A short note on “Group theoretic approach to rationally extended shape invariant potentials” [Ann. Phys. 359 (2015) 46–54]

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Abstract

It is proved the equivalence of the compatibility condition of [A. Ramos, J. Phys. A 44 (2011) 342001, Phys. Lett. A 376 (2012) 3499] with a condition found in [Yadav et al., Ann. Phys. 359 (2015) 46]. The link of Shape Invariance with the existence of a Potential Algebra is reinforced for the rationally extended Shape Invariant potentials. Some examples on X_1 and X_ℓ Jacobi and Laguerre cases are given.

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1. Introduction

The concept of Shape Invariance has by now a long tradition in Quantum Mechanics, in the search of exactly solvable potentials. It started in the classical

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work of Infeld and Hull [1] or even in the seminal works of Schrödinger himself [2, 3, 4]. Later, the concept was reformulated by Gendenshteïn and Krive [5, 6] as it is known today. See the relatively recent monographes [7, 8] for an overview. The list of Shape Invariant potentials remained unchanged until Gómez-Ullate, Kamran and Milson [9] realized that the classical orthogonal polynomials can be generalized to a situation in which the lowest degree of the polynomials of the family need not to be zero. This fostered the key developments of Quesne and collaborators [10, 11, 12, 13, 14, 15], who have shown that it is possible to rationally extend some types of the standard Shape Invariant potentials in order to give isospectral ones. This line of research has been followed by many authors, for example important contributions by Grandati and collaborators [16, 17, 18, 19, 20, 21, 22] and by Odake and Sasaki [23, 24, 25, 26, 27, 28]. On his side, Ramos [29, 30] has found a compatibility condition that the new extended Shape Invariant potentials have to satisfy. Apart from that, in [31] it has been considered the complex Lie algebra $sl(2, \mathbb{C})$ when dealing with non-Hermitian Hamiltonians with real eigenvalues. Later on, in [32] a group theoretical approach to some extended Shape Invariant potentials has been developed, in which another condition has to be satisfied by the seed superpotential and the functions defining the extension. This technique was further employed by Yadav et al. in [33]. However, they did not discuss the relation of their condition with the previously mentioned compatibility condition of [29, 30].

The aim of this short note is to show that the compatibility condition of [29, 30] and the condition of [32] are indeed equivalent. This has the important consequence that the former compatibility condition is given in this way a group theoretical sense, and that the overall picture becomes unified in a common setting. In Section 2 we prove the mentioned equivalence. In Section 3 we provide some examples. In the final section we provide an Outlook and some Conclusions.

2. Equivalence of the conditions

2.1. The compatibility condition

We briefly recall here the compatibility condition approach of [29, 30]. Given a superpotential of the type

$$W(x, m) = W_0(x, m) + W_{1+}(x, m) - W_{1-}(x, m) \tag{1}$$

where x is the coordinate of the problem under study, m is a parameter that is transformed by translation (by $f(m) = m - 1$, without loss of generality), and

$W_0(x,m) = k_0(x) + mk_1(x)$ is a superpotential of the affine in m type treated by Infeld and Hull [1]. $W_{1+}(x,m)$, $W_{1-}(x,m)$ are logarithmic derivatives which moreover satisfy

$$W_{1-}(x,m) = W_{1+}(x,m-1) \tag{2}$$

In [30, Theorem 1] it has been proved that the superpotential (1) defines a Shape Invariant pair of partner potentials through the usual Riccati equations if and only if it is satisfied

$$\begin{aligned}
&W_{1+}^2(x,m) + W_{1+}'(x,m) + W_{1-}^2(x,m) + W_{1-}'(x,m) \\
&-2W_0(x,m)W_{1-}(x,m) \\
&+2W_0(x,m)W_{1+}(x,m) - 2W_{1-}(x,m)W_{1+}(x,m) = \epsilon(x)
\end{aligned} \tag{3}$$

where $\epsilon(x)$ is a function of x only. Since (3) holds for all allowed m's, in particular it holds as well for $m-1$.

2.2. *The group theory approach*

On its side, Yadav et al. [32] have developed a group theoretical approach to some rationally extended Shape Invariant potentials. They were inspired by the well-known paper of Wu and Alhassid [34] and in this way not all the possible cases of affine in m Shape Invariant potentials are considered. Therefore, in this paper we consider a slightly more general approach inspired by Miller [35] combined with the previous two papers.

That is, we now consider the $G(a,b)$ Potential Algebra by means of the operators

$$\begin{aligned}
J_\pm &= \mathrm{e}^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - \left(\left(-i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)F(x) - G(x)\right)\right. \\
&\qquad \left. -U\left(x, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)\right]
\end{aligned} \tag{4}$$

$$J_3 = -i\frac{\partial}{\partial\phi} \tag{5}$$

$$E = 1 \tag{6}$$

where $F(x), G(x)$ are functions of x only, a and b are real numbers, and $U\left(x, -i\frac{\partial}{\partial\phi} \pm \frac{1}{2}\right)$ is a functional operator. The commutation relations

$$[J_+, E] = 0\,, \quad [J_-, E] = 0\,, \quad [J_3, E] = 0\,, \tag{7}$$

$$[J_3, J_+] = J_+\,, \quad [J_3, J_-] = -J_-\,, \tag{8}$$

are satisfied automatically. The commutation relation

$$[J_+, J_-] = -2aJ_3 - 2bE \tag{9}$$

is satisfied (when evaluated in a basis of eigenfunctions of the Casimir [35] $C = aJ_3^2 - aJ_3 + 2bJ_3E - J_+J_- = aJ_3^2 + aJ_3 + 2bJ_3E + 2bE - J_-J_+$ and J_3, $\psi_{cm}(x)\mathrm{e}^{im\phi}$) if and only if three conditions do. The first two are analogous to the conditions of [35], and the third one is an additional condition. They are:

$$F'(x) + F^2(x) = a\,, \quad G'(x) + F(x)G(x) = b \tag{10}$$

and

$$\begin{aligned}
&U^2\left(x, m-\frac{1}{2}\right) - U'\left(x, m-\frac{1}{2}\right)\\
&+2U\left(x, m-\frac{1}{2}\right)\left(F(x)\left(m-\frac{1}{2}\right) - G(x)\right)\\
&-U^2\left(x, m+\frac{1}{2}\right) - U'\left(x, m+\frac{1}{2}\right)\\
&-2U\left(x, m+\frac{1}{2}\right)\left(F(x)\left(m+\frac{1}{2}\right) - G(x)\right) = 0
\end{aligned} \tag{11}$$

2.3. *Relation between the two approaches*

The aim of this note is to establish the relation between (11) and (2), (3). So let us start from the left hand side of relation (11). Performing a change of parameter m, without loss of generality, $m \to m - \frac{1}{2}$, we obtain

$$\begin{aligned}
&U^2(x, m-1) - 2G(x)(U(x, m-1) - U(x, m)) - U^2(x, m)\\
&+2F(x)((m-1)U(x, m-1) - mU(x, m))\\
&-U'(x, m-1) - U'(x, m)
\end{aligned} \tag{12}$$

Then, linking the notations in the Subsection 2.1 with those of Subsection 2.2 in the following way (see eqs. (34) and (35) in [32])

$$F(x) = k_1(x) \tag{13}$$
$$G(x) = -k_0(x) \tag{14}$$
$$U(x, m) = W_{1+}(x, m) - W_{1-}(x, m) \tag{15}$$
$$U(x, m-1) = W_{1+}(x, m-1) - W_{1-}(x, m-1) \tag{16}$$

we obtain

$$\begin{aligned}
&-2(k_0(x)+(m-1)k_1(x))(W_{1-}(x,m-1)-W_{1+}(x,m-1))\\
&+(W_{1-}(x,m-1)-W_{1+}(x,m-1))^2\\
&+2(k_0(x)+mk_1(x))(W_{1-}(x,m)-W_{1+}(x,m))-(W_{1-}(x,m)-W_{1+}(x,m))^2\\
&+W'_{1-}(x,m-1)+W'_{1-}(x,m)-W'_{1+}(x,m-1)-W'_{1+}(x,m) \qquad (17)
\end{aligned}$$

Then, using (3) for m and $m-1$, the expression simply reduces to

$$-2W'_{1+}(x,m-1)+2W'_{1-}(x,m) \qquad (18)$$

which, by virtue of (2), vanishes identically. The steps can be reversed in a natural way so it is proved the equivalence of (2), (3) with (11).

3. Examples

In this Section we illustrate the applicability of the previous relation in several instances, including extensions of Shape Invariant potentials with X_1 and X_ℓ Jacobi and Laguerre polynomials.

3.1. Case of $k_0(x)+mk_1(x)=-\frac{\beta}{c}\coth(cx)+\frac{d}{\sinh(cx)}+mc\coth(cx)$*,* X_1 *extension*

For this case $k_0(x)=-G(x)=-\frac{\beta}{c}\coth(cx)+\frac{d}{\sinh(cx)}$ and $k_1(x)=F(x)=c\coth(cx)$ where $x\in(0,\infty)$, $c>0$, β, d are constants. This example is a slight generalization of one appeared first in [11, 12] and is a slight correction of one appeared in [29]. We can take

$$W_{1+}(x,m) = \frac{2c^2d\sinh(cx)}{-2\beta+c^2(2m+1)+2cd\cosh(cx)} \qquad (19)$$

$$W_{1-}(x,m) = \frac{2c^2d\sinh(cx)}{-2\beta+c^2(2m-1)+2cd\cosh(cx)} \qquad (20)$$

and then with the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a=c^2$ and $b=\beta$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular if β is real, $d<0$ and $m<\frac{2\beta-c^2-2cd}{2c^2}$ or if $d>0$ when $m>\frac{2\beta+c^2-2cd}{2c^2}$.

3.2. *Case of* $k_0(x) + mk_1(x) = \frac{\omega x}{2} + \frac{d}{x} + \frac{m}{x}$*, radial oscillator with* X_1 *extension*

For this case $k_0(x) = -G(x) = \frac{\omega x}{2} + \frac{d}{x}$ and $k_1(x) = F(x) = \frac{1}{x}$ where $x \in (0, \infty)$, and $\omega > 0$, $d > 0$ are two constants. This example is a slight generalization of one appeared first in [10] and is a modification of one appeared in [29]. We can take

$$
\begin{aligned}
W_{1+}(x,m) &= -\frac{2\omega x}{1 + 2d + 2m - \omega x^2} && (21)\\
W_{1-}(x,m) &= -\frac{2\omega x}{-1 + 2d + 2m - \omega x^2} && (22)
\end{aligned}
$$

and then with the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a = 0$ and $b = -\omega$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular if $m < -\frac{1}{2}(1 + 2d)$.

3.3. *Case of* $k_0(x) + mk_1(x) = -\frac{\beta}{c}\tan(cx) + \frac{d}{\cos(cx)} - mc\tan(cx)$*,*$X_1$ *extension*

For this case $k_0(x) = -G(x) = -\frac{\beta}{c}\tan(cx) + \frac{d}{\cos(cx)}$ and $k_1(x) = F(x) = -c\tan(cx)$ where $x \in \left(-\frac{\pi}{2c}, \frac{\pi}{2c}\right)$, $c > 0$, β, d are constants. This example is a slight generalization of one appeared first in [10] and is a modification of one appeared in [29]. We can take

$$
\begin{aligned}
W_{1+}(x,m) &= -\frac{2c^2 d\cos(cx)}{2\beta + c^2(1 + 2m) - 2cd\sin(cx)} && (23)\\
W_{1-}(x,m) &= \frac{2c^2 d\cos(cx)}{-2\beta + c^2(1 - 2m) + 2cd\sin(cx)} && (24)
\end{aligned}
$$

and then with the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a = -c^2$ and $b = \beta$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular if β is real, $d > 0$ and $m < \frac{-2\beta - c^2 - 2cd}{2c^2}$ or if $d < 0$ when $m > \frac{2\beta + c^2 - 2cd}{2c^2}$, or also if β is real, $d > 0$ and $m > \frac{-2\beta + c^2 + 2cd}{2c^2}$ or if $d < 0$ when $m < \frac{-2\beta - c^2 + 2cd}{2c^2}$.

3.4. *Case of* $k_0(x) + mk_1(x) = -\frac{B}{\sinh(x)} + m\coth(x)$*, Generalized Pösch–Teller potential with* X_ℓ *extension*

For this case $k_0(x) = -G(x) = -\frac{B}{\sinh(x)}$ and $k_1(x) = F(x) = \coth(x)$ where $x \in (0, \infty)$, B is a real constant. This example appeared in [32], inspired in

[11, 12]. We can take

$$W_{1+}(x,\ell,m) = \frac{1}{2}(\ell-2B-1)\sinh(x)\frac{P_{\ell-1}^{(-B+m+1/2,-B-m-1/2)}(\cosh(x))}{P_{\ell}^{(-B+m-1/2,-B-m-3/2)}(\cosh(x))} \tag{25}$$

$$W_{1-}(x,\ell,m) = \frac{1}{2}(\ell-2B-1)\sinh(x)\frac{P_{\ell-1}^{(-B+m-1/2,-B-m+1/2)}(\cosh(x))}{P_{\ell}^{(-B+m-3/2,-B-m-1/2)}(\cosh(x))} \tag{26}$$

where $P_\ell^{(\alpha,\beta)}(x)$ denotes the ordinary ℓ-th Jacobi polynomial. Then, with the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a=1$ and $b=0$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular if $B<-\frac{1}{2}$ and $\frac{1}{2}(1+2B)<m<-\frac{1}{2}(1+2B)$ (with these conditions it is ensured that the roots of the Jacobi polynomials in the denominators above are on the interval $(-1,1)$ and then $\cosh(x)$ takes values in $[1,\infty)$).

3.5. Case of $k_0(x)+mk_1(x)=\frac{iB}{\cosh(x)}+m\tanh(x)$, PT symmetric complex Scarf-II with X_ℓ extension

For this case $k_0(x)=-G(x)=\frac{iB}{\cosh(x)}$ and $k_1(x)=F(x)=\tanh(x)$ where $x\in(-\infty,\infty)$, B is a real constant and i is the imaginary unit. This example appeared in [32], inspired in [11, 12, 31, 36]. We can take

$$W_{1+}(x,\ell,m) = \frac{1}{2}i(\ell-2B-1)\cosh(x)\frac{P_{\ell-1}^{(-B+m+1/2,-B-m-1/2)}(i\sinh(x))}{P_{\ell}^{(-B+m-1/2,-B-m-3/2)}(i\sinh(x))} \tag{27}$$

$$W_{1-}(x,\ell,m) = \frac{1}{2}i(\ell-2B-1)\cosh(x)\frac{P_{\ell-1}^{(-B+m-1/2,-B-m+1/2)}(i\sinh(x))}{P_{\ell}^{(-B+m-3/2,-B-m-1/2)}(i\sinh(x))} \tag{28}$$

Then, with the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a=1$ and $b=0$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular (except maybe at $x=0$) because the argument of the Jacobi polynomials above is purely imaginary.

3.6. *Case of* $k_0(x) + mk_1(x) = \frac{\omega x}{2} + \frac{m}{x}$*, radial oscillator with* X_ℓ *extension*

For this case $k_0(x) = -G(x) = \frac{\omega x}{2}$ and $k_1(x) = F(x) = \frac{1}{x}$ where $x \in (0, \infty)$ and $\omega > 0$. This example is a slight modification of one appeared first in [10, 23]. We can take

$$W_{1+}(x, \ell, m) = \omega x \frac{L_{\ell-1}^{(-m-1/2)}\left(-\frac{\omega x^2}{2}\right)}{L_{\ell}^{(-m-3/2)}\left(-\frac{\omega x^2}{2}\right)} \tag{29}$$

$$W_{1-}(x, \ell, m) = \omega x \frac{L_{\ell-1}^{(-m+1/2)}\left(-\frac{\omega x^2}{2}\right)}{L_{\ell}^{(-m-1/2)}\left(-\frac{\omega x^2}{2}\right)} \tag{30}$$

where now $L_\ell^{(\alpha)}(x)$ denotes the ℓ-th (associated) Laguerre polynomial. With the identifications above it is readily checked that (2), (3) are satisfied. Likewise, (10) is satisfied with $a = 0$ and $b = -\omega$, and (11) is also satisfied. This example leads to a pair of Shape Invariant partner potentials which are non-singular since the roots of the Laguerre polynomials of the denominators above lie in $(0, \infty)$, and we have taken explicitly negative arguments in them by means of $-\frac{\omega x^2}{2}$, see also [23].

4. Conclusions and outlook

We have demonstrated, in a general way and by means of examples, the validity of the equivalence of the compatibility conditions (2), (3) with the group theoretical condition (11). Thus both approaches are linked in a clear way. The first two relations establish the Shape Invariance condition of the by now well-known rationally extended potentials, and the last one is one of the conditions for the closing of the extended potential algebra $G(a, b)$, inspired by Miller [35]. Thus extended Shape Invariance is linked with the closing of a Potential Lie Algebra, initially being an approach known at least since the works [35, 34] for some of the classical cases of Infeld and Hull [1]. Thus the classical results are shown to be valid in a new situation. As a possible extension of the methods employed here, we could try to model rationally extended Shape Invariant potentials with two [23, 24, 25] or more parameters subject to translation with a Potential Algebra, using perhaps the insight of [37]. This is work to be done in another paper(s).

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