# THE VARIANCE CONJECTURE ON PROJECTIONS OF THE CUBE

DAVID ALONSO-GUTIÉRREZ AND JULIO BERNUÉS

ABSTRACT. We prove that the uniform probability measure $\mu$ on every $(n-k)$-dimensional projection of the $n$-dimensional unit cube verifies the variance conjecture with an absolute constant $C$

$$\mathrm{Var}_\mu|x|^2 \le C \sup_{\theta\in S^{n-1}} \mathbb{E}_\mu\langle x,\theta\rangle^2 \mathbb{E}_\mu|x|^2,$$

provided that $1 \le k \le \sqrt{n}$. We also prove that if $1 \le k \le n^{\frac{2}{3}}(\log n)^{-\frac{1}{3}}$, the conjecture is true for the family of uniform probabilities on its projections on random $(n-k)$-dimensional subspaces.

## 1. Introduction and notation

The (generalized) variance conjecture states that there exists an absolute constant $C$ such that for every centered log-concave probability $\mu$ on $\mathbb{R}^n$ (i.e. of the form $d\mu = e^{-v(x)}dx$ for some convex function $v : \mathbb{R}^n \to (-\infty,\infty])$

$$\mathrm{Var}_\mu|x|^2 \le C\lambda_\mu^2\mathbb{E}_\mu|x|^2,$$

where $\mathbb{E}_\mu$ and $\mathrm{Var}_\mu$ denote the expectation and the variance with respect to $\mu$ and $\lambda_\mu$ is the largest eigenvalue of the covariance matrix, i.e. $\lambda_\mu^2 = \max_{\theta\in S^{n-1}} \mathbb{E}_\mu\langle x,\theta\rangle^2$ where $S^{n-1}$ denotes the unit Euclidean sphere in $\mathbb{R}^n$.

This conjecture was first considered in the context of the so called Central Limit Problem for isotropic convex bodies in [BK] and it is a particular case of a more general statement, known as the Kannan, Lovász, and Simonovits or KLS-conjecture, see [KLS], which conjectures the existence of an absolute constant $C$ such that for any centered log-concave probability in $\mathbb{R}^n$ and any locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{Var}_\mu g(x)$ is finite

$$\mathrm{Var}_\mu\, g(x) \le C\lambda_\mu^2\mathbb{E}_\mu\,|\nabla g(x)|^2.$$

In recent years a number of families of measures have been proved to verify these conjectures (see [AB2] for a recent review on the subject). For instance, the KLS-conjecture is known to be true for the Gaussian probability and the uniform probability measures on the $\ell_p^n$-balls, some revolution bodies, the simplex and, with an extra $\log n$ factor, on unconditional bodies and log-concave probabilities with many symmetries (see [BaC], [BaW], [B], [H], [K], [LW], [S]). The best general known result for the KLS-conjecture adds a factor $\sqrt{n}$ and is due to Lee and Vempala (see [LV]). Besides, the variance conjecture is known to be true for uniform

*Date*: May 24, 2017.
2010 *Mathematics Subject Classification.* Primary 52B09, Secondary 52A23.
*Key words and phrases.* Variance conjecture, log-concave measures, convex bodies.
Partially supported by MINECO Spanish grant MTM2016-77710-P and DGA grant E-64.

probabilities on unconditional bodies and on hyperplane projections of the cross-polytope and the cube (see [K] and [AB1]). The best general estimate for the variance conjecture is the one given by Lee and Vempala for the KLS-conjecture.

We would like to remark that, while in the case of the KLS-conjecture one can assume without loss of generality that $\mu$ is isotropic (since then every linear transformation of the measure verifies it) this is not the case when we restrict to the variance conjecture, as we are considering only the function $g(x) = |x|^2$.

Before stating our results let us introduce some more notation. Let

$$B_\infty^n := \{x \in \mathbb{R}^n \,:\, |x_i| \leq 1, \forall\, 1 \leq i \leq n\}$$

denote the $n$-dimensional unit cube and, for any $1 \leq k \leq n$, let $G_{n,n-k}$ be the set of all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. For any $E \in G_{n,n-k}$ we will denote by $K := P_E B_\infty^n$ the orthogonal projection of $B_\infty^n$ onto $E$ and by $\mu$ the uniform probability on $K$. $\{e_i\}_{i=1}^n$ will denote the standard canonical basis in $\mathbb{R}^n$. As mentioned before, it was proved in [AB1] that the family of uniform probabilities on any $(n-1)$-dimensional projection of $B_\infty^n$ verifies the variance conjecture.

In this paper we will prove the following

**Theorem 1.1.** *There exists an absolute constant $C$ such that for any $1 \leq k \leq \sqrt{n}$ and any $E \in G_{n,n-k}$, if $\mu$ denotes the uniform probability measure on $K = P_E B_\infty^n$, then*

$$Var_\mu |x|^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

We will also prove the following theorem, which shows that for $k$ in a larger range, the variance conjecture is true for the family of uniform probabilities on the projections of $B_\infty^n$ on a random $(n-k)$-dimensional subspace. For that matter, we denote by $\mu_{n,n-k}$ the Haar probability measure on $G_{n,n-k}$.

**Theorem 1.2.** *There exist absolute constants $C, c_1, c_2$ such that for any $1 \leq k \leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, if $\mu$ denotes the uniform probability measure on $K = P_E(B_\infty^n)$, the measure $\mu_{n,n-k}$ of the subspaces $E \in G_{n,n-k}$ for which*

$$Var_\mu |x|^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu |x|^2$$

*is greater than $1 - c_1 e^{-c_2 n^{\frac{2}{3}} (\log n)^{\frac{2}{3}}}$.*

The main tool to prove both theorems will be to decompose an integral on $K$ as the sum of the integrals on the projections of some $(n-k)$-dimensional faces. It was proved in [ABBW] that for any $E \in G_{n,n-k}$ there exist $F_1, \dots, F_l$ a set of $(n-k)$-dimensional faces of $B_\infty^n$ such that for any integrable function $f$ on $K$

$$\begin{aligned}
\mathbb{E}_\mu f &:= \frac{1}{|K|}\int_K f(x)dx = \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\mathbb{E}_{P_E(F_i)} f(x)\\
&= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\frac{1}{|P_E(F_i)|}\int_{P_E(F_i)} f(x)dx\\
&= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\frac{1}{|F_i|}\int_{F_i} f(P_E x)dx\\
&= \sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\mathbb{E}_{F_i} f(P_E x), \qquad (1)
\end{aligned}$$

where we have denoted by $|\cdot|$ the relative volume of a convex body to the affine subspace in which it lies, $\mathbb{E}_{F_i}$ and by $\mathbb{E}_{P_E(F_i)}$ the expectation with respect to the uniform probability on the face $F_i$ and on its projection $P_E(F_i)$. In particular

$$\sum_{i=1}^{l} \frac{|P_E(F_i)|}{|K|} = 1.$$

In this way, $\mathrm{Var}_\mu |x|^2$ is written as the sum of two averages on some faces (see Lemma 2.1 below). The restrictions on the range of $k$ in both theorems arise from the fact that in order to give estimates for these averages we do it for *each* face of the cube separately. With this method we provide an upper bound for $\mathrm{Var}_\mu|x|^2$ and a lower bound for $\lambda_\mu^2 \mathbb{E}_\mu |x|^2$ of the same order. It would be interesting to know whether one could bound these two terms from above directly by $\lambda_\mu^2\mathbb{E}_\mu|x|^2$.

Notice that the $(n-k)$-dimensional faces of $B_\infty^n$ are the sets of the form

$$F_{(i_1,\varepsilon_1,\ldots,i_k,\varepsilon_k)} = \{x \in B_\infty^n \,:\, x_{i_j} = \varepsilon_j, j = 1,\ldots,k\},$$

where $1 \leq i_j \leq n$, $i_{j_1} \neq i_{j_2}$ and $\varepsilon_j = \pm 1$. In other words, if we divide $\{1,\ldots,n\}$ into two disjoint sets $\{i_j\}_{j=1}^k$ and $\{l_j\}_{j=1}^{n-k}$ of cardinality $k$ and $n-k$, then the $(n-k)$-dimensional face $F_{(i_1,\varepsilon_1,\ldots,i_k,\varepsilon_k)}$ is

$$F_{(i_1,\varepsilon_1,\ldots,i_k,\varepsilon_k)} = \sum_{j=1}^{k} \varepsilon_j e_{i_j} + I_F(B_\infty^{n-k}), \tag{2}$$

where $I_F$ is the linear map from $\mathbb{R}^{n-k}$ to $\mathbb{R}^n$ given by $I_F x = \sum_{j=1}^{n-k} x_j e_{l_j}$.

For any $E \in G_{n,n-k}$ we write $S_E = S^{n-1} \cap E$ and denote by $\sigma_E$ the Haar probability measure on $S_E$.

## 2. The variance conjecture on $(n-k)$-dimensional projections of the cube

In this section we shall prove Theorem 1.1. We will use the aforementioned representation of $\mathbb{E}_\mu f$ in order to write $\mathrm{Var}_\mu|x|^2$ as the sum of two terms. One of them will be an average of the variances of $|x|^2$ on the projections of the faces involved in (1). The other one will be an average of the distances from the expected value of $|x|^2$ on the projections of such faces to $\mathbb{E}_\mu|x|^2$. We will give estimates for these terms valid for every face and prove

$$\mathrm{Var}_\mu|x|^2 \leq Cn \leq C\lambda_\mu^2\mathbb{E}_\mu|x|^2.$$

We start with the following lemma, which can be proved by direct computation:

**Lemma 2.1.** *Let $E \in G_{n,n-k}$, $\mu$ the uniform probability on $K = P_E(B_\infty^n)$ and $\{F_i\}_{i=1}^l$ the set of $(n-k)$-dimensional faces described in* (1)*. Then*

$$Var_\mu|x|^2 = \sum_{i=1}^{l} \frac{|P_E(F_i)|}{|K|} Var_{P_E(F_i)}|x|^2 + \sum_{i=1}^{l} \frac{|P_E(F_i)|}{|K|} \left(\mathbb{E}_{P_E(F_i)}|x|^2 - \mathbb{E}_\mu|x|^2\right)^2. \tag{3}$$

We will estimate the two summands appearing in (3). The following lemma provides upper and lower bounds to some of parameters involved.

**Lemma 2.2.** *Let $E \in G_{n,n-k}$. Then, for any $\theta \in S_E$ and any $(n-k)$-dimensional face $F = F_{(i_1,\varepsilon_1,\dots,i_k,\varepsilon_k)}$ of $B_\infty^n$ we have,*

$$\mathbb{E}_{P_E(F)}\langle x,\theta\rangle^2 = \mathbb{E}_F\langle P_E x,\theta\rangle^2 = \frac{1}{3} + \left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 - \frac{1}{3}\sum_{j=1}^k \theta_{i_j}^2$$

*and*

$$\mathbb{E}_{P_E(F)}|x|^2 = \mathbb{E}_F|P_E x|^2 = \frac{n-k}{3} + \left|P_E\left(\sum_{j=1}^k \varepsilon_j e_{i_j}\right)\right|^2 - \frac{1}{3}\sum_{j=1}^k |P_E(e_{i_j})|^2.$$

*Consequently,*

$$\frac{n-2k}{3} \le \mathbb{E}_{P_E(F)}|x|^2 \le \frac{n+2k}{3}, \qquad \frac{n-2k}{3} \le \mathbb{E}_\mu|x|^2 \le \frac{n+2k}{3},$$

*and*

$$\lambda_\mu^2 \ge \frac{n-2k}{3(n-k)}.$$

*Proof.* Using (2), we have a random vector $x$ uniformly distributed in $F$ can be written as

$$x = \sum_{j=1}^k \varepsilon_j e_{i_j} + I_F(y)$$

where $y$ is uniformly distributed in $B_\infty^{n-k}$. Then, for every $\theta \in S_E$, straightforward computations yield

$$\begin{aligned}
\mathbb{E}_F\langle P_E x,\theta\rangle^2 &= \mathbb{E}_F\langle x,\theta\rangle^2 \\
&= \frac{1}{|B_\infty^{n-k}|}\int_{B_\infty^{n-k}}\left(\sum_{j=1}^k \varepsilon_j\theta_{i_j} + \sum_{j=1}^{n-k} y_j\theta_{l_j}\right)^2 dy \\
&= \left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 + \frac{1}{3}\left(\sum_{j=1}^{n-k}\theta_{l_j}^2\right) \\
&= \left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 + \frac{1}{3}\left(1 - \sum_{j=1}^k \theta_{i_j}^2\right).
\end{aligned}$$

This proves the first identity. Now, by integrating on $\theta \in S_E$ with respect to the uniform probability measure on $S_E$, using the fact that for every $x \in \mathbb{R}^n$

$$|P_E x|^2 = (n-k)\int_{S_E}\langle P_E x,\theta\rangle^2 d\sigma_E(\theta) = (n-k)\int_{S_E}\langle x,\theta\rangle^2 d\sigma_E(\theta),$$

and using Fubini's theorem, we obtain

$$\frac{1}{n-k}\mathbb{E}_F|P_E x|^2 = \frac{1}{3} + \frac{1}{n-k}\left|P_E\left(\sum_{j=1}^k \varepsilon_j e_{i_j}\right)\right|^2 - \frac{1}{3(n-k)}\sum_{j=1}^k |P_E(e_{i_j})|^2,$$

since $\left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 = \left\langle \sum_{j=1}^k \varepsilon_j e_{i_j},\theta\right\rangle^2$ and $\theta_{i_j}^2 = \langle e_{i_j},\theta\rangle^2$. This proves the second identity.

The bounds

$$0 \leq \left| P_E \left( \sum_{j=1}^{k} \varepsilon_j e_{i_j} \right) \right|^2 \leq \left| \sum_{j=1}^{k} \varepsilon_j e_{i_j} \right|^2 = k$$

and

$$0 \leq |P_E(e_{i_j})|^2 \leq |e_{i_j}|^2 = 1,$$

prove the upper and lower bound for $\mathbb{E}_F|P_E x|^2$ and by using formula (1) we deduce the estimates for $\mathbb{E}_\mu|x|^2$. Finally, notice that

$$\lambda_\mu^2 \geq \int_{S_E} \mathbb{E}_\mu \langle x, \theta \rangle^2 \, d\sigma_E(\theta) = \frac{1}{n-k} \mathbb{E}_\mu |x|^2 \geq \frac{n-2k}{3(n-k)},$$

which proves the last inequality. □

We now focus on the first summand in (3). We take into account the fact that for any $(n-k)$-dimensional face $F = F_{(i_1,\varepsilon_1,\ldots,i_k,\varepsilon_k)}$ we can write

$$P_E(F) = a_F + T_F(B_\infty^{n-k}),$$

where $a_F = P_E\left(\sum_{j=1}^k \varepsilon_j e_{i_j}\right)$ and $T_F : \mathbb{R}^{n-k} \to E$ is the linear map obtained as the composition of $I_F$ in (2) and $P_E$.

The effect of the translation map in our problem is the content of the next

**Lemma 2.3.** *Let $\nu$ be a symmetric measure in $\mathbb{R}^n$ (i.e., $\nu(A) = \nu(-A)$ for every measurable set $A$), $a \in \mathbb{R}^n$ and $\nu_a$ the translate measure $\nu_a(A) := \nu(A - a)$ (or equivalently, $\int_{\mathbb{R}^n} f(x)\, d\nu_a(x) = \int_{\mathbb{R}^n} f(x+a)\, d\nu(x)$ for any integrable measurable function $f$). Then,*

$$Var_{\nu_a}|x|^2 = Var_\nu |x|^2 + 4\mathbb{E}_\nu \langle a, x\rangle^2.$$

*Proof.*

$$|x+a|^2 = |x|^2 + 2\langle a, x\rangle + |a|^2$$

and

$$|x+a|^4 = |x|^4 + 4|x|^2\langle a, x\rangle + 4|a|^2\langle a, x\rangle + 2|x|^2|a|^2 + 4\langle a, x\rangle^2 + |a|^4.$$

Taking expectations and using symmetry we have,

$$\mathrm{Var}_{\nu_a}|x|^2 = \mathbb{E}_{\nu_a}|x|^4 - \left(\mathbb{E}_{\nu_a}|x|^2\right)^2 = \mathbb{E}_\nu|x+a|^4 - \left(\mathbb{E}_\nu|x+a|^2\right)^2 =$$

$$= \mathbb{E}_\nu|x+a|^4 - \left(\mathbb{E}_\nu|x|^2 + |a|^2\right)^2 = \mathrm{Var}_\nu|x|^2 + 4\mathbb{E}_\nu\langle a, x\rangle^2.$$

□

Taking in the previous lemma $\nu$ as the uniform probability measure on $T_F(B_\infty^{n-k})$ we have

**Corollary 2.1.** *Let $F = F_{(i_1,\varepsilon_1,\ldots,i_k,\varepsilon_k)}$ be an $(n-k)$-dimensional face of $B_\infty^n$ and let $P_E(F) = a_F + T_F(B_\infty^{n-k})$ as above. Then*

$$Var_{P_E(F)}|x|^2 = Var_{T_F(B_\infty^{n-k})}|x|^2 + 4\mathbb{E}_{T_F(B_\infty^{n-k})}\langle a_F, x\rangle^2.$$

**Lemma 2.4.** *Let $E \in G_{n,n-k}$. Then, for any $\theta \in S_E$ and any $(n-k)$-dimensional face of $B_\infty^n$, $F = F_{(i_1,\varepsilon_1,\dots,i_k,\varepsilon_k)}$, if $P_E(F) = a_F + T_F(B_\infty^{n-k})$ as above, we have*

$$\mathbb{E}_{T_F(B_\infty^{n-k})}\langle x,\theta\rangle^2 = \frac{1}{3} - \frac{1}{3}\sum_{j=1}^k \theta_{i_j}^2 \quad (\leq \frac{1}{3}),$$

*and*

$$\mathbb{E}_{T_F(B_\infty^{n-k})}|x|^2 = \frac{n-k}{3} - \frac{1}{3}\sum_{j=1}^k |P_E(e_{i_j})|^2 \quad (\leq \frac{n-k}{3}).$$

*Proof.* Notice that

$$\begin{aligned}\mathbb{E}_{P_E(F)}\langle x,\theta\rangle^2 &= \mathbb{E}_{T_F(B_\infty^n)}\langle a_F + x,\theta\rangle^2 = \langle a_F,\theta\rangle^2 + \mathbb{E}_{T_F(B_\infty^n)}\langle x,\theta\rangle^2 \\ &= \left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 + \mathbb{E}_{T_F(B_\infty^n)}\langle x,\theta\rangle^2.\end{aligned}$$

On the other hand, by Lemma 2.2

$$\mathbb{E}_{P_E(F)}\langle x,\theta\rangle^2 = \mathbb{E}_F\langle P_E x,\theta\rangle^2 = \frac{1}{3} + \left(\sum_{j=1}^k \varepsilon_j\theta_{i_j}\right)^2 - \frac{1}{3}\sum_{j=1}^k \theta_{i_j}^2,$$

and we obtain the result. By integrating in $\theta \in S_E$ with respect to the uniform measure and using Fubini's theorem we obtain the second identity. □

As a consequence we have the following lemma, which gives an upper bound for the first term in (3) of the right order for the variance conjecture to be true as long as $k \leq \frac{n}{3}$.

**Lemma 2.5.** *Let $E \in G_{n,n-k}$. Then, for any $(n-k)$-dimensional face $F$ of $B_\infty^n$ we have,*

$$Var_{P_E(F)}|x|^2 \leq Cn.$$

*Consequently, there exists an absolute constant $C$ such that if $k \leq \frac{n}{3}$ and $\{F_i\}_{i=1}^l$ is the set of $(n-k)$-dimensional faces described in* (1) *then*

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|} Var_{P_E(F_i)}|x|^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu|x|^2.$$

*Proof.* By Corollary 2.1, we have that for any such $F$

$$\mathrm{Var}_F|P_E x|^2 = \mathrm{Var}_{T_F(B_\infty^{n-k})}|x|^2 + 4\mathbb{E}_{T_F(B_\infty^{n-k})}\langle a_F, x\rangle^2.$$

Since $B_\infty^{n-k}$ verifies the Kannan-Lovász-Simonovits conjecture (see [LW]), every linear transform of it verifies the variance conjecture and therefore there exists an absolute constant $C$ such that

$$\mathrm{Var}_{T_F(B_\infty^{n-k})}|x|^2 \leq C\lambda^2_{T_F(B_\infty^{n-k})}\mathbb{E}_{T_F(B_\infty^{n-k})}|x|^2.$$

Since by Lemma 2.4 the two factors involved are bounded by $\frac{1}{3}$ and $\frac{n-k}{3}$ respectively, we have $\mathrm{Var}_{T_F(B_\infty^{n-k})}|x|^2 \leq C(n-k)$.

On the other hand, by Lemma 2.4,

$$\mathbb{E}_{T_F(B_\infty^{n-k})}\langle a_F, x\rangle^2 \leq \frac{1}{3}|a_F|^2 = \frac{1}{3}\left|P_E\left(\sum_{j=1}^k \varepsilon_j e_{i_j}\right)\right|^2 \leq \frac{1}{3}\left|\left(\sum_{j=1}^k \varepsilon_j e_{i_j}\right)\right|^2 = \frac{1}{3}k.$$

Therefore, there exists an absolute constant $C$ such that

$$\mathrm{Var}_{P_E(F)}|x|^2 \leq C(n-k+k) = Cn,$$

which proves the first part of the Lemma.

For the second part, notice that by Lemma 2.2 we have

$$\lambda_\mu^2 \mathbb{E}_\mu |x|^2 \geq \frac{(n-2k)^2}{9(n-k)} \geq \frac{n}{54}$$

when $1 \leq k \leq \frac{n}{3}$ and now the second part of the Lemma easily follows. □

For the second summand of (3) we invoke once again Lemma 2.2. The estimates therein provide an upper bound of the right order for the variance conjecture to hold as long as $k \leq \sqrt{n}$.

**Lemma 2.6.** *Let $E \in G_{n,n-k}$ and let $\mu$ be the uniform probability on $K = P_E(B_\infty^n)$. Then for any $(n-k)$-dimensional face $F$ of $B_\infty^n$ we have,*

$$\left|\mathbb{E}_F|P_E x|^2 - \mathbb{E}_\mu|x|^2\right| \leq \frac{4k}{3}.$$

*Consequently, there exists an absolute constant $C$ such that if $k \leq \sqrt{n}$ and $\{F_i\}_{i=1}^l$ is the set of $(n-k)$-dimensional faces described in* (1)*,*

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\left(\mathbb{E}_{F_i}|P_E x|^2 - \mathbb{E}_\mu|x|^2\right)^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu|x|^2.$$

*Proof.* By Lemma 2.2 we have that for any $(n-k)$ dimensional face $F$

$$-\frac{4k}{3} \leq \mathbb{E}_F|P_E x|^2 - \mathbb{E}_\mu|x|^2 \leq \frac{4k}{3}.$$

Therefore,

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\left(\mathbb{E}_{F_i}|P_E x|^2 - \mathbb{E}_\mu|x|^2\right)^2 \leq \frac{16k^2}{9}.$$

On the other hand, using as above the bound $\lambda_\mu^2 \mathbb{E}_\mu|x|^2 \geq \frac{n}{54}$ we have that if $k \leq \sqrt{n}$

$$\sum_{i=1}^l \frac{|P_E(F_i)|}{|K|}\left(\mathbb{E}_{F_i}|P_E x|^2 - \mathbb{E}_\mu|x|^2\right)^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu|x|^2.$$

□

Lemmas 2.5 and 2.6, together with formula (3) prove Theorem 1.1.

## 3. The variance conjecture on random $(n-k)$-dimensional projections of the cube

We will show that we can improve the range of the codimension $k$ for which the variance conjecture remains true on a random subspace $E \in G_{n,n-k}$. In order to do that we will consider, for any $(n-k)$-dimensional face $F$ of $B_\infty^n$, the function $f : G_{n,n-k} \to \mathbb{R}$ given by $f(E) = \mathbb{E}_F |P_E x|^2$ and make use of the concentration of measure theorem, proved by Gromov and Milman, on $G_{n,n-k}$ (see, for instance, [MS]). As a consequence, since the value of $\mathbb{E}_{\mu_{n,n-k}} f(E)$ does not depend on $F$, we will obtain that, as long as $k$ is in the range considered in the statement of Theorem 1.2, for a set of subspaces with large measure $f(E)$ is very close to its expected value and, therefore, to $\mathbb{E}_\mu |x|^2$ for every $(n-k)$-dimensional face $F$. This will imply that the second term in (3) is bounded by $Cn$ for every subspace $E$ in this set. Then, since we have seen before that the first term in (3) is also bounded by $Cn$ for a larger range of $k$, we have that

$$\mathrm{Var}_\mu |x|^2 \le Cn \le C\lambda_\mu^2 \mathbb{E}_\mu |x|^2.$$

We will denote by $O(n)$ the orthogonal group equipped with the Hilbert-Schmidt distance $\|\cdot\|_{HS}$ and we represent any $U \in O(n)$ by $U = (u_1, \dots, u_n)$, where $(u_i)$ is an orthonormal basis of $\mathbb{R}^n$. Let us recall that for any linear map $T = (t_{ij})_{i,j=1}^n$, its Hilbert-Schmidt norm is defined by

$$\|T\|_{HS} := \left( \sum_{i,j=1}^n t_{ij}^2 \right)^{\frac{1}{2}}$$

and that $\|T\| \le \|T\|_{HS}$, where $\|T\|$ denotes the usual operator norm $\|T\| = \sup_{x \in S^{n-1}} |Tx|$.

**Theorem 3.1** (Concentration of measure). *Let $f : G_{n,n-k} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $\sigma$ with respect to the distance*

$$\begin{aligned} d(E_1, E_2) = \inf \{ &\|U - V\|_{HS} : U, V \in O(n),\, E_1 = \mathit{span}\{u_1, \dots, u_{n-k}\}\,, \\ &E_2 = \mathit{span}\{v_1, \dots, v_{n-k}\}\}\,. \end{aligned}$$

*Then, for every $\lambda > 0$*

$$\mu_{n,n-k} \{ E \in G_{n,n-k} : |f(E) - \mathbb{E} f(E)| > \lambda | \} \le c_1 e^{-\frac{c_2 \lambda^2 n}{\sigma^2}},$$

*where $c_1$ and $c_2$ are positive absolute constants.*

In the following lemma we compute the expected value of $f$. Let us point out that what matters to us for our purposes is that, due to the symmetries of $B_\infty^n$, its value does not depend on the face $F$. Nevertheless, we compute its exact value.

**Lemma 3.1.** *Let $F$ be an $(n-k)$-dimensional face of $B_\infty^n$. Then*

$$\int_{G_{n,n-k}} \mathbb{E}_F |P_E x|^2 d\mu_{n,n-k}(E) = \frac{(n-k)(n+2k)}{3n}.$$

*Proof.* Notice that, by Fubini's theorem and the uniqueness of the Haar measure $\sigma$ on $S^{n-1}$ we have

$$\int_{G_{n,n-k}} \mathbb{E}_F |P_E x|^2 d\mu_{n,n-k}(E) \quad = \quad (n-k) \int_{G_{n,n-k}} \mathbb{E}_F \int_{S_E} \langle P_E x, \theta \rangle^2 d\sigma_E(\theta) d\mu_{n,n-k}(E)$$

$$\begin{aligned}
&= (n-k)\int_{G_{n,n-k}}\mathbb{E}_F\int_{S_E}\langle x,\theta\rangle^2 d\sigma_E(\theta)d\mu_{n,n-k}(E)\\
&= (n-k)\mathbb{E}_F\int_{G_{n,n-k}}\int_{S_E}\langle x,\theta\rangle^2 d\sigma_E(\theta)d\mu_{n,n-k}(E)\\
&= (n-k)\mathbb{E}_F\int_{S^{n-1}}\langle x,\theta\rangle^2 d\sigma(\theta)\\
&= \frac{n-k}{n}\mathbb{E}_F|x|^2.
\end{aligned}$$

Using the description of any $(n-k)$ dimensional face $F$ given in (2), a random vector uniformly distributed in the face $F$ has $k$ coordinates equal to $\pm1$ and the other $n-k$ coordinates are given by a random vector uniformly distributed on $B_\infty^{n-k}$. Thus, the latter expectation equals

$$\begin{aligned}
\frac{n-k}{n}\left(k+\mathbb{E}_{B_\infty^{n-k}}|x|^2\right) &= \frac{n-k}{n}\left(k+\frac{n-k}{3}\right)\\
&= \frac{n-k}{n}\frac{n+2k}{3}.
\end{aligned}$$

□

In the following lemma we estimate the Lipschitz constant of $f$ with respect to the distance defined in Theorem 3.1. Notice that, as before, its value does not depend on $F$.

**Lemma 3.2.** *Let $F=F_{(i_1,\varepsilon_1,\dots,i_k,\varepsilon_k)}$ be an $(n-k)$-dimensional face of $B_\infty^n$ and let $f: G_{n,n-k}\to\mathbb{R}$ be the function defined as $f(E)=\mathbb{E}_F|P_Ex|^2$. For any $E_1,E_2\in G_{n,n-k}$ we have*

$$|f(E_1)-f(E_2)|\le\frac{8\sqrt{2}k}{3}d(E_1,E_2).$$

*Proof.* Let $E_1,E_2\in G_{n,n-k}$. By Lemma 2.2 we have

$$\begin{aligned}
&|f(E_1)-f(E_2)| = \left|\mathbb{E}_F|P_{E_1}x|^2-\mathbb{E}_F|P_{E_2}x|^2\right|\\
=\ & \left|\left|P_{E_1}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|^2-\left|P_{E_2}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|^2-\frac{1}{3}\left(\sum_{j=1}^k|P_{E_1}(e_{i_j})|^2-|P_{E_2}(e_{i_j})|^2\right)\right|\\
\le\ & \left|\left|P_{E_1}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|^2-\left|P_{E_2}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|^2\right|\\
+\ & \frac{1}{3}\left|\left(\sum_{j=1}^k|P_{E_1}(e_{i_j})|^2-|P_{E_2}(e_{i_j})|^2\right)\right|\\
=\ & \left|\left|P_{E_1}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|+\left|P_{E_2}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|\right|\left|\left|P_{E_1}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|-\left|P_{E_2}\left(\sum_{j=1}^k\varepsilon_je_{i_j}\right)\right|\right|\\
+\ & \frac{1}{3}\left(\sum_{j=1}^k\left||P_{E_1}(e_{i_j})|+|P_{E_2}(e_{i_j})|\right|\left||P_{E_1}(e_{i_j})|-|P_{E_2}(e_{i_j})|\right|\right)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\sqrt{k}\left|(P_{E_1}-P_{E_2})\left(\sum_{j=1}^{k}\varepsilon_j e_{i_j}\right)\right| \\
&+ \frac{2}{3}\sum_{j=1}^{k}|\,(P_{E_1}-P_{E_2})\,(e_{i_j})| \\
&\leq 2k\|P_{E_1}-P_{E_2}\|+\frac{2k}{3}\|P_{E_1}-P_{E_2}\| \\
&= \frac{8k}{3}\|P_{E_1}-P_{E_2}\|.
\end{aligned}$$

Notice that for any $U,V\in O(n)$ such that $E_1=\mathrm{span}\{u_1,\dots,u_{n-k}\}$ and $E_2=\mathrm{span}\{v_1,\dots,v_{n-k}\}$, the vectors $\{u_j\}_{j=1}^{n-k}$ and $\{v_j\}_{j=1}^{n-k}$ for orthonormal basis of $E_1$ and $E_2$ respectively and for any $x\in\mathbb{R}^n$

- $P_{E_1}x=\sum_{j=1}^{n-k}\langle P_{E_1}x,u_j\rangle u_j=\sum_{j=1}^{n-k}\langle x,u_j\rangle u_j$,
- $P_{E_2}x=\sum_{j=1}^{n-k}\langle P_{E_2}x,v_j\rangle v_j=\sum_{j=1}^{n-k}\langle x,v_j\rangle v_j$.

Then, for any such $U,V$, the projections onto $E_1$ and $E_2$, $P_{E_1}$ and $P_{E_2}$ are given by the matrices (with respect to the canonical basis in $\mathbb{R}^n$) $(\sum_{j=1}^{n-k}\langle u_j,e_k\rangle\langle u_j,e_l\rangle)_{k,l=1}^n$ and $(\sum_{j=1}^{n-k}\langle v_j,e_k\rangle\langle v_j,e_l\rangle)_{k,l=1}^n$ respectively. Thus,

$$\begin{aligned}
\|P_{E_1}-P_{E_2}\|^2 &\leq \|P_{E_1}-P_{E_2}\|_{HS}^2=2(n-k)-2\sum_{i,j=1}^{n-k}\langle u_i,v_j\rangle^2 \\
&\leq 2\sum_{j=1}^{n-k}(1-\langle u_j,v_j\rangle^2)\leq 2\sum_{j=1}^{n-k}|u_j-v_j|^2\leq 2\sum_{j=1}^{n}|u_j-v_j|^2 \\
&= 2\|U-V\|_{HS}^2,
\end{aligned}$$

since $1-\langle u_j,v_j\rangle^2\leq 2(1-\langle u_j,v_j\rangle)=|u_j-v_j|^2$. Consequently $\|P_{E_1}-P_{E_2}\|\leq\sqrt{2}d(E_1,E_2)$ and we obtain the result. □

**Lemma 3.3.** *Let $1\leq k\leq\frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$. There exist positive absolute constants $C,c_1,c_2$ such that the set*

$$\left\{E\in G_{n,n-k}\ :\ \left|\mathbb{E}_F|P_Ex|^2-\frac{(n-k)(n+2k)}{3n}\right|>C\sqrt{n}\,,\ \text{for some } F\right\}$$

*has measure $\mu_{n,n-k}$ smaller than $c_1e^{-c_2n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$.*

*Proof.* Let $F$ be a fixed $(n-k)$-dimensional face of $B_\infty^n$. Then, taking $\lambda=C\sqrt{n}$ we obtain, using Theorem 3.1 that

$$\mu_{n,n-k}\left\{E\in G_{n,n-k}\ :\ \left|\mathbb{E}_F|P_Ex|^2-\frac{(n-k)(n+2k)}{3n}\right|>C\sqrt{n}\right\}\leq c_1e^{-\frac{c_2C^2n^2}{k^2}}.$$

Since the number of $(n-k)$-dimensional faces of $B_\infty^n$ equals $2^k\binom{n}{k}\leq 2^k\left(\frac{en}{k}\right)^k$, using the union bound we have that for any $C>0$

$$\begin{aligned}
&\mu_{n,n-k}\left\{E\in G_{n,n-k}\ :\ \left|\mathbb{E}_F|P_Ex|^2-\frac{(n-k)(n+2k)}{3n}\right|>C\sqrt{n}\,,\ \text{for some } F\right\} \\
&\leq c_1e^{-\frac{c_2C^2n^2}{k^2}+k\log 2+k\log\frac{en}{k}}\leq c_1e^{-\frac{c_2C^2n^2}{k^2}+c_3k\log n} \\
&= c_1e^{-\frac{c_2C^2n^2-c_3k^3\log n}{k^2}}\leq c_1e^{-(c_2C^2-c_3)n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}},
\end{aligned}$$

Taking into account that $\frac{c_2C^2n^2-c_3k^3\log n}{k^2}$ is decreasing in $k$ and that $1\leq k\leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, the latter term is bounded above by $c_1e^{-(c_2C^2-c_3)n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$. Choosing $C$ a constant big enough we obtain the result. $\square$

As a consequence, we obtain the following lemma, which gives an estimate of the right order for most subspaces, for the second term in (3).

**Lemma 3.4.** *There exists absolute constants $C,c_1,c_2$ such that for any $1\leq k\leq \frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, the measure $\mu_{n,n-k}$ of the set of subspaces $E\in G_{n,n-k}$ for which*

$$\left|\mathbb{E}_{P_E(F)}|x|^2-\mathbb{E}_\mu|x|^2\right|\leq C\sqrt{n}$$

*for every $(n-k)$-dimensional face $F$ of $B_\infty^n$ is greater than $1-c_1e^{-c_2n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$.*

*Proof.* Recall that $\mu$ denotes the uniform probability measure on $K=P_E(B_\infty^n)$ and $\mathbb{E}_{P_E(F)}$ denotes the expectation with respect to the uniform probability on $P_E(F)$. By Lemma 3.3 there exists positive absolute constants $C,c_1,c_2$ and a set of subspaces with measure $\mu_{n,n-k}$ greater than $1-c_1e^{-c_2n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$ such that for every $E$ in such set and every $(n-k)$-dimensional face $F$ of $B_\infty^n$,

$$\left|\mathbb{E}_{P_E(F)}|x|^2-\frac{(n-k)(n+2k)}{3n}\right|=\left|\mathbb{E}_F|P_Ex|^2-\frac{(n-k)(n+2k)}{3n}\right|\leq C\sqrt{n}.$$

Then, for every $F_1,F_2$, $(n-k)$-dimensional faces

$$\left|\mathbb{E}_{P_E(F_1)}|x|^2-\mathbb{E}_{P_E(F_2)}|x|^2\right|\leq 2C\sqrt{n}.$$

Consequently, since $\mathbb{E}_\mu|x|^2=\displaystyle\sum_{i=1}^{l}\frac{|P_E(F_i)|}{|K|}\mathbb{E}_{P_EF_i}|x|^2$, we have that for every $E$ in this set and every face $F$

$$\mathbb{E}_{P_E(F)}|x|^2-\mathbb{E}_\mu|x|^2\leq\mathbb{E}_{P_E(F)}|x|^2-\min_{i=1,\dots,l}\mathbb{E}_{P_E(F_i)}|x|^2\leq 2C\sqrt{n}$$

and

$$\mathbb{E}_F|P_Ex|^2-\mathbb{E}_\mu|x|^2\geq\mathbb{E}_F|P_Ex|^2-\max_{i=1,\dots,l}\mathbb{E}_{F_i}|x|^2\geq -2C\sqrt{n}.$$

$\square$

Now we are able to prove Theorem 1.2.

*Proof of Theorem 1.2.* If $1\leq k\leq\frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$ by Lemma 2.2 we have that there exists an absolute constant $C$ such that $\lambda_\mu^2\mathbb{E}_\mu|x|^2\geq Cn$. By equation (3), if $\{F_i\}_{i=1}^l$ are the $(n-k)$-dimensional faces of $B_\infty^n$ described in (1) we have,

$$\mathrm{Var}_\mu|x|^2\leq\max_{i=1\dots,l}\mathrm{Var}_{P_E(F_i)}|x|^2+\max_{i=1,\dots,l}\left(\mathbb{E}_{P_E(F_i)}|x|^2-\mathbb{E}_\mu|x|^2\right)^2.$$

By Lemma 2.5 the first maximum is bounded from above by $C_1n$ and by Lemma 3.4 there exists a set of $(n-k)$-dimensional subspaces with measure larger than $1-c_1e^{-c_2n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$ such that the second maximum is bounded from above by $C_2n$, where $C_1$ and $C_2$ are absolute constants. This proves Theorem 1.2. $\square$

## References

[AB1] D. Alonso-Gutiérrez, J. Bastero *The variance conjecture on some polytopes.* In Asymptotic Geometric Analysis, Proceedings of the Fall 2010, Fields Institute Thematic Program, pp. 1–20, (Springer, 2013)

[AB2] D. Alonso-Gutiérrez, J. Bastero *Approaching the Kannan-Lovász-Simonovits and Variance Conjectures*, Lecture Notes in Math. **2131**, Springer, (2015).

[ABBW] D. Alonso-Gutiérrez, J. Bastero, J. Bernués, P. Wolff *On the isotropy constant of projections of polytopes.* Journal of Functional Analysis **258**, no. 5 (2010), pp. 1452–1465.

[BaC] F. Barthe, D. Cordero-Erausquin *Invariances in variance estimates.* Proc. London Math. Soc. **106**, no. 1 (2013), pp. 33–64.

[BaW] F. Barthe, P. Wolff *Remarks on non-interacting conservative spin systems: The case of gamma distributions.* pp. 37–43. Stochastic Process. App. **119** (2009), pp. 2711–2723.

[B] S. Bobkov *Spectral gap and concentration for some spherically symmetric probability measures.* Geometric Aspects of Functional Analysis, Lecture Notes in Math. **1807** (2003), pp. 37–43

[BK] S. Bobkov, A. Koldobsky *On the central limit property of convex bodies.* Geometric Aspects of Functional Analysis, Lecture Notes in Math. **1807** Springer (2003), pp. 44–52

[H] N. Huet *Spectral gap for some invariant log-concave probability measures.* Mathematika **57**, no. 1 (2011), pp. 51–62.

[KLS] R. Kannan, L. Lovsz, M. Simonovits *Isoperimetric problems for convex bodies and a localization lemma.* Discrete Comput. Geom. **13**, no. 3-4 (1995), pp. 541–559.

[K] B. Klartag *A Berry-Esseen type inequality for convex bodies with an unconditional basis.* Probab. Theory Related Fields **145**, no. 1-2 (2009), pp. 1–33.

[LW] R. Latała and J. O. Wojtaszczyk *On the infimum convolution inequality.* Studia Math. **189**, no. 2 (2008), pp. 147–187.

[LV] Y.T. Lee and S.S. Vempala *Eldans Stochastic Localization and the KLS Hyperplane Conjecture: An Improved Lower Bound for Expansion* arXiv:1612.01507

[MS] V.Milman, G. Schechtman. *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, (1986).

[S] S. Sodin *An isoperimetric inequality on the $\ell_p$ balls.* Ann. Inst. H. Poincar é Probab. Statist. **44**, no. 2 (2008), pp. 362–373.

Área de análisis matemático, Departamento de matemáticas, Facultad de Ciencias, Universidad de Zaragoza, Pedro cerbuna 12, 50009 Zaragoza (Spain), IUMA

*E-mail address*, (David Alonso): `alonsod@unizar.es`

Área de análisis matemático, Departamento de matemáticas, Facultad de Ciencias, Universidad de Zaragoza, Pedro cerbuna 12, 50009 Zaragoza (Spain), IUMA

*E-mail address*, (Julio Bernués): `bernues@unizar.es`