An efficient numerical scheme for 1D parabolic singularly perturbed problems with an interior and boundary layers

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Abstract

In this paper we consider a 1D parabolic singularly perturbed reaction-convection-diffusion problem, which has a small parameter in both the diffusion term (multiplied by the parameter ε^2) and the convection term (multiplied by the parameter μ) in the differential equation ($\varepsilon \in (0,1]$, $\mu \in [0,1]$, $\mu \leq \varepsilon$). Moreover, the convective term degenerates inside the spatial domain, and also the source term has a discontinuity of first kind on the degeneration line. In general, for sufficiently small values of the diffusion and the convection parameters, the exact solution exhibits an interior layer in a neighborhood of the interior degeneration point and also a boundary layer in a neighborhood of both end points of the spatial domain. We study the asymptotic behavior of the exact solution with respect to both parameters and we construct a monotone finite difference scheme, which combines the implicit Euler method, defined on a uniform mesh, to discretize in time, together with the classical upwind finite difference scheme, defined on an appropriate nonuniform mesh of Shishkin type, to discretize in space. The numerical scheme converges in the maximum norm uniformly in ε and μ, having first order in time and almost first order in space. Illustrative numerical results corroborating in practice the theoretical results are showed.

Key words: 1D parabolic singularly perturbed problems, two parameters, discontinuous source, interior and boundary layers, uniform convergence.
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1 Introduction

We consider the singularly perturbed initial-boundary value problem

$$\begin{aligned} &L\,u(x,t) = f(x,t), && (x,t) \in G \setminus S^{\pm}, \\ &u(x,t) = \varphi(x,t), && (x,t) \in S, \\ &l^{\pm}u(x,t) \equiv \varepsilon \left[\frac{\partial}{\partial x} u(x+0,\,t) - \frac{\partial}{\partial x} u(x-0,\,t) \right] = 0, && (x,t) \in S^{\pm}, \end{aligned} \qquad (1)$$

where the differential operator L is given by

$$L\,u(x,t) \equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial x^2} + \mu x^{2p+1} a(x) \frac{\partial}{\partial x} - b(x,t) \frac{\partial}{\partial t} - r(x,t) \right\} u(x,t),$$

with $G = D \times (0,T]$, $D = (-d,d)$, $d > 0$, $S = \overline{G} \setminus G$ and $S^{\pm} = \{x = 0\} \times (0,T]$. We assume that

$$0 \le \mu \le 1, \quad 0 < \varepsilon \le 1, \quad \mu \le \varepsilon, \qquad (2)$$

p is a nonnegative integer, a, b and r are sufficiently smooth functions such that

$$a(x) > 0,\ b(x,t) \ge \beta > 0,\ r(x,t) \ge 2r_0^2, \text{ with } r_0 > 0 \text{ for } (x,t) \in \overline{G}, \qquad (3)$$

and the source function $f(x,t)$ is continuous on $\overline{G}^{+}$ and $\overline{G}^{-}$, where $\overline{G}^{-} = [-d,\,0] \times [0,\,T]$, $\overline{G}^{+} = [0,\,d] \times [0,\,T]$, and it has a first kind discontinuity on the set $S^{\pm}$. Moreover, we assume that the data of problem (1) satisfy sufficient regularity conditions that guarantee the smoothness of the solution on the sets $\overline{G}^{+}$ and $\overline{G}^{-}$ required in the analysis below.

We denote by $S = S^L \cup S_0$, $S_0 = S_0^{+} \cup S_0^{-}$, $S^L = S^l \cup S^r$, where

$$S_0 = [-d,\,d] \times \{t = 0\}, \quad S_0^{+} = [0,\,d] \times \{t = 0\}, \quad S_0^{-} = [-d,\,0] \times \{t = 0\},$$
$$S^l = \{x = -d\} \times (0,\,T], \quad S^r = \{x = d\} \times (0,\,T].$$

The sets described above are displayed in Figure 1.

Moreover, compatibility conditions at the corners $(-d,0)$, $(0,0)$ and $(d,0)$ are satisfied that guarantee the required smoothness of the solution in the neighborhoods of these points. We refer to [1] for a discussion of the required regularity and compatibility conditions in our analysis.

Problems for partial differential equations with discontinuous data are simple models of diffraction problems. In [2,3] the case of regular equations was discussed. The analysis of special methods for singularly perturbed problems

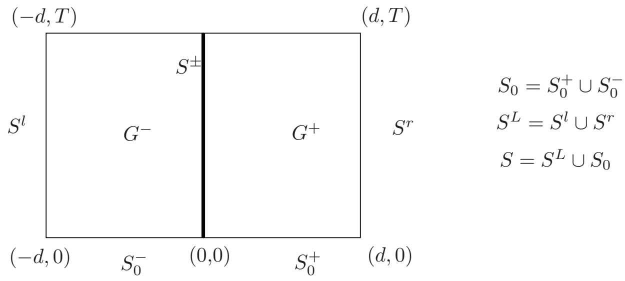

Fig. 1. Sets of the domain

with discontinuous data and degenerating convective terms has been little developed in the literature. In [4–6] a problem with discontinuous data in differential equations was analyzed; in [7] a problem with a convective term degenerating on the domain boundary for a parabolic convection-diffusion equation was studied. An experimental technique to analyze the ε-uniform convergence of numerical schemes defined on piecewise-uniform meshes, for a singularly perturbed elliptic equation when the convective term degenerates on the boundary, was considered in [8, Chapter 7].

The problem (1), (2) for the simpler case $\mu = 0$ was considered in [9]; in [1,10,11] a similar problem to (1) for $\mu = 1$ was studied; in those works, it was proved that the numerical scheme, combining the implicit Euler method on a uniform mesh in time and the classical upwind scheme on a piecewise uniform mesh in space, gives a scheme converging ε-uniformly in the maximum norm, or, in short, uniformly convergent scheme. The analysis is based on a discrete minimum principle and appropriate estimates of the solution and its partial derivatives.

In this paper we are interested into extending those results to the more general case where the small parameters ε and μ affect both the convection and the diffusion coefficients in the differential equation. This class of problems, but with smooth coefficients and in the absence of turning point, has been analyzed in [12].

The paper is structured as follows. In Section 2, we analyze the asymptotic behavior of the exact solution of the continuous problem (1), proving appropriate bounds for its derivatives, which will be used in the analysis of the uniform convergence of the numerical scheme. In Section 3, we construct the numerical method, combining the implicit Euler to discretize in time and the upwind finite difference scheme to discretize in space. We prove that, if a uniform mesh in time and a special Shishkin type mesh in space are used, then the scheme is uniformly convergent and it has first order in time and almost first order in space. Finally, in Section 4, some numerical results are shown, which corroborate in practice the efficiency of the method and the order of uniform convergence with respect to both parameters ε, μ, according to the theoretical results.

Henceforth, we denote by $C^{k,k/2}$ the space of functions with continuous derivatives with respect to x up to order k and continuous derivatives with respect to t up to order $k/2$, and by M a generic positive constant independent of the parameters ε, μ and also of the discretization parameters N and N_0, where N and N_0 are the number of mesh intervals in the variables x and t, respectively.

2 The continuous problem: asymptotic behavior

In this section we study the asymptotic behavior, with respect to ε and μ, of the exact solution of the continuous problem (1), and we establish appropriate bounds for its partial derivatives. Using a similar technique to this one in [6], the following comparison principle can be proved.
Lemma 1. *Let assume that the functions $u^1(x,t)$, $u^2(x,t)$ satisfy*

$$\begin{aligned} L\, u^1(x,\, t) &\leq L\, u^2(x,t), && (x,\, t) \in G \setminus S^{\pm}, \\ l^{\pm}\, u^1(x,\, t) &\leq l^{\pm}\, u^2(x,\, t), && (x,\, t) \in S^{\pm}, \\ u^1(x,\, t) &\geq u^2(x,t), && (x,\, t) \in S. \end{aligned}$$

Then, it holds that $u^1(x,t) \geq u^2(x,t), \quad (x,t) \in \overline{G}$.

We assume that the data of the problem are sufficiently smooth functions and also that they satisfy compatibility conditions in the corner points $(-d, 0)$, $(d, 0)$ and $(0, 0)$ in order that the exact solution of (1) belongs to the space $C(\overline{G}) \cap \{C^{4,2}(\overline{G}^-) \cup C^{4,2}(\overline{G}^+)\}$. We use a truncation error argument to prove the uniform convergence of the numerical scheme defined below; thus, we need appropriate bounds of the derivatives of the exact solution which are deduced in this section.

We begin with a lemma where some coarse bounds for the derivatives of u are given.
Lemma 2. *The solution of the problem* (1) *satisfies*

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t) \right| \leq M\varepsilon^{-k}, \quad (x,t) \in \overline{G}^- \cup \overline{G}^+, \quad 0 \leq k + 2k_0 \leq 4. \qquad (4)$$

PROOF. Using the stretching variables $\xi = \varepsilon^{-1} x$, $\tau = t$, the solution,

$\widehat{u}(\xi,\tau)=\widehat{u}(\xi(x),\tau(t))=u(x,t)$, of the transformed problem satisfies

$$\left\{\tfrac{\partial^2}{\partial\xi^2}+\tfrac{\mu}{\varepsilon}(\xi\varepsilon)^{2p+1}\widehat{a}(\xi)\,\tfrac{\partial}{\partial\xi}-\widehat{b}(\xi,\tau)\tfrac{\partial}{\partial\tau}-\widehat{r}(\xi,\tau)\right\}\widehat{u}(\xi,\tau)=\widehat{f}(\xi,\tau),$$
$$(\xi,\tau)\in\left(-\tfrac{d}{\varepsilon},\tfrac{d}{\varepsilon}\right)\setminus\{0\}\times(0,T],$$
$$\frac{\partial}{\partial\xi}\widehat{u}(\xi+0,t)-\frac{\partial}{\partial\xi}\widehat{u}(\xi-0,t)=0,\quad \xi=0,\,t\in(0,T].$$

Noting that $\mu/\varepsilon\le 1$, a classical theory (see [13] for full details) brings to the estimate
$$\left|\frac{\partial^{k+k_0}}{\partial\xi^k\partial\tau^{k_0}}\widehat{u}(\xi,\tau)\right|\le M,\quad 0\le k+2k_0\le 4,$$
and transforming back to the original variables, the bound (4) follows. From estimate (4), it follows that the parameter μ in problem (1), (2) is regular.

Nevertheless, these bounds are not sufficient to deduce the uniform convergence of the numerical scheme. Thus, we decompose the exact solution as

$$\begin{aligned} u(x,\,t)&=U^+(x,\,t)+V^+(x,\,t)+W_R(x,\,t),\quad (x,\,t)\in\overline{G}^+,\\ u(x,\,t)&=U^-(x,\,t)+V^-(x,\,t)+W_L(x,\,t),\quad (x,\,t)\in\overline{G}^-,\end{aligned}\tag{5}$$

where $U^{\pm}$ is the regular component, $V^{\pm}$ is the interior layer component, and W_R, W_L are the right and left boundary layer components, respectively.

Note that the function $u(x,\,t)$, considered on the set $\overline{G}^+$, is the solution of the initial-boundary value problem

$$\begin{aligned} Lu(x,t)&=f(x,t),\quad (x,t)\in G^+,\\ u(x,t)&=\varphi(x,t),\quad (x,t)\in S_0^+\cup S^r,\\ u(x,t)&=\varphi_u(x,t),\quad (x,t)\in S^{\pm},\end{aligned}\tag{6}$$

where $\varphi_u(x,t)=u(x,\,t)$, $(x,\,t)\in S^{\pm}$, and u is the solution of the problem (1).

From (4), it is straightforward that the derivatives with respect to t of the function $\varphi_u(x,t)$, $(x,t)\in S^{\pm}$ are ε-uniformly bounded. Moreover, we assume that the necessary compatibility conditions are fulfilled in the corner points of the set $\overline{G}^+$.

First, to obtain the required bounds for the regular component U^+, we use the idea of extending the domain problem (see, for example, [1,6]). Let $\overline{G}^{+e}$ be an extension of the domain $\overline{G}^+$ (for example, one can choose $\overline{G}^{+e}=[-1,2]\times[0,1]$). The functions a,b,r,f and φ are smoothly extended to $\overline{G}^{+e}$ and the new differential operator is denoted by L^e. We define $U^+(x,t)$ as the restriction to $\overline{G}^+$ of the function $U^{+e}(x,t)$, $(x,t) \in \overline{G}^{+e}$, i.e., $U^+(x,t) = U^{+e}(x,t)$, $(x,t) \in \overline{G}^+$, where $U^{+e}(x,t)$ is the solution of the initial-boundary value problem

$$\begin{aligned} L^e U^{+e}(x,t) &= f^{+e}(x,t), \quad (x,t) \in G^{+e}, \\ U^{+e}(x,t) &= \varphi^{+e}(x,t), \quad (x,t) \in S^{+e}, \end{aligned} \tag{7}$$

which is an extension of the problem (6) beyond the sets $S^{\pm}$, S^r, considered on $\overline{G}^+$.

The data of the problem (7), the functions $f^{+e}(x,t)$, $(x,t) \in \overline{G}^+$ and $\varphi^{+e}(x,t)$, $(x,t) \in S_0^+ \cup S^r$, on the set $\overline{G}^+$ are the same data as in (1). In the rest of $\overline{G}^{+e}$, they are smooth extensions of the data of (1) prescribed on $\overline{G}^+$.

In the second place, the function $V^+(x,t)$, $(x,t) \in \overline{G}^+$, i.e., the singular component of the interior layer, is the solution of the problem

$$\begin{aligned} L\,V^+(x,t) &= 0, & (x,t) &\in G^+, \\ V^+(x,t) &= \varphi_{V^+}(x,t), & (x,t) &\in S^{\pm}, \\ V^+(x,t) &= 0, & (x,t) &\in S_0^+ \cup S^r, \end{aligned} \tag{8}$$

where $\varphi_{V^+} = \varphi_u(x,t) - U^+(x,t)$, $(x,t) \in S^{\pm}$.

Third, the left layer component $W_L(x,t)$, $(x,t) \in \overline{G}^-$ is the solution of the problem

$$\begin{aligned} L\,W_L(x,t) &= 0, & (x,t) &\in G^-, \\ W_L(x,t) &= 0, & (x,t) &\in S^{\pm} \cup S_0^-, \\ W_L(x,t) &= u - U^-, & (x,t) &\in S^l, \end{aligned} \tag{9}$$

and finally, the right layer component $W_R(x,t)$, $(x,t) \in \overline{G}^+$ is the solution of the problem

$$\begin{aligned} L\,W_R(x,t) &= 0, & (x,t) &\in G^+, \\ W_R(x,t) &= 0, & (x,t) &\in S^{\pm} \cup S_0^+, \\ W_R(x,t) &= u - U^+, & (x,t) &\in S^r. \end{aligned} \tag{10}$$

Now, we analyze the function $U^{+e}(x,t)$, i.e., the solution of (7). To do that, we decompose it as

$$U^{+e}(x,t) = U_0(x,t) + \varepsilon U_1(x,t) + \varepsilon^2 v_U(x,t), \quad (x,t) \in \overline{G},$$

where $U_i(x,t)$, for $i = 0, 1$, are solutions of the problems

$$\begin{aligned}
L_0\, U_0(x,t) &\equiv \left\{ -b(x,t)\frac{\partial}{\partial t} - r(x,t) \right\} U_0(x,t) = f^{+e}(x,t), \; (x,t) \in \overline{G} \setminus S_0, \\
U_0(x,t) &= \varphi^{+e}(x,t), \qquad\qquad\qquad\qquad\qquad (x,t) \in S_0; \\
L_0 U_1(x,t) &= \left\{ -\varepsilon\frac{\partial^2}{\partial x^2} - \frac{\mu}{\varepsilon} x^{2p+1} a(x) \frac{\partial}{\partial x} \right\} U_0(x,t), \qquad (x,t) \in \overline{G} \setminus S_0, \\
U_1(x,t) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in S_0,
\end{aligned} \tag{11}$$

and $v_U(x,t)$ is the solution of the problem

$$\begin{aligned}
L^e v_U(x,t) &= \left\{ -\varepsilon\frac{\partial^2}{\partial x^2} - \frac{\mu}{\varepsilon} x^{2p+1} a(x) \frac{\partial}{\partial x} \right\} U_1(x,t), \; (x,t) \in G, \\
v_U(x,t) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in S.
\end{aligned} \tag{12}$$

Taking into account that the derivatives of $U_i(x,t)$, $i = 0, 1$, are bounded uniformly with respect to ε and μ, and using the coarse bounds (4) in (12), we deduce bounds for the partial derivatives of $U^{+e}(x,t)$ on $\overline{G}^{+e}$, and therefore it is straightforward that it holds

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U^+(x,t) \right| \le M\,[1 + \varepsilon^{2-k}], \quad (x,t) \in \overline{G}^+, \quad 0 \le k + 2k_0 \le 4. \tag{13}$$

Now we analyze the interior layer component $V^+(x,\, t)$. Using (4) and (13), we obtain

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} \varphi_{V^+}(x,t) \right| \le M, \quad (x,t) \in S^{\pm}, \quad k_0 = 0, 1, 2.$$

Defining the barrier function

$$\phi_0(x,t) = M\, \exp(-m_1 \varepsilon^{-1} x), \quad (x,\, t) \in \overline{G}^+,$$

where m_1 is an arbitrary positive constant such that

$$m_1^2 \le 2 r_0^2, \tag{14}$$

and M is a constant sufficiently large. Note that

$$\begin{aligned} L\phi_0(x,t) &= M\exp(-m_1\varepsilon^{-1}x)\left(m_1^2 - m_1\frac{\mu}{\varepsilon}x^{2p+1}a(x) - r(x,t)\right) \\ &\le M\exp(-m_1\varepsilon^{-1}x)(m_1^2 - 2r_0^2) \\ &\le 0. \end{aligned}$$

It follows from the classical minimum principle applied to the domain $\overline{G}^+$ that

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}V^+(x,\, t)\right| \le M\ \exp(-m_1\varepsilon^{-1}x), \quad (x,\, t)\in\overline{G}^+, \quad k_0 = 0,1,2, \tag{15}$$

proving bounds for the derivatives with respect to the time variable.

Lemma 2 provides the following crude bounds for the derivatives of V^+ in x

$$\left|\frac{\partial^{k}}{\partial x^{k}}V^+(x,\, t)\right| \le M\varepsilon^{-k} \quad (x,\, t)\in\overline{G}^+, \quad 1\le k\le 4. \tag{16}$$

Although these bounds are not sharp, they are enough for the analysis of the uniform convergence of our finite difference scheme, since the barrier function $\phi_0(x,t)$ decays exponentially from $x=0$.

Similarly, for components $U^-(x,t)$ and $V^-(x,t)$ and theirs derivatives, on the set $\overline{G}^-$ we can obtain the bounds

$$\begin{aligned} &\left|\frac{\partial^{k+k_0}}{\partial x^k\partial t^{k_0}}\,U^-(x,t)\right| \le M\left[1+\varepsilon^{2-k}\right],\ (x,t)\in\overline{G}^-, \quad 0\le k+2k_0\le 4, \\ &\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\,V^-(x,t)\right| \le M\exp(m_1\varepsilon^{-1}x) \quad (x,t)\in\overline{G}^-, \quad k_0=0,1,2, \\ &\left|\frac{\partial^{k}}{\partial x^{k}}\,V^-(x,t)\right| \le M\varepsilon^{-k}, \qquad (x,t)\in\overline{G}^-, \quad 1\le k\le 4. \end{aligned} \tag{17}$$

Now, we study the left singular component W_L. We define the function

$$\phi_1(x,t) = M\ \exp(-m_1\varepsilon^{-1}(d+x))\ \exp\left(\frac{m_1}{\beta}d^{2p+1}\|a\|_{[-d,0]}t\right),\ (x,\, t)\in\overline{G}^-,$$

where $\|a\|_{[-d,0]} = \max_{-d\le x\le 0} a(x)$. It satisfies $\phi_1(x,0)\ge 0$, $\phi_1(-d,t)\ge M$, $\phi_1(0,t)\ge 0$, and

$$L\phi_1(x,t) = \phi_1(x,t)\left\{\left[m_1^2 - r(x,t)\right] + \left[-m_1\frac{\mu}{\varepsilon}x^{2p+1}a(x) - m_1\frac{b(x,t)}{\beta}d^{2p+1}\|a\|_{[-d,0]}\right]\right\}.$$

Note that the expression in the first square brackets is not positive since $m_1^2\le 2r_0^2$ and $r(x,t)\ge 2r_0^2$ and the expression in the second square brackets is not positive since $-d < x < 0$ and $\mu \leq \varepsilon$. Hence,

$$L\phi_1(x,t) \leq 0$$

and, from Lemma 1, it follows

$$|W_L(x,t)| \leq \phi_1(x,t) \leq M\,\exp(-m_1\varepsilon^{-1}(d+x)), \quad (x,\,t) \in \overline{G}^-.$$

Using the same argument as this one for the problems associated to

$$\frac{\partial^{k_0} W_L}{\partial t^{k_0}},\ k_0 = 1, 2,$$

we obtain

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}} W_L(x,t)\right| \leq M\,\exp(-m_1\varepsilon^{-1}(d+x)), \quad (x,\,t) \in \overline{G}^-, \quad k_0 = 1, 2. \quad (18)$$

Finally, we consider the right singular component W_R. Now the barrier function is given by

$$\phi_2(x,t) = M\,\exp(-m_1\varepsilon^{-1}(d-x))\,\exp\left(\frac{m_1}{\beta} d^{2p+1}\|a\|_{[0,d]}t\right),\ (x,\,t) \in \overline{G}^+,$$

with $\|a\|_{[0,d]} = \max_{x\in[0,d]} a(x)$. Then, similarly to the analysis made for W_L, we can obtain

$$|W_R(x,t)| \leq \phi_2(x,t) \leq M\,\exp(-m_1\varepsilon^{-1}(d-x)), \quad (x,\,t) \in \overline{G}^+,$$

and

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}} W_R(x,t)\right| \leq M\,\exp(-m_1\varepsilon^{-1}(d-x)), \quad (x,\,t) \in \overline{G}^+, \quad k_0 = 1, 2. \quad (19)$$

We will use again in the analysis of the convergence the crude bounds of the partial derivatives w.r.t. x for both layer components

$$\left|\frac{\partial^k}{\partial x^k} W_L(x,t)\right| \leq M\,\varepsilon^{-k},\ (x,\,t) \in \overline{G}^-,\ \left|\frac{\partial^k}{\partial x^k} W_R(x,t)\right| \leq M\,\varepsilon^{-k}, \quad (x,\,t) \in \overline{G}^+, \quad (20)$$

with $1 \leq k \leq 4$.

Theorem 1. *Let $u(x,t)$ be the solution of* (1) *and $U^\pm$ be the regular component, $V^\pm$ be the interior layer component, and W_R, W_L be the right and left boundary layer components given in* (5). *Then, the estimates* (13), (15) (16), (17), (18), (19) *and* (20) *hold, showing the asymptotic behavior of $u(x,t)$ and its derivatives.*

3 The numerical scheme: uniform convergence

In this section we construct and analyze the uniform convergence of a finite difference scheme to solve the problem (1). Let us denote by N_0 and N the number of intervals in t and x, respectively. The rectangular grid is given by $\overline{G}_h = \overline{\omega} \times \overline{\omega}_0$, where $\overline{\omega}_0$ is a uniform mesh for the time variable, where the step size is $\tau = T/N_0$, and $\overline{\omega}$ is a piecewise uniform mesh of Shishkin type for the space variable, such that $x = 0 \in \overline{\omega}$, which condenses in a neighborhood of the interior and boundary layers (see, e.g., [1,6]). We divide the interval $[-d,\, d]$ into five parts $[-d,\, -d+\sigma]$, $[-d+\sigma,\, -\sigma]$, $[-\sigma,\, \sigma]$, $[\sigma,\, d-\sigma]$ and $[d-\sigma,\, d]$, where the transition parameter σ is defined by

$$\sigma = \min\left[4^{-1}\, d,\; m^{-1}\,\varepsilon\, \ln N\right], \tag{21}$$

with $0 < m \le r_0$. We approximate on $\overline{G}_h$ the problem (1) by the finite difference scheme

$$\begin{aligned}
\Lambda\, z(x,t) &= f(x,t), && (x,t) \in G_h \setminus S_h^{\pm},\\
z(x,t) &= \varphi(x,t), && (x,t) \in S_h,\\
\Lambda^{\pm}\, z(x,t) &\equiv \varepsilon\left[\delta_x\, z(x,t) - \delta_{\overline{x}}\, z(x,t)\right] = 0, && (x,t) \in S_h^{\pm},
\end{aligned} \tag{22}$$

where

$$\Lambda\, z(x,t) \equiv \left\{\varepsilon^2\, \delta_{\overline{x}\widehat{x}} + \mu x^{2p+1} a(x)\, \delta_x^* - b(x,t)\delta_{\overline{t}} - r(x,t)\right\} z(x,t),$$

$G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, $S_h^{\pm} = S^{\pm} \cap \overline{G}_h$, and

$$\delta_x^*\, z(x,t) = \begin{cases} \delta_x\, z(x,t), & \text{if } x > 0,\\ \delta_{\overline{x}}\, z(x,t), & \text{if } x < 0, \end{cases}$$

is the monotone approximation of the first-order derivative $\dfrac{\partial}{\partial x} u(x,t)$ in the differential equation, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ is the second-order central differences on a nonuniform grid, given by

$$\delta_{\overline{x}\widehat{x}}\, z(x^i,t) = 2\left(h^i + h^{i-1}\right)^{-1}\left[\delta_x\, z(x^i,t) - \delta_{\overline{x}}\, z(x^i,t)\right], \quad (x^i,t) \in G_h,$$

$\delta_x\, z(x,t)$ and $\delta_{\overline{x}}\, z(x,t)$ are the first-order (forward and backward respectively) difference derivatives

$$\delta_x\, z(x^i,t) = \left(h^i\right)^{-1}\left[z(x^{i+1},t) - z(x^i,t)\right],\;\; \delta_{\overline{x}}\, z(x,t) = (h^{i-1})^{-1}\left[z(x^i,t) - z(x^{i-1},t)\right],$$

with $h^i = x^{i+1} - x^i$, $h^{i-1} = x^i - x^{i-1}$, x^{i-1}, x^i, $x^{i+1} \in \overline{\omega}$ and $\delta_{\overline{t}}\, z(x,t) = \tau^{-1}[z(x,t) - z(x,t-\tau)]$, $(x,t) \in G_h$.

To prove the uniform convergence of the finite difference scheme (22), we will use the following discrete comparison principle.

Lemma 3. *The finite difference scheme* (22) *is ε-uniformly monotone (see, e.g., [2,14]). Moreover, if the functions $z^1(x,t)$, $z^2(x,t)$, $(x,t) \in \overline{G}_h$ satisfy the conditions*

$$
\begin{aligned}
\Lambda z^1(x,\, t) \quad &\le \Lambda z^2(x,t), \qquad (x,\, t) \in G_h \setminus S_h^{\pm},\\
\Lambda^{\pm}\, z^1(x,\, t) &\le \Lambda^{\pm}\, z^2(x,\, t), \quad (x,\, t) \in S_h^{\pm},\\
z^1(x,\, t) \qquad &\ge z^2(x,\, t), \qquad (x,\, t) \in S_h,
\end{aligned}
$$

then $z^1(x,\, t) \ge z^2(x,\, t), \quad (x,t) \in \overline{G}_h$.

The main result of the paper is the following.

Theorem 2. *Let $u(x,t)$ be the solution of* (1) *and $z(x,t)$ be the solution of the difference scheme* (22) *on the grid $\overline{G}_h$. Assume that $\mu \le \varepsilon$. Then, the error satisfies the estimate*

$$
|u(x,t) - z(x,t)| \le M\left[N^{-1}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \tag{23}
$$

i.e., the difference scheme converges uniformly in ε and μ with first order in time and almost first order in space.

PROOF. In the proof we distinguish two cases.

In the first one, we assume that $\sigma = d/4$, and therefore $\varepsilon^{-1} \le C \ln N$. The truncation error at the interior points of G_h, excluding the interface $S_h^{\pm}$, is given by

$$
\Lambda(u - z) = \varepsilon^2(\delta_{\overline{x}\widehat{x}}u - u_{xx}) + \mu a(x)x^{2p+1}(\delta_x^* u - u_x) - b(x,t)(\delta_{\overline{t}}u - u_t).
$$

Taking Taylor expansions and using the coarse estimates (4), it is straightforward to prove that

$$
\begin{aligned}
|\Lambda(u - z)| &\le M(N^{-1}\ln N + N_0^{-1}), \quad \text{on } G_h \setminus S_h^{\pm},\\
|\Lambda^{\pm}(u - z)| &\le M\, N^{-1}\, \ln N, \quad \text{on } S_h^{\pm}.
\end{aligned}
$$

Then, from Lemma 3, it follows

$$
|u - z| \le M\left(N^{-1}\ln N + N_0^{-1}\right). \tag{24}
$$

In the second case, we assume that $\sigma \ne d/4$. Now, we write the components of the continuous problem as $u = U + V + W_L + W_R,\ U = U^+ \cup U^-,\ V =$

$V^+ \cup V^-$. Similarly to the continuous problem, we consider a decomposition of the numerical solution as $z = \upsilon + w + w_L + w_R$, $\upsilon = \upsilon^+ \cup \upsilon^-$, $w = w^+ \cup w^-$, where the discrete regular component υ is the solution of the problem

$$\begin{aligned} \Lambda \upsilon &= f, && \text{in } \ G_h^+ \cup G_h^-, \\ \upsilon &= U, && \text{on } \ (S_0)_h \cup S_h^l \cup S_h^r, \\ \Lambda^\pm \upsilon &= \varepsilon \left[\delta_x U^+ - \delta_{\overline{x}} U^- \right], && \text{on } \ S_h^\pm, \end{aligned} \tag{25}$$

the discrete singular component w is the solution of the problem

$$\begin{aligned} &\Lambda w = 0, && \text{in } \ G_h^+ \cup G_h^-, \\ &w = V = 0, && \text{on } \ (S_0)_h \cup S_h^l \cup S_h^r, \\ &\Lambda^\pm w = -\Lambda^\pm \upsilon && \text{on } \ S_h^\pm, \end{aligned} \tag{26}$$

the discrete singular component w_L satisfies

$$\begin{aligned} &\Lambda w_L = 0, && \text{in } \ G_h^-, \\ &w_L = 0, && \text{on } \ (S_0^-)_h \cup S_h^\pm, \\ &w_L = W_L && \text{on } \ S_h^l, \end{aligned} \tag{27}$$

and the discrete singular component w_R satisfies

$$\begin{aligned} &\Lambda w_R = 0, && \text{in } \ G_h^+, \\ &w_R = 0, && \text{on } \ (S_0^+)_h \cup S_h^\pm, \\ &w_R = W_R && \text{on } \ S_h^r. \end{aligned} \tag{28}$$

Using Taylor expansions, the local error associated to the regular component satisfies

$$\begin{aligned} &|\Lambda(\upsilon - U)| \leq M(\varepsilon N^{-1} + N_0^{-1}), && \text{in } G_h^+ \cup G_h^-, \\ &\upsilon - U = 0, && \text{on } (S_0)_h \cup S_h^l \cup S_h^r, \\ &|\Lambda^\pm(\upsilon - U)| \leq M \varepsilon N^{-1}, && \text{on } S_h^\pm, \end{aligned}$$

and therefore, using again the discrete comparison principle, it follows

$$|(\upsilon - U)| \leq M(\varepsilon N^{-1} + N_0^{-1}), \qquad \text{on } \ \overline{G}_h. \tag{29}$$

Next, we consider the interior layer component. At the boundary, this component satisfies

$$|(w - V)| = 0, \quad \text{on } (S_0)_h \cup S_h^l \cup S_h^r,$$

and at $S_h^{\pm}$, we have the estimate

$$\left|\Lambda^{\pm}(w-V)\right| \le M\,N^{-1}\,\ln N. \tag{30}$$

We study the error for the grid points in $G^+ \cup G^-$. First, we assume that the grid point is such that $|x| \ge \sigma$. Recall that $\sigma = m^{-1}\varepsilon \ln N$. From (15), (17) and choosing $m_1 \ge m$, we obtain

$$|V(x,t)| \le M \exp(-m_1|x|/\varepsilon) \le MN^{-1}.$$

To bound its discrete counterpart, we consider the discrete barrier function

$$\phi(x^j,t) = \begin{cases} \displaystyle\prod_{i=j+1}^{N/2}\left(1+r_0\frac{h^i}{\varepsilon}\right)^{-1}, & \text{if } 0 \le j < N/2, \\ 1, & \text{if } j = N/2, \\ \displaystyle\prod_{i=N/2+1}^{j}\left(1+r_0\frac{h^i}{\varepsilon}\right)^{-1}, & \text{if } N/2 < j \le N. \end{cases}$$

It satisfies

$$\begin{array}{llll} \varepsilon^2\delta_{\overline{x}\widehat{x}}\phi(x^j,t) < 2r_0^2\phi(x^j,t), & \delta_{\overline{x}}\phi(x^j,t) > 0, & \delta_{\overline{t}}\phi(x^j,t) = 0, & \text{in } G_h^-, \\ \varepsilon^2\delta_{\overline{x}\widehat{x}}\phi(x^j,t) < 2r_0^2\phi(x^j,t), & \delta_{x}\phi(x^j,t) < 0, & \delta_{\overline{t}}\phi(x^j,t) = 0, & \text{in } G_h^+. \end{array}$$

Thus,

$$\Lambda\phi < 0, \text{ in } G_h^+ \cup G_h^-, \quad \Lambda^{\pm}\phi < 0, \text{ in } S_h^{\pm},$$

and therefore the comparison principle proves the following estimate

$$|w(x,t)| \le \phi(x,t) \le MN^{-r_0/m} \le MN^{-1},$$

where we have used that $m \le r_0$.

From the triangular inequality, we have $|w-V| \le |w| + |V|$, and therefore

$$|(w-V)(x,t)| \le MN^{-1}, \quad |x| \ge \sigma, \quad (x,t) \in G_h^- \cup G_h^+. \tag{31}$$

If the grid point is such that $|x| < \sigma$, taking Taylor expansions, we deduce that

$$\begin{array}{ll} |\Lambda(w-V)(x,t)| \le M(N^{-1}\ln N + N_0^{-1}), & \text{for } |x| < \sigma, \quad (x,t) \in G_h^- \cup G_h^+, \\ |\Lambda^{\pm}(w-V)(x,t)| \le MN^{-1}\ln N, & (x,t) \in S_h^{\pm}. \end{array}$$

Using the discrete comparison principle, now on the interval $[-\sigma,\sigma]$, it follows

$$|(w-V)(x,t)| \le M(N^{-1}\ln N + N_0^{-1}), \quad (x,t) \in [-\sigma,\sigma] \cap (G_h^- \cup G_h^+). \tag{32}$$

The boundary layer components w_L and w_R are defined in G^- and G^+, respectively. Then, the analysis of the convergence of the finite difference scheme on the Shishkin mesh is standard and we have the estimates

$$\begin{aligned} |(W_L - w_L)| &\leq M(N^{-1}\ln N + N_0^{-1}), \quad \text{in } G_h^-\,, \\ |(W_R - w_R)| &\leq M(N^{-1}\ln N + N_0^{-1}), \quad \text{in } G_h^+\,. \end{aligned} \tag{33}$$

From the estimates (29), (30), (31), (32) and (33), it follows

$$|u - z| \leq M(N^{-1}\ln N + N_0^{-1}),$$

which is the required result.

4 Numerical experiments

In this section, we show the numerical results obtained for the test problem

$$\begin{cases} \varepsilon^2 u_{xx} + \mu x^3(1+x^2)u_x - u_t - (3+xt)u = f(x,t), & (x,t) \in G \setminus S^{\pm}, \\ u_x(x+0,t) - u_x(x-0,t) = 0, & (x,t) \in S^{\pm}, \\ u(x,t) = 0, & (x,t) \in S, \end{cases} \tag{34}$$

where $\overline{G} = [-1,1] \times [0,1]$, $S^{\pm} = \{x = 0\} \times (0,1]$, taking different values of μ and

$$f(x,t) = \begin{cases} (1-t^2)(x+e^x), & \text{if } x > 0,\ t \in (0,1], \\ -(1+t)(x^2+1), & \text{if } x < 0,\ t \in (0,1]. \end{cases} \tag{35}$$

Figure 2 displays the numerical approximation on the piecewise-uniform Shishkin mesh for $\varepsilon = 10^{-2}, N = N_0 = 32$ when $\mu = \varepsilon^{3/2} = 10^{-3}$; from it we see the interior and the boundary layers.

To approximate the numerical errors we use a variant of the double mesh principle (see [8]): the approximated error $D_{i,n}^{\varepsilon,N,N_0}$ at the grid point (x_i, t_n) is calculated by

$$D_{i,n}^{\varepsilon,N,N_0} = \left| U_{i,n}^{\varepsilon,N,N_0} - U_{i,n}^{\varepsilon,2N,2N_0} \right|,\ i = 0, 1 \ldots, N,\ n = 0, 1, \ldots, N_0,$$

where $U_{i,n}^{\varepsilon,N,N_0}$ is the numerical solution obtained on $\overline{G}_h$ by using the constant time step $\tau = 1/N_0$, and $(N+1)$ points in the spatial mesh, and $U_{i,n}^{\varepsilon,2N,2N_0}$ is the numerical solution when the time step size is $\tau/2$, and we take $(2N+1)$ points in the spatial mesh, but with the same transition parameter as in the

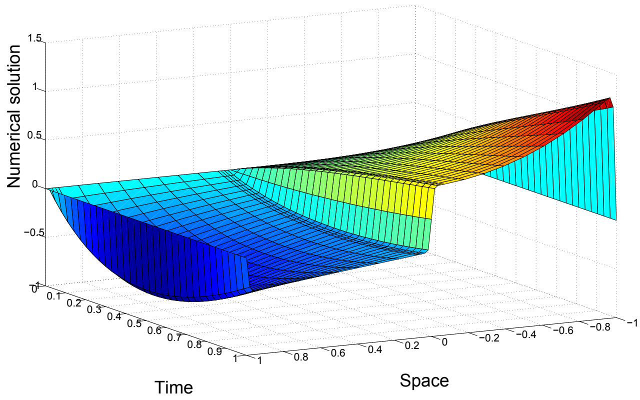

Fig. 2. Numerical solution of problem (34), (35) for $\varepsilon = 10^{-2}$, $\mu = 10^{-3}$ with $N = N_0 = 32$ on the Shishkin mesh

original mesh $\overline{G}_h$. Both numerical solutions are compared in the coarse grid $\overline{G}_h$. For each fixed value of ε, the maximum global errors D^{ε,N,N_0} are estimated by

$$D^{\varepsilon,N,N_0} = \max_{i,n} D_{i,n}^{\varepsilon,N,N_0},$$

and therefore, the numerical orders of convergence q are given by

$$q = q(\varepsilon, N, N_0) = \frac{\log\left(D^{\varepsilon,N,N_0}/D^{\varepsilon,2N,2N_0}\right)}{\log 2}.$$

From these values we obtain the ε-uniform errors D^{N,N_0} and the ε-uniform orders of convergence q_{uni}, respectively, by

$$D^{N,N_0} = \max_{\varepsilon} D^{\varepsilon,N,N_0}, \quad q_{uni} = q_{uni}(N, N_0) = \frac{\log\left(D^{N,N_0}/D^{2N,2N_0}\right)}{\log 2}. \tag{36}$$

Results of numerical experiments are given in Tables 1–4, where, for simplicity, we take $N = N_0$.

Table 1 displays the results on a uniform mesh and $\mu = \varepsilon^{3/2}$; from it, we see that the solution of a difference scheme on a uniform mesh does not converge ε-uniformly.

Table 2 displays the numerical results for errors in the solution of the difference scheme (22) on the piecewise uniform Shishkin mesh in x taking $m = 1/2$ with $\mu = \varepsilon^{3/2}$. This table shows that the solution of the scheme converges ε-uniformly with order of the convergence rate close to one in agreement with the theoretical estimate.

In Tables 3 and 4, the numerical results are given for the same difference scheme as in Table 2 but with $\mu = \varepsilon$ in Table 3 and with $\mu = \varepsilon^2$ in Table

Table 1
Maximum errors and orders of convergence on a uniform mesh when $\mu = \varepsilon^{3/2}$

ε^2	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-5}	0.8244E-2 1.073	0.3919E-2 1.042	0.1903E-2 1.022	0.9367E-3 1.012	0.4646E-3 1.006	0.2313E-3
2^{-7}	0.1388E-1 1.209	0.6002E-2 1.137	0.2730E-2 1.077	0.1294E-2 1.041	0.6286E-3 1.021	0.3097E-3
2^{-9}	0.3161E-1 1.341	0.1248E-1 1.294	0.5090E-2 1.183	0.2242E-2 1.112	0.1037E-2 1.061	0.4973E-3
2^{-11}	0.5521E-1 0.884	0.2991E-1 1.428	0.1111E-1 1.372	0.4293E-2 1.240	0.1817E-2 1.149	0.8191E-3
2^{-13}	0.4255E-1 -0.300	0.5237E-1 0.909	0.2789E-1 1.509	0.9800E-2 1.453	0.3579E-2 1.305	0.1448E-2
2^{-15}	0.1754E-1 -1.170	0.3947E-1 -0.328	0.4955E-1 0.927	0.2606E-1 1.581	0.8710E-2 1.535	0.3006E-2
2^{-17}	0.8315E-2 -0.926	0.1579E-1 -1.219	0.3675E-1 -0.360	0.4717E-1 0.941	0.2457E-1 1.643	0.7866E-2
2^{-19}	0.6745E-2 0.379	0.5185E-2 -1.466	0.1432E-1 -1.272	0.3460E-1 -0.389	0.4531E-1 0.952	0.2342E-1
2^{-21}	0.6314E-2 0.774	0.3693E-2 -0.263	0.4432E-2 -1.573	0.1319E-1 -1.322	0.3298E-1 -0.414	0.4393E-1
2^{-23}	0.6185E-2 0.903	0.3307E-2 0.666	0.2085E-2 -0.933	0.3979E-2 -1.634	0.1235E-1 -1.364	0.3179E-1
2^{-25}	0.6145E-2 0.942	0.3198E-2 0.886	0.1731E-2 0.460	0.1258E-2 -1.536	0.3647E-2 -1.687	0.1174E-1
D^{N,N_0} q_{uni}	0.5521E-1 0.076	0.5237E-1 0.080	0.4955E-1 0.071	0.4717E-1 0.058	0.4531E-1 0.045	0.4393E-1

4. We have chosen $m = 1/2$ in (21) to define the Shishkin mesh. From these results it follows that the difference schemes converge ε-uniformly with order of the convergence rate not lower than 0.68 in Table 3 and 1.04 in Table 4.

From results in Tables 2, 3 and 4, it is seen that the rate of the ε-uniform convergence decreases with growth of μ.

Thus, the results of numerical experiments are in agreement with the theoretical estimate.

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Table 2
Maximum errors and uniform orders of convergence on the Shishkin mesh when $\mu = \varepsilon^{3/2}$

ε^2	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-5}	0.8244E-2	0.3919E-2	0.1903E-2	0.9367E-3	0.4646E-3	0.2313E-3
	1.073	1.042	1.022	1.012	1.006	
2^{-7}	0.1388E-1	0.6002E-2	0.2730E-2	0.1294E-2	0.6286E-3	0.3097E-3
	1.341	1.294	1.183	1.112	1.061	
2^{-9}	0.3161E-1	0.1248E-1	0.5090E-2	0.2242E-2	0.1037E-2	0.4973E-3
	1.213	1.163	1.114	1.061	1.033	
2^{-11}	0.4610E-1	0.2397E-1	0.1082E-1	0.4293E-2	0.1817E-2	0.8191E-3
	0.944	1.148	1.334	1.240	1.149	
2^{-13}	0.4364E-1	0.2218E-1	0.9537E-2	0.4131E-2	0.1873E-2	0.8960E-3
	0.977	1.217	1.207	1.141	1.064	
2^{-15}	0.4126E-1	0.2057E-1	0.8469E-2	0.3506E-2	0.1517E-2	0.6992E-3
	1.004	1.280	1.272	1.208	1.118	
2^{-17}	0.3929E-1	0.1927E-1	0.7641E-2	0.3028E-2	0.1248E-2	0.5509E-3
	1.028	1.335	1.335	1.278	1.180	
2^{-19}	0.3776E-1	0.1828E-1	0.7118E-2	0.2753E-2	0.1068E-2	0.4443E-3
	1.047	1.361	1.370	1.366	1.266	
2^{-21}	0.3661E-1	0.1761E-1	0.7011E-2	0.2654E-2	0.1011E-2	0.4084E-3
	1.056	1.328	1.401	1.393	1.307	
2^{-23}	0.3577E-1	0.1737E-1	0.6937E-2	0.2601E-2	0.9740E-3	0.3852E-3
	1.042	1.324	1.415	1.417	1.338	
2^{-25}	0.3517E-1	0.1721E-1	0.6884E-2	0.2571E-2	0.9497E-3	0.3700E-3
	1.031	1.322	1.421	1.437	1.360	
D^{N,N_0}	0.4610E-1	0.2397E-1	0.1082E-1	0.4293E-2	0.1873E-2	0.8960E-3
q_{uni}	0.944	1.148	1.334	1.197	1.064	

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Table 3
Maximum errors and uniform orders of convergence on the Shishkin mesh when $\mu = \varepsilon$

ε^2	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
2^{-5}	0.1035E-1	0.5178E-2	0.2585E-2	0.1291E-2	0.6450E-3	0.3224E-3
	1.000	1.002	1.002	1.001	1.001	
2^{-7}	0.2012E-1	0.1037E-1	0.5217E-2	0.2615E-2	0.1309E-2	0.6547E-3
	0.956	0.991	0.996	0.998	0.999	
2^{-9}	0.4077E-1	0.2216E-1	0.1115E-1	0.5603E-2	0.2801E-2	0.1400E-2
	0.879	0.992	0.992	1.001	1.001	
2^{-11}	0.6248E-1	0.3873E-1	0.2267E-1	0.1160E-1	0.5796E-2	0.2894E-2
	0.690	0.773	0.966	1.001	1.002	
2^{-13}	0.6387E-1	0.3970E-1	0.2312E-1	0.1301E-1	0.7223E-2	0.3968E-2
	0.686	0.780	0.830	0.848	0.864	
2^{-15}	0.6455E-1	0.4018E-1	0.2333E-1	0.1311E-1	0.7290E-2	0.4000E-2
	0.684	0.784	0.832	0.847	0.866	
2^{-17}	0.6489E-1	0.4042E-1	0.2344E-1	0.1317E-1	0.7322E-2	0.4016E-2
	0.683	0.786	0.832	0.846	0.867	
2^{-19}	0.6506E-1	0.4053E-1	0.2349E-1	0.1320E-1	0.7339E-2	0.4024E-2
	0.683	0.787	0.832	0.847	0.867	
2^{-21}	0.6514E-1	0.4059E-1	0.2352E-1	0.1322E-1	0.7347E-2	0.4028E-2
	0.682	0.787	0.832	0.847	0.867	
2^{-23}	0.6518E-1	0.4062E-1	0.2353E-1	0.1323E-1	0.7351E-2	0.4031E-2
	0.682	0.788	0.831	0.847	0.867	
2^{-25}	0.6520E-1	0.4064E-1	0.2354E-1	0.1323E-1	0.7353E-2	0.4032E-2
	0.682	0.788	0.831	0.847	0.867	
D^{N,N_0}	0.6520E-1	0.4064E-1	0.2354E-1	0.1323E-1	0.7353E-2	0.4032E-2
q_{uni}	0.682	0.788	0.831	0.847	0.867	

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Table 4
Maximum errors and uniform orders of convergence on the Shishkin mesh when $\mu = \varepsilon^2$

ε^2	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
2^{-5}	0.7113E-2	0.3311E-2	0.1593E-2	0.7803E-3	0.3861E-3	0.1921E-3
	1.103	1.056	1.029	1.015	1.007	
2^{-7}	0.1135E-1	0.4570E-2	0.1977E-2	0.9131E-3	0.4382E-3	0.2147E-3
	1.313	1.209	1.114	1.059	1.030	
2^{-9}	0.2363E-1	0.8474E-2	0.3053E-2	0.1196E-2	0.5169E-3	0.2393E-3
	1.479	1.473	1.352	1.210	1.111	
2^{-11}	0.3575E-1	0.1739E-1	0.6954E-2	0.2289E-2	0.7817E-3	0.3042E-3
	1.040	1.322	1.603	1.550	1.361	
2^{-13}	0.3481E-1	0.1711E-1	0.6857E-2	0.2557E-2	0.9380E-3	0.3626E-3
	1.024	1.319	1.423	1.447	1.371	
2^{-15}	0.3426E-1	0.1696E-1	0.6808E-2	0.2529E-2	0.9169E-3	0.3502E-3
	1.014	1.317	1.428	1.464	1.389	
2^{-17}	0.3397E-1	0.1688E-1	0.6783E-2	0.2516E-2	0.9077E-3	0.3443E-3
	1.009	1.315	1.431	1.471	1.399	
2^{-19}	0.3382E-1	0.1684E-1	0.6770E-2	0.2509E-2	0.9037E-3	0.3413E-3
	1.006	1.315	1.432	1.473	1.405	
2^{-21}	0.3374E-1	0.1682E-1	0.6764E-2	0.2505E-2	0.9016E-3	0.3399E-3
	1.004	1.314	1.433	1.474	1.407	
2^{-23}	0.3370E-1	0.1681E-1	0.6761E-2	0.2504E-2	0.9006E-3	0.3392E-3
	1.003	1.314	1.433	1.475	1.409	
2^{-25}	0.3368E-1	0.1680E-1	0.6759E-2	0.2503E-2	0.9001E-3	0.3388E-3
	1.003	1.314	1.433	1.475	1.410	
D^{N,N_0}	0.3575E-1	0.1739E-1	0.6954E-2	0.2557E-2	0.9380E-3	0.3626E-3
q_{uni}	1.040	1.322	1.443	1.447	1.371	

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