Ann Oper Res
DOI 10.1007/s10479-014-1637-z

Systemic decision making in AHP: a Bayesian approach

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Abstract *Systemic decision making* is a new approach for dealing with complex multiactor decision making problems in which the actors' individual preferences on a fixed set of alternatives are incorporated in a holistic view in accordance with the "principle of tolerance". The new approach integrates all the preferences, even if they are encapsulated in different individual theoretical models or approaches; the only requirement is that they must be expressed as some kind of probability distribution. In this paper, assuming the analytic hierarchy process (AHP) is the multicriteria technique employed to rank alternatives, the authors present a new methodology based on a Bayesian analysis for dealing with AHP systemic decision making in a local context (a single criterion). The approach integrates the individual visions of reality into a collective one by means of a *tolerance distribution*, which is defined as the weighted geometric mean of the individual preferences expressed as probability distributions. A mathematical justification of this distribution, a study of its statistical properties and a Monte Carlo method for drawing samples are also provided. The paper further presents a number of decisional tools for the evaluation of the acceptance of the tolerance distribution, the construction of tolerance paths that increase representativeness and the extraction of the relevant knowledge of the subjacent multiactor decisional process from a cognitive perspective. Finally, the proposed methodology is applied to the AHP-multiplicative model with lognormal errors and a case study related to a real-life experience in local participatory budgets for the Zaragoza City Council (Spain).

Keywords Multiactor decision making · Systemic decision making · Tolerance distribution · AHP · Bayesian inference · Participatory budgets

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Journal: **10479** Article No.: **1637** TYPESET DISK LE CP Disp.:**2014/5/29** Pages: **24** Layout: **Small**

1 Introduction

Some of the most significant characteristics of the knowledge society (KS) are: the participation and interdependencies of multiple actors; the consideration of intangible, subjective and emotional aspects; the interrelation between determinants; and the holistic vision of reality that is considered in decision making processes. This new societal context requires scientific approaches which provide an appropriate response to new needs and requirements, in particular, those needs associated with the key component of the Knowledge Society: the human factor in multiactor settings.

Moreno-Jiménez (2003a) and Escobar and Moreno-Jiménez (2007) identified three multiple actor decision making situations: (1) group decision making (GDM), (2) negotiated decision making (NDM); and (3) systemic decision making (SDM).

In the first situation (GDM), individuals work together in pursuit of a common goal under the *principle of consensus*. Consensus refers to the approach, model, tools, and procedures for deriving the final group priority vector. In the second situation (NDM), assuming that all the actors follow the same scientific approach, each individual resolves the problem separately, the zones of agreement and disagreement between the actors are identified and agreement paths (sometimes known as consensus paths) are constructed by changing one or several judgements. In the third situation (SDM), in accordance with the *principle of tolerance,* each individual acts independently and the individual preferences, expressed as probability distributions, are aggregated to form a collective one, denominated as the *tolerance distribution*. This new approach integrates all the preferences, even if they are encapsulated in different "individual theoretical models"; the only requirement is that they must be expressed as some kind of probability distribution. This means that the systemic situation for dealing with multiactor decision making allows the capturing of the holistic vision of reality and the subjacent ideas of lateral thinking (Bono 1970). The information provided by the tolerance distribution can be used to construct *tolerance paths* to gain a more democratic and representative final decision, that is to say, a decision will be accepted, by a greater number of actors or by a number of actors with greater weighting in the decisional process.

Due to its flexibility and adaptability in complex decision making contexts, one of the most widely used techniques in decisional processes involving multiple actors, scenarios and criteria is Saaty's analytic hierarchy process (AHP) (Saaty 1972, 1980). AHP contemplates the philosophical changes (from mechanistic reductionism to evolutionist holism), methodological changes (from the search for truth to the search for knowledge) and technological changes (from information communication to knowledge generation and diffusion) that have been taking place since the end of the twentieth century (Moreno-Jiménez 2003a; Altuzarra et al. 2007).

AHP methodology constructs an absolute scale associated with the priorities of the elements being compared. There are four steps: (1) Modelling - the decision making problem as a hierarchy in which criteria, subcriteria (several levels if necessary), attributes and alternatives are incorporated; (2) Valuation – the incorporation into the hierarchy of the individual preferences by means of the judgements elicited to fill the pairwise comparison matrices. The judgements belong to Saaty's fundamental scale (Saaty 1980); (3) Prioritisation of the elements of the hierarchy using any of the existing prioritisation procedures (local priorities) and the hierarchical composition principle (global priorities); (4) Synthesis of the global priorities of the alternatives to obtain their total or final priorities using an aggregation procedure. In contrast to other multicriteria techniques, AHP allows an assessment of inconsistency in the judgement elicitation process. Two of the most widely used procedures in the AHP literature are Saaty's Consistency Ratio (Saaty 1980) and the Geometric Consistency Index (Aguarón

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and Moreno-Jiménez 2003), used with the Eigenvector Prioritization Method (EGVM) and the Row Geometric Mean Method (RGMM), respectively.

With AHP-Group Decision Making (AHP-GDM), the two procedures conventionally employed to obtain the group priorities in a determinist context (Saaty 1980; Ramanathan and Ganesh 1994; Forman and Peniwati 1998) are: (1) the *Aggregation of Individual Judgements* (AIJ) and (2) the *Aggregation of Individual Priorities* (AIP). The first is used when the group works as a synergistic unit and the second when the group functions as a collective of individuals (Forman and Peniwati 1998). These traditional (deterministic) approaches and some more recent proposals for the stochastic context have been discussed in the literature: Altuzarra et al. (2007) presented a more efficient Bayesian prioritisation procedure for AHP-GDM, than (the commonly employed) AIJ and AIP; Escobar and Moreno-Jiménez (2007) developed the *Aggregation of Individual Preference Structures* (AIPS) which captures the vision and uncertainty of decision makers and the contextual interdependences of the alternatives. Other AHP-GDM approaches include: Goal Programming (Bryson 1996; Bryson and Jones 1999); Interval Judgements (Hämäläinen and Pöyhönen 1996); Stochastic Preference Modelling (Honert 1998); Fuzzy Preference Programming (Mikhailov 2004); Taguchi's Loss Function (Cho and Cho 2008); Nonlinear Least Squares Regression (Lipovetsky 2009); Linear Programming (Hosseinian et al. 2012); and the Dong et al. (2010) idea for two new AHP consensus models that improve original inconsistency and satisfy the Pareto Principle of Social Choice. A comparison of different AHP-GDM methods can be seen in Peniwati (2007), Saaty and Peniwati (2008) and Huang et al. (2009).

Using the property of consistency, Moreno-Jiménez et al. (2005, 2008) advanced a consensus searching decisional tool, the *Consistent Consensus Matrix* (CCM), which has been recently extended (*Precise Consistent Consensus Matrix*) in order to increase the number of entries considered in the CCM and the accuracy of the estimations (Aguarón et al. 2014).

There are also a number of approaches to AHP Negotiated Decision Making (AHP-NDM): Gargallo et al. (2007) put forward a Bayesian procedure based on the use of mixtures in cases with a large number of actors where a prior consensus is not required. They further developed graphic tools and clustering algorithms to identify homogeneous groups of actors with different patterns of behaviours for the priority rankings; Altuzarra et al. (2010), working in a local context and with a small number of actors, introduced a semi-automatic procedure for the search for consensus that works with complete and incomplete matrices. They use a hierarchical Bayesian regression linear model with log-normal errors and Monte Carlo Markov Chain (MCMC) methods to estimate the agreement priorities. In the same paper, these authors also advocate criteria for measuring the degree of agreement or compatibility between individual and collective priority vectors and use optimisation procedures based on genetic algorithms for developing consensus paths among the actors.

In the context of AHP-NDM: Honert and Lootsma (2000) developed the relative strength of the negotiating position of each of the bargaining parties; Hämäläinen's (2003) *Decisionarium* (http://www.decisionarium.hut.fi) is a public site for interactive multicriteria decision support with tools for individual decision making and group collaboration and negotiation; Bellucci and Zeleznikow (2005) Negotiation Decision Support Systems is based on the use of trade-off manipulations; Chen and Huang (2007) published a scheme aimed at the uncertainty and imprecision of identifying suitable supplier offers, evaluating the offers and choosing the best alternatives in bi-negotiation; and Altuzarra et al. (2013) have recently compiled a taxonomy for criteria, taking into account their influence and relevance in the final ranking of the alternatives.

In this paper, the authors consider the third and most original situation in the AHP context - AHP systemic decision making (AHP-SDM). The situation assumes that the actors indepen-

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dently elicit their judgements and the individual preferences within a fixed set of alternatives are given a type of probability distribution that reflects the intensity of the preferences. Once the actors' individual preferences are established, they look for a holistic decision, based on the principle of tolerance which attempts to link multiactor decision making with one of the main ideas of lateral thinking (Bono 1970): the parallel integration of the visions of reality of all the actors involved in the resolution process. This systemic decision making context is addressed by a Bayesian procedure similar to that which is considered by Altuzarra et al. (2007, 2010).

With the aim of reaching a joint position for the group, the first step is to define a tolerance distribution as the weighted geometric mean of the individual priorities distribution. The tolerance distribution allows the integration of the actors' vision of reality by minimising a weighted average of the *Kullback-Leibler distances* between it and each decision maker's individual priorities distribution. The statistical properties of this distribution are also examined and as it is not usually analytically tractable, the authors have designed an algorithm to draw samples, that will be used (from a cognitive perspective - Moreno-Jiménez et al. 2001) in the search for the relevant knowledge from the subjacent decision making process.

The remainder of this paper is structured as follows: Sect. 2 describes the problem, defines the group tolerance distribution and analyses its statistical properties; Sect. 3 presents decisional tools for exploiting (using a cognitive perspective) the information provided by the tolerance distribution; Sect. 4 applies the tools to the multiplicative model with lognormal errors conventionally used in the stochastic AHP; Sect. 5 illustrates the procedure with a case study; Sect. 6 sets out the main conclusions and offers some possibilities for future research.

2 Tolerance distribution

2.1 Problem formulation

Assuming a set of *n* alternatives $\{A_1, \ldots, A_n\}$ in a local context (a single criterion), let $\mathbf{D} = \{D_1, \ldots, D_K\}$ be a group of K decision makers ($K \geq 2$) and let D_0 be the supra decision maker (analyst) in charge of solving the problem. Let $\{\alpha_k; k = 1, \ldots, K, \alpha_k > 0; \sum_{k=1}^{K} \alpha_k = 1\}$ be a set of weights fixed by D_0 that reflects the relative importance of each decision maker $D_1, \ldots, D_K$ in the joint decision making process.

To solve the group decision making problem using AHP, the decision makers $\{D_1, \ldots, D_K\}$ express their preferences by means of K reciprocal pairwise comparison matrices $\{\mathbf{R}^{(k)}, k = 1, \ldots, K\}$. Without loss of generality and with the aim of simplifying the notation, it is assumed that $\mathbf{R}^{(k)}_{nxn} = \left(r^{(k)}_{ij}\right)$ is a complete reciprocal positive square matrix (nxn), where $r^{(k)}_{ii} = 1,\ r^{(k)}_{ji} = \frac{1}{r^{(k)}_{ij}} > 0$ for $i, j = 1, \ldots, n$.

The judgements $r^{(k)}_{ij}$ represent the relative preference between alternatives i and j for the decision maker D_k, according to Saaty's fundamental scale (Saaty 1980). Despite the fact that the "reference" points of the categories (equal, moderate, strong, very strong and extreme) used in this scale are a discrete set $\{1/9, \ldots, 1/2, 1, 2, \ldots, 9\}$, the judgements considered in this proposal belong to the continuous interval [1/9, 9].

Let $\left\{\mathbf{v}^{(k)} = \left(v^{(k)}_1, ..., v^{(k)}_n\right)'; k = 1, ..., K\right\}$, $\left(v^{(k)}_1 > 0, ..., v^{(k)}_n > 0\right)$ be the individual's (unnormalised) priorities of the alternatives for each decision maker and let $\left\{\mathbf{w}^{(k)} = \left(w^{(k)}_1\right.\right.$

$, ..., w_n^{(k)}\Big)'$; $k = 1, ..., K\}$ be their normalised values according to a distributive mode: $w_i^{(k)} = \frac{v_i^{(k)}}{\sum_{i=1}^{n} v_i^{(k)}}, i = 1, \ldots, n$ with $\sum_{i=1}^{n} w_i^{(k)} = 1, k = 1, \ldots, K$.

Let us adopt a stochastic approach for AHP, and assume that the judgements $(r_{ij}^{(k)})$ elicited by the decision makers $D_k, k = 1, \ldots, K$ can be described by means of general Bayesian models

$$g_k\left(\mathbf{r}^{(k)}, \mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right) = f_k\left(\mathbf{r}^{(k)}|\mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right) \pi_k\left(\mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right), \ k = 1, \ldots, K \qquad (1)$$

where $\mathbf{r}^{(k)} = \left(r_{ij}^{(k)}; 1 \le i < j \le n\right)'$ is the judgements vector, $f_k\left(\mathbf{r}^{(k)}|\mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right)$ is the likelihood function of the model, $\mathbf{w}^{(k)}$ is the priorities vector of decision maker D_k, $\boldsymbol{\theta}^{(k)}$ is a vector of nuisance parameters (usually related to the inconsistency level of each decision maker, see Sect. 4), $\pi_k\left(\mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right)$ is the prior distribution of these parameters and $g_k\left(\mathbf{r}^{(k)}, \mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right)$ the joint distribution of judgements and parameters.

Applying Bayes Theorem, the inferences about the priority vectors $\mathbf{w}^{(k)}$ would be made from their posterior distribution given by the expression:

$$\pi_k\left(\mathbf{w}^{(k)}|\mathbf{r}^{(k)}\right) = \frac{\int g_k\left(\mathbf{r}^{(k)}, \mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right) d\boldsymbol{\theta}^{(k)}}{\int g_k\left(\mathbf{r}^{(k)}, \mathbf{w}^{(k)}, \boldsymbol{\theta}^{(k)}\right) d\mathbf{w}^{(k)} d\boldsymbol{\theta}^{(k)}}; \ k = 1, \ldots, K \qquad (2)$$

Note that if some of the matrices $\mathbf{R}^{(k)}$ are incomplete, the mathematical calculus should be modified in an appropriate manner, taking into account that the posterior distribution (2) must be proper.

Distribution (2) contains, for each decision maker D_k, the relevant information on the priorities, $\mathbf{w}^{(k)}$, which reflects their preferences on the alternatives $\{A_1, \ldots, A_n\}$ of the problem. From this distribution, point estimations and Bayesian credibility intervals of $\mathbf{w}^{(k)}$ can be calculated, respectively, by using the posterior mean or median of the components and the appropriate quantiles. Furthermore, using Roy's decisional problem taxonomy (Roy 1985), [2]
inference about the best alternative (P.α problem), the second best (P.α 2) problem), the two best alternatives (P.α 1, 2) problem) and the preferred preference structure (P.γ problem) can be made using their corresponding posterior distributions and the posterior probabilities of rank reversal can also be obtained (Altuzarra et al. 2010, 2013).

The information about the relevant aspects of the decision making process allows the extraction of the knowledge from the cognitive perspective that are followed in the resolution of the problem (Moreno-Jiménez et al. 2001; Moreno-Jiménez 2003a). This information can also be very useful to initiate a subsequent tolerance process that concludes with a collective decision accepted by the majority of the actors involved in the resolution process. In the following section the tolerance distribution is defined and its properties are analysed.

2.2 Tolerance distribution for a set of decision makers

In order to solve the decision problem, it is assumed that D_0 acts under a principle of tolerance where a permissive and democratic attitude toward the different visions and preferences of decision makers in $\mathbf{D}$ (expressed by their individual distributions $\{\pi_k; k = 1, \ldots, K\}$) is adopted. Therefore, a collective probability distribution which highlights the priority vectors $\mathbf{w}$ that are well supported, i.e. have a non-negligible density value $\pi_k(\mathbf{w})$, for all the members of the collective is sought and the following definition is introduced:

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Definition 2.1 The Tolerance Distribution for **D** is defined as the probability distribution given by:

$$\pi_{tol}\left(\mathbf{w}\,\middle|\{\pi_k\}_{k=1}^{K}\right) \alpha \prod_{k=1}^{K}\left[\pi_k\left(\mathbf{w}\right)\right]^{\alpha_k} \tag{3}$$

where $\pi_k(\mathbf{w}) = \pi_k\left(\mathbf{w}|\mathbf{r}^{(k)}\right)$ for k = 1,…,K. □

The following proposition proves that the tolerance distribution is well defined.

Proposition 2.1 *Assuming that* $\{\pi_k(\mathbf{w});\ k = 1, \ldots, K\}$ *are proper probability distributions with their respective supports* $\mathrm{SUPP}_k \subseteq S_n = \left\{\mathbf{w} = (w_1, \ldots, w_n)' : w_i \geq 0;\ i = 1, \ldots, n;\ \sum_{i=1}^{n} w_i = 1\right\}$*; and to avoid dogmatic positions among the decision makers of* **D***, that SUPP* $= \bigcap_{k=1}^{K}$ *SUPP*$_k$ *is not a null measure set, then the tolerance distribution is proper and its support is SUPP.*

Proof It is sufficient to show that this is a density function; firstly, it is not negative because each density $\{\pi_k(\mathbf{w});\ k = 1, \ldots, K\}$ is not negative, and SUPP ≠ Ø because it is not null measure. In addition, it is a proper density (Davidson 1994: Corollary 9.26) as:

$$0 < \int \prod_{k=1}^{K}\left[\pi_k\left(\mathbf{w}\right)\right]^{\alpha_k} d\mathbf{w} \ \leq\ \prod_{k=1}^{K}\left(\int \pi_k\left(\mathbf{w}\right) d\mathbf{w}\right)^{\alpha_k} = 1$$

□

Remark 2.1 The tolerance distribution aims to incorporate the opinion of all the actors implicated in the resolution process. The density of the tolerance distribution π_{tol} will be higher for those priority vectors **w** that are well supported, i.e. have a non-negligible density value $\pi_k(\mathbf{w})$, for all the members of the collective. In contrast, if a priority vector **w** is rejected by at least one of the actors (i.e. $\pi_k(\mathbf{w}) \approx 0$ for at least one k) then **w** will tend to be rejected by the tolerance distribution even though **w** will be well supported by the rest of the collective. The tolerance distribution will provide a probability distribution that is more democratic and in accordance with the tolerance principle, by highlighting those **w** where there is a greater probability of reaching a final agreement for all the members of **D**. □

Furthermore, the tolerance distribution is a synthesis (weighted geometric mean) of the individual preferences of the decision makers of **D**, which is optimal in the following sense.

Definition 2.2 Let $\pi(\mathbf{w})$ and $\{\pi_k(\mathbf{w});\ k = 1, \ldots, K\}$ be a set of (1+K) probability distributions of **w**. The *Collective Kullback-Leibler (CKL) distance* is defined as the distance between d and the set $\{\pi_k(\mathbf{w});\ k = 1, \ldots, K\}$ as the weighted arithmetic mean of the individual KL distances given by:

$$\mathrm{CKL}\left(\pi\ \{\pi_k\}_{k=1}^{K}\right) = \mathrm{D}\left(\pi\ \{\pi_k\}_{k=1}^{K}\right) = \sum_{k=1}^{K} \alpha_k\ \mathrm{KL}\left(\pi, \pi_k\right), \tag{4}$$

where $\mathrm{KL}(\pi, \pi_k) = \int \log\left(\frac{\pi(\mathbf{w})}{\pi_k(\mathbf{w})}\right)\pi(\mathbf{w})d\mathbf{w}$ is the Kullback-Leibler distance between π and π_k, $k = 1, \ldots, K$. □

Theorem 2.1 *The tolerance distribution* π_{tol} *defined in (3) minimises the CKL distance (4).*

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Proof Given that

$$\mathrm{CKL}(\pi\,\{\pi_k\}_{k=1}^{K}) = \sum_{k=1}^{K} \int \log\left(\frac{[\pi(\mathbf{w})]^{\alpha_k}}{[\pi_k(\mathbf{w})]^{\alpha_k}}\right)\pi(\mathbf{w})\,d\mathbf{w} = \int \log\left(\frac{\prod_{k=1}^{K}[\pi(\mathbf{w})]^{\alpha_k}}{\prod_{k=1}^{K}[\pi_k(\mathbf{w})]^{\alpha_k}}\right)\pi(\mathbf{w})\,d\mathbf{w} = \\ = \int \log\left(\frac{\pi(\mathbf{w})}{\prod_{k=1}^{K}[\pi_k(\mathbf{w})]^{\alpha_k}}\right)\pi(\mathbf{w})\,d\mathbf{w} = \mathrm{KL}(\pi, \pi_{tol}) + C \tag{5}$$

where $C = -\log\left(\int \prod_{k=1}^{K}[\pi_k(\mathbf{w})]^{\alpha_k}\right) d\mathbf{w}$ does not depend on d. From (5), it follows that $\mathrm{Min}_\pi\ \mathrm{CKL}(\pi, \{\pi_k\}_{k=1}^{K}) \equiv \mathrm{Min}_\pi \mathrm{KL}(\pi, \pi_{tol}) = \mathrm{KL}(\pi_{tol}, \pi_{tol}) = 0.$ □

Remark 2.2 The *CKL distance* (4) adopts the point of view of a supra decision maker who looks to integrate the preferences of all the decision makers $\{D_k; k = 1, \ldots, K\}$ under a principle of tolerance (collective perspective). According to this principle (permissive attitude towards individual preferences), the CKL distance takes the collective distribution d as the anchor with respect to the individual distributions $\{\pi_k\}_{k=1}^{K}$ that are compared. This, and the fact that the KL distance is not symmetric, justify that the selected KL distance was $\mathrm{KL}(\pi, \pi_k)$ and not $\mathrm{KL}(\pi_k, \pi)$. The last distance adopts an individual perspective in the sense that each decision maker considers its individual distribution π_k as the anchor and compares the collective distribution π with respect to it. This favours the selection of collective distributions where the decision makers with greater influence will impose their opinions. In fact, if we consider the collective distance given by

$$\mathrm{CKL}_1\left(\{\pi_k\}_{k=1}^{K}, \pi\right) = D_1\left(\{\pi_k\}_{k=1}^{K}, \pi\right) = \sum_{k=1}^{K} \alpha_k\,\mathrm{KL}(\pi_k, \pi) \tag{6}$$

it can be proved that its minimum is achieved in the mixture $\pi = \sum_{k=1}^{K} \alpha_k\,\pi_k$ where the decision makers with larger weights α_k will be more determinant in the selection of the joint priority vector $\mathbf{w}$. □

To conclude this analysis of the tolerance distribution, it is worth mentioning that it is essentially unique and invariant to re-parameterisations of the priority vector $\mathbf{w}$, as shown by the following proposition:

Proposition 2.2 *Let* $\mathbf{v} = h(\mathbf{w})$ *be a one-to-one re-parameterisation of the priorities vector* $\mathbf{w}$. *Then*

$$\pi_{tol}\left(\mathbf{v} \middle| \{\pi_k\}_{k=1}^{K}\right) \propto \prod_{k=1}^{K}[\pi_k(\mathbf{v})]^{\alpha_k} \tag{7}$$

$\{\pi_k(\mathbf{v}); k = 1, \ldots, r\}$ *are the individual distributions obtained from the distributions (2) by the transformation* $\mathbf{v} = h(\mathbf{w})$.

Proof If $\left|\frac{d\mathbf{w}}{d\mathbf{v}}\right|$ denotes the Jacobian of the transformation $\mathbf{w} = h^{-1}(\mathbf{v})$ it is therefore verified that:

$$\pi_{tol}\left(\mathbf{v} \middle| \{\pi_k\}_{k=1}^{K}\right) \propto \pi_{tol}(\mathbf{w})\left|\frac{d\mathbf{w}}{d\mathbf{v}}\right| = \prod_{k=1}^{K}[\pi_k(\mathbf{w})]^{\alpha_k}\left|\frac{d\mathbf{w}}{d\mathbf{v}}\right| =$$

$$= \prod_{k=1}^{K}\left[\pi_k(\mathbf{w})\left|\frac{d\mathbf{w}}{d\mathbf{v}}\right|\right]^{\alpha_k} = \prod_{k=1}^{K}[\pi_k(\mathbf{v})]^{\alpha_k}$$

□

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3 Knowledge extraction from the tolerance distribution

As demonstrated in Sect. 2, the tolerance distribution provides a synthesis of the individual priority vector distributions and highlights the priority vectors that are compatible with the judgements elicited by the members of the group. For these reasons it seems logical to use it to construct decisional tools that favour the extraction of knowledge related with the scientific resolution of the decision problem. The following section describes several of these tools, depending on the problem that is to be resolved.

3.1 Selection of the best alternative

For the selection of the best alternative, known in the literature as the P.α problem (Roy 1985), it is possible to use the distribution of the most preferred alternative $A_{(1)}$, a discrete distribution with support $\{A_1, \ldots, A_n\}$ and a probability function given by:

$$
\begin{aligned}
P(A_{(1)} = A_i) &= P\left(w_i = \max_{1 \le j \le n} \{w_j\}\right) \\
&= \int\limits_{\{\mathbf{w}: w_i = \max_{1 \le j \le n}\{w_j\}\}} \pi_{tol}(\mathbf{w})\, d\mathbf{w};\ i = 1, \ldots, n \qquad (8)
\end{aligned}
$$

The best alternative will be that which maximises the probabilities (8).

3.2 Selection of the k-best alternatives

Generalising the previous idea (8), the k most preferred alternatives can be determined by using the joint distribution of the k first alternatives $(A_{(1)}, A_{(2)}, \ldots, A_{(k)})$ where $A_{(j)}$ denotes the j-th most preferred alternative for j = 1,…, k. In particular, taking k = n the distribution of the preference structures (Moreno-Jiménez and Vargas 1993) used to select the most preferred ranking of alternatives can also be determined; a problem that is known in the literature as a gamma type problem or P.γ problem.

These distributions can be employed for the analysis of the most preferred and the most rejected alternatives and this is information that can be very valuable for designing strategies (tolerance paths) to achieve more democratic or representative decision processes.

3.3 Pairwise dominance probability matrix

The Pairwise Dominance Probabilities Matrix (PDPM) given by Altuzarra et al. (2013) can be very useful for analysing the knowledge extraction process:

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$$\begin{aligned} &P(A_i \succ A_j) = P\left(w_i > w_j\right) + \tfrac{1}{2}P\left(w_i = w_j\right) = \\ &= \int\limits_{\{\mathbf{w}:w_i>w_j\}} \pi_{tol}\left(\mathbf{w}\right) d\mathbf{w} + \frac{1}{2} \int\limits_{\{\mathbf{w}:w_i=w_j\}} \pi_{tol}\left(\mathbf{w}\right) d\mathbf{w};\ 1 \le i \ne j \le n \\ &P(Ai \succ Ai) = 1 \end{aligned} \tag{9}$$

where $A_i \succ A_j$ means "A_i is as least as preferred as A_j".

From these probabilities, the rankings of alternatives can be established that take into account, not only the two first positions, but also if they are located in any other places compatible with the dominance criterion "$\succ$" (Altuzarra et al. 2013). The consideration of this information will increase the robustness of the ranking that is ultimately selected. This information should also be used to evaluate the representativeness of the tolerance distribution.

4 Tolerance distribution in AHP multiplicative models with logarithmic-normal errors

This section contemplates the multiplicative model with logarithm-normal errors usually employed in the stochastic analysis of AHP (Ramsay 1977; Genest and Rivest 1994; Alho and Kangas 1997; Laininen and Hämäläinen 2003, Altuzarra et al. 2007, 2010) which will be used to illustrate the methodology described in the previous sections. However, it is worth noting that other kinds of Bayesian models can also be used, for example, the categorical data models proposed by Hahn (2003, 2006).

In this case, the individual models are given by the expressions:

$$r_{ij}^{(k)} = \frac{v_i^{(k)}}{v_j^{(k)}} e_{ij}^{(k)}, \quad i = 1, ..., n-1;\ j = i+1, ..., n;\ k = 1, ..., K \tag{10}$$

where we assume that $\left\{e_{ij}^{(k)}; i = 1, ..., n-1; j = i+1, ..., n; k = 1, ..., K\right\}$ are independent errors with $e_{ij}^{(k)} \sim LN\left(0, \sigma^{(k)2}\right)$, being $LN(\mu, \sigma^2)$ the log-normal distribution with location parameter μ and scale parameter σ^2.

Taking these logarithms, we have a regression model with normal errors given by the equations:

$$y_{ij}^{(k)} = \mu_i^{(k)} - \mu_j^{(k)} + \varepsilon_{ij}^{(k)}; i = 1, \ldots, n-1;\ j = i+1, \ldots, n;\ k = 1, .., K \tag{11}$$

where $y_{ij}^{(k)} = \log\left(r_{ij}^{(k)}\right)$, $\mu_i^{(k)} = \log\left(v_i^{(k)}\right)$ and $\varepsilon_{ij}^{(k)} = \log\left(e_{ij}^{(k)}\right) \sim N\left(0, \sigma^{(k)2}\right)$ for $k = 1, \ldots, K$. In addition, and in order to avoid identification problems, we take $\mu_n = 0$, that is to say, we take A_n as a reference alternative.

Let $\mathbf{y}^{(k)} = \left(y_{12}^{(k)}, y_{13}^{(k)}, ..., y_{n-1n}^{(k)}\right)'$ be the vector of judgements elicited by the decision maker D_k, $k = 1, \ldots, K$, and let $J = \frac{n(n-1)}{2}$ be the number of these judgements.

Let $\mathbf{X} = (x_{ij})$ be the $Jx(n-1)$ matrix in such a way that if the i^{th} component of these vectors $\{\mathbf{y}^{(k)}; k = 1, \ldots, K\}$ corresponds to the comparison among alternatives A_j and A_ℓ with $1 \le j < \ell < m$ then $x_{ij} = 1$, $x_{i\ell} = -1$ and $x_{is} = 0$ for $s \ne j, \ell$, and if the i^{th} component corresponds to a comparison between the alternatives $A_j 1 \le j < n$ and A_n, then $x_{ij} = 1$ and $x_{is} = 0$ for $s \ne j$.

Equation (11) can be written in a matrix form as:

$$\mathbf{y}^{(k)} = \mathbf{X}\boldsymbol{\mu}^{(k)} + \overset{(k)}{\varepsilon};\ k = 1, \ldots, K \tag{12}$$

with $\boldsymbol{\varepsilon}^{(k)} = \left(\varepsilon_{12}^{(k)}, \varepsilon_{13}^{(k)}, ..., \varepsilon_{n-1n}^{(k)}\right)' \sim N_J\left(\mathbf{0}_J, \sigma^{(k)2}\,\mathbf{I}_J\right)$ and $\mathbf{I}_J$ is the JxJ identity matrix.

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It must be decided if the error variances are known or unknown. In the first case, it is possible to calculate exactly the tolerance distribution, whilst in the second case, the tolerance distribution is analytically intractable and Monte Carlo methods are employed. A general procedure to obtain a sample of this distribution is provided below.

4.1 Tolerance distribution with known variances

If the variances of the error terms $\{\sigma^{(1)2}, \ldots, \sigma^{(K)2}\}$ are known, and we take the non-informative uniform distribution in $\mathbf{R}^{n-1}$ as the prior distribution on $\boldsymbol{\mu}^{(k)} = \left(\mu_1^{(k)}, \ldots, \mu_{n-1}^{(k)}\right)'$ (Gelman et al. 2004; Altuzarra et al. 2007), the posterior distributions of $\{\boldsymbol{\mu}^{(k)}; k = 1, \ldots, K\}$ are given by:

$$\boldsymbol{\mu}^{(k)}|\mathbf{y}^{(k)} \sim N_{n-1}\left(\hat{\boldsymbol{\mu}}^{(k)}, \sigma^{(k)2}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right) \tag{13}$$

where $\hat{\boldsymbol{\mu}}^{(k)} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{y}^{(k)}\right)$.

Using standard calculus and Proposition 2.2 ($\boldsymbol{\mu} = h(\mathbf{w}) = \log \mathbf{w}$), the tolerance distribution (3) will be given by:

$$\pi_{tol}(\mu) \propto \prod_{k=1}^{K} [\pi_k(\mu)]^{\alpha_k} \sim N_{n-1}\left(\hat{\boldsymbol{\mu}}, \hat{\sigma}^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right) \tag{14}$$

where $\pi_k(\mu)$ is given by (4.4) and $\hat{\boldsymbol{\mu}} = \frac{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma^{(k)2}} \hat{\boldsymbol{\mu}}^{(k)}}{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma^{(k)2}}}$ and $\hat{\sigma}^2 = \frac{1}{\sum_{k=1}^{K} \frac{\alpha_k}{\sigma^{(k)2}}}$.

Altuzarra et al. (2007) proved that $\hat{\boldsymbol{\mu}}$ (the posterior mean of the tolerance distribution of the parameter $\boldsymbol{\mu}$) behaves better in terms of the mean square estimation error than the estimators of $\boldsymbol{\mu}$ applying the aggregation of individual judgements (AIJ) and the aggregation of individual priorities (AIP) procedures traditionally considered in the literature.

Using (14) it is possible to make inferences about $\mathbf{w}$, as described in Sect. 2.1, and to calculate the probabilities presented in Sect. 3.

4.2 Tolerance distribution with unknown variances

Assuming the non-informative uniform distribution in $\mathbf{R}^{n-1}$ as the prior distribution on $\boldsymbol{\mu}^{(k)} = \left(\mu_1^{(k)}, \ldots, \mu_{n-1}^{(k)}\right)'$, and taking as prior distributions for the precisions "$\tau^{(k)}; k = 1, \ldots, K$ the usual conjugates given by:

$$\tau^{(k)} = \frac{1}{\sigma^{2(k)}} \sim \text{Gamma}\left(\frac{n_0}{2}, \frac{n_0 s_0^2}{2}\right) \quad k = 1, \ldots, K \text{ with } n_0,\ s_0^2 > 0 \tag{15}$$

with n_0 small in order to make it diffuse and s_0^2 equal to the desirable values of the inconsistency levels (Genest and Rivest 1994).

Standard calculations show that the individual posterior distributions are given by:

$$\tau^{(k)}|\mathbf{y}^{(k)} \sim \text{Gamma}\left(\frac{n_0 + J - n + 1}{2}, \frac{(n_0 + J - n + 1)s^{2(k)}}{2}\right) \tag{16}$$

$$\boldsymbol{\mu}^{(k)}|y^{(k)} \sim T_{n-1}\left(\hat{\boldsymbol{\mu}}^{(k)}, s^{2(k)}\left(\mathbf{X}'\mathbf{X}\right)^{-1}, n_0 + J - n + 1\right), k = 1, \ldots, K \text{ independents}$$

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where

$$\hat{\mu}^{(k)} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{y}^{(k)}\right), \quad s^{2(k)} = \frac{n_0 s_0^2 + \left(\mathbf{y}^{(k)} - \mathbf{X}\hat{\mu}^{(k)}\right)'\left(\mathbf{y}^{(k)} - \mathbf{X}\hat{\mu}^{(k)}\right)}{n_0 + J - n + 1}$$

and $T_n(\mu, \sigma^2, \nu)$ denotes the multivariate n-dimensional T of Student[1] with location parameter μ, scale parameter σ^2 and ν degrees of freedom.

Taking into account (16), the tolerance distribution will be given by:

$$\pi_{tol}(\mu) \propto \prod_{k=1}^{K}\left[\pi_k\left(\mu|y^{(k)}\right)\right]^{\alpha_k} = \prod_{k=1}^{K}\left[T_{n-1}\left(\hat{\mu}^{(k)}, s^{2(k)}\left(\mathbf{X}'\mathbf{X}\right)^{-1}, n_0 + \text{J-n} + 1\right)(\mu)\right]^{\alpha_k} \tag{17}$$

This distribution is not a standard form and it is necessary to use Monte Carlo methods to calculate it. A general algorithm to solve this situation follows.

4.2.1 Algorithm to draw a sample from the tolerance distribution

This section describes a general procedure for obtaining a sample of the tolerance distribution. The procedure can be used when it is necessary to calculate analytically intractable probabilities, posterior moments, quantiles, etc. and it is possible to draw samples from the individual distributions $\{\pi_k(\mathbf{w}); k = 1, \ldots, K\}$. The process uses importance sampling and, more specifically, the sampling-importance re-sampling procedure or SIR (Rubin 1987), taking the mixture $\sum_{k=1}^{K} \alpha_k\, \pi_k(\mathbf{w})$ as an importance distribution. Note that this distribution has heavier tails than the tolerance distribution (3) and, therefore, the asymptotic results of Geweke (1989) can be applied.

Algorithm 1 Extraction of samples from the tolerance distribution

Step 0 Fix the number of simulations (S) and the number of samples (S')

Step 1 Draw S' samples ($S' >> S$), $\{\mathbf{u}^{(s)}; s = 1, \ldots, S'\}$, from the mixture $\sum_{k=1}^{K} \alpha_k\, \pi_k(\mathbf{w})$ using, for example, a composition method.

Step 2 Assign importance weights $\{\beta^{(s)}; s = 1, \ldots, S'\}$ to the sample $\{\mathbf{u}^{(s)}; s = 1, \ldots, S'\}$ where:

$$\beta^{(s)} = \frac{\prod_{k=1}^{K}\left[\pi_k\left(\mathbf{u}^{(s)}\right)\right]^{\alpha_k}}{\sum_{k=1}^{K} \alpha_k\, \pi_k\left(\mathbf{u}^{(s)}\,\middle|\, r^{(k)}\right)}; \; s = 1, \ldots, S'$$

Step 3 Draw S samples $\{\mathbf{w}^{(s)}; s = 1, \ldots, S\}$ from the discrete distribution $\{(\mathbf{u}^{(s)}, p^{(s)}); s = 1, \ldots, KS\}$ with $p^{(s)} = \frac{\beta^{(s)}}{\sum_{i=1}^{S'} \beta^{(i)}}; s = 1, \ldots, S'$.

□

From these samples it is possible to make inferences about **w**, as explained in Sect. 2.1, and to calculate the probabilities presented in Sect. 3 using their corresponding Monte Carlo estimates.

[1] The stability of the priorities given by (16) against small judgement changes is guaranteed by having the T of Student with a reduced number of degrees of freedom (heavy-tailed distributions).

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Table 1 Pairwise comparison judgments for each decision maker

DM	Type	Weights (%)	r_{12}	r_{13}	r_{14}	r_{23}	r_{24}	r_{34}
D_1	Political	10	1	5	3	6	5	1
D_2	Political	10	7	4	4	1/5	1/5	2
D_3	Political	10	9	1	7	1/7	3	8
D_4	Political	10	7	2	7	1/5	1/5	5
D_5	Association	16	1/6	1/3	1/3	3	3	1
D_6	Association	16	1	1	1	3	3	1
D_7	Association	4	9	1/2	6	1/7	1	8
D_8	Association	4	2	9	9	9	8	1
D_9	Association	8	9	7	7	1/3	1/2	1
D_{10}	Citizen	4	1	4	1	5	5	1
D_{11}	Citizen	4	1/2	4	6	5	8	5
D_{12}	Citizen	4	4	9	9	9	9	1

5 Case study: e-participatory budgets

The methodology is applied to a case study, adapted from a real-life experience (http://www.zaragoza.es/presupuestosparticipativos/ElRabal/) developed by the "Zaragoza Multicriteria Decision Making Group" (GDMZ) for the Zaragoza City Council (Spain). The experience was based on a new democratic system, known as *e-cognocracy* (Moreno-Jiménez 2003b, 2006; Moreno-Jiménez and Polasek 2003), applied to an e-participatory budget allocation problem. The budget that the municipal district of El Rabal (Zaragoza) assigns to each one of four alternatives proposed by the Neighbourhood Associations and the Members of the District Council was determined by using AHP as the multicriteria methodological support and Internet as the communication tool for the extraction of the individuals' preferences. The four alternatives were ($n = 4$): A_1: the *Longares* Avenue tunnel; A_2: the renovation of *Puente del Pilar* Avenue; A_3: the shortening of *Pacuala Peire* Street; and A_4: the renovation of *Ignacio Zapata* Street. They were prioritised by taking into account a total of three criteria and six subcriteria.

The study contemplated the preferences elicited by 12 actors or decision makers (4 politicians, 5 representatives of neighbourhood associations and 3 citizens) with respect to one of the most important aspects of the problem (a local context[2]): the environmental subcriterion called "*Prevention*". A weighting was assigned to each decision maker, based on the number of citizens represented (the authors acted as the supra decision maker). The weightings and the pairwise preference judgements elicited by each of them are shown in Table 1. For each of the $K = 12$ decision makers, a 4x4 pairwise comparison matrix (six judgements) was obtained from the initial data. The matrices reflect the preferences of the actors between the four alternatives with respect to the single criterion (Prevention).

The methodology discussed in Sects. 2 and 3 was applied (assuming unknown variances) by taking $n_0 = 0.0001$ and $s_0 = 0.1$[3].

[2] Extension to a global context (hierarchy) will be the subject of a future paper.

[3] These values correspond to a diffuse prior centred on the level of inconsistency, as suggested by Genest and Rivest (1994).

Table 2 Estimations of three quantiles ($Q_{2.5}$, $Q_{50.0}$, $Q_{97.5}$) of the priorities and consistency levels

DM	w_1			w_2			w_3			w_4			Consistency[a]
	$Q_{2.5}$	Mean	$Q_{97.5}$	$Q_{2.5}$	Mean	$Q_{97.5}$	$Q_{2.5}$	Mean	$Q_{97.5}$	$Q_{2.5}$	Mean	$Q_{97.5}$	
D_1	0.0080	0.3564	0.8359	0.0283	0.4288	0.9093	0.0017	0.1029	0.4640	0.0021	0.1119	0.4393	0.0245
D_2	0.0476	0.5125	0.9257	0.0014	0.0679	0.3267	0.0125	0.2414	0.7076	0.0063	0.1782	0.6710	0.1319
D_3	0.0309	0.4304	0.8873	0.0013	0.0928	0.3914	0.0214	0.4158	0.8844	0.0017	0.0611	0.2716	0.1193
D_4	0.0472	0.5005	0.9105	0.0022	0.0935	0.4212	0.0099	0.3128	0.7909	0.0027	0.0932	0.4187	0.0106
D_5	0.0033	0.0915	0.3477	0.0408	0.4796	0.9076	0.0062	0.2157	0.6894	0.0082	0.2132	0.6542	0.0158
D_6	0.0080	0.2331	0.7144	0.0196	0.3874	0.8515	0.0063	0.1884	0.5928	0.0058	0.1911	0.6596	0.1006
D_7	0.0338	0.3712	0.8587	0.0022	0.0750	0.3727	0.0484	0.4749	0.8935	0.0023	0.0789	0.3706	0.0498
D_8	0.0463	0.4971	0.9142	0.0153	0.3656	0.8366	0.0013	0.0714	0.3824	0.0018	0.0659	0.2952	0.0344
D_9	0.0660	0.6133	0.9540	0.0023	0.0760	0.3148	0.0041	0.1590	0.5633	0.0035	0.1517	0.6328	0.0449
D_{10}	0.0161	0.2946	0.8030	0.0350	0.4294	0.8758	0.0047	0.1166	0.4397	0.0090	0.1594	0.5538	0.1901
D_{11}	0.0225	0.3244	0.7881	0.0354	0.4729	0.9153	0.0049	0.1450	0.5623	0.0019	0.0577	0.2830	0.1305
D_{12}	0.0421	0.5357	0.9322	0.0175	0.3268	0.8378	0.0010	0.0704	0.3770	0.0023	0.0671	0.3063	0.1602
Tolerance	0.0008	0.3172	0.9425	0.0009	0.2746	0.9154	0.0007	0.2480	0.9155	0.0006	0.1603	0.7534	

[a] The values of consistency are measured by the Geometric Consistency Index (GCI)

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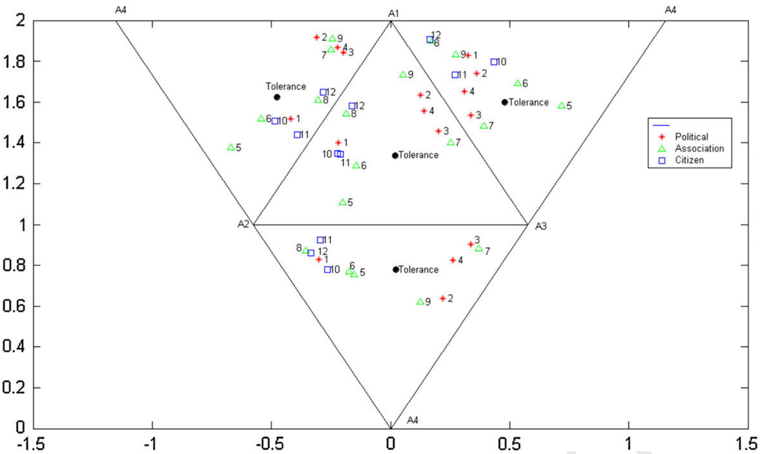

Fig. 1 Quaternary graph associated with the mean priorities of the decision makers and the tolerance distribution

5.1 Individual priorities

Table 2 shows the posterior means and the 95 % Bayesian credibility intervals constructed from the posterior quantiles 2.5 % ($Q_{2.5}$) and 97.5 % ($Q_{97.5}$) of the individual priorities $\{w_i^{(k)}; i = 1, \ldots, 4\}$ of each of the 12 decision makers and the posterior means of the variances $\left\{\sigma^{2(k)}; k = 1, ..., 12\right\}$ that can be used to measure the individuals' levels of consistency. The consistency values in Table 2 have been measured by the Geometric Consistency Index (GCI) and all of them fall under the permitted threshold (0.35 for n = 4). Figure 1 represents, by means of a *quaternary graph* (Aitchison 1986: p. 45, exercise 2.3), the posterior mean of the individual priorities and the tolerance distribution projected over the 4 different, 3-dimensional simplex; Fig. 2 shows the box plots of the individual posterior distributions of the decision makers' priorities and the tolerance distribution calculated from the samples of these distributions. All the moments and quantiles were calculated by using the Monte Carlo method (10000 simulations) from the individual posterior distributions (16).

Tables 3, 4 and 5 show the posterior distributions of the ordered alternatives, the two most preferred alternatives and the rankings of the alternatives for each decision maker. Table 6 presents the dominance probabilities (9) and Table 7 the posterior mean of the quotients of priorities $\frac{w_i}{w_j}$ for each pair of alternatives that measure the strength of the relative preference of the decision maker of A_i over A_j estimated by the priorities vector **w**. These distributions were obtained by using the Monte Carlo method (10000 simulations) from the posterior distributions (16).

Figure 1 and the individual priorities of Table 2 show the existence of 4 groups of decision makers. The first group, with a total weight (representativeness) of 42 % (Table 1), is formed by the decision makers D_2, D_3, D_4, D_7 and D_9, who seem to prefer alternatives A_1 and A_3 over the rest of alternatives. In this group the majority (D_2, D_3, D_4 and D_9) show a higher preference for the alternative A_1 while D_7 prefers alternative A_3. The second group, with a total weight of 34 %, consists of the decision makers D_1, D_6, D_{10} and D_{11}, who support alternatives A_2 as the most preferred and A_1 as the second most preferred. The third group, with a total weight of 16 %, is D_5 who set alternative A_2 as the most preferred; this individual clearly rejects the alternative A_1 and is, essentially, indifferent with regards to A_3 and A_4

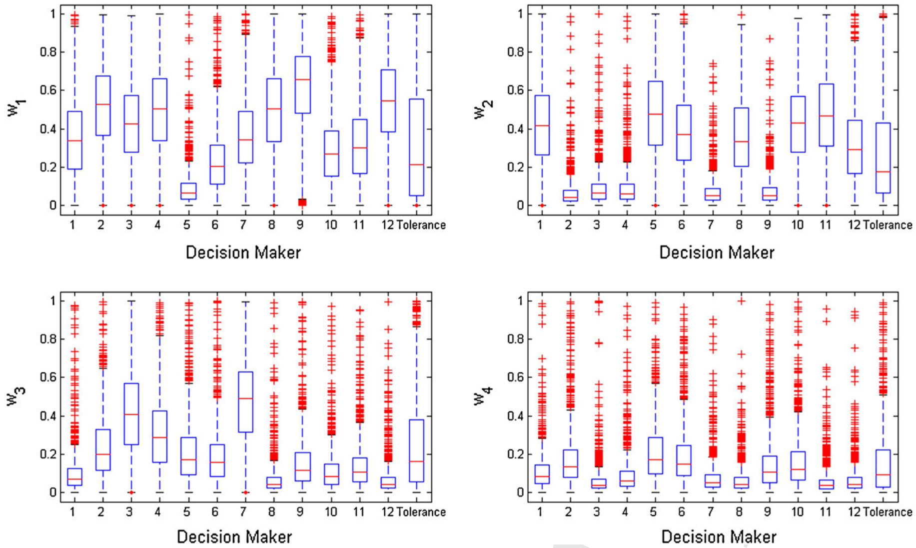

Fig. 2 Boxplot of the individual posterior distributions of decision makers' priorities and the tolerance distribution

(Tables 3, 6 and 7). The fourth group has a total weight of 8 % and contains decision makers D_8 and D_{12} who set alternatives A_1 and A_2 as the most and the second most preferred alternatives. All the decision makers manifested a high degree of consistency in the judgement elicitation process (Table 2) and provided well determined rankings for the alternatives.

5.2 Tolerance distribution

Tables 2, 3, 4, 5, 6, and 7 and Figs. 1 and 2 also show, under Tolerance, the inferences made about the groups' joint priorities using a sample drawn from the tolerance distribution (17). The algorithm described in Sect. 4.2 was used with S = 1000 and S' = 10000. It can be observed that this distribution represents a compromise opinion among the various preferences given in Sect. 3.1. Tables 3, 4, and 5 show that the tolerance distribution favors the selection of alternative A_1 as the most preferred and A_4 as the least preferred.

The proposal reflects the existence of a majority of decision makers who show strong affinity to A_1. With the exception of D_5, all the decision makers prefer A_1 as the first or second most preferred alternative with a majority (D_2, D_3, D_4, D_8, D_9 and D_{12}, total weight 46 %) who consider it to be the most suitable (see implied rankings of Table 4) and with strong intensity (see relative preferences w_1/w_i i = 2, 3, 4 in Table 7). Alternative A_4 is the least suitable, with the only exception of D_5, all the decision makers tend place it third or fourth (Tables 3, 6) with middle/strong intensity for most of the decision makers (see relative preferences w_4/w_i i = 1, 2, 3 in Table 7). There is no clear difference between alternatives A_2 and A_3. If we consider the results of Table 3, A_3 is selected as the second most preferred by the tolerance distribution, reflecting that decision makers D_2, D_3, D_4 and D_9 (total weight 38 %) selected it in second place while only D_8 and D_{12} (total weight 8 %) selected A_2 as the second most preferred. However, (Table 3) decision makers D_1, D_5, D_6, D_{10} and D_{11} (total weight 50 %) selected A_2 as the most preferred alternative while only D_7 (weight 4 %) preferred A_3. This fact is reflected in the results shown in Tables 6 and 7 from which it is concluded that A_2 dominates A_3, but with a high probability of rank reversal ($P(A_3 \succ A_2)$

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Table 3 Tolerance and individual posterior distributions of the ordered alternatives

Ordered Alternative [a]	Alternative	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
$A_{(1)}$	A_1	38.70	*72.00*	*50.60*	*66.10*	2.60	19.50	34.80	*61.70*	*82.70*	27.00	31.40	*69.90*	*36.70*
	A_2	*54.90*	2.10	3.00	2.80	*67.10*	*54.50*	2.20	34.70	1.70	*60.90*	*61.20*	25.90	27.40
	A_3	3.00	16.90	44.90	28.00	15.40	12.20	*60.50*	2.30	7.50	4.40	6.10	2.40	24.40
	A_4	3.40	9.00	1.50	3.10	14.90	13.80	2.50	1.30	8.10	7.70	1.30	1.80	11.50
	Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$A_{(2)}$	A_1	45.40	18.80	41.80	25.50	8.80	30.30	55.80	32.40	11.50	43.80	49.10	24.30	21.30
	A_2	32.40	5.40	7.80	9.60	21.50	23.60	6.30	56.10	11.30	25.10	29.60	62.10	24.60
	A_3	9.90	45.90	45.40	55.40	34.40	23.90	31.00	4.80	41.30	10.40	17.10	6.60	29.10
	A_4	12.30	29.90	5.00	9.50	35.30	22.20	6.90	6.70	35.90	20.70	4.20	7.00	25.00
	Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
$A_{(3)}$	A_1	11.40	6.50	6.50	6.30	20.40	27.70	7.20	4.50	4.00	19.40	15.60	3.80	17.50
	A_2	9.60	14.90	***58.60***	***42.60***	8.70	14.10	39.80	6.20	25.10	10.00	7.30	9.60	***30.10***
	A_3	36.20	31.10	7.10	11.80	***36.00***	***29.80***	6.90	44.40	***35.80***	31.60	***61.20***	***43.30***	25.90
	A_4	***42.80***	***47.50***	27.80	39.30	34.90	28.40	***46.10***	***44.90***	35.10	***39.00***	15.90	43.30	26.50
	Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

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Table 3 continued

Ordered Alternative [a]	Alternative	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
$A_{(4)}$	A_1	4.50	2.70	1.10	2.10	**68.20**	22.50	2.20	1.40	1.80	9.80	3.90	2.00	24.50
	A_2	3.10	**77.60**	30.60	45.00	2.70	7.80	**51.70**	3.00	**61.90**	4.00	1.90	2.40	17.90
	A_3	**50.90**	6.10	2.60	4.80	15.30	34.10	1.60	**48.50**	15.40	**53.60**	15.60	47.70	20.60
	A_4	41.50	13.60	**65.70**	**48.10**	13.80	**35.60**	44.50	47.10	20.90	32.60	**78.60**	**47.90**	**37.00**
	Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Ranking[b]		**2 > 1 > 4 > 3**	**1 > 3 > 4 > 2**	**1 > 3 > 2 > 4**	**1 > 3 > 2 > 4**	**2 > 4 > 3 > 1**	**2 > 1 > 3 > 4**	**3 > 1 > 4 > 2**	**1 > 2 > 4 > 3**	**1 > 3 > 4 > 2**	**2 > 1 > 4 > 3**	**2 > 1 > 3 > 4**	**1 > 2 > 3 > 4**	**1 > 3 > 2 > 4**

The most probable alternatives for each distribution are in bold.

Those corresponding to $A_{(1)}$ are in italic values; those corresponding to $A_{(2)}$ are in underlined values; those corresponding to $A_{(3)}$ are in bold with italic values; those corresponding to $A_{(4)}$ are in bold with underlined values

[a] $A_{(1)}$ denotes the most preferred alternative. $A_{(2)}$ denotes the second most preferred alternative and so on

[b] Ranking implied by the ordered alternative distributions. For instance for the decision maker D_8 the most preferred alternative is A_1 (blue probability); the second most preferred alternative is A_2 (red probability) and the third most preferred alternative is A_4 (cyan probability). Hence the implied ranking is $1 > 2 > 4 > 3$

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Table 4 Tolerance and individual posterior distributions of the two most preferred alternatives $A_{(1)}$ and $A_{(2)}$

$(A_{(1)}.A_{(2)})$	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
$(A_1.A_2)$	29.60	3.10	4.50	6.40	1.40	10.70	2.30	**54.40**	9.40	19.00	25.80	**60.30**	11.40
$(A_1.A_3)$	4.10	**43.00**	**43.80**	**53.30**	0.60	3.90	29.50	2.90	**39.00**	2.70	4.50	4.90	**16.50**
$(A_1.A_4)$	5.00	25.90	2.30	6.40	0.60	4.90	3.00	4.40	34.30	5.30	1.10	4.70	8.80
$(A_2.A_1)$	**42.80**	0.90	1.50	1.50	5.90	**22.40**	1.10	30.70	0.90	**39.70**	**46.60**	22.40	7.90
$(A_2.A_3)$	5.30	0.50	1.40	1.00	30.20	17.10	0.90	1.90	0.30	7.00	12.20	1.40	9.40
$(A_2.A_4)$	6.80	0.70	0.10	0.30	**31.00**	15.00	0.20	2.10	0.50	14.20	2.40	2.10	10.10
$(A_3.A_1)$	1.20	12.20	39.30	22.50	1.70	4.30	**53.30**	1.10	5.30	1.50	2.30	1.10	9.90
$(A_3.A_2)$	1.30	1.40	3.00	2.70	10.00	5.60	3.50	1.00	1.10	1.70	3.10	1.10	8.40
$(A_3.A_4)$	0.50	3.30	2.60	2.80	3.70	2.30	3.70	0.20	1.10	1.20	0.70	0.20	6.10
$(A_4.A_1)$	1.40	5.70	1.00	1.50	1.20	3.60	1.40	0.60	5.30	2.60	0.20	0.80	3.50
$(A_4.A_2)$	1.50	0.90	0.30	0.50	10.10	7.30	0.50	0.70	0.80	4.40	0.70	0.70	4.80
$(A_4.A_3)$	0.50	2.40	0.20	1.10	3.60	2.90	0.60	0.00	2.00	0.70	0.40	0.30	3.20
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

The most probable pair of alternatives selected for each decision maker and the tolerance distribution are in bold

Table 5 Tolerance and individual posterior distributions of the preference rankings

Rankings	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
1234[a]	13.00	1.70	3.20	4.90	0.50	6.00	2.30	**27.30**	5.80	7.60	20.90	29.90	7.40
1243	16.60	1.40	1.30	1.50	0.90	4.70	0.00	27.10	3.60	11.40	4.90	**30.40**	4.00
1324	3.10	7.50	**29.60**	**28.40**	0.30	2.70	13.00	2.00	12.10	2.00	3.80	3.90	**11.30**
1342	1.00	**35.50**	14.20	24.90	0.30	1.20	16.50	0.90	**26.90**	0.70	0.70	1.00	5.20
1423	4.30	2.80	0.50	2.10	0.30	2.60	0.60	3.10	8.60	3.70	0.60	3.90	5.20
1432	0.70	23.10	1.80	4.30	0.30	2.30	2.40	1.30	25.70	1.60	0.50	0.80	3.60
2134	19.70	0.70	1.30	1.10	3.30	10.30	0.70	15.20	0.50	16.60	**38.80**	11.40	4.20
2143	**23.10**	0.20	0.20	0.40	2.60	**12.10**	0.40	15.50	0.40	**23.10**	7.80	11.00	3.70
2314	4.20	0.30	1.10	0.80	6.00	9.70	0.80	1.30	0.10	4.20	10.40	0.90	3.90
2341	1.10	0.20	0.30	0.20	**24.20**	7.40	0.10	0.60	0.20	2.80	1.80	0.50	5.50
2413	4.80	0.40	0.10	0.20	7.30	7.80	0.00	1.80	0.30	11.30	1.80	1.30	3.30
2431	2.00	0.30	0.00	0.10	23.70	7.20	0.20	0.30	0.20	2.90	0.60	0.80	6.80
3124	0.70	2.50	27.70	10.70	0.80	2.80	24.90	0.60	1.80	1.50	2.10	0.90	5.40
3142	0.50	9.70	11.60	11.80	0.90	1.50	**28.40**	0.50	3.50	0.00	0.20	0.20	4.50
3214	0.80	0.90	2.80	2.20	2.90	4.10	2.80	0.70	0.60	0.70	2.60	0.90	4.80
3241	0.50	0.50	0.20	0.50	7.10	1.50	0.70	0.30	0.50	1.00	0.50	0.20	3.60
3412	0.30	2.80	2.10	1.90	0.40	0.70	2.90	0.20	1.10	0.50	0.30	0.10	2.30
3421	0.20	0.50	0.50	0.90	3.30	1.60	0.80	0.00	0.00	0.70	0.40	0.10	3.80
4123	0.90	0.70	0.30	0.40	0.70	2.60	0.40	0.50	2.00	1.80	0.10	0.60	2.40
4132	0.50	5.00	0.70	1.10	0.50	1.00	1.00	0.10	3.30	0.80	0.10	0.20	1.10
4213	1.20	0.60	0.20	0.20	3.50	4.30	0.20	0.50	0.50	2.30	0.40	0.50	2.00
4231	0.30	0.30	0.10	0.30	6.60	3.00	0.30	0.20	0.30	2.10	0.30	0.20	2.80
4312	0.10	1.50	0.20	1.00	0.30	1.10	0.50	0.00	1.40	0.40	0.10	0.10	1.20
4321	0.40	0.90	0.00	0.10	3.30	1.80	0.10	0.00	0.60	0.30	0.30	0.20	2.00
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

The most probable rankings corresponding to each distribution are in bold

[a] 1234 denotes the ranking $A_1 > A_2 > A_3 > A_4$. where ">" means "is preferred to"

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Table 6 Tolerance and individual posterior dominance probabilities between pair of alternatives

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
$A_1 > A_2$[a]	41.70	*94.20*	*93.20*	*93.00*	6.20	29.20	92.90	*63.60*	*95.80*	32.00	34.30	*72.00*	*53.60*
$A_1 > A_3$	88.90	*79.60*	*53.40*	*69.50*	20.50	57.60	37.50	*95.30*	*89.70*	82.90	80.40	*94.90*	*53.40*
$A_1 > A_4$	87.70	*86.30*	*95.30*	*93.10*	19.10	60.00	92.80	*95.50*	*89.60*	73.10	93.30	*95.20*	*63.20*
$A_2 > A_1$	*58.30*	5.80	6.80	7.00	*93.80*	*70.80*	7.10	36.40	4.20	*68.00*	*65.70*	28.00	46.40
$A_2 > A_3$	*91.20*	9.60	8.60	12.20	*79.60*	*77.70*	6.00	93.40	22.50	*89.80*	*88.40*	91.40	51.20
$A_2 > A_4$	*89.60*	16.60	***67.80***	***51.00***	*79.60*	*76.30*	45.90	92.70	26.10	*85.10*	*96.00*	92.10	63.90
$A_3 > A_1$	11.10	20.40	46.60	30.50	***79.50***	42.40	*62.50*	4.70	10.30	17.10	19.60	5.10	46.60
$A_3 > A_2$	8.80	90.40	91.40	87.80	20.40	22.30	*94.00*	6.60	77.50	10.20	11.60	8.60	48.80
$A_3 > A_4$	45.10	62.80	94.60	88.30	50.00	49.50	*93.90*	49.60	53.10	38.30	***82.50***	***50.00***	***61.90***
$A_4 > A_1$	12.30	13.70	4.70	6.90	80.90	40.00	7.20	4.50	10.40	26.90	6.70	4.80	36.80
$A_4 > A_2$	10.40	***83.40***	32.20	49.00	20.40	23.70	***54.10***	7.30	***73.90***	14.90	4.00	7.90	36.10
$A_4 > A_3$	**54.90**	37.20	5.40	11.70	50.00	***50.50***	6.10	***50.40***	46.90	***61.70***	17.50	50.00	38.10
weights	10.00	10.00	10.00	10.00	16.00	16.00	4.00	4.00	8.00	4.00	4.00	4.00	100.00
Ranking[b]	**2 > 1 > 4 > 3**	**1 > 3 > 4 > 2**	**1 > 3 > 2 > 4**	**1 > 3 > 2 > 4**	**2 > 4 > 3 > 1**	**2 > 1 > 3 > 4**	**3 > 1 > 4 > 2**	**1 > 2 > 4 > 3**	**1 > 3 > 4 > 2**	**2 > 1 > 4 > 3**	**2 > 1 > 3 > 4**	**1 > 2 > 3 > 4**	**1 > 2 > 3 > 4**

The dominance probabilities larger than 50 % are in bold; The dominance probabilities that determine the most preferred alternative are in Italic values; The dominance probabilities that determine the second most preferred alternative are in underlined values; The dominance probabilities that determine the third most preferred alternative are in bold with italic values

[a] $A_1 > A_2$ denotes A_1 is as least as preferred as A_2

[b] Ranking implied by the dominance probabilities. For instance, for the decision maker D_8 the sums of the rows of the PDCM are 3.544 (for A_1), 3.225 (for A_2), 1.609 (for A_3) and 1.622 (for A_4). So the implied ranking is 1 > 2 > 4 > 3

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Table 7 Tolerance and individual posterior medians of the quotient of priorities between pair of alternatives

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	Tolerance
w_1/w_2	0.790	*11.510*	*5.987*	*7.800*	0.137	0.566	6.743	*1.507*	*12.554*	0.611	0.630	*1.862*	*1.289*
w_1/w_3	4.532	*2.595*	*1.083*	*1.754*	0.384	1.288	0.692	*10.900*	*5.559*	2.983	2.590	*11.955*	*1.269*
w_1/w_4	3.811	*3.653*	*9.841*	*7.748*	0.382	1.322	6.551	*10.940*	*5.751*	2.115	7.848	*12.591*	*2.556*
w_2/w_1	*1.267*	0.087	0.167	0.128	*7.276*	*1.766*	0.148	0.664	0.080	*1.636*	*1.586*	0.537	0.776
w_2/w_3	*5.530*	0.219	0.165	0.237	*2.650*	*2.223*	0.112	7.433	0.448	*4.807*	*4.329*	6.541	1.041
w_2/w_4	*4.519*	0.332	***1.644***	***1.024***	*2.722*	*2.378*	0.882	7.636	0.515	*3.402*	*13.134*	6.430	1.789
w_3/w_1	0.221	0.385	0.924	0.570	***2.603***	0.776	*1.446*	0.092	0.180	0.335	0.386	0.084	0.788
w_3/w_2	0.181	4.568	6.062	4.218	0.377	0.450	*8.903*	0.135	2.233	0.208	0.231	0.153	0.961
w_3/w_4	0.845	1.409	10.003	4.615	1.000	0.987	*8.512*	0.996	1.120	0.703	***2.920***	***1.001***	***1.760***
w_4/w_1	0.262	0.274	0.102	0.129	2.618	0.756	0.153	0.091	0.174	0.473	0.127	0.079	0.391
w_4/w_2	0.221	***3.012***	0.608	0.977	0.367	0.420	***1.133***	0.131	***1.942***	0.294	0.076	0.156	0.559
w_4/w_3	***1.183***	0.710	0.100	0.217	1.001	***1.014***	0.118	***1.005***	0.893	***1.423***	0.343	0.999	0.568
weights	10.00	10.00	10.00	10.00	16.00	16.00	4.00	4.00	8.00	4.00	4.00	4.00	100.00
Ranking[a]	**2 > 1 > 4 > 3**	**1 > 3 > 4 > 2**	**1 > 3 > 2 > 4**	**1 > 3 > 2 > 4**	**2 > 4 > 3 > 1**	**2 > 1 > 3 > 4**	**3 > 1 > 4 > 2**	**1 > 2 > 4 > 3**	**1 > 3 > 4 > 2**	**2 > 1 > 4 > 3**	**2 > 1 > 3 > 4**	**1 > 2 > 3 > 4**	**1 > 2 > 3 > 4**

The dominance probabilities larger than 50 % are in bold; The quotients that determine the most preferred alternative are in italic values; The quotients that determine the second most preferred alternative are in underlined values; The quotients that determine the third most preferred alternative are in bold with italic valuea

[a] Ranking implied by the dominance probabilities

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= 0.488 with the tolerance distribution (Table 6) and a weak relative preference of A_2 with respect to A_3 ($\frac{w_2}{w_3} \approx 1.041$, $\frac{w_3}{w_2} \approx 0.961$, Table 7).

Alternative A_1 could therefore be selected as the most suitable alternative and A_4 as the least preferred. With respect to the alternatives A_2 and A_3, there is no consensus in the group about the arrangement between them and it would be necessary to start a subsequent tolerance process that would conclude in a preference ranking accepted by the majority of the actors involved in the resolution process.

6 Conclusions

This paper presents a new approach to multi-actor decision making (systemic decision making - SDM), which has been applied, with a Bayesian perspective, in the specific context of AHP. In accordance with the principle of tolerance that characterises this new approach, SDM allows the holistic integration of the visions of reality associated with the actors involved in the resolution process. A tolerance distribution for the group's priorities vector has been defined. The distribution minimises a weighted average of the Kullback-Leibler distances to every posterior distribution of the individual priorities vector and provides a democratic tool which highlights the more probable priority vectors for reaching a final agrement by all the members of **D**. The methodology has been illustrated by applying it to the multiplicative model usually employed with stochastic AHP, for known and unknown variances. Furthermore, an e-participatory budget allocation problem has been analysed in which several resolution proposals were made using the decision tools introduced in the paper.

As with any aggregation procedure or synthesis measure, some of the actors involved in the construction of the tolerance distribution may not be in agreement or hold opinions compatible with the final result. In these situations, it would be necessary to identify maximum compatible sets of actors and to provide (changing the initial priorities) tolerance paths between them in order to increase the representativeness of the tolerance distribution. These two issues (compatibility and tolerance paths) will be the subject of another paper (Salvador et al. 2014). The representativeness of the tolerance distribution, that is to say, the weight of the actors that are compatible with it, guarantees that the conclusions (patterns of behaviour of the alternatives) derived from it will be accepted by a representative or qualified number of actors. In order to measure this representativeness, measurements of discrepancy of the preference distribution of each decision maker (quantified by the individual posterior distributions (2)) such as that introduced in Altuzarra et al. (2010) could be used.

Even though this paper only considers a local context, the new approach can be extended to AHP hierarchies. In that case, the components of the priority vector **w** would be the global priorities of each alternative and it would not be necessary for the decision makers to use the same hierarchy to establish them. Moreover, given that the tolerance distribution is a joint multivariate distribution of the components of **w**, it takes into account the existing statistical dependencies among them in order to analyse the preference ranking of the alternatives. This allows both the evaluation of the probabilities of rank reversal and the extraction of the multivariate preference patterns, and this could be very useful for establishing new tolerance paths. All these aspects reflect the flexibility and generality of the new approach with respect to other methodologies detailed in the literature (Ramanathan 1997; Stam and Silva 1997). Finally, it should be mentioned that although in this paper the AHP context has been adopted, the SDM framework provides a general and flexible methodology which allows the actors to employ different multicriteria approaches, the only requisite being that the preferences of

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each actor can be expressed by a probability distribution. All this gives the proposal a high level of realism, flexibility and generality that will become more apparent in future papers. 5

Acknowledgments This work was partially financed by the project "Social Cognocracy Network" (Ref. ECO2011-24181), supported by the Spanish Ministry of Science and Innovation.

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Journal: **10479** Article No.: **1637** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2014/5/29** Pages: **24** Layout: **Small**