ON GENERALIZED GAUDUCHON NILMANIFOLDS

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Abstract. We construct invariant generalized Gauduchon metrics on the product of two complex nilmanifolds that do not necessarily admit this kind of metrics. In particular, we prove that the product of a locally conformal Kähler nilmanifold and a balanced nilmanifold admits a generalized Gauduchon metric. In complex dimension 4, generalized Gauduchon nilmanifolds with (the highest possible) nilpotency step $s = 5$ are given, as well as 3-step and 4-step examples for which the center of their underlying Lie algebras does not contain any non-trivial J-invariant ideal. These examples show strong differences between the SKT and the generalized Gauduchon geometries of nilmanifolds.

1. Introduction

Let X be a compact complex manifold of complex dimension n, and let F be a Hermitian metric on X. When the Lee form is co-closed, or equivalently F^{n-1} is $\partial\overline{\partial}$-closed, the metric F is called *standard* or *Gauduchon*. By [13] there is a standard metric in the conformal class of every Hermitian metric on X. Fu, Wang, and Wu investigate in [11] the following generalization of Gauduchon metrics. For $1 \leq k \leq n-1$, a Hermitian metric F on X is called *k-th Gauduchon* if $\partial\overline{\partial}F^k \wedge F^{n-k-1} = 0$. In [11] a unique constant $\gamma_k(F)$ is associated to any F on X. This constant is invariant by biholomorphisms and depends smoothly on the metric F. Moreover, it is proved that $\gamma_k(F) = 0$ if and only if there exists a k-th Gauduchon metric in the conformal class of F. For $k = n-1$, $(n-1)$-th Gauduchon metrics are by definition the usual Gauduchon metrics, and it is showed in [11] that $\gamma_{n-1}(F) = 0$, accordingly to [13].

In this paper we are mainly concerned with generalized Gauduchon metrics F for $k = 1$, i.e. those satisfying $\partial\overline{\partial}F \wedge F^{n-2} = 0$. Some compact complex manifolds with 1-st Gauduchon metrics are constructed in [9, 11, 15] by different methods. A particularly interesting subclass of 1-st Gauduchon metrics is that constituted by the *pluriclosed* or *strong Kähler with torsion* (*SKT* for short) metrics. They are defined by the condition $\partial\bar{\partial}F = 0$ and have been studied by many authors on non-Kähler compact complex manifolds (see for instance [4, 6, 7, 8, 10, 20] and the references therein).

The class of complex nilmanifolds has proved to be an important source of compact complex manifolds admitting these types of special Hermitian metrics. Here by a *complex nilmanifold* we mean a compact complex manifold of the form $X = (\Gamma\backslash G, J)$, where $\Gamma\backslash G$ is a compact quotient of a simply-connected nilpotent Lie group G by a uniform discrete subgroup Γ, and J is an *invariant* complex structure. For instance, Fino, Parton, and Salamon find in [7] the complex nilmanifolds of complex dimension 3 with invariant SKT metrics, and they show that the existence of such metrics only depends on the complex structure J. Moreover, it is proved in [9] that any invariant 1-st Gauduchon metric on a complex nilmanifold of complex dimension 3 is necessarily SKT.

Our first goal in this paper is to construct invariant 1-st Gauduchon metrics on the product of two complex nilmanifolds that do not necessarily admit this type of metrics. For this purpose, in Proposition 2.3 we study the constant $c_1(F + F')$ given by (1) that measures the 1-st Gauduchon condition for the product of two Hermitian metrics F and F'. As a consequence, we conclude

Key words and phrases. Nilmanifold; complex structure; Hermitian metrics.

that the product of two Hermitian nilmanifolds with constants $c_1(F)$ and $c_1(F')$ of opposite signs admits a 1-st Gauduchon metric (see Corollary 2.4).

As a first application, we show in Theorem 2.5 that the product of a locally conformal Kähler nilmanifold and a balanced nilmanifold always admits a 1-st Gauduchon metric. Recall that a *locally conformal Kähler* (*LCK* for short) metric is a Hermitian metric that is conformal to some local Kähler metric in a neighborhood of each point of the manifold, and a Hermitian metric F is *balanced* if the Lee form vanishes, or equivalently F^{n-1} is a closed form. Balanced nilmanifolds are studied in [1, 22], whereas the complex nilmanifolds admitting LCK metrics are classified in [19].

A second application is given in Theorem 2.11, where we consider the product of two Hermitian nilmanifolds X and X' of complex dimension 3. In this case, the sign of the constant $c_1(F)$ of an invariant J-Hermitian metric F on X (respectively, X') only depends on the complex structure J, as seen in [9] (see also Proposition 2.7 and Table 1). Thanks to it, we can provide classifications of those complex structures for which $c_1(F)$ is negative, zero (SKT case), or positive (see Propositions 2.9 and 2.10). Using these classifications, one can easily choose X and X' with constants c_1's of opposite signs in order to construct invariant 1-st Gauduchon metrics on the product nilmanifold $Y = X \times X'$.

Enrietti, Fino, and Vezzoni proved in [4, Proposition 3.1] that the existence of an SKT metric on a complex nilmanifold implies that the center of its underlying Lie algebra is J-invariant. Furthermore, [4, Theorem 1.2] asserts that an SKT nilmanifold (not a torus) is necessarily 2-step. The aim of Section 3 is to show that the generalized Gauduchon geometry of nilmanifolds is much more flexible that the SKT geometry. Since in complex dimension 3 the invariant 1-st Gauduchon metrics coincide with the invariant SKT metrics [9], we are led to study complex nilmanifolds of complex dimension 4. Although there are some examples of 1-st Gauduchon nilmanifolds of complex dimension 4 in the literature [9, 16], we notice that all of them are 2-step.

In Proposition 3.1 we construct for $s = 3, 4$ an s-step nilmanifold of complex dimension 4 with invariant 1-st Gauduchon metrics such that the center $Z(\mathfrak{g})$ of the underlying Lie algebra $\mathfrak{g}$ is not J-invariant; moreover, $Z(\mathfrak{g})$ does not contain any non-trivial J-invariant ideal. In Proposition 3.2 we give 5-step nilmanifolds of complex dimension 4 having invariant 1-st Gauduchon metrics. Note that this result is optimal, since for $s \geq 6$ there do not exist s-step 8-dimensional nilmanifolds admitting invariant complex structures by [2, 12, 14] (see Remark 3.3).

The examples constructed in Propositions 3.1 and 3.2 are irreducible. By taking appropriate products, in Theorem 3.5 we conclude that for $3 \leq s \leq 5$ and for any $n \geq 4$, there exists an s-step nilmanifold of complex dimension n admitting invariant 1-st Gauduchon metrics. Furthermore, by [16, Proposition 2.2] all the 1-st Gauduchon metrics given in this paper also satisfy the k-th Gauduchon property for every $2 \leq k \leq n-1$ (see Remark 3.4).

2. Generalized Gauduchon metrics on product nilmanifolds

In this section we study the existence of generalized Gauduchon metrics on a product of complex nilmanifolds endowed with invariant Hermitian metrics. As a consequence, we obtain many examples of generalized Gauduchon nilmanifolds.

Let us start reviewing the definition and some of the main properties of the generalized Gauduchon metrics obtained by Fu, Wang, and Wu in [11].

Definition 2.1. [11] *Let X be a compact complex manifold of complex dimension n, and let $1 \leq k \leq n-1$ be an integer. A Hermitian metric F on X is called* k-th Gauduchon *if it satisfies the condition*

$$\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0.$$

From the definition, one can see that the value $k = n-1$ recovers the classical standard (Gauduchon) metrics. Moreover, it is clear that any SKT metric is a 1-st Gauduchon metric.

Extending the result proved by Gauduchon in [13] for standard metrics, in [11] it is shown that, for any $1 \le k \le n-1$, there exists a unique constant $\gamma_k(F)$ and a (unique up to a constant) function $v \in \mathcal{C}^\infty(X)$ such that

$$\frac{i}{2}\,\partial\overline{\partial}(e^v F^k)\wedge F^{n-k-1} = \gamma_k(F)\, e^v F^n.$$

It is seen that for $k = n-1$ one always has $\gamma_{n-1}(F) = 0$. Moreover, if X admits a Kähler metric F, then $\gamma_k(F) = 0$ and v is a constant function for any $1 \le k \le n-1$.

Furthermore, the constant $\gamma_k(F)$ is invariant under biholomorphisms, and by [11, Proposition 11] the sign of $\gamma_k(F)$ is invariant in the conformal class of F.

To compute the sign of the constant $\gamma_k(F)$ one can use the following result:

Proposition 2.2. [11] *For a Hermitian metric F on a compact complex manifold X of complex dimension n, the number $\gamma_k(F)$ is > 0 ($= 0$, or < 0) if and only if there exists a metric $\tilde{F}$ in the conformal class of F such that*

$$\frac{i}{2}\,\partial\overline{\partial}\tilde{F}^k\wedge \tilde{F}^{n-k-1} > 0 \ (= 0,\ or < 0).$$

In this paper we focus our attention on these Hermitian metrics in the special class of nilmanifolds. We recall that a *complex nilmanifold* is a compact complex manifold of the form $X = (\Gamma\backslash G, J)$, where $\Gamma\backslash G$ is a compact quotient of a simply-connected nilpotent Lie group G by a uniform discrete subgroup Γ, and J is an *invariant* complex structure. This means that J is an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G such that $J^2 = -\mathrm{Id}$ and it is integrable, that is, the i-eigenspace $\mathfrak{g}_{1,0}$ of J in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We will be mainly concerned with Hermitian metrics F on X which are also *invariant*.

For any invariant Hermitian metric F on a complex nilmanifold of complex dimension $n \ge 2$, the real (n,n)-form $\frac{i}{2}\,\partial\bar{\partial}F\wedge F^{n-2}$ is proportional to the volume form F^n. Therefore,

$$\frac{i}{2}\,\partial\bar{\partial}F\wedge F^{n-2} = c_1(F)\, F^n, \tag{1}$$

for some constant $c_1(F) \in \mathbb{R}$. Observe that F is 1-st Gauduchon if and only if $c_1(F) = 0$.

Let us notice that by Proposition 2.2 the sign of $c_1(F)$ coincides with the sign of the constant $\gamma_1(F)$. In our study of generalized Gauduchon metrics we will consider $c_1(F)$ instead of $\gamma_1(F)$, because its precise value on nilmanifolds can be determined easily, and thus its sign.

Proposition 2.3. *Let X and X' be complex nilmanifolds endowed with invariant Hermitian metrics F and F', respectively.*

(i) *For any real $\lambda > 0$, we have*

$$c_1(\lambda\, F) = \frac{c_1(F)}{\lambda}. \tag{2}$$

(ii) *Let $Y = X \times X'$ be the product nilmanifold endowed with the product Hermitian metric $F + F'$. Then,*

$$c_1(F+F') = \frac{n\,(n-1)}{(n+n')(n+n'-1)}\, c_1(F) + \frac{n'(n'-1)}{(n+n')(n+n'-1)}\, c_1(F'), \tag{3}$$

where $n = \dim_{\mathbb{C}} X$ and $n' = \dim_{\mathbb{C}} X'$.

Proof. Let us start with part (i). At the sight of (1), one has the following expression for the Hermitian metric $\lambda\, F$:

$$\frac{i}{2}\,\partial\bar{\partial}(\lambda\, F)\wedge(\lambda\, F)^{n-2} = c_1(\lambda\, F)\,(\lambda\, F)^n.$$

If we expand the left-hand side of this equality, we obtain

$$\frac{i}{2}\,\partial\bar\partial(\lambda\,F)\wedge(\lambda\,F)^{n-2}=\lambda^{n-1}\,\frac{i}{2}\,(\partial\bar\partial F\wedge F^{n-2})=\lambda^{n-1}\,c_1(F)\,F^n=\lambda^{-1}\,c_1(F)\,(\lambda\,F)^n,$$

and the result comes straightforward.

We next prove (ii). On the one hand, the equation (1) for the Hermitian metric $F+F'$ on the $(n+n')$-dimensional complex nilmanifold $Y=X\times X'$ reads as

$$\frac{i}{2}\,\partial\bar\partial(F+F')\wedge(F+F')^{n+n'-2}=c_1(F+F')\,(F+F')^{n+n'}.$$

On the other hand, we consider

$$\begin{aligned}\frac{i}{2}\,\partial\bar\partial(F+F')\wedge(F+F')^{n+n'-2}&=\frac{i}{2}\,(\partial\bar\partial F+\partial\bar\partial F')\wedge(F+F')^{n+n'-2}\\&=\frac{i}{2}\,(\partial\bar\partial F+\partial\bar\partial F')\wedge(\alpha\,F^{n-2}\wedge F'^{\,n'}+\zeta\,F^{n-1}\wedge F'^{\,n'-1}+\beta\,F^n\wedge F'^{\,n'-2})\\&=\frac{i}{2}\,\partial\bar\partial F\wedge(\alpha\,F^{n-2}\wedge F'^{\,n'})+\frac{i}{2}\,\partial\bar\partial F'\wedge(\beta\,F^n\wedge F'^{\,n'-2})\\&=\alpha\,(\frac{i}{2}\,\partial\bar\partial F\wedge F^{n-2})\wedge F'^{\,n'}+\beta\,F^n\wedge(\frac{i}{2}\,\partial\bar\partial F'\wedge F'^{\,n'-2})\\&=\big(\alpha\,c_1(F)+\beta\,c_1(F')\big)\;F^n\wedge F'^{\,n'}\\&=\left(\frac{\alpha}{\nu}\,c_1(F)+\frac{\beta}{\nu}\,c_1(F')\right)\,(F+F')^{n+n'},\end{aligned}$$

where $\alpha=\binom{n+n'-2}{n'}$, $\zeta=\binom{n+n'-2}{n-1}$, $\beta=\binom{n+n'-2}{n}$, and $\nu=\binom{n+n'}{n}$. Therefore, $\frac{\alpha}{\nu}=\frac{n\,(n-1)}{(n+n')(n+n'-1)}$ and $\frac{\beta}{\nu}=\frac{n'(n'-1)}{(n+n')(n+n'-1)}$. This gives us (3). □

As a direct consequence of Proposition 2.3, the product of 1-st Gauduchon nilmanifolds is again a 1-st Gauduchon nilmanifold. Nonetheless, we can also consider Hermitian nilmanifolds with opposite signs for their constants c_1 in order to produce examples of 1-st Gauduchon nilmanifolds. Indeed:

Corollary 2.4. *Let X and X' be complex nilmanifolds endowed with invariant Hermitian metrics F and F', respectively, such that $c_1(F)>0$ and $c_1(F')<0$. Then, the product nilmanifold $X\times X'$ has a 1-st Gauduchon metric.*

Proof. Let n, n' be the complex dimensions of X and X', respectively. Let us consider

$$\lambda=-\frac{n'(n'-1)\,c_1(F')}{n\,(n-1)\,c_1(F)}>0,$$

which is in fact a positive real number, since $c_1(F)>0$ and $c_1(F')<0$ by hypothesis. Now, we can take the Hermitian metric $\tilde{F'}=\lambda\,F'$ on the complex nilmanifold X'. A direct calculation using (2) and (3) shows that the Hermitian metric $F+\tilde{F'}$ on the complex product nilmanifold $X\times X'$ satisfies

$$\begin{aligned}(n+n')(n+n'-1)\,c_1(F+\tilde{F'})&=n(n-1)\,c_1(F)+n'(n'-1)\,\tfrac{c_1(F')}{\lambda}\\&=n(n-1)\,c_1(F)-n(n-1)\,c_1(F)\\&=0.\end{aligned}$$

Hence, $F+\tilde{F'}$ is a 1-st Gauduchon metric. □

Let us recall that a Hermitian manifold (X, F) of complex dimension n is called *locally conformal Kähler* (LCK for short) if F is conformal to some local Kähler metric in a neighborhood of each point of X. Another important class of Hermitian manifolds is that of *balanced* manifolds, characterized by the condition $dF^{n-1} = 0$. Nilmanifolds with balanced metrics are given in [1, 22] and LCK nilmanifolds are studied in [19].

In the following result, the complex structures on the nilmanifolds are invariant, but the LCK metric and the balanced metric are not necessarily of invariant type.

Theorem 2.5. *The product of a locally conformal Kähler nilmanifold by a balanced nilmanifold admits a* 1*-st Gauduchon metric.*

Proof. Let X be a complex nilmanifold of complex dimension n admitting a balanced metric. By [5, Theorem 4.1], the existence of a balanced metric on X implies the existence of an invariant one. Let us denote F an invariant balanced metric on X. By [15, Lemma 3.7], the constant $c_1(F) > 0$.

Let X' be a complex nilmanifold of complex dimension n' admitting an LCK metric. As a consequence of [21, Proposition 34], there must exist an invariant LCK metric F' on X'. Using [15, Proposition 3.8] we have that the constant $c_1(F') < 0$.

Now, it suffices to apply Corollary 2.4 to the pair (X, F) and (X', F') to ensure the existence of a 1-st Gauduchon metric on the product nilmanifold $X \times X'$. □

Next, we will apply the previous results to the product of low dimensional complex nilmanifolds. In complex dimension 2, all the invariant Hermitian metrics F satisfy $c_1(F) = 0$; in fact, for the complex torus every metric F is Kähler, and on the Kodaira-Thurston manifold any invariant Hermitian metric F satisfies $\partial\bar\partial F = 0$, so $c_1(F) = 0$. Therefore, in order to construct generalized Gauduchon metrics on nilmanifolds making use of Corollary 2.4 we need to consider two complex nilmanifolds of complex dimension at least 3.

Recall that in complex dimension 3, the nilpotent Lie algebras underlying the nilmanifolds that admit an invariant complex structure are classified by Salamon in [18], whereas the classification of invariant complex structures is carried out in [3]. For the description of the complex structures we use a complex basis of (invariant) forms $\{\omega^j\}_{j=1}^3$ of bidegree (1,0) with respect to the complex structure. Remember that there exist two complex-parallelizable nilmanifolds, defined by the equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12}, \tag{4}$$

where $\rho \in \{0, 1\}$. One is the torus ($\rho = 0$) and the other one is the Iwasawa manifold ($\rho = 1$). We next use the description of the remaining invariant complex structures obtained in [3], where they are divided into three families:

Family (I): $d\omega^1 = d\omega^2 = 0,\;\; d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar1} + \lambda\,\omega^{1\bar2} + D\,\omega^{2\bar2}$,
where $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}^{\geq 0}$, and $D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$;

Family (II): $d\omega^1 = 0,\; d\omega^2 = \omega^{1\bar1},\; d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar2} + c\,\omega^{2\bar1}$,
where $\rho \in \{0, 1\}$, $B \in \mathbb{C}$, $c \in \mathbb{R}^{\geq 0}$, with $(\rho, B, c) \neq (0, 0, 0)$;

Family (III): $d\omega^1 = 0,\;\; d\omega^2 = \omega^{13} + \omega^{1\bar3},\;\; d\omega^3 = \varepsilon\, i\,\omega^{1\bar1} + i\,\delta(\omega^{1\bar2} - \omega^{2\bar1})$,
where $\varepsilon \in \{0, 1\}$ and $\delta = \pm 1$.

Any invariant Hermitian metric F on X is given in terms of a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ by $F = \sum_{j,k=1}^3 x_{j\bar k}\,\omega^{j\bar k}$, where $x_{j\bar k} \in \mathbb{C}$ and $\overline{x_{k\bar j}} = -x_{j\bar k}$. That is, F can be written as

$$F = x_{1\bar1}\,\omega^{1\bar1} + x_{2\bar2}\,\omega^{2\bar2} + x_{3\bar3}\,\omega^{3\bar3} + x_{1\bar2}\,\omega^{1\bar2} - \overline{x_{1\bar2}}\,\omega^{2\bar1} + x_{1\bar3}\,\omega^{1\bar3} - \overline{x_{1\bar3}}\,\omega^{3\bar1} + x_{2\bar3}\,\omega^{2\bar3} - \overline{x_{2\bar3}}\,\omega^{3\bar2}. \tag{5}$$

Notice that the positive definiteness of the metric F implies that in particular

$$-i\,x_{j\bar{j}} \in \mathbb{R}^+, \qquad i\det(x_{j\bar{k}}) > 0.$$

Remark 2.6. When J is a complex structure given by (4), i.e. J is complex-parallelizable, the sign of the constant $c_1(F)$ is well known for any J-Hermitian metric F. Indeed, it is clear that if $\rho = 0$ then F is closed, so $c_1(F) = 0$. If $\rho = 1$, i.e. the complex nilmanifold is the Iwasawa manifold, then F is balanced and thus $c_1(F) > 0$.

The following result shows the sign of any invariant Hermitian metric for the families of complex structures (I), (II), and (III).

Proposition 2.7. *Let X be a complex nilmanifold of complex dimension 3, and let F be any invariant Hermitian metric on X. Suppose that the complex structure J on X is not complex-parallelizable. We have:*

(i) *If J is a complex structure in* Family (I)*, then*

$$\frac{i}{2}\,\partial\bar{\partial}F \wedge F = \frac{i\,x_{3\bar{3}}^2}{12\,\det(x_{j\bar{k}})}\left(\rho + \lambda^2 - 2\,\mathfrak{Re}\,D\right)\,F^3,$$

and thus the sign of $c_1(F)$ only depends on the complex structure. Indeed, $c_1(F)$ is > 0 ($= 0$, or < 0) if and only if $2\,\mathfrak{Re}\,D - \rho - \lambda^2 < 0$ ($= 0$, or > 0).

(ii) *If J is a complex structure in* Family (II)*, then*

$$\frac{i}{2}\,\partial\bar{\partial}F \wedge F = \frac{i\,x_{3\bar{3}}^2}{12\,\det(x_{j\bar{k}})}\left(\rho + |B|^2 + c^2\right)\,F^3,$$

and $c_1(F) > 0$, for any F.

(iii) *If J is a complex structure in* Family (III)*, then*

$$\frac{i}{2}\,\partial\bar{\partial}F \wedge F = \frac{i\,(x_{2\bar{2}}^2 + x_{3\bar{3}}^2)}{6\,\det(x_{j\bar{k}})}\,F^3,$$

and thus $c_1(F) > 0$, for any F.

Proof. The result is a direct application of [9, Lemma 3.2] taking into account the reduced complex structure equations obtained in [3] and given in the families (I), (II), and (III) above. □

As a consequence of the previous proposition, the sign of $c_1(F)$ only depends on the complex structure. Furthermore, as it was noticed in [9, Proposition 3.3], an invariant Hermitian metric F on a complex nilmanifold X of complex dimension 3 satisfies $c_1(F) = 0$, i.e. it is 1-st Gauduchon, if and only if F is SKT. Recall that the classification of nilmanifolds admitting an invariant SKT metric was given in [7].

In Table 1 we analyze the sign of c_1 for invariant Hermitian metrics on complex nilmanifolds of complex dimension 3, i.e. on 6-dimensional nilmanifolds endowed with invariant complex structures J. The algebras in the first column correspond to those nilpotent Lie algebras underlying such nilmanifolds. Here, we follow the notation given in the paper [18] to name and describe the different Lie algebras. For instance, $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$ means that there is a basis of real 1-forms $\{e^j\}_{j=1}^6$ satisfying $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$, and $de^6 = e^3 \wedge e^4$.

Note that the nilpotent Lie algebras admitting a complex-parallelizable structure (4) are $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$). Moreover, a Lie algebra admitting a complex structure in Family (I) is isomorphic to $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$, or $\mathfrak{h}_8$, and it is always 2-step nilpotent. The nilpotent Lie algebras having complex structures in Family (II) are $\mathfrak{h}_7$ and $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$. Finally, the nilpotent Lie algebras corresponding to Family (III) are $\mathfrak{h}_{19}^-$ (for $\varepsilon = 0$) and $\mathfrak{h}_{26}^+$ (for $\varepsilon = 1$).

In the second column of Table 1, we indicate the nilpotency step of the nilmanifolds. For the other columns, we use the following convention. The symbol "✓" means that **for any** invariant complex structure J on the corresponding nilmanifold and **for any** invariant J-Hermitian metric F, the sign of $c_1(F)$ is always as indicated in the table (remember that the sign of $c_1(F)$ only depends on the complex structure, and $c_1(F) = 0$ if and only if F is SKT). The symbol "$\checkmark_{(J)}$" means that there exist invariant complex structures J on the corresponding nilmanifold admitting invariant J-Hermitian metrics F with the given sign of $c_1(F)$, but there are also other complex structures with invariant Hermitian metrics of different sign. In contrast, the symbol "–" means that none of the invariant complex structures admits invariant Hermitian metrics of the given sign.

	step	$c_1 < 0$	SKT	$c_1 > 0$
$\mathfrak{h}_1 = (0,0,0,0,0,0)$	1	–	✓	–
$\mathfrak{h}_2 = (0,0,0,0,12,34)$	2	$\checkmark_{(J)}$	$\checkmark_{(J)}$	$\checkmark_{(J)}$
$\mathfrak{h}_3 = (0,0,0,0,0,12{+}34)$	2	$\checkmark_{(J)}$	–	$\checkmark_{(J)}$
$\mathfrak{h}_4 = (0,0,0,0,12,14{+}23)$	2	$\checkmark_{(J)}$	$\checkmark_{(J)}$	$\checkmark_{(J)}$
$\mathfrak{h}_5 = (0,0,0,0,13{+}42,14{+}23)$	2	$\checkmark_{(J)}$	$\checkmark_{(J)}$	$\checkmark_{(J)}$
$\mathfrak{h}_6 = (0,0,0,0,12,13)$	2	–	–	✓
$\mathfrak{h}_7 = (0,0,0,12,13,23)$	2	–	–	✓
$\mathfrak{h}_8 = (0,0,0,0,0,12)$	2	–	✓	–
$\mathfrak{h}_9 = (0,0,0,0,12,14{+}25)$	3	–	–	✓
$\mathfrak{h}_{10} = (0,0,0,12,13,14)$	3	–	–	✓
$\mathfrak{h}_{11} = (0,0,0,12,13,14{+}23)$	3	–	–	✓
$\mathfrak{h}_{12} = (0,0,0,12,13,24)$	3	–	–	✓
$\mathfrak{h}_{13} = (0,0,0,12,13{+}14,24)$	3	–	–	✓
$\mathfrak{h}_{14} = (0,0,0,12,14,13{+}42)$	3	–	–	✓
$\mathfrak{h}_{15} = (0,0,0,12,13{+}42,14{+}23)$	3	–	–	✓
$\mathfrak{h}_{16} = (0,0,0,12,14,24)$	3	–	–	✓
$\mathfrak{h}_{19}^{-} = (0,0,0,12,23,14{-}35)$	3	–	–	✓
$\mathfrak{h}_{26}^{+} = (0,0,12,13,23,14{+}25)$	4	–	–	✓

TABLE 1. Sign of c_1 for invariant Hermitian metrics on 6-nilmanifolds

The existence of locally conformal Kähler and balanced metrics on 6-dimensional nilmanifolds is studied in [21]. On the one hand, it is seen that, apart from the torus, the nilmanifolds admitting balanced metrics have underlying Lie algebras $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$, or $\mathfrak{h}_{19}^{-}$. On the other hand, if a 6-dimensional nilmanifold has an LCK metric then its underlying Lie algebra is $\mathfrak{h}_3$. Therefore, the only 6-nilmanifold having invariant complex structures with LCK and balanced metrics is the one with $\mathfrak{h}_3$ as underlying Lie algebra. In the following example we apply Theorem 2.5 to this nilmanifold.

Example 2.8. Let us start recalling that the nilpotent Lie algebra $\mathfrak{h}_3 = (0,0,0,0,0,12+34)$ admits, up to equivalence, only two complex structures $J^{\pm}$, which correspond to $\rho = \lambda = 0$ and $D = \pm 1$ in Family (I). The structure J^{+} admits LCK metrics, whereas J^{-} admits balanced ones.

Let us denote by $X^{\pm}$ the complex nilmanifolds respectively associated to $(\mathfrak{h}_3, J^{\pm})$. Consider the $J^{\pm}$-Hermitian metrics F^+ and F_t^-, with $t>0$, given as follows:

$$(X^+, F^+): \begin{cases} d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar 1} + \omega^{2\bar 2}, \\ F^+ = \frac{i}{2}\,(\omega^{1\bar 1} + \omega^{2\bar 2} + \omega^{3\bar 3}), \end{cases}$$

$$(X^-, F_t^-): \begin{cases} d\sigma^1 = d\sigma^2 = 0,\ d\sigma^3 = \sigma^{1\bar 1} - \sigma^{2\bar 2}, \\ F_t^- = \frac{i}{2}\,(\sigma^{1\bar 1} + \sigma^{2\bar 2} + t\,\sigma^{3\bar 3}), \quad t>0. \end{cases}$$

We observe that $dF^+ = \theta \wedge F^+$ with $\theta = \omega^3 + \omega^{\bar 3}$ (i.e. F^+ is an LCK metric on X^+) and $d(F_t^-)^2 = 0$ (i.e. F_t^- is a balanced metric on X^-, for any $t>0$). Now, we can apply Theorem 2.5 to ensure that the product manifold $Y = X^+ \times X^-$ admits a 1-st Gauduchon metric.

More concretely, by Proposition 2.7 one has

$$c_1(F^+) = -\frac{1}{3}, \quad c_1(F_t^-) = \frac{t}{3},$$

so the metrics

$$F_t = \frac{1}{t}\,F^+ + F_t^-$$

defined on Y are 1-st Gauduchon for each $t>0$.

Next we describe the complex nilmanifolds of complex dimension 3 for which the invariant Hermitian metrics F satisfy $c_1(F) \le 0$. As the $\mathfrak{h}_3$ case is explained in Example 2.8, it remains to study the Lie algebras $\mathfrak{h}_2$, $\mathfrak{h}_4$, and $\mathfrak{h}_5$:

Case $\mathfrak{h}_2$: In [3] it is proved that any complex structure J on $\mathfrak{h}_2$ is isomorphic to one and only one in the following families:

$(\mathfrak{h}_2, J_1)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar 1} + D\,\omega^{2\bar 2}, \quad D \in \mathbb{C}$ with $\mathfrak{Im}\, D = 1$;

$(\mathfrak{h}_2, J_2)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar 1} + \omega^{1\bar 2} + D\,\omega^{2\bar 2}, \quad D \in \mathbb{C}$ with $\mathfrak{Im}\, D > 0$.

By Proposition 2.7 (i), any J-Hermitian metric F on $\mathfrak{h}_2$ satisfies

$$c_1(F) < 0\ (=0) \text{ if and only if } J \text{ is given by } \begin{cases} J_1 \text{ with } \mathfrak{Re}\, D > 0\ (=0), \\ \text{or} \\ J_2 \text{ with } \mathfrak{Re}\, D > 1\ (=1). \end{cases}$$

Case $\mathfrak{h}_4$: As it is seen in [3], any complex structure J on $\mathfrak{h}_4$ is isomorphic to one and only one of the following ones:

$(\mathfrak{h}_4, J_3)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar 1} + \omega^{1\bar 2} + \frac{1}{4}\omega^{2\bar 2}$;

$(\mathfrak{h}_4, J_4)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar 1} + \omega^{1\bar 2} + D\,\omega^{2\bar 2}, \quad D \in \mathbb{R} \setminus \{0\}$.

Notice that for J_3 we have $2\,\mathfrak{Re}\, D = 1/2 < 1 = \rho + \lambda^2$. Therefore, by Proposition 2.7 (i) the sign of c_1 of any J_3-Hermitian metric F is positive, i.e. $c_1(F) > 0$. Similarly, for any complex structure J given by J_4, Proposition 2.7 (i) implies that $c_1(F) > 0$ if and only if $D < 1$. Therefore, for any J-Hermitian metric F on $\mathfrak{h}_4$ we have:

$$c_1(F) < 0\ (=0) \text{ if and only if } J \text{ is given by } J_4 \text{ with } D > 1\ (=1).$$

Case $\mathfrak{h}_5$: Using again [3], any complex structure J on $\mathfrak{h}_5$ is isomorphic to one and only one in the following list:

$(\mathfrak{h}_5, J_5)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{12}$;

$(\mathfrak{h}_5, J_6)$: $d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}},\ D \in [0, \frac{1}{4})$;

$(\mathfrak{h}_5, J_7)$: $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}$, with $(\lambda, D) \in \mathbb{R}^{\geq 0} \times \mathbb{C}$ such that:

- $\lambda = 0 \leq \mathfrak{Im}\,D,\ 4(\mathfrak{Im}\,D)^2 < 1 + 4\,\mathfrak{Re}\,D$;
- $0 < \lambda^2 < \frac{1}{2},\ 0 \leq \mathfrak{Im}\,D < \frac{\lambda^2}{2},\ \mathfrak{Re}\,D = 0$;
- $\frac{1}{2} \leq \lambda^2 < 1,\ 0 \leq \mathfrak{Im}\,D < \frac{1-\lambda^2}{2},\ \mathfrak{Re}\,D = 0$; or
- $\lambda^2 > 1,\ 0 \leq \mathfrak{Im}\,D < \frac{\lambda^2-1}{2},\ \mathfrak{Re}\,D = 0$.

We observe that J_5 is the complex-parallelizable structure on the Iwasawa manifold and any J_5-Hermitian metric F satisfies $c_1(F) > 0$ (see Remark 2.6). For any complex structure J given by J_6 we have $0 \leq 2\,\mathfrak{Re}\,D < \frac{1}{2} < 1 = \rho + \lambda^2$, hence by Proposition 2.7 (i) we also get $c_1(F) > 0$ for any J_6-Hermitian metric F.
Using again Proposition 2.7 (i) one concludes that for any J-Hermitian metric F on $\mathfrak{h}_5$:

$$c_1(F) < 0 \text{ if and only if } J \text{ is given by } J_7 \text{ with } \begin{cases} \lambda = 0 \leq \mathfrak{Im}\,D, \\ 4(\mathfrak{Im}\,D)^2 < 1 + 4\,\mathfrak{Re}\,D, \text{ and } \mathfrak{Re}\,D > \frac{1}{2}, \end{cases}$$

and

$$c_1(F) = 0 \text{ if and only if } J \text{ is given by } J_7 \text{ with } \begin{cases} \lambda = 0 \leq \mathfrak{Im}\,D \text{ and} \\ 4(\mathfrak{Im}\,D)^2 < 1 + 4\,\mathfrak{Re}\,D = 3. \end{cases}$$

As a direct consequence of the previous discussion, we get the following classification of the complex structures J for which $c_1(F) < 0$ for any J-Hermitian metric F.

Proposition 2.9. *Let X be a complex nilmanifold of complex dimension 3 admitting an invariant Hermitian metric F with $c_1(F) < 0$. Denote $(\mathfrak{g}, J)$ the Lie algebra and the complex structure underlying X. Then, the pair $(\mathfrak{g}, J)$ is isomorphic to one (and only one) in the following families:*

$(\mathfrak{h}_2, J_1)$ *with* $\mathfrak{Re}\,D > 0$ (**1-parameter family** *of non-equivalent complex structures*),

$(\mathfrak{h}_2, J_2)$ *with* $\mathfrak{Re}\,D > 1$ (**2-parameter family**),

$(\mathfrak{h}_3, J^+)$ *given in* Example 2.8 (*this is a* **unique** *complex structure*),

$(\mathfrak{h}_4, J_4)$ *with* $D > 1$ (**1-parameter family**),

$(\mathfrak{h}_5, J_7)$ *with* $\lambda = 0$, $\mathfrak{Im}\,D \geq 0$ *and* $\mathfrak{Re}\,D > \min\left\{\frac{1}{2}, (\mathfrak{Im}\,D)^2 - \frac{1}{4}\right\}$ (**2-parameter family**).

A second direct consequence of the discussion above is the classification of complex structures J admitting SKT metrics.

Proposition 2.10. *Let X be a complex nilmanifold of complex dimension 3 admitting an invariant SKT metric F (i.e. $c_1(F) = 0$). Let $(\mathfrak{g}, J)$ be the Lie algebra and the complex structure underlying X. Then, the pair $(\mathfrak{g}, J)$ is isomorphic to one (and only one) in the following families:*

$(\mathfrak{h}_1, J)$ *given by* (4) *with* $\rho = 0$ (**unique**),

$(\mathfrak{h}_2, J_1)$ *with* $\mathfrak{Re}\,D = 0$ (**unique**),

$(\mathfrak{h}_2, J_2)$ *with* $\mathfrak{Re}\,D = 1$ (**1-parameter family**),

$(\mathfrak{h}_4, J_4)$ *with* $D = 1$ (**unique**),

$(\mathfrak{h}_5, J_7)$ *with* $\lambda = 0$, $\mathfrak{Re}\,D = \frac{1}{2}$, *and* $0 \leq \mathfrak{Im}\,D < \frac{\sqrt{3}}{2}$ (**1-parameter family**),

$(\mathfrak{h}_8, J)$ *given by* Family (I) with $\rho = \lambda = D = 0$ (**unique**).

In the following result we give many examples of complex nilmanifolds with invariant 1-st Gauduchon metrics, constructed as products of nilmanifolds that do not admit any such metric.

Theorem 2.11. *Let X and X' be complex nilmanifolds of complex dimension* 3*. Suppose that the pair $(\mathfrak{g}, J)$ underlying X is isomorphic to one in the list given in Proposition* 2.9 *and that the pair $(\mathfrak{g}', J')$ underlying X' is not isomorphic to any of the pairs given in Propositions* 2.9 *or* 2.10*. Then, the product nilmanifold $Y = X \times X'$ has an invariant* 1*-st Gauduchon metric.*

Proof. On the one hand, by Proposition 2.9 we know that any invariant Hermitian metric F on X satisfies $c_1(F) < 0$. On the other hand, we have $c_1(F') > 0$ for any invariant Hermitian metric F' on X' whose pair $(\mathfrak{g}', J')$ is not isomorphic to any of those given in the lists in Propositions 2.9 and 2.10. Now, the existence of an invariant 1-st Gauduchon metric on the product nilmanifold $Y = X \times X'$ follows directly from Corollary 2.4. □

Notice that in the conditions of Theorem 2.11, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_3$, $\mathfrak{h}_4$, or $\mathfrak{h}_5$, whereas the Lie algebra $\mathfrak{g}'$ is isomorphic to $\mathfrak{h}_2, \ldots, \mathfrak{h}_7$, $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, or $\mathfrak{h}_{26}^+$ (see also Table 1).

3. Generalized Gauduchon metrics in complex dimension 4

Our goal in this section is to construct 1-st Gauduchon nilmanifolds of complex dimension 4 satisfying some special properties that allow to illustrate the strong differences between SKT and generalized Gauduchon geometries on nilmanifolds.

As we recalled in Section 2, the invariant 1-st Gauduchon metrics on complex nilmanifolds of complex dimension 3 coincide with the invariant SKT metrics. In [4, Proposition 3.1] it is proved that the existence of an SKT metric implies that the center of the underlying algebra is J-invariant, and in [4, Theorem 1.2] it is shown that an SKT nilmanifold (not a torus) is necessarily 2-step.

Let $Y = X \times X'$ be a 1-st Gauduchon nilmanifold given by Theorem 2.11. We observe that it suffices to choose X' with Lie algebra $\mathfrak{g}'$ isomorphic to $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, or $\mathfrak{h}_{26}^+$ in order to find 1-st Gauduchon nilmanifolds Y of complex dimension 6 whose underlying Lie algebras have nilpotency steps 3 or 4 (see Table 1). Nonetheless, the pair $(\mathfrak{g}, J)$ underlying X must be isomorphic to one in the list given in Proposition 2.9, and all these pairs have a non-trivial J-invariant ideal contained in the center $Z(\mathfrak{g})$. Therefore, the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}'$ underlying Y still has a non-trivial $(J \oplus J')$-invariant ideal contained in its center.

In this section, we are able to construct two examples in complex dimension 4 with nilpotency steps $s = 3$ and $s = 4$ for which the center $Z(\mathfrak{g})$ does not contain any non-trivial J-invariant ideal. Moreover, we also provide a 1-st Gauduchon nilmanifold of complex dimension 4 with the highest possible nilpotency step, namely $s = 5$. All these new examples are irreducible and have the minimal dimension in which the differences between invariant SKT and invariant 1-st Gauduchon metrics on nilmanifolds can arise.

In order to construct our first kind of examples, we consider the nilpotent Lie algebras $\mathfrak{g}_\nu$ of (real) dimension 8 defined by the following structure equations:

$$
\left\{
\begin{array}{rcl}
de^1 &=& de^2 = de^3 = de^4 = 0,\\
de^5 &=& 2\, e^{17},\\
de^6 &=& 2\, e^{27},\\
de^7 &=& 2\, \nu\, e^{12},\\
de^8 &=& 2\, e^{15} + 2\, e^{26} - 4\, e^{34},
\end{array}
\right.
\tag{6}
$$

where ν is a real number. Observe that ν can be reduced to the values 0 or 1 (if $\nu \neq 0$, just take the new basis given by $e'^i = e^i$, for $i = 1, 2, 3$ and $e'^i = \frac{1}{\nu}\, e^i$, for $i = 4, \ldots, 8$). For this reason, in what follows we will consider (6) with $\nu = 0, 1$. Since all the structure constants are rational numbers, it follows from [17] that the simply-connected nilpotent Lie group G_ν associated to $\mathfrak{g}_\nu$ has a uniform discrete subgroup. Hence, the structure equations (6) define a nilmanifold.

Let $\{E_i\}_{i=1}^8$ be the dual basis of $\{e^i\}_{i=1}^8$. From (6) we get that the non-zero basic Lie brackets $[E_i, E_j]$, $1 \leq i < j \leq 8$, are

$$[E_1, E_2] = -2\,\nu\, E_7, \qquad [E_1, E_7] = -2\, E_5, \qquad [E_2, E_7] = -2\, E_6,$$
$$[E_1, E_5] = -2\, E_8, \qquad [E_2, E_6] = -2\, E_8, \qquad [E_3, E_4] = 4\, E_8.$$

The ascending central series $\{(\mathfrak{g}_\nu)_k\}_{k\geq 1}$ of the Lie algebra $\mathfrak{g}_\nu$ is

$$(\mathfrak{g}_\nu)_1 = Z(\mathfrak{g}_\nu) = \langle E_8\rangle, \quad (\mathfrak{g}_\nu)_2 = \langle E_3, E_4, E_5, E_6, E_8\rangle,$$

and

$$\begin{cases} \text{for } \nu = 0\text{:} \quad (\mathfrak{g}_0)_3 = \mathfrak{g}_0, \\ \text{for } \nu = 1\text{:} \quad (\mathfrak{g}_1)_3 = \langle E_3, E_4, E_5, E_6, E_7, E_8\rangle, \quad (\mathfrak{g}_1)_4 = \mathfrak{g}_1. \end{cases}$$

Thus, for $\nu = 0$ the ascending central series has dimensions $(1, 5, 8)$ and the Lie algebra $\mathfrak{g}_0$ is nilpotent in step 3, whereas for $\nu = 1$ the ascending central series has dimensions $(1, 5, 6, 8)$ and the Lie algebra $\mathfrak{g}_1$ is 4-step.

Let us define the almost complex structure J given by

$$Je^1 = -e^2, \quad Je^3 = -e^4, \quad Je^5 = -e^6, \quad Je^7 = -e^8.$$

Since the center $Z(\mathfrak{g}_\nu)$ is 1-dimensional, it is clear that it cannot contain any non-trivial J-invariant ideal.

Let us consider the following basis of complex 1-forms of bidegree (1,0) with respect to J:

$$\omega^1 = e^1 + i\, e^2, \quad \omega^2 = e^3 + i\, e^4, \quad \omega^3 = e^5 + i\, e^6, \quad \omega^4 = e^7 + i\, e^8.$$

It follows from (6) that the differentials of these $(1, 0)$-forms are

$$\begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^4 = i\,\nu\,\omega^{1\bar{1}} + i\,\omega^{1\bar{3}} + 2\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}. \end{cases} \tag{7}$$

Observe that the differential of each ω^j has no component of bidegree (0,2), thus the almost complex structure J is integrable both on $\mathfrak{g}_0$ and on $\mathfrak{g}_1$.

Let F be the J-Hermitian metric on $\mathfrak{g}_\nu$ given by

$$F = i\,(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}} + \omega^{4\bar{4}}) + \frac{1}{2}\,(\omega^{1\bar{3}} - \omega^{3\bar{1}}).$$

A direct calculation using the complex structure equations (7) shows that the metric F satisfies $\partial\bar{\partial}F \wedge F^2 = 0$, namely, it is a 1-st Gauduchon metric. Therefore, we have proved:

Proposition 3.1. *For each $s \in \{3, 4\}$, there exists an s-step nilmanifold of complex dimension 4 with an invariant 1-st Gauduchon metric such that the center $Z(\mathfrak{g})$ of its underlying Lie algebra $\mathfrak{g}$ does not contain any non-trivial J-invariant ideal.*

Our next purpose is to construct 1-st Gauduchon nilmanifolds with nilpotency step $s = 5$. For this aim, we consider the family of 8-dimensional nilpotent Lie algebras $\mathfrak{g}$ defined as the following extensions of $\mathfrak{h}_{26}^{+}$ (see Table 1):

$$
(8)\qquad \begin{cases} de^1 &= de^2 = 0, \\ de^3 &= e^{12}, \\ de^4 &= e^{13}, \\ de^5 &= e^{23}, \\ de^6 &= e^{14} + e^{25}, \\ de^7 &= b_2\, e^{12} + \delta\, e^{13} + a_1\,(e^{14} - e^{25}) - a_2\,(e^{15} + e^{24}) + 2\, e^{26} - 2\, e^{34}, \\ de^8 &= -b_1\, e^{12} + a_2\,(e^{14} - e^{25}) + a_1\,(e^{15} + e^{24}) - 2\, e^{16} + \delta\, e^{23} - 2\, e^{35}, \end{cases}
$$

where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, and δ is equal to 1 or -1. The coefficients a_1, a_2, b_1, b_2 are rational numbers to ensure the existence of a uniform discrete subgroup for the associated simply-connected nilpotent Lie group G (see [17]). In this way, the equations (8) define a nilmanifold.

Let $\{E_i\}_{i=1}^8$ be the dual basis of $\{e^i\}_{i=1}^8$. From (8) one has that the non-zero basic Lie brackets $[E_i, E_j]$, $1 \le i < j \le 8$, are

$$
\begin{array}{lll}
[E_1,E_2] = -E_3 - b_2\, E_7 + b_1\, E_8, & [E_1,E_6] = 2\, E_8, & [E_2,E_6] = -2\, E_7, \\
[E_1,E_3] = -E_4 - \delta\, E_7, & [E_2,E_3] = -E_5 - \delta\, E_8, & [E_3,E_4] = 2\, E_7, \\
[E_1,E_4] = -E_6 - a_1\, E_7 - a_2\, E_8, & [E_2,E_4] = a_2\, E_7 - a_1\, E_8, & [E_3,E_5] = 2\, E_8. \\
[E_1,E_5] = a_2\, E_7 - a_1\, E_8, & [E_2,E_5] = -E_6 + a_1\, E_7 + a_2\, E_8, &
\end{array}
$$

The ascending central series $\{\mathfrak{g}_k\}_{k\ge1}$ of $\mathfrak{g}$ is given by

$$
\begin{array}{lll}
\mathfrak{g}_1 = Z(\mathfrak{g}) = \langle E_7, E_8\rangle, & \mathfrak{g}_2 = \langle E_6, E_7, E_8\rangle, & \mathfrak{g}_3 = \langle E_4, E_5, E_6, E_7, E_8\rangle, \\
\mathfrak{g}_4 = \langle E_3, E_4, E_5, E_6, E_7, E_8\rangle, & \mathfrak{g}_5 = \mathfrak{g}. &
\end{array}
$$

Hence, any Lie algebra $\mathfrak{g}$ defined by (8) is nilpotent in step $s = 5$.

For $\delta = \pm 1$, let J_δ be the almost complex structure on $\mathfrak{g}$ defined by:

$$
J_\delta(e^1) = -e^2, \quad J_\delta(e^3) = -2\,\delta\, e^6, \quad J_\delta(e^4) = -e^5, \quad J_\delta(e^7) = -e^8.
$$

Notice that the center $Z(\mathfrak{g})$ is J_δ-invariant.

It is clear that the forms

$$
\omega^1 = \frac{\sqrt{2}}{2}\,(e^1 + i\, e^2), \quad \omega^2 = \sqrt{2}\,(e^4 + i\, e^5), \quad \omega^3 = e^3 + 2\,\delta\, i\, e^6, \quad \omega^4 = \sqrt{2}\,(e^7 + i\, e^8),
$$

constitute a basis of type $(1,0)$ with respect to J_δ.

Now, it follows from (8) that their differentials are

$$
(9)\qquad \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{13} + \omega^{1\bar3}, \\ d\omega^3 &= i\,\omega^{1\bar1} + i\,\delta(\omega^{1\bar2} - \omega^{2\bar1}), \\ d\omega^4 &= \sqrt{2}\, A\,\omega^{12} + \omega^{23} + \sqrt{2}\, B\,\omega^{1\bar1} + 2\,\delta\,\omega^{1\bar3} + \omega^{2\bar3}, \end{cases}
$$

where $A = a_1 + i\, a_2$ and $B = b_1 + i\, b_2$ belong to $\mathbb{Q}[i]$. Since the differentials of each ω^j have no component of bidegree (0,2), we conclude that J_δ is integrable and defines a complex structure on $\mathfrak{g}$.

In the result below we show that it is possible to choose appropriate numbers A and B in $\mathbb{Q}[i]$ that allow to construct complex nilmanifolds having 1-st Gauduchon metrics.

Proposition 3.2. *There exist* 5*-step nilmanifolds of complex dimension* 4 *with invariant* 1*-st Gauduchon metrics.*

Proof. Let $(\mathfrak{g}, J)$ be a nilpotent Lie algebra given by (8) endowed with the complex structure $J = J_\delta$. We consider a J-Hermitian metric F on $\mathfrak{g}$ of the form

$$F = \Omega + \frac{i}{2}\,\omega^{4\bar{4}},$$

where Ω is given by

$$\Omega = i\,r\,\omega^{1\bar{1}} + i\,s\,\omega^{2\bar{2}} + i\,t\,\omega^{3\bar{3}} + x_{1\bar{2}}\,\omega^{1\bar{2}} - \overline{x_{1\bar{2}}}\,\omega^{2\bar{1}} + x_{1\bar{3}}\,\omega^{1\bar{3}} - \overline{x_{1\bar{3}}}\,\omega^{3\bar{1}} + x_{2\bar{3}}\,\omega^{2\bar{3}} - \overline{x_{2\bar{3}}}\,\omega^{3\bar{2}},$$

where $r, s, t \in \mathbb{R}^+$ and $x_{j\bar{k}} \in \mathbb{C}$ satisfy the conditions ensuring that Ω (and thus F) is positive definite. Notice that Ω is equivalently given by (5), simply writing $x_{1\bar{1}} = i\,r$, $x_{2\bar{2}} = i\,s$, and $x_{3\bar{3}} = i\,t$, for $r, s, t \in \mathbb{R}^+$.

For the computation of the form $\partial\bar{\partial}F \wedge F^2$, we observe that

$$\begin{aligned}
\partial\bar{\partial}F \wedge F^2 &= \Big(\partial\bar{\partial}\,\Omega + \tfrac{i}{2}\,\partial\bar{\partial}(\omega^{4\bar{4}})\Big) \wedge (\Omega^2 + i\,\Omega \wedge \omega^{4\bar{4}}) \\
&= \partial\bar{\partial}\,\Omega \wedge \Omega^2 + i\,\partial\bar{\partial}\,\Omega \wedge \Omega \wedge \omega^{4\bar{4}} + \tfrac{i}{2}\,\Omega^2 \wedge \partial\bar{\partial}(\omega^{4\bar{4}}) - \tfrac{1}{2}\,\Omega \wedge \partial\bar{\partial}(\omega^{4\bar{4}}) \wedge \omega^{4\bar{4}} \\
&= i\,\partial\bar{\partial}\,\Omega \wedge \Omega \wedge \omega^{4\bar{4}} - \tfrac{1}{2}\,\Omega \wedge \partial\bar{\partial}(\omega^{4\bar{4}}) \wedge \omega^{4\bar{4}} \\
&= -\tfrac{1}{2}\Big(\Omega \wedge \partial\bar{\partial}(\omega^{4\bar{4}}) - 4\,c_1(\Omega)\,\Omega^3\Big) \wedge \omega^{4\bar{4}}.
\end{aligned} \tag{10}$$

Simply note that in the third equality of (10) we have used that $\partial\bar{\partial}\,\Omega \wedge \Omega^2 = 0$, as this is a form of degree 8 on the 6-dimensional algebra $\mathfrak{h}_{26}^+$. Moreover, we have also applied that $\Omega^2 \wedge \partial\bar{\partial}(\omega^{4\bar{4}}) = 0$, because

$$\partial\bar{\partial}(\omega^{4\bar{4}}) = \partial\bar{\partial}\omega^4 \wedge \omega^{\bar{4}} + \bar{\partial}\omega^4 \wedge \partial\omega^{\bar{4}} - \partial\omega^4 \wedge \bar{\partial}\omega^{\bar{4}} + \omega^4 \wedge \partial\bar{\partial}\omega^{\bar{4}} \tag{11}$$

and from the complex equations (9) it follows that $\partial\bar{\partial}\omega^4$ and $\partial\bar{\partial}\omega^{\bar{4}}$ are forms of degree 3 on the Lie algebra $\mathfrak{h}_{26}^+$, and $\bar{\partial}\omega^4 \wedge \partial\omega^{\bar{4}}$ and $\partial\omega^4 \wedge \bar{\partial}\omega^{\bar{4}}$ are forms of degree 4 on $\mathfrak{h}_{26}^+$. Finally, notice that the last equality of (10) makes use of the definition of $c_1(\Omega)$ given in (1).

From (10) and taking into account (11), it follows that F is 1-st Gauduchon if and only if

$$\Omega \wedge \big(\bar{\partial}\omega^4 \wedge \partial\omega^{\bar{4}} - \partial\omega^4 \wedge \bar{\partial}\omega^{\bar{4}}\big) - 4\,c_1(\Omega)\,\Omega^3 = 0. \tag{12}$$

The value of the constant $c_1(\Omega)$ is given by Proposition 2.7 (iii), i.e. $c_1(\Omega) = \frac{s^2+t^2}{6\,i\,\det(x_{k\bar{l}})}$. A long but direct calculation shows that condition (12) is equivalent to

$$0 = r + 2\,s + t\,|A|^2 - 2\,\delta\,\mathfrak{Im}(x_{1\bar{2}}) + \sqrt{2}\,\mathfrak{Im}(x_{1\bar{3}}\,\bar{A}) + \sqrt{2}\,\mathfrak{Im}(x_{2\bar{3}}\,B) + 2\,s^2 + 2\,t^2.$$

We can now produce many explicit examples of 1-st Gauduchon nilmanifolds. For instance, if we choose $x_{1\bar{2}} = x_{1\bar{3}} = 0$ and $x_{2\bar{3}} = 1/\sqrt{2}$, then it is enough to make $b_2 = \mathfrak{Im}(B) = -r - 2\,s - t\,|A|^2 - 2\,s^2 - 2\,t^2$ to solve the previous equation. Taking the metric coefficients r, s, t in $\mathbb{Q}^+$, we ensure that b_2 is a rational number. Thus, this 1-st Gauduchon metric F has the form

$$F = i\,r\,\omega^{1\bar{1}} + i\,s\,\omega^{2\bar{2}} + i\,t\,\omega^{3\bar{3}} + \frac{i}{2}\,\omega^{4\bar{4}} + \frac{1}{\sqrt{2}}\,\omega^{2\bar{3}} - \frac{1}{\sqrt{2}}\,\omega^{3\bar{2}},$$

where r, s, t belong to $\mathbb{Q}^+$, and $st > 1/2$ in order to ensure that F is positive definite. The associated complex nilmanifold corresponds to the complex structure equations (9) where $a_1, a_2, b_1 \in \mathbb{Q}$ and $b_2 = -r - 2\,s - t\,(a_1^2 + a_2^2) - 2\,s^2 - 2\,t^2$. □

Remark 3.3. Notice that the result in Proposition 3.2 is optimal, in the sense that we have obtained 1-st Gauduchon metrics on complex nilmanifolds with the highest possible nilpotency step. In fact, by [14] (see also [2]) filiform Lie algebras, i.e. nilpotent Lie algebras with maximal nilpotency step, never admit a complex structure; and by [12] an 8-dimensional quasi-filiform nilpotent Lie algebra, i.e. with nilpotency step $s = 6$, cannot admit any complex structure. Hence, the Lie algebra underlying a complex nilmanifold of complex dimension 4 must be nilpotent in step $s \leq 5$.

Remark 3.4. In [16, Proposition 2.2] it is proved that if an invariant Hermitian metric F on a complex nilmanifold X of complex dimension $n \geq 4$ is k-th Gauduchon for some $1 \leq k \leq n-2$, then F is k-th Gauduchon for any other k. Notice also that any invariant Hermitian metric is $(n-1)$-Gauduchon. Therefore, all the 1-st Gauduchon metrics constructed in this paper are also k-th Gauduchon for every $2 \leq k \leq n-1$.

In the following result we summarize the main differences that we have obtained between the SKT and the generalized Gauduchon geometries of nilmanifolds in complex dimension ≥ 4.

Theorem 3.5. *For $3 \leq s \leq 5$ and for any $n \geq 4$, there exists an s-step nilmanifold of complex dimension n admitting invariant k-th Gauduchon metrics for every $1 \leq k \leq n-1$. Furthermore, for $s = 3$ or $s = 4$ there exist such nilmanifolds with the additional property that the center of their underlying Lie algebras is not J-invariant.*

Proof. Let (X, F) be a 1-st Gauduchon nilmanifold constructed in Propositions 3.1 or 3.2. Let X' be any other nilmanifold with an invariant 1-st Gauduchon metric F' (for instance, a torus of complex dimension ≥ 1). By Proposition 2.3, the nilmanifold $Y = X \times X'$ has invariant 1-st Gauduchon metrics. Since X has nilpotency step $3 \leq s \leq 5$, the nilmanifold Y is at least s-step, too. When (X, F) is given by Proposition 3.1, one can always ensure that the center of the Lie algebra underlying Y is not J-invariant (although it might contain a non-trivial J-invariant ideal, depending on the complex geometry of the factor X').

Finally, by Remark 3.4, the invariant 1-st Gauduchon metrics on Y are also k-th Gauduchon for every $1 \leq k \leq n-1$, where n is the complex dimension of Y. □

Observe that the nilmanifolds in Theorem 3.5 do not admit any (invariant or not) SKT metric, as a consequence of [4].

Acknowledgments

The authors would like to thank the referee for interesting comments that have improved the final version of this paper. This work has been partially supported by the projects MINECO (Spain) MTM2014-58616-P and Gobierno de Aragón/Fondo Social Europeo, grupo consolidado E15-Geometría.

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