

SIX DIMENSIONAL SOLVMANIFOLDS WITH HOLOMORPHICALLY TRIVIAL CANONICAL BUNDLE

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ABSTRACT. We determine the 6-dimensional solvmanifolds admitting an invariant complex structure with holomorphically trivial canonical bundle. Such complex structures are classified up to isomorphism, and the existence of strong Kähler with torsion (SKT), generalized Gauduchon, balanced and strongly Gauduchon metrics is studied. As an application we construct a holomorphic family (M, J_a) of compact complex manifolds such that (M, J_a) satisfies the $\partial\bar{\partial}$ -lemma and admits a balanced metric for any $a \neq 0$, but the central limit neither satisfies the $\partial\bar{\partial}$ -lemma nor admits balanced metrics.

In memory of Sergio Console

1. INTRODUCTION

Any compact complex surface with holomorphically trivial canonical bundle is isomorphic to a K3 surface, a torus, or a Kodaira surface; the first two are Kähler, and the latter is a nilmanifold, i.e. a compact quotient of a nilpotent Lie group by a lattice. It is well known that in any real dimension $2n$ the canonical bundle of a nilmanifold $\Gamma \backslash G$ endowed with an invariant complex structure is holomorphically trivial, where by invariant complex structure we mean one induced by a complex structure on the Lie algebra of the nilpotent Lie group G . In fact, Salamon proved in [27] the existence of a closed non-zero invariant $(n, 0)$ -form (see also [5] for some applications of this fact to hypercomplex nilmanifolds). For 6-dimensional nilmanifolds the classification of invariant complex structures J as well as the existence of some special Hermitian metrics (as SKT, generalized first Gauduchon, balanced and strongly Gauduchon metrics) with respect to such J 's have been studied in [7, 12, 13]. In this paper we are interested in the Hermitian geometry of 6-dimensional solvmanifolds admitting an invariant complex structure with holomorphically trivial canonical bundle, i.e. with an *invariant* non-zero closed $(3, 0)$ -form (see Proposition 2.1 for details). Throughout the paper, by a solvmanifold we mean a compact quotient $\Gamma \backslash G$ of a *non-nilpotent* solvable Lie group G by a lattice Γ .

Since the complex structures we consider are invariant, they come from complex structures on the Lie algebra \mathfrak{g} of G . Given an almost complex structure $J: \mathfrak{g} \rightarrow \mathfrak{g}$ on a 6-dimensional Lie algebra \mathfrak{g} , the existence of a non-zero closed form of bidegree $(3, 0)$ implies that J is integrable, i.e. that J is a complex structure. If the Lie algebra is nilpotent then both conditions are equivalent [27]. The 6-dimensional nilpotent Lie algebras admitting a complex structure were classified in [27] and recently these

complex structures have been classified up to equivalence [7]. For solvable Lie algebras admitting a complex structure there exists a general classification only in real dimension four [23], but no general result is known in higher dimension.

In this paper we consider the bigger class of 6-dimensional solvable Lie algebras \mathfrak{g} and look for almost complex structures admitting a non-zero closed $(3,0)$ -form. Notice that the latter condition implies that $b_3(\mathfrak{g}) \geq 2$. We will consider both the indecomposable and the decomposable solvable Lie algebras which are unimodular as we aim to find invariant complex structures with non-zero closed $(3,0)$ -form on solvmanifolds [20]. We recall that the 6-dimensional indecomposable solvable Lie algebras have been classified by Turkowski [31], Mubarakzyanov [21] and Shabanskaya, who recovered and corrected in [29] the work of Mubarakzyanov (see the Appendix for more details).

In Section 2 we consider the formalism of stable forms [17], together with ideas in [9, 14, 15, 28], and we explain in detail the method that we follow to classify unimodular solvable Lie algebras admitting a complex structure with non-zero closed $(3,0)$ -form. The classification is obtained in Propositions 2.6 and 2.7 for the decomposable and indecomposable cases, respectively. As a consequence we have that if a solvmanifold admits a complex structure arising from an invariant non-vanishing holomorphic $(3,0)$ -form, then its underlying Lie algebra \mathfrak{g} must be isomorphic to one in the list of Theorem 2.8, i.e. $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha$ ($\alpha \geq 0$), $\mathfrak{g}_3, \dots, \mathfrak{g}_8$ or \mathfrak{g}_9 . The Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2^\alpha$ ($\alpha \geq 0$) and \mathfrak{g}_3 are decomposable, whereas $\mathfrak{g}_4, \dots, \mathfrak{g}_8$ and \mathfrak{g}_9 are indecomposable. The Lie algebra \mathfrak{g}_8 is precisely the real Lie algebra underlying Nakamura manifold [22]. Moreover, using some results in [6], we ensure in Proposition 2.10 the existence of a lattice for the simply-connected Lie groups associated to the Lie algebras in the list, although for \mathfrak{g}_2^α we are able to find a lattice only for a countable number of different values of α (note that one cannot expect a lattice to exist for any real $\alpha > 0$ according to [32, Prop. 8.7]).

In Section 3 we consider the whole space of complex structures having closed $(3,0)$ -form and classify them up to equivalence (see Theorem 3.10). It turns out that the classification is finite, except for \mathfrak{g}_3 (Proposition 3.4) and \mathfrak{g}_8 (Proposition 3.7). As a consequence of this complex classification, we study in Section 4 the existence of several special Hermitian metrics on the corresponding complex solvmanifolds. The SKT geometry is studied in Theorem 4.1 and provides a new example of compact SKT manifold based on the Lie algebra \mathfrak{g}_4 . In Theorem 4.3 we investigate the existence of *invariant* first Gauduchon metrics, in the sense of [16], and find that a solvmanifold corresponding to \mathfrak{g}_6 has invariant first Gauduchon metrics which are not SKT, moreover it does not admit any SKT metric (notice that any invariant first Gauduchon metric on a 6-nilmanifold is necessarily SKT as proved in [13, Proposition 3.3]). Finally, Theorems 4.5 and 4.6 deal with the existence of balanced metrics [19] and strongly Gauduchon metrics [24, 25, 26], respectively, and provide new compact examples of such Hermitian geometries.

Our goal in Section 5 is to construct a holomorphic family of compact complex manifolds $(M, J_a)_{a \in \Delta}$, $\Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, such that (M, J_a) satisfies the $\partial\bar{\partial}$ -lemma and admits balanced metric for any $a \neq 0$, but (M, J_0) neither satisfies the $\partial\bar{\partial}$ -lemma nor admits balanced metric. The construction is based on the balanced Hermitian geometry studied in Theorem 4.5 for \mathfrak{g}_8 , the Lie algebra underlying Nakamura manifold, together with a recent result by Angella and Kasuya [3] on deformations of the holomorphically parallelizable Nakamura manifold. Notice that the central limit

(M, J_0) admits strongly Gauduchon metric by Theorem 4.6, as it must happen according to Popovici's result [26, Proposition 4.1]. We recall that there exists [7] a holomorphic family (N, J_a) , N being a 6-dimensional nilmanifold, such that (N, J_a) has balanced metrics for any $a \neq 0$, but the central limit (N, J_0) does not admit strongly Gauduchon metric.

2. THE CLASSIFICATION

We first show that the existence of a holomorphic form of bidegree $(n, 0)$ with respect to an invariant complex structure on a $2n$ -dimensional solvmanifold implies the existence of an invariant non-zero closed $(n, 0)$ -form. Furthermore:

Proposition 2.1. *Let $M = \Gamma \backslash G$ be a $2n$ -dimensional solvmanifold endowed with an invariant complex structure J . If Ω is a nowhere vanishing holomorphic $(n, 0)$ -form on (M, J) , then Ω is necessarily invariant.*

Proof. Since J is invariant, we consider a global basis $\{\omega^1, \dots, \omega^n\}$ of invariant $(1, 0)$ -forms on (M, J) . Then, there is a nowhere vanishing complex-valued function $f: M \rightarrow \mathbb{C}$ such that $\Omega = f \omega^1 \wedge \dots \wedge \omega^n$. Since Ω is holomorphic, we have $\bar{\partial}\Omega = \bar{\partial}f \wedge \omega^1 \wedge \dots \wedge \omega^n + f \bar{\partial}(\omega^1 \wedge \dots \wedge \omega^n) = 0$, that is, $\bar{\partial}(\omega^1 \wedge \dots \wedge \omega^n) = -\bar{\partial}(\log f) \wedge \omega^1 \wedge \dots \wedge \omega^n$. The latter form is an invariant $(n, 1)$ -form on (M, J) , so there is an invariant form α of bidegree $(0, 1)$ on (M, J) such that

$$(1) \quad \bar{\partial}(\log f) = \alpha.$$

Now we apply the well-known symmetrization process, which assigns an invariant k -form $\tilde{\gamma}$ to any k -form γ on M by integration with respect to a volume element on M induced by a bi-invariant one on the Lie group G (its existence is guaranteed by [20]). By extending the symmetrization to the space of complex forms, since J is invariant we have that symmetrization preserves the bidegree and commutes with $\bar{\partial}$, because it commutes with the differential d (see [7, 11] for more details). Applying this to (1), we get

$$\bar{\partial}(\widetilde{\log f}) = \tilde{\alpha} = \alpha,$$

because α is invariant. But $\widetilde{\log f}$ is the symmetrization of a function, so it is a constant and then $\bar{\partial}(\widetilde{\log f}) = 0$. Therefore, $\alpha = 0$ and by (1) we get $\bar{\partial}(\log f) = 0$. This means that $\log f$ is a holomorphic function on a compact complex manifold, which implies that $\log f = c$, where c is a constant. In conclusion, $f = \exp(c)$ is a constant function, and Ω is necessarily invariant. \square

As a consequence of this result, in order to describe the solvmanifolds $M = \Gamma \backslash G$ admitting an invariant complex structure with holomorphically trivial canonical bundle, we are led to study the unimodular solvable Lie algebras \mathfrak{g} admitting a complex structure J with non-zero closed $(n, 0)$ -form. Next we classify such Lie algebras in dimension 6.

We will recall first the formalism of stable forms which enable us to construct the space of almost complex structures on a given Lie algebra [9, 17]. Let (V, ν) be an oriented six-dimensional vector space, a 3-form ρ is *stable* if its orbit under the action of the group $\mathrm{GL}(V)$ is open. Let $\kappa: \Lambda^5 V^* \rightarrow V$ be the isomorphism defined by $\kappa(\eta) = X$ where X is such that $\iota_X \nu = \eta$, and let $K_\rho: V \rightarrow V$ be the endomorphism given by $K_\rho(X) = \kappa(\iota_X \rho \wedge \rho)$. The square of this endomorphism is proportional to the

identity map so it is an automorphism of the vector space. Let $\lambda(\rho)$ be the constant of proportionality, that is,

$$\lambda(\rho) = \frac{1}{6} \text{tr}(K_\rho^2).$$

Then, the stability of ρ is equivalent to the open condition $\lambda(\rho) \neq 0$. When $\lambda(\rho) < 0$ the endomorphism $J_\rho := \frac{1}{\sqrt{|\lambda(\rho)|}} K_\rho$ gives rise to an almost complex structure on V .

Moreover, $\lambda(\rho)$ enables to construct a specific volume form $\phi(\rho) := \sqrt{|\lambda(\rho)|} \nu \in \Lambda^6 V^*$ such that the action of the dual endomorphism J_ρ^* on 1-forms is given by the formula

$$(2) \quad ((J_\rho^* \alpha)(X)) \phi(\rho) = \alpha \wedge \iota_X \rho \wedge \rho,$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

As a consequence there is a natural mapping

$$(3) \quad \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) < 0\} \rightarrow \{J: V \rightarrow V \mid J^2 = -I\}$$

assigning each ρ to $J = J_\rho$ through relation (2). Although this map is not injective, as for example $J_{\rho'} = J_\rho$ when ρ' is proportional to ρ , it is onto and therefore it covers the space of almost complex structures on V .

We are interested in the complex structures on a Lie algebra \mathfrak{g} admitting non-zero closed (3,0)-form. Let $Z^3(\mathfrak{g}) = \{\rho \in \Lambda^3 \mathfrak{g}^* \mid d\rho = 0\}$. The map (3) restricts to the surjective mapping

$$\{\rho \in Z^3(\mathfrak{g}) \mid \lambda(\rho) < 0, d(J_\rho^* \rho) = 0\} \rightarrow \{J: \mathfrak{g} \rightarrow \mathfrak{g} \mid J^2 = -I, \exists \Psi \in \Lambda^{3,0} \text{ closed}\}.$$

The closed (3,0)-form is given by $\Psi = \rho + iJ_\rho^* \rho$.

The next result provides an equivalent condition to determine the existence of such complex structures on \mathfrak{g} .

Lemma 2.2. *Let \mathfrak{g} be a Lie algebra and ν a volume form on \mathfrak{g} . Then, \mathfrak{g} admits an almost complex structure with a non-zero closed (3,0)-form if and only if there exists $\rho \in Z^3(\mathfrak{g})$ such that the endomorphism $\tilde{J}_\rho^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ defined by*

$$(4) \quad \left((\tilde{J}_\rho^* \alpha)(X) \right) \nu = \alpha \wedge \iota_X \rho \wedge \rho,$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, satisfies that $\tilde{J}_\rho^* \rho$ is closed and $\text{tr}(\tilde{J}_\rho^{*2}) < 0$.

Proof. Let $J: \mathfrak{g} \rightarrow \mathfrak{g}$ be an almost complex structure admitting a non-zero (3,0)-form $\Psi = \Psi_+ + i\Psi_-$ which is closed. Let $\rho = \Psi_+$. Then, $\lambda(\rho) < 0$, $J = J_\rho$ is determined by (2) and the form $J_\rho^* \rho = \Psi_-$ is closed. Since the associated form $\phi(\rho)$ is a volume form on \mathfrak{g} , we have that $\nu = s \phi(\rho)$ for some $s \neq 0$. Now, for the endomorphism $\tilde{J}_\rho^*: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ given by (4) we get

$$s \left((\tilde{J}_\rho^* \alpha)(X) \right) \phi(\rho) = \left((\tilde{J}_\rho^* \alpha)(X) \right) \nu = \alpha \wedge \iota_X \rho \wedge \rho = ((J_\rho^* \alpha)(X)) \phi(\rho),$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. This implies that $J_\rho^* = s \tilde{J}_\rho^*$. Therefore, $\text{tr}(\tilde{J}_\rho^{*2}) < 0$ if and only if $\text{tr}(J_\rho^{*2}) < 0$, and moreover, $d(\tilde{J}_\rho^* \rho) = 0$ if and only if $d(J_\rho^* \rho) = 0$. \square

For the computation of the endomorphism \tilde{J}_ρ^* we will use the simplest volume form $\nu = e^{123456}$, where $\{e^1, \dots, e^6\}$ is the basis of \mathfrak{g}^* in which the Lie algebra is expressed. Notice that we will follow the notation given in [14, 15, 28] and [31] to name the Lie algebras; for instance, the notation $\mathfrak{e}(2) \oplus \mathfrak{e}(1, 1) = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45})$ means that $\mathfrak{e}(2) \oplus \mathfrak{e}(1, 1)$ is the (decomposable) Lie algebra determined by a basis $\{e^i\}_{i=1}^6$ such that $de^1 = 0$, $de^2 = -e^{13}$, $de^3 = e^{12}$, $de^4 = 0$, $de^5 = -e^{46}$, $de^6 = -e^{45}$.

The next two concrete examples show how we will proceed in general in the proofs of Propositions 2.6 and 2.7 below in order to exclude candidates. Some of the computations in these examples, as well as in Propositions 2.6 and 2.7 and in Section 3, were carried out using Mathematica and the differential forms package scalarEDC by S. Bonanos available at www.inp.demokritos.gr/sbonano/.

Example 2.3. Let us consider the indecomposable solvable Lie algebra $\mathfrak{g} = A_{6,25}^{0,-1} = (e^{23}, e^{26}, -e^{36}, 0, e^{46}, 0)$. Any $\rho \in Z^3(\mathfrak{g})$ is given by

$$\begin{aligned} \rho = & a_1 e^{123} + a_2 e^{126} + a_3 e^{136} + a_4 e^{234} + a_5 (e^{235} - e^{146}) + a_6 e^{236} + a_7 e^{246} \\ & + a_8 e^{256} + a_9 e^{346} + a_{10} e^{356} + a_{11} e^{456}, \end{aligned}$$

for $a_1, \dots, a_{11} \in \mathbb{R}$. Fix $\nu = e^{123456}$, and let \tilde{J}_ρ^* be the endomorphism given by (4). A direct calculation shows that

$$\text{tr}(\tilde{J}_\rho^{*2}) = 6(a_5^2 - a_1 a_{11})^2 \geq 0.$$

In this case it is not worth evaluating the closedness of $\tilde{J}_\rho^* \rho$ because by Lemma 2.2 there is no almost complex structure J_ρ^* coming from a closed 3-form $\rho \in Z^3(\mathfrak{g})$ and in particular \mathfrak{g} does not admit a closed complex volume form.

Example 2.4. Let us consider the $5 \oplus 1$ decomposable solvable Lie algebra $\mathfrak{g} = A_{5,15}^{-1} \oplus \mathbb{R} = (e^{15} + e^{25}, e^{25}, -e^{35} + e^{45}, -e^{45}, 0, 0)$. Any $\rho \in Z^3(\mathfrak{g})$ is given by

$$\begin{aligned} \rho = & a_1 e^{125} + a_2 e^{135} + a_3 e^{145} + a_4 e^{156} + a_5 e^{235} + a_6 (e^{236} - e^{146}) + a_7 e^{245} \\ & + a_8 e^{246} + a_9 e^{256} + a_{10} e^{345} + a_{11} e^{356} + a_{12} e^{456}, \end{aligned}$$

for $a_1, \dots, a_{12} \in \mathbb{R}$. Take $\nu = e^{123456}$, and let \tilde{J}_ρ^* be the endomorphism given by (4). Then, we have

$$\frac{1}{6} \text{tr}(\tilde{J}_\rho^{*2}) = (a_3 + a_5)^2 a_6^2 + 4(a_1 a_{10} - a_2 a_7) a_6^2 - 2(a_3 - a_5) a_2 a_6 a_8 + a_2^2 a_8^2$$

and

$$d(\tilde{J}_\rho^* \rho) = 2a_6^2 (2a_1 e^{1256} + a_2 (e^{1456} + e^{2356})) + (a_3 + a_5) e^{2456} - 2a_{10} e^{3456}.$$

Since the form $\tilde{J}_\rho^* \rho$ must be closed, we distinguish two cases depending on the vanishing of the coefficient a_6 . If $a_6 = 0$ then $\text{tr}(\tilde{J}_\rho^{*2}) = 6(a_2 a_8)^2 \geq 0$, and if $a_6 \neq 0$ then $a_1 = a_2 = a_3 + a_5 = a_{10} = 0$ and so $\text{tr}(\tilde{J}_\rho^{*2}) = 0$. Consequently, Lemma 2.2 assures that there is no almost complex structure J_ρ^* admitting a non-zero closed (3,0)-form.

From now on we shall denote by $b_k(\mathfrak{g})$ the dimension of the k -th Chevalley-Eilenberg cohomology group $H^k(\mathfrak{g})$ of the Lie algebra. The next lemma shows a simple but useful obstruction to the existence of complex structures with non-zero closed (3,0)-volume forms in the unimodular case involving b_3 . We remind that the unimodularity of \mathfrak{g} is equivalent to $b_6(\mathfrak{g}) = 1$.

Lemma 2.5. *If \mathfrak{g} is an unimodular Lie algebra admitting a complex structure with a non-zero closed (3,0)-form Ψ , then $b_3(\mathfrak{g}) \geq 2$.*

Proof. Let $\Psi_+, \Psi_- \in \Lambda^3(\mathfrak{g}^*)$ be the real and imaginary parts of Ψ , that is, $\Psi = \Psi_+ + i\Psi_-$. Since Ψ is closed we have that $d(\Psi_+) = d(\Psi_-) = 0$ and therefore $[\Psi_+], [\Psi_-] \in H^3(\mathfrak{g})$. It is sufficient to see that both classes are non-zero and, moreover, that they are not cohomologous.

Suppose that there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$ such that $a\Psi_+ + b\Psi_- = d\alpha$ for some $\alpha \in \Lambda^2(\mathfrak{g}^*)$. Since $0 \neq \frac{i}{2}\Psi \wedge \bar{\Psi} = \Psi_+ \wedge \Psi_- \in \Lambda^6(\mathfrak{g}^*)$, we get

$$d(\alpha \wedge (-b\Psi_+ + a\Psi_-)) = (a\Psi_+ + b\Psi_-) \wedge (-b\Psi_+ + a\Psi_-) = (a^2 + b^2)\Psi_+ \wedge \Psi_- \neq 0.$$

But this is in contradiction to the unimodularity of \mathfrak{g} . \square

As a consequence of Lemma 2.5 we will concentrate on unimodular (non nilpotent) solvable Lie algebras \mathfrak{g} with $b_3(\mathfrak{g}) \geq 2$. We will tackle the classification problem first in the decomposable case.

Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$. The unimodularity and solvability of \mathfrak{g} and Lemma 2.5 imply restrictions on the factors. In fact, \mathfrak{g} is unimodular, resp. solvable, if and only if \mathfrak{b} and \mathfrak{c} are unimodular, resp. solvable. Moreover, by Lemma 2.5 and the well known formula relating the cohomology of \mathfrak{g} with the cohomologies of the factors, we have

$$(5) \quad b_3(\mathfrak{b})b_0(\mathfrak{c}) + b_2(\mathfrak{b})b_1(\mathfrak{c}) + b_1(\mathfrak{b})b_2(\mathfrak{c}) + b_0(\mathfrak{b})b_3(\mathfrak{c}) = b_3(\mathfrak{g}) \geq 2.$$

Proposition 2.6. *Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$ be a six-dimensional decomposable unimodular (non nilpotent) solvable Lie algebra admitting a complex structure with a non-zero closed $(3,0)$ -form. Then, \mathfrak{g} is isomorphic to $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$, $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ or $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$.*

Proof. Since \mathfrak{g} is decomposable, we divide the proof in the three cases $3 \oplus 3$, $4 \oplus 2$ and $5 \oplus 1$. In the $3 \oplus 3$ case the inequality (5) is always satisfied. The $3 \oplus 3$ decomposable unimodular (non nilpotent) solvable Lie algebras are $\mathfrak{e}(2) \oplus \mathfrak{e}(2)$, $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(2) \oplus \mathfrak{h}_3$, $\mathfrak{e}(2) \oplus \mathbb{R}^3$, $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$ and $\mathfrak{e}(1,1) \oplus \mathbb{R}^3$ (see Table 1 in the Appendix for a description of the Lie algebras). An explicit computation shows that there is no $\rho \in Z^3$ satisfying the conditions $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$, except for $\mathfrak{g} = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$. We give an example of a closed complex volume form for $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$ in Table 1.

Since \mathbb{R}^2 is the only 2-dimensional unimodular Lie algebra, the $4 \oplus 2$ case is reduced to the study of $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}^2$ for any 4-dimensional unimodular (non nilpotent) solvable Lie algebra \mathfrak{b} satisfying (5), i.e. $b_3(\mathfrak{b}) + 2b_2(\mathfrak{b}) + b_1(\mathfrak{b}) \geq 2$. The resulting Lie algebras are: $A_{4,2}^{-2} \oplus \mathbb{R}^2$, $A_{4,5}^{\alpha,-1-\alpha} \oplus \mathbb{R}^2$ with $-1 < \alpha \leq -\frac{1}{2}$, $A_{4,6}^{\alpha,-\frac{\alpha}{2}} \oplus \mathbb{R}^2$, $A_{4,8} \oplus \mathbb{R}^2$ and $A_{4,10} \oplus \mathbb{R}^2$ (see Table 1). However, all of them satisfy $\lambda(\rho) \geq 0$ for any $\rho \in Z^3(\mathfrak{g})$.

Finally, the $5 \oplus 1$ case consists of Lie algebras of the form $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}$ for any 5-dimensional unimodular (non nilpotent) solvable Lie algebra \mathfrak{b} such that $(b_2(\mathfrak{b}), b_3(\mathfrak{b})) \neq (0,0), (1,0), (0,1)$. Therefore, the Lie algebras are: $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$, $A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ with $0 < \beta < 1$, $A_{5,8}^{-1} \oplus \mathbb{R}$, $A_{5,9}^{-1,-1} \oplus \mathbb{R}$, $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ with $\gamma > 0$, $A_{5,14}^0 \oplus \mathbb{R}$, $A_{5,15}^{-1} \oplus \mathbb{R}$, $A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$ with $0 < \gamma < 1$, $A_{5,17}^{\alpha,-\alpha,1}$ with $\alpha \geq 0$, $A_{5,18}^0 \oplus \mathbb{R}$, $A_{5,19}^{-1,2} \oplus \mathbb{R}$, $A_{5,19}^{1,-2} \oplus \mathbb{R}$, $A_{5,20}^0 \oplus \mathbb{R}$, $A_{5,26}^{0,\pm 1} \oplus \mathbb{R}$, $A_{5,33}^{-1,-1} \oplus \mathbb{R}$ and $A_{5,35}^{0,-2} \oplus \mathbb{R}$. The explicit computation of each case allows us to distinguish the following three situations:

- If $\mathfrak{g} = A_{5,9}^{-1,-1} \oplus \mathbb{R}$ or $A_{5,26}^{0,\pm 1} \oplus \mathbb{R}$, then $\lambda(\rho) \geq 0$ for all $\rho \in Z^3(\mathfrak{g})$.
- The Lie algebras $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$ admit a closed complex volume form (see Table 1 for a concrete example).
- For the rest of Lie algebras there is no $\rho \in Z^3(\mathfrak{g})$ satisfying $d(J_\rho^* \rho) = 0$ and $\lambda(\rho) < 0$ simultaneously.

In conclusion, in the $5 \oplus 1$ case the only possibilities are $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and the family $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$. \square

Next we obtain the classification in the indecomposable case.

Proposition 2.7. *Let \mathfrak{g} be a six-dimensional indecomposable unimodular (non nilpotent) solvable Lie algebra admitting a complex structure with a non-zero closed (3,0)-form. Then, \mathfrak{g} is isomorphic to $N_{6,18}^{0,-1,-1}$, $A_{6,37}^{0,0,1}$, $A_{6,82}^{0,1,1}$, $A_{6,88}^{0,0,1}$, $B_{6,4}^1$ or $B_{6,6}^1$.*

Proof. The Lie algebras \mathfrak{g} such that $b_3(\mathfrak{g}) \geq 2$ are listed in Table 2 in the Appendix. The indecomposable case is long to analyze because of the amount of Lie algebras, but after performing the computations we distinguish the following three situations:

- Let \mathfrak{g} be one of the following Lie algebras: $A_{6,13}^{a,-2a,2a-1}$ ($a \in \mathbb{R} - \{-1, 0, \frac{1}{3}, \frac{1}{2}\}$), $A_{6,13}^{a,-a,-1}$ ($a > 0, a \neq 1$), $A_{6,14}^{\frac{1}{3},-\frac{2}{3}}$, $A_{6,18}^{a,b}$ with $(a,b) \in \{(-\frac{1}{2}, -2), (-2, 1)\}$, $A_{6,25}^{a,b}$ with $(a,b) \in \{(0, -1), (-\frac{1}{2}, -\frac{1}{2})\}$, $A_{6,32}^{0,b,-b}$ ($b > 0$), $A_{6,34}^{0,0,\epsilon}$ ($\epsilon = 0, 1$), $A_{6,35}^{a,b,c}$ with $a > 0$ and $(b,c) \in \{(-2a, a), (-a, 0)\}$ and $A_{6,37}^{0,0,c}$ ($c > 0, c \neq 1$). Then, $\lambda(\rho) \geq 0$ for any $\rho \in Z^3(\mathfrak{g})$.
- The Lie algebras $N_{6,18}^{0,-1,-1}$, $A_{6,37}^{0,0,1}$, $A_{6,82}^{0,1,1}$, $A_{6,88}^{0,0,1}$, $B_{6,4}^1$ and $B_{6,6}^1$ admit a non-zero closed (3,0)-form (see Table 2 for a concrete example).
- For the rest of Lie algebras there is no $\rho \in Z^3(\mathfrak{g})$ such that $d(J_\rho^*\rho) = 0$ and $\lambda(\rho) < 0$. \square

From Propositions 2.6 and 2.7 it follows the following classification:

Theorem 2.8. *Let \mathfrak{g} be an unimodular (non nilpotent) solvable Lie algebra of dimension 6. Then, \mathfrak{g} admits a complex structure with a non-zero closed (3,0)-form if and only if it is isomorphic to one in the following list:*

$$\begin{aligned} \mathfrak{g}_1 &= A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0), \\ \mathfrak{g}_2^\alpha &= A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0), \quad \alpha \geq 0, \\ \mathfrak{g}_3 &= \mathfrak{e}(2) \oplus \mathfrak{e}(1, 1) = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45}), \\ \mathfrak{g}_4 &= A_{6,37}^{0,0,1} = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0), \\ \mathfrak{g}_5 &= A_{6,82}^{0,1,1} = (e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0), \\ \mathfrak{g}_6 &= A_{6,88}^{0,0,1} = (e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0), \\ \mathfrak{g}_7 &= B_{6,6}^1 = (e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0), \\ \mathfrak{g}_8 &= N_{6,18}^{0,-1,-1} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0), \\ \mathfrak{g}_9 &= B_{6,4}^1 = (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0). \end{aligned}$$

From now on, we will use only the simplified notation $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \dots, \mathfrak{g}_9$ when referring to the Lie algebras listed in the previous theorem.

2.1. Existence of lattices. In this section we show that the simply-connected solvable Lie groups G_k corresponding to the Lie algebras \mathfrak{g}_k in Theorem 2.8 admit lattices Γ_k of maximal rank. Therefore, we get compact complex solvmanifolds $\Gamma_k \backslash G_k$ with holomorphically trivial canonical bundle.

Let H be a n -dimensional nilpotent Lie group. We remind that a connected and simply-connected Lie group G with nilradical H is called *almost nilpotent* (resp. *almost abelian*) if it can be written as $G = \mathbb{R} \ltimes_\mu H$ (resp. $G = \mathbb{R} \ltimes_\mu \mathbb{R}^n$, that is, $H = \mathbb{R}^n$). If we denote by e the identity element of H then following [6] we have that $d_e(\mu(t)) = \exp^{\text{GL}(n, \mathbb{R})}(t\varphi)$, φ being a certain derivation of the Lie algebra \mathfrak{h} of H . Although it is not easy in general to know whether a solvable Lie group G admits a lattice, the next result allows to construct one in the case that G is almost nilpotent.

Lemma 2.9. [6] *Let $G = \mathbb{R} \ltimes_{\mu} H$ be a $(n+1)$ -dimensional almost nilpotent Lie group with nilradical H and \mathfrak{h} the Lie algebra of H . If there exists $0 \neq t_1 \in \mathbb{R}$ and a rational basis $\{X_1, \dots, X_n\}$ of \mathfrak{h} such that the coordinate matrix of $d_e(\mu(t_1))$ in such basis is integer, then $\Gamma = t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle X_1, \dots, X_n \rangle)$ is a lattice in G .*

Now we will use the former lemma to prove the following result:

Proposition 2.10. *For any $k \neq 2$, the connected and simply-connected Lie group G_k with underlying Lie algebra \mathfrak{g}_k admits a lattice.*

For $k = 2$, there exists a countable number of distinct α 's, including $\alpha = 0$, for which the connected and simply-connected Lie group with underlying Lie algebra \mathfrak{g}_2^{α} admits a lattice.

Proof. The Lie algebra \mathfrak{g}_8 is not almost nilpotent, but its corresponding connected and simply-connected Lie group G_8 admits a lattice by [33]. It is not hard to see that for $k \neq 8$ the Lie algebra \mathfrak{g}_k of Theorem 2.8 is either almost nilpotent or a product of almost nilpotent Lie algebras. In fact, we find the following correspondence with some of the Lie algebras studied in [6] (we use the notation in that paper in order to compare directly with the Lie algebras therein): $\mathfrak{g}_1 \cong \mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$, $\mathfrak{g}_2^0 \cong \mathfrak{g}_{5,17}^{0,0,1} \oplus \mathbb{R}$, $\mathfrak{g}_3 \cong \mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,4}^{-1}$, $\mathfrak{g}_4 \cong \mathfrak{g}_{6,37}^{0,0,-1}$, $\mathfrak{g}_5 \cong \mathfrak{g}_{6,88}^{0,-1,0}$, $\mathfrak{g}_6 \cong \mathfrak{g}_{6,92}^{0,-1,-1}$ and $\mathfrak{g}_7 \cong \mathfrak{g}_{6,92}^*$. The simply connected Lie group G_3 admits a lattice, since it is product of two 3-dimensional Lie groups and every 3-dimensional completely solvable simply connected Lie group has a lattice. For $\mathfrak{g}_1, \mathfrak{g}_2^0, \mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6$ and \mathfrak{g}_7 , the existence of lattices in the corresponding Lie groups is already proved in [6]. In fact, a lattice is respectively given by:

$$\begin{aligned} \Gamma_1 &= (t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle e_1, \dots, e_4 \rangle)) \times \mathbb{Z}, \quad H = \mathbb{R}^4, \quad t_1 = \log\left(\frac{3+\sqrt{5}}{2}\right); \\ \Gamma_2^0 &= (t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle e_1, \dots, e_4 \rangle)) \times \mathbb{Z}, \quad H = \mathbb{R}^4, \quad t_1 = \pi; \\ \Gamma_4 &= t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle e_1, \dots, e_5 \rangle), \quad H = \text{Heis}_3 \times \mathbb{R}^2, \quad t_1 = \pi; \\ \Gamma_5 &= t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle -\frac{1}{\sqrt{5}}e_1, \frac{1}{\sqrt{5}}(e_3 - e_5), \alpha e_3 - \frac{1}{5\alpha}e_5, \frac{1}{\sqrt{5}}(e_2 - e_4), \alpha e_2 - \frac{1}{5\alpha}e_4 \rangle), \\ &\quad \alpha = \frac{\sqrt{5}(3+\sqrt{5})}{10}, \quad H = \text{Heis}_5, \quad t_1 = \log\left(\frac{3+\sqrt{5}}{2}\right); \\ \Gamma_6 &= t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle e_1, \dots, e_5 \rangle), \quad H = \text{Heis}_5, \quad t_1 = \pi; \\ \Gamma_7 &= t_1 \mathbb{Z} \ltimes_{\mu} \exp^H(\mathbb{Z}\langle 2e_1, -e_2 - e_3, e_4 + e_5, e_2 - e_3, e_4 - e_5 \rangle), \quad H = \text{Heis}_5, \quad t_1 = \pi, \end{aligned}$$

where Heis_3 and Heis_5 are the real Heisenberg group of dimension 3 and 5. So, it remains to study \mathfrak{g}_2^{α} with $\alpha > 0$, and \mathfrak{g}_9 .

We will show first that there exists a countable subfamily of \mathfrak{g}_2^{α} with $\alpha > 0$ whose corresponding Lie group G_2^{α} admits lattice. The 5-dimensional factor $A_{5,17}^{\alpha,-\alpha,1}$ in the decomposable Lie algebra $\mathfrak{g}_2^{\alpha} = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ is given by

$$[e_1, e_5] = -\alpha e_1 + e_2, \quad [e_2, e_5] = -e_1 - \alpha e_2, \quad [e_3, e_5] = \alpha e_3 + e_4, \quad [e_4, e_5] = -e_3 + \alpha e_4,$$

which is almost abelian since $A_{5,17}^{\alpha,-\alpha,1} = \mathbb{R} \ltimes_{\text{ad}_{e_5}} \mathbb{R}^4$. If we denote by B_{α} the coordinate matrix of the derivation $\text{ad}_{e_5} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ in the basis $\{e_1, e_2, e_3, e_4\}$, then the coordinate matrix of $d_e(\mu(t))$ is the exponential

$$e^{tB_{\alpha}} = \begin{pmatrix} e^{\alpha t} \cos(t) & e^{\alpha t} \sin(t) & 0 & 0 \\ -e^{\alpha t} \sin(t) & e^{\alpha t} \cos(t) & 0 & 0 \\ 0 & 0 & e^{-\alpha t} \cos(t) & e^{-\alpha t} \sin(t) \\ 0 & 0 & -e^{-\alpha t} \sin(t) & e^{-\alpha t} \cos(t) \end{pmatrix}.$$

If $t_l = l\pi$ with $l \in \mathbb{Z}$ and $l > 0$, then the characteristic polynomial of the matrix $e^{t_l B_\alpha}$ is $p(\lambda) = (1 - (-1)^l (e^{\alpha t_l} + e^{-\alpha t_l})\lambda + \lambda^2)^2$, which is integer if $\alpha_{l,m} = \frac{1}{l\pi} \log\left(\frac{m + \sqrt{m^2 - 4}}{2}\right)$ with $m \in \mathbb{Z}$ and $m > 2$. Moreover, $e^{t_l B_\alpha} = P^{-1} C_{l,m} P$, where

$$P^{-1} = \begin{pmatrix} 0 & 0 & \epsilon & \beta^+ \\ \epsilon & \beta^+ & 0 & 0 \\ 0 & 0 & -\epsilon & \beta^- \\ -\epsilon & \beta^- & 0 & 0 \end{pmatrix}, \quad C_{l,m} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & m(-1)^l & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & m(-1)^l \end{pmatrix},$$

with $\epsilon = \frac{1}{\sqrt{m^2 - 4}}$ and $\beta^\pm = \frac{m^2 - 4 \pm (-1)^l \sqrt{m^2 - 4}}{2(m^2 - 4)}$. Taking the basis $X_1 = \epsilon(e_2 - e_4)$, $X_2 = \beta^+ e_2 + \beta^- e_4$, $X_3 = \epsilon(e_1 + e_3)$ and $X_4 = \beta^+ e_1 + \beta^- e_3$ of \mathbb{R}^4 and using Lemma 2.9 we have that $\Gamma' = l\pi\mathbb{Z} \times_\mu \mathbb{Z}\langle X_1, \dots, X_4 \rangle$ is a lattice of the simply-connected Lie group associated to $A_{5,17}^{\alpha, -\alpha, 1}$ with $\alpha = \alpha_{l,m}$. Hence, $\Gamma = \Gamma' \times \mathbb{Z}$ is a lattice in $G_2^{\alpha_{l,m}}$.

The Lie algebra \mathfrak{g}_9 can be seen as an almost nilpotent Lie algebra $\mathfrak{g} = \mathbb{R} \times_{\text{ad}_{e_6}} \mathfrak{h}$, where $\mathfrak{h} = \langle e_1, \dots, e_5 \mid [e_1, e_4] = -e_3, [e_1, e_5] = -e_2, [e_4, e_5] = -e_1 \rangle$ is a 5-dimensional nilpotent Lie algebra. Proceeding in a similar manner as for \mathfrak{g}_2^α , we compute the characteristic polynomial of $d_e(\mu(t))$ getting that $p(\lambda) = (\lambda^2 - 2\lambda \cos(t) + 1)^2$. If $t_1 = \pi$ then $p(\lambda) \in \mathbb{Z}[\lambda]$ and the coordinate matrix of $d_e(\mu(t_1))$ in the basis $X_1 = \frac{\pi}{2}e_1$, $X_2 = \sqrt{\frac{\pi}{2}}e_4$, $X_3 = \sqrt{\frac{\pi}{2}}e_5$, $X_4 = (\frac{\pi}{2})^{3/2}e_2 + \sqrt{\frac{\pi}{2}}e_4$ and $X_5 = -(\frac{\pi}{2})^{3/2}e_3 + \sqrt{\frac{\pi}{2}}e_5$ of \mathfrak{h} is

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Moreover, $\{X_1, \dots, X_5\}$ is a rational basis of \mathfrak{h} because $[X_1, X_2] = [X_1, X_4] = -X_3 + X_5$, $[X_1, X_3] = [X_1, X_5] = X_2 - X_4$, $[X_2, X_3] = [X_2, X_5] = -[X_3, X_4] = -X_1$. Hence, if we denote by H the simply-connected Lie group corresponding to \mathfrak{h} , then using Lemma 2.9 we have that $\Gamma = \pi\mathbb{Z} \times_\mu \exp^H(\mathbb{Z}\langle X_1, \dots, X_5 \rangle)$ is a lattice in the Lie group G_9 . \square

Remark 2.11. Bock found a lattice for the Lie group associated to $A_2^{\alpha, -\alpha, 1}$ with $\alpha = \alpha_{1,3} = \frac{1}{\pi} \log \frac{3 + \sqrt{5}}{2}$, that is, for $l = 1$ and $m = 3$. Notice that our result for $k = 2$ is consistent with the result obtained by Witte in [32, Prop. 8.7], where it is showed that only countably many non-isomorphic simply-connected Lie groups admit a lattice, so that one cannot expect a lattice to exist for any real $\alpha > 0$.

The Lie algebra \mathfrak{g}_9 does not appear in [6] because its nilradical is the 5-dimensional Lie algebra \mathfrak{h} , which is isomorphic to $\mathfrak{g}_{5,3}$ (in the notation of [6]), but there are only two solvable and unimodular Lie algebras with nilradical $\mathfrak{g}_{5,3}$ considered in that paper (namely $\mathfrak{g}_{6,76}^{-1}$ and $\mathfrak{g}_{6,78}$) which are both completely solvable, but \mathfrak{g}_9 is not.

3. MODULI OF COMPLEX STRUCTURES

In this section we classify, up to equivalence, the complex structures having closed (3,0)-form on the Lie algebras of Theorem 2.8. Recall that two complex structures J and J' on \mathfrak{g} are said to be *equivalent* if there exists an automorphism $F: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $F \circ J = J' \circ F$.

3.1. The decomposable case. We consider firstly the $5 \oplus 1$ decomposable Lie algebras.

Lemma 3.1. *Let J be any complex structure on \mathfrak{g}_1 or \mathfrak{g}_2^α , $\alpha \geq 0$, with closed volume (3,0)-form, then there is a non-zero closed (1,0)-form. More concretely, the (1,0)-form $e^5 - iJ^*e^5$ is closed.*

Proof. Let us consider first $\mathfrak{g} = \mathfrak{g}_1$ with structure equations given as in Theorem 2.8. Any $\rho \in Z^3(\mathfrak{g})$ is given by

$$\begin{aligned} \rho &= a_1e^{125} + a_2e^{126} + a_3e^{135} + a_4e^{136} + a_6e^{156} + a_8e^{245} + a_9e^{246} \\ &\quad + a_{10}e^{256} + a_{11}e^{345} + a_{12}e^{346} + a_{13}e^{356} + a_{14}e^{456}, \end{aligned}$$

where $a_i \in \mathbb{R}$. We use the equation (2) to compute the space of almost complex structures corresponding to $\rho \in Z^3(\mathfrak{g})$. When we compute the images of e^5, e^6 by J_ρ^* we find that the subspace spanned by e^5, e^6 is J_ρ^* -invariant, because

$$\begin{aligned} J_\rho^*e^5 &= \frac{1}{\sqrt{|\lambda(\rho)|}}((a_1a_{12} + a_{11}a_2 - a_4a_8 - a_3a_9)e^5 + 2(a_{12}a_2 - a_4a_9)e^6), \\ J_\rho^*e^6 &= \frac{1}{\sqrt{|\lambda(\rho)|}}(-2(a_1a_{11} - a_3a_8)e^5 - (a_1a_{12} + a_{11}a_2 - a_4a_8 - a_3a_9)e^6). \end{aligned}$$

Therefore, the (1,0)-form $\eta = e^5 - iJ_\rho^*e^5$ is closed for every $\rho \in Z^3(\mathfrak{g})$.

The same situation appears for $\mathfrak{g} = \mathfrak{g}_2^\alpha$, $\alpha \geq 0$, because again the subspace spanned by e^5, e^6 is found to be J_ρ^* -invariant for all $\rho \in Z^3(\mathfrak{g})$. \square

Lemma 3.2. *Let J be any complex structure on \mathfrak{g}_1 or \mathfrak{g}_2^α , $\alpha \geq 0$, with closed volume (3,0)-form. Then, there is a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the following reduced equations*

$$(6) \quad \begin{cases} d\omega^1 = A\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^2 = -A\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^3 = 0, \end{cases}$$

where $A = \cos \theta + i \sin \theta$, $\theta \in [0, \pi)$.

Proof. By Lemma 3.1 we can consider a basis of (1,0)-forms $\{\eta^1, \eta^2, \eta^3\}$ such that $\eta^3 = e^5 - iJ^*e^5$ is closed. The structure equations of \mathfrak{g}_1 and \mathfrak{g}_2^α with $\alpha \geq 0$ force the differential of any 1-form to be a multiple of $e^5 = \frac{1}{2}(\eta^3 + \eta^{\bar{3}})$, so there exist $A, B, C, D, E, F \in \mathbb{C}$ such that

$$\begin{cases} d\eta^1 = (A\eta^1 + B\eta^2 + E\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^2 = (C\eta^1 + D\eta^2 + F\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^3 = 0. \end{cases}$$

Moreover, since $d(\eta^{123}) = 0$ necessarily $D = -A$.

Let us consider the non-zero 1-form $\tau^1 = A\eta^1 + B\eta^2 + E\eta^3$. Notice that

$$d\tau^1 = ((A^2 + BC)\eta^1 + (AE + BF)\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}),$$

which implies that $A^2 + BC \neq 0$ because otherwise $d\tau^1$ would be a multiple of e^{56} . Then, with respect to the new (1,0)-basis $\{\tau^1, \tau^2, \tau^3\}$ given by

$$\tau^1 = A\eta^1 + B\eta^2 + E\eta^3, \quad \tau^2 = C\eta^1 - A\eta^2 + F\eta^3, \quad \tau^3 = \eta^3,$$

the complex structure equations are

$$(7) \quad \begin{cases} d\tau^1 = (A\tau^1 + B\tau^2) \wedge (\tau^3 + \tau^{\bar{3}}), \\ d\tau^2 = (C\tau^1 - A\tau^2) \wedge (\tau^3 + \tau^{\bar{3}}), \\ d\tau^3 = 0. \end{cases}$$

Now we distinguish two cases:

- If $B \neq 0$ then we consider the new basis $\{\omega^1, \omega^2, \omega^3\}$ given by

$$\begin{aligned} \omega^1 &= (A + \sqrt{A^2 + BC})\tau^1 + B\tau^2, & \omega^2 &= (A - \sqrt{A^2 + BC})\tau^1 + B\tau^2, \\ \omega^3 &= |\sqrt{A^2 + BC}|\tau^3. \end{aligned}$$

With respect to this basis, the equations (7) reduce to

$$d\omega^1 = \frac{\sqrt{A^2 + BC}}{|\sqrt{A^2 + BC}|} \omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -\frac{\sqrt{A^2 + BC}}{|\sqrt{A^2 + BC}|} \omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = 0,$$

that is, the equations are of the form (6) where the new complex coefficient has modulus equal to 1.

- If $C \neq 0$ then with respect to the basis $\{\omega^1, \omega^2, \omega^3\}$ given by

$$\begin{aligned} \omega^1 &= C\tau^1 - (A - \sqrt{A^2 + BC})\tau^2, & \omega^2 &= C\tau^1 - (A + \sqrt{A^2 + BC})\tau^2, \\ \omega^3 &= |\sqrt{A^2 + BC}|\tau^3, \end{aligned}$$

the equations (7) again reduce to equations of the form (6) where the new complex coefficient has modulus 1.

Finally, notice that in the equations (6) one can change the sign of A by changing the sign of ω^3 , so we can suppose that $A = \cos \theta + i \sin \theta$ with angle $\theta \in [0, \pi)$. \square

Proposition 3.3. *Up to isomorphism, there is only one complex structure with closed $(3, 0)$ -form on the Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2^0 , whereas \mathfrak{g}_2^α has two such complex structures for any $\alpha > 0$. More concretely, the complex structures are:*

$$(8) \quad (\mathfrak{g}_1, J): \quad d\omega^1 = \omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = 0;$$

$$(9) \quad (\mathfrak{g}_2^0, J): \quad d\omega^1 = i\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = 0;$$

$$(10) \quad (\mathfrak{g}_2^{\alpha = \frac{\cos \theta}{\sin \theta}}, J_\pm): \quad \begin{cases} d\omega^1 = (\pm \cos \theta + i \sin \theta) \omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^2 = -(\pm \cos \theta + i \sin \theta) \omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^3 = 0, \end{cases}$$

where $\theta \in (0, \pi/2)$.

Proof. A real Lie algebra underlying the equations (6) is isomorphic to \mathfrak{g}_1 or \mathfrak{g}_2^α for some $\alpha \geq 0$. In fact, in terms of the real basis β^1, \dots, β^6 given by $\omega^1 = \beta^1 + i\beta^2$, $\omega^2 = \beta^3 + i\beta^4$ and $\omega^3 = \frac{1}{2}(\beta^5 + i\beta^6)$, we have

$$\begin{aligned} d\beta^1 &= \cos \theta \beta^{15} - \sin \theta \beta^{25}, & d\beta^3 &= -\cos \theta \beta^{35} + \sin \theta \beta^{45}, & d\beta^5 &= 0, \\ d\beta^2 &= \sin \theta \beta^{15} + \cos \theta \beta^{25}, & d\beta^4 &= -\sin \theta \beta^{35} - \cos \theta \beta^{45}, & d\beta^6 &= 0. \end{aligned}$$

In particular:

- If $\theta = 0$ then taking $e^1 = \beta^1$, $e^2 = \beta^4$, $e^3 = \beta^3$, $e^4 = \beta^2$, $e^5 = \beta^5$ and $e^6 = \beta^6$ the resulting structure equations are precisely those of the Lie algebra \mathfrak{g}_1 .

• If $\theta \in (0, \pi)$ then $\sin \theta \neq 0$ and taking $e^1 = \beta^1$, $e^2 = -\beta^2$, $e^3 = \beta^3$, $e^4 = \beta^4$, $e^5 = \sin \theta \beta^5$ and $e^6 = \beta^6$ we get the structure equations of $\mathfrak{g}_2^{|\alpha|}$ with $\alpha = -\frac{\cos \theta}{\sin \theta}$. Notice that α takes any real value when θ varies in $(0, \pi)$, and if $\theta \neq \frac{\pi}{2}$ then θ and $\pi - \theta$ correspond to two complex structures on the same Lie algebra. By a standard argument one can prove that these two complex structures are non-equivalent. \square

Let us consider now the $3 \oplus 3$ decomposable Lie algebra \mathfrak{g}_3 . With respect to the structure equations given in Theorem 2.8, any closed 3-form $\rho \in Z^3(\mathfrak{g}_3)$ is given by

$$\begin{aligned} \rho = & a_1 e^{123} + a_2 e^{124} + a_3 e^{134} + a_4 e^{145} + a_5 e^{146} + a_6 e^{156} + a_7 e^{234} \\ & + a_8(e^{136} - e^{245}) + a_9(e^{135} - e^{246}) + a_{10}(e^{126} + e^{345}) \\ & + a_{11}(e^{125} + e^{346}) + a_{12} e^{456}, \end{aligned}$$

where $a_1, \dots, a_{12} \in \mathbb{R}$. By imposing the closedness of $J_\rho^* \rho$ together with the condition $\text{tr}(J_\rho^{*2}) < 0$, one arrives by a long computation to an explicit description of the complex structure J_ρ , which allows us to prove that $\{e^1, e^2, e^3, J_\rho^* e^1, J_\rho^* e^2, J_\rho^* e^3\}$ are always linearly independent. Therefore, the forms

$$\omega^1 = e^1 - iJ_\rho^* e^1, \quad \omega^2 = e^2 - iJ_\rho^* e^2, \quad \omega^3 = e^3 - iJ_\rho^* e^3,$$

constitute a (1,0)-basis for the complex structure J_ρ . Moreover, one can show that with respect to this basis the complex structure equations have the form

$$(11) \quad \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} + b\omega^{1\bar{1}} + fi\omega^{1\bar{2}} - fi\omega^{2\bar{1}} - (\frac{1}{2} + gi)\omega^{1\bar{3}} + gi\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + c\omega^{1\bar{1}} + (\frac{1}{2} + hi)\omega^{1\bar{2}} - hi\omega^{2\bar{1}} - fi\omega^{1\bar{3}} + fi\omega^{3\bar{1}}, \end{cases}$$

where the coefficients b, c, f, g, h are real and satisfy $4gh = 4f^2 - 1$. (Explicit expression of each coefficient in terms of a_1, \dots, a_{12} can be given, but this information is not relevant and so we omit it.) Notice that the condition $4gh = 4f^2 - 1$ is equivalent to the Jacobi identity $d^2 = 0$. Furthermore, these equations can be reduced as follows:

Proposition 3.4. *Up to isomorphism, the complex structures with closed (3,0)-form on the Lie algebra \mathfrak{g}_3 are*

$$(12) \quad (\mathfrak{g}_3, J_x): \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} - (\frac{1}{2} + xi)\omega^{1\bar{3}} + xi\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\omega^{2\bar{1}}, \end{cases}$$

where $x \in \mathbb{R}^+$.

Proof. Observe first that with respect to the (1,0)-basis $\{\omega^1, \omega^2 + 2c\omega^1, \omega^3 - 2b\omega^1\}$, the complex structure equations express again as in (11) but with $b = c = 0$, that is to say, one can suppose that the coefficients b and c both vanish.

Let us prove next that we can also take the coefficient f to be zero. To see this, let $\{\omega^1, \omega^2, \omega^3\}$ be a (1,0)-basis satisfying (11) with $b = c = 0$ and $f \neq 0$, and let us consider the (1,0)-basis $\{\eta^1, \eta^2, \eta^3\}$ given by

$$\eta^1 = \omega^1, \quad \eta^2 = \omega^2 - \frac{g-h-\sqrt{1+(g+h)^2}}{2f}\omega^3, \quad \eta^3 = \frac{g-h-\sqrt{1+(g+h)^2}}{2f}\omega^2 + \omega^3.$$

A direct calculation shows that with respect to $\{\eta^1, \eta^2, \eta^3\}$ the corresponding coefficient f vanishes. Therefore, since $4gh = -1$ we are led to the reduced equations (12), where we have written x instead of g .

Finally, let J_x and $J_{x'}$ be two complex structures corresponding to $x, x' \in \mathbb{R}$. It is easy to see that the structures are equivalent if and only if $xx' = -\frac{1}{4}$. This represents an hyperbola in the (x, x') -plane, so the equivalence class is given by one of the branches of the hyperbola, that is, we can take $x > 0$. \square

3.2. The indecomposable case. Next we classify the complex structures on the indecomposable Lie algebras $\mathfrak{g}_4, \dots, \mathfrak{g}_9$.

Lemma 3.5. *Let J be any complex structure on \mathfrak{g}_k ($4 \leq k \leq 7$) with closed $(3, 0)$ -form. Then, there is a $(1, 0)$ -basis $\{\omega^1, \omega^2, \omega^3\}$ such that*

$$(13) \quad \begin{cases} d\omega^1 = A\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^2 = -A\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^3 = G_{11}\omega^{1\bar{1}} + G_{12}\omega^{1\bar{2}} + \bar{G}_{12}\omega^{2\bar{1}} + G_{22}\omega^{2\bar{2}}, \end{cases}$$

where $A, G_{12} \in \mathbb{C}$ and $G_{11}, G_{22} \in \mathbb{R}$, with $(G_{11}, G_{12}, G_{22}) \neq (0, 0, 0)$, satisfy

$$(14) \quad |A| = 1, \quad (A + \bar{A})G_{11} = 0, \quad (A + \bar{A})G_{22} = 0, \quad (A - \bar{A})G_{12} = 0.$$

Proof. Let us consider first the Lie algebra \mathfrak{g}_4 with structure equations given as in Theorem 2.8. Any element $\rho \in Z^3(\mathfrak{g}_4)$ is given by

$$\begin{aligned} \rho = & a_1 e^{123} + a_2 e^{126} + a_3 (e^{125} - e^{134}) + a_4 (e^{124} + e^{135}) + a_5 e^{136} \\ & + a_6 (e^{156} + e^{234}) + a_7 (e^{146} - e^{235}) + a_8 e^{236} + a_9 e^{246} + a_{10} e^{256} \\ & + a_{11} e^{346} + a_{12} e^{356} + a_{13} e^{456}, \end{aligned}$$

where $a_1, \dots, a_{13} \in \mathbb{R}$. A direct calculation shows that if $a_3^2 + a_4^2 = 0$ then there do not exist closed 3-forms ρ satisfying the conditions $d(J_\rho^* \rho) = 0$ and $\lambda(\rho) < 0$.

Suppose that $a_3^2 + a_4^2 \neq 0$. Then, an element $\rho \in Z^3(\mathfrak{g}_4)$ satisfies the condition $d(J_\rho^* \rho) = 0$ if and only if $a_{10} = \frac{a_3(a_6^2 - a_7^2) + 2a_4 a_6 a_7 - a_{11}(a_3^2 + a_4^2)}{a_3^2 + a_4^2}$, $a_{12} = \frac{2a_3 a_6 a_7 - a_4(a_6^2 - a_7^2) + a_9(a_3^2 + a_4^2)}{a_3^2 + a_4^2}$ and $a_{13} = 0$. Moreover, under these relations one has that $\lambda(\rho) = -4(a_3 a_9 - a_4 a_{11} + a_6 a_7)^2 \leq 0$.

Let $\rho \in Z^3(\mathfrak{g}_4)$ be such that $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$. A direct calculation shows that $\tilde{J}_\rho^* e^6$ is given by

$$\begin{aligned} \tilde{J}_\rho^* e^6 = & 2(a_3^2 + a_4^2)e^1 + 2(a_3 a_6 + a_4 a_7)e^2 + 2(a_3 a_7 - a_4 a_6)e^3 \\ & + 2(a_3 a_{11} + a_4 a_9 + a_7^2)e^6. \end{aligned}$$

Therefore, the coefficient of $J_\rho^* e^6$ in e^1 is nonzero for any ρ .

A similar computation for the Lie algebras $\mathfrak{g}_5, \mathfrak{g}_6$ and \mathfrak{g}_7 shows that for any complex structure J_ρ with closed $(3, 0)$ -form, we also have that

$$J_\rho^* e^6 = c_1 e^1 + c_2 e^2 + c_3 e^3 + c_4 e^4 + c_5 e^5 + c_6 e^6,$$

where the coefficient c_1 is non-zero.

Let us consider the $(1, 0)$ -form $\eta^3 = e^6 - iJ_\rho^* e^6$. From the structure equations of \mathfrak{g}_k ($4 \leq k \leq 7$) in Theorem 2.8, it follows that

$$\begin{aligned} d\eta^3 = & ic_1 e^{23} - i\alpha \wedge e^6, & \text{if } \mathfrak{g} = \mathfrak{g}_4, \\ d\eta^3 = & ic_1 (e^{24} + e^{35}) - i\alpha \wedge e^6, & \text{if } \mathfrak{g} = \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_7, \end{aligned}$$

where α is a 1-form. Since $c_1 \neq 0$ we can write the 2-forms e^{23} and $e^{24} + e^{35}$ as

$$(15) \quad \begin{aligned} e^{23} &= -\frac{i}{c_1} d\eta^3 + \frac{1}{c_1} \alpha \wedge e^6, & \text{if } \mathfrak{g} = \mathfrak{g}_4, \\ e^{24} + e^{35} &= -\frac{i}{c_1} d\eta^3 + \frac{1}{c_1} \alpha \wedge e^6, & \text{if } \mathfrak{g} = \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_7. \end{aligned}$$

Now, let η^1, η^2 be such that $\{\eta^1, \eta^2, \eta^3\}$ is a basis of $(1,0)$ -forms. Since e^6 is closed and $\eta^3 + \eta^{\bar{3}} = 2e^6$, the integrability of the complex structure implies that $d\eta^3$ has no component of type $(2,0)$ and

$$d\eta^3 = G_{11} \eta^{1\bar{1}} + G_{12} \eta^{1\bar{2}} + G_{13} \eta^{1\bar{3}} + \bar{G}_{12} \eta^{2\bar{1}} + G_{22} \eta^{2\bar{2}} + G_{23} \eta^{2\bar{3}} + \bar{G}_{13} \eta^{3\bar{1}} + \bar{G}_{23} \eta^{3\bar{2}} + G_{33} \eta^{3\bar{3}},$$

for some $G_{11}, G_{22}, G_{33} \in \mathbb{R}$ and $G_{12}, G_{13}, G_{23} \in \mathbb{C}$.

From the structure of the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$), the relation (15) and taking into account that $d\eta^3$ is of type $(1,1)$, it follows that there exist $\lambda, \mu \in \mathbb{C}$ such that

$$(16) \quad \begin{cases} d\eta^1 = \lambda d\eta^3 + (A\eta^1 + B\eta^2 + E\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^2 = \mu d\eta^3 + (C\eta^1 + D\eta^2 + F\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^3 = G_{11} \eta^{1\bar{1}} + G_{12} \eta^{1\bar{2}} + G_{13} \eta^{1\bar{3}} \\ \quad + \bar{G}_{12} \eta^{2\bar{1}} + G_{22} \eta^{2\bar{2}} + G_{23} \eta^{2\bar{3}} \\ \quad + \bar{G}_{13} \eta^{3\bar{1}} + \bar{G}_{23} \eta^{3\bar{2}} + G_{33} \eta^{3\bar{3}}, \end{cases}$$

for some $A, B, C, D, E, F \in \mathbb{C}$.

We will see next that these complex equations can be reduced to equations of the form (13). Notice first that with respect to the $(1,0)$ -basis $\{\eta^1 - \lambda\eta^3, \eta^2 - \mu\eta^3, \eta^3\}$ we get complex equations of the form (16) with $\lambda = \mu = 0$. So, without loss of generality we can suppose $\lambda = \mu = 0$. Moreover, the coefficients E and F also vanish. In fact, suppose for example that $E \neq 0$ (the case $F \neq 0$ is similar). Using (16) with $\lambda = \mu = 0$, the condition $d(d\eta^1) = 0$ is equivalent to

$$EG_{11} = EG_{12} = EG_{13} = EG_{22} = EG_{23} = 0,$$

so $E \neq 0$ implies $d\eta^3 = G_{33} \eta^{3\bar{3}} = G_{33} \eta^3 \wedge (\eta^3 + \eta^{\bar{3}})$. But this is a contradiction with the structure of the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$), because $d(\mathfrak{g}_k^*)$ would be annihilated by the real 1-form $\eta^3 + \eta^{\bar{3}}$.

From now on, we suppose that $\lambda = \mu = E = F = 0$ in the equations (16). A direct calculation shows that

$$d\eta^{123} = \bar{G}_{13} \eta^{123\bar{1}} + \bar{G}_{23} \eta^{123\bar{2}} + (A + D + G_{33}) \eta^{123\bar{3}},$$

so η^{123} is closed if and only if $G_{13} = G_{23} = 0$ and $D = -A - G_{33}$. Moreover, the unimodularity of the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$) implies that $G_{33} = 0$. In fact, taking the real basis $\{f^1, \dots, f^6\}$ of \mathfrak{g}_k^* given by

$$\eta^1 = f^2 + if^3, \quad \eta^2 = f^4 + if^5, \quad \eta^3 = f^6 + if^1,$$

we get that the trace of ad_{f_6} is zero if and only if $G_{33} = -2\Re A - 2\Re D$, which implies, using that $G_{33} = -A - D$, that the coefficient $G_{33} = 0$.

Summing up, we have proved the existence of a $(1, 0)$ -basis $\{\eta^1, \eta^2, \eta^3\}$ satisfying the reduced complex equations

$$(17) \quad \begin{cases} d\eta^1 = (A\eta^1 + B\eta^2) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^2 = (C\eta^1 - A\eta^2) \wedge (\eta^3 + \eta^{\bar{3}}), \\ d\eta^3 = G_{11}\eta^{1\bar{1}} + G_{12}\eta^{1\bar{2}} + \overline{G}_{12}\eta^{2\bar{1}} + G_{22}\eta^{2\bar{2}}, \end{cases}$$

where $A, B, C, G_{12} \in \mathbb{C}$ and $G_{11}, G_{22} \in \mathbb{R}$.

Notice that $A^2 + BC \neq 0$ because otherwise the $(1, 0)$ -form $A\eta^1 + B\eta^2$ would be closed, but this is a contradiction to $b_1(\mathfrak{g}_k) = 1$, for $4 \leq k \leq 7$. Therefore, arguing as in the proof of Lemma 3.2 we can suppose that $B = C = 0$ and $|A| = 1$ in (17). Finally, the condition $d(d\eta^3) = 0$ is satisfied if and only if $(A + \overline{A})G_{11} = (A + \overline{A})G_{22} = (A - \overline{A})G_{12} = 0$. \square

As a consequence of the previous lemma, we have the following classification of complex structures on \mathfrak{g}_k , for $4 \leq k \leq 7$.

Proposition 3.6. *Up to isomorphism there is only one complex structure J with closed $(3, 0)$ -form on the Lie algebras \mathfrak{g}_5 and \mathfrak{g}_6 , and two such complex structures on the Lie algebras \mathfrak{g}_4 and \mathfrak{g}_7 . More concretely, the complex structures are:*

$$(18) \quad (\mathfrak{g}_4, J_{\pm}): d\omega^1 = i\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = \pm\omega^{1\bar{1}};$$

$$(19) \quad (\mathfrak{g}_5, J): d\omega^1 = \omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = \omega^{1\bar{2}} + \omega^{2\bar{1}};$$

$$(20) \quad (\mathfrak{g}_6, J): d\omega^1 = i\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = \omega^{1\bar{1}} + \omega^{2\bar{2}};$$

$$(21) \quad (\mathfrak{g}_7, J_{\pm}): d\omega^1 = i\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}}).$$

Proof. First notice that in the equations (13), after changing the sign of ω^3 if necessary, we can always suppose that $A = \cos \theta + i \sin \theta$ with angle $\theta \in [0, \pi)$. We have the following cases:

- If $\cos \theta \neq 0$, then (14) implies $G_{11} = G_{22} = 0$ and $\sin \theta G_{12} = 0$, so $\sin \theta = 0$ because $(G_{11}, G_{12}, G_{22}) \neq (0, 0, 0)$ is satisfied if and only if $G_{12} \neq 0$. Therefore, in this case $A = 1$ and, moreover, we can normalize the coefficient G_{12} (it suffices to consider $G_{12}\omega^1$ instead of ω^1). So the complex structure equations take the form (19), and in terms of the real basis $\{e^1, \dots, e^6\}$ defined by $\omega^1 = e^2 - ie^3$, $\omega^2 = e^5 + ie^4$ and $\omega^3 = \frac{1}{2}e^6 - 2ie^1$, one has

$$de^1 = e^{24} + e^{35}, \quad de^2 = e^{26}, \quad de^3 = e^{36}, \quad de^4 = -e^{46}, \quad de^5 = -e^{56}, \quad de^6 = 0,$$

that is, the underlying Lie algebra is \mathfrak{g}_5 .

- If $\cos \theta = 0$, then (14) implies that $A = i$ and $G_{12} = 0$. Therefore, the complex structure equations become

$$d\omega^1 = i\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = G_{11}\omega^{1\bar{1}} + G_{22}\omega^{2\bar{2}},$$

where $(G_{11}, G_{22}) \neq (0, 0)$. We have the following possibilities:

- When $G_{22} = 0$ we can suppose that $G_{11} = \pm 1$ (it suffices to consider $\sqrt{|G_{11}|}\omega^1$ instead of ω^1), and then the complex structure equations reduce to (18). In terms of

the real basis $\{e^1, \dots, e^6\}$ given by $\omega^1 = e^2 - ie^3$, $\omega^2 = e^4 + ie^5$ and $\omega^3 = -\frac{1}{2}e^6 \pm 2ie^1$, we arrive at

$$de^1 = e^{23}, \quad de^2 = -e^{36}, \quad de^3 = e^{26}, \quad de^4 = -e^{56}, \quad de^5 = e^{46}, \quad de^6 = 0,$$

that is, the underlying Lie algebra is \mathfrak{g}_4 . A standard argument allows to conclude that the two complex structures in (18) are non-isomorphic.

- The case $G_{11} = 0$ easily reduces to the previous case, so it does not produce any non-isomorphic complex structure.

- Finally, if $G_{11} \neq 0$ and $G_{22} \neq 0$ then we can suppose $G_{11} = \pm 1$ and $G_{22} = \pm 1$ (it suffices to consider $\sqrt{|G_{kk}|}\omega^k$ instead of ω^k for $k = 1, 2$). It is clear that the case $G_{11} = G_{22} = -1$ is equivalent to $G_{11} = G_{22} = 1$, so it remains to study the following three cases: $(G_{11}, G_{22}) = (1, 1), (1, -1), (-1, 1)$. In terms of the real basis $\{\beta^1, \dots, \beta^6\}$ defined by $\omega^1 = \beta^2 + i\beta^4$, $\omega^2 = \beta^3 + i\beta^5$ and $\omega^3 = \frac{1}{2}\beta^6 + 2i\beta^1$, one has $d\beta^1 = -G_{11}\beta^{24} - G_{22}\beta^{35}$, $d\beta^2 = -\beta^{46}$, $d\beta^3 = \beta^{56}$, $d\beta^4 = \beta^{26}$, $d\beta^5 = -\beta^{36}$, $d\beta^6 = 0$.

When $(G_{11}, G_{22}) = (1, 1)$, taking the basis $e^1 = -2\beta^1$, $e^2 = \beta^2 + \beta^3$, $e^3 = -\beta^4 + \beta^5$, $e^4 = \beta^4 + \beta^5$, $e^5 = \beta^2 - \beta^3$ and $e^6 = -\beta^6$, the real structure equations are

$$de^1 = e^{24} + e^{35}, \quad de^2 = -e^{36}, \quad de^3 = e^{26}, \quad de^4 = -e^{56}, \quad de^5 = e^{46}, \quad de^6 = 0,$$

so the underlying Lie algebra is \mathfrak{g}_6 and the complex structure is given by (20).

The cases $(G_{11}, G_{22}) = (1, -1)$ and $(G_{11}, G_{22}) = (-1, 1)$ both correspond to the same Lie algebra (in fact, a change in the sign of β^1 gives an isomorphism), so we suppose next that $(G_{11}, G_{22}) = (1, -1)$, i.e.

$$d\beta^1 = -\beta^{24} + \beta^{35}, \quad d\beta^2 = -\beta^{46}, \quad d\beta^3 = \beta^{56}, \quad d\beta^4 = \beta^{26}, \quad d\beta^5 = -\beta^{36}, \quad d\beta^6 = 0.$$

Taking $e^1 = -\beta^1$, $e^3 = -\beta^3$ and $e^6 = -\beta^6$, we conclude that \mathfrak{g}_7 is the underlying Lie algebra. Therefore, the complex structures on \mathfrak{g}_7 are given by (18), and it can be proved that they are non-isomorphic. \square

Next we find that there are infinitely many non-isomorphic complex structures on the Lie algebra \mathfrak{g}_8 .

Proposition 3.7. *Let J be any complex structure on \mathfrak{g}_8 with closed volume $(3, 0)$ -form. Then, there is a $(1, 0)$ -basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying one of the following reduced equations:*

$$(22) \quad (\mathfrak{g}_8, J): \quad d\omega^1 = 2i\omega^{13} + \omega^{3\bar{3}}, \quad d\omega^2 = -2i\omega^{23}, \quad d\omega^3 = 0;$$

$$(23) \quad (\mathfrak{g}_8, J'): \quad d\omega^1 = 2i\omega^{13} + \omega^{3\bar{3}}, \quad d\omega^2 = -2i\omega^{23} + \omega^{3\bar{3}}, \quad d\omega^3 = 0;$$

$$(24) \quad (\mathfrak{g}_8, J_A): \quad \begin{cases} d\omega^1 = -(A - i)\omega^{13} - (A + i)\omega^{1\bar{3}}, \\ d\omega^2 = (A - i)\omega^{23} + (A + i)\omega^{2\bar{3}}, \\ d\omega^3 = 0, \end{cases}$$

where $A \in \mathbb{C}$ with $\Im A \neq 0$.

Moreover, the complex structures above are non-isomorphic.

Proof. With respect to the structure equations of \mathfrak{g}_8 given in Theorem 2.8, any closed 3-form $\rho \in Z^3(\mathfrak{g}_8)$ is given by

$$\begin{aligned} \rho = & a_1 e^{126} + a_2 e^{135} + a_3 e^{145} + a_4 e^{156} + a_5 e^{235} + a_6(e^{146} + e^{236}) \\ & + a_7 e^{245} + a_8(e^{136} - e^{246}) + a_9 e^{256} + a_{10} e^{346} + a_{11} e^{356} + a_{12} e^{456}, \end{aligned}$$

where $a_1, \dots, a_{12} \in \mathbb{R}$. A direct calculation shows that such a ρ satisfies the conditions $d(J_\rho^* \rho) = 0$ and $\lambda(\rho) < 0$ if and only if $a_1 = 0$, $a_2 = -a_7$, $a_3 = a_5$, $a_{10} = 0$ and $a_6 a_7 - a_5 a_8 \neq 0$. Moreover, in this case $\lambda(\rho) = -4(a_6 a_7 - a_5 a_8)^2$.

The associated complex structures J_ρ^* express in terms of the real basis $\{e^1, \dots, e^6\}$ as

$$\begin{aligned} J_\rho^* e^1 &= e^2 + \frac{a_5 a_{12} - a_7 a_{11}}{a_6 a_7 - a_5 a_8} e^5 + \frac{a_6 a_{12} - a_8 a_{11}}{a_6 a_7 - a_5 a_8} e^6, \\ J_\rho^* e^2 &= -e^1 + \frac{a_5 a_{11} + a_7 a_{12}}{a_6 a_7 - a_5 a_8} e^5 + \frac{a_6 a_{11} + a_8 a_{12}}{a_6 a_7 - a_5 a_8} e^6, \\ J_\rho^* e^3 &= e^4 + \frac{a_4 a_7 - a_5 a_9}{a_6 a_7 - a_5 a_8} e^5 + \frac{a_4 a_8 - a_6 a_9}{a_6 a_7 - a_5 a_8} e^6, \\ J_\rho^* e^4 &= -e^3 - \frac{a_4 a_5 + a_7 a_9}{a_6 a_7 - a_5 a_8} e^5 - \frac{a_4 a_6 + a_8 a_9}{a_6 a_7 - a_5 a_8} e^6, \\ J_\rho^* e^5 &= \frac{a_5 a_6 + a_7 a_8}{a_6 a_7 - a_5 a_8} e^5 + \frac{a_6^2 + a_8^2}{a_6 a_7 - a_5 a_8} e^6, \\ J_\rho^* e^6 &= -\frac{a_5^2 + a_7^2}{a_6 a_7 - a_5 a_8} e^5 - \frac{a_5 a_6 + a_7 a_8}{a_6 a_7 - a_5 a_8} e^6. \end{aligned}$$

Let us consider the basis of $(1,0)$ -forms $\{\omega^1, \omega^2, \omega^3\}$ given by

$$\begin{aligned} \omega^1 &= e^1 - i J_\rho^* e^1 = e^1 - i(e^2 + k_1 e^5 + k_2 e^6), \\ \omega^2 &= e^3 - i J_\rho^* e^3 = e^3 - i(e^4 + k_3 e^5 + k_4 e^6), \\ \omega^3 &= \frac{1}{2c}(e^5 - i J_\rho^* e^5) = \frac{1}{2c} e^5 - i\left(\frac{b}{2c} e^5 + \frac{1}{2} e^6\right), \end{aligned}$$

where $k_1 = \frac{a_5 a_{12} - a_7 a_{11}}{a_6 a_7 - a_5 a_8}$, $k_2 = \frac{a_6 a_{12} - a_8 a_{11}}{a_6 a_7 - a_5 a_8}$, $k_3 = \frac{a_4 a_7 - a_5 a_9}{a_6 a_7 - a_5 a_8}$, $k_4 = \frac{a_4 a_8 - a_6 a_9}{a_6 a_7 - a_5 a_8}$, $b = \frac{a_5 a_6 + a_7 a_8}{a_6 a_7 - a_5 a_8}$ and $c = \frac{a_6^2 + a_8^2}{a_6 a_7 - a_5 a_8}$. Notice that $c \neq 0$, and $-2(a_6 + i a_8) \omega^{123} = \rho + i J_\rho^* \rho$.

With respect to this basis, the complex structure equations are

$$(25) \quad \begin{cases} d\omega^1 = -(A - i)\omega^{13} - (A + i)\omega^{1\bar{3}} + B\omega^{3\bar{3}}, \\ d\omega^2 = (A - i)\omega^{23} + (A + i)\omega^{2\bar{3}} + C\omega^{3\bar{3}}, \\ d\omega^3 = 0, \end{cases}$$

where $A = b + ic$, $B = 2c(k_1 + ik_2)$ and $C = -2c(k_3 + ik_4)$. Notice that $\Im A = c \neq 0$. Now, we will reduce the complex equations (25) as follows:

- If $A \neq -i$, then with respect to the $(1,0)$ -basis $\{\eta^1, \eta^2, \eta^3\}$ given by

$$\eta^1 = -(A + i)\omega^1 + B\omega^3, \quad \eta^2 = (A + i)\omega^2 + C\omega^3, \quad \eta^3 = \omega^3,$$

the complex structure equations are of the form (24).

- If $A = -i$, the equations (25) reduce to

$$J_{(B,C)}: d\omega^1 = 2i\omega^{13} + B\omega^{3\bar{3}}, \quad d\omega^2 = -2i\omega^{23} + C\omega^{3\bar{3}}, \quad d\omega^3 = 0.$$

Notice that the structures $J_{(B,C)}$ and $J_{(C,B)}$ are equivalent, since it suffices to consider the change of basis $\eta^1 = \omega^2$, $\eta^2 = \omega^1$, $\eta^3 = -\omega^3$. Now:

- if $B = C = 0$ then the complex equations are of the form (24) with $A = -i$;
- if only one of the coefficients B, C is nonzero, for instance B , then taking $\frac{1}{B}\omega^1$ instead of ω^1 , we arrive at the complex equations (22);
- finally, if $B, C \neq 0$ then we can normalize both coefficients and the corresponding complex equations are (23).

It is straightforward to check that the complex structures given in equations (22)–(24) are non-isomorphic. \square

Remark 3.8. Note that on \mathfrak{g}_8 there exists a unique complex structure J that is abelian [2], i.e. satisfying $[JX, JY] = [X, Y]$, which corresponds to the value $A = i$ in equations (24), and a unique bi-invariant complex structure [22], corresponding to $A = -i$ in (24).

Finally, let us consider now the Lie algebra \mathfrak{g}_9 . With respect to the structure equations given in Theorem 2.8, any closed 3-form $\rho \in Z^3(\mathfrak{g}_9)$ is given by

$$\begin{aligned} \rho &= a_1(e^{124} - e^{135}) + a_2 e^{145} + a_3 e^{146} + a_4 e^{156} + a_5(e^{136} - e^{245}) \\ &\quad + a_6(e^{125} + e^{134} - e^{246}) + a_7 e^{256} + a_8(e^{126} + e^{345}) + a_9 e^{346} \\ &\quad + a_{10}(e^{125} + e^{134} + e^{356}) + a_{11} e^{456}. \end{aligned}$$

By imposing the closedness of $J_\rho^* \rho$ together with the condition $\text{tr}(J_\rho^{*2}) < 0$, one can arrive after a long computation to an explicit description of the complex structure J_ρ , which allows us to prove that $\{e^2, e^4, e^6, J_\rho^* e^2, J_\rho^* e^4, J_\rho^* e^6\}$ are always linearly independent. Therefore, the forms

$$\omega^1 = e^6 - iJ_\rho^* e^6, \quad \omega^2 = e^2 - iJ_\rho^* e^2, \quad \omega^3 = e^4 - iJ_\rho^* e^4,$$

constitute a (1,0)-basis for the complex structure J_ρ , and one can show that with respect to this basis the complex structure equations have the form

$$(26) \quad \begin{cases} d\omega^1 = -c^2 \omega^{1\bar{1}} - c \omega^{3\bar{1}} - c \omega^{1\bar{3}} - \omega^{3\bar{3}}, \\ d\omega^2 = \left(\frac{c}{2} + cE - \frac{i}{2}G\right) \omega^{1\bar{1}} - \frac{i}{2} \omega^{2\bar{1}} + E \omega^{3\bar{1}} + \left(\frac{1}{2} + cG\right) \omega^{1\bar{3}} \\ \quad + G \omega^{3\bar{3}} + \frac{i}{2} \omega^{1\bar{2}} + (cG - E) \omega^{1\bar{3}}, \\ d\omega^3 = c(c^2 + \frac{i}{2}) \omega^{1\bar{1}} + (c^2 + \frac{i}{2}) \omega^{3\bar{1}} + c^2 \omega^{1\bar{3}} + c \omega^{3\bar{3}} - \frac{i}{2} \omega^{1\bar{3}}, \end{cases}$$

where c is real and $E, G \in \mathbb{C}$. (Explicit expression of each coefficient in terms of a_1, \dots, a_{11} can be given, but this information will not be relevant in what follows and so we omit it.) In the following result we prove that all the complex structures are equivalent.

Proposition 3.9. *Up to isomorphism, there is only one complex structure with closed (3,0)-form on the Lie algebra \mathfrak{g}_9 , whose complex equations are*

$$(27) \quad (\mathfrak{g}_9, J): d\omega^1 = -\omega^{3\bar{3}}, \quad d\omega^2 = \frac{i}{2} \omega^{1\bar{2}} + \frac{1}{2} \omega^{1\bar{3}} - \frac{i}{2} \omega^{2\bar{1}}, \quad d\omega^3 = -\frac{i}{2} \omega^{1\bar{3}} + \frac{i}{2} \omega^{3\bar{1}}.$$

Proof. First, notice that one can suppose that $G = 0$ by taking $\omega^2 + G\omega^1$ instead of ω^2 in the equations (26). Now, let $\{\omega^1, \omega^2, \omega^3\}$ be a (1,0)-basis satisfying (26) with $G = 0$, and consider the new (1,0)-basis

$$\{\sigma^1 = \omega^1, \quad \sigma^2 = icE\omega^1 + \omega^2 + iE\omega^3, \quad \sigma^3 = c\omega^1 + \omega^3\}.$$

A direct calculation shows that this basis satisfies equations (26) with $c = 0$ and $E = G = 0$, that is, the complex equations can always be reduced to (27). In particular, all the complex structures are equivalent. \square

In the following theorem we sum up the classification of invariant complex structures with closed (3,0)-form on solvmanifolds obtained in this section.

Theorem 3.10. *Let $M = \Gamma \backslash G$ be a 6-dimensional solvmanifold and denote by \mathfrak{g} the Lie algebra of G . If J is an invariant complex structure on M with closed (3,0)-form, then the pair (\mathfrak{g}, J) is isomorphic to one and only one of the complex structures given in Propositions 3.3, 3.4, 3.6, 3.7 or 3.9.*

4. EXISTENCE OF SPECIAL HERMITIAN METRICS

In this section we use the classification of complex structures obtained in the previous section to study the existence of several special Hermitian metrics. We will center our attention on SKT, generalized Gauduchon, balanced and strongly Gauduchon metrics.

Let (M, J, g) be a Hermitian manifold of real dimension $2n$ with fundamental 2-form $F(\cdot, \cdot) = g(J\cdot, \cdot)$. Since the metric g is determined by the form F , from now on we will denote a Hermitian metric also by F . An *SKT* (*strong Kähler with torsion*) metric is a Hermitian metric satisfying $\partial\bar{\partial}F = 0$ (for more details see e.g. [12] and the references therein).

Recently, Fu, Wang and Wu [16] introduced and studied *generalized k -th Gauduchon* metrics, defined by the condition $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, where $1 \leq k \leq n-1$. Notice that any SKT metric is a 1-st Gauduchon metric.

For any compact Hermitian manifold (M, J, F) and for any integer $1 \leq k \leq n-1$, in [16] it is proved the existence of a unique constant $\gamma_k(F)$ and a (unique up to a constant) function $v \in C^\infty(M)$ such that $\frac{i}{2}\partial\bar{\partial}(e^v F^k) \wedge F^{n-k-1} = \gamma_k(F)e^v F^n$. The constant $\gamma_k(F)$ is invariant under biholomorphisms and its sign is an invariant of the conformal class of F . Thus, $\gamma_k(F)$ is > 0 ($= 0$, or < 0) if and only if there exists \tilde{F} in the conformal class of F such that $\frac{i}{2}\partial\bar{\partial}\tilde{F}^k \wedge \tilde{F}^{n-k-1} > 0$ ($= 0$, or < 0).

On the other hand, a Hermitian metric is called *balanced* if the fundamental form F satisfies that F^{n-1} is a closed form, and it is said to be *strongly Gauduchon* (sG for short) if the $(n, n-1)$ -form ∂F^{n-1} is $\bar{\partial}$ -exact. It is obvious from the definitions that balanced implies sG. Strongly Gauduchon metrics have been introduced recently in [24], whereas balanced metrics were previously considered in [19].

Next we study the existence of special Hermitian metrics on 6-dimensional solvmanifolds $(M = \Gamma \backslash G, J)$ endowed with an invariant complex structure J with holomorphically trivial canonical bundle. By Proposition 2.1 the latter condition is equivalent to the existence of an invariant non-zero closed $(3,0)$ -form. Notice that the symmetrization process can be applied to this situation to conclude that the existence of SKT, balanced or sG metrics on M reduces to the level of the Lie algebra \mathfrak{g} of G (see [7, 11] for more details). Thus, our strategy will consist in starting with the classification of pairs (\mathfrak{g}, J) obtained in Theorem 3.10 and then find the J -Hermitian structures F on \mathfrak{g} that satisfy the required conditions.

Notice that given a $(1,0)$ -basis $\{\omega^1, \omega^2, \omega^3\}$ for the complex structure J , a generic Hermitian structure F on the Lie algebra \mathfrak{g} is expressed as

$$(28) \quad 2F = i(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + u\omega^{1\bar{2}} - \bar{u}\omega^{2\bar{1}} + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}},$$

where the coefficients r^2, s^2, t^2 are non-zero real numbers and $u, v, z \in \mathbb{C}$ satisfy $r^2s^2 > |u|^2$, $s^2t^2 > |v|^2$, $r^2t^2 > |z|^2$ and $r^2s^2t^2 + 2\Re(i\bar{u}\bar{v}z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2$.

Firstly we study the SKT geometry.

Theorem 4.1. *Let $(M = \Gamma \backslash G, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, and denote by \mathfrak{g} the Lie algebra of G . Then, (M, J) has an SKT metric if and only if $\mathfrak{g} \cong \mathfrak{g}_2^0$ or \mathfrak{g}_4 .*

Proof. Let F be a J -Hermitian metric given by (28). We study first the existence of SKT metrics on \mathfrak{g}_1 and \mathfrak{g}_2^α . The equations (6) parametrize all the complex structures

J on \mathfrak{g}_1 or \mathfrak{g}_2^α , from which we get

$$(29) \quad \begin{aligned} \partial\bar{\partial}F &= -2ir^2(\Re \mathbf{e} A)^2\omega^{13\bar{1}\bar{3}} + 2u(\Im A)^2\omega^{13\bar{2}\bar{3}} \\ &\quad - 2\bar{u}(\Im A)^2\omega^{23\bar{1}\bar{3}} - 2is^2(\Re \mathbf{e} A)^2\omega^{23\bar{2}\bar{3}}. \end{aligned}$$

Thus, $\partial\bar{\partial}F = 0$ implies $\Re \mathbf{e} A = 0$, and so necessarily $A = i$. In this case F is SKT if and only if $u = 0$. By Proposition 3.3 the corresponding Lie algebra is \mathfrak{g}_2^0 .

For the Lie algebra \mathfrak{g}_3 , by Proposition 3.4 any J on \mathfrak{g}_3 is equivalent to one complex structure J_x given by (12). The (3,3)-form $\partial\bar{\partial}F \wedge F$ is given by

$$(30) \quad \partial\bar{\partial}F \wedge F = \frac{1+4x^2}{16x^2} (4x^2s^4 + t^4) \omega^{123\bar{1}\bar{2}\bar{3}}.$$

Since this form is never zero, there is no SKT metric on \mathfrak{g}_3 .

For the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$), using the equations (13), which parametrize all the complex structures J on \mathfrak{g}_k , we get

$$(31) \quad \begin{aligned} \partial\bar{\partial}F &= it^2(G_{11}G_{22} - |G_{12}|^2)\omega^{12\bar{1}\bar{2}} - 2ir^2(\Re \mathbf{e} A)^2\omega^{13\bar{1}\bar{3}} - 2is^2(\Re \mathbf{e} A)^2\omega^{23\bar{2}\bar{3}} \\ &\quad + 2u(\Im A)^2\omega^{13\bar{2}\bar{3}} - 2\bar{u}(\Im A)^2\omega^{23\bar{1}\bar{3}}. \end{aligned}$$

Thus, $\partial\bar{\partial}F = 0$ implies $\Re \mathbf{e} A = 0$, and from the conditions (14) we have $G_{12} = 0$. Now, $\partial\bar{\partial}F = 0$ also implies $G_{11}G_{22} = 0$, and from Proposition 3.6 it follows that only \mathfrak{g}_4 admits SKT structures: in fact, a generic F given by (28) is SKT if and only if $u = 0$.

For the study of SKT metrics on \mathfrak{g}_8 , instead of using the complex structure equations (22), (23) and (24), we use the equations (25) obtained in the proof of Proposition 3.7. A direct calculation shows that

$$(32) \quad \partial\bar{\partial}F \wedge F = 2(r^2s^2(1 + \Re \mathbf{e}(A)^2) + |u|^2\Im \mathbf{m}(A)^2) \omega^{123\bar{1}\bar{2}\bar{3}},$$

in particular, this form does not depend on the complex coefficients B, C in (25). The form $\partial\bar{\partial}F \wedge F$ never vanishes, so there is no SKT metric on \mathfrak{g}_8 .

Finally, for the Lie algebra \mathfrak{g}_9 , from the complex equations (27) in Proposition 3.9 it follows

$$(33) \quad \partial\bar{\partial}F \wedge F = \left(|v|^2 + \frac{s^4}{8}\right) \omega^{123\bar{1}\bar{2}\bar{3}} \neq 0,$$

so the Lie algebra \mathfrak{g}_9 does not admit SKT metrics. \square

Remark 4.2. In the previous theorem we have proved that any complex structure with non-zero closed (3,0)-form on \mathfrak{g}_2^0 or \mathfrak{g}_4 admits SKT metrics. Moreover, a generic metric F given by (28) satisfies the SKT condition with respect to the complex equations (9) for (\mathfrak{g}_2^0, J) , or (18) for (\mathfrak{g}_4, J_\pm) , if and only if $u = 0$, so in both cases the SKT metrics are given by

$$2F = i(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}},$$

where the coefficients r^2, s^2, t^2 are non-zero real numbers and $v, z \in \mathbb{C}$ satisfy $r^2s^2t^2 > r^2|v|^2 + s^2|z|^2$.

Whereas it is known that the Lie algebra \mathfrak{g}_2^0 admits SKT metrics (actually it admits Calabi-Yau metrics, for instance any $F = \frac{i}{2}(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}})$ is Kähler), however a solvmanifold based on \mathfrak{g}_4 provides, as far as we know, a new example of 6-dimensional compact SKT manifold.

We recall that a complex structure J on a symplectic manifold (M, ω) is said to be tamed by the symplectic form ω if $\omega(X, JX) > 0$ for any non-zero vector field X on M . The pair (ω, J) is also called a *Hermitian-symplectic structure* in [30]. By

[10, Proposition 2.1] the existence of a Hermitian-symplectic structure on a complex manifold (M, J) is equivalent to the existence of a J -compatible SKT metric g whose fundamental form F satisfies $\partial F = \bar{\partial}\beta$ for some ∂ -closed $(2, 0)$ -form β . As a consequence of Theorem 4.1 we have that a 6-dimensional solvmanifold $(M = \Gamma \backslash G, J)$ with J invariant and holomorphically trivial canonical bundle, has a symplectic form ω taming J if and only if $\mathfrak{g} \cong \mathfrak{g}_2^0$ and (J, ω) is a Kähler structure. By [10] if $(\Gamma \backslash G, J)$ admits a non-invariant symplectic form taming J , then there exists an invariant one. So we can immediately exclude the solvmanifolds $\Gamma \backslash G$ with $\mathfrak{g} \cong \mathfrak{g}_4$ since \mathfrak{g}_4 does not admit any symplectic form. For the solvmanifolds $\Gamma \backslash G$ with $\mathfrak{g} \cong \mathfrak{g}_2^0$ by a direct computation we have that $\partial F = \bar{\partial}\beta$, for some ∂ -closed $(2, 0)$ -form β , if and only if $dF = 0$.

In the following result we study the existence of 1-st Gauduchon metrics.

Theorem 4.3. *Let $(M = \Gamma \backslash G, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, and let F be an invariant J -Hermitian metric on M . If \mathfrak{g} denotes the Lie algebra of G , then we have:*

- (i) *If $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha (\alpha > 0), \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7, \mathfrak{g}_8$ or \mathfrak{g}_9 , then $\gamma_1 > 0$ for any (J, F) .*
- (ii) *If $\mathfrak{g} \cong \mathfrak{g}_2^0$ or \mathfrak{g}_4 , then $\gamma_1 \geq 0$ for any (J, F) ; moreover, an invariant Hermitian metric is 1-st Gauduchon if and only if it is SKT.*
- (iii) *If $\mathfrak{g} \cong \mathfrak{g}_6$ then there exist invariant Hermitian metrics such that $\gamma_1 > 0, = 0$ or < 0 ; in particular, there are invariant 1-st Gauduchon metrics which are not SKT.*

Proof. Let F be an invariant J -Hermitian metric given by (28). Then, $F^3 = -\frac{3}{4} \det(F) \omega^{123\bar{1}\bar{2}\bar{3}}$, where

$$\det(F) = \begin{vmatrix} i r^2 & u & z \\ -\bar{u} & i s^2 & v \\ -\bar{z} & -\bar{v} & i t^2 \end{vmatrix}.$$

Notice that $i \det(F) > 0$. Now, if

$$\partial \bar{\partial} F \wedge F = \mu \omega^{123\bar{1}\bar{2}\bar{3}}$$

then $\frac{i}{2} \partial \bar{\partial} F \wedge F = \frac{2\mu}{3i \det(F)} F^3$, which implies that

$$\gamma_1(F) > 0, = 0 \text{ or } < 0 \quad \text{if and only if} \quad \mu > 0, = 0 \text{ or } < 0.$$

In what follows we will compute μ for any triple (\mathfrak{g}, J, F) , and study its possible signs.

For the Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2^α , from (29) it follows

$$\partial \bar{\partial} F \wedge F = 2 (r^2 s^2 (\Re A)^2 + |u|^2 (\Im A)^2) \omega^{123\bar{1}\bar{2}\bar{3}}.$$

Therefore, $\gamma_1(F) \geq 0$ for any F . Moreover, $\gamma_1(F) \geq 0$ if and only if $\Re A = 0$ and $u = 0$, which corresponds precisely to SKT metric on \mathfrak{g}_2^0 .

From (30), (32) and (33) it follows that $\gamma_1 > 0$ for any (J, F) on $\mathfrak{g}_3, \mathfrak{g}_8$ and \mathfrak{g}_9 .

For the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$), using (31) we get

$$2\partial \bar{\partial} F \wedge F = [4r^2 s^2 (\Re A)^2 + 4|u|^2 (\Im A)^2 - t^4 (G_{11} G_{22} - |G_{12}|^2)] \omega^{123\bar{1}\bar{2}\bar{3}}.$$

Let us consider first \mathfrak{g}_4 . By (18) we can take $A = i$, $G_{11} = \pm 1$ and $G_{12} = G_{22} = 0$, so $2\partial \bar{\partial} F \wedge F = 4|u|^2 \omega^{123\bar{1}\bar{2}\bar{3}}$. This implies that $\gamma_1 \geq 0$, and it is equal to zero if and only if the structure is SKT. This completes the proof of (i).

For the Lie algebra \mathfrak{g}_5 , by (19) we have that $A = G_{12} = 1$ and $G_{11} = G_{22} = 0$, so $2\partial\bar{\partial}F \wedge F = (4r^2s^2 + t^4)\omega^{123\bar{1}\bar{2}\bar{3}}$ and $\gamma_1 > 0$.

Similarly, using (21), for \mathfrak{g}_7 we can take $A = i$, $G_{12} = 0$ and $(G_{11}, G_{22}) = (-1, 1)$ or $(1, -1)$. Therefore, $2\partial\bar{\partial}F \wedge F = (t^4 + 4|u|^2)\omega^{123\bar{1}\bar{2}\bar{3}}$ and thus $\gamma_1 > 0$. This completes the proof of (ii).

Finally, to prove (iii), by (20) we consider $A = i$, $G_{12} = 0$ and $G_{11} = G_{22} = 1$. Since $2\partial\bar{\partial}F \wedge F = (4|u|^2 - t^4)\omega^{123\bar{1}\bar{2}\bar{3}}$, we conclude that on \mathfrak{g}_6 there exist Hermitian metrics such that $\gamma_1 > 0$, $= 0$ or < 0 , depending on the sign of $4|u|^2 - t^4$. \square

Remark 4.4. Notice that the symmetrization process cannot be applied to the 1-st Gauduchon condition on the solvmanifold $M = \Gamma \backslash G$ in order to reduce the problem to the Lie algebra level, so the previous theorem studies only the existence of *invariant* 1-st Gauduchon metrics.

On the other hand, it is worthy to remark that on the solvmanifold $M = \Gamma \backslash G$ with Lie algebra $\mathfrak{g} \cong \mathfrak{g}_6$ there exist invariant 1-st Gauduchon metrics, although M does not admit any SKT metric. In fact, with respect to the complex equations (20), any invariant Hermitian metric F given by (28) with $|u| = \frac{t^2}{2}$ is 1-st Gauduchon, however there is no SKT metric by Theorem 4.1. This is in deep contrast with the nilpotent case, because any invariant 1-st Gauduchon metric on a 6-nilmanifold is necessarily SKT (see [13, Proposition 3.3]).

In the following result we study the existence of balanced Hermitian metrics. In particular, new examples of balanced solvmanifolds are found.

Theorem 4.5. *Let $(M = \Gamma \backslash G, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, and denote by \mathfrak{g} the Lie algebra of G . If (M, J) has a balanced metric then $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7$ or \mathfrak{g}_8 .*

Moreover, in such cases, any J admits balanced metrics except for the complex structures which are isomorphic to (22) or (23) on \mathfrak{g}_8 .

Proof. Since a J -Hermitian metric F given by (28) is balanced if and only if $\partial F^2 = 0$, next we compute the (3,2)-form ∂F^2 for each Lie algebra \mathfrak{g} .

For the existence of balanced metrics on \mathfrak{g}_1 and \mathfrak{g}_2^α , from the complex structure equations (6) it follows

$$(34) \quad 2\partial F^2 = (ir^2z + \bar{u}v)\bar{A}\omega^{123\bar{1}\bar{3}} + (is^2v - uz)\bar{A}\omega^{123\bar{2}\bar{3}}.$$

Since A is non-zero, this form vanishes if and only if $is^2v - uz = 0$ and $ir^2z + \bar{u}v = 0$. Now, $r^2s^2 - |u|^2 > 0$ implies that these conditions are equivalent to $v = z = 0$.

For the Lie algebra \mathfrak{g}_3 , a direct calculation using the complex equations (12) shows

$$(35) \quad \begin{aligned} 2\partial F^2 = & -\frac{1}{2x}(t^2\Re u + \Im(\bar{v}z) - x(it^2u + \bar{v}z))\omega^{123\bar{1}\bar{2}} \\ & + 2x\left(s^2\Re z - \Im(uv) + \frac{is^2z - uv}{4x}\right)\omega^{123\bar{1}\bar{3}}. \end{aligned}$$

Thus, the form F^2 is closed if and only if

$$\begin{cases} it^2u + \bar{v}z = (t^2\Re u + \Im(\bar{v}z))/x, \\ is^2z - uv = -4x(s^2\Re z - \Im(uv)). \end{cases}$$

Notice that since x is real, we have that both $it^2u + \bar{v}z$ and $is^2z - uv$ are also real numbers. But this implies that $t^2\Re u + \Im(\bar{v}z) = 0$ and $s^2\Re z - \Im(uv) = 0$, and so the system above is homogeneous. Finally, since $s^2t^2 - |v|^2 > 0$ necessarily $u = z = 0$.

For the Lie algebras \mathfrak{g}_k ($4 \leq k \leq 7$), from equations (13) we have

$$(36) \quad \begin{aligned} 2\partial F^2 = & [(s^2t^2 - |v|^2)G_{11} + (r^2t^2 - |z|^2)G_{22} \\ & + (v\bar{z} - it^2\bar{u})G_{12} + (\bar{v}z + it^2u)\overline{G}_{12}] \omega^{123\bar{1}\bar{2}} \\ & + (ir^2v + \bar{u}z)\overline{A}\omega^{123\bar{1}\bar{3}} + (is^2z - uv)\overline{A}\omega^{123\bar{2}\bar{3}}. \end{aligned}$$

Since A is non-zero and $r^2s^2 - |u|^2 > 0$, the coefficients of $\omega^{123\bar{1}\bar{3}}$ and $\omega^{123\bar{2}\bar{3}}$ vanish if and only if $v = z = 0$. The latter conditions reduce the expression of the form to

$$2\partial F^2 = t^2(s^2G_{11} + r^2G_{22} - i\bar{u}G_{12} + iu\overline{G}_{12})\omega^{123\bar{1}\bar{2}}.$$

Now, we can use the complex classification given in Proposition 3.6 to conclude that the only possibilities to get a closed form F^2 are, either $G_{12} = 0$ and $(G_{11}, G_{22}) = (1, -1), (-1, 1)$, or $G_{11} = G_{22} = 0$ and $G_{12} = 1$. The first case corresponds to \mathfrak{g}_7 and the coefficients r^2 and s^2 in the metric must be equal, whereas the second case corresponds to \mathfrak{g}_5 with metric coefficient $u \in \mathbb{R}$.

For the study of balanced Hermitian metrics on \mathfrak{g}_8 , by the complex equations (25), a direct calculation shows that

$$(37) \quad \begin{aligned} 2\partial F^2 = & -[(ir^2v + \bar{u}z)(\overline{A} - i) + (r^2s^2 - |u|^2)\overline{C}] \omega^{123\bar{1}\bar{3}} \\ & + [(uv - is^2z)(\overline{A} - i) + (r^2s^2 - |u|^2)\overline{B}] \omega^{123\bar{2}\bar{3}}. \end{aligned}$$

Since $r^2s^2 - |u|^2 \neq 0$, the structure (J, F) is balanced if and only if

$$B = -\frac{is^2\bar{z} + \bar{u}\bar{v}}{r^2s^2 - |u|^2}(A + i), \quad C = \frac{ir^2\bar{v} - u\bar{z}}{r^2s^2 - |u|^2}(A + i).$$

It follows from Proposition 3.7 that the complex structures (22) and (23) do not admit balanced metrics, because $A = -i$ but B is not zero. However, any complex structure in the family (24) has balanced Hermitian metrics because $B = C = 0$. In fact, if $A \neq -i$ then the metric (28) is balanced if and only if $v = z = 0$, and for $A = -i$ (i.e. the complex structure is bi-invariant) any metric is balanced.

In the case of the Lie algebra \mathfrak{g}_9 , from the complex equations (27) it follows

$$(38) \quad \begin{aligned} 4\partial F^2 = & (i\bar{u}\bar{v} - s^2\bar{z})\omega^{123\bar{1}\bar{2}} - (iv\bar{z} + t^2\bar{u} - uv + is^2z)\omega^{123\bar{1}\bar{3}} \\ & + 2(|u|^2 - r^2s^2)\omega^{123\bar{2}\bar{3}}, \end{aligned}$$

which implies that the component of ∂F^2 in $\omega^{123\bar{2}\bar{3}}$ is nonzero, so there are not balanced Hermitian metrics.

Finally, notice that for the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5$ and \mathfrak{g}_7 we have proved above that any complex structure J admits balanced Hermitian metrics. However, for the Lie algebras \mathfrak{g}_8 , a complex structure J admits balanced metric if and only if it is isomorphic to one in the family (24). \square

Next we prove that the non-sufficient necessary condition for the existence of balanced metrics found in Theorem 4.5 is necessary and sufficient for the existence of sG metrics.

Theorem 4.6. *Let $(M = \Gamma \backslash G, J)$ be a 6-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, and denote by \mathfrak{g} the Lie algebra of G . Then, (M, J) has an sG metric if and only if $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7$ or \mathfrak{g}_8 .*

Moreover, if $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3$ or \mathfrak{g}_8 , then any invariant Hermitian metric is sG.

Proof. Since balanced implies sG, by Theorem 4.5 we know that if $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7$ or \mathfrak{g}_8 , then there exist sG metrics. Moreover, any J on the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3, \mathfrak{g}_5$ and \mathfrak{g}_7 admits sG metrics. We prove next that there are not sG metrics on $\mathfrak{g}_4, \mathfrak{g}_6$ and \mathfrak{g}_9 .

From (13) we have $\bar{\partial}(\Lambda^{3,1}) = \langle \omega^{123\bar{1}\bar{3}}, \omega^{123\bar{2}\bar{3}} \rangle$, and by (36) the (3,2)-form ∂F^2 is a combination of $\omega^{123\bar{1}\bar{2}}, \omega^{123\bar{1}\bar{3}}$ and $\omega^{123\bar{2}\bar{3}}$. Hence, the existence of sG metric is equivalent to the vanishing of the coefficient of $\omega^{123\bar{1}\bar{2}}$ in ∂F^2 . By (18), the Lie algebra \mathfrak{g}_4 corresponds to $A = i, G_{11} = \pm 1$ and $G_{12} = G_{22} = 0$, so the coefficient of $\omega^{123\bar{1}\bar{2}}$ is equal to $\pm(s^2t^2 - |v|^2)$, which is never zero. On the other hand, by (20) the Lie algebra \mathfrak{g}_6 corresponds to $A = i, G_{11} = G_{22} = 1$ and $G_{12} = 0$, and the coefficient of $\omega^{123\bar{1}\bar{2}}$ is $(s^2t^2 - |v|^2) + (r^2t^2 - |z|^2)$, which is strictly positive. In conclusion, there do not exist sG metrics for \mathfrak{g}_4 or \mathfrak{g}_6 .

For the Lie algebra \mathfrak{g}_9 , equations (27) imply

$$\bar{\partial}\omega^{123\bar{1}} = 0, \quad \bar{\partial}\omega^{123\bar{2}} = (i/2)\omega^{123\bar{1}\bar{2}}, \quad \bar{\partial}\omega^{123\bar{3}} = -(i/2)\omega^{123\bar{1}\bar{3}},$$

therefore $\bar{\partial}\Lambda^{3,1} = \langle \omega^{123\bar{1}\bar{2}}, \omega^{123\bar{1}\bar{3}} \rangle$. By (38) we have that the component of ∂F^2 in $\omega^{123\bar{2}\bar{3}}$ is nonzero, so $\partial F^2 \notin \bar{\partial}\Lambda^{3,1}$ and F is never sG. Thus, there do not exist sG metrics for \mathfrak{g}_9 .

To finish the proof it remains to see that any pair (J, F) on $\mathfrak{g}_1, \mathfrak{g}_2^\alpha, \mathfrak{g}_3$ and \mathfrak{g}_8 is sG. By Proposition 3.3 and (34) a direct calculation implies $\partial F^2 \in \bar{\partial}(\Lambda^{3,1})$, so any (J, F) on \mathfrak{g}_1 or \mathfrak{g}_2^α is sG. For \mathfrak{g}_3 (resp. \mathfrak{g}_8) we also have $\partial F^2 \in \bar{\partial}(\Lambda^{3,1})$ for any Hermitian structure (J, F) by Proposition 3.4 and (35) (resp. Proposition 3.7 and (37)). \square

5. HOLOMORPHIC DEFORMATIONS

In this section we study some properties related to the existence of balanced metrics under deformation of the complex structure. In what follows, $(M, J_a)_{a \in \Delta}$, Δ being an open disc around the origin in \mathbb{C} , will denote a holomorphic family of compact complex manifolds. We briefly recall that a property is said to be *open* under holomorphic deformations if when it holds for a given compact complex manifold (M, J_0) , then (M, J_a) also has that property for all $a \in \Delta$ sufficiently close to 0. On the other hand, a property is said to be *closed* under holomorphic deformations if whenever (M, J_a) has that property for all $a \in \Delta \setminus \{0\}$ then the property also holds for the central limit (M, J_0) .

Concerning the property of existence of balanced Hermitian metrics, Alessandrini and Bassanelli proved in [1] (see also [11]) that it is not deformation open. In contrast to the balanced case, the sG property is open under holomorphic deformations [25]. However, in [7] it is shown that the sG property and the balanced property of compact complex manifolds are not closed under holomorphic deformations. More concretely, there exists a holomorphic family of compact complex manifolds $(M, J_a)_{a \in \Delta}$ such that (M, J_a) has balanced metric for any $a \neq 0$ but the central limit (M, J_0) does not admit any sG metric, which provides a counterexample to the Popovici and Demailly closedness conjectures formulated in [25].

On the other hand, recall that a compact complex manifold M is said to satisfy the $\partial\bar{\partial}$ -lemma if for any d -closed form α of pure type on M , the following exactness properties are equivalent:

$$\alpha \text{ is } d\text{-exact} \iff \alpha \text{ is } \partial\text{-exact} \iff \alpha \text{ is } \bar{\partial}\text{-exact} \iff \alpha \text{ is } \partial\bar{\partial}\text{-exact.}$$

Under this strong condition, the existence of sG metric in the central limit is guaranteed:

Proposition 5.1. [26, Proposition 4.1] *If the $\partial\bar{\partial}$ -lemma holds on (M, J_a) for every $a \in \Delta \setminus \{0\}$, then (M, J_0) has an sG metric.*

An interesting problem is if the conclusion in the above proposition holds under weaker conditions than the $\partial\bar{\partial}$ -lemma. In [18, Corollary 4.5] it is proved that the vanishing of some complex invariants, which are closely related to the $\partial\bar{\partial}$ -lemma, is not sufficient to ensure the existence of an sG metric in the central limit.

Another problem related to Proposition 5.1 is if the central limit admits a Hermitian metric, stronger than sG, under the $\partial\bar{\partial}$ -lemma condition. Our aim in this section is to construct a holomorphic family of compact complex manifolds $(M, J_a)_{a \in \Delta}$ such that (M, J_a) satisfies the $\partial\bar{\partial}$ -lemma and admits balanced metric for any $a \neq 0$, but the central limit neither satisfies the $\partial\bar{\partial}$ -lemma nor admits balanced metric. As far as we know, this is the first known complex deformation with this behaviour. The construction is based on the balanced Hermitian geometry of \mathfrak{g}_8 studied in Theorem 4.5, which is the Lie algebra underlying Nakamura manifold.

Concerning the $\partial\bar{\partial}$ -lemma property, it is known that it is an open property (see for instance [4] for a proof of this fact), and recently Angella and Kasuya have proved in [3] that the $\partial\bar{\partial}$ -lemma is not a closed property under holomorphic deformations. The construction in [3] consists in a suitable deformation (M, I_t) of the holomorphically parallelizable Nakamura manifold (M, I_0) (notice that (M, I_0) has balanced metrics). We will use their result on the $\partial\bar{\partial}$ -lemma for (M, I_t) , $t \neq 0$, as a key ingredient in the proof of the following result.

Theorem 5.2. *There exists a solvmanifold M with a holomorphic family of complex structures J_a , $a \in \Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, such that (M, J_a) satisfies the $\partial\bar{\partial}$ -lemma and admits balanced metric for any $a \neq 0$, but the central limit (M, J_0) neither satisfies the $\partial\bar{\partial}$ -lemma nor admits balanced metrics.*

Proof. Let J be the complex structure on the Lie algebra \mathfrak{g}_8 defined by (22) in Proposition 3.7. By Theorem 4.5, any complex solvmanifold (M, J) with underlying Lie algebra \mathfrak{g}_8 and complex structure J does not admit balanced metrics. Next we will deform J to an analytic family J_a defined for any a in the open unit disc Δ in \mathbb{C} centered at 0, so that J is the central limit of the holomorphic deformation, i.e. $J_0 = J$.

For each $a \in \mathbb{C}$ such that $|a| < 1$, we consider the complex structure J_a on M defined by the (1,0)-basis

$$\Phi^1 = \omega^1, \quad \Phi^2 = \omega^2, \quad \Phi^3 = \omega^3 + a\omega^{\bar{3}}.$$

It is easy to check that the complex structure equations are

$$(39) \quad \begin{cases} d\Phi^1 = \frac{2i}{1-|a|^2}\Phi^{13} - \frac{2ia}{1-|a|^2}\Phi^{1\bar{3}} + \frac{1}{1-|a|^2}\Phi^{3\bar{3}}, \\ d\Phi^2 = -\frac{2i}{1-|a|^2}\Phi^{23} + \frac{2ia}{1-|a|^2}\Phi^{2\bar{3}}, \\ d\Phi^3 = 0. \end{cases}$$

Using these equations, the (2,3)-form $\bar{\partial}F^2$ for a generic metric (28) with respect to the basis $\{\Phi^1, \Phi^2, \Phi^3\}$ reads as

$$2\bar{\partial}F^2 = \left[\frac{2ia(ir^2\bar{v} - u\bar{z})}{1-|a|^2} \right] \Phi^{13\bar{1}\bar{2}\bar{3}} + \left[\frac{2ia(is^2\bar{z} + \bar{u}v)}{1-|a|^2} + (r^2s^2 - |u|^2) \right] \Phi^{23\bar{1}\bar{2}\bar{3}}.$$

Suppose that $a \neq 0$ with $|a| < 1$. If $u = v = 0$ then the balanced condition reduces to solve

$$\frac{2a\bar{z}}{1-|a|^2} = r^2, \quad \text{with } r^2 t^2 > |z|^2.$$

Thus, taking $z = (1-|a|^2)r^2/(2\bar{a})$, the condition $r^2 t^2 > |z|^2$ is satisfied for any t such that $t^2 > \frac{(1-|a|^2)^2 r^2}{4|a|^2}$.

Therefore, we have proved that for any J_a , $a \in \Delta - \{0\}$, the structures

$$(40) \quad 2F = i(r^2\Phi^{1\bar{1}} + s^2\Phi^{2\bar{2}} + t^2\Phi^{3\bar{3}}) + \frac{(1-|a|^2)r^2}{2\bar{a}}\Phi^{1\bar{3}} - \frac{(1-|a|^2)r^2}{2a}\Phi^{3\bar{1}},$$

with $r, s \neq 0$ and $t^2 > \frac{(1-|a|^2)^2 r^2}{4|a|^2}$, are balanced.

Notice that the previous argument is valid for the quotient M of any lattice in the simply-connected Lie group G associated to $\mathfrak{g} = \mathfrak{g}_8$. However, to ensure the $\partial\bar{\partial}$ -lemma for the complex structures J_a with $a \neq 0$ we need to consider the lattice Γ considered in [3]. In fact, in [3] the authors consider the holomorphically parallelizable Nakamura manifold $X = (\Gamma \backslash G, I_0)$, whose complex structure I_0 corresponds to the complex structure J_{-i} in our family (24) in Proposition 3.7, and they consider a small deformation I_t given by

$$t \frac{\partial}{\partial z_3} \otimes dz_3 \in H^{0,1}(X; T^{1,0}X),$$

where z_3 is a complex coordinate such that $\omega^3 = dz_3$. By [3, Proposition 4.1] (see also Tables 7 and 8 in [3]) one has that $X_t = (\Gamma \backslash G, I_t)$ satisfies the $\partial\bar{\partial}$ -lemma for any $t \neq 0$. Since $I_0 = J_{-i}$, in terms of complex structure equations (24) for $A = -i$ the deformation I_t is defined by the $(1,0)$ -basis

$$\Upsilon^1 = \omega^1, \quad \Upsilon^2 = \omega^2, \quad \Upsilon^3 = \omega^3 - t\omega^{\bar{3}},$$

and the structure equations for I_t are

$$(41) \quad \begin{cases} d\Upsilon^1 = \frac{2i}{1-|t|^2}\Upsilon^{13} + \frac{2it}{1-|t|^2}\Upsilon^{1\bar{3}}, \\ d\Upsilon^2 = -\frac{2i}{1-|t|^2}\Upsilon^{23} - \frac{2it}{1-|t|^2}\Upsilon^{2\bar{3}}, \\ d\Upsilon^3 = 0. \end{cases}$$

On the other hand, it is easy to see that for any $a \neq 0$ the equations (39) express with respect to the $(1,0)$ -basis $\{\Theta^1 = \Phi^1 + \frac{i}{2a}\Phi^3, \Theta^2 = \Phi^2, \Theta^3 = \Phi^3\}$ as

$$(42) \quad \begin{cases} d\Theta^1 = \frac{2i}{1-|a|^2}\Theta^{13} - \frac{2ia}{1-|a|^2}\Theta^{1\bar{3}}, \\ d\Theta^2 = -\frac{2i}{1-|a|^2}\Theta^{23} + \frac{2ia}{1-|a|^2}\Theta^{2\bar{3}}, \\ d\Theta^3 = 0. \end{cases}$$

Now, from (41) and (42) we conclude that for $a \neq 0$ the complex structure J_a is precisely the complex structure I_t with $t = -a$. Therefore, for any $a \neq 0$ the compact complex manifold $(M, J_a) = (\Gamma \backslash G, J_a)$ satisfies the $\partial\bar{\partial}$ -lemma because $X_{t=-a}$ does by [3, Proposition 4.1].

To finish the proof, it remains to see that the central limit J_0 does not satisfy the $\partial\bar{\partial}$ -lemma. By the symmetrization process, it suffices to prove that it is not satisfied at the Lie algebra level (\mathfrak{g}_8, J_0) . But this is clear from the equations (22), because the form ω^{23} is ∂ -closed, $\bar{\partial}$ -closed and d -exact, however it is not $\partial\bar{\partial}$ -exact. \square

Remark 5.3. For any $a \in \Delta - \{0\}$, the equations (42) imply that the (3,0)-form Θ^{123} is closed, so the complex structure J_a must be equivalent to a complex structure J_A in (24). In fact, taking $\omega^1 = \Theta^1$, $\omega^2 = \Theta^2$ and $\omega^3 = \frac{1}{1-a}\Theta^3$, the complex equations (42) express in terms of the (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ by

$$d\omega^1 = -(A-i)\omega^{13} - (A+i)\omega^{1\bar{3}}, \quad d\omega^2 = (A-i)\omega^{23} + (A+i)\omega^{2\bar{3}}, \quad d\omega^3 = 0,$$

with $A = \frac{i}{|a|^2-1}(1+|a|^2-2a)$. Notice that this correspondence, together with Theorem 4.5, assures the existence of balanced metric for all $a \in \Delta - \{0\}$, however in the proof of Theorem 5.2 we have provided in (40) an explicit family of balanced metrics. Of course, the central limit of any metric (40) does not exist, since J_0 does not admit any balanced metric by Theorem 4.5.

Remark 5.4. We can construct another deformation with central limit the complex structure J' given by (23), and it turns out that the deformation has the same behaviour as for the deformation of the complex structure J given by (22) constructed in Theorem 5.2. Therefore, the complex structures J and J' given by (22) and (23), respectively, are the central limits of complex structures that satisfy the $\partial\bar{\partial}$ -lemma. Notice that this is consistent with Proposition 5.1 because by our Theorem 4.6 both complex structures admit sG metric.

6. APPENDIX

In this appendix we include the classification of (non nilpotent) solvable Lie algebras which are unimodular and have b_3 at least 2. Table 1 contains the decomposable case, whereas Table 2 refers to the indecomposable case.

For Table 1 we use mainly the list of low dimensional Lie algebras of [14]. The $3 \oplus 3$ case is the product of two 3-dimensional unimodular (non nilpotent) solvable Lie algebras (notice that in this case b_3 is always ≥ 2 by (5)). The $4 \oplus 2$ case is the product by \mathbb{R}^2 of a 4-dimensional unimodular (non nilpotent) solvable Lie algebra \mathfrak{h} satisfying $b_1(\mathfrak{h}) + 2b_2(\mathfrak{h}) + b_3(\mathfrak{h}) \geq 2$. Finally, the $5 \oplus 1$ case is the product by \mathbb{R} of a 5-dimensional unimodular (non nilpotent) solvable Lie algebra \mathfrak{h} with $b_2(\mathfrak{h}) + b_3(\mathfrak{h}) \geq 2$.

In Table 2, the first two Lie algebras labeled as $N_{6,18}^{0,-1,-1}$ and $N_{6,20}^{-1,-1}$ comes from the classification in [31], and they are the only unimodular solvable Lie algebras with $b_3 \geq 2$ and nilradical of dimension 4. The other Lie algebras in Table 2 are taken from [15]. In Table 2 we also include the column “ $\lambda(\rho) \geq 0$ ” in which the symbol \checkmark means that any closed 3-form ρ on the Lie algebra satisfies $\lambda(\rho) \geq 0$, in particular, ρ does not give rise to an almost complex structure (a similar study was done in [14] for any decomposable Lie algebra).

Lie algebra \mathfrak{g}	structure equations	closed (3,0)-form Ψ
$\mathfrak{e}(2) \oplus \mathfrak{e}(2)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, e^{45})$	—
$\mathfrak{e}(2) \oplus \mathfrak{e}(1, 1)$	$(0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45})$	$(e^1 - ie^4) \wedge (e^2 - 2i(e^2 - e^6)) \wedge (e^3 + i(\frac{e^3}{2} + e^5))$
$\mathfrak{e}(2) \oplus \mathfrak{h}_3$	$(0, -e^{13}, e^{12}, 0, 0, e^{45})$	—
$\mathfrak{e}(2) \oplus \mathbb{R}^3$	$(0, -e^{13}, e^{12}, 0, 0, 0)$	—
$\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)$	$(0, -e^{13}, -e^{12}, 0, -e^{46}, -e^{45})$	—
$\mathfrak{e}(1, 1) \oplus \mathfrak{h}_3$	$(0, -e^{13}, -e^{12}, 0, 0, e^{45})$	—
$\mathfrak{e}(1, 1) \oplus \mathbb{R}^3$	$(0, -e^{13}, -e^{12}, 0, 0, 0)$	—
$A_{4,2}^{-2} \oplus \mathbb{R}^2$	$(-2e^{14}, e^{24} + e^{34}, e^{34}, 0, 0, 0)$	—
$A_{4,5}^{\alpha, -1-\alpha} \oplus \mathbb{R}^2$ $-1 < \alpha \leq -\frac{1}{2}$	$(e^{14}, \alpha e^{24}, -(1+\alpha)e^{34}, 0, 0, 0)$	—
$A_{4,6}^{\alpha, -\frac{\alpha}{2}} \oplus \mathbb{R}^2$ $\alpha > 0$	$(\alpha e^{14}, -\frac{\alpha}{2}e^{24} + e^{34}, -e^{24} - \frac{\alpha}{2}e^{34}, 0, 0, 0)$	—
$A_{4,8} \oplus \mathbb{R}^2$	$(e^{23}, e^{24}, -e^{34}, 0, 0, 0)$	—
$A_{4,10} \oplus \mathbb{R}^2$	$(e^{23}, e^{34}, -e^{24}, 0, 0, 0)$	—
$A_{5,7}^{-1, -1, 1} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$	$(e^1 - ie^4) \wedge (e^2 - ie^3) \wedge (e^5 - ie^6)$
$A_{5,7}^{-1, \beta, -\beta} \oplus \mathbb{R}$ $0 < \beta < 1$	$(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45}, 0, 0)$	—
$A_{5,8}^{-1} \oplus \mathbb{R}$	$(e^{25}, 0, e^{35}, -e^{45}, 0, 0)$	—
$A_{5,9}^{-1, -1} \oplus \mathbb{R}$	$(e^{15} + e^{25}, e^{25}, -e^{35}, -e^{45}, 0, 0)$	—
$A_{5,13}^{-1, 0, \gamma} \oplus \mathbb{R}$ $\gamma > 0$	$(e^{15}, -e^{25}, \gamma e^{45}, -\gamma e^{35}, 0, 0)$	—
$A_{5,14}^0 \oplus \mathbb{R}$	$(e^{25}, 0, e^{45}, -e^{35}, 0, 0)$	—
$A_{5,15}^{-1} \oplus \mathbb{R}$	$(e^{15} + e^{25}, e^{25}, -e^{35} + e^{45}, -e^{45}, 0, 0)$	—
$A_{5,17}^{0, 0, \gamma} \oplus \mathbb{R}$ $0 < \gamma < 1$	$(e^{25}, -e^{15}, \gamma e^{45}, -\gamma e^{35}, 0, 0)$	—
$A_{5,17}^{\alpha, -\alpha, 1} \oplus \mathbb{R}$ $\alpha \geq 1$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25},$ $- \alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0)$	$(e^1 - ie^2) \wedge (e^4 - ie^3) \wedge (e^6 - ie^5)$
$A_{5,18}^0 \oplus \mathbb{R}$	$(e^{25} + e^{35}, -e^{15} + e^{45}, e^{45}, -e^{35}, 0, 0)$	—
$A_{5,19}^{-1, 2} \oplus \mathbb{R}$	$(-e^{15} + e^{23}, e^{25}, -2e^{35}, 2e^{45}, 0, 0)$	—
$A_{5,19}^{1, -2} \oplus \mathbb{R}$	$(e^{15} + e^{23}, e^{25}, 0, -2e^{45}, 0, 0)$	—
$A_{5,20}^0 \oplus \mathbb{R}$	$(e^{23} + e^{45}, e^{25}, -e^{35}, 0, 0, 0)$	—
$A_{5,26}^{0, \pm 1} \oplus \mathbb{R}$	$(e^{23} \pm e^{45}, -e^{35}, e^{25}, 0, 0, 0)$	—
$A_{5,33}^{-1, -1} \oplus \mathbb{R}$	$(e^{14}, e^{25}, -e^{34} - e^{35}, 0, 0, 0)$	—
$A_{5,35}^{0, -2} \oplus \mathbb{R}$	$(-2e^{14}, e^{24} + e^{35}, -e^{25} + e^{34}, 0, 0, 0)$	—

Table 1. Decomposable unimodular (non-nilpotent) solvable Lie algebras with $b_3 \geq 2$.

Lie algebra \mathfrak{g}	structure equations	closed (3,0)-form Ψ	$\lambda(\rho) \geq 0$
$N_{6,18}^{0,-1,-1}$	$(e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0)$	$(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 - ie^6)$	—
$N_{6,20}^{-1,-1}$	$(-e^{56}, -e^{25} - e^{26}, -e^{36}, e^{45}, 0, 0)$	—	—
$A_{6,13}^{-1,b,-2b+1}$ $b \notin \{-1, 0, \frac{1}{2}, 1, 2\}$	$((b-1)e^{16} + e^{23}, -e^{26}, be^{36}, e^{46}, (1-2b)e^{56}, 0)$	—	—
$A_{6,13}^{a,-2a,2a-1}$ $a \notin \{-1, 0, \frac{1}{3}, \frac{1}{2}\}$	$(-ae^{16} + e^{23}, ae^{26}, -2ae^{36}, e^{46}, (2a-1)e^{56}, 0)$	—	✓
$A_{6,13}^{a,-a,-1}$ $a > 0, a \neq 1$	$(e^{23}, ae^{26}, -ae^{36}, e^{46}, -e^{56}, 0)$	—	✓
$A_{6,13}^{a,b,c}$, $(a, b, c) \in \{(0, -1, 1), (-1, 1, -1), (-1, -1, 3), (-1, 2, -3)\}$	$((a+b)e^{16} + e^{23}, ae^{26}, be^{36}, e^{46}, ce^{56}, 0)$	—	—
$A_{6,14}^{\frac{1}{3},-\frac{2}{3}}$	$(-\frac{1}{3}e^{16} + e^{23} + e^{56}, \frac{1}{3}e^{26}, -\frac{2}{3}e^{36}, e^{46}, -\frac{1}{3}e^{56}, 0)$	—	✓
$A_{6,14}^{-1,\frac{2}{3}}$	$(-\frac{1}{3}e^{16} + e^{23} + e^{56}, -e^{26}, \frac{2}{3}e^{36}, e^{46}, -\frac{1}{3}e^{56}, 0)$	—	—
$A_{6,15}^{-1}$	$(e^{23}, e^{26}, -e^{36}, e^{26} + e^{46}, e^{36} - e^{56}, 0)$	—	—
$A_{6,17}^{0,-\frac{1}{2}}$	$(-\frac{1}{2}e^{16} + e^{23}, -\frac{1}{2}e^{26}, 0, e^{36}, e^{56}, 0)$	—	—
$A_{6,18}^{a,b}$ $(a, b) \in \{(-\frac{1}{2}, -2), (-2, 1)\}$	$((a+1)e^{16} + e^{23}, ae^{26}, e^{36}, e^{36} + e^{46}, be^{56}, 0)$	—	✓
$A_{6,18}^{a,b}$ $(a, b) \in \{(-1, -1), (-3, 3)\}$	$((a+1)e^{16} + e^{23}, ae^{26}, e^{36}, e^{36} + e^{46}, be^{56}, 0)$	—	—
$A_{6,21}^{a,b}$, $(a, b) \in \{(0, -1), (-1, 3), (-\frac{1}{3}, \frac{1}{3})\}$	$(2ae^{16} + e^{23}, ae^{26}, e^{26} + ae^{36}, e^{46}, be^{56}, 0)$	—	—
$A_{6,25}^{a,b}$ $(a, b) \in \{(0, -1), (-\frac{1}{2}, -\frac{1}{2})\}$	$((b+1)e^{16} + e^{23}, e^{26}, be^{36}, ae^{46}, e^{46} + ae^{56}, 0)$	—	✓
$A_{6,25}^{-1,0}$	$(e^{16} + e^{23}, e^{26}, 0, -e^{46}, e^{46} - e^{56}, 0)$	—	—
$A_{6,26}^{-1}$	$(e^{23} + e^{56}, e^{26}, -e^{36}, 0, e^{46}, 0)$	—	—
$A_{6,32}^{0,b,-b}$ $b > 0$	$(e^{23}, -e^{36}, e^{26}, be^{46}, -be^{56}, 0)$	—	✓
$A_{6,34}^{0,0,\epsilon}$ $\epsilon \in \{0, 1\}$	$(e^{23} + \epsilon e^{56}, -e^{36}, e^{26}, 0, e^{46}, 0)$	—	✓
$A_{6,35}^{a,b,c}$, $a > 0$, $(b, c) \in \{(-2a, a), (-a, 0)\}$	$((a+b)e^{16} + e^{23}, ae^{26}, be^{36}, ce^{46} - e^{56}, e^{46} + ce^{56}, 0)$	—	✓
$A_{6,36}^{0,0}$	$(e^{23}, 0, e^{26}, -e^{56}, e^{46}, 0)$	—	—
$A_{6,37}^{0,0,c}$ $c > 0, c \neq 1$	$(e^{23}, -e^{36}, e^{26}, -ce^{56}, ce^{46}, 0)$	—	✓
$A_{6,37}^{0,0,1}$	$(e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0)$	$(e^1 + ie^6) \wedge (e^2 - ie^3) \wedge (e^4 + ie^5)$	—
$A_{6,39}^{a,b}$, $(a, b) \in \{(-1, -1), (-\frac{5}{2}, -\frac{1}{2}), (5, -3), (2, -2)\}$	$((b+1)e^{16} + e^{45}, e^{15} + (b+2)e^{26}, ae^{36}, be^{46}, e^{56}, 0)$	—	—

Table 2. Indecomposable unimodular (non-nilpotent) solvable Lie algebras with $b_3 \geq 2$.

Lie algebra \mathfrak{g}	structure equations	closed $(\mathbf{3},0)$ -form Ψ	$\lambda(\rho) \geq 0$
$A_{6,41}^{-1}$	$(e^{45}, e^{15} + e^{26}, -e^{36} + e^{46}, -e^{46}, e^{56}, 0)$	—	—
$A_{6,54}^{a,b}$ $(a, b) \in \{(0, -1), (-1, -\frac{3}{2}), (2, 0)\}$	$(e^{16} + e^{35}, be^{26} + e^{45}, (1-a)e^{36}, (b-a)e^{46}, ae^{56}, 0)$	—	—
$A_{6,63}^{-1}$	$(e^{16} + e^{35}, -e^{26} + e^{45} + e^{46}, e^{36}, -e^{46}, 0, 0)$	—	—
$A_{6,70}^{0,0}$	$(-e^{26} + e^{35}, e^{16} + e^{45}, -e^{46}, e^{36}, 0, 0)$	—	—
$A_{6,76}^{-1}$	$(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36}, e^{46}, -e^{56}, 0)$	—	—
$A_{6,78}$	$(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36} + e^{46}, e^{46}, -e^{56}, 0)$	—	—
$B_{6,3}^0$	$(e^{45}, e^{15} + e^{36}, e^{14} - e^{26}, -e^{56}, e^{46}, 0)$	—	—
$B_{6,4}^1$	$(e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0)$	$(e^1 - i\frac{e^6}{2}) \wedge (e^2 + ie^3) \wedge (e^4 - ie^5)$	—
$A_{6,82}^{0,1,b}$ $0 \leq b < 1$	$(e^{24} + e^{35}, e^{26}, be^{36}, -e^{46}, -be^{56}, 0)$	—	—
$A_{6,82}^{0,1,1}$	$(e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0)$	$(e^1 - i\frac{e^6}{2}) \wedge (e^2 - ie^3) \wedge (e^4 - ie^5)$	—
$A_{6,83}^{0,1}$	$(e^{24} + e^{35}, e^{26}, e^{26} + e^{36}, -e^{46} - e^{56}, -e^{56}, 0)$	—	—
$A_{6,88}^{0,1,b}$ $b > 0$	$(e^{24} + e^{35}, e^{26} - be^{36}, be^{26} + e^{36}, -e^{46} - be^{56}, be^{46} - e^{56}, 0)$	—	—
$A_{6,88}^{0,0,1}$	$(e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0)$	$(e^1 + i\frac{e^6}{2}) \wedge (e^2 + ie^4) \wedge (e^3 + ie^5)$	—
$A_{6,89}^{0,1,b}$ $b \in \mathbb{R}$	$(e^{24} + e^{35}, be^{26}, -e^{56}, -be^{46}, e^{36}, 0)$	—	—
$A_{6,90}^{0,\pm 1}$	$(e^{24} + e^{35}, e^{46}, \pm e^{56}, 0, \mp e^{36}, 0)$	—	—
$A_{6,93}^{0,1}$	$(e^{24} + e^{35}, -e^{56}, -e^{46} - e^{56}, e^{26} + e^{36}, e^{26}, 0)$	—	—
$B_{6,6}^a$, $a \neq 0$ $-1 < a < 1$	$(e^{24} + e^{35}, e^{46}, ae^{56}, -e^{26}, -ae^{36}, 0)$	—	—
$B_{6,6}^1$	$(e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0)$	$(e^1 - i\frac{e^6}{2}) \wedge (e^2 + ie^4) \wedge (e^3 - ie^5)$	—
$A_{6,94}^{-2}$	$(e^{25} + e^{34}, -e^{26} + e^{35}, -2e^{36}, 2e^{46}, e^{56}, 0)$	—	—

Table 2 (continued). Indecomposable unimodular (non-nilpotent) solvable Lie algebras with $b_3 \geq 2$.

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