2-Engel Relations between Subgroups

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Abstract

In this paper we study groups G generated by two subgroups A and B such that $\langle a, b\rangle$ is nilpotent of class at most 2 for all $a \in A$ and $b \in B$. A detailed description of the structure of such groups is obtained, generalizing the classical result of Hopkins and Levi on 2-Engel groups.

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1 Introduction

A well-known result, usually attributed to Levi [19] though already proved by Hopkins [15] and in parts by Burnside [4], states that a group G satisfying the 2-Engel condition $[x, y, y] = 1$ for all $x, y \in G$ is nilpotent of class at most 3 with $[G', G]$ of exponent dividing 3, where G' denotes the derived subgroup of G. (For the history of this result and a presentation of the general theory of n-Engel groups we refer to the survey of Traustason [22].)

In the present paper we assume the 2-Engel condition only for certain pairs of elements. More precisely, we investigate groups generated by two subgroups A and

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B such that $[a, b, b] = [b, a, a] = 1$ for all $a \in A$ and $b \in B$. Of course, this is equivalent to saying that $\langle a, b\rangle$ is nilpotent of class at most 2 for all $a \in A$ and $b \in B$. In the terminology of [5] this property can be stated as A and B being $\mathcal{N}_2$-connected where $\mathcal{N}_2$ denotes the class of nilpotent groups of class at most 2. Therefore this paper can also be seen as a contribution to the study of $\mathcal{C}$-connected subgroups for various classes of groups $\mathcal{C}$ (cf. [2], [6], [8], [9], [10], [14]).

For $\mathcal{N}_2$-connected subgroups A and B, our main objective is to obtain information about the structure and embedding of $[A, B]$ and certain distinguished subgroups of A and B.

One of the basic results is that $[A, B]$ centralizes A' and B' (Proposition 3.2 (i)). Moreover, $[A', B]$ centralizes A whence, setting $G = \langle A, B\rangle$, $[A', B'] \leq Z(G)$ (Theorem 4.8 (i)).

Denoting by A^2 the group $\langle a^2 | a \in A\rangle$, we show in Theorem 4.1 and Theorem 4.8 that $[A^2, B] \leq Z([A, B]) \cap Z_3(G)$. In particular, if $A^2 = A$ or $B^2 = B$, the commutator $[A, B]$ is abelian. The same is true if $[A, B]'$ does not contain an element of order 2 (Corollary 4.3) or if A and B are normal in G (Theorem 6.1). However, in general this need not be the case. In Section 7 we construct examples of finite 2-groups generated by $\mathcal{N}_2$-connected subgroups A and B where $[A, B]$ is nilpotent of arbitrarily large derived length (and hence also of arbitrarily large class).

It is easy to see that for infinite $\mathcal{N}_2$-connected subgroups A and B the commutator subgroup $[A, B]$ need not be nilpotent (Remark 4.7). In yet unpublished work, R. Dark has constructed examples of groups generated by two $\mathcal{N}_2$-connected elementary abelian subgroups A and B, each of order 4, where in one of these $[A, B]$ is infinite (and hence not nilpotent) and in the other $[A, B]$ is finite and not nilpotent.

For finite groups, nilpotency of $[A, B]$ is equivalent to the subnormality of A and B in G (see Theorem 2.3). In light of Dark's examples just mentioned it is interesting that for arbitrary $\mathcal{N}_2$-connected subgroups A and B of $G = \langle A, B\rangle$ certain characteristic subgroups of A and B are subnormal in G. It is shown in Theorem 5.1 that A^2 is subnormal in G of defect at most 3, A' is subnormal in G of defect at most 2 and the higher terms of the lower central series of A centralize B and are therefore normal in G.

Theorem 5.3 is concerned with the intersection of A and B: $A \cap B$ is a nilpotent subgroup of class at most 3; it is subnormal in G of defect at most 3. Moreover, $[(A \cap B)', G]$ is a central elementary abelian 3-subgroup of G. For $A = B = G$, this is just the result of Hopkins and Levi.

We conclude these introductory remarks by pointing out that for normal $\mathcal{N}_2$-connected subgroups A and B significantly stronger statements are possible than those in the general situation. We mentioned already that in this case $[A, B]$ is abelian. Furthermore, A' and B' centralize each other (Theorem 6.1). Finally, if A and B are nilpotent of class a and b, respectively, then $G = AB$ has nilpotency class at most $\max(a, b) + 1$ (Theorem 6.3).

2 Preliminaries

The group theoretical notation used in this paper is usually consistent with the one used in [7].

In addition, if G and H are groups and G is isomorphic to a subgroup of H, we write $G \lessapprox H$.

Moreover, given a group G, we set $G^2 = \langle g^2 \mid g \in G \rangle$, the smallest normal subgroup of G whose factor group is an elementary abelian 2-group (possibly infinite).

By $O^2(G)$ we denote the subgroup generated by all elements of (finite) odd order in G; clearly, $O^2(G) \leq G^2$. If G is periodic (that is, all elements of G have finite order), then $O^2(G)$ is the smallest normal subgroup of G whose factor group is a 2-group (that is, all elements have 2-power order).

The terms of the lower central series of a group G are denoted by $\Gamma^n(G)$ where $\Gamma^1(G) = G$ and $\Gamma^{n+1}(G) = [\Gamma^n(G), G]$ for $n \geq 1$. Also $G^{(n)}$ denote the terms of the derived series of G, i.e., $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 0$. Occasionally we write G' for $G^{(1)}$ and G'' for $G^{(2)}$.

We set $\Gamma^\infty(G) = \bigcap_{n \geq 1} \Gamma^n(G)$. Note that for finite G, $\Gamma^\infty(G) = G^{\mathcal{N}}$, the $\mathcal{N}$-residual of G, i.e. the smallest normal subgroup of G with nilpotent factor group.

For subsets S and T of a group G we set $[S, T] = \langle [s, t] \mid s \in S,\ t \in T \rangle$ and $\langle S^T \rangle = \langle s^t \mid s \in S,\ t \in T \rangle$. If X is a subgroup of G, then $\langle S^X \rangle$ is the smallest X-invariant subgroup of G containing S, that is $\langle S^X \rangle = \langle S \rangle [X, \langle S \rangle]$. For $s \in G$ we write $\langle s^X \rangle$ for $\langle \{s\}^X \rangle$.

Lemma 2.1. *Let S, T be subsets and X, Y, C subgroups of a group. Then:*

(i) ([21, 5.1.6, 5.1.7]) $[X, T]$ *is X-invariant.*

If $X = \langle S \rangle$, then $[X, T] = \langle [S, T]^X \rangle$.

If $X = \langle S \rangle$ and $Y = \langle T \rangle$, then $[X, Y] = \langle [S, T]^{\langle X, Y \rangle} \rangle = \langle [S, T]^{XY} \rangle$.

(ii) *If C is normal in $\langle X, Y \rangle$ and centralizes $[X, Y]$, then $[C, X, Y] = [C, Y, X]$.*

Proof (ii) We have $[C, X] = [\,C, X[X, Y]\,]$, a normal subgroup of $\langle X, Y \rangle$ contained in C. Analogously, $[C, Y]$, $[C, X, Y]$ and $[C, Y, X]$ are normal subgroups of $\langle X, Y \rangle$. Since $[X, Y, C] = 1$ and $[Y, C, X] = [C, Y, X]$, it follows that $[C, X, Y] \leq [C, Y, X]$ by the Three Subgroups Lemma. Similarly, $[C, Y, X] \leq [C, X, Y]$. $\square$

The following concept, due to Carocca [5], is central for this paper.

Definition. For a non-empty class of groups $\mathcal{C}$, two subgroups A and B of a group G are said to be $\mathcal{C}$-connected if $\langle a, b \rangle \in \mathcal{C}$ for all $a \in A$ and $b \in B$.

We will focus here on the case $\mathcal{C} = \mathcal{N}_2$, where we denote by $\mathcal{N}_c$ the class of nilpotent groups of class at most c. The structure of products of finite $\mathcal{N}$-connected groups, $\mathcal{N}$ the class of nilpotent groups, is rather well understood (cf. [2], [5], [14]) and occasionally we will refer to the corresponding results. However, our main interest is in $\mathcal{N}_2$-connected subgroups that do not necessarily permute, and in this situation there is not much known in the $\mathcal{N}$-connected case. We will need some simple properties about the commutator of $\mathcal{N}$-connected periodic subgroups that are presented in the following lemma.

Lemma 2.2. *Let A and B be periodic $\mathcal{N}$-connected subgroups of $\langle A, B\rangle$. Set*
$X = \langle [a, b] \mid a \in A, b \in B, a, b \text{ } 2-\text{elements}\rangle$ *and*
$Y = \langle [a, b] \mid a \in A, b \in B, a, b \text{ } 2'\text{-elements}\rangle$.
Then:

(i) *X and Y are normal subgroups of $\langle A, B\rangle$ and $[A, B] = XY$.*

(ii) $Y = [O^2(A), O^2(B)] = [O^2(A), B] = [A, O^2(B)]$
$= \langle [a, b] \mid a \in A, b \in B, a \text{ } 2'\text{-element}\rangle$
$= \langle [a, b] \mid a \in A, b \in B, b \text{ } 2'\text{-element}\rangle$.

(iii) *If A and B are finite, then $[A, B] = [A_2, B_2]Y$ for any $A_2 \in \mathrm{Syl}_2(A), B_2 \in \mathrm{Syl}_2(B)$.*

(iv) *If $[A, B]$ is finite and nilpotent, then $[A, B] = X \times Y$, $X = O_2([A, B])$, $Y = O_{2'}([A, B])$.*

Proof (i) Given x in A or in B, we use the decomposition $x = x_2x_{2'}$ where x_2 is a 2-element in $\langle x\rangle$ and $x_{2'}$ is a $2'$-element in $\langle x\rangle$.

Let $a \in A$ and $b \in B$. It follows from $\mathcal{N}$-connection of A and B that $[a_2, b_{2'}] = 1$ and $[b_2, a_{2'}] = 1$ whence

$$[a, b] = [a_{2'}, b]^{a_2}[a_2, b] = [a_{2'}, b_{2'}]^{a_2}[a_2, b_2] = [a_{2'}, b_{2'}][a_2, b_2].$$

Therefore $[A, B] = \langle X, Y\rangle$ and

$$\langle [a, b] \mid a \in A, b \in B, a \text{ } 2'\text{-element}\rangle = Y = \langle [a, b] \mid a \in A, b \in B, b \text{ } 2'\text{-element}\rangle. \quad (*)$$

Since A and B are subgroups, Y is normal in $\langle A, B\rangle$.

Analogously, X is normal in $\langle A, B\rangle$. Finally, $[A, B] = XY$.

(ii) The assertion follows from (*) and normality of Y.

(iii) For $A_2 \in \mathrm{Syl}_2(A), B_2 \in \mathrm{Syl}_2(B)$ we have $A = A_2O^2(A)$ and $B = B_2O^2(B)$. From this the assertion follows with (ii) and the normality of Y (by (i)).

(iv) By definition of X and $\mathcal{N}$-connection of A and B, X is generated by 2-elements. Since $[A, B]$ is finite nilpotent, $X \leq O_2([A, B])$. Similarly, $Y \leq O_{2'}([A, B])$. Part (i) yields the assertion. □

There is one important result about (not necessarily permuting) $\mathcal{N}$-connected subgroups which follows from [1, Theorem 2] and [9, Corollary 1]:

Theorem 2.3. *If A and B are $\mathcal{N}$-connected subgroups of a finite group G, then $[A, B]$ is nilpotent if and only if A and B are subnormal in $\langle A, B\rangle$.*

Remark 2.4. For $\mathcal{N}$-connected subgroups A and B of a finite group, nilpotency of $[A, B]$ is in fact equivalent to $[A, B] \leq Z_\infty(\langle A, B\rangle)$ by [1, Theorem 2] or [9, Corollary 1].

3 $\mathcal{N}_2$-connected subgroups: commutator relations

Let A and B be two $\mathcal{N}_2$-connected subgroups of a group. Since $[a, b] \in Z(\langle a, b\rangle)$ for all $a \in A$ and all $b \in B$, the following statements are clear (cf. [16, III, 1.3 a)]):

1. $[a^n, b^m] = [a, b]^{nm}$ for all $n, m \in \mathbb{Z}$;
2. $\langle [a, b]\rangle = [\langle a\rangle, \langle b\rangle]$.

We will use these facts repeatedly without further reference.

Lemma 3.1. *Let G be a group, A, B subgroups of G.*

(i) *The following properties are equivalent:*

(a) *A and B are $\mathcal{N}_2$-connected.*

(b) *$\langle a^B\rangle$ and $\langle b^A\rangle$ are abelian for all $a \in A$ and $b \in B$.*

(ii) *Assume that A and B are $\mathcal{N}_2$-connected , $a \in A$, $\alpha \in A$, $b \in B$, $\beta \in B$. Then: $[a, b^\beta] = [a, b]^{\beta^2}$ and $[a, b][a, \beta]$ centralizes $b\beta$ and $[\alpha, b][\alpha, \beta]$.*

Proof (i) That A and B are $\mathcal{N}_2$-connected is equivalent to the 2-Engel conditions $[a, b, b] = [b, a, a] = 1$ for all $a \in A$ and all $b \in B$. It is easy to see (cf. [16, III, 6.4]) that this is equivalent to $[b^a, b] = [a^b, a] = 1$ for all $a \in A$ and all $b \in B$. Clearly, this is equivalent to statement (b).
(ii) Since $[a, \beta b] = [a, b][a, \beta]^b = ([a, b][a, \beta])^b$, it follows that

$$[a, b][a, \beta] = [a, \beta b]^{b^{-1}} = [a^{b^{-1}}, b\beta].$$

By $\mathcal{N}_2$-connection $\langle a^{b^{-1}}, b\beta\rangle = \langle a, (b\beta)^b\rangle^{b^{-1}} \in \mathcal{N}_2$, whence $[a, b][a, \beta] = [a^{b^{-1}}, b\beta]$ centralizes $b\beta$. Similarly, $[\alpha, b][\alpha, \beta] = [\alpha, \beta b]^{b^{-1}}$. By (i), the subgroup $\langle(\beta b)^A\rangle$ is abelian, whence $[a, \beta b]$ centralizes $[\alpha, \beta b]$ and thus $[a, b][a, \beta]$ centralizes $[\alpha, b][\alpha, \beta]$. Since $\langle a^B\rangle$ is abelian, we have $[a, b][a, \beta] = [a, \beta][a, b] = [a, b\beta]^{\beta^{-1}}$. Therefore

$$[a, b\beta]^{\beta^{-1}} = [a, \beta b]^{b^{-1}}. \tag{*}$$

Equation (*) holds for all $b \in B$ and all $\beta \in B$ and thus

$$[a, (\beta^{-1}b)\beta]^{\beta^{-1}} = [a, \beta(\beta^{-1}b)]^{(\beta^{-1}b)^{-1}} = [a, b]^{b^{-1}\beta} = [a, b]^\beta.$$

Conjugating by β, $[a, b^\beta] = [a, b]^{\beta^2}$. □

In the following we prove various results about commutators of elements in $\mathcal{N}_2$-connected subgroups. They constitute the core for the proofs of the structural results in the subsequent sections.

Proposition 3.2. *Let G be a group, A, B $\mathcal{N}_2$-connected subgroups of G, $a \in A$, $b \in B$. Then:*

(i) *$[A, B]$ centralizes A' and B'. In particular $[a, b]$ centralizes $\langle a^A\rangle$ and $\langle b^B\rangle$.*

(ii) $\langle a^B \rangle$ *and* B *are* $\mathcal{N}_2$*-connected.*

Proof (i) Let $b_1 \in B$, $b_2 \in B$ and note that $[b_1, b_2] = ({b_1}^{-1})^2 (b_1 {b_2}^{-1})^2 \, {b_2}^2$. By Lemma 3.1 (ii) we have $[a, b]^{[b_1, b_2]} = [a, b^{({b_1}^{-1})(b_1 {b_2}^{-1}) b_2}] = [a, b]$. Thus $[A, B]$ centralizes A' and B'. In particular, $[a, b]$ centralizes $\langle a \rangle A'$ and $\langle b \rangle B'$ and (i) follows.

(ii) Since $\langle a^B \rangle$ is abelian, $\langle a^B \rangle$ centralizes $\langle [a^\beta, b] \mid \beta \in B \rangle$. Therefore $[\langle a^B \rangle, b] = \langle [a^\beta, b] \mid \beta \in B \rangle$. By Lemma 3.1 (ii), if $\beta \in B$, then $[a, b^{\beta^{-1}}] = [a, b]^{\beta^{-2}}$ and conjugating by β we have $[a^\beta, b] = [a, b]^{\beta^{-1}}$. Consequently, $[\langle a^B \rangle, b] = \langle [a, b]^B \rangle$ centralizes b by (i) and centralizes $\langle a^B \rangle$, which proves (ii). $\square$

Proposition 3.3. *Let* A *and* B *be* $\mathcal{N}_2$*-connected subgroups of* $\langle A, B \rangle$*,* $a, a_1, a_2 \in A$ *and* $b_1, b_2 \in B$*. Then:*

(i) $[a, b_2, b_1] = [a, b_1, b_2]^{-1} = [a, b_1^{-1}, b_2] = [a, b_1, b_2^{-1}] = [a^{-1}, b_1, b_2] = [b_1, a, b_2]$ *centralizes* a*,* b_1 *and* b_2*.*

(ii) $[[a_1, b_1], [a_2, b_2]] = [[a_1, b_2], [a_2, b_1]] = [[a_1, b_1], [a_2, b_2]]^{-1}$ *centralizes* $[a_1, b_1]$ *and* $[a_2, b_2]$*.*

(iii) $[a, b_1, b_2]^2 = [a, [b_1, b_2]] = [b_2, b_1, a]$*. If* $[a_2, b_2] \in B$*, then* $[a_2, b_2, b_1, a_1] = 1$*.*

The corresponding statements hold with the rôles of A *and* B *interchanged.*

Proof Let x_1, x_2, y_1, y_2 be elements of a group such that $1 = [x_1, y_1] = [x_2, y_2] = [x_1 x_2, y_1 y_2]$. Then $y_1 y_2 = (y_1 y_2)^{x_1 x_2} = {y_1}^{x_1 x_2} {y_2}^{x_1 x_2} = {y_1}^{x_2} {y_2}^{{x_2}^{-1} x_1 x_2}$ and thus

$$[x_2, y_1] = [{x_1}^{x_2}, {y_2}^{-1}]. \qquad (*)$$

(i) By Lemma 3.1 (ii) we can take $x_1 = [a, b_1]$, $x_2 = [a, b_2]$, $y_1 = b_1$ and $y_2 = b_2$. By (*) it follows that $[a, b_2, b_1] = [[a, b_1]^{[a, b_2]}, {b_2}^{-1}] = [a, b_1, b_2^{-1}]$ since $\langle a^B \rangle$ is abelian. By Proposition 3.2 (ii), $\langle a^B \rangle$ and B are $\mathcal{N}_2$-connected, whence $[a, b_1, {b_2}^{-1}] = [a, b_1, b_2]^{-1} = [[a, b_1]^{-1}, b_2] = [a, b_1^{-1}, b_2] = [a^{-1}, b_1, b_2] = [b_1, a, b_2]$. Now the last assertion is a consequence of the abelianness of $\langle a^B \rangle$ and $\mathcal{N}_2$-connection of $\langle a^B \rangle$ and B.

(ii) By Lemma 3.1 (i) and (ii) we can take $x_1 = [a_1, b_2]$, $x_2 = [a_1, b_1]$, $y_1 = [a_2, b_2]$ and $y_2 = [a_2, b_1]$. By (*) we have that $[x_2, y_1] = [{x_1}^{x_2}, {y_2}^{-1}] = [x_1, {y_2}^{-1}]$. Similarly $[x_1, x_2] = 1 = [y_1, y_2] = [x_1 y_1, x_2 y_2]$ and therefore $[y_1, x_2] = [{x_1}^{y_1}, {y_2}^{-1}] = [x_1, {y_2}^{-1}]$. It follows that $[x_2, y_1] = [x_1, {y_2}^{-1}] = [y_1, x_2] = [x_2, y_1]^{-1}$. Consequently, since x_2 centralizes x_1 and y_2, x_2 centralizes $[x_2, y_1]$. Similarly, y_1 centralizes $[x_2, y_1]$ and y_2 centralizes $[x_1, {y_2}^{-1}]$. Hence, $[x_1, y_2] = [x_1, {y_2}^{-1}]^{-1} = [x_2, y_1]$ which proves part (ii).

(iii) By $\mathcal{N}_2$-connection we have:

$$\begin{aligned} [b_2, b_1, a] &= [a, [b_2, b_1]^{-1}] = [a, [b_1, b_2]] = [a, {b_1}^{-1} {b_1}^{b_2}] \\ &= [a, {b_1}^{b_2}][a, {b_1}^{-1}]^{({b_1}^{b_2})} = [a, {b_1}^{b_2}][a, b_1]^{-1}, \end{aligned}$$

since $[a, {b_1}^{-1}] = [a, b_1]^{-1}$ centralizes $\langle {b_1}^B \rangle$ by Proposition 3.2 (i). By Lemma 3.1 (ii),

$$[a, {b_1}^{b_2}] = [a, b_1]^{{b_2}^2}.$$

Since $\langle a^B \rangle$ is abelian, it follows that

$$[b_2, b_1, a] = [a, b_1]^{-1}[a, b_1]^{b_2{}^2} = [a, b_1, b_2{}^2] = [a, b_1, b_2]^2$$

by $\mathcal{N}_2$-connection of $\langle a^B \rangle$ and B (Proposition 3.2 (ii)). If $[a_2, b_2] \in B$, then

$$[a_2, b_2, b_1, a_1] = [a_1, b_1, [a_2, b_2]]^2 = 1$$

by (ii). □

Proposition 3.4. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B \rangle$, $m, n \in \mathbb{Z}$, $n > 0$. Then for all $a \in A$ and $b_1, \ldots, b_n \in B$ the following hold:*

(i) $[a^m, b_1, \ldots, b_n] = [[a, b_1, \ldots, b_i]^m, b_{i+1}, \ldots, b_n] =$
$= [a, b_1, \ldots, b_{i-1}, b_i^m, b_{i+1}, \ldots, b_n] = [a, b_1, \ldots, b_n]^m$ *for all* $i = 1, \ldots, n$.

(ii) *If* $n \geq 3$, *then* $[a, b_1, \ldots, b_n]^2 = 1$.

(iii) *If* $n \geq 3$, *then* $[a, b_1, \ldots, b_n] = [a, b_{\sigma(1)}, \ldots, b_{\sigma(n)}]$ *for all* $\sigma \in \mathrm{Sym}(n)$.

(iv) *If* $n \geq 2$ *and* $\langle b_i \rangle \leq \langle b_j \rangle$ *for some* $i \neq j$, *then* $[a, b_1, \ldots, b_n] = 1$.

The corresponding statements hold with the rôles of A and B interchanged.

Proof (i) This follows from repeated use of $\mathcal{N}_2$-connection of $\langle a^B \rangle$ and B (Proposition 3.2 (ii)).
(ii) Let $n \geq 3$. By Proposition 3.3 (iii) and $\mathcal{N}_2$-connection of $\langle a^B \rangle$ and B we have $[a, b_1, \ldots, b_n]^2 = [a, b_1, \ldots, b_{n-2}, [b_{n-1}, b_n]] = 1$ since $[A, B]$ centralizes B' (Proposition 3.2 (i)).
(iii) Let $n \geq 3$. Since the transpositions $(1, 2), (2, 3), \ldots, (n-1, n)$ generate $\mathrm{Sym}(n)$, we may assume that $\sigma = (i, i+1)$, $i = 1, \ldots, n-1$. By (i) and Proposition 3.3 (i) we have $[a, b_1, \ldots, b_n]^{-1} = [a, b_{\sigma(1)}, \ldots, b_{\sigma(n)}]$ and the result follows from (ii).
(iv) If $n = 2$, then this follows from $\mathcal{N}_2$-connection.

If $n \geq 3$, choose a permutation σ such that $\sigma(i) = 1$ and $\sigma(j) = 2$. Then by (iii) and case $n = 2$, $[a, b_1, \ldots, b_n] = [a, b_i, b_j, \ldots, b_n] = 1$. □

Corollary 3.5. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B \rangle$, $x \in A \cap B$, $a \in A$, $b \in B$. Then:*

$$[x, a, b] = [a, b, x] = [b, x, a] = [x, b, a]^{-1} = [b, a, x]^{-1} = [a, x, b]^{-1} \in Z(\langle A, B \rangle),$$

and $[x, a, b]^3 = 1$.

Proof Since $x \in A$, Proposition 3.3 (iii) yields (with rôles of A and B interchanged) $[b, a, x]^2 = [x, a, b]$. As $x \in B$, $[x, a, b] = [a, x^{-1}, b] = [a, x, b]^{-1}$ and Proposition 3.3 (i) yields $[x, a, b] = [a, b, x] = [b, a, x]^{-1}$. Hence $[x, a, b]^3 = [b, a, x]^{-3} = 1$. Moreover, $[x, b, a]^{-1} = [b, x, a] = [b, a, x]^{-1}$ since $x \in A$. Finally, we prove that $[b, x, a] \in Z(\langle A, B \rangle)$:

Let $\alpha \in A$ and $\beta \in B$. We have $[b, x, a, \alpha]^3 = [[b, x, a]^3, \alpha] = 1$. By Proposition 3.4 (ii), $[b, x, a, \alpha]^2 = 1$, whence $[b, x, a, \alpha] = 1$. Since $[b, x, a] = [a, b, x]$, a similar argument shows that $[b, x, a, \beta] = 1$. Thus $[b, x, a] \in Z(\langle A, B \rangle)$. □

Remark 3.6. We note that some of the arguments in this section have been used frequently before by different authors in investigations on 2-Engel-elements or generalizations thereof. Also some special cases of the results above have been obtained (usually when at least one of the subgroups A or B is the whole group). We just mention the classical papers of Hopkins [15] and Levi [19] (see also [16, III.6]) and the papers [12], [13] of Gruenberg and [17], [18] of W. Kappe.

4 Structure and embedding of $[A, B]$

Since $\langle A, B\rangle = AB[A, B]$, it is necessary to investigate properties of $[A, B]$ for $\mathcal{N}_2$-connected subgroups A and B in order to obtain information about $\langle A, B\rangle$. This is the aim of this section. We first consider the subgroup $[A^2, B]$ of $[A, B]$.

Theorem 4.1. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B\rangle$. Then:*

(i) $[A^2, B] = \langle [a^2, b] \mid a \in A, b \in B\rangle = \langle [a, b]^2 \mid a \in A, b \in B\rangle$
$= \langle [a, b^2] \mid a \in A, b \in B\rangle = [A, B^2]$.

(ii) $[A^2, B] \leq Z([A, B])$.

(iii) $[A^2, B] \trianglelefteq \langle A, B\rangle$.

Proof Let $a_1, a \in A$ and $b_1, b \in B$. Proposition 3.3 (ii) implies that $1 = [[a_1, b_1], [a, b]]^2 = [[a_1, b_1], [a, b]^2]$, that is, $[a, b]^2 = [a^2, b] = [a, b^2] \in Z([A, B])$ for all $a \in A, b \in B$.

Since $\langle [a^2, b] \mid a \in A, b \in B\rangle = \langle [a, b^2] \mid a \in A, b \in B\rangle$, we deduce from Lemma 2.1 that this subgroup is A-invariant and B-invariant and

$$\begin{aligned}[A^2, B] =&\langle \langle [a^2, b] \mid a \in A, b \in B\rangle^{A^2}\rangle = \langle [a^2, b] \mid a \in A, b \in B\rangle\\ =&\langle [a, b^2] \mid a \in A, b \in B\rangle = \langle \langle [a, b^2] \mid a \in A, b \in B\rangle^{B^2}\rangle = [A, B^2].\end{aligned}$$

□

Corollary 4.2. *Let A and B be periodic $\mathcal{N}_2$-connected subgroups of $\langle A, B\rangle$. Let $Y = \langle [a, b] \mid a \in A, b \in B, a, b\ 2'$-elements$\rangle$ as in Lemma 2.2. Then:*

(i) $Y = [O^2(A), O^2(B)] = [O^2(A), B] = [A, O^2(B)] \leq Z([A, B])$.
In particular, if A and B are finite, $A_2 \in \mathrm{Syl}_2(A), B_2 \in \mathrm{Syl}_2(B)$, then

$$[A, B] = [A_2, B_2] \cdot Z([A, B]).$$

(ii) *If $[A, B]$ is finite and nilpotent, then $Y = O_{2'}([A, B]) = O_{2'}(Z([A, B]))$.*

Proof (i) Since $O^2(A) \leq A^2$, this follows from Lemma 2.2 (ii) and Theorem 4.1 (ii). Now the second assertion is a consequence of Lemma 2.2 (iii).
(ii) This follows from (i) and Lemma 2.2 (iv). □

Corollary 4.3. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B\rangle$. Suppose that $A^2 = A$ or $B^2 = B$ or that $[A, B]'$ contains no element of order 2.*

Then $[A, B]$ is abelian.

Proof If $A^2 = A$ or $B^2 = B$, the assertion follows from Theorem 4.1 (i) and (ii). If $[A, B]'$ contains no element of order 2, Proposition 3.3 (ii) yields the assertion. $\square$

Remark 4.4. a) In connection with Theorem 4.1 (i) we note that for $\mathcal{N}_2$-connected subgroups A and B in general

$$[A, B]^2 \neq \langle [a, b]^2 \mid a \in A, b \in B\rangle = [A^2, B] = [A, B^2].$$

In Section 7, Example 7.1 we present a finite 2-group $G = AB$, A and B $\mathcal{N}_2$-connected, where $[A^2, B] = 1$ but $[A, B]^2 \neq 1$.

b) The same example also shows that for A and B $\mathcal{N}_2$-connected, $[A, B]$ need not be abelian (cf. Corollary 4.3). In this example even one of the factors can be chosen to be normal. In contrast, if a group is the product of two $\mathcal{N}_2$-connected normal subgroups A, B, then we will show in Section 6 that $[A, B]$ is abelian (Theorem 6.1 (i)).

c) The example mentioned before has the property that $[A, B]$ has nilpotency class 2. In Section 7 we will construct a series of finite 2-groups (Example 7.2) generated by $\mathcal{N}_2$-connected subgroups A and B such that $[A, B]$ has arbitrarily high derived length.

d) With regard to the hypotheses in Corollary 4.3 we note that if A is finite and A and B are $\mathcal{N}_2$-connected, $A = A^2$ implies that $[A, B]$ contains no element of order 2:

For, if $A = A^2$ is finite, then $A = O^2(A)$ is generated by elements of odd order. Hence every element in $[A, B]$ can be written as a product of conjugates of commutators of the form $[a, b]$ where $a \in A$, $b \in B$ and a has odd order. By $\mathcal{N}_2$-connection, each such commutator has odd order. Since $A = A^2$, $[A, B]$ is abelian by Corollary 4.3. Thus every element in $[A, B]$ has odd order.

The next result gives information about the structure of $[A, B]/\Gamma^\infty([A, B])$ for $\mathcal{N}_2$-connected subgroups A, B.

Proposition 4.5. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B\rangle$. Then:*

(i) $[A, B]/[A, B]'Z([A, B])$ *is an elementary abelian 2-group.*

(ii) $\Gamma^n([A, B]')/\Gamma^{n+1}([A, B]')$ *is an elementary abelian 2-group for all $n \geq 1$.*

In particular, the residually nilpotent group $[A, B]'/\Gamma^\infty([A, B]')$ is residually a 2-group.

(iii) *If $[A, B]$ is finite, then $[A, B]^{\mathcal{N}} = ([A, B]')^{\mathcal{N}}$ and $[A, B]/[A, B]^{\mathcal{N}}Z([A, B])$ is a 2-group.*

Proof (i) This follows from Theorem 4.1 (i) and (ii).

(ii) By Lemma 2.1, the subgroup $[A, B]'$ is generated by all $[[a_1, b_1], [a_2, b_2]]^c$ with $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $c \in [A, B]$. These generators all have order at most 2 by Proposition 3.3 (ii). Hence $[A, B]'/[A, B]''$ is an elementary abelian 2-group. The assertion follows now from [16, III, 2.13 b)].

(iii) Since $[A, B]/([A, B]')^{\mathcal{N}}Z([A, B])$ is a 2-group by (i) and (ii), $[A, B]/([A, B]')^{\mathcal{N}} \in \mathcal{N}$ whence $[A, B]^{\mathcal{N}} \leq ([A, B]')^{\mathcal{N}}$. The reverse inclusion being clear, (iii) follows. $\square$

Corollary 4.6. *Let A and B be $\mathcal{N}_2$-connected subgroups of $\langle A, B\rangle$. If $[A, B]$ is finite and $[A, B]'$ is nilpotent, then $[A, B]$ is nilpotent.*

Proof This follows from Proposition 4.5 (iii). $\square$

Remark 4.7. If $G = AB$ is finite with $\mathcal{N}$-connected subgroups A and B, then $[A, B]$ is nilpotent; even more, $[A, B] \leq Z_\infty(G)$ ([1, Theorem 2] or [14, Proposition 1(8)]). This need not be the case anymore if A and B do not permute. In fact, there is a finite group $\langle A, B\rangle$ with $\mathcal{N}_3$-connected subgroups A and B such that $[A, B]$ is not nilpotent ([1, Example], [14, Example]).

For $\mathcal{N}_2$-connected (not necessarily permuting) subgroups A, B, Corollaries 4.3, 4.6 show nilpotency (or even abelianness) of $[A, B]$ in special situations.

However, for infinite $\mathcal{N}_2$-connected subgroups A and B, $[A, B]$ need not be nilpotent. This follows from Example 7.2 in Section 7 where we construct finite 2-groups $G_i = \langle A_i, B_i\rangle$ with $\mathcal{N}_2$-connected subgroups A_i and B_i such that $[A_i, B_i]$ is nilpotent of class at least 2^{i-1}, $i \in \mathbb{N}$. Taking G to be the direct product of the G_i, A the direct product of the A_i and B analogously, then $G = \langle A, B\rangle$, A and B are $\mathcal{N}_2$-connected, but $[A, B]$ is not nilpotent (albeit residually nilpotent and hypercentral).

But even if A and B are finite $\mathcal{N}_2$-connected subgroups, $[A, B]$ need not be nilpotent: In yet unpublished work, R. Dark constructed an example of a group generated by two $\mathcal{N}_2$-connected elementary abelian groups of order 4 such that $[A, B]$ is not finite. In particular, $[A, B]$ is not nilpotent (since $[A, B]$ is generated by a finite number of elements of finite order and nilpotent groups with this property are finite (cf. [21, 5.2.6])).

R. Dark also constructed an example of a finite group $\langle A, B\rangle$ with $\mathcal{N}_2$-connected elementary abelian subgroups of order 4 where $[A, B]$ is not nilpotent.

In the following we prove a series of results on the embedding of $[A, B]$ in $\langle A, B\rangle$ for $\mathcal{N}_2$-connected subgroups A and B. They yield rather precise insight into the structure of groups generated by $\mathcal{N}_2$-connected subgroups. We obtain results showing that $[A, B]$, if not nilpotent, is not too far from being nilpotent. For instance, Proposition 3.2 (i) and Theorem 4.8 (i),(ii) say that $[A, B]$ centralizes A' and B', $[A', B']$ is contained in $Z(\langle A, B\rangle)$, and $[A, B]'$ centralizes A^2 and B^2.

Theorem 4.8. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B\rangle$. Then:*

(i) $[A, B'] = [A, B^2, B] = [A^2, B, B] = [A, B, B^2] = \langle [a, b_1, b_2]^2 \mid a \in A, b_1, b_2 \in B\rangle$ *centralizes $B[A, B]$. The corresponding statements hold with the rôles of A and B interchanged. In particular, $[A, B']$ is normal in G and $[A', B'] \leq [A, B'] \cap [A', B] \leq Z(G)$.*

(ii) $[A, \Gamma^3(B)] = 1 = [B, \Gamma^3(A)]$ *and* $[A, B]'$ *centralizes A^2 and B^2.*

(iii) $[B', A, A] = [A, B^2, B, A] = [A, B^2, A, B] = [A', B, B] \leq Z(G)$.

(iv) $[A, B'][B, A'] \leq Z_2(G)$ *and* $[A, B^2] \leq Z_3(G)$.

Proof (i) Let $a \in A$ and $b, b_1, b_2 \in B$. By Proposition 3.3 (iii) and Proposition 3.4 (i) and (ii), it follows that $[a, [b_1, b_2]] = [a, b_1, b_2]^2 = [[a, b_1]^2, b_2] = [a, b_1, {b_2}^2]$ and $[[a, b_1, b_2]^2, b] = [a, b_1, b_2, b]^2 = 1$. We set $H = \langle [a, b_1, b_2]^2 \mid a \in A, b_1, b_2 \in B\rangle$. By Theorem 4.1 we have that $[A^2, B] = \langle [a, b]^2 \mid a \in A, b \in B\rangle = [A, B^2]$ is a normal subgroup of G and centralizes $[A, B]$. Therefore H centralizes $B[A, B]$. Moreover, using Lemma 2.1 (i), it follows that $[A, B'] = \langle H^{B'}\rangle = H$, $[A^2, B, B] = [A, B^2, B] = \langle H^{[A,B^2]}\rangle = H$ and $[A, B, B^2] = \langle H^{[A,B]B^2}\rangle = H$. In particular, we have that $[A, B']$ is normal in G and centralizes B. Similarly $[B, A']$ centralizes A, whence $[A', B'] \le Z(G)$.

(ii) Since $[A, B', B] = 1$ by (i) and $[B, A, B'] = 1$ by Proposition 3.2 (i), the Three Subgroups Lemma yields $[B', B, A] = 1$. Similarly, $[A', A, B] = 1$. Furthermore, by (i), $[B^2, [A, B], [A, B]] = [A, B, B^2, [A, B]] = [A, B', [A, B]] = 1$ and part (ii) follows.

(iii) By Theorem 4.1 we have that $[A^2, B] = [A, B^2]$ is a normal subgroup of G and centralizes $[A, B]$. Using Lemma 2.1 (ii), it follows that $[A, B^2, B, A] = [A, B^2, A, B]$.

Since $[A, B^2, B]$ is A-invariant, $[A, B^2, B, A]$ centralizes B by (i). Analogously, $[A, B^2, A, B] = [B, A^2, A, B]$ centralizes A. Hence, $[A, B^2, A, B] \le Z(G)$. The remaining statements follow from (i).

(iv) By (i) and (iii), $[A, B', G] = [A, B', B[A, B]A] = [A, B', A] = [B', A, A] \le Z(G)$ and similarly $[B, A', G] \le Z(G)$. By Theorem 4.1 (ii) and parts (i) and (iii), $[A, B^2, G] = [A, B^2, A][A, B^2, B] = [B, A'][[A, B'] \le Z_2(G)$ and part (iv) follows. □

Corollary 4.9. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B\rangle$. Then:*

(i) $[A, G, G] = \Gamma^3(A)[A', B][A, B, G]$, $[A, G, G, G] = \Gamma^4(A)[A', B, B][A, B, G, G]$, *and*

$$[A, G, \overset{(n)}{\ldots}, G] = \Gamma^{n+1}(A)[A, B, G, \overset{(n-1)}{\ldots}, G] \quad \textit{for all } n \ge 4.$$

(ii) $G'' = A''B''[A, B]'[A', B']$ *and* $G^{(n)} = A^{(n)}B^{(n)}[A, B]^{(n-1)}$ *for all* $n \ge 3$.

Proof (i) Note that $[A, G] = A'[A, B]$ and $[A, G, G] = [A', G][A, B, G]$. The first equation and the second one follow from the fact that $[A, B]$ centralizes A' and Theorem 4.8 (i) and (ii). Using Theorem 4.8 (iii) and induction, the last part follows.

(ii) This follows by the same arguments as in the proof of (i). □

Theorem 4.10. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B\rangle$. Assume that $[A, B]$ contains no element of order 2. Then:*

(i) *If, in addition, $[A, B]$ is periodic, then $[A, B'] = [A, B, B]$ and $[A', B] = [A, B, A]$.*

(ii) $[A, B, B, B] = [A, B, A, A] = 1$.

(iii) $[A, B, A, B] = [A, B, B, A]$. *If, in addition, $[A, B]$ is periodic, then*

$$[A^2, B, B, A] = [A, B, B, A] = [A, B, A, B] = [A^2, B, A, B].$$

(iv) $[A, B] \le Z_3(G)$.

Proof We notice first that if $[A, B]$ has no element of order 2, then $[A, B]$ is abelian by Corollary 4.3. Thus, Lemma 2.1 (i) yields

$$[A, B, B] = \langle [a, b_1, b_2] \mid a \in A, b_1, b_2 \in B \rangle \text{ and}$$
$$[A, B, B, B] = \langle [a, b_1, b_2, b_3] \mid a \in A, b_1, b_2, b_3 \in B \rangle.$$

(i) By Theorem 4.8 (i), $[A, B'] = \langle [a, b_1, b_2]^2 \mid a \in A, b_1, b_2 \in B \rangle$. Since by hypothesis $[a, b_1, b_2]$ has odd order, $\langle [a, b_1, b_2] \rangle = \langle [a, b_1, b_2]^2 \rangle$, and the assertion follows.

(ii) By Proposition 3.4 (ii), each generator $[a, b_1, b_2, b_3]$ of $[A, B, B, B]$ has order 1 or 2. By hypothesis, $[A, B, B, B] = 1$.

(iii) The first part follows from Lemma 2.1 (ii) using the fact that $[A, B]$ is abelian. The second part follows from (i) and Theorem 4.8 (i).

(iv) We have to show $[A, B, W_1, W_2, W_3] = 1$ for all choices $W_i \in \{A, B\}$, $i = 1, 2, 3$. If $W_1 = W_2$, then $[A, B, W_1, W_2, W_3] = 1$ by (ii). If $W_1 \neq W_2$, we may assume that $W_2 = W_3$ by (iii). Then $[A, B, W_1, W_2, W_3] \leq [A, B, W_2, W_3] = 1$ by (ii). □

Corollary 4.11. *Assume that A and B are $\mathcal{N}_2$-connected subgroups of $G = \langle A, B \rangle$ and that $[A, B]$ has no element of order 2. Then:*

$\Gamma^3(G) = \Gamma^3(A)\Gamma^3(B)[A, B, A][A, B, B]$, $\quad \Gamma^4(G) = \Gamma^4(A)\Gamma^4(B)[A, B, A, B]$, *and* $\Gamma^n(G) = \Gamma^n(A)\Gamma^n(B)$ *for all* $n \geq 5$.

Proof Clearly, $G' = A'B'[A, B]$. By Corollary 4.3 and Proposition 3.2 (i), $[A, B]$ is abelian and centralizes A' and B'. Using this and the fact that $[A', B] \leq [A, B, A]$ (Theorem 4.8 (i)), the first equation follows.

By the same reasoning and additionally using Theorem 4.10 (ii) and (iii), the second equation follows.

By Theorem 4.10 (iv), $[A, B, A, B] \leq Z(G)$ which yields the final assertion. □

Corollary 4.12. *Assume that A and B are $\mathcal{N}_2$-connected subgroups of $G = \langle A, B \rangle$ and that $[A, B]$ has no element of order 2. Then:*

(i) *If A and B are nilpotent of class at most 3, then G is nilpotent of class at most 4.*

If A and B are nilpotent of class at most $n \geq 4$, then G is nilpotent of class at most n.

(ii) *If A and B are abelian, then G is metabelian.*

If A and B are metabelian, then G is central-by-metabelian.

If A and B are soluble of derived length at most $n \geq 3$, then G is soluble of derived length at most n.

Proof Using the fact that $[A, B]$ is abelian (Corollary 4.3) and that both $[A, B, A, B]$ and $[A', B']$ are contained in the center of G (Theorem 4.10 (iv) and Theorem 4.8 (i)), the assertions follow from Corollary 4.11 and Corollary 4.9 (ii), respectively. □

Remark 4.13. a) In connection with Theorem 4.10 (i) we remark that the hypothesis that $[A, B]$ contains no elements of order 2 cannot be avoided. Example 7.1

in Section 7 presents a finite 2-group with $\mathcal{N}_2$-connected subgroups A, B satisfying $[A', B] = 1$ and $[A, B, A] \neq 1$.

b) Theorems 4.8 (i) and 4.10 (ii) cannot be improved to $[A, B'] = [A^2, B, B] = 1$ (or $[A, B, B] = 1$ in the case when $[A, B]$ contains no element of order 2) as is shown in Example 7.3 of Section 7. Here, for any odd prime p, a finite p-group $G = AB$ with $\mathcal{N}_2$-connected subgroups A and B is constructed such that $[A, B, B] \neq 1$. In this example $[A, B, A] = 1$. However, with a simple extension it is also possible to obtain a p-group example with $[A, B, A] \neq 1$ and $[A, B, B] \neq 1$: Take two copies G_1 and G_2 of the group G above. Setting $\tilde{G} = G_1 \times G_2$, $\tilde{A} = A_1 \times B_2$ and $\tilde{B} = B_1 \times A_2$, $\tilde{A}$ and $\tilde{B}$ are $\mathcal{N}_2$-connected, $[\tilde{A}, \tilde{B}, \tilde{A}] \neq 1$ and $[\tilde{A}, \tilde{B}, \tilde{B}] \neq 1$.

c) In Theorem 4.10 (ii) and (iv), the hypothesis that $[A, B]$ contains no element of order 2 cannot be omitted. Given $n \in \mathbb{N}$, we present a simple example (already contained in [13]) of a finite 2-group $G = AB$ with $\mathcal{N}_2$-connected subgroups A, B such that $[[A, B], A, \overset{(n)}{\ldots}, A] \neq 1$; in particular, $[A, B]$ is not contained in $Z_n(G)$:

Consider G the regular wreath product of the cyclic group Z_2 of order 2 with an elementary abelian group E of order 2^{n+1} and let B the base group of G. Then $G = BE$, $A := G$ and B are $\mathcal{N}_2$-connected and satisfy $[[A, B], A, \overset{(n)}{\ldots}, A] \neq 1$.

d) The statements of Theorem 4.8 (i),(iv) and Theorem 4.10 (iv) are all best possible. In Section 7 the p-group G (p any odd prime) constructed in Example 7.4 is generated by two $\mathcal{N}_2$-connected subgroups A and B, both of nilpotency class 2, such that $[A', B]$ is not contained in $Z(G)$, $[A', B'] \neq 1$ and $[A, B]$ is not contained in $Z_2(G)$. This also shows that in the second statement of Corollary 4.12 (ii) one cannot conclude that G is metabelian.

e) With regard to Theorem 4.8 (ii), $[A, B]'$ need not be contained in $Z(G)$. Actually, the finite 2-groups constructed in Example 7.2 of Section 7 show that there is no fixed n such that $[A, B]'$ is contained in $Z_n(G)$.

5 Particular subgroups of A and B

The aim of this section is to investigate the embedding of certain distinguished subgroups of A and B in $\langle A, B\rangle$ for $\mathcal{N}_2$-connected subgroups A and B.

Theorem 5.1. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B\rangle$. Then:*

(i) *A^2 is subnormal in G of defect at most 3.*

(ii) *A' is subnormal in G of defect at most 2.*

(iii) *$\Gamma^n(A)$ and $A^{(m)}$ are normal in G for all $n \geq 3$ and $m \geq 2$.*

Proof (i) $A^2[A^2, B]$ is normal in G since $[A^2, B]$ is normal in G by Theorem 4.1 (iii). Clearly, $A^2[A^2, B, A^2]$ is normal in $A^2[A^2, B]$. Since $[A^2, B, A^2]$ centralizes A^2 according to Theorem 4.8 (i), A^2 is normal in $A^2[A^2, B, A^2]$, and the assertion follows.

(ii) Clearly, $A'[A, B]$ is normal in G. Since A' centralizes $[A, B]$, part (ii) holds.

(iii) This follows from Theorem 4.8 (ii). $\square$

Remark 5.2. a) The statement in Theorem 5.1 (i) cannot be improved, that is, A^2 need not have defect at most 2 in G. This can be seen by taking the group G of Example 7.3 in Section 7 with $p = 3$. This group has class 3 and exponent 3 and is therefore a 2-Engel group by a result of Burnside [3], reproved by Levi and van der Waerden [20] (see also [16, III, 6.6]); that is, every pair of elements generates a group in $\mathcal{N}_2$. In the notation of Example 7.3, taking $A = \langle g_1, g_2\rangle$, an extraspecial group of order 27 with center $\langle g_4\rangle$, and $B = G$, A and B are $\mathcal{N}_2$-connected subgroups of $G = AB$. Since $A^2 = A$ is not normalized by $g_5 \in [A, G] = \langle g_4, g_5, g_6, g_7\rangle$, A has defect exactly 3 in G.

b) The statement in Theorem 5.1 (ii) cannot be improved, that is, A' need not be normal in G. This will be shown with Example 7.4 in Section 7.

It follows from Remark 4.7 and Theorem 2.3 that in a finite group $G = \langle A, B\rangle$ the $\mathcal{N}_2$-connected subgroups A and B need not be subnormal in G. Nevertheless, we show in the next result, among others, that $A\cap B$ is subnormal in G; even more, it is contained in $Z_\infty(G)$.

Theorem 5.3. *Let A and B be $\mathcal{N}_2$-connected subgroups of $G = \langle A, B\rangle$. Then:*

(i) *$Z := \langle [x, a, b] \mid x \in A\cap B, a \in A, b \in B\rangle = [A\cap B, A, B] = [A\cap B, B, A] = [A\cap B, [A, B]]$ is an elementary abelian 3-subgroup of $Z(G)$.*

In particular, if $[A\cap B, [A, B]]$ contains no elements of order 3, then $[A, B]$ centralizes $A\cap B$.

(ii) *$A\cap B$ is subnormal in G of defect at most 3 and is nilpotent of class at most 3.*

(iii) *$[(A\cap B)', G] \le Z$. In particular, $(A\cap B)' \le Z_2(G)$ and $(A\cap B)'' = 1$. If $[A, B]$ does not contain elements of order 3 or if $(A\cap B)'$ is periodic and does not contain elements of order 3, then $(A\cap B)' \le Z(G)$.*

(iv) *If $n \ge 3$, then $[A\cap B, G, \overset{(n)}{\ldots}, G] = [A\cap B, A, \overset{(n)}{\ldots}, A][A\cap B, B, \overset{(n)}{\ldots}, B]$.*

(v) *If G is finite, then $A\cap B \le Z_\infty(G)$.*

Proof (i) Let $x \in A\cap B$, $a \in A$ and $b \in B$. By Corollary 3.5, $[x, a, b] = [a, b, x] = [x, b, a]^{-1} \in Z(G)$ and $[x, a, b]^3 = 1$. We set $Z := \langle [x, a, b] \mid x \in A\cap B, a \in A, b \in B\rangle$. Thus, $Z \le Z(G)$ and Z is an elementary abelian 3-group. Using Lemma 2.1, it follows that $[A\cap B, A, B] = \langle Z^{[A\cap B, A]}\rangle = Z$, $[A\cap B, B, A] = \langle Z^{[A\cap B, B]}\rangle = Z$ and $[A\cap B, [A, B]] = [A, B, A\cap B] = \langle Z^{[A,B]}\rangle = Z$.

(ii) Clearly, $[G, A\cap B] \le [A, B]$. Hence $[G, A\cap B, A\cap B] \le [A, B, A\cap B] \le Z(G)$ by (i). In particular, $[G, A\cap B, A\cap B, A\cap B] = 1 \le A\cap B$ and $[A\cap B, A\cap B, A\cap B] \le Z(A\cap B)$, which proves part (ii).

(iii) By (i), $[(A\cap B)', A] \le Z$ and $[(A\cap B)', B] \le Z$, whence $[(A\cap B)', G] \le Z \le Z(G)\cap [A, B]$. Hence $(A\cap B)' \le Z_2(G)$ which implies $(A\cap B)'' = 1$ by the Three Subgroups Lemma. If $[A, B]$ does not contain elements of order 3, then $Z = 1$ and $(A\cap B)' \le Z(G)$. Let $x \in (A\cap B)'$, $u \in A\cup B$. If $(A\cap B)'$ is periodic and does not contain elements of order 3, then $\langle x\rangle = \langle x^3\rangle$ centralizes A and B since $[x,u] \in Z$ and $1 = [x,u]^3 = [x^3,u]$ by $\mathcal{N}_2$-connection. This completes the proof of part (iii).

(iv) Since $[A \cap B, A] \leq [A,B] \cap A'$, we have $[A \cap B, A] \leq Z([A,B])$ by Proposition 3.2 (i). By (i), $\langle [A\cap B,A]^B\rangle = [A \cap B, A]Z$, a normal subgroup of G contained in $Z([A,B])$. It is clear that $[\langle [A\cap B,A]^B\rangle, G] = [A\cap B,A,A][A\cap B,A,B] = [A\cap B,A,A]Z$, a normal subgroup of G contained in $Z(B[A,B])$ by Theorem 4.8 (ii). Therefore

$$[A\cap B,G] = \langle [A\cap B,A]^B\rangle\langle [A\cap B,B]^A\rangle = [A\cap B,A]Z[A\cap B,B],$$
$$[A\cap B,G,G] = [A\cap B,A,A]Z[A\cap B,B,B],$$
$$[A\cap B,G,G,G] = [A\cap B,A,A,A][A\cap B,B,B,B].$$

Using the previous arguments again, part (iv) is proven by induction.

(v) As we have mentioned in Remark 4.7, if $H = XY$ is finite with $\mathcal{N}$-connected subgroups X and Y, then $[X,Y] \leq Z_\infty(H)$. Since $A = A(A\cap B)$ is finite and A and $A \cap B$ are $\mathcal{N}$-connected, we have that $[A\cap B,A] \leq Z_\infty(A)$. Similarly, $[A\cap B,B] \leq Z_\infty(B)$. Thus, part (v) follows from (iv). □

As a consequence of Theorem 5.3 we obtain the well-known Hopkins-Levi result ([15, 19], cf. [16, III, 6.5]) on 2-Engel-groups:

Corollary 5.4. *If G is a group such that $[g,h,h] = 1$ for all $g,h \in G$, then G is nilpotent of class at most 3. $\Gamma^3(G)$ is an elementary abelian 3-group.*

Proof This follows from Theorem 5.3 (i) with $A = B = G$. □

Remark 5.5. a) The example given in Remark 5.2 a) shows that the first statement in Theorem 5.3 (ii) cannot be improved to defect 2.

b) The same group, this time with $A = B = G$, shows that $(A\cap B)'$ need not be contained in $Z(G)$ (cf. Theorem 5.3 (iii)). Also the last statement in Theorem 5.3 (iii) cannot be improved to $(A\cap B)' = 1$ as any nilpotent $3'$-group G of class 2 with $A = B = G$ shows.

c) With respect to Theorem 5.3 (iv),(v) we mention that the example in Remark 4.13 c) shows that for every n there exists a group $G = AB$ with normal $\mathcal{N}_2$-connected subgroups A, B such that $[[A,B],B,\overset{(n)}{\ldots},B] \neq 1$. In particular, $[A,B]$ is not contained in $Z_n(G)$ whence $A\cap B$ is not contained in $Z_n(G)$.

6 The case of normal subgroups A and B

If $G = AB$ with normal $\mathcal{N}_2$-connected subgroups A and B, then the results of Section 4 can be considerably sharpened.

Theorem 6.1. *Let $G = AB$ with normal $\mathcal{N}_2$-connected subgroups A and B. Then:*

(i) *$[A,B]$ is abelian.*

(ii) *$[A,B,A,B] = [A,B,B,A] = 1$.*

(iii) *$[A,B^2] \leq Z_2(G)$ and $[A,B'] \leq Z(G)$.*

(iv) $[A', B'] = 1$.

Proof (i) This follows from Theorem 5.3 (i) and Proposition 4.5 (ii) with $n = 1$.

(ii) By (i) and Lemma 2.1 (i) we have $[A, B, B] = \langle [a, b_1, b_2] \mid a \in A, b_1, b_2 \in B \rangle$. If $a_1, a_2 \in A$ and $b_1, b_2 \in B$, then $[a_2, b_2, b_1, a_1] = 1$ by Proposition 3.3 (iii). Therefore $[A, B, B]$ centralizes A. The result $[A, B, A, B] = [B, A, A, B] = 1$ follows from symmetry.

(iii) $[A, B^2] \leq Z_2(G)$ follows from (ii) and Theorem 4.8 (i). Since $[A, B'] = [A, B^2, B]$ by Theorem 4.8 (i), the second assertion follows from the first one.

(iv) Since $[A, B', A] = [B', A, A] = 1$ by (iii), the assertion follows from the Three-Subgroups-Lemma. □

Remark 6.2. a) The example given in Remark 5.5 b) also shows that the results in Theorem 6.1 are best possible.

For part (iii) of Theorem 6.1, also the p-groups (p an odd prime) of Example 7.3 show that $[A, B]$ need not be contained in $Z(G)$ and that A' need not centralize B.

b) If A and B are normal $\mathcal{N}_2$-connected subgroups of $G = AB$, then by Theorem 6.1 (ii) we have that $[A, B, A, B] = [A, B, B, A] = 1$. However, it is not true that $[A, B, A, A] = [A, B, B, B] = 1$ (unless $A = A^2$ or $B = B^2$ by Theorem 4.8 (i)).

In fact, for every $n \in \mathbb{N}$ there is a finite 2-group $G = AB$ with normal $\mathcal{N}_2$-connected subgroups A, B such that $[[A, B], B, \overset{(n)}{\ldots}, B] \neq 1$ and $[[A, B], A, \overset{(n)}{\ldots}, A] \neq 1$ (but note that, of course, $[A, B] \leq Z_\infty(G)$ as this is true for any product of finite $\mathcal{N}$-connected groups): Using the group G of Remark 4.13 c) (for the given n) and performing the same kind of construction as at the end of Remark 4.13 b) yields such an example.

By the well-known result of Fitting (cf. [16, III, 4.1]), a product of two normal nilpotent subgroups is nilpotent and the nilpotency class of the product is bounded by the sum of the nilpotency classes of the subgroups. We shall finish this section with a result that shows that for nilpotent $\mathcal{N}_2$-connected normal subgroups the bound on the nilpotency class of the product is significantly lower.

Theorem 6.3. *Let A and B be $\mathcal{N}_2$-connected normal subgroups of $G = AB$. Then:*

(i) $\Gamma^{n+1}(G) \leq \Gamma^n(A)\Gamma^n(B)$ *for all* $n \geq 1$.

(ii) *If A and B are nilpotent of nilpotency class a and b, respectively, then G has nilpotency class at most* $\max(a, b) + 1$.

Proof (i) The statement is trivially true for $n = 1$.

Clearly, $\Gamma^{n+1}(G) = [A, G, \overset{(n)}{\ldots}, G][B, G, \overset{(n)}{\ldots}, G]$. By symmetry, it suffices to prove that $[A, G, \overset{(n)}{\ldots}, G] \leq \Gamma^n(A)\Gamma^n(B)$ for all $n \geq 2$.

Since $[A', B] \leq \Gamma^2(A)$ and $[A', B, B] = 1$ (by Theorem 6.1 (iii)), Corollary 4.9 (i) yields
$$[A, G, \overset{(n)}{\ldots}, G] \leq \Gamma^n(A)[A, B, G, \overset{(n-1)}{\ldots}, G] \quad \text{for all } n \geq 2.$$
Hence, it suffices to prove that (*): $[A, B, G, \overset{(n-1)}{\ldots}, G] \leq \Gamma^n(A)\Gamma^n(B)$ for all $n \geq 2$. Note that we have $[A, B, G] = [A, B, A][A, B, B] \leq \Gamma^2(A)\Gamma^2(B)$. By Theorem 6.1 (ii)

$[A, B, G, G] = [[A, B, A][A, B, B], A][[A, B, A][A, B, B], B] = [A, B, A, A][A, B, B, B]$, proving (*) for $n = 3$. Finally, by Theorem 5.3 (iv), (*) holds for all $n \geq 4$.

(ii) This follows from (i). $\square$

Remark 6.4. a) The bound in Theorem 6.3 (ii) is sharp for every nilpotency class of G: This can be seen by taking the group $G = BE$ (for the given nilpotency class) of Remark 4.13 c), but this time with $A = [B, E]E$.

b) If $G = AB$, A, B nilpotent and $\mathcal{N}_2$-connected, then G is nilpotent (already by $\mathcal{N}$-connection). However, if not both factors are normal, there is no general bound for the nilpotency class of G in terms of the nilpotency classes of A and B:

To show this, we use again the groups $G = BE$ of Remark 4.13 c), this time with $A = E$. They have arbitrary high nilpotency class and are products of two abelian $\mathcal{N}_2$-connected subgroups, one of them normal.

7 Examples

This section contains the construction of groups to which we have referred in the previous sections. Some of the calculations have been performed with GAP [11].

Example 7.1. We present an example of a group $G = AB$ of order 2^{11} where A and B are $\mathcal{N}_2$-connected subgroups, A normal in G, $[A, B]$ not abelian:

G is generated by g_i, $i = 1, \ldots, 11$, subject to the following defining relations:

$$g_i^2 = 1,\ i = 1, \ldots, 11,$$

$$[g_1, g_3] = g_7,\ [g_1, g_4] = g_5,\ [g_1, g_{10}] = g_{11},$$

$$[g_2, g_3] = g_6,\ [g_2, g_4] = g_8,\ [g_2, g_9] = g_{11},$$

$$[g_3, g_5] = g_9,\ [g_3, g_8] = g_{10},$$

$$[g_4, g_6] = g_{10},\ [g_4, g_7] = g_9,$$

$$[g_5, g_6] = g_{11},\ [g_7, g_8] = g_{11},$$

$$\text{all other } [g_i, g_j] = 1.$$

It's straightforward to verify that $\mid G \mid = 2^{11}$ and

$$Z(G) = \langle g_{11} \rangle,\ Z_2(G) = \langle g_9, g_{10} \rangle Z(G),\ Z_3(G) = \langle g_5, g_6, g_7, g_8 \rangle Z_2(G),\ Z_4(G) = G$$

$$G' = Z_3(G),\ \Gamma^3(G) = Z_2(G),\ G'' = \Gamma^4(G) = Z(G),\ G''' = \Gamma^5(G) = 1.$$

We set

$$A = \langle g_1, g_2, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11} \rangle,\ \ B = \langle g_3, g_4 \rangle.$$

Then $\mid A \mid = 2^9$ and $\mid B \mid = 2^2$, $G = AB$ and $A \cap B = 1$. $A' = Z(A) = \langle g_{11} \rangle$ whence A has nilpotency class 2 and B is abelian.

Moreover,

$$[A, B] = \langle g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11} \rangle,\ [A, B]^2 = [A, B]' = \langle g_{11} \rangle.$$

Hence A is normal in G and $[A,B]$ is not abelian (cf. Remark 4.4 b)). Note also that $[A^2,B]=[A,B^2]=1\neq[A,B]^2$ (cf. Remark 4.4 a)).

We show that A and B are $\mathcal{N}_2$-connected. Let $a\in A$ and $b\in B$. To prove that $\langle a,b\rangle\in\mathcal{N}_2$ it suffices to consider the case

$$a=g_1^{d_1}g_2^{d_2}g_5^{d_5}g_6^{d_6}g_7^{d_7}g_8^{d_8}\ \text{ and }\ b=g_3^{e_3}g_4^{e_4}\ \text{ with }\ d_i,e_j\in\{0,1\}$$

since $g_9,g_{10},g_{11}\in Z_2(G)$. A little calculation shows

$$[a,b]=g_5^{d_1e_4}g_6^{d_2e_3}g_7^{d_1e_3}g_8^{d_2e_4}g_9^{d_7e_4+d_1e_3e_4+d_5e_3}g_{10}^{d_6e_4+d_2e_3e_4+d_8e_3}z$$

with $z\in Z(G)$. Now

$$[a,b,a]=$$
$$g_{11}^{d_1e_4d_6}g_{11}^{d_2e_3d_5}g_{11}^{d_1e_3d_8}g_{11}^{d_2e_4d_7}g_{11}^{d_7e_4d_2+d_1e_3e_4d_2+d_5e_3d_2}g_{11}^{d_6e_4d_1+d_2e_3e_4d_1+d_8e_3d_1}=1,$$
$$[a,b,b]=g_9^{d_1e_4e_3}g_{10}^{d_2e_3e_4}g_9^{d_1e_3e_4}g_{10}^{d_2e_4e_3}=1.$$

This proves the assertion.

We conclude with one remark. This example is a product of a nilpotent subgroup A of class 2 and an abelian subgroup B that are $\mathcal{N}_2$-connected with non-abelian $[A,B]$. With regard to the structure of A and B this is the simplest possible example: For abelian A and B, the commutator $[A,B]$ is abelian by the well known result of Ito on products of abelian groups (cf. [16, VI, 4.4]).

However, we will see in Example 7.2 that for groups generated by elementary abelian $\mathcal{N}_2$-connected 2-groups A, B, the commutator $[A,B]$ can have arbitrary derived length.

Example 7.2. We construct a series of 2-groups $G_n=\langle A_n,B_n\rangle$ such that A_n and B_n are $\mathcal{N}_2$-connected elementary abelian subgroups and $[A_n,B_n]$ has derived length at least n and nilpotency class at least 2^{n-1}.

In particular, $[A_n,B_n]'$ is not contained in $Z_m(G_n)$ with $m=2^{n-1}-2$ (cf. Remark 4.13 e) of Section 4):

For the following we note the fact that in the regular wreath product $G\wr Z_2$ the base group and Z_2 are $\mathcal{N}_2$-connected if (and only if) G is an elementary abelian 2-group.

With a slight abuse of notation we write the base group of $G\wr Z_2$ as $G\times G^z$ if $Z_2=\langle z\rangle$. In particular, if Y is a subgroup of G, we view Y also as a subgroup of the first factor in the base group of $G\wr Z_2$.

For the construction we need the following result:

Lemma. *Assume that X, Y are $\mathcal{N}_2$-connected subgroups of a group G, Y an elementary abelian 2-group. Let $Z_2=\langle z\rangle$ and $D(X)=\{xx^z\mid x\in X\}$, $\overline{X}=D(X)\times\langle z\rangle\leq G\wr Z_2$.*
Then $\overline{X}$ and Y are $\mathcal{N}_2$-connected in $G\wr Z_2$.

Proof Let $xx^z \in D(X)$, $x \in X$, $y \in Y$. Then, since $[x^z, y] = 1$, we deduce that $[xx^z, y] = [x, y]$ centralizes both xx^z and y because X and Y are $\mathcal{N}_2$-connected. We prove next that $[xx^z z, y]$ centralizes y and $xx^z z$.

We notice that

$$[xx^z z, y] = [xx^z, y]^z [z, y] = [x, y]^z [z, y].$$

But

$$[x, y]^{zy} = [x, y]^{y^z z} = [x, y]^z.$$

Hence y centralizes both $[x, y]^z$ and $[z, y]$, which implies that y centralizes $[xx^z z, y]$.

On the other hand, using that xx^z centralizes z and the $\mathcal{N}_2$-connection of Y with Z_2 and with X, it follows that

$$\begin{aligned}[xx^z z, y]^{xx^z z} =& ([x, y]^z)^{xx^z z} [z, y]^{xx^z z} = [x, y][z, y^x] \\ =& [z, y^x][x, y] = [z, y^{xx^z}][xx^z, y] = [z, y]^{xx^z}[xx^z, y] = [zxx^z, y],\end{aligned}$$

because $[z, y^x] = (y^x)^z y^x$ centralizes $[x, y]$. This proves that $xx^z z$ centralizes $[xx^z z, y]$ and concludes the proof. $\square$

We are now ready for the construction:

Define, inductively,

$$\begin{aligned} &Z_2 \wr_0 Z_2 = Z_2 \\ &Z_2 \wr_n Z_2 = (Z_2 \wr_{n-1} Z_2) \wr Z_2, \ n \geq 1 \end{aligned}$$

Let A_1, B_1 be two copies of Z_2 and consider

$$\langle A_1, B_1 \rangle \leq A_1 \wr B_1 = Z_2 \wr Z_2 = Z_2 \wr_1 Z_2$$

For $n \geq 1$, assume inductively subgroups $A_n, B_n \cong Z_2 \times \overset{(n)}{\cdots} \times Z_2$ such that

$$\langle A_n, B_n \rangle \leq Z_2 \wr_{2n-1} Z_2.$$

We consider the group

$$((Z_2 \wr_{2n-1} Z_2) \wr \langle a \rangle) \wr \langle b \rangle = ((Z_2 \wr_{2n-1} Z_2) \wr Z_2) \wr Z_2 = Z_2 \wr_{2n+1} Z_2$$

and construct subgroups as follows:

$$\begin{aligned} &A = \{xx^a \mid x \in A_n\} \leq (Z_2 \wr_{2n-1} Z_2) \wr \langle a \rangle \\ &A_{n+1} = A \times \langle a \rangle \leq (Z_2 \wr_{2n-1} Z_2) \wr \langle a \rangle \\ &A_{n+1} = A \times \langle a \rangle \cong A_n \times \langle a \rangle \cong Z_2 \times \overset{(n+1)}{\cdots} \times Z_2 \end{aligned}$$

and

$$\begin{aligned} &B = \{yy^b \mid y \in B_n\} \leq ((Z_2 \wr_{2n-1} Z_2) \wr \langle a \rangle) \wr \langle b \rangle \\ &B_{n+1} = B \times \langle b \rangle \leq ((Z_2 \wr_{2n-1} Z_2) \wr \langle a \rangle) \wr \langle b \rangle \\ &B_{n+1} = B \times \langle b \rangle \cong B_n \times \langle b \rangle \cong Z_2 \times \overset{n+1)}{\cdots} \times Z_2 \end{aligned}$$

Hence $A_{n+1}, B_{n+1} \cong Z_2 \times \overset{(n+1)}{\cdots} \times Z_2$ and

$$\langle A_{n+1}, B_{n+1} \rangle \leq Z_2 \wr_{2n+1} Z_2.$$

It should be noticed that

$$[\langle A_n, B_n \rangle, A_n^a] = [\langle A_n, B_n \rangle, B_n^b] = [A_n^a, B_n^b] = 1.$$

From the construction and the lemma we can deduce:

(i) A_{n+1} and B_n are $\mathcal{N}_2$-connected for all $n \geq 1$.

(ii) A_n and B_n are $\mathcal{N}_2$-connected for all $n \geq 1$.

We now show:

With the previous notation, $Z_2 \wr_{n-1} Z_2 \lessapprox [A_n, B_n]$ for all $n \geq 1$.

This yields, together with [16, III, 15.3], that the groups $G_n = \langle A_n, B_n \rangle$ satisfy the properties stated at the beginning of the example.

To prove the assertion above, we assume inductively that $Z_2 \wr_{n-1} Z_2 \lessapprox [A_n, B_n]$ for some $n \geq 1$ and show that $Z_2 \wr_n Z_2 \lessapprox [A_{n+1}, B_{n+1}]$.

We notice first that

$$[A_{n+1}, B_{n+1}] \geq [A_{n+1}, B] = \langle [A, B], [\langle a \rangle, B], [ta, r] \mid t \in A, r \in B \rangle.$$

Now,

$$[A, B] = \langle [xx^a, yy^b] \mid x \in A_n, y \in B_n \rangle = \langle [x, y] \mid x \in A_n, y \in B_n \rangle = [A_n, B_n].$$

On the other hand, for $t \in A, r \in B$,

$$[ta, r] = [t, r]^a [a, r],$$

which implies that

$$\langle [\langle a \rangle, B], [ta, r] \mid t \in A, r \in B \rangle = \langle [\langle a \rangle, B], [A, B]^a \rangle = \langle [\langle a \rangle, B], [A_n, B_n]^a \rangle.$$

Consequently,

$$[A_n, B_n], [A_n, B_n]^a = [A_n, B_n]^{aa^b} \leq [A_{n+1}, B_{n+1}],$$

and also $\langle aa^b \rangle \leq [A_{n+1}, B_{n+1}]$.

Therefore,

$$Z_2 \wr_n Z_2 = (Z_2 \wr_{n-1} Z_2) \wr Z_2 \lessapprox [A_n, B_n] \wr Z_2 \cong ([A_n, B_n] \times [A_n, B_n]^{aa^b}) \cdot \langle aa^b \rangle \leq [A_{n+1}, B_{n+1}],$$

and we are done.

Example 7.3. Let p be an odd prime. We give an example of a group $G = AB$ of order p^7 with $\mathcal{N}_2$-connected subgroups A and B, both normal in G, such that $[A, B]$ is not contained in the center of G:

G is generated by $g_1, \ldots, g_7$ subject to the following relations:

$$g_i^p = 1,\ i = 1, \ldots, 7,$$

$$[g_1, g_2] = g_4,\ [g_1, g_3] = g_5,\ [g_1, g_6] = g_7^2,$$
$$[g_2, g_3] = g_6^{-1},\ [g_2, g_5] = g_7^{-1},\ [g_3, g_4] = g_7,$$
$$\text{all other } [g_i, g_j] = 1.$$

It is easy to check that $\mid G \mid = p^7$ and

$$Z(G) = \Gamma^3(G) = \langle g_7 \rangle,\ Z_2(G) = G' = \langle g_4, g_5, g_6, g_7 \rangle,\ Z_3(G) = G,\ G'' = 1.$$

We set $A = \langle g_1, g_4, g_5, g_7 \rangle$ and $B = \langle g_2, \ldots, g_7 \rangle$. Then $G = AB$, A and B are normal in G, A is abelian and B is of nilpotency class 2 (with $B' = \langle g_6, g_7 \rangle$).

Moreover,

$$[A, B] = \langle g_4, g_5, g_7 \rangle,\ [A, B, B] = \langle g_7 \rangle \neq 1 \ \text{ and } \ [A, B'] = \langle g_7 \rangle$$

(cf. Remark 4.13 b) and Remark 6.2 a)).

We show that A and B are $\mathcal{N}_2$-connected. Let $a \in A$ and $b \in B$. To prove that $\langle a, b \rangle \in \mathcal{N}_2$ it suffices to consider the case $a = g_1$ and $b = g_2^d g_3^e$ with $d, e \in \{0, \ldots, p-1\}$ since $g_4, g_5, g_6, g_7 \in Z_2(G)$.

Then $[a, b] = g_5^e g_4^d g_7^{-de}$ and $[a, b, a] = 1$, $[a, b, b] = g_7^{ed} g_7^{-de} = 1$.

We note that for $p = 3$ the group constructed above is just the free Burnside group $B(3, 3)$ on 3 generators of exponent 3 which is a 2-Engel group ([3, 20]).

Example 7.4. Let p be an odd prime. We construct a group $G = \langle A, B \rangle$ of order p^{15} with N_2-connected subgroups A and B where A' and B' are not normal in G, $[A, B, A, B] \neq 1$, $[A, B, B, A] \neq 1$, $[A', B]$ not contained in $Z(G)$ and $[A', B'] \neq 1$:

G is generated by $g_1, \ldots, g_{15}$ subject to the following relations:

$$g_i^p = 1,\ i = 1, \ldots, 15,$$

$$[g_1, g_2] = g_5,\ [g_1, g_3] = g_6,\ [g_1, g_4] = g_8,\ [g_1, g_7] = g_{11},$$
$$[g_1, g_9] = g_{12},\ [g_1, g_{10}] = g_{13},\ [g_1, g_{14}] = g_{15},$$
$$[g_2, g_3] = g_7^{-1},\ [g_2, g_4] = g_9^{-1},\ [g_2, g_6] = g_{11},$$
$$[g_2, g_8] = g_{12},\ [g_2, g_{10}] = g_{14}^{-1},\ [g_2, g_{13}] = g_{15},$$
$$[g_3, g_4] = g_{10}^{-1},\ [g_3, g_5] = g_{11}^2, [g_3, g_8] = g_{13}^{(p-1)/2},$$
$$[g_3, g_9] = g_{14}^{(p-1)/2},\ [g_3, g_{12}] = g_{15}^{(p-1)/2},$$

$$[g_4, g_5] = g_{12}^2,\ [g_4, g_6] = g_{13}^{(p+1)/2},\ [g_4, g_7] = g_{14}^{(p+1)/2},\ [g_4, g_{11}] = g_{15}^{(p+1)/2},$$
$$[g_5, g_{10}] = g_{15}^{-2},$$
$$\text{all other } [g_i, g_j] = 1.$$

With some calculations one checks that actually $\mid G \mid = p^{15}$ and

$$Z_3(G) = G' = \langle g_5, \ldots, g_{15} \rangle, \ \mid Z_3(G) \mid = p^{11},$$

$$Z_2(G) = \Gamma^3(G) = \langle g_{11}, \ldots, g_{15} \rangle, \ \mid Z_2(G) \mid = p^5,$$
$$Z(G) = G'' = \Gamma^4(G) = \langle g_{15} \rangle, \ \mid Z(G) \mid = p.$$

We set $A = \langle g_1, g_2, g_5 \rangle$ and $B = \langle g_3, g_4, g_{10} \rangle$. A and B are extraspecial of exponent p and order p^3, $G = \langle A, B \rangle$ and $A \cap B = 1$.

Moreover, $A' = \langle g_5 \rangle$ and $B' = \langle g_{10} \rangle$. Thus A' and B' are not normal in G (cf. Remark 5.5 b)). Also $[A', B] = \langle g_{11}, g_{12}, g_{15} \rangle \not\leq Z(G)$ and $[A', B'] = \langle g_{15} \rangle \neq 1$.

Furthermore,

$$[A, B] = \langle g_6, g_7, g_8, g_9, g_{11}, g_{12}, g_{13}, g_{14}, g_{15} \rangle,$$

$$[A, B, A] = \langle g_{11}, g_{12}, g_{15} \rangle, \ [A, B, B] = \langle g_{13}, g_{14}, g_{15} \rangle,$$
$$[A, B, A, B] = [A, B, B, A] = Z(G) \neq 1$$

(cf. Remark 4.13 d)).

We show that A and B are $\mathcal{N}_2$-connected. An easy calculation shows that

$$[g_1^a g_2^b, g_3^c g_4^d] = g_6^{ac} g_7^{-bc} g_8^{ad} g_9^{-bd} g_{11}^{-abc} g_{12}^{-abd} g_{13}^{(p-1)acd/2} g_{14}^{(p+1)bcd/2} g_{15}^{(p+1)abcd/2}.$$

It follows that $[g_1^a g_2^b, g_3^c g_4^d]$ is centralized by $g_1^a g_2^b$, $g_3^c g_4^d$, g_5 and g_{10}.

Now
$$[g_1^a g_2^b g_5^e, g_3^c g_4^d g_{10}^f] = g_{11}^{2ec} g_{12}^{2ed} g_{13}^{af} g_{14}^{-bf} g_{15}^{2af} [g_1^a g_2^b, g_3^c g_4^d].$$

One checks that $g_{11}^{2ec} g_{12}^{2ed} g_{13}^{af} g_{14}^{-bf} g_{15}^{2af}$ is centralized both by $g_1^a g_2^b g_5^e$ and $g_3^c g_4^d g_{10}^f$.

Hence A and B are $\mathcal{N}_2$-connected.

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