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Generalized Jacobi–Weierstrass operators and Jacobi expansions

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Abstract

We present a realization for some K-functionals associated with Jacobi expansions in terms of generalized Jacobi–Weierstrass operators. Fractional powers of the operators as well as results concerning simultaneous approximation and Nikolskii–Stechkin type inequalities are also considered.

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1 Introduction

In this note, we work with two fixed real parameters α and β satisfying $\alpha \geq \beta \geq -1/2$. We use the following notations:

$$\varrho^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad x \in (-1,1), \tag{1}$$

and, for $1 \leq p < \infty$,

$$L^p_{(\alpha,\beta)} = \left\{ f : [-1,1] \to \mathbb{R} : \|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \varrho^{\alpha,\beta}(x)\, dx \right)^{1/p} < \infty \right\}.$$

Moreover, for each $n \in \mathbb{N}_0$, $\mathbb{P}_n$ is the family of all algebraic polynomials of degree not greater than n,

$$w_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)(\Gamma(\alpha+1))^2} \tag{2}$$

(Γ stands for the gamma function) and

$$\lambda_n = n(n+\alpha+\beta+1). \tag{3}$$

Since α and β are fixed, we set X for one of the spaces $C[-1,1]$ or $L^p_{(\alpha,\beta)}$.

For $n \in \mathbb{N}$, the Jacobi polynomial $R_n^{(\alpha,\beta)}$ is the unique polynomial of degree n which satisfies

$$R_n^{(\alpha,\beta)}(1) = 1 \quad \text{and} \quad \int_{-1}^{1} Q_{n-1}(x) R_n^{(\alpha,\beta)}(x) \varrho^{\alpha,\beta}(x)\, dx = 0$$

for all $Q_{n-1} \in \mathbb{P}_{n-1}$. We also take $R_0^{(\alpha,\beta)}(x) = 1$.

For $f \in X$, the Fourier–Jacobi coefficients are defined by

$$\langle f, R_n^{(\alpha,\beta)} \rangle = \int_{-1}^{1} f(x) R_n^{(\alpha,\beta)}(x) \varrho^{\alpha,\beta}(x)\, dx, \quad n \in \mathbb{N}_0,$$

and the associated expansion is

$$f(x) \sim \sum_{n=0}^{\infty} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x). \tag{4}$$

It is known that each $f \in L^1_{(\alpha,\beta)}$ is completely determined a.e. by its Fourier–Jacobi coefficients.

Definition 1.1 For fixed $\gamma > 0$ and $t > 0$, the generalized Jacobi–Weierstrass kernel is defined by

$$W_{t,\gamma}(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n^{\gamma}} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), \quad x \in [-1,1]. \tag{5}$$

For $f \in X$, the generalized Jacobi–Weierstrass (or Abel–Cartwright) operator is defined by

$$C_{t,\gamma}(f,x) = \int_{-1}^{1} \tau_y(f,x) W_{t,\gamma}(y) \varrho^{\alpha,\beta}(y)\, dy, \quad x \in [-1,1], \tag{6}$$

where $\tau_y(f,x)$ is the translation given in Theorem 2.1 below.

Of course the kernel $W_{t,\gamma}$ and the operator $C_{t,\gamma}$ also depend on α and β but, for simplicity, we omit these indexes. The (classical) Jacobi–Weierstrass operators correspond to $\gamma = 1$.

The generalized Jacobi–Weierstrass operators have been studied in different papers, but only for parameters satisfying $0 < \gamma \leq 1$. This restriction was considered because in such a case the kernels $W_{t,\gamma}$ are positive and the family $\{C_{t,\gamma}\}$ can be considered as formed by positive operators (see [2, 3], [7], pp. 96–97) and/or as a semigroup of contractions (see [2], pp. 49–52, and [18]). For $\gamma > 1$, one cannot expect the positivity of $W_{t,\gamma}$. For instance, it is known that the analogous generalized Weierstrass kernels for trigonometric expansion are not positive when $\gamma > 1$ (see [6], p. 263).

In this paper we will prove that the operators $C_{t,\gamma}$ can be used as a realization of some K-functionals which usually appear in some approximation problems related to Jacobi expansions.

For fixed real $\gamma > 0$, let $\Phi^\gamma(X)$ denote the family of all $f \in X$ for which there exists $\Psi^\gamma(f) \in X$ satisfying

$$\Psi^\gamma(f)(x) \sim \sum_{n=0}^{\infty} \lambda_n^\gamma \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x).$$

The associated K-functional is defined by

$$K_\gamma(f,t) = K_\gamma(f,t)_{\alpha,\beta} = \inf_{g \in \Psi^\gamma(X)} \bigl\{ \|f-g\|_X + t \bigl\| \Psi^\gamma(g) \bigr\|_X \bigr\} \tag{7}$$

for $f \in X$ and $t > 0$. For different realizations of these K-functionals, see [8], Theorem 7.1, and [10], Lemma 2.3. We will not use the characterization of these K-functionals in terms of moduli of smoothness. We will show that, for any $\gamma > 0$,

$$\sup_{0<s\le t} \bigl\| (I - C_{s,\gamma})(f) \bigr\|_X \approx K_\gamma(f,t).$$

The notation $A(f,t) \approx B(f,t)$ means that there exists a positive constant C such that $C^{-1}A(f,t) \le B(f,t) \le CA(f,t)$ with C independent of f and t.

Following [19], for $\gamma > 0$, define

$$(I - C_{t,1})^\gamma = \sum_{j=0}^{\infty} (-1)^j \binom{\gamma}{j} C_{jt,1}, \tag{8}$$

where

$$\binom{\gamma}{0} = 1 \quad \text{and} \quad \binom{\gamma}{j} = \prod_{k=1}^{j} \frac{\gamma - k + 1}{k} \quad \text{for } j \in \mathbb{N}.$$

For these operators, we will show the relations

$$K_\gamma\bigl(f,t^\gamma\bigr) \approx \sup_{0<s\le t} \bigl\| (I - C_{s,1})^\gamma(f) \bigr\|_X \approx \sup_{0<s\le t^\gamma} \bigl\| (I - C_{s,\gamma})(f) \bigr\|_X.$$

It is known that, if Q_n is a trigonometric polynomial of degree not greater than n and $r \in \mathbb{N}$, then

$$\bigl\| Q_n^{(r)} \bigr\|_p \le \biggl(\frac{n}{2\sin(nh)} \biggr)^r \bigl\| (1 - T_h)^r(Q_n) \bigr\|_p, \quad h \in (0, \pi/n),$$

where $\|\cdot\|_p$ denotes the L^p-norm of 2π-periodic functions and T_h is the translation operator. That is, $T_hQ(x) = Q(x+h)$. These inequalities are due to Nikolskii [11] and Stechkin [13]. For similar inequalities for algebraic polynomials, see [4] and the references given there. Here we will verify an analogous inequality by considering the operators Ψ^r and the linear combination of the Jacobi–Weierstrass operators $C_{t,1}$.

In Sect. 2 we collect some definitions and results which will be needed later. The main results are given in Sect. 3, where the result concerning simultaneous approximation is also included. Finally, in Sect. 4 we present a Nikolskii–Stechkin type inequality.

2 Auxiliary results

We need a convolution structure due to Askey and Wainger (see [1]).

Theorem 2.1 *For each $h \in [-1,1)$, there exists a function $\tau_h : X \to X$ with the following properties*:

(i) *For each $f \in X$, one has*

$$\|\tau_h f\|_X \le \|f\|_X, \qquad \lim_{h\to 1-} \|\tau_h(f) - f\|_X = 0$$

and

$$\langle \tau_h(f), R_n^{(\alpha,\beta)} \rangle = R_n^{(\alpha,\beta)}(h)\langle f, R_n^{(\alpha,\beta)} \rangle, \quad n \in \mathbb{N}_0.$$

(ii) *For $f \in X$ and $g \in L^1_{\alpha,\beta}$, the integral*

$$(f * g)(x) := \int_{-1}^{1} \tau_y(f,x) g(y) \varrho^{\alpha,\beta}(y)\, dy$$

exists a.e. in $[-1.1]$,

$$f * g = g * f, \qquad f * g \in X, \qquad \|f * g\|_p \le \|g\|_1 \|f\|_X$$

and

$$\langle f * g, R_n^{(\alpha,\beta)} \rangle = \langle f, R_n^{(\alpha,\beta)} \rangle \langle g, R_n^{(\alpha,\beta)} \rangle, \quad n \in \mathbb{N}_0. \tag{9}$$

For $j > \alpha + 1/2$ and $f \in X$, let

$$S_m^j(f) = \sum_{k=0}^{m} \frac{A_{m-k}^j}{A_m^j} \langle f, R_k^{(\alpha,\beta)} \rangle w_k^{\alpha,\beta} R_k^{(\alpha,\beta)}(x), \quad A_m^j = \binom{m+j}{m},$$

be the mth Cesàro means of order j. It is known that there exists a constant C such that

$$\|S_m^j\| \le C, \tag{10}$$

and, for each $f \in X$, one has ([2], Corollary 3.3.3, or [7], Theorem A)

$$\lim_{m\to\infty} \|f - S_m^j(f)\|_X = 0. \tag{11}$$

We need some classical results related to Banach spaces.

Definition 2.2 Let Y be a real Banach space and $B(Y)$ be the Banach algebra of all bounded linear operators $B : Y \to Y$. A uniformly bounded family of operators $\{T(t) : t \ge 0\}$ in $B(Y)$ is called an equi-bounded semigroup of class (C_0) if

$$T(s)T(t) = T(s+t) \quad \text{for } s,t \ge 0, \qquad T(0) = I, \tag{12}$$

and $\lim_{t\to 0+} \|f - T(t)f\|_Y = 0$ for each $f \in Y$.

Let Y, $B(Y)$ and $\{T(t) : t > 0\}$ be an equi-bounded semigroup as in Definition 2.2. Let $D(Q)$ be the family of all $g \in Y$, for which there exists $Q(g) \in Y$ such that

$$Q(g) = \lim_{t\to 0+} \frac{1}{t}\big[T(t) - I\big]g \tag{13}$$

(the limit is considered in the norm of Y). The operator $Q : D(Q) \to Y$ is called the infinitesimal generator of the semigroup $\{T(t) : t \geq 0\}$. It is known that Q is a closed linear operator and $D(Q)$ is dense in Y. For properties of semigroups of operators, see [5].

For $r \in \mathbb{N}$, set

$$D\big(Q^{r+1}\big) = \big\{f \in Y : f \in D\big(Q^r\big) \text{ and } Q^r(f) \in D(Q)\big\}$$

and, for $f \in D(Q^{r+1})$,

$$Q^{r+1}(f) = Q\big(Q^r(f)\big). \tag{14}$$

A family of operators $S = \{S_t, : t > 0\}$, $S_t \in B(Y)$ for each $t > 0$ is called a (commutative) strong approximation process for Y if, for all $f \in Y$ and $s, t > 0$,

$$S_s\big(S_t(f)\big) = S_t\big(S_s(f)\big), \qquad \big\|S_t(f)\big\|_Y \leq \Lambda \|f\|_Y \quad \text{and} \quad \lim_{t\to 0+} \big\|f - S_t(f)\big\|_Y = 0,$$

where Λ is a constant. In such a case, we set

$$\theta_S(f,t) = \sup_{0<s\leq t} \big\|f - S_s(f)\big\|_Y.$$

Let $\phi : [0, 1) \to \mathbb{R}^+$ be a positive increasing function, $\phi(t) \to 0$ as $t \to 0$, and Y_0 be a subspace of Y. We say that S is saturated with order ϕ and with trivial subspace Y_0 if every $f \in Y$ satisfying

$$\lim_{t\to 0+} \frac{\theta_S(f,t)}{\phi(t)} = 0$$

belongs to Y_0 and there exists $f \in Y \setminus Y_0$ satisfying $\theta_S(f,t) \leq C(f)\phi(t)$. The following assertion is known (for instance, see [2], Theorem 2.4.2).

Theorem 2.3 *Assume that Y is a Banach space, $D(B)$ is a dense subspace of Y, and $B : D(B) \to Y$ is a closed linear operator. Let $S = \{S_t : t > 0\}$ be a strong approximation process in Y satisfying $S_t(f) \in D(B)$ for any $f \in Y$ and each $t > 0$. If there exists a constant γ_0 such that, for all $g \in D(B)$,*

$$\lim_{t\to 0+} \left\| \frac{S_t(g) - g}{t^{\gamma_0}} - B(g) \right\|_Y = 0, \tag{15}$$

then the strong approximation process S is saturated with order t^{γ_0} and the trivial space is the kernel of B.

3 The operators $C_{t,\gamma}$ as a semigroup

In fact, it is known that, for $x \in (-1,1)$, $|R_n^{(\alpha,\beta)}(x)| < 1$, [14], pp. 163–164, and there exists a constant C such that, for each $n \in \mathbb{N}_0$,

$$w_n^{(\alpha,\beta)} \leq Cn^{2\alpha+1}. \tag{16}$$

These relations can be used to prove that the series in (5) converges absolutely and uniformly in $[-1,1]$. Thus $W_{t,\gamma} \in L^1_{(\alpha,\beta)}$ and, for each $f \in L^1_{(\alpha,\beta)}$, the series $C_{t,\gamma}(f)$ converges absolutely and uniformly in $[-1,1]$. Moreover,

$$C_{t,\gamma}(f,x) = (W_{t,\gamma} * f)(x) = \sum_{n=0}^{\infty} e^{-t\lambda_n^\gamma} \langle f, R_n^{(\alpha,\beta)} \rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x).$$

For these assertions, see [2], p. 30.

Our first result seems to be known. For convenience of the reader, we include a proof.

Theorem 3.1 *For each $\gamma > 0$, the family of operators $\{C_{t,\gamma} : t > 0\}$ is an equi-bounded semigroup of operators in X.*

Proof. It follows from Theorem 3.9 of [15] that the family of operators $\{C_{t,\gamma} : t > 0\}$ is uniformly bounded.

Condition (12) is derived from the properties of the convolution. In fact, it follows from (9) that, for each $f \in X$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \langle C_{s+t}(f), R_k^{(\alpha,\beta)} \rangle &= e^{-(s+t)\lambda_n^\gamma} \langle f, R_k^{(\alpha,\beta)} \rangle = e^{-s\lambda_n^\gamma} \langle C_{t,\gamma}(f), R_k^{(\alpha,\beta)} \rangle \\ &= \langle C_{s,\gamma}(C_{t,\gamma}(f)), R_k^{(\alpha,\beta)} \rangle \end{aligned}$$

and this implies $C_{s+t}(f) = (C_{s,\gamma} \circ C_{t,\gamma})(f)$.

Finally, for each $k \in \mathbb{N}_0$,

$$C_{t,\gamma}(R_k^{(\alpha,\beta)})(x) = e^{-t\lambda_n^\gamma} R_k^{(\alpha,\beta)}(x). \tag{17}$$

Hence

$$\lim_{t\to 0+} \| R_k^{(\alpha,\beta)} - C_{t,\gamma}(R_k^{(\alpha,\beta)}) \|_X = 0.$$

Since the operators $C_{t,\gamma}$ are linear and uniformly bounded and the polynomials are dense in X, the last equation holds for every $f \in X$.

Taking into account Theorem 3.1, we denote by A_γ the infinitesimal generator of $C_{t,\gamma}$ and by $D(A_\gamma) = D(A_\gamma(\alpha,\beta))$ the domain of A_γ. In the next result we give a description of the infinitesimal generator.

Theorem 3.2 *If $\gamma, t > 0$ and $A_\gamma : D(A_\gamma) \to X$ is the infinitesimal generator of $C_{t,\gamma}$, then*

$$D(A_\gamma) = \Psi^\gamma(X) \quad \textit{and} \quad -A_\gamma(f) = \Psi^\gamma(f)$$

for each $f \in \Psi^\gamma(X)$.

Moreover, for each $r \in \mathbb{N}$ and $f \in D(A_\gamma^r)$,

$$D(A_\gamma^r) = \Psi^{r\gamma}(X) \quad \textit{and} \quad (-1)^r A_\gamma^r(f) = \Psi^{r\gamma}(f), \tag{18}$$

where A_γ^r is defined as in (14).

Proof Since A_γ is the infinitesimal generator of the semi-group (see (13)), $A_\gamma : D(A_\gamma) \to X$ is a closed operator.

If $f \in D(A_\gamma)$, then

$$\langle A_\gamma(f), R_n^{(\alpha,\beta)} \rangle = \lim_{t\to 0+} \frac{1}{t}\left(e^{-t\lambda_n^\gamma} - 1\right)\langle f, R_n^{(\alpha,\beta)} \rangle = -\lambda_n^\gamma \langle f, R_n^{(\alpha,\beta)} \rangle. \tag{19}$$

Thus $f \in \Psi^\gamma(X)$ and

$$\Psi^\gamma(f) = -A_\gamma(f).$$

In particular, for each polynomial P, one has $P \in D(A_\gamma)$ and $\Psi^\gamma(P) = -A_\gamma(P)$.

On the other hand, fix an integer $j > \alpha + 1/2$. For $f \in \Psi^\gamma(X)$, let $S_m^j(f)$ and $S_m^j(\Psi^\gamma(f))$ be the mth Cesàro means of order j of f and $\Psi^\gamma(f)$, respectively. We know that (see (11))

$$S_m^j(f) \to f, \quad m \to \infty$$

and

$$-A_\gamma\left(S_m^j(f)\right) = \Psi^\gamma\left(S_m^j(f)\right) = S_m^j\left(\Psi^\gamma(f)\right) \to \Psi^\gamma(f).$$

Since $-A_\gamma$ is a closed operator, $f \in D(A_\gamma)$ and $-A_\gamma(f) = \Psi^\gamma(f)$.

Equations (18) can be proved by recurrence. For instance, (19) can be written as

$$\langle A_\gamma^2(f), R_n^{(\alpha,\beta)} \rangle = \langle A_\gamma\left(A_\gamma(f)\right), R_n^{(\alpha,\beta)} \rangle = -\lambda_n^\gamma \langle A_\gamma(f), R_n^{(\alpha,\beta)} \rangle = \lambda_n^{2\gamma} \langle f, R_n^{(\alpha,\beta)} \rangle. \qquad \square$$

Theorem 3.3 (i) *If for $\gamma, t > 0$, and $f \in X$*

$$\theta_\gamma(f,t) = \theta_\gamma(f,t)_{\alpha,\beta} = \sup_{0<s\leq t} \left\| (I - C_{s,\gamma})(f) \right\|,$$

and $K_\gamma(f,t)$ is defined by (7), then

$$\theta_\gamma(f,t) \approx K_\gamma(f,t).$$

(ii) *The strong approximation process $\{C_{t,\gamma}; t > 0\}$ is saturated with order t and the trivial class consists of the constant functions.*

Proof (i) From Theorem 3.2 we know that $-\Psi^\gamma$ is the infinitesimal generator of $\{C_{t,\gamma}\}$ and $D(A_\gamma) = \Psi^\gamma(X)$. Thus, the result is a simple consequence of [17], Theorem 1.1, or [5], p. 192.

(ii) We will derive the result from Theorem 2.3, with $B = \Psi^\gamma$ and $D(B) = D(A_\gamma)$. We should verify that $C_{t,\gamma}(f) \in D(A_\gamma)$ for any $f \in X$ and each $t > 0$.

For any $f \in X$, the Fourier–Jacobi coefficients of f are bounded by $\|f\|_{L^1_{(\alpha,\beta)}}$. Taking into account (16), for every $x \in [-1,1]$,

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n\}\langle f, R_n^{(\alpha,\beta)}\rangle w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x)\right| \\
&\quad \le \|f\|_{L^1_{(\alpha,\beta)}} \sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\} w_n^{(\alpha,\beta)} \\
&\quad \le C\|f\|_{L^1_{(\alpha,\beta)}} \sum_{n=1}^{\infty} \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\} n^{2\alpha+1} < \infty.
\end{aligned}$$

Since the series converges absolutely and uniformly, it defines a function $g_t \in X$ satisfying

$$\langle g_t, R_n^{(\alpha,\beta)}\rangle = \lambda_n^\gamma \exp\{-t\lambda_n^\gamma\}\langle f, R_n^{(\alpha,\beta)}\rangle = \lambda_n^\gamma \langle C_{t,\gamma}(f), R_n^{(\alpha,\beta)}\rangle, \quad n \in \mathbb{N}.$$

By definition of the operator Ψ^γ, $C_{t,\gamma}(f) \in \Psi^\gamma(X)$ (Theorem 3.2) and

$$\Psi^\gamma\left(C_{t,\gamma}(f)\right) = g_t.$$

We have proved that $C_{t,\gamma}(X) \in D(A_\gamma)$.

If $g \in \Psi^\gamma(X) = D(A_\gamma)$, by definition of the infinitesimal generator,

$$\lim_{t\to 0+}\left\|\frac{C_{t,\gamma}(g) - g}{t} - A_\gamma(g)\right\|_Y = 0.$$

If $f \in \Psi^\gamma(X)$ and $A_\gamma(f) = -\Psi^\gamma(f) = 0$, then $\langle f, R_n^{(\alpha,\beta)}\rangle = 0$ for all $n \in \mathbb{N}$. Therefore f is a constant.

From part (i), if $g \in \Psi^\gamma(X)$, then

$$\theta_\gamma(g,t) \le CK_\gamma(g,t) \le Ct\left\|\Psi^\gamma(g)\right\|_X.$$

Hence, the family

$$\left\{f \in X : \exists C(f) \text{ such that } \theta_\gamma(f,t) \le C(f)t\right\}$$

contains nonconstant functions.

Now, from Theorem 2.3, we know that the strong approximation process $\{C_{t,\gamma} : t > 0\}$ is saturated with order t. □

Remark 3.4 Some characterizations of the saturation class of the strong approximation process $\{C_{t,\gamma} : t > 0\}$ can be given as in [2], Theorems 5.1.1 and 7.4.1, where the case $\gamma = 1$ was considered. When $\gamma > 0$ is not an integer, fractional derivatives should be considered. This task would lead us far from our main topic.

Remark 3.5 A relation similar to (i) in Theorem 3.3 is asserted in [16], p. 2885, for the discrete case and Gauss–Weierstrass type means

$$\widetilde{W}_{\Omega(n),\gamma}(f,x)=\sum_{n=0}^{\infty}e^{-(\Omega(k)/\Omega(n))^{\gamma}}\langle f,R_n^{(\alpha,\beta)}\rangle w_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(x),$$

with Ω varying in a specified class of functions. The proof suggested there is different from the one given here (it does not use the semi-group structure). The main argument in [16] is that some abstract Riesz means are equivalent (as approximation processes) to some Gauss–Weierstrass type means. This kind of equivalence can also be derived by using Corollary 5.4 of [9]. Anyway, the arguments of [16] and the proof given here are related because both use [15], Theorem 3.9, to obtain a uniformly bounded family of multipliers. Apart from this, other topics considered here are not connected with [16].

The arguments used in the proof of Theorem 3.2 can be used to derive similar relations concerning the fractional powers of the Jacobi–Weierstrass operators $\{C_{t,1}\}$.

Recall that $A_1 : D(A_1) \to X$ is the infinitesimal generator of $\{C_{t,1}, t > 0\}$. For $\gamma > 0$, let $D((-A_1)^{\gamma}, X)$ be the family of all $f \in X$, for which there exists an element $(-A_1)^{\gamma}(f) \in X$ satisfying

$$\lim_{t\to 0+}\left\|(-A_1)^{\gamma}(f)-\frac{1}{t^{\gamma}}(I-C_{t,1})^{\gamma}(f)\right\|_X=0, \tag{20}$$

where $(I - C_{t,1})^{\gamma}(f)$ is defined by (8). This induces a map

$$\left(-A^1\right)^{\gamma}:D\left(\left(-A^1\right)^{\gamma},X\right)\to X$$

which is called the fractional power of order γ of $-A_1$.

Proposition 3.6 *If $\gamma > 0$ and $(-A_1)^{\gamma}$ is the fractional power of order γ of $-A_1$, then*

$$D\left((-A_1)^{\gamma},X\right)=\Psi^{\gamma}(X)$$

and, for each $f \in \Psi^{\gamma}(X)$,

$$\Psi^{\gamma}(f)=\lim_{t\to 0+}\frac{1}{t^{\gamma}}(I-C_{t,1})^{\gamma}(f)=\lim_{t\to 0+}\frac{1}{t}\left(f-C_{t,\gamma}(f)\right). \tag{21}$$

Proof If γ is a positive integer or $|a| < 1$, the Taylor expansion gives

$$(1-a)^{\gamma}=\sum_{j=0}^{\infty}(-1)^j\binom{\gamma}{j}a^j.$$

Notice that

$$\left\langle (I-C_{t,1})^{\gamma}(f),R_n^{(\alpha,\beta)}\right\rangle=\sum_{k=0}^{\infty}(-1)^k\binom{\gamma}{k}\left\langle C_{kt,1}(f),R_n^{(\alpha,\beta)}\right\rangle$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \langle W_{kt}, R_n^{(\alpha,\beta)} \rangle \langle f, R_n^{(\alpha,\beta)} \rangle$$
$$= \langle f, R_n^{(\alpha,\beta)} \rangle \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} \exp(-kt\lambda_n))$$
$$= \langle f, R_n^{(\alpha,\beta)} \rangle \bigl(1 - \exp(-t\lambda_n)\bigr)^{\gamma}. \tag{22}$$

Therefore, if $f \in D((-A_1)^\gamma, X)$, then

$$\langle (-A_1)^\gamma (f), R_n^{(\alpha,\beta)} \rangle = (\lambda_n)^\gamma \langle f, R_n^{(\alpha,\beta)} \rangle.$$

Hence $f \in \Psi^\gamma(X)$ and $(-A_1)^\gamma(f) = \Psi^\gamma(f)$.

It is clear that, for each polynomial P, one has $P \in D((-A_1)^\gamma, X)$ and

$$(-A_1)^\gamma(P) = \Psi^\gamma(P).$$

On the other hand, fix an integer $j > \alpha + 1/2$. For $f \in \Psi^\gamma(X)$, let $S_m^j(f)$ and $S_m^j(\Psi^\gamma(f))$ be the mth Cesàro means of order j of f and $\Psi^\gamma(f)$, respectively. From (11), as in the proof of Theorem 3.2, one has $\lim_{m\to\infty} \|S_m^j(f) - f\|_X = 0$ and

$$\lim_{m\to\infty} \bigl\| (-A^1)^\gamma \bigl(S_m^j(f)\bigr) - \Psi^\gamma(f) \bigr\|_X = \lim_{m\to\infty} \bigl\| \Psi^\gamma \bigl(S_m^j(f)\bigr) - \Psi^\gamma(f) \bigr\|_X$$
$$= \lim_{m\to\infty} \bigl\| S_m^j \bigl(\Psi^\gamma(f)\bigr) - \Psi^\gamma(f) \bigr\|_X = 0.$$

It was proved in [19], Theorem 4, that $D((-A_1)^\gamma, X)$ is dense in X and $(-A_1)^\gamma$ is a closed operator. Hence $f \in D((-A_1)^\gamma, X)$ and $(-A_1)^\gamma(f) = \Psi^\gamma(f)$.

The last equality in (21) was proved in Theorem 3.2, because Ψ^γ is the infinitesimal generator of $\{C_{t,\gamma}, t > 0\}$. □

Theorem 3.7 *For fixed $\gamma > 0$, one has*

$$K_\gamma\bigl(f, t^\gamma\bigr) \approx \sup_{0<s\le t} \bigl\| (I - C_{s,1})^\gamma (f) \bigr\|_X \approx \theta_\gamma\bigl(f, t^\gamma\bigr)$$

for each $f \in X$ and $t > 0$.

Proof From Theorems 3.1 and 3.2 we know that the family $\{C_{t,1}, t \ge 0\}$ is a semi-group of operators of class (C_0) with the infinitesimal generator $A_1 = -\Psi^1$. From Theorem 1.1 of [17], we know that, for all $f \in X$ and $t > 0$,

$$\inf_{g \in D((-A_1)^\gamma, X)} \bigl(\|f - g\|_X + t^\gamma \bigl\| (-A_1)^\gamma (g) \bigr\|_X \bigr) \approx \sup_{0<s\le t} \bigl\| (I - C_{s,1})^\gamma (f) \bigr\|_X,$$

where $(-A_1)^\gamma$ is given as in (20). But it was verified in Proposition 3.6 that $\Psi^\gamma(X) = D((-A_1)^\gamma, X)$ and $(-A_1)^\gamma(g) = \Psi^\gamma(g)$ for each $g \in \Psi^\gamma(X)$.

The equivalence with $\theta_\gamma(f, t^\gamma)$ follows from Theorem 3.3. □

Remark 3.8 When γ is an integer, Theorem 3.7 is similar to the Main Theorem in [18], p. 390, but the authors assumed that the operators are positive (plus other conditions).

Remark 3.9 The results of Theorem 3.7 allow us to obtain equivalent relations between fractional powers $(I - C_{s,1})^\gamma$ and some Riesz means as in Theorem 5.1 of [9].

Some result concerning simultaneous approximation can be derived from the ones given above.

Theorem 3.10 *If γ, σ, and t are positive real numbers and $f \in \Psi^\sigma(X)$, then*

$$C_{t,\gamma}(f), (I - C_{t,1})^\gamma(f) \in \Psi^\sigma(X),$$
$$\left\| \Psi^\sigma(f) - \Psi^\sigma\left(C_{t,\gamma}(f)\right) \right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t\right)$$

and

$$\left\| \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right) \right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t^\gamma\right),$$

where the constant C is independent of f and t.

Proof If $f \in \Psi^\sigma(X)$ and $n \in \mathbb{N}_0$, from (17) we obtain

$$\begin{aligned} \left\langle C_{t,\gamma}\left(\Psi^\sigma(f)\right), R_n^{(\alpha,\beta)} \right\rangle &= \exp\left(-t\lambda_n^\gamma\right)\left\langle \Psi^\sigma(f), R_n^{(\alpha,\beta)} \right\rangle \\ &= \lambda_n^\sigma \exp\left(-t\lambda_n^\gamma\right)\left\langle f, R_n^{(\alpha,\beta)} \right\rangle = \lambda_n^\sigma \left\langle C_{t,\gamma}(f), R_n^{(\alpha,\beta)} \right\rangle \end{aligned}$$

and from (22) one has

$$\begin{aligned} \left\langle (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right), R_n^{(\alpha,\beta)} \right\rangle &= \left(1 - \exp(-t\lambda_n)\right)^\gamma \left\langle \Psi^\sigma(f); R_n^{(\alpha,\beta)} \right\rangle \\ &= \lambda_n^\sigma \left(1 - \exp(-t\lambda_n)\right)^\gamma \left\langle f, R_n^{(\alpha,\beta)} \right\rangle = \lambda_n^\sigma \left\langle (I - C_{t,1})^\gamma(f), R_n^{(\alpha,\beta)} \right\rangle. \end{aligned}$$

Therefore $C_{t,\gamma}(f), (I - C_{t,1})^\gamma(f) \in \Psi^\sigma(X)$,

$$\Psi^\sigma\left(C_{t,\gamma}(f)\right) = C_{t,\gamma}\left(\Psi^\sigma(f)\right) \quad \text{and} \quad \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right) = (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right).$$

Now, from Theorem 3.3 one has

$$\left\| \Psi^\sigma(f) - \Psi^\sigma(C_{t,\gamma}) \right\|_X = \left\| (I - C_{t,\gamma})\left(\Psi^\sigma(f)\right) \right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t\right),$$

and using Theorem 3.7 we obtain

$$\left\| \Psi^\sigma\left((I - C_{t,1})^\gamma(f)\right) \right\|_X = \left\| (I - C_{t,1})^\gamma\left(\Psi^\sigma(f)\right) \right\|_X \leq C\theta_\gamma\left(\Psi^\sigma(f), t^\gamma\right). \qquad \square$$

4 A Nikolskii–Stechkin type inequality

Theorem 4.1 *For each $r \in \mathbb{N}$, there exists a constant C, depending upon r, such that, for every $\lambda \geq 1$ and for each polynomial $P \in \mathbb{P}_{\xi(\lambda)}$,*

$$\left\| \Psi^r(P) \right\|_X \leq C\lambda^r \sup_{0<h\leq 1/\lambda} \left\| (I - C_{h,1})^r(P) \right\|_X,$$

where

$$\xi(\lambda) = \max\{k \in \mathbb{N}_0 : k(k+\alpha+\beta+1) < \lambda\}.$$

Proof In this proof the infinitesimal generator of $\{C_{t,1} : t > 0\}$ is denoted by A.

From the proof of Lemma 1 in [12] we know that, given $r \in \mathbb{N}$, there exists a constant $C_1 = C(r)$ such that, for each $f \in X$ and $t > 0$, there is $g_t \in D(A^{r+1})$ satisfying

$$\|f - g_t\|_X \leq \sup_{0<h\leq t} \|(I - C_{h,1})^r f\|_X, \tag{23}$$

$$\|A^{r+1}(g_t)\|_X \leq C_1 \frac{1}{t^{r+1}} \sup_{0<h\leq t} \|(I - C_{h,1})^r f\|_X \tag{24}$$

and

$$\|(-A)^r(g_t)\|_X \leq C_1 \frac{1}{t^r} \sup_{0<h\leq t} \|(I - C_{h,1})^r f\|_X. \tag{25}$$

As in [9], for $\lambda > 0$ and $f \in X$, consider the best approximation

$$E_\lambda(f) = \inf\{\|f - P\|_X : P \in \mathbb{P}_{\xi(\lambda)}\}.$$

It was proved there (Theorem 6.1) that there exists a constant $C_2 = C(r,\alpha,\beta)$ such that, for $\lambda > 0$ and $f \in X$,

$$E_\lambda(f) \leq C_2 K_{r+1}(f, \lambda^{-r-1}), \tag{26}$$

and (Theorem 3.2) for each $Q \in \mathbb{P}_{\xi(\lambda)}$,

$$\|\Psi^r(Q)\|_X \leq C_2 \lambda^r \|Q\|_X. \tag{27}$$

Now, fix $\lambda > 0$ and $P \in \mathbb{P}_{\xi(\lambda)}$. Let $g_t \in D(A^{r+1}) = \Psi^{r+1}(X)$ (see (18)) be given as (23)–(25) with $t = 1/\lambda$ and $f = P$.

For $\varepsilon > 0$ and $k \in \mathbb{N}_0$, choose

$$q(g_t, k) \in \mathbb{P}_{\xi(2^k\lambda)} \tag{28}$$

such that

$$\|g_t - q(g_t,k)\|_X \leq (1+\varepsilon) E_{2^k\lambda}(g_t). \tag{29}$$

From (26), (18), and (24) we know that

$$\begin{aligned}\|g_t - q(g_t,k)\|_X &\leq C_2(1+\varepsilon) K_{r+1}\big(g_t, (2^k\lambda)^{-r-1}\big)\\ &\leq \frac{C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}} \|\Psi^{r+1}(g_t)\|_X\\ &= \frac{C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}} \|A^{r+1}(g_t)\|_X\end{aligned}$$

$$\leq \frac{C_1C_2(1+\varepsilon)}{(2^k\lambda)^{r+1}}\frac{1}{t^{r+1}}\sup_{0<h\leq t}\left\|(I-C_{h,1})^rP\right\|_X$$
$$=\frac{C_1C_2(1+\varepsilon)}{(2^k)^{r+1}}\sup_{0<h\leq 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.$$

On the other hand, from the identity

$$q(g_t,0)-g_t=\sum_{k=0}^{\infty}\bigl(q(g_t,k)-q(g_t,k+1)\bigr),$$

(28), (27), (29), and (26), one has

$$\begin{aligned}
\left\|\Psi^r\bigl(q(g_t,0)-g_t\bigr)\right\|_X &\leq \sum_{k=0}^{\infty}\left\|\Psi^r\bigl(q(g_t,k)-q(g_t,k+1)\bigr)\right\|_X\\
&\leq C_2\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\left\|q(g_t,k)-q(g_t,k+1)\right\|_X\\
&\leq C_2\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\bigl(\left\|q(g_t,k)-g_t\right\|_X+\left\|g_t-q(g_t,k+1)\right\|_X\bigr)\\
&\leq 2C_1C_2^2(1+\varepsilon)\sup_{0<h\leq 1\lambda}\left\|(I-C_{h,1})^rP\right\|_X\sum_{k=0}^{\infty}\bigl(2^{k+1}\lambda\bigr)^r\frac{1}{(2^k)^{r+1}}\\
&=2^{r+1}C_1C_2^2(1+\varepsilon)\lambda^r\sup_{0<h\leq 1\lambda}\left\|(I-C_{h,1})^rP\right\|_X\sum_{k=0}^{\infty}\frac{1}{2^k}\\
&=C_3(1+\varepsilon)\lambda^r\sup_{0<h\leq 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.
\end{aligned}$$

We also need the inequality (see (18) and (25))

$$\begin{aligned}
\left\|\Psi^r(g_t)\right\|_X=\left\|A^r(g_t)\right\|_X &\leq C_1\frac{1}{t^r}\sup_{0<h\leq t}\left\|(I-C_{h,1})^rP\right\|_X\\
&=C_1\lambda^r\sup_{0<h\leq 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.
\end{aligned}$$

From the inequalities given above, for $P\in\mathbb{P}_{\xi(\lambda)}$, we obtain

$$\begin{aligned}
\left\|\Psi^r(P)\right\|_X &\leq \left\|\Psi^r\bigl(P-q(g_t,0)\bigr)\right\|_X+\left\|\Psi^r\bigl(q(g_t,0)\bigr)\right\|_X\\
&\leq C_2\lambda^r\left\|P-q(g_t,0)\right\|_X+\left\|\Psi^r(g_t)\right\|_X+\left\|\Psi^r\bigl(g_t-q(g_t,0)\bigr)\right\|_X\\
&\leq C_1\lambda^r(\|P-g_t\|_X+\left\|g_t-q(g_t,0)\right\|_{X_{\alpha,\beta}}+C_4\lambda^r\sup_{0<h\leq 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X\\
&\leq C_5\lambda^r\sup_{0<h\leq 1/\lambda}\left\|(I-C_{h,1})^rP\right\|_X.
\end{aligned}$$
□

Remark 4.2 The problem of obtaining a Nikolskii–Stechkin inequality for fractional derivatives is open.

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