-3D}$ ($P_2 = 0$) (recall the inequality $0 \leqslant P_1 \leqslant P_2 \leqslant 1$), are obvious from Secs. I and II.

A. States with $P_2 = 1$ and $0 < P_1 < 1$

In this case, the characteristic decomposition of Eq. (11) takes the form

$$\mathbf{R}_{pm} = I[P_1\hat{\mathbf{R}}_p + (1 - P_1)\hat{\mathbf{R}}_m]. \quad (40)$$

The corresponding degree of polarimetric purity is given by $P_{3D}^2 = (3P_1^2 + 1)/4$ and thus $P_{3D} > 1/2$, showing that the absence of the 3D unpolarized component (with $0 < P_1$) entails a contribution of $1/2$ to polarimetric purity.

In accordance to the analysis performed in Sec. II, the physical interpretation of the state in Eq. (40) depends strongly on m (see Table IV). When $m = 2$, $\mathbf{R}_{pm}$ represents a partially polarized 2D state, that is, an incoherent composition of a pure state $\mathbf{R}_p$ and a 2D unpolarized state $\mathbf{R}_{u-2D}$ with common direction of propagation. When $m = 3$, $\mathbf{R}_{pm}$ represents a type of 3D nonregular state given by an incoherent composition of a pure state $\mathbf{R}_p$ and a nonregular state $\mathbf{R}_m$. Observe that regardless of the value of m, $\mathbf{R}_p$ can be real or not, while the component $\mathbf{R}_m$ is real if and only if $m = 2$. Notice that an incoherent composition of a pure state and a 2D unpolarized state with different directions of propagation constitutes a regular state that satisfies $P_2 = P_d < 1$ (i.e., $r = t = 3$ with $m = 2$). Such states necessarily fall out of the case under consideration.

B. States with $0 < P_1 = P_2 < 1$

In this case, the characteristic decomposition has the form

$$\mathbf{R}_{pu} = I[P_1\hat{\mathbf{R}}_p + (1 - P_1)\hat{\mathbf{R}}_{u-3D}] \quad (41)$$

and corresponds to a 3D regular *decoupled state* composed of an incoherent mixture of a pure state and a 3D unpolarized state (see Table V). This is a particular type of 3D regular state for which the following relations hold:

$$0 < P_{3D} = P_1 = P_2 = P_d = P_e < 1. \quad (42)$$

This simplicity is derived from the fact that, in this case, the polarimetric randomness is exclusively provided by the 3D unpolarized component $\mathbf{R}_{u-3D}$ (while the polarimetric purity is exclusively provided by the pure component $\mathbf{R}_p$).

C. States with $P_1 = 0$ and $0 < P_2 < 1$

The characteristic decomposition of these states has the form

$$\mathbf{R}_{mu} = I[P_2\hat{\mathbf{R}}_m + (1 - P_2)\hat{\mathbf{R}}_{u-3D}]. \quad (43)$$

The corresponding degree of polarimetric purity is given by $P_{3D} = P_2/2$ and thus $P_{3D} < 1/2$, showing that the absence of the pure component entails this upper limit of $1/2$ for the achievable values of P_{3D}.

As expected, the physical interpretation of $\mathbf{R}_{mu}$ depends strongly on m (see Table VI). When $m = 2$, $\mathbf{R}_{mu}$ represents a

TABLE V. States of polarization with $0 < P_1 = P_2 < 1$.

Integer descriptors	P_2, P_d	P_1, P_e	P_l	P_c	P_{3D}	2D-3D	Regularity	Characteristic components
$r = 3$ $(t = 3)$ $m = 2$	$0 < P_d = P_2 < 1$	$0 < P_e = P_1 < 1$	$0 \leqslant P_l < 1$	$0 \leqslant P_c < 1$	$0 < P_{3D} = P_i < 1$ $(i = 1, 2, p, d)$	3D	R	$\mathbf{R}_p$ $\mathbf{R}_{u-3D}$

TABLE VI. Classification of states of polarization with $P_1 = 0$ and $0 < P_2 < 1$.

Integer descriptors	P_2, P_d	P_1, P_e	P_l	P_c	P_{3D}	2D-3D	Regularity	Characteristic components
$r = 3$ $m = 2$	$0 < P_d = P_2 < 1$	$P_e = P_1 = 0$	$P_l = 0$	$P_c = 0$	$0 < P_{3D} = P_2/2 = P_d/2 < 12$	3D	R	$\mathbf{R}_{u-2D}$ $\mathbf{R}_{u-3D}$
$r = 3$ $m = 3$	$0 \leqslant P_d < P_2 < 1$	$0 = P_1 < P_e < 1$	$0 \leqslant P_l < 1$	$0 < P_c < 1$	$0 < P_{3D} = P_2/2 < 12$	3D	NR	$\mathbf{R}_m$ $\mathbf{R}_{u-3D}$

type of 3D regular state given by an incoherent composition of a 2D unpolarized state $\mathbf{R}_{u-2D}$ and a 3D unpolarized state $\mathbf{R}_{u-3D}$. When $m = 3$, $\mathbf{R}_{mu}$ is a kind of 3D nonregular state given by an incoherent composition of a nonregular state $\mathbf{R}_m$ and a 3D unpolarized state $\mathbf{R}_{u-3D}$.

IV. CONCLUSIONS

All the second-order properties of a general state of polarization (3D state) are determined by the corresponding polarization matrix $\mathbf{R}$. Nevertheless, the relevant polarimetric information does not appear explicitly in the mere elements of $\mathbf{R}$, but it has been shown that the physical interpretation, characterization, and classification of 3D polarization states require the use of appropriate sets of parameters derived from the elements of $\mathbf{R}$, such as the rank descriptors r, t, m in Eq. (37); the indices of polarimetric purity P_1, P_2 in Eq. (12); the components of purity P_l, P_c, P_d in Eqs. (17)–(20); and the degree of polarimetric purity P_{3D} of Eq. (9). These sets of parameters are closely linked to the possible decompositions of $\mathbf{R}$ as convex sums of incoherent components, such as the arbitrary decomposition given in Eq. (4), which includes the spectral decomposition as a particular case [Eq. (7)], and the characteristic decomposition of Eq. (11).

While all the said decompositions are useful for a complete analysis of the different types of polarization states and their properties, the characteristic decomposition of a given $\mathbf{R}$ as a convex sum of a pure state $\mathbf{R}_p$, a state of the form $\mathbf{R}_m$ (dealt with in detail in Sec. II), and a fully random polarization state $\mathbf{R}_{u-3D}$, provides the optimum framework for the study of the structure of polarimetric purity (or conversely, the structure of polarimetric randomness) of 3D states. The characterization and interpretation of states $\mathbf{R}_m$ arise as key issues for solving the problem of understanding physically the 3D polarization phenomena.

States $\mathbf{R}_m$ cannot always be identified with conventional 2D unpolarized states (i.e., states whose electric field evolves fully randomly but in a fixed plane). In fact, there exists a category of states $\mathbf{R}_m$, which we named *nonregular*, that can be interpreted as equiprobable mixtures of two mutually orthogonal totally polarized (or pure) states with different directions of propagation (at least one of them with nonzero ellipticity). Once the said states $\mathbf{R}_m$, in their alternative forms, regular and nonregular, have been characterized and illustrated with appropriate examples, a general classification of the 3D states of polarization has been performed in an objective and unambiguous way in Sec. II (see Table III).

The physical significance and the interrelations among the degree of polarimetric purity, the indices of polarimetric purity, and the components of purity have been studied and illustrated with meaningful examples in Sec. III, providing a complete framework for the understanding of the structure of polarimetric purity of any kind of polarization state.

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APPENDIX: ARBITRARY DECOMPOSITION OF A POLARIZATION MATRIX

Given a polarization matrix $\mathbf{R}$ with $\operatorname{rank}\mathbf{R} = r$, let us consider its diagonalization,

$$\mathbf{R} = \mathbf{U}\,\mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3)\mathbf{U}^\dagger, \quad 0 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1, \tag{A1}$$

where $\mathbf{U}$ is the unitary matrix whose columns are the mutually orthonormal eigenvectors of $\mathbf{R}$, and let us define

$$\sqrt{\mathbf{D}} \equiv \mathrm{diag}(\sqrt{\lambda_1},\sqrt{\lambda_2},\sqrt{\lambda_3}), \tag{A2}$$

whose last $3 - r$ diagonal elements are zero.

By introducing the 3×3 complex matrix

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{A3}$$

where all elements are zero except for the arbitrary $r \times r$ unitary submatrix $\mathbf{V}_r$, we find that the following equalities are always satisfied,

$$\begin{aligned} \mathbf{R} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger &= \mathbf{U}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{U}^\dagger = \mathbf{U}\sqrt{\mathbf{D}}\mathbf{I}_r\sqrt{\mathbf{D}}\mathbf{U}^\dagger \\ &= \mathbf{U}\sqrt{\mathbf{D}}\mathbf{V}\mathbf{V}^\dagger\sqrt{\mathbf{D}}\mathbf{U}^\dagger \\ &= \mathbf{U}\sqrt{\mathbf{D}}\left(\sum_{i=1}^{r}\mathbf{v}_i \otimes \mathbf{v}_i^\dagger\right)\sqrt{\mathbf{D}}\mathbf{U}^\dagger \\ &= \sum_{i=1}^{r}(\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i) \otimes (\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i)^\dagger, \end{aligned} \tag{A4}$$

where the matrix

$$\mathbf{I}_r = \mathbf{V}\,\mathbf{V}^\dagger = \left(\sum_{i=1}^{r}\mathbf{v}_i \otimes \mathbf{v}_i^\dagger\right) \tag{A5}$$

is the diagonal matrix whose first r diagonal elements are 1 and its last $3-r$ diagonal elements are 0, while $\mathbf{v}_i (i=1,\ldots,r)$ are the orthonormal column vectors of $\mathbf{V}_r$.

Let us now write the vector $\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i$ that generates each Hermitian matrix $(\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i)\otimes(\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i)^\dagger$ in expansion (A4) in the following manner:

$$\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i = |\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i|\mathbf{w}_i = |\sqrt{\mathbf{D}}\mathbf{v}_i|\mathbf{w}_i, \tag{A6}$$

where the equality $|\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i| = |\sqrt{\mathbf{D}}\mathbf{v}_i|$ holds because $\mathbf{U}$ is a unitary matrix and hence it does not affect the absolute value of vector $\sqrt{\mathbf{D}}\mathbf{v}_i$, while $\mathbf{w}_i$ is the unit vector obtained through the normalization of $\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i$.

From Eqs. (A4) and (A6) we deduce that any polarization matrix $\mathbf{R}$ can always be expressed as the following convex sum:

$$\mathbf{R} = \sum_{i=1}^{r} p_i \mathbf{R}_{pi}; \quad \mathbf{R}_{pi} \equiv (\mathrm{tr}\,\mathbf{R})(\mathbf{w}_i \otimes \mathbf{w}_i^\dagger),$$
$$\mathbf{w}_i \equiv \frac{\mathbf{U}\sqrt{\mathbf{D}}\mathbf{v}_i}{|\sqrt{\mathbf{D}}\mathbf{v}_i|}, \quad p_i \equiv \frac{|\sqrt{\mathbf{D}}\mathbf{v}_i|^2}{\mathrm{tr}\,\mathbf{R}} \quad (i=1,\ldots,r), \quad \sum_{i=1}^{r} p_i = 1. \tag{A7}$$

Based on the fact that the only exigency on $\mathbf{V}_r$ is that it is unitary, expansion (A7) provides expressions for generating arbitrary complete sets of the covariance matrices $(\mathrm{tr}\,\mathbf{R})(\mathbf{w}_i \otimes \mathbf{w}_i^\dagger)$ of the pure components as well as their corresponding coefficients p_i in terms of arbitrary sets of r orthonormal vectors $\mathbf{v}_i$ whose last $3-r$ components are zero. When $r=3$, any arbitrary three-dimensional orthonormal complex basis $\mathbf{v}_i$ $(i=1,2,3)$ can be chosen. In the other limiting case, where $r=1$, $\mathbf{R}$ represents a pure state and consequently the arbitrary decomposition has no physical interest because it becomes a tautology.

As a preliminary step to determine the expression of p_i directly in terms of the set of r independent vectors $\mathbf{w}_i$, we note that each of the r orthonormal vectors $\mathbf{v}_i$ constituting the columns of the unitary matrix $\mathbf{V}_r$ introduced in Eq. (A3) can be written as follows:

$$\mathbf{v}_i \equiv \sqrt{\mathrm{tr}\,\mathbf{R}}\sqrt{p_i}(\sqrt{\mathbf{D}})^- \mathbf{U}^\dagger \mathbf{w}_i \quad (i=1,\ldots,r),$$
$$(\sqrt{\mathbf{D}})^- \equiv (1/\sqrt{\lambda_0},\ldots,\, 1/\sqrt{\lambda_r}, 0,\ldots,0), \tag{A8}$$

$(\sqrt{\mathbf{D}})^-$ being the pseudoinverse of $\sqrt{\mathbf{D}}$. Consequently, given a set of r independent unit vectors $\mathbf{w}_i$ belonging to the image subspace of a given polarization matrix $\mathbf{R}$ with a given rank $\mathbf{R}=r$, the r matrices $(\mathrm{tr}\,\mathbf{R})(\mathbf{w}_i \otimes \mathbf{w}_i^\dagger)$ can be considered as a complete set of generators of the pure components of $\mathbf{R}$. That is to say, providing $\mathbf{w}_i \in \mathrm{Range}\,(\mathbf{R})$, the independence of the unit vectors $\mathbf{w}_i$ in Eq. (A7) is equivalent to the orthonormality of their corresponding vectors $\mathbf{v}_i$. We finally observe that since $\mathbf{v}_i$ is a unit vector and $\lambda_j > 0\,(j=1,\ldots,r)$, the following holds:

$$\begin{aligned} 1 &= \mathrm{tr}(\mathbf{v}_i \otimes \mathbf{v}_i^\dagger) \\ &= (\mathrm{tr}\,\mathbf{R})p_i \mathrm{tr}\{(\sqrt{\mathbf{D}})^- \mathbf{U}^\dagger \mathbf{w}_i \otimes [(\sqrt{\mathbf{D}})^- \mathbf{U}^\dagger \mathbf{w}_i]^\dagger\} \\ &= (\mathrm{tr}\,\mathbf{R})p_i \sum_{j=1}^{r} \left| \frac{1}{\sqrt{\lambda_j}} (\mathbf{U}^\dagger \mathbf{w}_i)_j \right|^2 \\ &= (\mathrm{tr}\,\mathbf{R})p_i \sum_{j=1}^{r} \frac{1}{\lambda_j} \left| (\mathbf{U}^\dagger \mathbf{w}_i)_j \right|^2 \quad (i=1,\ldots,r), \end{aligned} \tag{A9}$$

and therefore

$$p_i = \frac{1}{(\mathrm{tr}\,\mathbf{R}) \sum_{j=1}^{r} \frac{1}{\lambda_j} \left| (\mathbf{U}^\dagger \mathbf{w}_i)_j \right|^2}. \tag{A10}$$

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