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Adela Latorre Larrodé

# Geometry of nilmanifolds with invariant complex structure

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Ugarte Vilumbrales, Luis
Villacampa Gutiérrez, Raquel



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# GEOMETRY OF NILMANIFOLDS WITH INVARIANT COMPLEX STRUCTURE


Autor

Adela Latorre Larrodé

Director/es

Ugarte Vilumbrales, Luis
Villacampa Gutiérrez, Raquel

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# GEOMETRY OF NILMANIFOLDS WITH INVARIANT COMPLEX STRUCTURE

**(Geometría de nilvariedades con estructura compleja invariante)**

Memoria presentada por

**ADELA LATORRE LARRODÉ**

para optar al Grado de

Doctora por la Universidad de Zaragoza

Dirigida por los doctores

Luis Ugarte Vilumbrales, Universidad de Zaragoza (España)

Raquel Villacampa Gutiérrez, Centro Universitario de la Defensa
Zaragoza (España)

*A mis padres*

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# Resumen

En esta tesis centramos nuestra atención en una clase especial de variedades compactas, conocidas como *nilvariedades*, y estudiamos su *geometría compleja* hasta dimensión (real) ocho. Nuestros principales objetivos son construir todas las estructuras complejas *invariantes* sobre dichos espacios, analizar diversos aspectos cohomológicos en dimensión compleja 3 e investigar estructuras geométricas y métricas Hermíticas que son específicas de la dimensión compleja 4.

El estudio de variedades complejas compactas representa hoy en día una de las principales ramas de la Geometría, y tiene su origen en la teoría de superficies de Riemann (dimensión compleja 1) y en la teoría de superficies complejas compactas (dimensión compleja 2). En estas dimensiones se conocen numerosos resultados de clasificación gracias al trabajo de Enriques y Kodaira, entre otros, pero en dimensiones más altas queda mucho por investigar. No obstante, las variedades complejas compactas de dimensión compleja mayor o igual que 3 son de especial relevancia no solo en Geometría sino también en Física Matemática. De hecho, es bien conocido que las variedades Calabi-Yau, así como las variedades complejas compactas dotadas de otras métricas Hermíticas especiales, juegan actualmente un papel central en Geometría Diferencial y Teoría de Cuerdas.

Las variedades complejas son espacios que localmente se parecen a $\mathbb{C}^n$ y cuyos cambios de cartas son biholomorfos. Cada variedad compleja $n$-dimensional es de hecho una variedad real $2n$-dimensional. Sin embargo, decidir cuándo una variedad real $M$ de dimensión par $2n$ admite estructura de variedad compleja no es una tarea fácil. La respuesta a esta cuestión viene dada por el Teorema de Newlander-Nirenberg [NN57]. Este requiere la búsqueda de una estructura casi-compleja $J$ (un endomorfismo en el espacio de campos de vectores diferenciables sobre $M$ que *"imita"* a la unidad imaginaria) que cumple una condición de integrabilidad. De hecho, el teorema caracteriza las variedades complejas $X$ como los pares $(M, J)$. Es importante notar que dos estructuras complejas sobre una misma variedad diferenciable $M$ pueden dar lugar a dos variedades complejas distintas. Por tanto, el problema de encontrar estructuras complejas constituye un paso previo al estudio de las variedades complejas.

Una de las clases de variedades complejas más conocida es la de las variedades Kähler. Estas se pueden ver como variedades diferenciables $M$ dotadas de tres estructuras geométricas *compatibles* entre sí: una estructura compleja, una métrica Riemanniana y una forma simpléctica. En el caso compacto, esta interconexión impone

fuertes obstrucciones topológicas a la variedad. Por ejemplo, sus números de Betti pares no pueden anularse (como consecuencia de ser simpléctica) y sus números de Betti impares son pares (por el Teorema de descomposición de Hodge). En el caso particular de superficies complejas compactas, la condición Kähler es puramente topológica ya que es equivalente a tener primer número de Betti par (véase [Kod64, Miy74, Siu83]). El primer ejemplo de una estructura simpléctica sobre una superficie compleja compacta no-Kähler fue proporcionado por Thurston en 1976 [Thu76]. Esta superficie se conoce como la *variedad de Kodaira-Thurston*.

Los grupos de cohomología constituyen importantes invariantes de las variedades complejas compactas $X$ (véase [Ang14] y las referencias que allí aparecen). Mientras que la cohomología de De Rham $H^k_{\mathrm{dR}}(X)$ es un invariante topológico, las cohomologías de Dolbeault $H^{p,q}_{\bar\partial}(X)$, Bott-Chern $H^{p,q}_{\mathrm{BC}}(X)$ y Aeppli $H^{p,q}_{\mathrm{A}}(X)$ dependen fuertemente de la estructura compleja de la variedad. En general, la cohomología de Aeppli se puede obtener a partir de la de Bott-Chern. Además, en el caso Kähler los grupos de cohomología de Dolbeault y Bott-Chern son isomorfos, y la cohomología de De Rham se puede recuperar haciendo uso del Teorema de descomposición de Hodge: $H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X)$. Por consiguiente, todos estos invariantes son de alguna manera coincidentes sobre las variedades Kähler compactas. En el caso no-Kähler, existen requisitos más débiles que aseguran las mismas relaciones entre los grupos de cohomología anteriores. Por ejemplo la condición del $\partial\bar\partial$-lema, que está relacionada con el concepto de formalidad y fue inicialmente estudiada en [DGMS75]. Recientemente, esta condición ha sido caracterizada en términos de los números de Bott-Chern, Aeppli y Betti en [AT13].

En cuanto a la descomposición de la cohomología de De Rham, un resultado similar al caso Kähler se obtiene cuando la sucesión espectral de Frölicher [Frö55] degenera en primer paso. La generalización de tal descomposición a una variedad casi-compleja $(M, J)$ ha llevado al concepto de *complex-$\mathcal{C}^\infty$-pure-and-fullness*, donde los grupos de cohomología de Dolbeault se reemplazan por ciertos subespacios $H^{p,q}_J(M)$ del grupo de cohomología de De Rham $H^k_{\mathrm{dR}}(M;\mathbb{C})$, donde $k = p + q$. Los análogos reales de los espacios $H^{p,q}_J(M)$ previos son de especial interés en Geometría Simpléctica cuando $p + q = 2$. De hecho, motivados por una pregunta de Donaldson [Don06], Li y Zhang estudian en [LZ09] los espacios $H^+_J(M) = H^{(1,1)}_J(M)_{\mathbb{R}}$ y $H^-_J(M) = H^{(2,0),(0,2)}_J(M)_{\mathbb{R}}$, mostrando que permiten comparar los conos simplécticos *tamed* y compatible. Además, se tiene por [DLZ10] que toda variedad casi-compleja compacta $(M, J)$ de dimensión 4 es $\mathcal{C}^\infty$*-pure-and-full*, es decir, cumple $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$. La descomposición cohomológica en dimensiones más altas ha sido estudiada por diversos autores (véase [AT11, FT10]).

Las restricciones a la existencia de una métrica Kähler que hemos mencionado anteriormente motivan la búsqueda de métricas Hermíticas más débiles que una variedad no-Kähler pueda admitir. En particular, es posible generalizar la condición Kähler haciendo uso de los dos operadores diferenciales $\partial$ y $\bar\partial$ en los que se descompone la diferencial exterior $d$ bajo la presencia de una estructura compleja. En este sentido, se pueden considerar las condiciones $\partial\bar\partial F^k = 0$, donde $1 \leq k \leq n-1$ y $F$ es la 2-forma fundamental de una métrica Hermítica. Cabe observar que si $k = n - 1$, entonces se recupera la definición de *métrica Gauduchon*. Esta es particularmente importante, ya que siempre existe una métrica de tal tipo en la clase conforme de cualquier métrica Hermítica sobre una variedad compleja compacta [Gau84]. Por tanto, nuestra atención se centra en $1 \leq k \leq n - 2$. Más en concreto, estamos principalmente interesados en los dos casos extremos: $k = 1$ y $k = n - 2$.

Una métrica Hermítica cuya 2-forma fundamental $F$ cumple $\partial\bar{\partial}F = 0$ se conoce como *métrica fuertemente Kähler con torsión*, o simplemente *métrica SKT*, de sus siglas en inglés. Equivalentemente, se caracteriza por el hecho de que la torsión de la conexión de Bismut asociada es cerrada. Estas estructuras juegan un papel central en el flujo de Ricci Hermítico introducido por Streets y Tian [ST10]. Las métricas Hermíticas tales que $\partial\bar{\partial}F^{n-2} = 0$ se llaman *astheno-Kähler*, y fueron introducidas por Jost y Yau en [JY93] en relación con ciertos teoremas de rigidez en geometría no-Kähler. Notemos que en dimensión compleja $n = 3$ las métricas astheno-Kähler coinciden con las SKT. Así, se necesita $n \geq 4$ para separar las dos condiciones de una manera efectiva. Muchos autores han investigado propiedades y construcciones relativas a las condiciones astheno-Kähler y SKT (véase [EFV12, FFUV11, FT09, FT11, Swa10], entre otros).

Más recientemente, Fu, Wang y Wu han introducido en [FWW13] una nueva clase de métricas Hermíticas. Se conocen como *métricas k-Gauduchon* y cumplen la condición $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, donde $1 \leq k \leq n - 1$. Cabe notar que se recuperan las métricas Gauduchon para $k = n - 1$. Además, las métricas SKT son tipos especiales de métricas 1-Gauduchon, y las astheno-Kähler son una clase particular de las métricas $(n - 2)$-Gauduchon. Estas estructuras Gauduchon generalizadas también han sido estudiadas en [IP13] y [FU13].

Otra manera de generalizar la condición Kähler es eliminar el requisito de que la métrica asociada a la forma fundamental $F$ sea definida positiva. En este sentido, la atención se centra en las *estructuras pseudo-Kähler*, dadas por la condición $dF = 0$. Estas 2-formas $F$ son de hecho formas simplécticas sobre $M$ compatibles con la estructura compleja $J$ y han sido estudiadas en diferentes artículos, por ejemplo [CFU04] y [Yam05].

Cabe destacar que cada estructura pseudo-Kähler $F$ define una clase de cohomología no degenerada en el subgrupo $H^+_J(M)$ de $H^2_{\mathrm{dR}}(M;\mathbb{R})$. Por consiguiente, se puede pensar en considerar otras estructuras geométricas que definan una clase en $H^-_J(M)$. Una *estructura simpléctica holomorfa* es una $(2,0)$-forma $\Omega$ cerrada y no degenerada sobre $(M, J)$. En este caso, resulta que la parte real $\omega$ de $\Omega$ es una forma simpléctica tal que $[\omega] \in H^-_J(M)$. Observemos que las estructuras simplécticas holomorfas solo viven en variedades complejas de dimensión compleja par. Además, han sido ampliamente estudiadas por Guan en relación a la condición Kähler [Gua94, Gua95a, Gua95b] (véase también [Bog96]).

En esta tesis nos centramos en una case particular de variedades complejas, concretamente, las *nilvariedades dotadas de estructura compleja invariante*. Recordemos que una nilvariedad $\Gamma\backslash G$ es un cociente compacto de un grupo de Lie $G$ conexo, simplemente conexo y nilpotente, por un subgrupo discreto $\Gamma$ de rango máximo. Cabe destacar que la elección de estudiar nilvariedades complejas no es arbitraria. Por un lado, las nilvariedades complejas han proporcionado interesantes ejemplos de variedades que admiten estructuras geométricas con comportamientos poco usuales, siendo la variedad de Kodaira-Thurston un ejemplo. Es bien conocido que las únicas nilvariedades Kähler son los toros complejos (véase [BG88, Has89]). Este hecho hace de las nilvariedades complejas una clase apropiada de variedades complejas compactas donde estudiar geometría no-Kähler. Por otro lado, la investigación de estructuras complejas sobre $\Gamma\backslash G$ puede simplificarse cuando nos restringimos a aquellas que son *invariantes*, esto es, definidas a nivel del álgebra de Lie nilpotente $\mathfrak{g}$ de $G$. Bajo este supuesto y teniendo en cuenta el Teorema de Malcev [Mal49], el problema queda reducido a encontrar los pares $(\mathfrak{g}, J)$, donde $J$ es una estructura compleja sobre un álgebra de Lie nilpotente racional $\mathfrak{g}$. El tener una estructura compleja invariante sobre una nilvariedad permite estudiar la existencia de ciertos tipos de métricas Hermíticas a nivel de $\mathfrak{g}$, gracias al proceso de simetrización, introducido en [Bel00] y desarrollado en [FG04]. Además, también existen resultados de tipo Nomizu para los grupos de cohomología de Dolbeault y Bott-Chern, bajo hipótesis adicionales sobre la estructura compleja (véase [Ang13, CF01, CFGU00, Rol09a, Sak76]); esto es, estas cohomologías sobre la nilvariedad son isomorfas a las cohomologías correspondientes definidas sobre el álgebra de Lie subyacente.

Cuando la dimensión (real) de $\mathfrak{g}$ es igual a 4, es bien conocido que hay tres álgebras de Lie nilpotentes no isomorfas. Solo dos de ellas admiten estructuras complejas. Es más, para cada tal $\mathfrak{g}$ la estructura compleja $J$ es única (salvo equivalencia), por lo que existen exactamente dos pares $(\mathfrak{g}, J)$ que dan lugar al toro complejo $\mathbb{T}^2$ y a la variedad de Kodaira-Thurston. Hasegawa probó en [Has05] que toda estructura compleja sobre una nilvariedad 4-dimensional es invariante. Así, las dos variedades anteriores cubren toda la geometría compleja sobre nilvariedades de dimensión 4.

El problema de encontrar estructuras complejas invariantes sobre nilvariedades de dimensión 6 resulta ser más complicado, debido a varias razones. Entre ellas, destacamos que por [Mag86, Mor58] existen 34 álgebras de Lie nilpotentes de dimensión 6 no isomorfas sobre las que estudiar la existencia de $J$. A pesar de ello, la clasificación de aquellas que admiten estructura compleja fue obtenida por Salamon en [Sal01]. En cuanto a la equivalencia de estructuras complejas sobre estas álgebras de Lie nilpotentes, nos remitimos a [ABD11] y [COUV16], donde se obtienen clasificaciones completas. La riqueza de estructuras contrasta con la obtenida en el caso 4-dimensional. De especial interés es la variedad de Iwasawa, que es el cociente del grupo de Heisenberg complejo por un subgrupo discreto adecuado, y cuyas deformaciones holomorfas fueron estudiadas por Nakamura en [Nak75]. Muchos aspectos geométricos de las variedades complejas han sido analizados sobre 6-nilvariedades: métricas SKT en [FPS04, FT09], métricas 1-Gauduchon en [FU13] o estructuras pseudo-Kähler en [CFU04]. Otras métricas también han sido investigadas, como las fuertemente Gauduchon [COUV16] y las *balanced* [AB90, FG04, UV15], así como varias aplicaciones a la Teoría de Cuerdas [FIUV09, FY15, UV14]. Por este motivo, en este trabajo nos centramos principalmente en los aspectos cohomológicos de estas variedades complejas.

En dimensiones superiores a seis, todavía se sabe poco acerca de qué nilvariedades admiten estructuras complejas invariantes. Un motivo podría ser la ausencia de una lista completa de álgebras de Lie nilpotentes en dimensiones mayores o iguales a ocho sobre la que estudiar su existencia (la dimensión más alta para la cual se ha obtenido tal clasificación es siete [Gon98]). No obstante, hay diferentes resultados parciales que abordan este problema. Se ha probado que no existen estructuras complejas sobre álgebras de Lie filiformes [GR02] ni cuasi-filiformes [VR09]. En [Mil] se acota el paso de nilpotencia de aquellas álgebras que admiten estructuras complejas. También existen otros resultados en dimensión 8 bajo condiciones más fuertes, como la existencia de estructuras hipercomplejas [DF03], métricas SKT [EFV12], o métricas *balanced* con estructuras complejas abelianas [AV]. Algunos resultados parciales han sido recientemente obtenidos en [CSCO15]. Uno de los objetivos de este trabajo es proporcionar una descripción completa de la geometría compleja invariante sobre las nilvariedades 8-dimensionales. De hecho, desarrollamos un procedimiento general que arroja algo de luz sobre el problema en dimensiones mayores. Esto nos permite llevar a cabo un estudio más profundo de aquellas estructuras geométricas que no aparecen en dimensiones bajas, como las astheno-Kähler, las 2-Gauduchon y las simplécticas holomorfas.

A continuación describimos en detalle los contenidos de esta memoria.

En el Capítulo 1 presentamos las diferentes nociones sobre las que hemos hablado a lo largo de esta introducción: variedades complejas, grupos de cohomología y descomposición cohomológica, métricas Hermíticas especiales y otras estructuras geométricas, y nilvariedades dotadas de estructuras complejas invariantes. Además, recordamos el concepto de deformación holomorfa, como una herramienta para investigar la dependencia en la estructura compleja de las propiedades complejas y métricas.

El principal objetivo del Capítulo 2 es el estudio de propiedades cohomológicas de nilvariedades 6-dimensionales $M$ dotadas de estructura compleja invariante $J$. En la primera parte del capítulo, concretamente en la Sección 2.1, nos centramos en la cohomología de Bott-Chern y otros temas relacionados. Gracias a [Rol09a] es bien sabido que se puede calcular la cohomología de Dolbeault de cada par $(M, J)$ a nivel del álgebra de Lie $\mathfrak{g}$ subyacente a $M$ (con la única excepción de $\mathfrak{g} \cong \mathfrak{h}_7$). Aplicando el resultado principal de [Ang13], la cohomología de Bott-Chern también se puede calcular de manera similar. En la Sección 2.1.1 usamos la clasificación de [COUV16] para hallar los números de Bott-Chern de cada $(M, J)$ (Tablas 2.1, 2.2, 2.3). Además, investigamos la relación entre las métricas Hermíticas Gauduchon y fuertemente Gauduchon por medio de los grupos de cohomología de Dolbeault y Aeppli. En la Sección 2.1.2, introducimos ciertos invariantes relacionados con la condición del $\partial\bar{\partial}$-lema definidos en términos de los números de Bott-Chern. Más concretamente, por [AT13] estos invariantes son números enteros no negativos iguales a cero para $\partial\bar{\partial}$-variedades compactas. Su anulación nos permite definir las propiedades $\mathcal{F}_k$ y $\mathcal{K}$, para las que estudiamos el comportamiento por deformación holomorfa. En particular, mostramos que estas propiedades son abiertas. Además, probamos que $\mathcal{F}_2$ y $\mathcal{K}$ no son cerradas (Proposiciones 2.1.6 y 2.1.15). El hecho de que $\mathcal{F}_2$ no sea cerrada sugiere que la condición del $\partial\bar{\partial}$-lema también podría no serlo, resultado que fue finalmente demostrado en [AK]. Terminamos esta sección analizando la relación entre las propiedades $\mathcal{F}_k$ y la existencia de métricas fuertemente Gauduchon en los límites de deformación. Dada una familia analítica de variedades complejas compactas $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$, Popovici demuestra que si $X_{\mathbf{t}}$ cumple la condición del $\partial\bar{\partial}$-lema para $\mathbf{t}\neq\mathbf{0}$, entonces $X_{\mathbf{0}}$ admite una métrica fuertemente Gauduchon. Sin embargo, vemos que un resultado similar no se cumple cuando la fibra $X_{\mathbf{t}}$, con $\mathbf{t}\neq\mathbf{0}$, cumple la condición más débil $\mathcal{F}_2$ (Teorema 2.1.16).

La segunda parte de este capítulo (Sección 2.2) está dedicada al problema de la descomposición cohomológica. En primer lugar, en la Sección 2.2.1, examinamos la propiedad de ser complex-$\mathcal{C}^\infty$-pure-and-full para cada $(M,J)$ (Tablas 2.5, 2.6, 2.7). Esto nos permite encontrar un nuevo ejemplo de variedad no-Kähler en dimensión real 6 que cumple la condición en cada paso, así como recuperar el caso de la variedad de Iwasawa, ya estudiada en [AT11, FT10]. Es interesante notar que las dos poseen la misma álgebra de Lie subyacente (Teorema 2.2.6). Además, vemos que en la clase de nilvariedades $(M,J)$ de dimensión 6 se cumple el siguiente resultado de *dualidad*: $J$ es complex-$\mathcal{C}^\infty$-full en paso $k$ si y solo si es complex-$\mathcal{C}^\infty$-pure en paso $6-k$ (Proposición 2.2.8). En la Sección 2.2.2, nos centramos en la descomposición real en paso dos y estudiamos los espacios $H^+_J(M)$ y $H^-_J(M)$. Aparte de las dos nilvariedades complex-$\mathcal{C}^\infty$-pure-and-full anteriores, encontramos dos casos más para los cuales $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M)\oplus H^-_J(M)$ (Teorema 2.2.9). Gracias a estos ejemplos, demostramos que existen pequeñas deformaciones holomorfas $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ de una variedad compleja compacta $X_{\mathbf{0}}$ que cumple las propiedades $\mathcal{C}^\infty$-pure y $\mathcal{C}^\infty$-full a lo largo de las cuales una de estas propiedades se pierde en $X_{\mathbf{t}}$ para $\mathbf{t}\neq\mathbf{0}$ mientras que la otra se preserva (véanse las Proposiciones 2.2.14 y 2.2.16). Esto contrasta con las pequeñas deformaciones de la variedad de Iwasawa, donde $\mathcal{C}^\infty$-pure y $\mathcal{C}^\infty$-full fallaban al mismo tiempo [AT11]. Usando un resultado de [FOU15], mostramos también que la propiedad de "*ser $\mathcal{C}^\infty$-pure-and-full*" no es cerrada por deformaciones holomorfas (véase Teorema 2.2.17 y Corolario 2.2.18). Finalmente, en la Sección 2.2.3, vemos que tal propiedad no está relacionada con la existencia de ciertas estructuras geométricas especiales sobre variedades complejas compactas. En particular, nos centramos en las métricas SKT, localmente conforme Kähler, *balanced* y fuertemente Gauduchon (Corolario 2.2.20). Además, relacionamos el hecho de ser $\mathcal{C}^\infty$-pure-and-full con los invariantes introducidos en la Sección 2.1.2 relativos a la condición del $\partial\bar{\partial}$-lema, y respondemos también a una pregunta de Drăghici, Li y Zhang [DLZ12] referente al primer número de Betti en el producto de variedades $\mathcal{C}^\infty$-pure-and-full.

Los resultados de este capítulo son una selección de aquellos relacionados con los aspectos cohomológicos de las nilvariedades 6-dimensionales que se pueden encontrar en los artículos [ACL15, LOUV13, LU, LU15, LUV14a, LUV14b].

El Capítulo 3 se dedica al problema de construir estructuras complejas invariantes sobre nilvariedades de dimensión arbitraria $2n$. En la Sección 3.1 proporcionamos una estrategia para encontrar cualquier estructura compleja $J$ sobre cualquier álgebra de Lie nilpotente $2n$-dimensional $\mathfrak{g}$ sin necesidad de conocer de antemano las álgebras involucradas. De hecho, se introducen dos métodos de acuerdo al grado de nilpotencia de la estructura compleja $J$ a construir (Definición 3.1.1). De esta forma, el espacio de estructuras complejas sobre cualquier $\mathfrak{g}$ se divide en dos categorías, dependiendo de la existencia de un subespacio $J$-invariante en el centro $\mathfrak{g}_1$ de $\mathfrak{g}$. Si tal subespacio existe, entonces la estructura compleja $J$ se dice *cuasi-nilpotente*. Demostramos en la

Sección 3.1.1 que este tipo de estructuras se pueden obtener a través de un procedimiento de *extensión* desde cualquier estructura compleja definida sobre un álgebra de Lie nilpotente de dimensión $2(n-1)$ (Corolario 3.1.10). Si el centro del álgebra no contiene ningún subespacio $J$-invariante, entonces $J$ se dice *fuertemente no-nilpotente*. Estas estructuras no se pueden obtener desde otras, así que deben ser explícitamente construidas para cada dimensión compleja. En la Sección 3.1.2 presentamos un método para generar estructuras fuertemente no-nilpotentes. Se basa en una construcción simultánea de los corchetes que definen la estructura del álgebra de Lie y de su serie central ascendente, en términos de una base $J$-adaptada. La combinación de estas dos aproximaciones permite construir estructuras complejas invariantes sobre nilvariedades de cualquier dimensión. Como aplicación, en la Sección 3.2 recuperamos la clasificación de estructuras complejas sobre álgebras de Lie nilpotentes de dimensión cuatro (Proposiciones 3.2.1 y 3.2.2) y seis (Proposiciones 3.2.3 y 3.2.6). Estos resultados motivan el estudio de la dimensión 8 en la Sección 3.3. Más concretamente, extendemos las familias que clasifican las estructuras complejas salvo equivalencia en dimensión 6, usando el método explicado en la Sección 3.1.1. De esta manera, obtenemos una parametrización del espacio de estructuras complejas cuasi-nilpotentes en dimensión 8 (Lemma 3.3.1). Las estructuras complejas no-nilpotentes se reservan para el siguiente capítulo.

La finalidad del Capítulo 4 es investigar estructuras complejas fuertemente no-nilpotentes, prestando especial atención al caso 8-dimensional. La Sección 4.1 contiene varios lemas técnicos que permiten estudiar la serie central ascendente $\{\mathfrak{g}_k\}_k$ de cualquier álgebra de Lie nilpotente $2n$-dimensional $\mathfrak{g}$ admitiendo una estructura compleja fuertemente no-nilpotente $J$. En particular, cuando $n \geq 4$, demostramos que el centro de una tal $\mathfrak{g}$ debe cumplir $1 \leq \dim \mathfrak{g}_1 \leq n-3$ (Teorema 4.1.11). Así, se tiene de manera inmediata que $\dim \mathfrak{g}_1 = 1$, para $n = 4$. Esto hace que las estructuras complejas cuasi-nilpotentes y fuertemente no-nilpotentes no puedan coexistir sobre una nilvariedad 8-dimensional $M$ dada. No obstante, uno puede encontrar estructuras complejas nilpotentes y débilmente no-nilpotentes sobre la misma $M$ (Ejemplo 4.1.13). Cuando la dimensión compleja es $n = 5$, notamos que la cota superior puede alcanzarse (Ejemplo 4.1.14). Esto hace que las estructuras complejas nilpotentes y fuertemente no-nilpotentes puedan coexistir sobre una misma nilvariedad 10-dimensional (Ejemplo 4.1.15). Respecto al segundo término $\mathfrak{g}_2$ de la serie central ascendente, en general se tiene que $2 \leq \dim \mathfrak{g}_2 \leq 2n-3$ (Proposición 4.1.16). En el resto del capítulo nos centramos en la dimensión real ocho, con el propósito de construir cada par fuertemente no-nilpotente $(\mathfrak{g}, J)$. En la Sección 4.2, analizamos el término $\mathfrak{g}_2$ comenzando desde un centro $\mathfrak{g}_1$ de dimensión 1. Más concretamente, vemos que $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$ (Proposición 4.2.1) y, por consiguiente, $3 \leq \dim \mathfrak{g}_2 \leq 5$. De hecho, terminamos esta sección con un resultado de estructura para el término $\mathfrak{g}_2$ (Corolario 4.2.2). La construcción de los restantes términos de la serie central ascendente se completa en la Sección 4.3. Usando el resultado de Vergnolle y Remm [VR09] que dice que las álgebras de Lie cuasi-filiformes no admiten estructuras complejas, nos reducimos a aquellas $\mathfrak{g}$ que son nilpotentes a lo sumo en paso 5. Por ello, basta centrarse en los términos $\mathfrak{g}_3$ y $\mathfrak{g}_4$. Empezando desde el $\mathfrak{g}_2$ dado en el Corolario 4.2.2 y después de un largo estudio caso a caso, obtenemos las dimensiones admisibles de la serie central ascendente de cualquier $\mathfrak{g}$ admitiendo estructuras complejas fuertemente no-nilpotentes $J$ (Teorema 4.3.7). Puesto que nuestro método es constructivo, para cada caso detallado en el Teorema 4.3.7 conseguimos una descripción explícita de sus corchetes de Lie. Usándolos en la Sección 4.4, obtenemos las correspondientes ecuaciones de estructura complejas de cualquier $(\mathfrak{g}, J)$ (ver Teorema 4.4.6). Con este resultado cumplimos el objetivo del capítulo. Ya que las demostraciones de este capítulo son bastante técnicas, se proporciona el esquema de algunas de ellas en el Apéndice A, en aras de una mayor claridad. También hemos incluido el resumen de algunos resultados de estructura en el Apéndice B.

En el Capítulo 5 estudiamos algunas estructuras geométricas que no aparecen en dimensiones 4 y 6. Para ello, trabajamos con la descripción completa de nilvariedades 8-dimensionales dotadas de estructura compleja invariante obtenida en los Capítulos 3 y 4. La Sección 5.1 está dedicada a métricas Hermíticas especiales. En primer lugar, mostramos que la expresión de la 2-forma fundamental de cualquier métrica Hermítica invariante sobre una nilvariedad $2n$-dimensional $(M, J)$ se puede reducir cuando $J$ es cuasi-nilpotente (Lema 5.1.1). Como consecuencia, vemos que el procedimiento de extensión desarrollado en la Sección 3.1.1 es de gran utilidad en la caracterización de ciertos tipos de métricas Hermíticas especiales (Proposición 5.1.2). En la Sección 5.1.1 nos centramos en las métricas astheno-Kähler. Estamos interesados principalmente en las similitudes y diferencias entre estas métricas Hermíticas y las SKT, ya que para $n \geq 4$ dejan de ser coincidentes. En este sentido, merece la pena observar que la existencia de una métrica SKT sobre $(M, J)$ implica que el álgebra de Lie subyacente $\mathfrak{g}$ es nilpotente en paso 2 [EFV12]. Usando este resultado, concluimos que las estructuras complejas fuertemente no-nilpotentes no pueden admitir métricas SKT (Corolario 5.1.5). Además, en dimensión 8, las estructuras complejas fuertemente no-nilpotentes tampoco admiten métricas astheno-Kähler invariantes (Proposición 5.1.6). De hecho, hallamos aquellos pares 8-dimensionales $(M, J)$ donde existen métricas astheno-Kähler invariantes y vemos que $J$ es necesariamente nilpotente (Teorema 5.1.7). En contraste con el caso SKT, proporcionamos nilvariedades con paso de nilpotencia igual a 3 dotadas de estructuras astheno-Kähler (Corolario 5.1.9). En la Sección 5.1.2 nos centramos en las métricas Gauduchon generalizadas, ya que constituyen una clase más amplia a la que pertenecen las métricas SKT y astheno-Kähler. Más concretamente, mostramos que existen nilvariedades con pasos de nilpotencia 4 y 5 que admiten métricas Gauduchon generalizadas y cuyas estructuras complejas son no-nilpotentes (Proposición 5.1.13). Por tanto, las condiciones sobre la estructura compleja para la existencia de dichas métricas resultan ser mucho más débiles que para las anteriores.

En la última parte de esta tesis, concretamente en la Sección 5.2, investigamos las estructuras simplécticas holomorfas. En la Sección 5.2.1, reducimos el problema de su existencia a nivel del álgebra de Lie (Corolario 5.2.6) y probamos que las nilvariedades de dimensión 8 dotadas de estructuras complejas no-nilpotentes no admiten este tipo de geometría (Proposición 5.2.7). Este hecho reduce el espacio en el que analizar la propiedad de existencia de estructuras simplécticas holomorfas bajo deformaciones holomorfas. Se sabe [Gua95b] que tal propiedad no es en general abierta (véase también el Ejemplo 5.2.9), así que nos centramos en sus límites de deformación y demostramos que tampoco es cerrada (Teorema 5.2.10). En la Sección 5.2.2 nos interesamos por la relación entre las estructuras simplécticas holomorfas y las pseudo-Kähler. Primero, usamos la clase de estructuras complejas abelianas sobre nilvariedades para recuperar un resultado de Yamada que establece que la existencia de estas dos estructuras no está relacionada (Proposición 5.2.12). A continuación, mostramos que existen nilvariedades 8-dimensionales con estructuras complejas no-nilpotentes admitiendo estructuras pseudo-Kähler (Teorema 5.2.13). Esto proporciona un contraejemplo a una conjetura establecida en [CFU04] y constituye una diferencia importante con el caso 6-dimensional, para el cual su existencia implica que la estructura compleja es de tipo nilpotente.

# Introduction

In this thesis we focus our attention on a special class of compact manifolds, known as *nilmanifolds*, and study their *complex geometry* up to (real) dimension eight. Our main objectives are to construct all *invariant* complex structures on such spaces, analyze several cohomological aspects in complex dimension 3, and investigate geometric structures and Hermitian metrics which are particular of complex dimension 4. Next we explain in detail the motivation and the main results in the thesis.

The study of compact complex manifolds nowadays represents one of the main branches in Geometry, and it has its origins in the theory of Riemann surfaces (complex dimension 1) and the theory of compact complex surfaces (complex dimension 2). Unlike in these low dimensions, where many classification results are known due to the work of Enriques and Kodaira, among many others, much less is known in higher dimensions. Nonetheless, compact complex manifolds of complex dimension greater than or equal to 3 are of great interest not only in Geometry but also in Mathematical Physics. Indeed, it is well known that Calabi-Yau manifolds, as well as compact complex manifolds with other special Hermitian metrics, currently play a central role in Differential Geometry and String Theory.

Complex manifolds are spaces that locally look like $\mathbb{C}^n$ and whose changes of charts are biholomorphic. Every $n$-dimensional complex manifold is indeed a $2n$-dimensional real manifold. However, deciding when a real manifold $M$ of even dimension $2n$ admits a complex manifold structure is not an easy task. The answer to this question is given by the Newlander-Nirenberg Theorem [NN57]. It requires the search of an almost-complex structure $J$ (an endomorphism of the space of smooth vector fields of $M$ which *"imitates"* the imaginary unit) that is integrable. As a matter of fact, the theorem characterizes complex manifolds $X$ as those pairs $(M, J)$. It is important to remark that two complex structures on the same differentiable manifold $M$ might give rise to two different complex manifolds. Hence, the problem of finding complex structures constitutes the previous stage to the study of complex manifolds.

One of the best-known classes of complex manifolds is that of Kähler manifolds. They can be seen as differentiable manifolds $M$ endowed with three geometric structures which are *compatible* with each other: a complex structure, a Riemannian metric, and a symplectic form. In the compact case, this interconnection imposes strong topological conditions on the manifold. For instance, its even Betti numbers cannot vanish (as a consequence of being symplectic), and its odd Betti numbers are even (by the Hodge

Decomposition Theorem). In the particular case of compact complex surfaces, the Kähler condition is purely topological as it is equivalent to having an even first Betti number (see [Kod64, Miy74, Siu83]). The first example of a symplectic structure on a non-Kähler compact complex surface was provided by Thurston in 1976 [Thu76]. This surface is known as the *Kodaira-Thurston manifold*.

Cohomology groups constitute important invariants of compact complex manifolds $X$ (see [Ang14] and the references therein). While the de Rham cohomology $H^k_{\mathrm{dR}}(X)$ is a topological invariant, the Dolbeault $H^{p,q}_{\bar\partial}(X)$, Bott-Chern $H^{p,q}_{\mathrm{BC}}(X)$, and Aeppli $H^{p,q}_{\mathrm{A}}(X)$ cohomologies strongly depend on the complex structure of the manifold. In general, the Aeppli cohomology can be obtained from the Bott-Chern one. Moreover, in the Kähler case, the Dolbeault and the Bott-Chern cohomology groups are isomorphic, and the de Rham cohomology can be recovered via the Hodge Decomposition Theorem: $H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X)$. Hence, all these invariants are somehow coincident on compact Kähler manifolds. In the non-Kähler case, there are weaker requirements that ensure the same relations among the previous cohomology groups. For instance the $\partial\bar\partial$-lemma condition, which is related to the concept of formality and was initially studied in [DGMS75]. Recently, this condition has been characterized in terms of the Bott-Chern, Aeppli, and Betti numbers in [AT13].

Concerning the decomposition of the de Rham cohomology, one obtains a similar result to the Kähler case when the Frölicher spectral sequence [Frö55] degenerates at the first step. The generalization of such decomposition to an almost-complex manifold $(M, J)$ has led to the concept of *complex-$\mathcal{C}^\infty$-pure-and-fullness*, where the Dolbeault cohomology groups are replaced by certain subspaces $H^{p,q}_J(M)$ of the de Rham cohomology group $H^k_{\mathrm{dR}}(M;\mathbb{C})$, where $k = p + q$. The real counterparts of the previous $H^{p,q}_J(M)$ are of special interest in Symplectic Geometry when $p + q = 2$. Indeed, motivated by a Donaldson's question [Don06], Li and Zhang study in [LZ09] the spaces $H^+_J(M) = H^{(1,1)}_J(M)_{\mathbb{R}}$ and $H^-_J(M) = H^{(2,0),(0,2)}_J(M)_{\mathbb{R}}$, showing that they allow to compare the tamed and the compatible symplectic cones. In addition, one has by [DLZ10] that every compact 4-dimensional almost-complex manifold $(M, J)$ is $\mathcal{C}^\infty$*-pure-and-full*, i.e., it satisfies $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$. Cohomological decomposition has been studied in higher dimensions by different authors (see [AT11, FT10]).

The constraints to the existence of a Kähler metric that we mentioned above motivate the search for weaker Hermitian metrics that a non-Kähler manifold could still admit. In particular, it is possible to generalize the Kähler condition by making use of the two differential operators $\partial$ and $\bar\partial$ in which the exterior differential $d$ decomposes under the presence of a complex structure. In this sense, one can consider the conditions $\partial\bar\partial F^k = 0$, where $1 \leq k \leq n-1$ and $F$ is the fundamental 2-form of a Hermitian metric. Observe that if $k = n - 1$, then one recovers the definition of a *Gauduchon metric*. It is particularly important, as a metric of such type always exists in the conformal class of any Hermitian metric on any compact complex manifold [Gau84]. Therefore, our attention is focused on $1 \leq k \leq n - 2$. More concretely, we are mainly interested in the two extremal cases: $k = 1$ and $k = n - 2$.

A Hermitian metric whose fundamental 2-form $F$ satisfies $\partial\bar\partial F = 0$ is known as a *strong Kähler with torsion metric*, or simply *SKT metric*. Equivalently, it is characterized by the fact that the torsion of the associated Bismut connection is closed. These structures play a central role in the Hermitian-Ricci flow introduced by Streets and Tian [ST10]. The Hermitian metrics such that $\partial\bar{\partial}F^{n-2} = 0$ are called *astheno-Kähler*, and they were introduced by Jost and Yau in [JY93] in relation to some rigidity theorems in non-Kähler geometry. Notice that in complex dimension $n = 3$ astheno-Kähler metrics coincide with SKT ones. Thus, one needs $n \geq 4$ to effectively separate the two conditions. Many authors have investigated properties and constructions related to the astheno-Kähler and SKT conditions (see [EFV12, FFUV11, FT09, FT11, Swa10], among others).

More recently, Fu, Wang, and Wu have introduced a new class of Hermitian metrics in [FWW13]. They are known as *k-th Gauduchon metrics* and fulfill the condition $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, where $1 \leq k \leq n-1$. Note that one recovers Gauduchon metrics for $k = n-1$. Moreover, SKT metrics are special types of 1-st Gauduchon metrics, and astheno-Kähler metrics are a particular class of $(n-2)$-th Gauduchon metrics. These generalized Gauduchon structures have also been studied in [IP13] and [FU13].

Another different way of generalizing the Kähler condition is dropping the positive definiteness of the metric associated to the fundamental form $F$. In this sense, the attention is paid to *pseudo-Kähler structures*, which are defined by the condition $dF = 0$. These $F$'s are indeed symplectic forms on $M$ compatible with the complex structure $J$ and have been studied in some papers, for instance [CFU04] and [Yam05].

Let us remark that each pseudo-Kähler structure $F$ defines a non-degenerate cohomology class in the subgroup $H^+_J(M)$ of $H^2_{\mathrm{dR}}(M;\mathbb{R})$. Hence, one can think about considering other geometric structures defining a class in $H^-_J(M)$. A *holomorphic symplectic structure* is a closed non-degenerate $(2,0)$-form $\Omega$ on $(M, J)$. In this case, it turns out that the real part $\omega$ of $\Omega$ is a symplectic form such that $[\omega] \in H^-_J(M)$. We observe that holomorphic symplectic structures only live on complex manifolds of even complex dimension. Furthermore, they have been widely studied by Guan in relation to the Kähler condition [Gua94, Gua95a, Gua95b] (see also [Bog96]).

In this thesis, we focus on a particular class of complex manifolds, namely, *nilmanifolds endowed with invariant complex structures*. We recall that a nilmanifold $\Gamma\backslash G$ is a compact quotient of a connected, simply connected, nilpotent Lie group $G$ by a discrete subgroup $\Gamma$ of maximal rank. We note that the choice of studying complex nilmanifolds is not arbitrary. On the one hand, complex nilmanifolds have provided interesting examples of manifolds admitting geometric structures with unusual behaviours, the Kodaira-Thurston manifold being an example. It is well known that the only Kähler nilmanifolds are the complex tori (see [BG88, Has89]). This fact makes complex nilmanifolds an appropriate class of compact complex manifolds where studying non-Kähler geometry. On the other hand, the investigation of complex structures on $\Gamma\backslash G$ can be simplified when we restrict ourselves to *invariant* ones, that is, those defined at the level of the nilpotent Lie algebra (shortly, NLA) $\mathfrak{g}$ of $G$. Under this assumption and taking into account Malcev's Theorem [Mal49], the problem is reduced to finding the pairs $(\mathfrak{g}, J)$, where $J$ is a complex structure on a rational nilpotent Lie algebra $\mathfrak{g}$. Having an invariant complex structure on a nilmanifold allows to study the existence of certain types of Hermitian metrics at the level of $\mathfrak{g}$, thanks to the symmetrization process, introduced in [Bel00] and developed in [FG04]. Moreover, there also exist Nomizu-type results concerning Dolbeault and Bott-Chern cohomology groups, under additional hypothesis on the complex structure (see [Ang13, CF01, CFGU00, Rol09a, Sak76]); i.e., these cohomologies on the nilmanifold are isomorphic to the corresponding ones defined on the underlying Lie algebra.

When the (real) dimension of $\mathfrak{g}$ equals 4, it is well known that there are three non-isomorphic nilpotent Lie algebras. Only two of them admit complex structures. In fact, for each such $\mathfrak{g}$ the complex structure $J$ is unique (up to equivalence), so there exist exactly two pairs $(\mathfrak{g}, J)$ that give rise to the complex torus $\mathbb{T}^2$ and the Kodaira-Thurston manifold. Hasegawa proved in [Has05] that every complex structure on a 4-dimensional nilmanifold is invariant. Therefore, the previous two manifolds cover all the complex geometry on nilmanifolds of dimension 4.

The problem of finding invariant complex structures on nilmanifolds of dimension 6 turns out to be more complicated, due to several reasons. Among them, we underline that by [Mag86, Mor58] there are 34 non-isomorphic 6-dimensional NLAs on which studying the existence of $J$. In spite of it, the classification of NLAs that admit complex structures was achieved by Salamon in [Sal01]. Concerning the equivalence of complex structures on those NLAs, we refer to [ABD11] and [COUV16], where complete classifications are obtained. The richness of structures contrasts with that obtained in the 4-dimensional case. Of special interest is the Iwasawa manifold, which is the quotient of the complex Heisenberg group and an appropriate discrete subgroup, and whose holomorphic deformations were studied by Nakamura in [Nak75]. Many geometric aspects of complex manifolds have been analyzed on 6-nilmanifolds: SKT metrics in [FPS04, FT09], 1-st Gauduchon metrics in [FU13], or pseudo-Kähler structures in [CFU04]. Also other types of Hermitian metrics have been investigated, such as strongly Gauduchon [COUV16] and balanced [AB90, FG04, UV15], as well as several applications to String Theory [FIUV09, FY15, UV14]. For this reason, in this work we mainly focus on the cohomological aspects of these complex nilmanifolds.

In dimensions higher than six, little is yet known about which nilmanifolds admit invariant complex structures. A reason could be the lack of a full list of NLAs in dimensions greater than or equal to eight on which studying their existence (the highest dimension in which such a classification has been obtained is seven [Gon98]). Nonetheless, there are different partial results concerning this problem. It has been proved that complex structures do not exist on filiform Lie algebras [GR02] or quasi-filiform Lie algebras [VR09]. In [Mil] the nilpotency step of those algebras admitting complex structures has been bounded. There are also other results in dimension 8 under stronger conditions, such as the existence of hypercomplex structures [DF03], SKT metrics [EFV12], or balanced metrics with abelian complex structures [AV]. Some partial results have been recently obtained in [CSCO15]. One of our aims in this work is to provide a complete description of the invariant complex geometry on 8-dimensional nilmanifolds. Indeed, we develop a general procedure that casts some light into the higher dimensional problem.

This allows us to perform a deeper study of those geometrical structures that do not appear in low dimensions, such as astheno-Kähler, 2-nd Gauduchon, and holomorphic symplectic.

We next describe in detail the contents of this thesis.

In Chapter 1, we present the different notions about which we have discussed along this introduction: complex manifolds, cohomology groups and cohomological decomposition, special Hermitian metrics and other geometric structures, and nilmanifolds endowed with invariant complex structures. In addition, we recall the concept of holomorphic deformations, as a tool to investigate the dependence of complex and metric properties on the complex structure.

The main goal of Chapter 2 is studying cohomological properties of 6-dimensional nilmanifolds $M$ with invariant complex structure $J$. In the first part of the chapter, namely Section 2.1, we focus on the Bott-Chern cohomology and other related topics. Thanks to [Rol09a] it is well known that one can compute the Dolbeault cohomology of each pair $(M, J)$ at the level of the Lie algebra $\mathfrak{g}$ underlying $M$ (with the only exception of $\mathfrak{g} \cong \mathfrak{h}_7$). Applying the main result in [Ang13], also the Bott-Chern cohomology can be similarly computed. In Section 2.1.1 we use the classification in [COUV16] to provide the Bott-Chern numbers of each $(M, J)$ (Tables 2.1, 2.2, 2.3). Furthermore, we investigate the relation between Gauduchon and strongly-Gauduchon Hermitian metrics by means of the Dolbeault and Aeppli cohomology groups. In Section 2.1.2, we introduce some invariants related to the $\partial\bar{\partial}$-lemma condition and defined in terms of the Bott-Chern numbers. More precisely, by [AT13] these invariants are non-negative integer numbers that are equal to zero for compact $\partial\bar{\partial}$-manifolds. Their vanishing allows us to define the properties $\mathcal{F}_k$ and $\mathcal{K}$ for which we study the behaviour under holomorphic deformations. In particular, we show that these properties are open. Moreover, we prove that $\mathcal{F}_2$ and $\mathcal{K}$ are not closed (Propositions 2.1.6 and 2.1.15). The non-closedness of $\mathcal{F}_2$ suggests that the $\partial\bar{\partial}$-lemma condition might be non-closed, a fact that was finally proven in [AK]. We finish this section analyzing the possible relationship between the properties $\mathcal{F}_k$ and the existence of strongly-Gauduchon metrics in the deformation limits. Given an analytic family of compact complex manifolds $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$, Popovici proves that if $X_{\mathbf{t}}$ satisfies the $\partial\bar{\partial}$-lemma condition for $\mathbf{t} \neq \mathbf{0}$, then $X_{\mathbf{0}}$ admits a strongly-Gauduchon metric. However, we see that a similar result does not hold when the fiber $X_{\mathbf{t}}$, where $\mathbf{t} \neq \mathbf{0}$, satisfies the weaker condition $\mathcal{F}_2$ (Theorem 2.1.16).

The second half of this chapter (Section 2.2) is devoted to the problem of cohomological decomposition. First, in Section 2.2.1, we examine complex-$\mathcal{C}^\infty$-pure-and-fullness for each $(M, J)$ (Tables 2.5, 2.6, 2.7). This allows us to find a new example of a non-Kähler manifold in real dimension 6 satisfying the condition at every stage, as well as recovering the case of the Iwasawa manifold, already studied in [AT11, FT10]. It is interesting to note that both of them have the same underlying Lie algebra (Theorem 2.2.6). Moreover, we see that in the class of 6-dimensional nilmanifolds $(M, J)$ there exists the following *duality* result: $J$ is complex-$\mathcal{C}^\infty$-full at the $k$-stage if and only if it is complex-$\mathcal{C}^\infty$-pure at the $(6-k)$-stage (Proposition 2.2.8). In Section 2.2.2, we focus on the real decomposition at the second stage and study the spaces $H^+_J(M)$ and $H^-_J(M)$. Apart from the two complex-$\mathcal{C}^\infty$-pure-and-full nilmanifolds above, we find two more cases in which $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$ (Theorem 2.2.9). Thanks to these examples, we prove that there exist small holomorphic deformations $\{X_t\}_{t\in\Delta}$ of a compact complex manifold $X_0$ satisfying the $\mathcal{C}^\infty$-pure and $\mathcal{C}^\infty$-full properties, such that one of these properties is lost on $X_t$ for $t \neq 0$ while the other one is preserved (see Propositions 2.2.14 and 2.2.16). This contrasts with the small deformations of the Iwasawa manifold, where both pureness and fullness fail at the same time [AT11]. By using a result in [FOU15], we also show that the property of "*being* $\mathcal{C}^\infty$*-pure-and-full*" is not closed under holomorphic deformations (see Theorem 2.2.17 and Corollary 2.2.18). Finally, in Section 2.2.3, we see that this property is unrelated to the existence of some special geometric structures on compact complex manifolds. In particular, we focus on SKT, locally conformal Kähler, balanced, and strongly-Gauduchon metrics (Corollary 2.2.20). Moreover, we relate the $\mathcal{C}^\infty$-pure-and-full property with the invariants introduced in Section 2.1.2 concerning the $\partial\bar{\partial}$-condition, and we also answer to a question by Drăghici, Li, and Zhang [DLZ12] concerning the first Betti number in a product of $\mathcal{C}^\infty$-pure-and-full manifolds.

The results of this chapter are a selection of those related to cohomological aspects of 6-dimensional nilmanifolds that can be found in the papers [ACL15, LOUV13, LU, LU15, LUV14a, LUV14b].

Chapter 3 is devoted to the problem of constructing invariant complex structures on nilmanifolds of arbitrary dimension $2n$. In Section 3.1 we provide an strategy to find any complex structure $J$ on any $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ without the need of knowing the involved algebras in advance. Indeed, two methods are introduced according to the degree of nilpotency of the complex structure $J$ to be constructed (Definition 3.1.1). In this way, the space of complex structures defined on any $\mathfrak{g}$ splits into two categories, depending on the existence of a $J$-invariant subspace inside the center $\mathfrak{g}_1$ of $\mathfrak{g}$. If such a subspace exists, then the complex structure $J$ is called *quasi-nilpotent*. We prove in Section 3.1.1 that this type of structures can be obtained through an *extension* procedure starting from any complex structure defined on a nilpotent Lie algebra of dimension $2(n-1)$ (Corollary 3.1.10). If the center of the algebra does not contain any $J$-invariant subspace, then $J$ is called *strongly non-nilpotent* (shortly, SnN). These structures cannot be derived from others, so they should be explicitly constructed for each complex dimension. In Section 3.1.2 we present a method in order to produce strongly non-nilpotent complex structures. It is based on a simultaneous construction of the brackets defining the Lie algebra structure and of its ascending central series, in terms of a $J$-adapted basis. The combination of these two approaches allows to construct invariant complex structures on nilmanifolds of any even dimension. As an application, in Section 3.2 we recover the classification of complex structures on nilpotent Lie algebras of dimensions four (Propositions 3.2.1 and 3.2.2) and six (Propositions 3.2.3 and 3.2.6). These results motivate us to study dimension 8 in Section 3.3. More concretely, we extend the families that classify complex structures up to equivalence in dimension 6, using the method explained in Section 3.1.1. In this way, we obtain a parametrization of the space of quasi-nilpotent complex structures in dimension 8 (Lemma 3.3.1). Strongly non-nilpotent complex structures are left to the following chapter.

The aim of Chapter 4 is to investigate SnN complex structures, paying special attention to the 8-dimensional case. Section 4.1 contains some technical lemmas that allow to study the ascending central series $\{\mathfrak{g}_k\}_k$ of any $2n$-dimensional NLA $\mathfrak{g}$ admitting an SnN complex structure $J$. In particular, when $n \geq 4$, we prove that the center of such $\mathfrak{g}$ must satisfy $1 \leq \dim \mathfrak{g}_1 \leq n-3$ (Theorem 4.1.11). Hence, one immediately has $\dim \mathfrak{g}_1 = 1$, for $n = 4$. This makes that quasi-nilpotent and SnN complex structures cannot coexist on a given 8-dimensional nilmanifold $M$. Nonetheless, one can find nilpotent and weakly non-nilpotent complex structures on the same $M$ (Example 4.1.13). When the complex dimension is $n = 5$, we notice that the upper bound can be attained (Example 4.1.14). This fact makes that nilpotent and SnN complex structures can coexist on a 10-dimensional nilmanifold (Example 4.1.15). Concerning the second term $\mathfrak{g}_2$ of the ascending central series, one in general has that $2 \leq \dim \mathfrak{g}_2 \leq 2n-3$ (Proposition 4.1.16). For the rest of the chapter we focus on real dimension eight, with the purpose of constructing every SnN pair $(\mathfrak{g}, J)$. In Section 4.2, we analyze the term $\mathfrak{g}_2$ starting from a 1-dimensional center $\mathfrak{g}_1$. More concretely, we see that $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$ (Proposition 4.2.1) and, consequently, $3 \leq \dim \mathfrak{g}_2 \leq 5$. Indeed, we finish this section with a structure result for the term $\mathfrak{g}_2$ (Corollary 4.2.2). The construction of the remaining terms in the ascending central series is completed in Section 4.3. Using the result by Vergnolle and Remm [VR09] asserting that quasi-filiform Lie algebras do not admit complex structures, we are reduced to $\mathfrak{g}$'s that are at most 5-step nilpotent. Therefore, it suffices to focus on the terms $\mathfrak{g}_3$ and $\mathfrak{g}_4$. Starting from $\mathfrak{g}_2$ given in Corollary 4.2.2 and after a long case-by-case study, we get the admissible dimensions of the ascending central series of any $\mathfrak{g}$ admitting SnN complex structures $J$ (Theorem 4.3.7). Since our method is constructive, for each case detailed in Theorem 4.3.7 we get the explicit description of the Lie brackets. Using them, we obtain in Section 4.4 the corresponding complex structure equations for any $(\mathfrak{g}, J)$ (see Theorem 4.4.6). With this result we fulfill the objective of the chapter. Since the proofs of this chapter are quite technical, the sketch of some of them are provided in Appendix A, for the seek of clarity. We have also included the summary of some structural results in Appendix B.

In Chapter 5 we study some geometric structures that do not appear in dimensions 4 and 6. In order to do this, we work with the complete description of 8-dimensional nilmanifolds endowed with invariant complex structures attained in Chapters 3 and 4. Section 5.1 is devoted to special Hermitian metrics. First, we show that the expression of the fundamental 2-form of any invariant Hermitian metric on a $2n$-dimensional nilmanifold $(M, J)$ can be reduced when $J$ is quasi-nilpotent (Lemma 5.1.1). As a consequence, we see that the extension procedure developed in Section 3.1.1 can help in the characterization of certain types of special Hermitian metrics (Proposition 5.1.2). In Section 5.1.1 we focus on astheno-Kähler metrics. We are mainly interested in the similarities and differences between these Hermitian metrics and SKT ones, as they are no longer coincident for $n \geq 4$. In this sense, it is worth observing that the existence of an SKT metric on $(M, J)$ implies that the underlying Lie algebra $\mathfrak{g}$ is 2-step [EFV12]. Using this result, we conclude that SnN complex structures cannot admit SKT metrics (Corollary 5.1.5). Moreover, in dimension 8, strongly non-nilpotent complex structures neither admit invariant astheno-Kähler metrics (Proposition 5.1.6). Indeed, we find those 8-dimensional $(M, J)$ where invariant astheno-Kähler metrics exist and see that $J$ is necessarily nilpotent (Theorem 5.1.7). In contrast with the SKT case, we provide 3-step nilmanifolds with astheno-Kähler structures (Corollary 5.1.9). In Section 5.1.2 we focus on generalized Gauduchon metrics, as they constitute a larger class to which SKT and astheno-Kähler metrics belong. More concretely, we show that there are 4-step and 5-step nilmanifolds admitting generalized Gauduchon metrics and whose complex structures are non-nilpotent (Proposition 5.1.13). Hence, the conditions on the complex structure for the existence of such metrics turn out to be much weaker than for the previous ones.

In the last part of this thesis, namely, Section 5.2, we investigate holomorphic symplectic structures. In Section 5.2.1, we reduce their existence problem to the Lie algebra level (Corollary 5.2.6) and prove that 8-dimensional nilmanifolds with non-nilpotent complex structures cannot admit this sort of geometry (Proposition 5.2.7). This fact reduces the space in which analyzing the property of existence of holomorphic symplectic structures under holomorphic deformations. It is known [Gua95b] that such property is in general not open (see also Example 5.2.9), so we focus on its deformation limits and prove that it is also non-closed (Theorem 5.2.10). In Section 5.2.2 we are mainly concerned about the relation between holomorphic symplectic structures and pseudo-Kähler ones. First, we use the class of abelian complex structures on nilmanifolds in order to recover the result by Yamada [Yam05] asserting that the existence of these two structures is unrelated (Proposition 5.2.12). Then, we show that there are 8-dimensional nilmanifolds with non-nilpotent complex structures admitting pseudo-Kähler structures (Theorem 5.2.13). This provides a counterexample to a conjecture in [CFU04] and entails an important difference with the 6-dimensional case, for which their existence implies that the complex structure is of nilpotent type.

## Chapter 1

# Preliminaries on complex manifolds

Complex manifolds constitute the core of study in Complex Geometry. They are spaces that locally look like $\mathbb{C}^n$ and whose changes of charts are biholomorphic. Equivalently, they can be seen as even-dimensional differentiable manifolds endowed with a complex structure. Roughly speaking, this complex structure would be telling us how to organize the smooth charts in order to obtain holomorphic ones. Hence, it seems clear that two complex structures on the same differentiable manifold $M$ might give rise to two different complex manifolds. In this introductory chapter, we recall these ideas and present the basic concepts that will be used along this work. More precisely, Section 1.1 focuses on the general theory of complex manifolds, showing how almost-complex structures come into play. Holomorphic deformations are also considered, as a tool to study the dependence of certain properties on the complex structure. Once the basic notions are fixed, we concentrate on cohomological and geometric aspects of compact complex manifolds.

In Section 1.2 we examine some invariants defined as certain cohomology groups. More concretely, we review the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomologies. The search for relations among them leads to the Frölicher spectral sequence and the $\partial\bar{\partial}$-lemma condition. If the $\partial\bar{\partial}$-property holds, then the Frölicher spectral sequence degenerates at the first step and each de Rham cohomology group can be decomposed using the Dolbeault cohomology. In the last part of this section, we consider the cohomological decomposition of (almost) complex manifolds.

In Section 1.3, we present several geometric structures that are of great interest in Complex Geometry. As a starting point, we concentrate on special Hermitian metrics. Since we are mainly interested in non-Kähler geometry, we show different ways of weakening the Kähler condition that lead to new classes of metrics: balanced, strongly Gauduchon, strong Kähler with torsion (SKT), generalized Gauduchon,... We then drop the positive definiteness of the Hermitian metrics and introduce pseudo-Kähler structures, which in turn motivate the interest in holomorphic symplectic geometry.

Let us observe that the study of the previous concepts is not straightforward. One first needs to find complex structures on differentiable manifolds and then analyze the cohomological and metric aspects of the resulting complex manifolds. Nevertheless, the problem can be slightly simplified when the class of nilmanifolds is considered. These spaces have valuable features coming from their algebraic background, in such a way that many results on their associated Lie algebras have a counterpart on the nilmanifold. We review some of these aspects in Section 1.4. In particular, we present the classifications of *invariant* complex structures on nilmanifolds of dimensions four and six.

## 1.1 Almost-complex structures and integrability

This section contains some basic notions that will appear along this work, such as the concept of complex manifold, the integrability of almost-complex structures, and the bigraduation of forms. Although they are well known, they will serve to fix the notation and give a short overview of the topic. We will also introduce the notion of holomorphic deformations.

We start with a fundamental definition in Complex Geometry: that of a complex manifold.

**Definition 1.1.1.** *Let $X$ be a topological space which is Hausdorff and second countable. $X$ is said to be a* complex manifold *of complex dimension $n$ if:*

*i) for every $p \in X$, there exists an open set $p \in U \subset X$ and a homeomorphism $\varphi : U \to \mathbb{C}^n$ such that $\varphi(U)$ is open in $\mathbb{C}^n$; and*

*ii) given two pairs $(U, \varphi)$ and $(V, \psi)$ in the conditions above such that $U \cap V \neq \emptyset$, then the maps $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ and $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ are holomorphic.*

*Each pair $(U, \varphi)$ is known as a* local complex chart*. A set of complex charts $\{(U_i, \varphi_i)\}_{i \in I}$ satisfying $X = \bigcup_{i \in I} U_i$ is called a* holomorphic atlas *of $X$.*

Given a complex $n$-dimensional manifold $X$, let $\pi_j$ denote the projection of $\mathbb{C}^n$ on its $j$-th component, where $1 \leq j \leq n$. If $(U, \varphi)$ is a local complex chart, we can define the maps $z_j = \pi_j \circ \varphi$, for every $1 \leq j \leq n$, and write $\varphi = (z_1, \ldots, z_n)$. This is known as a *local coordinate complex chart* on $X$. Then, it is possible to consider $\tilde{\varphi} = (\mathfrak{Re}\, z_1, \mathfrak{Im}\, z_1, \ldots, \mathfrak{Re}\, z_n, \mathfrak{Im}\, z_n)$, which will give us a local coordinate (real) chart for $X$. Therefore, every complex manifold $X$ of complex dimension $n$ is also a differentiable manifold of real dimension $2n$.

The converse is in general not true. There are even-dimensional differentiable manifolds not admitting a complex manifold structure. For instance, every sphere $\mathbb{S}^{2n}$ with $n \neq 1, 3$. In contrast, $\mathbb{S}^2$ is complex, and $\mathbb{S}^6$ remains unknown. Hence, the problem of determining which differentiable manifolds of real dimension $2n$ can be seen as complex manifolds arises in a natural way.

Let $M$ be a $2n$-dimensional differentiable manifold. Denote $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on $M$ with the usual Lie bracket $[\cdot, \cdot]$. An *almost-complex structure* on $M$ is an endomorphism $J \in \mathrm{End}\,(\mathfrak{X}(M))$ satisfying $J^2 = -id$. If such $J$ exists, then the pair $(M, J)$ is said to be an *almost-complex manifold.*

One can extend $J$ by $\mathbb{C}$-linearity to the complexified Lie algebra $\mathfrak{X}_{\mathbb{C}}(M) = \mathfrak{X}(M)\otimes\mathbb{C}$, and obtain a decomposition

$$\mathfrak{X}_{\mathbb{C}}(M) = \mathfrak{X}_J^{1,0}(M) \oplus \mathfrak{X}_J^{0,1}(M),$$

where $\mathfrak{X}_J^{1,0}(M) = \{Z \in \mathfrak{X}_{\mathbb{C}}(M) \mid JZ = iZ\}$ and $\mathfrak{X}_J^{0,1}(M) = \{Z \in \mathfrak{X}_{\mathbb{C}}(M) \mid JZ = -iZ\}$. If there is no confusion, we will also denote them by $\mathfrak{X}^{1,0}(M)$ and $\mathfrak{X}^{0,1}(M)$, respectively. Observe that they are conjugate to each other.

Furthermore, $J$ can also be defined on the space of smooth 1-forms $\Omega^1(M)$ simply taking

$$(J\alpha)(V) = -\alpha(JV), \tag{1.1}$$

for every $\alpha \in \Omega^1(M)$ and $V \in \mathfrak{X}(M)$. Considering its $\mathbb{C}$-linear extension to $\Omega^1_{\mathbb{C}}(M) = \Omega^1(M) \otimes \mathbb{C}$, we get a decomposition

$$\Omega^1_{\mathbb{C}}(M) = \Omega_J^{1,0}(M) \oplus \Omega_J^{0,1}(M),$$

where $\Omega_J^{1,0}(M) = \{\omega \in \Omega^1_{\mathbb{C}}(M) \mid J\omega = i\omega\}$ and $\Omega_J^{0,1}(M) = \{\omega \in \Omega^1_{\mathbb{C}}(M) \mid J\omega = -i\omega\}$. Once again, these spaces are related by conjugation, i.e., $\overline{\Omega_J^{0,1}(M)} = \Omega_J^{1,0}(M)$. Similarly, $J$ can be extended to the complexification of $k$-forms, obtaining a bigraduation

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega_J^{p,q}(M).$$

For the seek of simplicity, we will write $\Omega^{p,q}(M)$, or simply $\Omega^{p,q}$, instead of $\Omega_J^{p,q}(M)$.

Recall that the space of smooth forms on $M$, $\Omega^*(M)$, is a differential graded algebra endowed with a product $\wedge$ and a differential $d$. It is well known that $d$ acts on every $\alpha \in \Omega^k(M)$ in such a way that $d\alpha \in \Omega^{k+1}(M)$. The existence of an almost-complex structure $J$ on $M$ induces a decomposition of $d : \Omega^{p,q}(M) \longrightarrow \Omega^{p+q+1}_{\mathbb{C}}(M)$ as follows:

$$d = A + \partial + \bar{\partial} + \bar{A},$$

where

$$A : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M), \qquad \partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),$$
$$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M), \qquad \bar{A} : \Omega^{p,q}(M) \to \Omega^{p-1,q+2}(M).$$

Simply note that $d^2 = 0$ induces some relations among the previous operators, namely,

$$\left\{\begin{array}{rcl} 0 &=& A^2, \\ 0 &=& A\partial + \partial A, \\ 0 &=& A\bar{\partial} + \partial^2 + \bar{\partial}A, \\ 0 &=& A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + A\bar{A}. \end{array}\right. \qquad \left\{\begin{array}{rcl} 0 &=& \bar{A}^2, \\ 0 &=& \bar{A}\bar{\partial} + \bar{\partial}\bar{A}, \\ 0 &=& \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial, \end{array}\right. \tag{1.2}$$

When we consider the $2n$-dimensional differentiable manifold $M$ underlying a complex manifold of complex dimension $n$, it turns out that $M$ naturally admits an almost-complex structure $J$ locally given by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i},$$

being $\varphi = (x_1 + iy_1, \ldots, x_n + iy_n)$ a local coordinate complex chart on $M$. The Cauchy-Riemann conditions for holomorphic functions of several complex variables ensure that this local definition of $J$ can be expanded to all $M$. In this way, if an even-dimensional differentiable manifold does not admit almost-complex structures, then it cannot be a complex manifold.

**Definition 1.1.2.** *An almost-complex structure $J$ on a differentiable manifold $M$ is said to be* integrable *if the* Nijenhuis tensor $N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$*, defined by*

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \tag{1.3}$$

*vanishes for every pair $X, Y \in \mathfrak{X}(M)$.*

**Theorem 1.1.3.** [NN57] *Let $(M, J)$ be an almost-complex manifold. The following statements are equivalent:*

*i)* $(M, J)$ *is a complex manifold,*

*ii) the differential $d$ decomposes as $d = \partial + \bar{\partial}$ (that is, $A = \bar{A} = 0$),*

*iii)* $\mathfrak{X}^{1,0}(M)$*, respectively* $\mathfrak{X}^{0,1}(M)$*, is a Lie subalgebra of* $\mathfrak{X}_{\mathbb{C}}(M)$*,*

*iv) $J$ is integrable.*

*If one of the previous conditions holds, we will refer to $J$ as a* complex structure *on $M$.*

When $X$ is a complex manifold, we will also write $\mathfrak{X}^{1,0}(X)$, $\mathfrak{X}^{0,1}(X)$, and $\Omega^{p,q}(X)$, assuming that it is the natural complex structure on $X$ which induces the bigraduation.

Although Theorem 1.1.3 solves our initial question, it opens new problems to keep in mind when working in Complex Geometry. On the one hand, one still needs to discover which differentiable manifolds can carry almost-complex structures. Moreover, on a given differentiable manifold $M$ there can exist both integrable and non-integrable almost-complex structures, so the appropriate ones should be found. On the other hand, the same $M$ endowed with two distinct complex structures can have very different properties as a complex manifold. In this sense it is useful to define a notion of equivalence between complex structures that gives an isomorphism between complex manifolds.

**Definition 1.1.4.** *Two (almost-) complex manifolds $(M, J)$ and $(M', J')$ are said to be* isomorphic *if there exists a diffeomorphism $F : M \longrightarrow M'$ such that $F_* \circ J = J' \circ F_*$.*

A concept that may help to better understand how complex properties depend on complex structures is that of holomorphic deformations. Let $\mathcal{B}$ and $\Delta$ be two complex manifolds such that $\pi : \mathcal{B} \longrightarrow \Delta$ is a proper holomorphic submersion. A *holomorphic family* of complex manifolds $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ is a collection of compact complex manifolds defined by the fibers of $\pi$, that is, $X_{\mathbf{t}} := \pi^{-1}(\mathbf{t})$ for each $\mathbf{t} \in \Delta$. The base manifold $\Delta$ will be assumed to be an open ball around the origin in $\mathbb{C}^k$, for some $k \in \mathbb{N}$. It turns out that the manifolds $X_{\mathbf{t}}$ are diffeomorphic, so they can be seen as $X_{\mathbf{t}} = (M, J_{\mathbf{t}})$, where $M$ is a compact differentiable manifold of even dimension and $J_{\mathbf{t}}$ the complex structure varying at each fiber. We will call $\{J_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ an *analytic family* of complex structures. Since the complex manifolds $X_{\mathbf{t}}$ are thought to be "*sufficiently close*" to each other, the idea is studying the behaviour of complex properties on them.

**Definition 1.1.5.** *Let $X$ be a complex manifold. A holomorphic family of complex manifolds $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ is said to be a* holomorphic deformation *of $X$ if one has $X_{\mathbf{0}} = X$.*

Every compact complex manifold $X$ admits a locally complete space of holomorphic deformations, known as the *Kuranishi space* and denoted by $\mathrm{Kur}(X)$. If we assume that $X$ satisfies some property $\mathcal{P}$, one would like to know if every sufficiently small holomorphic deformation also satisfies it. The following concept arises:

**Definition 1.1.6.** *A property $\mathcal{P}$ of a compact complex manifold is said to be* open *or* stable *under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ and for every $\mathbf{t}_0 \in \Delta$ the following implication holds:*

$$X_{\mathbf{t}_0} \textit{ has property } \mathcal{P} \Rightarrow X_{\mathbf{t}} \textit{ has property } \mathcal{P} \textit{ for all } \mathbf{t} \in \Delta \textit{ sufficiently close to } \mathbf{t}_0.$$

Nonetheless, there is another approach which complements the previous one. The idea is trying to extend a certain property to the central limit of a holomorphic deformation which satisfies it at every point.

**Definition 1.1.7.** *A property $\mathcal{P}$ of a compact complex manifold is said to be* closed *under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ and for every $\mathbf{t}_0 \in \Delta$ the following implication holds:*

$$X_{\mathbf{t}} \textit{ has property } \mathcal{P} \textit{ for every } \mathbf{t} \in \Delta \setminus \{\mathbf{t}_0\} \Rightarrow X_{\mathbf{t}_0} \textit{ also has property } \mathcal{P}.$$

We will give some specific examples of complex structures and complex properties along the rest of this chapter, in relation to their behaviour under holomorphic deformations.

## 1.2 Cohomology groups and related topics

In this section we introduce some invariants related to complex manifolds, defined as certain cohomology groups. In particular, we focus our attention on the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomologies. We also revise the Frölicher spectral sequence, the $\partial\bar{\partial}$-lemma property, and the notion of cohomological decomposition.

### 1.2.1 De Rham and Dolbeault cohomologies

We present here two classical cohomologies and recall some elementary results about them. In addition, we see that it is possible to connect these two invariants by means of the Frölicher spectral sequence.

Let us start with the de Rham cohomology, which is actually a topological invariant. That is, it only depends on the underlying topological space and thus is unable to distinguish between two non-isomorphic complex manifolds $(M, J)$ and $(M', J')$, let alone when $M \cong M'$. In fact, this cohomology can be simply described in terms of differentiable manifolds. Let us briefly see it.

Let $M$ be a differentiable manifold of (real) dimension $m$. Since $\Omega^*(M)$ is a differential graded algebra, one can consider the complex

$$(1.4) \qquad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{m-1}(M) \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0.$$

The fact that $d \circ d = 0$ implies that the subspace $\operatorname{im}\{d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)\}$ is contained in $\ker\{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\}$. The *k-th (real) de Rham cohomology group* is defined by

$$H^k_{\mathrm{dR}}(M;\mathbb{R}) = \frac{\ker\{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\}}{\operatorname{im}\{d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)\}}.$$

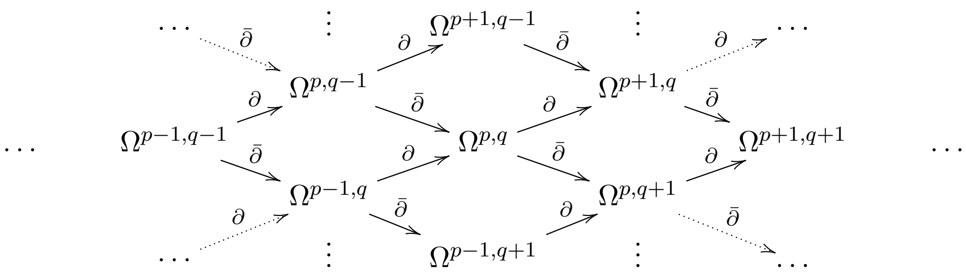
As a consequence of the de Rham Theorem, when $M$ is compact the previous groups are finite dimensional. Therefore, it makes sense to consider their dimensions, known as the *Betti numbers*: $b_k(M) = \dim H^k_{\mathrm{dR}}(M;\mathbb{R})$. Furthermore, if $M$ is both compact and orientable, then one has that $b_k(M) = b_{m-k}(M)$, by the Poincaré Duality Theorem.

Observe that one can complexify each $\Omega^k(M)$ in (1.4) and the differential $d$ will still satisfy the same properties as above. Therefore, it is possible to consider the *k-th complex de Rham cohomology group*:

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) = \frac{\ker\{d : \Omega^k_{\mathbb{C}}(M) \longrightarrow \Omega^{k+1}_{\mathbb{C}}(M)\}}{\operatorname{im}\{d : \Omega^{k-1}_{\mathbb{C}}(M) \longrightarrow \Omega^k_{\mathbb{C}}(M)\}}.$$

It is important to note that the real and the complex de Rham cohomology groups have the same dimension. It is just the representation of the cohomology classes what changes. Depending on our purposes, we will use the real or the complex description. In particular, the complex groups are particularly convenient when we work with complex manifolds.

In what follows, let $X$ be a complex manifold of complex dimension $n$. As the natural complex structure on $X$ induces a decomposition of the differential, $d = \partial + \bar{\partial}$, it seems natural to investigate other cohomologies involving the operators $\partial$ and $\bar{\partial}$. Notice that by (1.2) these operators satisfy $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} = -\bar{\partial}\partial$ (since $A = 0 = \bar{A}$). To clarify the new setting, one should bear in mind the following diagram:




Observe that each column contains the spaces whose forms have the same total degree. Therefore, the (complexified) de Rham sequence can be thought as the one linking these columns, one by one, from left to right.

Now, one could consider the diagonals (equivalently, antidiagonals) of the previous scheme, motivated by the fact that $\bar\partial \circ \bar\partial = 0$. The *Dolbeault cohomology* is given by the groups

$$H^{p,q}_{\bar\partial}(X) = \frac{\ker\{\bar\partial : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X)\}}{\operatorname{im}\{\bar\partial : \Omega^{p,q-1}(X) \longrightarrow \Omega^{p,q}(X)\}}.$$

Once again, when $X$ is compact the previous spaces are finite-dimensional (Hodge Theorem), so it is possible to define the *Hodge numbers*: $h^{p,q}(X) = \dim H^{p,q}_{\bar\partial}(X)$. By the Serre Duality Theorem one also has $H^{p,q}_{\bar\partial}(X) \cong H^{n-p,n-q}_{\bar\partial}(X)$, so in particular we get the relations $h^{p,q}(X) = h^{n-p,n-q}(X)$.

Let us notice that each $d$-closed form of pure degree $(p,q)$ is also $\bar\partial$-closed and thus defines a Dolbeault cohomology class. However, the opposite is in general not true. Nonetheless, following [Frö55] it is possible to achieve a connection by means of spectral sequences.

Every filtered differential graded module determines a spectral sequence satisfying certain properties (see for instance [McC01, Theorem 2.6]). In the case of complex differential forms, one can consider the decreasing filtration

$$\mathcal{F}^p\left(\Omega^*_{\mathbb{C}}(X)\right) = \bigoplus_{r\geq p,\, q\geq 0} \Omega^{r,q}(X), \text{ where } p \geq 0,$$

which can intuitively be thought as follows. Initially, we take $\mathcal{F}^0\left(\Omega^*_{\mathbb{C}}(X)\right) = \Omega^*_{\mathbb{C}}(X)$, which corresponds to the whole space described by the diagram above. At each step from $p=1$ to $p=n-1$, we progressively eliminate one by one diagonals of the diagram, starting from the down left corner. In this way, one finally gets $\mathcal{F}^n\left(\Omega^*_{\mathbb{C}}(X)\right) = \bigoplus_{q=0}^n \Omega^{n,q}(X)$ and $\mathcal{F}^p\left(\Omega^*_{\mathbb{C}}(X)\right) = \{0\}$, for every $p > n$. Then $(\Omega^*_{\mathbb{C}}(X), d, \mathcal{F}^*)$ induces the so-called *Frölicher spectral sequence* $\{(E_r^{*,*}(X), d_r)\}_{r\in\mathbb{N}}$, where for each $r \in \mathbb{N}$ one has

$$\cdots \xrightarrow{d_r} E_r^{p-r,q+r-1}(X) \xrightarrow{d_r} E_r^{p,q}(X) \xrightarrow{d_r} E_r^{p+r,q-r+1}(X) \xrightarrow{d_r} \cdots$$

such that $d_r \circ d_r = 0$ and

$$E^{p,q}_{r+1}(X) \cong \frac{\ker\{d_r : E_r^{p,q}(X) \longrightarrow E_r^{p+r,q-r+1}(X)\}}{\operatorname{im}\{d_r : E_r^{p-r,q+r-1}(X) \longrightarrow E_r^{p,q}(X)\}}.$$

We denote $\mathcal{F}^p\left(\Omega^k_{\mathbb{C}}(X)\right) = \{\omega \in \mathcal{F}^p\left(\Omega^*_{\mathbb{C}}(X)\right) \mid \omega \in \Omega^k_{\mathbb{C}}(X)\}$, i.e., we focus on a certain column of the diagram at the step $p$ of the filtration, and we can consider

$$\mathcal{F}^p Z^k_\infty(X) = \{\omega \in \mathcal{F}^p\big(\Omega^k_{\mathbb{C}}(X)\big) \mid d\omega = 0\},$$

$$\mathcal{F}^p B^k_\infty(X) = \{\omega \in \mathcal{F}^p\big(\Omega^k_{\mathbb{C}}(X)\big) \mid \omega = d\eta, \text{ for some } \eta \in \Omega^{k-1}_{\mathbb{C}}(X)\}.$$

Then, $\mathcal{F}^p\big(H^k_\infty(X)\big) = \frac{\mathcal{F}^p Z^k_\infty(X)}{\mathcal{F}^p B^k_\infty(X)}$ and the limit terms for $\{(E_r^{*,*}(X), d_r)\}_{r\in\mathbb{N}}$ are:

$$E^{p,q}_\infty(X) \cong \frac{\mathcal{F}^p\big(H^{p+q}_\infty(X)\big)}{\mathcal{F}^{p+1}\big(H^{p+q}_\infty(X)\big)}.$$

The key point about this sequence is that it satisfies [Frö55]:

- $E_1^{p,q}(X) \cong H^{p,q}_{\bar\partial}(X)$, for every pair $(p,q)$;
- $E_N^{p,q}(X) \cong E_\infty^{p,q}(X)$, for every $(p,q)$, when $N$ is sufficiently large; and
- $H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} E^{p,q}_\infty(X)$.

As we can see, it connects our two cohomologies, apart from providing new invariants for complex manifolds: the spaces $E_r^{p,q}(X)$. An explicit description of them can be found in [CFGU97a]. We recall it here.

**Theorem 1.2.1.** [CFGU97a] *Let $X$ be a complex manifold. Then,*

$$E_r^{p,q}(X) \cong \frac{X_r^{p,q}(X)}{Y_r^{p,q}(X)},$$

*where*

$$X_1^{p,q}(X) \;=\; \big\{\alpha \in \Omega^{p,q}(X) \mid \bar\partial\alpha = 0\big\}, \qquad Y_1^{p,q}(X) \;=\; \bar\partial\left(\Omega^{p,q-1}(X)\right),$$

*and for $r > 1$,*

$$\begin{aligned} X_r^{p,q}(X) \;=\; \big\{&\alpha^{p,q} \in \Omega^{p,q}(X) \mid \bar\partial\alpha^{p,q} = 0 \text{ and there exist } \alpha^{p+i,q-i} \in \Omega^{p+i,q-i}(X), \\ &\text{for } 1 \le i \le r-1, \text{ such that } \partial\alpha^{p+i-1,q-i+1} + \bar\partial\alpha^{p+i,q-i} = 0\big\}, \end{aligned}$$

$$\begin{aligned} Y_r^{p,q}(X) \;=\; \big\{&\partial\beta^{p-1,q} + \bar\partial\beta^{p,q-1} \in \Omega^{p,q}(X) \mid \text{ there exist } \beta^{p-i,q+i-1} \in \Omega^{p-i,q+i-1}(X), \\ &\text{for } 2 \le i \le r-1, \text{ satisfying } \partial\beta^{p-i,q+i-1} + \bar\partial\beta^{p-i+1,q+i-2} = 0, \\ &\text{and } \bar\partial\beta_{p-r+1,q+r-2} = 0\big\}. \end{aligned}$$

Observe that the differential $d_1$ is given by

$$\begin{array}{ccc} H^{p,q}_{\bar\partial}(X) & \xrightarrow{d_1} & H^{p+1,q}_{\bar\partial}(X), \\ [\alpha] & \mapsto & [\partial\alpha] \end{array}$$

whereas for $r > 1$, one has the following result:

**Theorem 1.2.2.** [CFGU97a] *For* $r \geq 2$ *the map* $d_r : E_r^{p,q}(X) \longrightarrow E_r^{p+r,q-r+1}(X)$ *is defined by*

$$d_r\left([\alpha^{p,q}]\right) = [\partial \alpha^{p+r-1,q-r+1}],$$

*where* $\alpha^{p+r-1,q-r+1}$ *is determined by the space* $X_r^{p,q}(X)$ *given in Theorem 1.2.1.*

As a consequence of the previous description, one can relate the Betti and the Hodge numbers of any compact complex manifold by the *Frölicher inequality.* We finish this section with it.

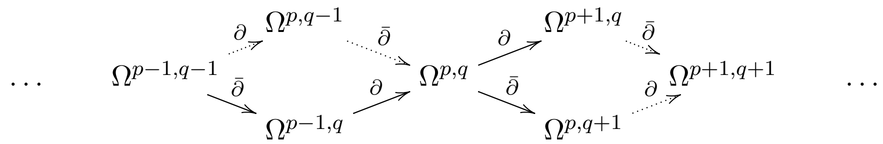
**Theorem 1.2.3.** [Frö55] *Let* $X$ *be a compact complex manifold. Then,*

$$b_k(X) \leq \sum_{p+q=k} h^{p,q}(X).$$

### 1.2.2 Bott-Chern and Aeppli cohomologies

In this section, we continue reviewing some cohomologies of complex manifolds defined in terms of the operators $\partial$ and $\bar{\partial}$.

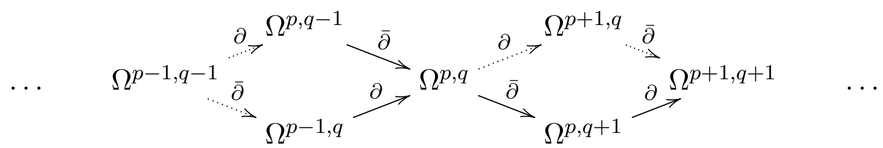
On the one hand, we can consider the sequence




Notice that any complex form $\beta$ such that $\beta = \partial\bar{\partial}\alpha$, for some $\alpha \in \Omega^*_{\mathbb{C}}(X)$, satisfies $d\beta = 0$. Therefore, the *Bott-Chen cohomology* is given by the groups:

$$H^{p,q}_{\mathrm{BC}}(X) = \frac{\ker\{\partial + \bar{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X) \oplus \Omega^{p,q+1}(X)\}}{\mathrm{im}\{\partial\bar{\partial} : \Omega^{p-1,q-1}(X) \longrightarrow \Omega^{p,q}(X)\}}.$$

On the other hand, one can take




Observe that any complex form $\beta$ such that $\beta = d\alpha$, for some $\alpha \in \Omega^*_{\mathbb{C}}(X)$, satisfies $\partial\bar{\partial}\beta = 0$. Hence, the *Aeppli cohomology* can be defined as follows

$$H^{p,q}_{\mathrm{A}}(X) = \frac{\ker\{\partial\bar{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q+1}(X)\}}{\mathrm{im}\{\partial : \Omega^{p-1,q}(X) \longrightarrow \Omega^{p,q}(X)\} \oplus \mathrm{im}\{\bar{\partial} : \Omega^{p,q-1}(X) \longrightarrow \Omega^{p,q}(X)\}}.$$

By conjugation, it is clear that $\overline{H^{q,p}_{\mathrm{BC}}(X)} \cong H^{p,q}_{\mathrm{BC}}(X)$ and $\overline{H^{q,p}_{\mathrm{A}}(X)} \cong H^{p,q}_{\mathrm{A}}(X)$. Furthermore, using the theory of elliptic operators Schweitzer proved [Sch] that if $X$ is compact, then the Bott-Chern and the Aeppli cohomology groups are finite-dimensional. In that case, one can denote $h^{p,q}_{\mathrm{BC}}(X) = \dim H^{p,q}_{\mathrm{BC}}(X)$ and $h^{p,q}_{\mathrm{A}}(X) = \dim H^{p,q}_{\mathrm{A}}(X)$. They will be respectively called *Bott-Chern* and *Aeppli numbers*.

Moreover, the Hodge star operator associated to a Hermitian metric induces the following isomorphism

$$H^{p,q}_{\mathrm{BC}}(X) \cong H^{n-q,n-p}_{\mathrm{A}}(X), \text{ for } 0 \leq p \leq q \leq n.$$

Thus, one can say that these two cohomologies are essentially the same.

A natural question to ask is whether the previous cohomologies are related to the de Rham and the Dolbeault ones. First, notice that every $\alpha \in \Omega^{p,q}$ such that $d\alpha = 0$ satisfies $\partial\alpha = \bar\partial\alpha = 0$ and hence, $\partial\bar\partial\alpha = 0$. Secondly, if $\alpha = \partial\bar\partial\beta$ for some $\beta \in \Omega^{p-1,q-1}$, then $\alpha = (\partial+\bar\partial)(\bar\partial\beta) = d(\bar\partial\beta)$. Therefore, there are natural maps defined in the following way:

$$\begin{array}{ccccc} & & H^{p,q}_{\mathrm{BC}}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H^{p,q}_{\partial}(X) & & H^{p+q}_{\mathrm{dR}}(X;\mathbb{C}) & & H^{p,q}_{\bar\partial}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{p,q}_{\mathrm{A}}(X) & & \end{array}$$

It is important to note that, in general, the maps above are neither injective nor surjective. However, it turns out [DGMS75, Remark 5.16] that if one of them is bijective for every $(p,q)$, then all of them are (see [Ang14, Theorem 2.1] for an explicit proof). Of particular interest is the injectivity of the natural map $H^{p,q}_{\mathrm{BC}}(X) \longrightarrow H^{p+q}_{\mathrm{dR}}(X;\mathbb{C})$:

**Definition 1.2.4.** [DGMS75] *A compact complex manifold is said to satisfy the* $\partial\bar\partial$-lemma condition*,* $\partial\bar\partial$-lemma property*, or simply* $\partial\bar\partial$-property*, if every* $\partial$*-closed,* $\bar\partial$*-closed, and* $d$*-exact form is also* $\partial\bar\partial$*-exact; that is,*

$$ker\,\partial \cap ker\,\bar\partial \cap im\,d = im\,\partial\bar\partial.$$

*When such condition holds, we will say that* $X$ *is a* $\partial\bar\partial$-manifold.

Observe that $\partial\bar\partial$-manifolds have some interesting properties. For instance, if $X$ is a compact complex manifold satisfying the $\partial\bar\partial$-lemma condition, then its Frölicher spectral sequence degenerates at the first step [DGMS75], i.e. $E^{p,q}_1(X) \cong E^{p,q}_\infty(X)$ for every pair $(p,q)$. According to the previous section, this implies

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X). \tag{1.5}$$

This gives the equality in Theorem 1.2.3. In addition, the $\partial\bar\partial$-property also implies that the conjugation induces an isomorphism $H^{p,q}_{\bar\partial}(X) \cong H^{q,p}_{\bar\partial}(X)$, which is known as the *Hodge symmetry*.

Recently, Angella and Tomassini give an inequality relating the dimensions of the Bott-Chern, Aeppli, and de Rham cohomology groups, in the spirit of Theorem 1.2.3. Their result also provides a characterization of the $\partial\bar\partial$-lemma property:

**Theorem 1.2.5.** [AT13] *Let $X$ be a compact complex manifold. For each $k \in \mathbb{N}$, one has*

$$b_k(X) \leq \frac{1}{2} \sum_{p+q=k} \left( h^{p,q}_{\mathrm{BC}}(X) + h^{p,q}_{\mathrm{A}}(X) \right).$$

*Moreover, the equality holds for every $k \in \mathbb{N}$ if and only if $X$ satisfies the $\partial\bar{\partial}$-property.*

Let us notice that Theorem 1.2.5 enables to "quantify" how far is a compact complex manifold of satisfying the $\partial\bar{\partial}$-lemma condition. In particular, Angella and Tomassini use their result to obtain a new proof of the stability of the $\partial\bar{\partial}$-property, by means of the duality between the Bott-Chern and Aeppli cohomology groups and the upper-semi-continuity of the numbers $h^{p,q}_{\mathrm{BC}}$ (see [Sch]):

**Theorem 1.2.6.** [AT13, Voi02, Wu06] *For compact complex manifolds, satisfying the $\partial\bar{\partial}$-lemma condition is an open property under holomorphic deformations.*

Other invariants related to Theorem 1.2.5 can be found in Section 2.1.2. Concerning the deformation limits of the $\partial\bar{\partial}$-lemma property, the following result was recently obtained by Angella and Kasuya:

**Theorem 1.2.7.** [AK] *For compact complex manifolds, the $\partial\bar{\partial}$-property is not closed under holomorphic deformations.*

### 1.2.3 The problem of cohomological decomposition

In this section, we deal with new spaces $H^{p,q}_J(M)$ that generalize the Dolbeault cohomology groups. The aim is studying whether a similar decomposition to (1.5) can hold even if the Frölicher spectral sequence does not degenerate at the first step.

Let $M$ be a $2n$-dimensional differentiable manifold endowed with an almost-complex structure $J$ (not necessarily integrable). Motivated by the Donaldson *"tamed to compatible conjecture"* [Don06, Question 2], Li and Zhang consider in [LZ09] the spaces

$$H^{\pm}_J(M) \;=\; \{\, [\alpha] \in H^2_{\mathrm{dR}}(M; \mathbb{R}) \mid J\alpha = \pm\alpha \,\},$$

in the context of Symplectic Geometry. Recall that when $M$ is compact the de Rham cohomology groups are finite dimensional, so also are $H^{\pm}_J(M)$, and one can denote $h^{\pm}_J(M) = \dim H^{\pm}_J(M)$.

The almost-complex structure $J$ is said to be $\mathcal{C}^\infty$*-pure* if $H^+_J(M) \cap H^-_J(M) = \{\mathbf{0}\}$, and it is $\mathcal{C}^\infty$*-full* if $H^+_J(M) + H^-_J(M) = H^2_{\mathrm{dR}}(M; \mathbb{R})$. In case that both properties are satisfied, i.e. the decomposition

$$H^2_{\mathrm{dR}}(M; \mathbb{R}) = H^+_J(M) \oplus H^-_J(M)$$

holds, $J$ is called $\mathcal{C}^\infty$*-pure-and-full.* It turns out [LZ09] that the $J$-compatible cone is an open (possibly empty) convex cone in $H^+_J(M)$, whereas $H^-_J(M)$ measures the difference between the tamed and the compatible cones as long as the latter is non-empty and the almost-complex structure is $\mathcal{C}^\infty$-full. Despite these interesting facts, our interest in the spaces $H_J^\pm(M)$ comes from the next observations.

In the integrable case, if the Frölicher spectral sequence of the complex manifold $(M, J)$ degenerates at the first step and there is a weight 2 formal Hodge decomposition, then $(M, J)$ is $\mathcal{C}^\infty$-pure-and-full (see [AT11, DLZ10, LZ09]). Indeed, in such case the subgroups $H_J^\pm(M)$ are nothing but the (real) Dolbeault cohomology groups, i.e.

$$H_J^+(M) = H_{\bar\partial}^{1,1}(M, J) \cap H_{\mathrm{dR}}^2(M; \mathbb{R}),$$
$$H_J^-(M) = \left(H_{\bar\partial}^{2,0}(M, J) \oplus H_{\bar\partial}^{0,2}(M, J)\right) \cap H_{\mathrm{dR}}^2(M; \mathbb{R}).$$

Hence, compact complex surfaces and compact complex manifolds satisfying the $\partial\bar\partial$-property [DGMS75] (in particular, compact Kähler manifolds) have the $\mathcal{C}^\infty$-pure-and-full property, but also any sufficiently small holomorphic deformation of a compact $\partial\bar\partial$-manifold [AT13, Wu06]. Furthermore, for the non-integrable case in dimension 4, one has:

**Theorem 1.2.8.** [DLZ10] *Every almost-complex structure on a compact 4-dimensional manifold is $\mathcal{C}^\infty$-pure-and-full.*

In contrast, a first example of a non-$\mathcal{C}^\infty$-pure 6-dimensional almost-complex manifold was given in [FT10]. Later, it was indeed shown [AT12] that any possible combination of these two properties could occur in six dimensions, making clear that the cohomological decomposition property was far from being trivial. In fact, it was seen that the Iwasawa manifold is $\mathcal{C}^\infty$-pure-and-full but there are small deformations which are neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full [AT11]. Hence, one has:

**Theorem 1.2.9.** [AT11] *"Being $\mathcal{C}^\infty$-pure-and-full", "being $\mathcal{C}^\infty$-pure", and "being $\mathcal{C}^\infty$-full" are not open properties for compact complex manifolds.*

These results motivate the interest in finding conditions ensuring $\mathcal{C}^\infty$-pure-and-fullness, as well as new manifolds satisfying the property. We will go back to this issue in Chapter 2, where we will also prove that the $\mathcal{C}^\infty$-pure-and-full property is not closed under holomorphic deformations.

One might also wonder if a similar notion of decomposability could be given for other de Rham cohomology groups. With this aim, Angella and Tomassini [AT11] look at the spaces

$$H_J^{(r,r)}(M)_{\mathbb{R}} = \left\{[\alpha] \in H_{\mathrm{dR}}^{2r}(M; \mathbb{R}) \mid \alpha \in \Omega^{r,r}(M)\right\},$$
$$H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} = \left\{[\alpha] \in H_{\mathrm{dR}}^{p+q}(M; \mathbb{R}) \mid \alpha \in \Omega^{p,q}(M) \oplus \Omega^{q,p}(M)\right\},$$

for $0 \le r \le n$ and $0 \le p < q \le n$, and they introduce the following:

**Definition 1.2.10.** [AT11] *Let $M$ be a $2n$-dimensional differentiable manifold. An almost-complex structure $J$ on $M$ is said to be:*

*i)* $\mathcal{C}^\infty$-pure-and-full at the $k$-th stage*, if*

$$H^k_{\mathrm{dR}}(M;\mathbb{R}) = \bigoplus_{\substack{p+q=k\\ p\leq q}} H_J^{(p,q),(q,p)}(M)_{\mathbb{R}}; \tag{1.6}$$

*ii)* $\mathcal{C}^\infty$-pure at the $k$-th stage*, if the sum on the right hand side of* (1.6) *is direct but not necessarily equal to* $H^k_{\mathrm{dR}}(M;\mathbb{R})$*; and*

*iii)* $\mathcal{C}^\infty$-full at the $k$-th stage*, if the previous sum equals* $H^k_{\mathrm{dR}}(M;\mathbb{R})$*, but it is not necessarily direct.*

In particular, $H_J^+(M) = H_J^{(1,1)}(M)_{\mathbb{R}}$ and $H_J^-(M) = H_J^{(2,0),(0,2)}(M)_{\mathbb{R}}$, so the initial notions by Li and Zhang are recovered. For this reason, we will omit the stage when we refer to the second one.

Since we also have the complex de Rham cohomology groups, one can define analogous notions in terms of them. Simply take

$$H_J^{p,q}(M) = \left\{[\alpha] \in H^{p+q}_{\mathrm{dR}}(M;\mathbb{C}) \mid \alpha \in \Omega^{p,q}(M)\right\},$$

and then:

**Definition 1.2.11.** [AT11, DLZ10] *Let $M$ be a $2n$-dimensional differentiable manifold. An almost-complex structure $J$ on $M$ is said to be:*

*i)* complex-$\mathcal{C}^\infty$-pure-and-full at the $k$-th stage*, if*

$$H^k_{\mathrm{dR}}(M;\mathbb{C}) = \bigoplus_{p+q=k} H_J^{p,q}(M); \tag{1.7}$$

*ii)* complex-$\mathcal{C}^\infty$-pure at the $k$-th stage*, if the sum on the right hand side of* (1.7) *is direct but not necessarily equal to* $H^k_{\mathrm{dR}}(M;\mathbb{C})$*; and*

*iii)* complex-$\mathcal{C}^\infty$-full at the $k$-th stage*, if the previous sum equals* $H^k_{\mathrm{dR}}(M;\mathbb{C})$*, but it is not necessarily direct.*

For analogy with the real case, we will be referring to the second stage when not specifically mentioned. In view of Theorem 1.2.8, one might ask if a similar result holds for the complex notions of $\mathcal{C}^\infty$-pureness and $\mathcal{C}^\infty$-fullness.

**Proposition 1.2.12.** [DLZ10] *Every almost-complex structure $J$ on a compact 4-dimensional manifold $M$ is complex-$\mathcal{C}^\infty$-pure. In addition, it is also complex-$\mathcal{C}^\infty$-full if and only if $J$ is integrable or $H_J^-(M) = \{\mathbf{0}\}$.*

Concerning other stages, there are examples of (non-integrable) almost-complex structures on compact 4-dimensional manifolds which are complex-$\mathcal{C}^\infty$-pure-and-full at the first stage and others which are not [AT12]. Hence, the problem becomes more complicated not only when the dimension of the manifold increases, but also when the stage at which the decomposition is considered varies.

Observe that (1.7) can be seen as a generalization of (1.5) for almost-complex manifolds. Indeed, it can give a decomposition of the de Rham cohomology for those complex manifolds whose Frölicher spectral sequence does not necessarily degenerate at the first stage. In this sense, it is worth noting that (1.5) and (1.7) can sometimes coincide. For instance when $(M, J)$ is a compact complex manifold satisfying the $\partial\bar{\partial}$-property, in which case $H_J^{p,q}(M) \cong H_{\bar{\partial}}^{p,q}(M, J)$, for every pair $(p, q)$. Hence, $\partial\bar{\partial}$-manifolds are always (complex-) $\mathcal{C}^\infty$-pure-and-full at every stage.

In general, if the $\partial\bar{\partial}$-lemma condition does not hold on a compact complex manifold $(M, J)$, we simply have [Ang14, Remark 3.1]

$$H_J^{p,q}(M) = \operatorname{im}\left(H_{\mathrm{BC}}^{p,q}(M, J) \longrightarrow H_{\mathrm{dR}}^{p+q}(M; \mathbb{C})\right),$$

where $H_{\mathrm{BC}}^{p,q}(M, J) \longrightarrow H_{\mathrm{dR}}^{p+q}(M; \mathbb{C})$ is the natural map induced by the inclusion seen in Section 1.2.2. Notice that in this situation $M$ can still be complex-$\mathcal{C}^\infty$-pure-and-full at every stage, the *Iwasawa manifold* being an example of this type. Since not many other cases are known, we will study cohomological decomposition for a large class of compact complex non-$\partial\bar{\partial}$-manifolds in Chapter 2.

Observe that one can somehow relate the previous complex and real spaces by the following expressions:

$$H_J^{(r,r)}(M)_{\mathbb{R}} = H_J^{(r,r)}(M) \cap H_{\mathrm{dR}}^{2r}(M; \mathbb{R}),$$

$$H_J^{(p,q),(q,p)}(M)_{\mathbb{R}} = H_J^{(p,q),(q,p)}(M) \cap H_{\mathrm{dR}}^{p+q}(M; \mathbb{R}),$$

for $0 \le r \le n$ and $0 \le p < q \le n$, where

$$H_J^{(r,r)}(M) = \left\{[\alpha] \in H_{\mathrm{dR}}^{2r}(M; \mathbb{C}) \mid \alpha \in \Omega^{r,r}(M)\right\},$$

$$H_J^{(p,q),(q,p)}(M) = \left\{[\alpha] \in H_{\mathrm{dR}}^{p+q}(M; \mathbb{C}) \mid \alpha \in \Omega^{p,q}(M) \oplus \Omega^{q,p}(M)\right\}.$$

Focusing on the second stage, it is well known that if the almost-complex structure $J$ is complex-$\mathcal{C}^\infty$-full, then it is $\mathcal{C}^\infty$-full. However, one needs an integrable $J$ to obtain $\mathcal{C}^\infty$-pureness from complex-$\mathcal{C}^\infty$-pureness. In fact, when integrability holds one has the following equality (see [Ang14, Remark 3.2], [DLZ10, Lemma 2.12]):

$$H_J^-(M) = \left(H_J^{2,0}(M) + H_J^{0,2}(M)\right) \cap H_{\mathrm{dR}}^2(M; \mathbb{R}).$$

However, Drăghici, Li, and Zhang provide in [DLZ10] a non-integrable almost-complex structure $J$ on a compact 4-dimensional manifold such that $H_J^-(M) \neq \{\mathbf{0}\}$, but satisfying $H_J^{2,0}(M) + H_J^{0,2}(M) = \{\mathbf{0}\}$.

Furthermore, there exists a relation between the notions of pureness and fullness at different stages.

**Theorem 1.2.13.** [AT11] *Let $(M, J)$ be a $2n$-dimensional almost-complex manifold. If $J$ is (complex-) $\mathcal{C}^\infty$-full at $k$-th stage, then $J$ is (complex-) $\mathcal{C}^\infty$-pure at $(2n-k)$-th stage.*

Similar notions of pureness and fullness can be defined in terms of currents instead of differential forms. However, we will not deal with them. Simply note that these properties and their relation with Definitions 1.2.10 and 1.2.11 are also being investigated by different authors (for more information see [AT11, FT10, LZ09]).

## 1.3 Hermitian metrics and other special geometries

We now move to some geometric aspects of complex manifolds. More concretely, we first focus on Hermitian metrics and introduce different types using the differential operators $d$, $\partial$, and $\bar{\partial}$. More concretely, we present Kähler, balanced, strongly Gauduchon, SKT, astheno-Kähler, and generalized Gauduchon metrics. Afterwards, we consider other interesting structures also characterized in terms of a 2-form, namely, pseudo-Kähler and holomorphic symplectic ones.

Let $M$ be a $2n$-dimensional differentiable manifold. A *pseudo-Riemannian metric* on $M$ is a map

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{C}^\infty(M)$$

such that $g_p$ is a non-degenerate symmetric bilinear product on the tangent space $T_pM$, for each $p \in M$. If this product is positive definite for every $p \in M$, then $g$ is a Riemannian metric.

**Definition 1.3.1.** *Let $(M, J)$ be a $2n$-dimensional differentiable manifold endowed with a complex structure. A pseudo-Riemannian metric $g$ on $M$ is said to be* compatible *with $J$ when*

$$g(JX, JY) = g(X, Y), \text{ for every } X, Y \in \mathfrak{X}(M).$$

*In this case, $g$ is known as a* pseudo-Hermitian metric *and the triple $(M, J, g)$ is called a* pseudo-Hermitian manifold*. Moreover, if $g$ is Riemannian then we say that $g$ is a* Hermitian metric *and $(M, J, g)$ is a* Hermitian manifold*.*

**Remark 1.3.2.** In the same way that every differentiable manifold admits a Riemannian metric, every complex manifold admits a Hermitian metric.

Let $(M, J, g)$ be a pseudo-Hermitian manifold of real dimension $2n$. For each $p \in M$, the tangent space $T_pM$ admits an adapted basis $\{u_1, \ldots, u_n, Ju_1, \ldots, Ju_n\}$ in terms of which the bilinear product $g_p$ can be characterized by the $2n \times 2n$ matrix

$$G = \left( \begin{array}{c|c} A & B \\ \hline B^t & A \end{array} \right),$$

where $A$, $B$ are real $n \times n$ matrices defined by

$$A = \Big( g_p(u_k, u_l) \Big)_{k,l}, \quad B = \Big( g_p(u_k, Ju_l) \Big)_{k,l}$$

and satisfying $A^t = A$, $B^t = -B$. In particular, the diagonal elements of $B$ are always zero, i.e. $g_p(u_k, Ju_k) = 0$ for every $1 \leq k \leq n$. Furthermore, if $g$ is Riemannian then the diagonal entries of $A$ are positive (among other conditions).

Notice that $g$ can be naturally extended to the complexified space of vector fields $\mathfrak{X}_{\mathbb{C}}(M)$. In order to give a local expression of it, it suffices to apply a change of basis that takes $G$ to a new matrix $\widetilde{G}$ in the complex basis $\{u_1 - iJu_1, \ldots, u_n - iJu_n, u_1 + iJu_1, \ldots, u_n + iJu_n\} \equiv \{Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n\}$. Observe that the first $n$ vectors correspond to the $i$-eigenspace of $J$ in $(T_pM)_{\mathbb{C}}$ and the rest, to the $(-i)$-eigenspace. The matrix for $g_p$ in the new basis is

$$\widetilde{G} = \left( \begin{array}{c|c} \mathbf{0} & 2\,(A + i\,B) \\ \hline 2\,(A - i\,B) & \mathbf{0} \end{array} \right).$$

If we denote $H = 2\,(A + i\,B)$, then it is clear that $\bar{H}^t = H$. Moreover, it is worth noting that $g_p(X, Y) = 0$ for every $X, Y \in T_p^{1,0}M$ or $X, Y \in T_p^{0,1}M$.

Let $\{\omega^1, \ldots, \omega^n, \omega^{\bar{1}}, \ldots, \omega^{\bar{n}}\}$ be the dual basis of $\{Z_1, \ldots, Z_n, \bar{Z}_1, \ldots, \bar{Z}_n\}$, where we denote $\omega^{\bar{k}} = \overline{\omega^k}$. One can see that $g_p$ can be expressed as a tensor in the following way:

$$\begin{aligned} g_p \;&=\; \sum_{k=1}^{n} 2\, g_p(u_k, u_k)\big(\omega^k \otimes \omega^{\bar{k}} + \omega^{\bar{k}} \otimes \omega^k\big) \\ &+\; \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} 2\,\big(g_p(u_k, u_l) + i\, g_p(u_k, Ju_l)\big)\big(\omega^k \otimes \omega^{\bar{l}} + \omega^{\bar{l}} \otimes \omega^k\big) \\ &+\; \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} 2\,\big(g_p(u_k, u_l) - i\, g_p(u_k, Ju_l)\big)\big(\omega^l \otimes \omega^{\bar{k}} + \omega^{\bar{k}} \otimes \omega^l\big)\,. \end{aligned} \tag{1.8}$$

**Definition 1.3.3.** *Let $(M, J, g)$ be a pseudo-Hermitian manifold. The* fundamental form*, or* Kähler form*, is the 2-form $F$ defined by*

$$F(X, Y) = g(JX, Y), \text{ for every } X, Y \in \mathfrak{X}(M).$$

*Notice that the real form $F$ has bidegree $(1,1)$ with respect to $J$ and satisfies $F^n \neq 0$.*

**Remark 1.3.4.** Since $F$ and $g$ are in one-to-one correspondence, we will indistinctly refer to a pseudo-Hermitian manifold as $(M, J, g)$ or $(M, J, F)$. In the same way, we will talk about a pseudo-Hermitian metric $F$ meaning a pseudo-Hermitian metric whose fundamental form is $F$.

A local expression of the fundamental form can be found using (1.8), simply introducing $J$ in the first component of each tensorial product and bearing in mind the definition of wedge product. We then obtain:

$$F_p = \sum_{k=1}^{n} i\, h_{k\bar{k}}\, \omega^k \wedge \omega^{\bar{k}} + \sum_{1 \leq k < l \leq n} \big(h_{k\bar{l}}\, \omega^k \wedge \omega^{\bar{l}} - \bar{h}_{k\bar{l}}\, \omega^l \wedge \omega^{\bar{k}}\big)\,, \tag{1.9}$$

where

$$h_{k\bar{k}} = 2\, g_p(u_k, u_k), \qquad h_{k\bar{l}} = 2\, i\, \big(g_p(u_k, u_l) + i\, g_p(u_k, Ju_l)\big).$$

The formula (1.9) clearly shows the local relation between the coefficients of the fundamental form $F$ and those of the pseudo-Hermitian metric $g$.

We now focus on $2n$-dimensional Hermitian manifolds $(M, J, F)$, i.e. those for which the pseudo-Hermitian metric $g$ is positive definite. Different families of metrics can be distinguished according to the behaviour of $F$ under certain conditions involving the differential $d$ or the differential operators $\partial$, $\bar{\partial}$.

One of the most important types of Hermitian metrics are Kähler ones, which lie in the intersection between Hermitian and Symplectic Geometry.

**Definition 1.3.5.** *A Hermitian metric on* $(M, J)$ *is* Kähler *if its fundamental form* $F$ *satisfies* $dF = 0$. *Then,* $(M, J, F)$ *is a* Kähler manifold.

**Remark 1.3.6.** Equivalently, one can define Kähler manifolds from the symplectic point of view. Let $\omega$ be a symplectic form on $(M, J)$. Assume that $\omega$ is compatible with $J$ in a similar sense to Definition 1.3.1. We say that $(M, J, \omega)$ is a Kähler manifold if the metric defined by $g(X, Y) = -\omega(JX, Y)$, for $X, Y \in \mathfrak{X}(M)$, is positive definite. That is, $g$ is a Riemannian metric. Otherwise, $(M, J, \omega)$ will be called pseudo-Kähler (see Definition 1.3.24).

It was proven in [DGMS75] that compact Kähler manifolds satisfy the $\partial\bar{\partial}$-lemma condition. Therefore, they share the cohomological properties of $\partial\bar{\partial}$-manifolds, such as the degeneration of the Frölicher spectral sequence at the first stage and the Hodge symmetry. Furthermore, the existence of a Kähler metric imposes strong topological conditions to the manifold. For instance, its even Betti numbers cannot vanish (as a consequence being symplectic), and its odd Betti numbers are even (by the Hodge Decomposition Theorem). In fact, for compact complex surfaces ($n = 2$) the Kähler property is strictly topological, since they are Kähler if and only if their first Betti number is even (see [Kod64, Miy74, Siu83]). Therefore, it seems clear that for $n = 2$ the property of being Kähler is both open and closed. More in general, one has:

**Theorem 1.3.7.** [KS60] *For compact complex manifolds, the property of being Kähler is open under holomorphic deformations.*

However, the property in the deformation limits behaves differently for $n \geq 3$:

**Theorem 1.3.8.** [Hir62] *The property of being Kähler is not closed for compact complex manifolds of complex dimension* $n \geq 3$.

The constraints for the existence of a Kähler metric motivate the search of weaker conditions on $F$ that a non-Kähler manifold could still satisfy.

A natural generalization of the Kähler condition consist on taking a certain power of $F$ and asking its differential to be zero. For any $2 \leq k \leq n-2$, Gray and Hervella prove that $dF^k = 0$ implies $dF = 0$, so just the case $k = n - 1$ remains:

**Definition 1.3.9.** *A Hermitian metric is called* balanced *if* $dF^{n-1} = 0$.

Although for $n = 2$ the Kähler and the balanced conditions overlap, for $n \geq 3$ it is possible to find balanced metrics which are not Kähler. This proves that, in general, these two classes do not coincide. The study of balanced metrics under holomorphic deformations reinforces this idea, as in contrast with Theorem 1.3.7 we have:

**Theorem 1.3.10.** [AB90] *The existence of balanced metrics on compact complex manifolds is not an open property under holomorphic deformations.*

Although it was conjectured in [Pop14] that these two types of metrics also differed in their behaviours at the deformation limits, the following was finally proven:

**Theorem 1.3.11.** [COUV16] *For compact complex manifolds, the balanced property is not closed under holomorphic deformations.*

At this point, we make use of the two differential operators $\partial$ and $\bar{\partial}$ in which the exterior differential $d$ decomposes. In the same way that $\partial$ and $\bar{\partial}$ served to define new cohomologies in Section 1.2, they can now be used to generalize Kähler metrics. In this sense, one could study the conditions $\partial\bar{\partial}F^k = 0$, for $1 \leq k \leq n-1$. In particular, three values of $k$ will be specially interesting for us:

**Definition 1.3.12.** *Let* $(M, J, F)$ *be a Hermitian manifold of complex dimension* $n$*. The metric is said to be:*

*i)* strong Kähler with torsion, SKT*, or* pluriclosed*, if* $\partial\bar{\partial}F = 0$*;*

*ii)* astheno-Kähler*, if* $\partial\bar{\partial}F^{n-2} = 0$*;*

*iii)* Gauduchon *or* standard*, if* $\partial\bar{\partial}F^{n-1} = 0$.

Let us start with Gauduchon metrics. We first need to mention the following result:

**Theorem 1.3.13.** [Gau84] *Let* $(M, J, F)$ *be a compact Hermitian manifold. Then, there is a Gauduchon metric* $\tilde{F}$ *in the conformal class of* $F$*, i.e., there exists* $f \in \mathcal{C}^\infty(M)$ *such that* $\tilde{F} = e^f F$ *satisfies* $\partial\bar{\partial}\tilde{F}^{n-1} = 0$.

Its importance comes from the fact that, in combination with Remark 1.3.2, it implies that any compact complex manifold admits a Gauduchon metric. Furthermore, one should note that every balanced metric is Gauduchon, although the converse is in general not true. Indeed, there exists another class of metrics between the previous two. It was introduced and studied by Popovici in [Popa, Popb, Pop13, Pop14].

**Definition 1.3.14.** *A Hermitian metric is said to be* strongly Gauduchon *if it satisfies the condition* $\partial F^{n-1} = \bar{\partial}\alpha$*, for some form* $\alpha$ *of type* $(n, n-2)$ *on* $(M, J)$.

It turns out that any compact complex surface ($n = 2$) admitting a strongly Gauduchon metric also admits a Kähler metric (hence, balanced) [Popb]. However, this no longer holds for $n \geq 3$, where one can find strongly Gauduchon manifolds which are not balanced. Moreover, their stability under holomorphic deformation also differs:

**Theorem 1.3.15.** [Popa] *The existence of strongly Gauduchon metrics on compact complex manifolds is an open property under holomorphic deformations.*

Concerning the closedness of this property, it was conjectured in [Pop14] that the existence of strongly Gauduchon metrics holds in the deformation limits. In the end, the following result was recently attained:

**Theorem 1.3.16.** [COUV16] *The strongly Gauduchon property for compact complex manifolds is not closed under holomorphic deformations.*

However, Popovici shows that the existence of strongly Gauduchon metrics in the central limit can be guaranteed under strong additional requirements. More concretely:

**Proposition 1.3.17.** [Popb] *Let $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ be a holomorphic family of compact complex manifolds, where $\Delta$ is an open disk around the origin in $\mathbb{C}^k$. If the $\partial\bar\partial$-lemma holds on $X_{\mathbf{t}}$ for every $\mathbf{t} \in \Delta \setminus \{\mathbf{0}\}$, then $X_{\mathbf{0}}$ has a strongly Gauduchon metric.*

With this new class between balanced and Gauduchon metrics, we have constructed a first line of generalizations of the Kähler condition:

$$\Big\{ \text{Kähler} \Big\} \Rightarrow \Big\{ \text{balanced} \Big\} \Rightarrow \Big\{ \text{strongly Gauduchon} \Big\} \Rightarrow \Big\{ \text{Gauduchon} \Big\}$$

Figure 1.1: Generalization of Kähler metrics I.

We should now go back to study the other metrics in Definition 1.3.12, namely, SKT and astheno-Kähler. We start noting that it is possible to relate both of them with the balanced condition. More concretely:

**Proposition 1.3.18.** *Let $(M, J, F)$ be a compact Hermitian manifold of complex dimension $n \geq 3$. Then:*

*i) if $F$ is both SKT and balanced, then $F$ is Kähler* [AI01]*;*

*ii) if $F$ is both astheno-Kähler and balanced, then $F$ is Kähler* [MT01].

Let us note that in the previous result it is the metric $F$ the one satisfying all the conditions. However, there is a "*folklore*" conjecture (see for instance [FV14]) asserting that the existence of an SKT metric and a balanced metric (not necessarily the same) on a complex manifold $(M, J)$ implies the existence of a Kähler metric on $(M, J)$.

Concerning the behaviour under holomorphic deformations of pluriclosed metrics, we can only guarantee their non-openness (the study of closedness is still open):

**Theorem 1.3.19.** [FT09] *For compact complex manifolds, the property of admitting an SKT metric is not open under holomorphic deformations.*

Since the counterexample was found for $n = 3$, the only dimension in which the SKT and the astheno-Kähler conditions overlap, one trivially has:

**Corollary 1.3.20.** *The astheno-Kähler property of compact complex manifolds is not open under holomorphic deformations.*

Nevertheless, we should mention that the two previous metrics does not necessarily coincide for $n \geq 4$, as shown in [RT12].

We would now like to complete those paths of generalizations of the Kähler condition initiated by SKT and astheno-Kähler metrics with the help of *generalized Gauduchon* metrics, recently introduced by Fu, Wang, and Wu in [FWW13]:

**Definition 1.3.21.** *A Hermitian metric $F$ is called* $k$-th Gauduchon *when it fulfills the condition $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, where $1 \leq k \leq n-1$.*

In particular, classical Gauduchon metrics are recovered for $k = n-1$. Furthermore, these new Hermitian metrics have some interesting properties, as we next show.

**Theorem 1.3.22.** [FWW13] *Let $(M, J, F)$ be a compact Hermitian manifold of complex dimension $n$. For each $1 \leq k \leq n-1$, there is a unique constant $\gamma_k(F)$ and a (unique up to a constant) function $f \in \mathcal{C}^\infty(M)$ such that $\frac{i}{2}\partial\bar{\partial}\left(e^f F^k\right) \wedge F^{n-k-1} = \gamma_k(F)\, e^f F^n$.*

For $k = n-1$, one can determine that $\gamma_k(F) = 0$, in such a way that Theorem 1.3.13 is recovered. Additionally, if $F$ is Kähler then $\gamma_k(F) = 0$ and $f$ is constant, for every $1 \leq k \leq n-1$. It is also shown in [FWW13] that the sign of $\gamma_k(F)$ remains constant in the conformal class of $F$.

The new lines that generalize the Kähler condition arise as follows:

$$
\begin{array}{cccccc}
 & \nearrow & \Big\{\text{SKT/pluriclosed}\Big\} & \Longrightarrow & \Big\{\text{1-st Gauduchon}\Big\} \\
\Big\{\text{Kähler}\Big\} & & & & \vdots \\
 & \searrow & \Big\{\text{astheno-Kähler}\Big\} & \Rightarrow & \Big\{(n-2)\text{-th Gauduchon}\Big\}
\end{array}
$$

Figure 1.2: Generalization of Kähler metrics II.

It is worth mentioning that some of them have not been yet deeply studied. In fact, one of the aims of this work is to cast some light into the topic. Nevertheless, similar results to Proposition 1.3.18 have already been obtained for this new type of metrics:

**Proposition 1.3.23.** *Let $(M, J, F)$ be a compact Hermitian manifold of complex dimension $n \geq 3$. Then:*

*i) if $F$ is both 1-st Gauduchon and balanced, then $F$ is Kähler* [FU13]*;*

*ii) for $2 \leq k \leq n-2$, if $F$ is both $k$-th Gauduchon and balanced, then $F$ is Kähler* [IP13]*.*

Observe that when working in non-Kähler geometry, Propositions 1.3.18 and 1.3.23 provide obstructions for a metric to be both balanced and SKT, astheno-Kähler, or generalized Gauduchon at the same time.

Another completely different way of generalizing the Kähler condition is dropping the positive definiteness of the metric. In this sense, we have already introduce the notion of *pseudo-Kähler manifold* in Remark 1.3.6 using symplectic forms. We here give an equivalent definition in terms of pseudo-Hermitian metrics.

**Definition 1.3.24.** *Let $(M, J, F)$ be a $2n$-dimensional pseudo-Hermitian manifold. The metric is called* pseudo-Kähler *if it satisfies $dF = 0$.*

Let us note that each pseudo-Kähler metric defines a class in $H_J^+(M)$. Hence, one could think about defining structures in terms of a 2-form belonging to $H_J^-(M)$. In this sense, the following definition arises.

**Definition 1.3.25.** *Let $X$ be a complex manifold of complex dimension $n = 2p$. A* holomorphic symplectic structure *$\Omega$ is a $(2,0)$-form on $X$ such that $d\Omega = 0$ and $\Omega^p \neq 0$.*

Therefore, if $(M, J, \Omega)$ is a holomorphic symplectic manifold of real dimension $4p$, then $\omega = \frac{1}{2}(\Omega + \bar{\Omega})$ defines a class in $H_J^-(M)$. In fact, $\omega$ is a symplectic form on $M$. However, unlike in the pseudo-Kähler case, this symplectic form is not compatible with the complex structure $J$ in the sense of Definition 1.3.1. In fact, due to the bidegree of $\Omega$, one has that

$$2\,\omega(J\cdot, J\cdot) = \Omega(J\cdot, J\cdot) + \bar{\Omega}(J\cdot, J\cdot) = -\Omega(\,\cdot\,,\,\cdot\,) - \bar{\Omega}(\,\cdot\,,\,\cdot\,) = -2\,\omega(\,\cdot\,,\,\cdot\,).$$

Therefore, $\omega$ is actually *anti-invariant*, as it fulfills the condition $\omega(JX, JY) = -\omega(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$.

There are examples in the literature showing that both pseudo-Kähler and holomorphic symplectic structures can exist on non-Kähler manifolds (see [CFU04, Gua95a, Gua95b]). Nonetheless, it is true that the second type of structures has been less studied, maybe due to the dimensional constraint. We will go back to this issue in Chapter 5.

## 1.4 Nilmanifolds with invariant complex structures

We now turn our attention to the main type of manifolds that we will consider in this work: nilmanifolds. In particular, we focus on those nilmanifolds endowed with *invariant* complex structures and present some well-known results about them. Special attention is paid to dimensions four and six, where classifications have been obtained.

### 1.4.1 Reduction to the Lie algebra level

In this part, we recall some basic notions about Lie groups and Lie algebras. We see how complex structures defined on Lie algebras give rise to *invariant* complex structures on certain quotients of Lie groups, paying special attention to the case in which this quotient is a nilmanifold.

Let us remind that a *Lie group $G$* of dimension $m$ is an $m$-dimensional differentiable manifold which is also a group in the algebraic sense whose product and inverse maps are differentiable. For each $g \in G$, one can consider the *left translation* map $L_g : G \longrightarrow G$, defined by $L_g(h) := gh$. Then, there is a Lie algebra $\mathfrak{g}$ naturally associated to $G$ given by the space of left-invariant vector fields on $G$, $\mathfrak{g} = \{X \in \mathfrak{X}(G) \mid (L_g)_* X = X,$ for every $g \in G\}$, with the usual bracket $[\cdot, \cdot]$. The key point about this space is that $\mathfrak{g} \cong T_eG$, being $e$ the neutral element of the group $G$. Hence, $\mathfrak{g}$ has finite dimension $m$ in contrast with $\mathfrak{X}(G)$, whose dimension is not finite.

Furthermore, it turns out that $\mathfrak{g}^* \cong T_e^*G$. As the space $\bigwedge^* \mathfrak{g}^*$, is a differential graded algebra, one can use its differential $d$ to relate left-invariant vector fields and left-invariant 1-forms. More precisely, one has the formula:

$$d\alpha(A, B) = -\alpha([A, B]), \quad \forall \alpha \in \mathfrak{g}^*, \ \forall A, B \in \mathfrak{g}. \tag{1.10}$$

If we let $\{e^k\}_{k=1}^m$ be a basis of $\mathfrak{g}^*$, then the Lie algebra $\mathfrak{g}$ is determined by the expressions:

$$de^k = \sum_{1 \leq i < j \leq m} c_{ij}^k \, e^i \wedge e^j, \quad 1 \leq k \leq m, \tag{1.11}$$

which are known as the *structure equations* of the Lie group $G$. The numbers $c_{ij}^k$ are the *structure constants* of $G$ with respect to the given basis for $\mathfrak{g}^*$.

Let $\Gamma$ be a subgroup of the $m$-dimensional Lie group $G$. If the topology induced by $G$ on $\Gamma$ is discrete, then $\Gamma$ is called a *discrete subgroup* of $G$. In this case the space $\Gamma\backslash G$ admits the structure of an $m$-dimensional differentiable manifold, in such a way that the natural projection $\pi : G \longrightarrow \Gamma\backslash G$ is smooth. Using the pushforward $\pi_*$ one can transfer any left-invariant vector field on $G$, i.e. $X \in \mathfrak{g}$, to a vector field $\pi_* X$ on $\Gamma\backslash G$ that will be called *invariant*. In fact, any basis of $\mathfrak{g}$ generates a basis of invariant vector fields on $\Gamma\backslash G$. Analogously, any basis of $\mathfrak{g}^*$ can be transmitted via the pullback $\pi^*$ to $\Gamma\backslash G$, so in particular, the basis $\{e^k\}_{k=1}^m$ determining the structure equations (1.11) of $G$. More precisely, there exists a basis $\{\alpha^k\}_{k=1}^m$ on $\Gamma\backslash G$ such that $\pi^*\alpha^k = e^k$, for every $1 \leq k \leq m$. Furthermore, one can see that the structure equations of $\Gamma\backslash G$ in terms of $\{\alpha^k\}_{k=1}^m$ have the same form as (1.11). For this reason, we will denote $\alpha^k$ by $e^k$ and indistinctly refer to (1.11) as the structure equations of $G$ and $\Gamma\backslash G$.

In the same way that vectors and 1-forms in $\mathfrak{g}$ can be transferred to the quotient $\Gamma\backslash G$, any other type of tensor on $\mathfrak{g}$ descends to $\Gamma\backslash G$. In particular, every $\alpha \in \bigwedge^k \mathfrak{g}^*$ and every pseudo-Riemannian metric on $\mathfrak{g}$ define *invariant* ones on $\Gamma\backslash G$. This can also be applied to almost-complex structures:

**Definition 1.4.1.** *Let $\mathfrak{g}$ be a Lie algebra of even dimension. An* almost-complex structure on $\mathfrak{g}$ *is an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -id$. If $\mathfrak{g}$ is the Lie algebra associated to a certain Lie group $G$ and $\Gamma \leq G$ is a discrete subgroup, then the induced almost-complex structure on $\Gamma\backslash G$, is called* invariant. *It will also be denoted by $J$.*

**Example 1.4.2.** Let $\mathbb{A}$ be a ring, and consider the set of matrices:

$$H_3(\mathbb{A}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{A} \right\}.$$

For $\mathbb{A} = \mathbb{R}$, this is known as the *Heisenberg group.* Observe that $G = H_3(\mathbb{R}) \times \mathbb{R}$ is a 4-dimensional Lie group with the matrix product that admits a global chart taking each $g \in G$ to an element $(x, y, z, t) \in \mathbb{R}^4$, where $t$ is the coordinate in $\mathbb{R}$. Its Lie algebra $\mathfrak{g}$ has a basis

$$X_1 = -\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\,\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = \frac{\partial}{\partial z},$$

where the only non-zero bracket is $[X_1, X_2] = -X_4$. Moreover, $\Gamma = H_3(\mathbb{Z}) \times \mathbb{Z}$ is a discrete subgroup of $G$, so one can consider the quotient $M = \Gamma\backslash G$. An invariant almost-complex structure on $M$ is induced by the following $J$ on $\mathfrak{g}$:

$$JX_1 = X_2, \quad JX_3 = X_4.$$

Since $J^2 = -id$, one directly has $JX_2 = -X_1$ and $JX_4 = -X_3$. ◇

The existence of an almost-complex structure $J$ on a $2n$-dimensional Lie algebra $\mathfrak{g}$ yields a decomposition of the complexified space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, in a similar way an almost-complex structure on a manifold $M$ generates it on $\mathfrak{X}_{\mathbb{C}}(M)$. More precisely,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1},$$

where $\mathfrak{g}_{1,0} = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid JZ = iZ\}$ and $\mathfrak{g}_{0,1} = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid JZ = -iZ\}$. Note that these spaces are conjugate to each other. Following the same ideas contained in Section 1.1, one can equivalently define $J$ on $\mathfrak{g}^*$ and obtain a decomposition of $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^* \otimes \mathbb{C}$,

$$\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} = \{\omega \in \mathfrak{g}^*_{\mathbb{C}} \mid J\omega = i\omega\}$ and $\mathfrak{g}^{0,1} = \overline{\mathfrak{g}^{1,0}} = \{\omega \in \mathfrak{g}^*_{\mathbb{C}} \mid J\omega = -i\omega\}$. Extending $J$ to the complexified space of $k$-forms $\left(\bigwedge^k \mathfrak{g}^*\right)_{\mathbb{C}} \cong \bigwedge^k(\mathfrak{g}^*_{\mathbb{C}}) = \bigwedge^k\left(\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}\right)$, for $1 \leq k \leq 2n$, there is a bigraduation:

$$\bigwedge^k(\mathfrak{g}^*_{\mathbb{C}}) \;=\; \bigoplus_{p+q=k} \bigwedge^{p,q}(\mathfrak{g}^*),$$

where $\bigwedge^{p,q}(\mathfrak{g}^*) = \bigwedge^p(\mathfrak{g}^{1,0}) \otimes \bigwedge^q(\mathfrak{g}^{0,1})$. The exterior differential $d$ on $\bigwedge^k(\mathfrak{g}^*_{\mathbb{C}})$ also decomposes as $d = A + \partial + \bar\partial + \bar A$, and

$$d\big(\bigwedge^{p,q}(\mathfrak{g}^*)\big) \subset \bigwedge^{p+2,q-1}(\mathfrak{g}^*) \oplus \bigwedge^{p+1,q}(\mathfrak{g}^*) \oplus \bigwedge^{p,q+1}(\mathfrak{g}^*) \oplus \bigwedge^{p-1,q+2}(\mathfrak{g}^*).$$

**Definition 1.4.3.** *Let $\mathfrak{g}$ be a Lie algebra of even dimension. An almost-complex structure $J$ on $\mathfrak{g}$ is called* integrable *if the Nijenhuis tensor* (1.3) *vanishes for every $X, Y \in \mathfrak{g}$. In this case, $J$ is also known as a* complex structure *on $\mathfrak{g}$.*

**Definition 1.4.4.** *Let $\mathfrak{g}$ be a Lie algebra endowed with two complex structures $J$ and $J'$. We say that $J$ and $J'$ are* equivalent *if there exists an automorphism of the Lie algebra $F : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $F \circ J' = J \circ F$. Equivalently, $F^*$ commutes with the differential $d$ and, extended to $\bigwedge^k(\mathfrak{g}^*_{\mathbb{C}})$, preserves the bigraduations induced by $J$ and $J'$.*

**Remark 1.4.5.** Let $\mathfrak{g}$ be the Lie algebra associated to an even-dimensional Lie group $G$ which has a discrete subgroup $\Gamma \leq G$. If $\mathfrak{g}$ admits an integrable almost-complex structure $J$, then the almost-complex structure induced on $\Gamma\backslash G$ is also integrable. It is then called *invariant complex structure* on $\Gamma\backslash G$. Furthermore, two equivalent complex structures on $\mathfrak{g}$ induce two equivalent invariant complex structures on $\Gamma\backslash G$.

The integrability of an almost-complex structure $J$ on a Lie algebra $\mathfrak{g}$ is equivalent to having $A = \bar{A} = 0$ in the decomposition of the exterior differential $d$ (recall Theorem 1.1.3). In particular, it suffices to check the condition:

$$d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*).$$

Two interesting types of complex structures arise from this expression:

- if $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*)$, equivalently $[JX, JY] = [X, Y]$ for every $X, Y \in \mathfrak{g}$, then $\mathfrak{g}^{1,0}$ is an abelian complex Lie algebra and $J$ is said to be an *abelian* complex structure;
- if $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*)$, equivalently $[JX, Y] = J[X, Y]$ for every $X, Y \in \mathfrak{g}$, then $\mathfrak{g}$ turns to be a complex Lie algebra and $J$ is called *complex-parallelizable* structure.

**Example 1.4.6. The Kodaira-Thurston manifold.** Let $M = \Gamma\backslash G$ be the manifold constructed in Example 1.4.2. If we consider the dual basis $\{e^k\}_{k=1}^4$ of $\{X_k\}_{k=1}^4$ and apply the formula (1.10), we get the following structure equations for $M$:

$$de^1 = de^2 = de^3 = 0, \qquad de^4 = e^1 \wedge e^2.$$

The almost-complex structure $J$ is equivalently defined by $Je^1 = e^2$, $Je^3 = e^4$, using (1.1). Take the basis for $\mathfrak{g}^{1,0}$ given by $\omega^1 = \frac{1}{2}(e^1 - ie^2)$, $\omega^2 = -\frac{1}{2}(e^3 - ie^4)$. Then, it is easy to see that the *complex* structure equations of $(M, J)$ are:

$$d\omega^1 = 0, \quad d\omega^2 = \omega^1 \wedge \omega^{\bar{1}}.$$

Hence, $J$ is integrable. In fact, it is an abelian complex structure. The complex manifold $\mathbb{KT} = (M, J)$ is called the *Kodaira-Thurston manifold.* ◇

**Example 1.4.7. The Iwasawa manifold.** Let $\mathfrak{g}$ be the 6-dimensional Lie algebra defined by the structure equations

$$de^1 = de^2 = de^3 = de^4 = 0, \quad de^5 = e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23},$$

where $e^{kl} = e^k \wedge e^l$. An almost-complex structure on $\mathfrak{g}$ can be defined by $Je^1 = e^2$, $Je^3 = e^4$, $Je^5 = e^6$. Then, there is a basis for $\mathfrak{g}^{1,0}$ given by $\omega^1 = \frac{1}{2}(e^1 - ie^2)$, $\omega^2 = \frac{1}{2}(e^3 - ie^4)$, and $\omega^3 = \frac{1}{4}(e^5 - ie^6)$ in terms of which the complex structure equations are

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12}.$$

It is clear that $J$ is a complex-parallelizable structure on $\mathfrak{g}$. Hence, $\mathfrak{g}$ can be seen as a complex Lie algebra. In fact, this is the Lie algebra associated to the *complex Heisenberg group* $G = H_3(\mathbb{C})$, which has a discrete subgroup $\Gamma = H_3(\mathbb{Z}[i])$. The complex manifold $\Gamma\backslash G$ is the *Iwasawa manifold.* The complex structure naturally associated to it coincides with that induced by our $J$. ◇

The differentiable manifolds in the Examples 1.4.6 and 1.4.7 belong to a special class: that of nilmanifolds. Another important examples in this class are tori, which can be seen as $\mathbb{Z}^m\backslash\mathbb{R}^m$, for every $m \geq 1$. Their associated Lie algebras are abelian, so it is trivial that every even-dimensional torus admits an invariant complex structure.

Let us next see the precise definition of nilmanifolds and some interesting properties that will make them our preferred object of study.

**Definition 1.4.8.** *A* nilmanifold *is a compact quotient $M = \Gamma\backslash G$, where $G$ is a connected, simply-connected, nilpotent Lie group and $\Gamma \leq G$ is a discrete subgroup of maximal rank, also called* lattice.

Recall that the dimension of the nilmanifold $M$ is the dimension of $G$ as a differentiable manifold. Also remember that a Lie group $G$ is said to be *nilpotent* when its associated Lie algebra $\mathfrak{g}$ is nilpotent, i.e., its descending central series $\{\mathfrak{g}^k\}_{k\geq 0}$ degenerates after a finite number $s$ of steps. The smallest integer $s$ is called the *nilpotency step* of $\mathfrak{g}$. We will see in Chapter 3 how the nilpotency of $\mathfrak{g}$ can help us to define invariant complex structures on nilmanifolds.

In general, given a Lie group $G$ of dimension $m$, it is not easy to find a discrete subgroup $\Gamma \leq G$ such that the quotient $\Gamma\backslash G$ is compact. However, the problem can be simplified for nilpotent Lie groups:

**Theorem 1.4.9.** [Mal49] *A connected, simply-connected, nilpotent Lie group $G$ admits compact quotients of the form $\Gamma\backslash G$ if and only if there is a basis for the dual of its associated Lie algebra in which the structure constants $c_{ij}^k$ are rational numbers.*

Furthermore, the integrability of an almost-complex structure $J$ on a nilpotent Lie algebra $\mathfrak{g}$ leads to a simplification of the complex structure equations:

**Theorem 1.4.10.** [Sal01] *Let $J$ be an almost-complex structure on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$. Then $J$ is integrable if and only if there is a basis $\{\omega^k\}_{k=1}^n$ for $\mathfrak{g}^{1,0}$ in terms of which the complex structure equations have the form*

$$d\omega^1 = 0, \quad d\omega^k \in \mathcal{I}(\omega^1, \ldots, \omega^{k-1}), \ \textit{for } 2 \leq k \leq n,$$

*where $\mathcal{I}(\omega^1, \ldots, \omega^{k-1})$ is the ideal of $\bigwedge^k(\mathfrak{g}_{\mathbb{C}}^*)$ generated by $\omega^1, \ldots, \omega^{k-1}$.*

We have seen that any tensor on $\mathfrak{g}$ can be transferred to an invariant one on the corresponding nilmanifold $M = \Gamma\backslash G$. This makes, for instance, that the existence of Hermitian metrics and other special geometries on $\mathfrak{g}$ implies their existence on $M$. However, there are also structures on $M$ which are not invariant, so cannot apparently be detected on $\mathfrak{g}$. For this reason, it would be interesting to know in which cases this detection is actually possible. The *symmetrization process*, introduced by Belgun [Bel00] and developed in [FG04], precisely allows to reduce some results on $M$ to the Lie algebra level. We finish this section with its description.

By [Mil76], every nilmanifold $M$ has a volume element $\nu$ induced by one on $G$ which is bi-invariant, i.e., both left- and right-invariant. Rescaling, one can suppose that the volume of $M$ is equal to one.

Given a covariant $k$-tensor $T : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ on $M$, it is possible to define a covariant $k$-tensor $T_\nu$ on the nilpotent Lie algebra $\mathfrak{g}$ associated to $M$ by:

$$T_\nu(X_1, \ldots, X_k) = \int_{p\in M} T_p(X_1|_p, \ldots, X_k|_p)\,\nu,$$

where $X_i \in \mathfrak{g}$ for every $1 \leq i \leq k$, and $X_i|_p$ denotes the projection on $M$ of the left-invariant vector field $X_j$ of $G$ evaluated at $p \in M$. Note that $T_\nu = T$ whenever $T$ is invariant.

Moreover, if $T = \alpha$ is a $k$-form on $M$, then it satisfies some interesting properties:

*a)* $(d\alpha)_\nu = d\alpha_\nu$ and

*b)* $(\alpha_\nu \wedge \beta)_\nu = \alpha_\nu \wedge \beta_\nu$, for any $\beta \in \Omega^l(M)$.

Furthermore, if there is an invariant complex structure $J$ on $M$, the symmetrization process can be extended to the space of complex forms and the bigraduation induced by $J$ is preserved. That is, if $\alpha$ is a $(p,q)$-form on $(M,J)$ then $\alpha_\nu$ is a $(p,q)$-form on $(\mathfrak{g}, J)$. Additionally, one can see that $(\partial\alpha)_\nu = \partial\alpha_\nu$ and $(\bar\partial\alpha)_\nu = \bar\partial\alpha_\nu$.

Let us simply remark that the symmetrization process can be applied to other compact quotients $M = \Gamma\backslash G$, where $G$ is a Lie group not necessarily nilpotent and $\Gamma \leq G$ is discrete. One just needs that $G$ is simply-connected and $M$ admits a bi-invariant volume form $\nu$ such that $\int_M \nu = 1$.

### 1.4.2 Computation of cohomologies

As we have seen, any invariant complex structure $J$ on a nilmanifold $M$ can be directly studied at the Lie algebra level. In this part, we discuss when something similar can be applied to the cohomologies of the complex manifold $(M,J)$.

Following Section 1.2, we start with the de Rham cohomology. In the same way that the de Rham cohomology of differentiable manifolds is a topological invariant, the Lie-algebra (de Rham) cohomology is an algebraic one. Let $\mathfrak{g}$ be an $m$-dimensional Lie algebra. The (real) de Rham cohomology groups of $\mathfrak{g}$ are naturally defined by

$$H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R}) = \frac{\ker\{d : \bigwedge^k(\mathfrak{g}^*) \longrightarrow \bigwedge^{k+1}(\mathfrak{g}^*)\}}{\mathrm{im}\{d : \bigwedge^{k-1}(\mathfrak{g}^*) \longrightarrow \bigwedge^{k}(\mathfrak{g}^*)\}},$$

where $1 \leq k \leq m$. If $\mathfrak{g}$ is the Lie algebra associated to a nilmanifold $M = \Gamma\backslash G$, note that there is a natural inclusion $\iota : H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R}) \hookrightarrow H^k_{\mathrm{dR}}(M;\mathbb{R})$. Let us take $[\alpha] \in H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R})$ such that $\mathbf{0} = \iota([\alpha]) \in H^k_{\mathrm{dR}}(M;\mathbb{R})$. Then, $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$ and applying the symmetrization process, it turns out that $\alpha = \alpha_\nu = (d\beta)_\nu = d\beta_\nu$. Hence, $\mathbf{0} = [\alpha] \in H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R})$ and $\iota$ is injective. The following result by Nomizu asserts that $\iota$ is actually an isomorphism.

**Theorem 1.4.11.** [Nom54] *Let $M = \Gamma\backslash G$ be an $m$-dimensional nilmanifold with associated Lie algebra $\mathfrak{g}$. Then, $H^k_{\mathrm{dR}}(M;\mathbb{R}) \cong H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R})$ for every $1 \leq k \leq m$.*

Let us remark that if $[\alpha] \in H^k_{\mathrm{dR}}(M;\mathbb{R})$, then one can obtain a representative of $\iota^{-1}[\alpha]$ by simply applying the symmetrization process. More precisely, let us consider $\nu : H^k_{\mathrm{dR}}(M;\mathbb{R}) \to H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R})$ given by $\nu([\alpha]) = [\alpha_\nu]$. Observe that $\nu \circ \iota = id_{H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{R})}$. Since $\iota$ is an isomorphism, also it is $\nu$, and one has $\nu = \iota^{-1}$. In particular, if $\alpha$ defines a de Rham cohomology class, its symmetrized $\alpha_\nu$ is another representative of the class which is, additionally, invariant. As a consequence of the previous theorem, it is possible to calculate the de Rham cohomologies of nilmanifolds, such as the tori or those presented in Examples 1.4.6 and 1.4.7, by simply studying that of their Lie algebras. As an illustration, we provide here the de Rham cohomology groups of the Kodaira-Thurston manifold.

**Example 1.4.12.** Using the structure equations given in Example 1.4.6 to compute $H^k_{\mathrm{dR}}(\mathfrak{g})$, for $k = 1, 2, 3, 4$, and applying Theorem 1.4.11, it is easy to see:

$$H^1_{\mathrm{dR}}(\mathbb{KT}) = \langle\, [e^1],\, [e^2],\, [e^3]\,\rangle, \quad H^2_{\mathrm{dR}}(\mathbb{KT}) = \langle\, [e^{13}],\, [e^{14}],\, [e^{23}],\, [e^{24}]\,\rangle,$$
$$H^3_{\mathrm{dR}}(\mathbb{KT}) = \langle\, [e^{124}],\, [e^{134}],\, [e^{234}]\,\rangle, \quad H^4_{\mathrm{dR}}(\mathbb{KT}) = \langle\, [e^{1234}]\,\rangle.$$

◇

We should mention that Theorem 1.4.11 also holds when we consider the complex de Rham cohomologies of a nilmanifold and its associated Lie algebra,

$$H^k_{\mathrm{dR}}(\mathfrak{g};\mathbb{C}) = \frac{\ker\{d : \bigwedge^k(\mathfrak{g}^*_{\mathbb{C}}) \longrightarrow \bigwedge^{k+1}(\mathfrak{g}^*_{\mathbb{C}})\}}{\mathrm{im}\{d : \bigwedge^{k-1}(\mathfrak{g}^*_{\mathbb{C}}) \longrightarrow \bigwedge^k(\mathfrak{g}^*_{\mathbb{C}})\}}.$$

In order to study the Lie-algebra counterparts of those cohomologies related to complex manifolds, one first needs to consider a $2n$-dimensional Lie algebra $\mathfrak{g}$ endowed with a complex structure $J$. Recall that $J$ induces a bigraduation in the space of complex forms, so the Dolbeault cohomogy groups of $(\mathfrak{g}, J)$ can be defined by

$$H^{p,q}_{\bar\partial}(\mathfrak{g}, J) = \frac{\ker\{\bar\partial : \bigwedge^{p,q}(\mathfrak{g}^*) \longrightarrow \bigwedge^{p,q+1}(\mathfrak{g}^*)\}}{\mathrm{im}\{\bar\partial : \bigwedge^{p,q-1}(\mathfrak{g}^*) \longrightarrow \bigwedge^{p,q}(\mathfrak{g}^*)\}},$$

where $1 \le p, q \le 2n$. If $\mathfrak{g}$ is the Lie algebra associated to a nilmanifold $M = \Gamma\backslash G$, one can endow $M$ with the induced invariant complex structure $J$. The inclusion $\bigwedge^{p,q}(\mathfrak{g}^*) \hookrightarrow \Omega^{p,q}(M)$, for $1 \le p, q \le 2n$, induces a natural map

$$\iota\colon H^{p,q}_{\bar\partial}(\mathfrak{g}, J) \longrightarrow H^{p,q}_{\bar\partial}(M, J). \tag{1.12}$$

Using the symmetrization process one can check that $\iota$ is injective. One would like see when this is an isomorphism, in order to obtain a similar result to Theorem 1.4.11 for the Dolbeault cohomology. Different authors have worked in the issue along the years. We collect here some of their conclusions:

**Theorem 1.4.13.** *Let $M = \Gamma\backslash G$ be a nilmanifold endowed with an invariant complex structure $J$, and let $\mathfrak{g}$ be the Lie algebra of $G$. Then, the map* (1.12) *is an isomorphism in the following cases:*

*i) J is abelian* [CF01]*;*

*ii) J is complex-parallelizable* [Sak76]*;*

*iii) J is rational, i.e.,* $J(\mathfrak{g}_{\mathbb{Q}}) \subset \mathfrak{g}_{\mathbb{Q}}$*, where* $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g} \otimes \mathbb{Q}$ [CF01]*;*

*iv) the manifold* $(M, J)$ *is an iterated principal holomorphic torus bundle* [CFGU00]*;*

*v) the Lie algebra* $\mathfrak{g}$ *admits a torus bundle series compatible with* $J$ *and the rational structure induced by* $\Gamma$ [Rol09a]*.*

It is clear that this theorem holds for all the examples of complex nilmanifolds presented up to this point, namely, complex tori, the Kodaira-Thurston manifold, and the Iwasawa manifold.

**Example 1.4.14.** The Dolbeault cohomology of the Kodaira-Thurston manifold can be obtained through the complex structure equations given in Example 1.4.6:

$$H^{1,0}_{\bar\partial}(\mathbb{KT}) = \langle\, [\omega^1]\,\rangle, \quad H^{0,1}_{\bar\partial}(\mathbb{KT}) = \langle\, [\omega^{\bar 1}],\, [\omega^{\bar 2}]\,\rangle,$$

$$H^{2,0}_{\bar\partial}(\mathbb{KT}) = \langle\, [\omega^{12}]\,\rangle, \quad H^{1,1}_{\bar\partial}(\mathbb{KT}) = \langle\, [\omega^{1\bar 2}],\, [\omega^{2\bar 1}]\,\rangle, \quad H^{2,2}_{\bar\partial}(\mathbb{KT}) = \langle\, [\omega^{12\bar 1\bar 2}]\,\rangle.$$

By the Serre Duality Theorem, the rest of the groups are isomorphic to those above. ⋄

**Remark 1.4.15.** The Hodge numbers of the Iwasawa manifold had been known for a long time. In fact, Nakamura computes in [Nak75] the Kuranishi space of the Iwasawa manifold and divides it into three classes according to the Hodge numbers $h^{p,q}$. This allowed him to show that the values of $h^{p,q}$ are not invariant under small deformations, unlike the Betti numbers. Moreover, he notes that a holomorphic deformation of a complex-parallelizable structure might not be complex-parallelizable.

In addition, notice that the isomorphism between the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(\mathfrak{g}, J)$ and $H^{p,q}_{\bar\partial}(M, J)$ leads to other interesting results. On the one hand, Angella proves the following Nomizu type theorem for the Bott-Chern cohomology:

**Theorem 1.4.16.** [Ang13] *Let* $M = \Gamma\backslash G$ *be a nilmanifold endowed with an invariant complex structure* $J$ *and* $\mathfrak{g}$ *the Lie algebra associated to* $G$*. If the map* (1.12) *is an isomorphism, then the natural map*

$$H^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J) \longrightarrow H^{p,q}_{\mathrm{BC}}(M, J)$$

*between the Lie-algebra Bott-Chern cohomology of* $(\mathfrak{g}, J)$ *and the Bott-Chern cohomology of* $M$ *is also an isomorphism.*

Simply note that the Bott-Chern cohomology of $(\mathfrak{g}, J)$ is defined in the usual way,

$$H^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J) = \frac{\ker\{\partial + \bar\partial : \bigwedge^{p,q}(\mathfrak{g}^*) \longrightarrow \bigwedge^{p+1,q}(\mathfrak{g}^*) \oplus \bigwedge^{p,q+1}(\mathfrak{g}^*)\}}{\mathrm{im}\{\bar\partial : \bigwedge^{p,q-1}(\mathfrak{g}^*) \longrightarrow \bigwedge^{p,q}(\mathfrak{g}^*)\}}.$$

We should remark that Theorem 1.4.16 indeed holds for a bigger class of manifolds, although the previous result will be enough for our purposes.

**Example 1.4.17.** The Bott-Chern cohomology groups of the Kodaira-Thurston manifold are given by:

$$H^{1,0}_{\mathrm{BC}}(\mathbb{KT}) = \langle\, [\omega^1]\,\rangle, \quad H^{1,1}_{\mathrm{BC}}(\mathbb{KT}) = \langle\, [\omega^{1\bar{1}}],\, [\omega^{1\bar{2}}],\, [\omega^{2\bar{1}}]\,\rangle,$$

$$H^{2,0}_{\mathrm{BC}}(\mathbb{KT}) = \langle\, [\omega^{12}]\,\rangle, \quad H^{2,1}_{\mathrm{BC}}(\mathbb{KT}) = \langle\, [\omega^{12\bar{1}}],\, [\omega^{12\bar{2}}]\,\rangle, \quad H^{2,2}_{\mathrm{BC}}(\mathbb{KT}) = \langle\, [\omega^{12\bar{1}\bar{2}}]\,\rangle.$$

The remaining groups are obtained by conjugation. ◇

On the other hand, such an equivalence for a nilmanifold $(M, J)$ has interesting consequences for the holomorphic deformations in $\mathrm{Kur}(M, J)$.

**Theorem 1.4.18.** [CF01, Rol09b] *Any sufficiently small deformation $\{J_{\mathbf{t}}\}$ of an invariant complex structure $J$ for which the canonical map* (1.12) *is an isomorphism, is still invariant.*

Concerning the problem of cohomological decomposition (Section 1.2.3) we have the following Nomizu type result, which comes straightforward from the symmetrization process (see also [ATZ14]):

**Proposition 1.4.19.** *Let $J$ be an invariant (almost-)complex structure on a nilmanifold $M$. The restriction to $H^{p,q}_J(\mathfrak{g})$ of the isomorphism $\iota\colon H^{p+q}_{\mathrm{dR}}(\mathfrak{g};\mathbb{C}) \longrightarrow H^{p+q}_{\mathrm{dR}}(M;\mathbb{C})$ is an isomorphism onto $H^{p,q}_J(M)$, with inverse mapping $\sim\colon H^{p,q}_J(M) \longrightarrow H^{p,q}_J(\mathfrak{g})$ given by the symmetrization process.*

We also recall that nilmanifolds endowed with invariant complex structures do not satisfy the $\partial\bar{\partial}$-lemma property, with the exception of complex tori. The reason is that $\partial\bar{\partial}$-manifolds are formal [DGMS75], but only complex tori satisfy this condition [Has89].

### 1.4.3 Classifications in dimensions four and six

In this section, we present the classifications of invariant complex structures on 4- and 6-dimensional nilmanifolds. We also recall some of the contributions that were attained thanks to these low dimensional complex nilmanifolds in the study of cohomologies and Hermitian metrics.

As a consequence of Section 1.4.1, the search of invariant complex structures on nilmanifolds can be accomplished at the Lie algebra level. In fact, it suffices to apply Theorem 1.4.10 in order to find the corresponding complex structure equations. However, this is not as easy as it might seem.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension $2n$. The first non-trivial case is $n = 2$. Applying the result by Salamon together with a change of basis, it is not difficult to check that every complex structure $J$ on a 4-dimensional nilpotent Lie algebra is given, up to equivalence, by:

$$d\omega^1 = 0, \quad d\omega^2 = \varepsilon\, \omega^{1\bar{1}},$$

where $\varepsilon \in \{0, 1\}$. In fact, $\varepsilon = 0$ corresponds to a complex torus and $\varepsilon = 1$ to the Kodaira-Thurston manifold (see Example 1.4.6). Hasegawa proves in [Has05] that every complex structure on a 4-dimensional nilmanifold is invariant, so few complex nilmanifolds exist in this dimension. Nonetheless, the Kodaira-Thurston manifold was the first example of a non-Kähler symplectic manifold. Later, it was indeed discovered that the only Kähler nilmanifolds are the complex tori (see [BG88]). Without any doubt, these facts motivated the interest in the class of complex nilmanifolds.

If we now consider the case $n = 3$ and span the equations in Theorem 1.4.10, fourteen complex parameters are obtained. In addition, these parameters are not free: they should satisfy some relations coming from the nilpotency of $\mathfrak{g}$ and $d \circ d = 0$. Nevertheless, the following was proved:

**Theorem 1.4.20.** [Sal01] *A 6-dimensional nilpotent Lie algebra admits a complex structure if and only if it is isomorphic to one of the following Lie algebras:*

$$\begin{aligned}
\mathfrak{h}_1 &= (0,0,0,0,0,0), & \mathfrak{h}_{10} &= (0,0,0,12,13,14),\\
\mathfrak{h}_2 &= (0,0,0,0,12,34), & \mathfrak{h}_{11} &= (0,0,0,12,13,14+23),\\
\mathfrak{h}_3 &= (0,0,0,0,0,12+34), & \mathfrak{h}_{12} &= (0,0,0,12,13,24),\\
\mathfrak{h}_4 &= (0,0,0,0,12,14+23), & \mathfrak{h}_{13} &= (0,0,0,12,13+14,24),\\
\mathfrak{h}_5 &= (0,0,0,0,13+42,14+23), & \mathfrak{h}_{14} &= (0,0,0,12,14,13+42),\\
\mathfrak{h}_6 &= (0,0,0,0,12,13), & \mathfrak{h}_{15} &= (0,0,0,12,13+42,14+23),\\
\mathfrak{h}_7 &= (0,0,0,12,13,23), & \mathfrak{h}_{16} &= (0,0,0,12,14,24),\\
\mathfrak{h}_8 &= (0,0,0,0,0,12), & \mathfrak{h}_{19}^- &= (0,0,0,12,23,14-35),\\
\mathfrak{h}_9 &= (0,0,0,0,12,14+25), & \mathfrak{h}_{26}^+ &= (0,0,12,13,23,14+25).
\end{aligned}$$

**Notation 1.4.21.** For instance, $\mathfrak{h}_2 = (0,0,0,0,12,34)$ means that there is a basis $\{e^i\}_{i=1}^6$ of real 1-forms such that $de^1 = de^2 = de^3 = de^4 = 0,\ de^5 = e^1 \wedge e^2,\ de^6 = e^3 \wedge e^4$.

Bearing in mind that the list of non-isomorphic 6-dimensional real Lie algebras contains 34 elements, the previous result supposed an important step. However, the problem of finding every complex structure on each of these algebras was open for some years. The main strategy consisted on applying subsequent changes of basis to the basis for $\mathfrak{g}^{1,0}$ obtained in Theorem 1.4.10, in order to get the most reduced version of the complex structure equations. Notice that this involves eliminating the cases in which one obtains equivalent complex structures on a Lie algebra and then, detecting which values of the complex structure constants correspond to each real Lie algebra in Theorem 1.4.20. Some papers dealt with the issue (see for instance [ABD11, Uga07, UV14]), until the classification was finally completed in [COUV16]. We briefly present it here, as it will be of special interest to us.

The only two complex-parallelizable structures are encoded in the equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12}, \tag{1.13}$$

with $\rho = 0$ or 1. The underlying Lie algebras are $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$). The former corresponds to the complex torus and the latter, to the Iwasawa manifold (recall Example 1.4.7).

The remaining complex structures are parametrized by the following three families:

$$\text{Family I}: \left\{ d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \right. \tag{1.14}$$

where $\rho \in \{0,1\}$, $\lambda \in \mathbb{R}^{\geq 0}$ and $D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$;

$$\text{Family II}: \left\{ d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \right. \tag{1.15}$$

where $\rho \in \{0,1\}$, $B \in \mathbb{C}$, $c \in \mathbb{R}^{\geq 0}$ and $(\rho, B, c) \neq (0,0,0)$;

$$\text{Family III}: \left\{ d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \right. \tag{1.16}$$

where $\varepsilon \in \{0,1\}$.

Note that Family I corresponds to complex structures on $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$, or $\mathfrak{h}_8$ and Family II, on $\mathfrak{h}_7$ or $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$. The concrete values of the parameters for these two families can be found, respectively, in Tables A and B. Family III defines complex structures on $\mathfrak{h}_{19}^-$ for $\varepsilon = 0$ and on $\mathfrak{h}_{26}^+$ for $\varepsilon = 1$. Furthermore, a complex structure $J$ is abelian if and only if it belongs to Families I or II with $\rho = 0$.

In contrast to dimension 4, it is clear that not every invariant complex structure $J$ on a 6-dimensional nilmanifold $M$ is either abelian or complex-parallelizable. For instance, this has obvious consequences in the computation of their Dolbeault cohomology, as one first needs to verify one of the other hypothesis in Theorem 1.4.13. In this sense, Rollenske proves in [Rol09a] that the fifth condition of the cited theorem turns to be fulfilled by any pair $(M, J)$ whose associated Lie algebra $\mathfrak{g}$ is non-isomorphic to $\mathfrak{h}_7$ (the case $\mathfrak{g} \cong \mathfrak{h}_7$ remaining open). Furthermore, an induction argument shows in [CFGU99] that the Frölicher spectral sequence of those $(M, J)$ with Lie algebra $\mathfrak{g} \ncong \mathfrak{h}_7$ is isomorphic to that defined on $\mathfrak{g}$. This made possible to calculate the Frölicher spectral sequence of the Families I, II, and III in [COUV16]. As $E_1^{p,q}(M,J) \cong H_{\bar{\partial}}^{p,q}(M,J)$ for every $(p,q)$, also their Dolbeault cohomology was found.

**Remark 1.4.22.** Thanks to the general study accomplished in [COUV16], a complex (nil)manifold was found with $E_1 \cong E_\infty$, satisfying Hodge symmetry $h^{p,q} = h^{q,p}$ for $1 \leq p, q \leq n$, but not fulfilling the $\partial\bar{\partial}$-condition. This allowed to answer a question posed in [AT13]. In addition, all the cases in which the Frölicher spectral sequence degenerates at the first stage were obtained, thus providing a decomposition of their de Rham cohomology groups.

Note that the result by Rollenske together with Theorem 1.4.16 also allows to compute the Bott-Chern cohomology of every $(M, J)$ with $\mathfrak{g} \ncong \mathfrak{h}_7$ at the Lie algebra level. We will go back to this issue in Chapter 2.

However, 6-dimensional nilmanifolds $M$ with invariant complex structures $J$ are not only interesting for their cohomological properties. As we have said, they are non-Kähler manifolds, so they constitute a natural class in which considering other more

| Family I | | | |
|---|---|---|---|
| $\mathfrak{g}$ | $\rho$ | $\lambda$ | $D$ |
| $\mathfrak{h}_2$ | 0 | 0 | $\mathfrak{Im}D = 1$ |
| | 1 | 1 | $\mathfrak{Im}D > 0$ |
| $\mathfrak{h}_3$ | 0 | 0 | $\pm 1$ |
| $\mathfrak{h}_4$ | 0 | 1 | $\frac{1}{4}$ |
| | 1 | 1 | $D \in \mathbb{R} - \{0\}$ |
| $\mathfrak{h}_5$ | 0 | 1 | $D \in [0, \frac{1}{4})$ |
| | 1 | 0 | $\mathfrak{Im}D \geq 0$<br>$4(\mathfrak{Im}D)^2 < 1 + 4\mathfrak{Re}D$ |
| | | $0 < \lambda^2 < \frac{1}{2}$ | $0 \leq \mathfrak{Im}D < \frac{\lambda^2}{2}$<br>$\mathfrak{Re}D = 0$ |
| | | $\frac{1}{2} \leq \lambda^2 < 1$ | $0 \leq \mathfrak{Im}D < \frac{1-\lambda^2}{2}$<br>$\mathfrak{Re}D = 0$ |
| | | $\lambda^2 > 1$ | $0 \leq \mathfrak{Im}D < \frac{\lambda^2-1}{2}$<br>$\mathfrak{Re}D = 0$ |
| $\mathfrak{h}_6$ | 1 | 1 | 0 |
| $\mathfrak{h}_8$ | 0 | 0 | 0 |

Table A

| Family II | | | |
|---|---|---|---|
| $\mathfrak{g}$ | $\rho$ | $B$ | $c$ |
| $\mathfrak{h}_7$ | 1 | 1 | 0 |
| $\mathfrak{h}_9$ | 0 | 1 | 1 |
| $\mathfrak{h}_{10}$ | 1 | 0 | 1 |
| $\mathfrak{h}_{11}$ | 1 | $B \in \mathbb{R} - \{0, 1\}$ | $\lvert B - 1\rvert$ |
| $\mathfrak{h}_{12}$ | 1 | $\mathfrak{Im}B \neq 0$ | $\lvert B - 1\rvert$ |
| $\mathfrak{h}_{13}$ | 1 | $(c, \lvert B\rvert) \neq (0, 1), \quad c \neq \lvert B - 1\rvert$<br>$\mathcal{S}(B, c) < 0$ | |
| $\mathfrak{h}_{14}$ | 1 | $(c, \lvert B\rvert) \neq (0, 1), \quad c \neq \lvert B - 1\rvert$<br>$\mathcal{S}(B, c) = 0$ | |
| $\mathfrak{h}_{15}$ | 0 | 0 | 1 |
| | | 1 | $c \neq 1$ |
| | 1 | $(c, \lvert B\rvert) \neq (0, 1), \quad c \neq \lvert B - 1\rvert$<br>$\mathcal{S}(B, c) > 0$ | |
| $\mathfrak{h}_{16}$ | 1 | $\lvert B\rvert = 1, \ B \neq 1$ | 0 |

where $\mathcal{S}(B, c) = c^4 - 2\,(|B|^2 + 1)\,c^2 + (|B|^2 - 1)^2$.

Table B

general types of Hermitian metrics. In fact, they have proved to be quite useful in dealing with some problems and conjectures involving the existence of these metrics. For instance, the counterexamples for the openness of the balanced and the SKT properties (Theorems 1.3.10 and 1.3.19) involved small deformations of the Iwasawa manifold. Those for the non-closedness of the balanced and strongly Gauduchon properties (Theorems 1.3.11 and 1.3.16) were based on an appropriate holomorphic deformation of the abelian complex structure on the Lie algebra $\mathfrak{h}_4$.

Also complete studies concerning the existence of Hermitian metrics have been accomplished for all complex nilmanifolds $(M, J)$ of real dimension 6. For instance, those involving balanced metrics can be found in [UV14] and [UV15], where they were applied to finding solutions to the Strominger system. The strongly Gauduchon case was considered in [COUV16, Proposition 5.3], and $\mathfrak{h}_{19}^-$ arised as the first space in which one could find metrics making all the inclusions in Figure 1.1 (p. 19) strict. Six-nilmanifolds endowed with invariant complex structures have also been of interest in relation to the metrics given in Figure 1.2. We give the details in Chapter 5, where we address some related questions in higher dimension. Also pseudo-Kähler and holomorphic symplectic structures will be considered.

## Chapter 2

# Cohomological aspects of six-dimensional nilmanifolds

As we have seen in Chapter 1, metric aspects of 6-dimensional nilmanifolds $M = G\backslash\Gamma$ endowed with invariant complex structures $J$ have been widely studied. In contrast, some features of their cohomological properties are less known. In this chapter we focus on them, showing that these compact complex manifolds provide examples of complex dimension 3 satisfying interesting properties.

Notice that the Betti numbers of 6-dimensional nilmanifolds have been known for a long time, as a consequence of the Nomizu theorem together with the classification of non-isomorphic 6-dimensional nilpotent Lie algebras [Mag86, Mor58, Nom54]. Without any doubt, the fact that the de Rham cohomology is indeed a topological invariant plays a crucial role. In contrast, the Dolbeault and the Bott-Chern cohomologies are complex invariants of the manifold, so one first needs to study the space of invariant complex structures $J$ on each 6-nilmanifold $M$ in order to compute them. An effective approach to find these $J$'s consists on *parametrizing* them as complex structures defined on the underlying six dimensional nilpotent Lie algebras.

The classification, up to equivalence, of every $J$ on the different nilpotent Lie algebras $\mathfrak{g}$ that can be associated to each 6-dimensional nilpotent Lie group $G$ was finally completed in [COUV16] (see also [ABD11, Uga07, UV14]). There, among other applications, the classification is used to compute the Frölicher spectral sequence $\{E_r(M, J)\}_{r\geq 1}$ of any pair $(M, J)$, showing interesting behaviours. In particular, their Dolbeault cohomology was explicitly obtained, since $E_1^{p,q}(M, J) \cong H_{\bar\partial}^{p,q}(M, J)$ for every $(p, q)$.

Thanks to a recent theorem by Angella [Ang13] and applying [Rol09a], the cited classification can also be used to calculate the Bott-Chern cohomology groups of these 6-dimensional complex nilmanifolds $(M, J)$. The results can be found in Section 2.1.1. In addition, we see how these groups might cast some light into the connection between strongly Gauduchon and Gauduchon metrics. In Section 2.1.2 we introduce some invariants involving the previous cohomology and study their stability under holomorphic deformation. Special attention is paid to the relation of these invariants with the existence of some special Hermitian metrics.

In the last part of the chapter, we address the problem of cohomological decomposition. Using the classification [COUV16], we study complex-$\mathcal{C}^\infty$-pure-and-fullness at every stage for 6-dimensional nilmanifolds with invariant complex structures $(M, J)$. On the one hand, this gives a complete overview of the topic for a large class of manifolds, in contrast to other approaches mainly based on concrete examples. On the other hand, this enables us to identify new (complex-) $\mathcal{C}^\infty$-pure-and-full manifolds and a new compact complex manifold for which the cohomological decomposition holds at every stage. Finally, we concentrate on the second stage and study pureness and fullness under holomorphic deformation of the complex structure. We also investigate some relations with other metric and complex properties.

Let us finally remark that the results of this chapter are a selection of those appearing in the papers [ACL15, LOUV13, LU, LU15, LUV14a, LUV14b].

## 2.1 Bott-Chern cohomology

We start our study of cohomological properties of 6-dimensional nilmanifolds endowed with invariant complex structures with the Bott-Chern cohomology. The results of this section are contained in [LUV14b], where some applications to Type IIB String Theory can also be found.

### 2.1.1 Cohomology groups and special Hermitian metrics

In this part, we provide the dimensions of the Bott-Chern cohomology groups for 6-dimensional nilmanifolds endowed with invariant complex structures. We also study the injectivity of the natural map between the Dolbeault and the Aeppli cohomologies in relation to the existence of some special Hermitian metrics.

Let $M = G\backslash\Gamma$ be a $2n$-dimensional nilmanifold endowed with an invariant complex structure $J$. We start with the following observation. As a consequence of Salamon's theorem [Sal01], it is possible to find a nowhere vanishing closed $(n, 0)$-form $\Psi$ on the nilmanifold $M$. Therefore, one immediately has $h^{n,0}_{\mathrm{BC}}(M, J) = 1$, for every invariant complex structure $J$ on $M$. In addition, if $\{\omega^k\}_{k=1}^n$ is a $(1, 0)$-basis for the nilpotent Lie algebra $\mathfrak{g}$ of $G$, then the $(p, 0)$-form $\omega^1 \wedge \cdots \wedge \omega^p$ is closed on $M$ for every $1 \leq p \leq n$. Since

$$H^{p,0}_{\mathrm{BC}}(M) = \ker\{d\colon \Omega^{p,0}(M) \longrightarrow \Omega^{p+1}(M)\},$$

this form necessarily defines a non-zero cohomology class, and one can then conclude $h^{p,0}_{\mathrm{BC}}(M, J) \geq 1$, for all $p$.

Recall that the Bott-Chern cohomology of $(M, J)$ can be computed at the level of the Lie algebra $\mathfrak{g}$ of $G$, according to Theorem 1.4.16. In the particular case of dimension six, one just needs to assume $\mathfrak{g} \ncong \mathfrak{h}_7$ in order to apply this result for any $J$ on $\mathfrak{g}$ (recall Section 1.4.3).

The Bott-Chern cohomology of invariant complex-parallelizable structures on 6-dimensional nilmanifolds is already known. While the case of the complex torus is straightforward, one can find the Bott-Chern cohomology groups of the Iwasawa manifold in [Sch]. Moreover, the paper [Ang13] contains the Bott-Chern cohomology of its small deformations, some of which are no longer complex-parallelizable manifolds.

Let us observe that these deformations are particular cases of complex structures living on the 6-dimensional nilpotent Lie algebra $\mathfrak{h}_5$, as a consequence of Theorem 1.4.18. Our aim is extending the previous study to any complex structure $J$ defined on any nilpotent Lie algebra $\mathfrak{g}$ of dimension six. The invariance of the Bott-Chern cohomology under equivalence of complex structures makes possible to use the three families obtained in [COUV16] (see Section 1.4) to do this.

The explicit description of the groups $H^{p,q}_{\mathrm{BC}}(\mathfrak{g})$ appears in [LUV14b] (the computations can be found in [Lat12]). In this work, we only give the dimensions $h^{p,q}_{\mathrm{BC}}(\mathfrak{g}, J)$ for each pair $(\mathfrak{g}, J)$. They are contained in Tables 2.1, 2.2, and 2.3. We omit those corresponding to the torus and the Iwasawa manifold. Simply recall that different values of the parameters correspond to non-isomorphic complex structures. Moreover, since $h^{3,0}_{\mathrm{BC}} = 1 = h^{3,3}_{\mathrm{BC}}$ and by the duality in the Bott-Chern cohomology, it suffices to show the dimensions $h^{p,q}_{\mathrm{BC}}$ for $(p, q) = (1, 0), (2, 0), (1, 1), (2, 1), (3, 1), (2, 2)$, and $(3, 2)$. Clearly, this also gives the dimensions of the Aeppli cohomology as a consequence of the duality $h^{p,q}_{\mathrm{A}} = h^{n-q,n-p}_{\mathrm{BC}}$.

**Remark 2.1.1.** Angella, Franzini, and Rossi obtain in [AFR15] similar computations for these cohomology groups. In their paper, the invariants $h^{p,q}_{\mathrm{BC}}$ are used to provide a measure of the degree of non-Kählerianity of 6-dimensional nilmanifolds with invariant complex structure. They also study the relation between Bott-Chern cohomological properties and the existence of SKT metrics.

Let $X$ be a compact complex manifold of complex dimension $n$, endowed with a Hermitian metric (whose fundamental form is) $F$. Many of the special metrics defined in Chapter 1 are characterized in terms of the closedness of some power of $F$ under $\partial$, $\bar\partial$, or some combination of the two. Hence, trying to relate their existence with some cohomological aspects of $X$ seems something natural.

In this sense, we focus on the natural map

$$H^{n,n-1}_{\bar\partial}(X) \longrightarrow H^{n,n-1}_{\mathrm{A}}(X) \tag{2.1}$$

and see that it can cast some light into the connection between strongly Gauduchon and Gauduchon metrics. Let us develop this idea in the next lines.

It is clear that any strongly Gauduchon metric is Gauduchon. Hence, it seems natural to look at the inverse problem. Let $F$ be a Gauduchon metric. Then $\partial\bar\partial F^{n-1} = 0$, so $\partial F^{n-1}$ defines a cohomology class in $H^{n,n-1}_{\bar\partial}(X)$. Applying (2.1), one gets $\mathbf{0} = [\partial F^{n-1}]_{\mathrm{A}} \in H^{n,n-1}_{\mathrm{A}}(X)$. If we assume the additional hypothesis of (2.1) being injective, we obtain $\mathbf{0} = [\partial F^{n-1}]_{\bar\partial}$ and there is some complex $(n, n-2)$-form $\alpha$ on $X$ such that $\partial F^{n-1} = \bar\partial\alpha$. That is, $F$ is strongly Gauduchon. Since there is a Gauduchon metric in the conformal class of any Hermitian metric, the injectivity of (2.1) leads to a similar conclusion for the existence of strongly Gauduchon metrics.

By the Serre duality and the dualities between the Aeppli and the Bott-Chern cohomologies, the injectivity of (2.1) also implies

$$h^{0,1}(X) = \dim H^{n,n-1}_{\bar\partial}(X) \leq \dim H^{n,n-1}_{\mathrm{A}}(X) = h^{0,1}_{\mathrm{BC}}(X).$$

From here and Angella's Theorem, we get the following.

<table>
<tr><th></th><th colspan="4">Family I</th><th colspan="7">Bott-Chern numbers</th></tr>
<tr><th>$\mathfrak{g}$</th><th>$\rho$</th><th>$\lambda$</th><th colspan="2">$D = x + iy$</th><th>$h^{1,0}_{\mathrm{BC}}$</th><th>$h^{2,0}_{\mathrm{BC}}$</th><th>$h^{1,1}_{\mathrm{BC}}$</th><th>$h^{2,1}_{\mathrm{BC}}$</th><th>$h^{3,1}_{\mathrm{BC}}$</th><th>$h^{2,2}_{\mathrm{BC}}$</th><th>$h^{3,2}_{\mathrm{BC}}$</th></tr>
<tr><td rowspan="5">$\mathfrak{h}_2$</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">$y = 1$</td><td>$x = 0$</td><td rowspan="2">2</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">6</td><td rowspan="2">3</td><td>7</td><td rowspan="2">3</td></tr>
<tr><td>$x \neq 0$</td><td>6</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td><td rowspan="3">$y > 0$</td><td>$x = -1 \pm \sqrt{1-y^2}$</td><td rowspan="3">2</td><td rowspan="3">1</td><td>5</td><td rowspan="3">6</td><td rowspan="3">2</td><td rowspan="2">6</td><td rowspan="3">3</td></tr>
<tr><td>$x \neq -1 \pm \sqrt{1-y^2}$<br>$x \neq 1$</td><td rowspan="2">4</td></tr>
<tr><td>$x = 1$</td><td>7</td></tr>
<tr><td>$\mathfrak{h}_3$</td><td>0</td><td>0</td><td colspan="2">$\pm 1$</td><td>2</td><td>1</td><td>4</td><td>6</td><td>3</td><td>7</td><td>3</td></tr>
<tr><td rowspan="4">$\mathfrak{h}_4$</td><td>0</td><td>1</td><td colspan="2">$\frac{1}{4}$</td><td>2</td><td>1</td><td>4</td><td>6</td><td>3</td><td>6</td><td>3</td></tr>
<tr><td rowspan="3">1</td><td rowspan="3">1</td><td colspan="2">$-2$</td><td rowspan="3">2</td><td rowspan="3">1</td><td>5</td><td rowspan="3">6</td><td rowspan="3">2</td><td rowspan="2">6</td><td rowspan="3">3</td></tr>
<tr><td colspan="2">$D \in \mathbb{R} - \{-2, 0, 1\}$</td><td rowspan="2">4</td></tr>
<tr><td colspan="2">1</td><td>7</td></tr>
<tr><td rowspan="16">$\mathfrak{h}_5$</td><td rowspan="2">0</td><td rowspan="2">1</td><td colspan="2">0</td><td rowspan="2">2</td><td>2</td><td>6</td><td rowspan="2">6</td><td rowspan="2">3</td><td rowspan="2">6</td><td rowspan="2">3</td></tr>
<tr><td colspan="2">$D \in \left(0, \frac{1}{4}\right)$</td><td>1</td><td>4</td></tr>
<tr><td rowspan="14">1</td><td rowspan="5">0</td><td rowspan="3">$y = 0$</td><td>$x = 0$</td><td rowspan="14">2</td><td>2</td><td rowspan="13">4</td><td rowspan="14">6</td><td rowspan="14">2</td><td>7</td><td rowspan="14">3</td></tr>
<tr><td>$x = \frac{1}{2}$</td><td rowspan="4">1</td><td>8</td></tr>
<tr><td>$x \neq 0, \frac{1}{2}, \quad x > -\frac{1}{4}$</td><td rowspan="2">7</td></tr>
<tr><td>$0 < y^2 < \frac{3}{4}$</td><td>$x = \frac{1}{2}$</td></tr>
<tr><td>$y > 0$</td><td>$x \neq \frac{1}{2}, \quad x > y^2 - \frac{1}{4}$</td><td rowspan="10">6</td></tr>
<tr><td rowspan="2">$0 < \lambda^2 < \frac{1}{2}$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2}{2}$</td><td>1</td></tr>
<tr><td rowspan="2">$\frac{1}{2} \leq \lambda^2 < 1$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{1-\lambda^2}{2}$</td><td>1</td></tr>
<tr><td rowspan="2">$1 < \lambda^2 \leq 5$</td><td rowspan="2">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2}$</td><td>1</td></tr>
<tr><td rowspan="3">$\lambda^2 > 5$</td><td rowspan="3">$x = 0$</td><td>$y = 0$</td><td>2</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2}, \ y \neq \sqrt{\lambda^2 - 1}$</td><td rowspan="2">1</td></tr>
<tr><td>$0 < y < \frac{\lambda^2-1}{2}, \ y = \sqrt{\lambda^2 - 1}$</td><td>5</td></tr>
<tr><td>$\mathfrak{h}_6$</td><td>1</td><td>1</td><td colspan="2">0</td><td>2</td><td>2</td><td>5</td><td>6</td><td>2</td><td>6</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_8$</td><td>0</td><td>0</td><td colspan="2">0</td><td>2</td><td>2</td><td>6</td><td>7</td><td>3</td><td>8</td><td>3</td></tr>
</table>

Table 2.1: Dimensions of the Bott-Chern cohomology groups for Family I.

<table>
<tr><th></th><th colspan="3">Family II</th><th colspan="7">Bott-Chern numbers</th></tr>
<tr><th>$\mathfrak{g}$</th><th>$\rho$</th><th>$B$</th><th>$c$</th><th>$h^{1,0}_{\mathrm{BC}}$</th><th>$h^{2,0}_{\mathrm{BC}}$</th><th>$h^{1,1}_{\mathrm{BC}}$</th><th>$h^{2,1}_{\mathrm{BC}}$</th><th>$h^{3,1}_{\mathrm{BC}}$</th><th>$h^{2,2}_{\mathrm{BC}}$</th><th>$h^{3,2}_{\mathrm{BC}}$</th></tr>
<tr><td>$\mathfrak{h}_7$</td><td>1</td><td>1</td><td>0</td><td>1</td><td>2</td><td>5</td><td>6</td><td>2</td><td>5</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_9$</td><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td><td>4</td><td>5</td><td>3</td><td>6</td><td>3</td></tr>
<tr><td>$\mathfrak{h}_{10}$</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td><td>4</td><td>5</td><td>2</td><td>5</td><td>3</td></tr>
<tr><td rowspan="2">$\mathfrak{h}_{11}$</td><td rowspan="2">1</td><td>$B \in \mathbb{R} - \{0, \frac{1}{2}, 1\}$</td><td>$|B-1|$</td><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">5</td><td rowspan="2">2</td><td>5</td><td rowspan="2">3</td></tr>
<tr><td>$\frac{1}{2}$</td><td>$\frac{1}{2}$</td><td>6</td></tr>
<tr><td rowspan="2">$\mathfrak{h}_{12}$</td><td rowspan="2">1</td><td>$\mathfrak{Re}B \neq \frac{1}{2},\ \mathfrak{Im}B \neq 0$</td><td rowspan="2">$|B-1|$</td><td rowspan="2">1</td><td rowspan="2">1</td><td rowspan="2">4</td><td rowspan="2">5</td><td rowspan="2">2</td><td>5</td><td rowspan="2">3</td></tr>
<tr><td>$\mathfrak{Re}B = \frac{1}{2},\ \mathfrak{Im}B \neq 0$</td><td>6</td></tr>
<tr><td rowspan="4">$\mathfrak{h}_{13}$</td><td rowspan="4">1</td><td rowspan="2">1</td><td>$0 < c < 2,\ c \neq 1$</td><td rowspan="4">1</td><td rowspan="4">1</td><td rowspan="2">5</td><td rowspan="4">5</td><td rowspan="4">2</td><td>5</td><td rowspan="4">3</td></tr>
<tr><td>1</td><td>6</td></tr>
<tr><td colspan="2">$B \neq 1,\ c \neq |B|, |B-1|,$<br>$(c, |B|) \neq (0,1),$<br>$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 < 0$</td><td rowspan="2">4</td><td>5</td></tr>
<tr><td colspan="2">$B \neq 1,\ c = |B| > \frac{1}{2},$<br>$|B| \neq |B-1|$</td><td>6</td></tr>
<tr><td rowspan="3">$\mathfrak{h}_{14}$</td><td rowspan="3">1</td><td>1</td><td>2</td><td rowspan="3">1</td><td rowspan="3">1</td><td>5</td><td rowspan="3">5</td><td rowspan="3">2</td><td>5</td><td rowspan="3">3</td></tr>
<tr><td>$|B| = \frac{1}{2}$</td><td>$\frac{1}{2}$</td><td rowspan="2">4</td><td>6</td></tr>
<tr><td colspan="2">$c \neq |B-1|,$<br>$(c, |B|) \neq (0,1),\ (\frac{1}{2}, \frac{1}{2}),\ (2,1),$<br>$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 = 0$</td><td>5</td></tr>
<tr><td rowspan="8">$\mathfrak{h}_{15}$</td><td rowspan="3">0</td><td>0</td><td>1</td><td rowspan="3">1</td><td rowspan="2">1</td><td>5</td><td rowspan="3">5</td><td rowspan="3">3</td><td rowspan="3">5</td><td rowspan="3">3</td></tr>
<tr><td rowspan="2">1</td><td>$c \neq 0,\ 1$</td><td rowspan="2">4</td></tr>
<tr><td>0</td><td>2</td></tr>
<tr><td rowspan="5">1</td><td>0</td><td>0</td><td rowspan="5">1</td><td rowspan="2">2</td><td rowspan="2">4</td><td rowspan="5">5</td><td rowspan="5">2</td><td>7</td><td rowspan="5">3</td></tr>
<tr><td>$|B| \neq 0, 1$</td><td>0</td><td rowspan="2">5</td></tr>
<tr><td>1</td><td>$c > 2$</td><td rowspan="3">1</td><td>5</td></tr>
<tr><td>$|B| = c$</td><td>$0 < c < \frac{1}{2}$</td><td rowspan="2">4</td><td>6</td></tr>
<tr><td colspan="2">$c \neq 0,\ |B-1|,$<br>$B \neq 1,\ |B| \neq c,$<br>$c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 > 0$</td><td>5</td></tr>
<tr><td>$\mathfrak{h}_{16}$</td><td>1</td><td>$|B| = 1,\ B \neq 1$</td><td>0</td><td>1</td><td>2</td><td>4</td><td>5</td><td>2</td><td>5</td><td>3</td></tr>
</table>

Table 2.2: Dimensions of the Bott-Chern cohomology groups for Family II.

| | Family III | Bott-Chern numbers | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\mathfrak{g}$ | $\varepsilon$ | $h^{1,0}_{\rm BC}$ | $h^{2,0}_{\rm BC}$ | $h^{1,1}_{\rm BC}$ | $h^{2,1}_{\rm BC}$ | $h^{3,1}_{\rm BC}$ | $h^{2,2}_{\rm BC}$ | $h^{3,2}_{\rm BC}$ |
| $\mathfrak{h}_{19}^-$ | 0 | 1 | 1 | 2 | 3 | 2 | 4 | 2 |
| $\mathfrak{h}_{26}^+$ | 1 | 1 | 1 | 2 | 3 | 2 | 3 | 2 |

Table 2.3: Dimensions of the Bott-Chern cohomology groups for Family III.

**Proposition 2.1.2.** *Let $M$ be a $2n$-dimensional nilmanifold (not a torus) endowed with an abelian complex structure $J$. Then, the map* (2.1) *is not injective.*

*Proof.* It suffices to show that if $J$ is abelian then $h^{0,1}(M,J) > h^{0,1}_{\rm BC}(M,J)$. By Theorem 1.4.16 we have

$$H^{0,1}_{\bar\partial}(M,J) \cong H^{0,1}_{\bar\partial}(\mathfrak{g},J) = \{\alpha_{0,1} \in \mathfrak{g}^{0,1} \mid \bar\partial\alpha_{0,1} = 0\} \cong \{\alpha_{1,0} \in \mathfrak{g}^{1,0} \mid \partial\alpha_{1,0} = 0\},$$

and also

$$H^{0,1}_{\rm BC}(M,J) \cong H^{0,1}_{\rm BC}(\mathfrak{g},J) = \{\alpha_{0,1} \in \mathfrak{g}^{0,1} \mid d\alpha_{0,1} = 0\} \cong \{\alpha_{1,0} \in \mathfrak{g}^{1,0} \mid d\alpha_{1,0} = 0\}.$$

As $J$ is abelian, it is clear that $\partial(\mathfrak{g}^{1,0}) = 0$ and thus $h^{0,1}(M,J) = n$. Hence, if $M$ is not a torus, we have $h^{0,1}_{\rm BC}(M,J) < n$ and (2.1) is not injective. □

In contrast, due to our study of the Bott-Chern cohomology of 6-dimensional nilmanifolds with invariant complex structure, we have the next result:

**Proposition 2.1.3.** *Let $M$ be a 6-dimensional nilmanifold endowed with a complex structure $J$ belonging to Family I. Then, the map* (2.1) *is injective if and only if $J$ is not abelian.*

*Proof.* Due to Proposition 2.1.2, it suffices to consider non-abelian complex structures. Hence, we notice that the Lie algebras underlying $M$ are $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$, because any $J$ on $\mathfrak{h}_1, \mathfrak{h}_3$, and $\mathfrak{h}_8$ is abelian. Since $\mathfrak{g} \ncong \mathfrak{h}_7$, we have $H^{3,2}_{\bar\partial}(M,J) \cong H^{3,2}_{\bar\partial}(\mathfrak{g},J)$ and $H^{3,2}_{\rm A}(M,J) \cong H^{3,2}_{\rm A}(\mathfrak{g},J)$. A direct calculation from the equations in Family I with $\rho = 1$ shows that

$$H^{3,2}_{\bar\partial}(\mathfrak{g},J) = \langle [\omega^{123\bar1\bar3}],\, [\omega^{123\bar2\bar3}] \rangle \quad \text{and} \quad H^{3,2}_{\rm A}(\mathfrak{g},J) = \langle [\omega^{123\bar1\bar3}],\, [\omega^{123\bar2\bar3}] \rangle,$$

so the natural map $H^{3,2}_{\bar\partial}(\mathfrak{g},J) \longrightarrow H^{3,2}_{\rm A}(\mathfrak{g},J)$ is injective. □

Let us recall that $M$ in the statement of Proposition 2.1.3 is 2-step and $J$ is always of nilpotent type. This result explains why on a nilmanifold with underlying Lie algebra isomorphic to $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$, or $\mathfrak{h}_6$, any invariant $J$-Hermitian metric with respect to a non-abelian nilpotent $J$ is strongly Gauduchon [COUV16, Proposition 7.3 and Remark 7.4]. In fact, as any invariant Hermitian metric is Gauduchon, the injectivity of (2.1) implies that it is automatically strongly Gauduchon.

### 2.1.2 Associated invariants and their deformation limits

We now introduce some complex invariants which are related to the $\partial\bar{\partial}$-lemma condition and defined in terms of the Bott-Chern cohomology groups. We use the Tables 2.1, 2.2, and 2.3 to show that the vanishing of some of these invariants is not a closed property under holomorphic deformations. In this way, we intended to suggest that satisfying the $\partial\bar{\partial}$-lemma might also be a non-closed property, a fact that was finally proven in [AK].

We start remarking that the invariants studied in this section involve not only the Bott-Chern numbers, but also the Betti numbers. Therefore, it is convenient to bear in mind the following table, containing their values for those 6-dimensional nilpotent Lie algebras which admit a complex structure (recall Theorem 1.4.10).

| | $\mathfrak{h}_1$ | $\mathfrak{h}_2$ | $\mathfrak{h}_3$ | $\mathfrak{h}_4$ | $\mathfrak{h}_5$ | $\mathfrak{h}_6$ | $\mathfrak{h}_7$ | $\mathfrak{h}_8$ | $\mathfrak{h}_9$ | $\mathfrak{h}_{10}$ | $\mathfrak{h}_{11}$ | $\mathfrak{h}_{12}$ | $\mathfrak{h}_{13}$ | $\mathfrak{h}_{14}$ | $\mathfrak{h}_{15}$ | $\mathfrak{h}_{16}$ | $\mathfrak{h}_{19}^-$ | $\mathfrak{h}_{26}^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b_1$ | 6 | 4 | 5 | 4 | 4 | 4 | 3 | 5 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 |
| $b_2$ | 15 | 8 | 9 | 8 | 8 | 9 | 8 | 11 | 7 | 6 | 6 | 6 | 5 | 5 | 5 | 5 | 5 | 4 |
| $b_3$ | 20 | 10 | 10 | 10 | 10 | 12 | 12 | 14 | 8 | 8 | 8 | 8 | 6 | 6 | 6 | 6 | 6 | 6 |

Table 2.4: Betti numbers of 6-dimensional NLAs admitting complex structures.

Let $X$ be a compact complex manifold of complex dimension $n$. In view of Theorem 1.2.5 and the duality between the Bott-Chern and Aeppli cohomologies, let us denote by $\mathbf{f}_k(X)$, $0 \le k \le n$, the non-negative integer given by

$$\mathbf{f}_k(X) = \sum_{p+q=k} \Big( h_{\mathrm{BC}}^{p,q}(X) + h_{\mathrm{BC}}^{n-p,n-q}(X) \Big) - 2b_k(X). \tag{2.2}$$

**Definition 2.1.4.** *A compact complex manifold $X$ is said to* satisfy the property $\mathcal{F}_k$, *where $0 \le k \le n$, if the complex invariant $\mathbf{f}_k(X)$ vanishes, i.e.,*

$$\mathcal{F}_k = \big\{ X \text{ satisfies } \mathbf{f}_k(X) = 0 \big\}.$$

By [AT13] the compact complex manifold $X$ satisfies the $\partial\bar{\partial}$-lemma if and only if $X$ has the property $\mathcal{F}_k$ for every $0 \le k \le n$.

A similar argument as in [AT13, Corollary 3.7] shows that the property $\mathcal{F}_k$ is open under holomorphic deformations:

**Proposition 2.1.5.** *Let $X$ be a compact complex manifold for which the property $\mathcal{F}_k$ holds, and let $X_t$ be a small deformation of $X$. Then, for sufficiently small $t$, the manifold $X_t$ has the property $\mathcal{F}_k$. Moreover, $h_{\mathrm{BC}}^{p,q}(X_t) = h_{\mathrm{BC}}^{p,q}(X)$ and $h_{\mathrm{BC}}^{n-p,n-q}(X_t) = h_{\mathrm{BC}}^{n-p,n-q}(X)$, for any $(p,q)$ such that $p+q=k$.*

*Proof.* Let $\Delta$ be an open disk around the origin in $\mathbb{C}$. Let $\{X_t\}_{t\in\Delta}$ be a small deformation of $X = X_0$. By [Sch] we known that the dimensions of the Bott-Chern cohomology groups are upper-semi-continuous functions at $t$. That is, one has $h_{\mathrm{BC}}^{p,q}(X_0) \ge h_{\mathrm{BC}}^{p,q}(X_t)$, for every pair $(p, q)$ and each $t \in \Delta \setminus \{0\}$. However, the Betti numbers are constant along $\{X_t\}_{t\in\Delta}$ so it is clear that

$$0 = \mathbf{f}_k(X_0) \geq \mathbf{f}_k(X_t) \geq 0, \quad \forall t \in \Delta \setminus \{0\}.$$

Necessarily, every $X_t$ in the deformation satisfies the property $\mathcal{F}_k$. The equalities come straightforward. □

Concerning the deformation limits of the properties $\mathcal{F}_k$, we have the following result for the case $k = 2$. One could expect a similar behaviour for other $k$'s.

**Proposition 2.1.6.** *Let $(M, J_0)$ be a compact nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with an abelian complex structure $J_0$. Then, there is a holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1/3\}$, such that $(M, J_t)$ satisfies the property $\mathcal{F}_2$ for each $t \in \Delta \setminus \{0\}$, but $(M, J_0)$ does not satisfy $\mathcal{F}_2$. Therefore, $\mathcal{F}_2$ is not a closed property.*

*Proof.* There is only one abelian complex structure on $M$ up to isomorphism, whose Bott-Chern cohomology numbers are given in Table 2.1 (case $\mathfrak{h}_4$ with $\rho = 0$, $\lambda = 1$, $D = 1/4$). Since $b_2(M) = 8$, one has that $\mathbf{f}_2(M, J_0) = 2$.

Using the Kuranishi's method, Maclaughlin, Pedersen, Poon, and Salamon proved in [MPPS06] that $J_0$ has a locally complete family of deformations entirely consisting of invariant complex structures. Indeed, they obtained the deformation parameter space in terms of invariant forms. Their results will help us to find an appropriate direction in which we should deform $(M, J_0)$ in order to get the $\mathcal{F}_2$-property. Writing the structure equations of $J_0$ as

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \frac{i}{2}\eta^{1\bar{1}} + \frac{1}{2}\eta^{1\bar{2}} + \frac{1}{2}\eta^{2\bar{1}},$$

by [KS04, MPPS06] any complex structure sufficiently near to $J_0$ has a basis of $(1,0)$-forms such that

$$\text{(2.3)} \qquad \begin{cases} \mu^1 \;=\; \eta^1 + \Phi^1_1\,\eta^{\bar{1}} + \Phi^1_2\,\eta^{\bar{2}}, \\ \mu^2 \;=\; \eta^2 + \Phi^2_1\,\eta^{\bar{1}} + \Phi^2_2\,\eta^{\bar{2}}, \\ \mu^3 \;=\; \eta^3 + \Phi^3_1\,\eta^{\bar{1}} + \Phi^3_2\,\eta^{\bar{2}} + \Phi^3_3\,\eta^{\bar{3}}, \end{cases}$$

where $i(1 + \Phi^3_3)\Phi^1_2 = (1 - \Phi^3_3)(\Phi^1_1 - \Phi^2_2)$ and the coefficients $\Phi^i_j$ are sufficiently small. The complex structure remains abelian if and only if $\Phi^1_2 = 0$ and $\Phi^1_1 = \Phi^2_2$.

We will consider the particular holomorphic deformation $J_t$ given in [COUV16] by conveniently shrinking the radius of the deformation disk. For each $t \in \mathbb{C}$ such that $|t| < 1$, we take the basis of (1,0)-forms $\{\mu^1, \mu^2, \mu^3\}$ given by

$$\mu^1 = \eta^1 + t\eta^{\bar{1}} - it\eta^{\bar{2}}, \quad \mu^2 = \eta^2, \quad \mu^3 = \eta^3.$$

Notice that this choice corresponds to $\Phi^1_1 = t$, $\Phi^1_2 = -it$ and $\Phi^2_1 = \Phi^2_2 = \Phi^3_1 = \Phi^3_2 = \Phi^3_3 = 0$ in the parameter space (2.3). Following [COUV16, Theorem 7.9], for any $t \in \mathbb{C}$ such that $0 < |t| < 1$ the complex structure $J_t$ is nilpotent but not abelian, and there is a (1,0)-basis $\{\tau^1, \tau^2, \tau^3\}$ such that the structure equations for $J_t$ are

$$d\tau^1 = d\tau^2 = 0, \quad d\tau^3 = \tau^{12} + \tau^{1\bar{1}} + \frac{1}{|t|}\,\tau^{1\bar{2}} + \frac{1-|t|^2}{4|t|^2}\,\tau^{2\bar{2}}. \tag{2.4}$$

Moreover, using [COUV16, Proposition 3.7] there is a basis $\{\omega^1, \omega^2, \omega^3\}$ of $(1,0)$-forms for which (2.4) can be reduced to the normalized structure equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + \frac{|t|^2-1}{4|t|^2}\,\omega^{2\bar{2}}. \tag{2.5}$$

That is, the coefficients $(\rho, \lambda, D)$ satisfy $\rho = \lambda = 1$ and $D = \frac{|t|^2-1}{4|t|^2} < 0$.

Now, let us compute $\mathbf{f}_2$ for any $J_t$ with $t \neq 0$. Since

$$\mathbf{f}_2(M, J_t) = 2\,h^{2,0}_{\mathrm{BC}}(M, J_t) + h^{1,1}_{\mathrm{BC}}(M, J_t) + 2\,h^{3,1}_{\mathrm{BC}}(M, J_t) + h^{2,2}_{\mathrm{BC}}(M, J_t) - 2b_2(M),$$

from Table 2.1 for $\mathfrak{h}_4$ we have $h^{2,0}_{\mathrm{BC}}(M, J_t) = 1$, $h^{3,1}_{\mathrm{BC}}(M, J_t) = 2$, and $h^{2,2}_{\mathrm{BC}}(M, J_t) = 6$. Moreover, $h^{1,1}_{\mathrm{BC}}(M, J_t) = 4$ if and only if $\frac{|t|^2-1}{4|t|^2} \neq -2$, that is, if and only if $|t| \neq \frac{1}{3}$. Therefore, for $0 < |t| < \frac{1}{3}$ we can conclude that $h^{1,1}_{\mathrm{BC}}(M, J_t) = 4$ and thus, $\mathbf{f}_2(M, J_t) = 0$. □

**Remark 2.1.7.** Notice that the previous holomorphic deformation can be actually defined for $|t| < 1$. In that case, the dimensions of the Bott-Chern cohomology groups vary with $t$ as follows:

$$h^{1,1}_{\mathrm{BC}}(M, J_t) = \begin{cases} 4, & \text{for } |t| \neq 1/3, \\ 5, & \text{for } |t| = 1/3, \end{cases} \qquad h^{3,1}_{\mathrm{BC}}(M, J_t) = \begin{cases} 2, & \text{for } 0 < |t| < 1, \\ 3, & \text{for } t = 0. \end{cases}$$

From [COUV16], we know that the complex manifold $(M, J_0)$ does not admit any strongly Gauduchon metric. Indeed, in this paper the deformation (2.3) is used to prove that the existence of strongly Gauduchon metrics is not a closed property. Next we see that, even under stronger additional hypothesis on a holomorphic family $\{X_t\}_{t\in\Delta}$ for $t \neq 0$, the deformation limit $X_0$ does not admit a strongly Gauduchon metric. For the proof, we make use of Proposition 2.1.6:

**Corollary 2.1.8.** *Let $(M, J_0)$ be a compact nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with abelian complex structure $J_0$. Denote $\Delta = \{t \in \mathbb{C} \mid |t| < 1/3\}$. There is a holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$ such that, for each $t \in \Delta \setminus \{0\}$, every Gauduchon metric is strongly Gauduchon and $(M, J_t)$ satisfies the property $\mathcal{F}_2$. However, $(M, J_0)$ does not admit any strongly Gauduchon metric.*

*Proof.* We use the same deformation as in the proof of Proposition 2.1.6, so it is clear that the property $\mathcal{F}_2$ holds for every $t \in \Delta \setminus \{0\}$. Now, recall that these $t \neq 0$ correspond to equations (2.5), which are nilpotent but non-abelian complex structures of Family I. It suffices to apply Proposition 2.1.3 in order to get the result. □

This corollary says that Proposition 1.3.17 does not hold in general if we weaken the hypothesis of the $\partial\bar{\partial}$-lemma. In other words, having a single property $\mathcal{F}_k$ is not sufficient to ensure the existence of a strongly Gauduchon metric in the deformation limit.

We now consider a second invariant related to the Bott-Chern cohomology and the $\partial\bar{\partial}$-lemma condition. For any compact complex manifold $X$, Schweitzer proved in [Sch, Lemma 3.3] that

$$h^{1,1}_{\rm BC}(X) + 2\, h^{0,2}(X) \geq b_2(X),$$

in such a way that if $X$ is Kähler then the equality holds. We extend this inequality as follows, using the terms in the Frölicher spectral sequence:

**Proposition 2.1.9.** *If $X$ is a compact complex manifold, then for any $r \geq 1$*

$$h^{1,1}_{\rm BC}(X) + 2 \dim E_r^{0,2}(X) \geq b_2(X),$$

*where $E_r^{0,2}(X)$ denotes the $r$-step $(0,2)$-term of the Frölicher spectral sequence. Furthermore, if $X$ satisfies the $\partial\bar{\partial}$-lemma then the above inequalities are all equalities.*

*Proof.* For $r = 1$, the term $E_1^{0,2}(X)$ is precisely the Dolbeault cohomology group $H^{0,2}_{\bar{\partial}}(X)$. According to Theorem 1.2.1, one has

$$E_2^{0,2}(X) = \frac{\{\alpha_{0,2} \in \Omega^{0,2}(X) \mid \bar{\partial}\alpha_{0,2} = 0,\ \partial\alpha_{0,2} = \bar{\partial}\alpha_{1,1}\}}{\bar{\partial}(\Omega^{0,1}(X))},$$

$$E_3^{0,2}(X) = \frac{\{\alpha_{0,2} \in \Omega^{0,2}(X) \mid \bar{\partial}\alpha_{0,2} = 0,\ \partial\alpha_{0,2} = \bar{\partial}\alpha_{1,1},\ \partial\alpha_{1,1} = \bar{\partial}\alpha_{2,0}\}}{\bar{\partial}(\Omega^{0,1}(X))},$$

and

$$E_r^{0,2}(X) = \frac{\{\alpha_{0,2} \in \Omega^{0,2}(X) \mid \bar{\partial}\alpha_{0,2} = 0,\ \partial\alpha_{0,2} = \bar{\partial}\alpha_{1,1},\ \partial\alpha_{1,1} = \bar{\partial}\alpha_{2,0},\ \partial\alpha_{2,0} = 0\}}{\bar{\partial}(\Omega^{0,1}(X))},$$

for any $r \geq 3$. Let us consider the sequence

$$0 \to Z^{1,1}(X) \hookrightarrow H^{1,1}_{\rm BC}(X) \xrightarrow{\ \iota\ } H^2_{\rm dR}(X,\mathbb{C}) \xrightarrow{\ \sigma_r\ } \overline{E_r^{0,2}(X)} \oplus E_r^{0,2}(X) \to \operatorname{coker}(\sigma_r) \to 0,$$

where $\iota$ is the natural map,

$$Z^{1,1}(X) = \frac{\ker\{d\colon \Omega^{1,1}(X) \longrightarrow \Omega^3(X)\} \cap d(\Omega^1)}{\partial\bar{\partial}(\Omega^0(X))},$$

and $\sigma_r$ is given by

$$\sigma_r([\alpha = \alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2}]_{\rm dR}) = ([\alpha_{2,0}], [\alpha_{0,2}]) \in \overline{E_r^{0,2}(X)} \oplus E_r^{0,2}(X).$$

The above sequence is exact because $\ker \sigma_r \subset \operatorname{im} \iota$. In fact, if $([\alpha_{2,0}], [\alpha_{0,2}]) = (0,0)$ in $\overline{E_r^{0,2}(X)} \oplus E_r^{0,2}(X)$ then $\alpha_{2,0} = \partial\beta_{1,0}$ and $\alpha_{0,2} = \bar{\partial}\beta_{0,1}$, for some (0,1)-forms $\overline{\beta_{1,0}}$ and $\beta_{0,1}$. Hence, the (1,1)-form $\gamma = \alpha_{1,1} - \bar{\partial}\beta_{1,0} - \partial\beta_{0,1}$ is closed and

$$\alpha - \gamma = d(\beta_{1,0} + \beta_{0,1}).$$

That is, $\gamma$ defines a class in $H^{1,1}_{\mathrm{BC}}(X)$ such that $\iota([\gamma]_{\mathrm{BC}}) = [\alpha]_{\mathrm{dR}}$.

Now, the exactness of the sequence implies

$$\begin{aligned} 0 &= \dim Z^{1,1}(X) - h^{1,1}_{\mathrm{BC}}(X) + b_2(X) - 2\dim E_r^{0,2}(X) + \dim\operatorname{coker}(\sigma_r) \\ &\geq b_2(X) - h^{1,1}_{\mathrm{BC}}(X) - 2\dim E_r^{0,2}(X). \end{aligned}$$

If the $\partial\bar\partial$-lemma holds, then the natural map $H^{p,q}_{\mathrm{BC}}(X) \longrightarrow H^{p,q}_{\bar\partial}(X)$ is an isomorphism. This means that $h^{2,0}(X) = h^{2,0}_{\mathrm{BC}}(X) = h^{0,2}_{\mathrm{BC}}(X) = h^{0,2}(X)$ and $h^{1,1}(X) = h^{1,1}_{\mathrm{BC}}(X)$. Moreover, as the Frölicher spectral sequence degenerates at the first step [DGMS75], one gets $\dim E_r^{0,2}(X) = h^{0,2}(X)$, for every $r$, and

$$b_2(X) = h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X) = h^{1,1}_{\mathrm{BC}}(X) + 2\,\dim E_r^{0,2}(X).$$

□

From now on, we will denote $\mathbf{k}_r(X)$ the non-negative integer given by

$$\mathbf{k}_r(X) = h^{1,1}_{\mathrm{BC}}(X) + 2\dim E_r^{0,2}(X) - b_2(X).$$

Therefore, $\mathbf{k}_r(X)$ are complex invariants that vanish if the manifold $X$ satisfies the $\partial\bar\partial$-lemma. Notice that $\mathbf{k}_1(X) \geq \mathbf{k}_2(X) \geq \mathbf{k}_3(X) = \mathbf{k}_r(X) \geq 0$, for any $r \geq 4$.

**Remark 2.1.10.** In general $\mathbf{k}_1(X)$, $\mathbf{k}_2(X)$, and $\mathbf{k}_3(X)$ do not coincide. For example, let $M$ be a nilmanifold with underlying Lie algebra $\mathfrak{h}_{15}$, for which $b_2(M) = 5$. From the cases for $\rho = 1$ on $\mathfrak{h}_{15}$ given in Table 2.2 and from [COUV16, Proposition 6.2] we get:

- if $J$ is a complex structure defined by $\rho = B = 1$ and $c > 2$, then $h^{1,1}_{\mathrm{BC}}(M,J) = 5$ and $\dim E_1^{0,2}(M,J) = \dim E_2^{0,2}(M,J) = 2 > 1 = \dim E_3^{0,2}(M,J)$. In this way,
$$\big(\mathbf{k}_1(M,J), \mathbf{k}_2(M,J), \mathbf{k}_3(M,J)\big) = (4,4,2);$$
- if $J'$ is a complex structure defined by $\rho = 1 \neq |B|$ and $c = 0$, then $h^{1,1}_{\mathrm{BC}}(M,J') = 4$ and $\dim E_1^{0,2}(M,J') = 2 > 1 = \dim E_2^{0,2}(M,J') = \dim E_3^{0,2}(M,J')$, so
$$\big(\mathbf{k}_1(M,J'), \mathbf{k}_2(M,J'), \mathbf{k}_3(M,J')\big) = (3,1,1).$$

Let us observe that there are compact complex manifolds $X$ with no Kähler metrics but satifying $\mathbf{k}_r(X) = 0$, the Iwasawa manifold being an example. In fact:

**Proposition 2.1.11.** *Let $(M,J)$ be a compact complex-parallelizable nilmanifold. Then, $\mathbf{k}_r(M,J) = 0$ for all $r$.*

*Proof.* Due to the relations among the invariants $\mathbf{k}_r(M,J)$, it suffices to prove the result for $r = 1$. Let $\mathfrak{g}$ be the Lie algebra underlying $M$. First, notice that all the involved cohomology groups can be computed at the Lie-algebra level, as a consequence of combining Theorems 1.4.13 and 1.4.16. Secondly, our complex structure is parallelizable, so $\bar{\partial}(\mathfrak{g}^{1,0}) = 0$. These two facts tell us that the sequence

$$0 \to H^{1,1}_{\rm BC}(\mathfrak{g}) = \ker\left\{d\colon {\textstyle\bigwedge}^{1,1}(\mathfrak{g}^*) \longrightarrow {\textstyle\bigwedge}^{3}(\mathfrak{g}^*)\right\} \stackrel{\iota}{\hookrightarrow} H^2_{dR}(\mathfrak{g};\mathbb{C}) \stackrel{\sigma_1}{\longrightarrow} \overline{H^{0,2}_{\bar\partial}(\mathfrak{g})} \oplus H^{0,2}_{\bar\partial}(\mathfrak{g}) \to 0$$

is indeed a short exact sequence. Therefore, $h^{1,1}_{\rm BC}(M) - b_2(M) + 2\,h^{0,2}(M) = 0$. □

Since the vanishing of $\mathbf{k}_1(X)$ implies the vanishing of any other $\mathbf{k}_r(X)$, we next consider:

**Definition 2.1.12.** *We say that a compact complex manifold $X$* satisfies the property *$\mathcal{K}$ if the complex invariant $\mathbf{k}_1(X)$ equals zero:*

$$\mathcal{K} = \big\{X \text{ satisfies } \mathbf{k}_1(X) = 0\big\}.$$

It is clear that any compact complex manifold satisfying the $\partial\bar\partial$-lemma fulfills $\mathcal{K}$.

By the same argument as for $\mathcal{F}_k$, the property $\mathcal{K}$ is open under holomorphic deformations. In this way, as a consequence of Proposition 2.1.11 we can conclude:

**Corollary 2.1.13.** *Let $(M, J_0)$ be a compact complex-parallelizable nilmanifold and let $(M, J_t)$ be a small deformation of $(M, J)$. Then, $(M, J_t)$ has the property $\mathcal{K}$ and satisfies both $h^{1,1}_{\rm BC}(M, J_t) = h^{1,1}_{\rm BC}(M, J_0)$ and $h^{0,2}(M, J_t) = h^{0,2}(M, J_0)$ for sufficiently small $|t|$.*

**Remark 2.1.14.** As a particular case of the previous result, we get that $h^{1,1}_{\rm BC}$ is stable under the small deformations of the Iwasawa manifold. For such deformations, Angella proved in [Ang13] that $h^{1,1}_{\rm BC} = 4$ and $h^{0,2} = 2$.

**Proposition 2.1.15.** *The property $\mathcal{K}$ is not closed under holomorphic deformations.*

*Proof.* The proof is similar to that of Proposition 2.1.6. For the holomorphic deformation $(M, J_t)_{t\in\Delta}$ given in the proof of that result, one can count the dimension of the Dolbeault cohomology group (see [COUV16, Proposition 6.1]) and get that $h^{0,2}(M, J_0) = 3$ and $h^{0,2}(M, J_t) = 2$, for any $t \neq 0$. Since $h^{1,1}_{\rm BC}(M, J_t) = 4$ for any $t$ such that $|t| < 1/3$, we conclude that $\mathbf{k}_1(M, J_t) = 0$ for $0 < |t| < 1/3$. However, $\mathbf{k}_1(M, J_0) = 2$. □

The deformation of the abelian complex structure $J_0$ on $\mathfrak{h}_4$ allowed us to show that in general the properties $\mathcal{F}_2$ and $\mathcal{K}$ are not closed under holomorphic deformations. In addition, combining Propositions 2.1.6 and 2.1.15 with the results on the existence of balanced metrics and the behaviour of the Frölicher spectral sequence obtained in [COUV16], we conclude:

**Theorem 2.1.16.** *Let $(M, J_0)$ be a compact nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with abelian complex structure $J_0$. Then, there is a holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1/3\}$, such that for each $t \in \Delta\backslash\{0\}$ every Gauduchon metric is strongly Gauduchon, $(M, J_t)$ admits balanced metric, satisfies the properties $\mathcal{F}_2$ and $\mathcal{K}$, and has degenerate Frölicher spectral sequence. However, the central limit $(M, J_0)$ does not admit strongly Gauduchon metrics, the properties $\mathcal{F}_2$ and $\mathcal{K}$ fail, and the Frölicher spectral sequence does not degenerate at the first step.*

## 2.2 Cohomological decomposition: pureness and fullness

In this section, we continue with the study of some properties related to cohomology groups. In particular, we focus on the cohomological decomposition of 6-dimensional nilmanifolds endowed with invariant complex structures. The results contained in this section can be found in [LOUV13] and [LU15] (see also [LU]).

### 2.2.1 Study of the complex stages

Let us start focusing on the complex notions of *pureness* and *fullness* at different stages (Definition 1.2.11). Apart from proving some partial results on nilmanifolds of arbitrary dimension, we study these properties for any 6-dimensional nilmanifold endowed with an invariant complex structure, using the classification by Ceballos, Otal, Ugarte, and Villacampa [COUV16].

We first concentrate on complex-$\mathcal{C}^\infty$-pure-and-fullness at the first stage.

**Lemma 2.2.1.** *Let $J$ be an invariant complex structure on a $2n$-dimensional nilmanifold $M$. Then:*

*i) The complex structure $J$ is complex-$\mathcal{C}^\infty$-pure at the first stage;*

*ii) If $b_1(M) = 2n-1$, then $J$ is not complex-$\mathcal{C}^\infty$-full at the first stage.*

*Proof.* Let $\alpha$ be a closed complex 1-form which is cohomologous to both a closed $(1,0)$-form $\beta$ and a closed $(0,1)$-form $\gamma$, i.e. $\beta + df = \alpha = \gamma + dg$, for some complex-valued functions $f$ and $g$ on $M$. Let us see that the class of $\alpha$ is zero. Using the symmetrization process, we find $\tilde{\alpha} \in \mathfrak{g}^*_{\mathbb{C}}$ such that $\tilde{\beta} = \tilde{\alpha} = \tilde{\gamma}$, because $\tilde{f}$ and $\tilde{g}$ are constant functions. As symmetrization preserves the bidegree, one has that $\mathfrak{g}^{1,0} \cap \mathfrak{g}^{0,1} = \{0\}$ implies $\tilde{\alpha} = 0$. By Nomizu's Theorem [Nom54] the form $\alpha$ is cohomologous to zero, so the class $[\alpha] = \mathbf{0}$. This proves *i)*.

For the proof of *ii)*, first notice that the condition $b_1(M) = 2n-1$ implies that $J$ is abelian. Therefore, there is a basis of $(1,0)$-forms $\{\omega^1, \ldots, \omega^n\}$ such that

$$d\omega^1 = \cdots = d\omega^{n-1} = 0, \qquad d\omega^n = \sum_{i,j=1}^{n-1} A_{ij}\omega^{i\bar{j}},$$

where $A_{ij} \in \mathbb{C}$ are not all equal to zero and satisfy $A_{ij} = \bar{A}_{ji}$. By Nomizu's Theorem we get $H^1_{\mathrm{dR}}(M;\mathbb{C}) = \langle [\omega^1], [\omega^{\bar{1}}], \ldots, [\omega^{n-1}], [\omega^{\overline{n-1}}], [\omega^n + \omega^{\overline{n}}] \rangle$. It is easy to see that $\omega^n + \omega^{\overline{n}}$ cannot be represented neither by a $(1,0)$-form nor by a $(0,1)$-form. $\square$

**Remark 2.2.2.** One has from [ABD11, Proposition 2.2] that if $b_1(M) = 2n-1$, then $\mathfrak{g}$ is isomorphic to the product of $\mathbb{R}^{2n-2k-1}$ by a $(2k+1)$-dimensional generalized Heisenberg algebra. In addition, there are exactly $[k/2] + 1$ equivalence classes of complex structures. Notice that in dimension 6 there are only two algebras satisfying this condition, namely, $\mathfrak{h}_3$ and $\mathfrak{h}_8$ [COUV16].

We next consider complex-$\mathcal{C}^\infty$-pureness at higher stages.

**Proposition 2.2.3.** *Let $J$ be an invariant complex structure on a $2n$-dimensional nilmanifold $M$. Then:*

*i)* $H_J^{n,0}(M) \cap H_J^{n-k,k}(M) = \{\mathbf{0}\} = H_J^{k,n-k}(M) \cap H_J^{0,n}(M)$*, for every* $1 \le k \le n$.

*ii) If $J$ is abelian or complex-parallelizable, then for any $2 \le k \le n$ the complex structure is complex-$\mathcal{C}^\infty$-pure at the $k$-th stage if and only if the sum*

$$H_J^{k-1,1}(M) + \cdots + H_J^{1,k-1}(M)$$

*is direct; in particular, $J$ is always complex-$\mathcal{C}^\infty$-pure at the second stage.*

*Proof.* By Proposition 1.4.19, we reduce the proof to the level of the Lie algebra $\mathfrak{g}$. Let us first see *i)*. Fix some $k = 1, \ldots, n$ and consider $\mathbf{a} \in H_J^{n,0}(\mathfrak{g}) \cap H_J^{n-k,k}(\mathfrak{g})$. We want to see that $\mathbf{a} = \mathbf{0}$. On the one hand, we have that $\mathbf{a} \in H_J^{n,0}(\mathfrak{g})$, so there is a closed element $\beta \in \bigwedge^{n,0}(\mathfrak{g}_{\mathbb{C}}^*)$ such that $\mathbf{a} = [\beta]$. On the other hand, as $\mathbf{a} \in H_J^{n-k,k}(\mathfrak{g})$, one can also find a closed element $\gamma \in \bigwedge^{n-k,k}(\mathfrak{g}_{\mathbb{C}}^*)$ satisfying $\mathbf{a} = [\gamma]$. Then, it is clear that $\beta - \gamma = d\alpha$, for some $\alpha = \alpha^{n-1,0} + \cdots + \alpha^{0,n-1} \in \bigwedge^{n-1}(\mathfrak{g}_{\mathbb{C}}^*)$. Due to the action of $d = \partial + \bar\partial$ on the elements of total degree $n-1$ in which $\alpha$ is decomposed, one necessarily has $\beta = \partial\alpha^{n-1,0}$. Notice that Theorem 1.4.10 implies that there is a $(1,0)$-basis $\{\omega^j\}_{j=1}^n$ such that the $(n,0)$-form $\omega^{1\cdots n}$ is closed, so there exists $\lambda \in \mathbb{C}$ such that $\beta = \lambda\,\omega^{1\cdots n}$. Therefore,

$$|\lambda|^2\,\omega^{1\cdots n\bar 1\cdots \bar n} = \beta \wedge \overline{\beta} = \partial\alpha^{n-1,0} \wedge \overline{\beta} = d(\alpha^{n-1,0} \wedge \overline{\beta}).$$

However, the Lie algebra $\mathfrak{g}$ is unimodular, so $b_{2n}(\mathfrak{g}) = 1$. Hence, there cannot exist a non-zero element of top degree which is exact, that is, necessarily $\lambda = 0$ and thus $\beta = 0$. This implies $\mathbf{a} = [\gamma] = -[d\alpha]$, i.e. $\mathbf{a} = \mathbf{0}$. This yields $H_J^{n,0}(\mathfrak{g}) \cap H_J^{n-k,k}(\mathfrak{g}) = \{\mathbf{0}\}$, for any $k = 1, \ldots, n$. Similarly, one gets $H_J^{k,n-k}(M) \cap H_J^{0,n}(M) = \{\mathbf{0}\}$, for $k = 1, \ldots, n$.

For the proof of *ii)*, first note that one of the implications is trivial due to the definition of complex-$\mathcal{C}^\infty$-pure. For the other one, we first check that for each $2 \le k \le n$ we have $H_J^{k,0}(\mathfrak{g}) \cap H_J^{0,k}(\mathfrak{g}) = \{\mathbf{0}\}$. The case $k = n$ is a consequence of *i)*. For $k = 2, \ldots, n-1$, let us see that any class $\mathbf{a} \in H_J^{k,0}(\mathfrak{g}) \cap H_J^{0,k}(\mathfrak{g})$ is zero. Let $\beta \in \bigwedge^{k,0}(\mathfrak{g}_{\mathbb{C}}^*)$ and $\gamma \in \bigwedge^{0,k}(\mathfrak{g}_{\mathbb{C}}^*)$ be closed forms such that $[\beta] = \mathbf{a} = [\gamma]$. Then, there exists $\alpha = \alpha^{k-1,0} + \cdots + \alpha^{0,k-1} \in \bigwedge^{k-1}(\mathfrak{g}_{\mathbb{C}}^*)$ satisfying $d\alpha = \beta - \gamma$, which implies $\beta = \partial\alpha^{k-1,0}$. However, when $J$ is abelian the Lie algebra differential $d$ satisfies $d(\mathfrak{g}_{\mathbb{C}}^*) \subset \bigwedge^{1,1}(\mathfrak{g}_{\mathbb{C}}^*)$, thus $d\alpha^{k-1,0} \in \bigwedge^{k-1,1}(\mathfrak{g}_{\mathbb{C}}^*)$ and $\beta = \partial\alpha^{k-1,0} = 0$. In the same way, when $J$ is complex-parallelizable one has $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}_{\mathbb{C}}^*)$, which in particular implies $\bar\partial(\bigwedge^{k-1,0}(\mathfrak{g}_{\mathbb{C}}^*)) = 0$, so we get $\beta = \partial\alpha^{k-1,0} = d\alpha^{k-1,0}$. One can then conclude that, in any case, the form $\beta$ is cohomologous to zero, i.e. $\mathbf{a} = [\beta] = \mathbf{0}$. Therefore, $H_J^{k,0}(M) \oplus H_J^{0,k}(M)$. We next need to see that

$$\big(H_J^{k,0}(\mathfrak{g}) \oplus H_J^{0,k}(\mathfrak{g})\big) \cap \big(H_J^{k-1,1}(\mathfrak{g}) \oplus \cdots \oplus H_J^{1,k-1}(\mathfrak{g})\big) = \{\mathbf{0}\}.$$

Let the class $\mathbf{a}$ be in the intersection situated on the left-hand side of the previous equality. There exist

$$[\beta] = [\beta^{k,0}] + [\beta^{0,k}] \in H_J^{k,0}(\mathfrak{g}) \oplus H_J^{0,k}(\mathfrak{g}),$$

$$[\gamma] = [\gamma^{k-1,1}] + \cdots + [\gamma^{1,k-1}] \in H_J^{k-1,1}(\mathfrak{g}) \oplus \cdots \oplus H_J^{1,k-1}(\mathfrak{g})$$

such that $[\beta] = \mathbf{a} = [\gamma]$. This implies that one can find $\alpha = \alpha^{k-1,0} + \cdots + \alpha^{0,k-1} \in \bigwedge^{k-1}(\mathfrak{g}_{\mathbb{C}}^*)$ satisfying $d\alpha = \beta - \gamma$. Observe that $\beta^{k,0} = \partial\alpha^{k-1,0}$, and the same arguments as above can be applied for the abelian and the complex-parallelizable cases. Hence, we get $\mathbf{a} = \mathbf{0}$, and the proof of part *ii)* is accomplished. □

**Remark 2.2.4.** Recall that the integrability of $J$ makes that being complex-$\mathcal{C}^\infty$-pure implies being $\mathcal{C}^\infty$-pure. Hence, as a consequence of Proposition 2.2.3 *ii)* one has that every abelian or complex-parallelizable structure is $\mathcal{C}^\infty$-pure.

Fino and Tomassini show in [FT10, Corollary 5.2] that every complex-parallelizable structure is $\mathcal{C}^\infty$-full. This observation together with Remark 2.2.4 allow us to conclude:

**Corollary 2.2.5.** *Every complex-parallelizable structure on a nilmanifold is $\mathcal{C}^\infty$-pure-and-full.*

Let us recall that $\partial\bar{\partial}$-manifolds are complex-$\mathcal{C}^\infty$-pure-and-full at every stage. In addition, 4-dimensional almost-complex manifolds are always $\mathcal{C}^\infty$-pure-and-full. Hence, when one wants to either support or discard the existence of relations among these and other properties, it is natural to focus on non-$\partial\bar{\partial}$-manifolds of dimension greater than 4. In particular, 6-dimensional nilmanifolds with invariant almost-complex structures have been proved to be quite useful (see [AT12, ATZ14, FT12], among others). Nonetheless, most of the examples deal with particular cases and focus on the first stages of the cohomological decomposition. Using the classification of invariant complex structures on 6-dimensional nilmanifolds [COUV16] and Proposition 1.4.19, we can give a wider view of the topic.

Let $M = G\backslash\Gamma$ be a 6-dimensional nilmanifold endowed with an invariant complex structure $J$. If $M$ is isomorphic to either the complex torus or the Iwasawa manifold, then $M$ is complex-$\mathcal{C}^\infty$-pure-and-full at every stage. The first case is a consequence of the torus being a Kähler manifold, whereas the second one comes from [AT11]. In the latter paper, also the small deformations of the Iwasawa manifold are investigated, although the attention is paid to complex-$\mathcal{C}^\infty$-pure-and-fullness at the second stage (the conclusion is that none of the properties is satisfied along a deformation which is not in class *(i)*, see p. 54 for details).

By Proposition 1.4.19, one can study pureness and fullness for any $(M, J)$ at the level of the Lie algebra $\mathfrak{g}$ associated to $G$, as long as $\mathfrak{g} \ncong \mathfrak{h}_7$. Hence, we now directly focus on those $(\mathfrak{g}, J)$ given in [COUV16] and discuss their complex-$\mathcal{C}^\infty$-pure-and-fullness at each stage. Note that the complete behaviour for the small deformations of the Iwasawa manifold will be covered by our analysis, since they are particular cases of complex structures living on the Lie algebra $\mathfrak{h}_5$.

| Family I | | | | Stages | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 1st | | 2nd | | 3rd | | 4th | | 5th | |
| $\mathfrak{g}$ | $\rho$ | $\lambda$ | $D = x + iy$ | pure | full | pure | full | pure | full | pure | full | pure | full |
| $\mathfrak{h}_2$ | 0 | 0 | $y = 1$ | ✓ | ✓ | ✓ | – | – | – | – | ✓ | ✓ | ✓ |
| | 1 | 1 | $y > 0$ | ✓ | ✓ | – | – | – | – | – | – | ✓ | ✓ |
| $\mathfrak{h}_3$ | 0 | 0 | $\pm 1$ | ✓ | – | ✓ | – | – | – | – | ✓ | – | ✓ |
| $\mathfrak{h}_4$ | 0 | 1 | $\frac{1}{4}$ | ✓ | ✓ | ✓ | – | – | – | – | ✓ | ✓ | ✓ |
| | 1 | 1 | $D \in \mathbb{R} \setminus \{0\}$ | ✓ | ✓ | – | – | – | – | – | – | ✓ | ✓ |
| $\mathfrak{h}_5$ | 0 | 1 | 0 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | 0 | 1 | $D \in \left(0, \frac{1}{4}\right)$ | ✓ | ✓ | ✓ | – | – | – | – | ✓ | ✓ | ✓ |
| | 1 | 0 | 0 | ✓ | ✓ | – | – | ✓ | ✓ | – | – | ✓ | ✓ |
| | 1 | $\lambda \neq 0$ | 0 | ✓ | ✓ | – | – | – | – | – | – | ✓ | ✓ |
| | 1 | any allowed structure satisfying $D \neq 0$ | | ✓ | ✓ | – | – | – | – | – | – | ✓ | ✓ |
| $\mathfrak{h}_6$ | 1 | 1 | 0 | ✓ | ✓ | – | – | – | – | – | – | ✓ | ✓ |
| $\mathfrak{h}_8$ | 0 | 0 | 0 | ✓ | – | ✓ | – | – | – | – | ✓ | – | ✓ |

Table 2.5: Complex-$\mathcal{C}^\infty$-pure-and-fullness at different stages for Family I.

Our conclusions are contained in Tables 2.5, 2.6, and 2.7, according to the parameters that define every non-isomorphic $J$ on each $\mathfrak{g}$. We omit the cases corresponding to the complex torus and the Iwasawa manifold. In order to illustrate the procedure followed to attain these results, here we briefly describe how to study complex-$\mathcal{C}^\infty$-pureness and fullness at the 2nd stage for the complex nilmanifolds corresponding to $(\mathfrak{h}_4, J)$.

First observe that the complex structures $J$ on $\mathfrak{h}_4$ belong to Family I. They correspond to the parameters $\lambda = 1$ and either $(\rho, D) = (0, 1/4)$ or $(\rho, D) = (1, x)$ with $x \in \mathbb{R} \setminus \{0\}$. A direct calculation shows that

$$\begin{aligned} H^2_{\mathrm{dR}}(\mathfrak{h}_4; \mathbb{C}) \;\; = \;\; & \langle\, [\omega^{1\bar{2}}],\, [\omega^{2\bar{2}}],\, [\omega^{\bar{1}\bar{2}}],\, [\omega^{13} + D\,\omega^{23} + D\,\omega^{2\bar{3}} - \rho\,\omega^{\bar{1}\bar{3}}], \\ & [\rho\,\omega^{13} + D\,\omega^{3\bar{2}} - \omega^{\bar{1}\bar{3}} - D\,\omega^{\bar{2}\bar{3}}],\, [\rho\,\omega^{23} - \omega^{3\bar{1}} - \omega^{3\bar{2}} + \omega^{\bar{1}\bar{3}}], \\ & [\omega^{13} + \omega^{1\bar{3}} + \omega^{2\bar{3}} + \rho\,\omega^{\bar{2}\bar{3}}],\, \delta_\rho\,[\omega^{12}],\, \delta_{\rho-1}[\omega^{2\bar{1}}]\,\rangle, \end{aligned}$$

where the notation $\delta_{\text{expression}}$ means that $\delta_{\text{expression}} = 1$ if expression $= 0$ is satisfied, and $\delta_{\text{expression}} = 0$ otherwise. Moreover, the cohomology classes satisfy some relations which depend on the value of $\rho$:

- if $\rho = 0$, i.e. $J$ is abelian, then $[\omega^{1\bar{1}}] = -[\omega^{1\bar{2}}] - D\,[\omega^{2\bar{2}}]$ and $[\omega^{2\bar{1}}] = [\omega^{1\bar{2}}]$;
- if $\rho = 1$, then $[\omega^{1\bar{1}}] = -[\omega^{2\bar{1}}] - D\,[\omega^{2\bar{2}}] + [\omega^{\bar{1}\bar{2}}]$ and $[\omega^{12}] = -[\omega^{1\bar{2}}] + [\omega^{2\bar{1}}] - [\omega^{\bar{1}\bar{2}}]$.

| Family II | | | | Stages | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | 1st | | 2nd | | 3rd | | 4th | | 5th | |
| $\mathfrak{g}$ | $\rho$ | $B$ | $c$ | pure | full | pure | full | pure | full | pure | full | pure | full |
| $\mathfrak{h}_7$ | 1 | 1 | 0 | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_9$ | 0 | 1 | 1 | ✓ | – | ✓ | – | – | – | – | ✓ | – | ✓ |
| $\mathfrak{h}_{10}$ | 1 | 0 | 1 | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{11}$ | 1 | $B \in \mathbb{R} \setminus \{0,1\}$ | $\lvert B-1\rvert$ | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{12}$ | 1 | $\mathfrak{Im}\, B \neq 0$ | $\lvert B-1\rvert$ | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{13}$ | 1 | $c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$<br>$\mathcal{S}(B,c) < 0$ | | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{14}$ | 1 | $c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$<br>$\mathcal{S}(B,c) = 0$ | | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{15}$ | 0 | 0 | 1 | ✓ | – | ✓ | – | ✓ | ✓ | – | ✓ | – | ✓ |
| | | 1 | 0 | ✓ | – | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | – | ✓ |
| | | | $c \neq 0,1$ | ✓ | – | ✓ | – | – | – | – | ✓ | – | ✓ |
| | 1 | 0 | 0 | ✓ | – | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | – | ✓ |
| | | | $c \neq 0,1$ | ✓ | – | – | – | – | – | – | – | – | ✓ |
| | | $\lvert B\rvert \neq 0,1$ | 0 | ✓ | – | – | ✓ | – | – | ✓ | – | – | ✓ |
| | | $c \neq \lvert B-1\rvert,\ (c,\lvert B\rvert) \neq (0,1)$,<br>$cB \neq 0,\quad \mathcal{S}(B,c) > 0$ | | ✓ | – | – | – | – | – | – | – | – | ✓ |
| $\mathfrak{h}_{16}$ | 1 | $\lvert B\rvert = 1,\ B \neq 1$ | 0 | ✓ | – | – | ✓ | – | – | ✓ | – | – | ✓ |

where $\mathcal{S}(B,c) = c^4 - 2\,(|B|^2+1)\,c^2 + (|B|^2-1)^2$.

Table 2.6: Complex-$\mathcal{C}^\infty$-pure-and-fullness at different stages for Family II.

| Family III | | Stages | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1st | | 2nd | | 3rd | | 4th | | 5th | |
| $\mathfrak{g}$ | $\varepsilon$ | pure | full | pure | full | pure | full | pure | full | pure | full |
| $\mathfrak{h}_{19}^-$ | 0 | ✓ | – | ✓ | – | – | – | – | ✓ | – | ✓ |
| $\mathfrak{h}_{26}^+$ | 1 | ✓ | ✓ | ✓ | – | – | – | – | ✓ | ✓ | ✓ |

Table 2.7: Complex-$\mathcal{C}^\infty$-pure-and-fullness at different stages for Family III.

In any case, the second Betti number is equal to 8.

Now, one can check that

$$H_J^{2,0}(\mathfrak{h}_4) = \langle\, [\omega^{12}]\,\rangle, \quad H_J^{0,2}(\mathfrak{h}_4) = \langle\, [\omega^{\bar{1}\bar{2}}]\,\rangle, \quad \text{and}$$

$$H_J^{1,1}(\mathfrak{h}_4) = \langle\, [\omega^{1\bar{2}}],\, [\omega^{2\bar{2}}],\, \delta_{\rho-1}[\omega^{1\bar{1}}],\, \delta_{\rho-1}[\omega^{2\bar{1}}],\, \delta_{\rho-1}\delta_{D+2}[\omega^{1\bar{3}} + 2\omega^{2\bar{3}} + \omega^{3\bar{1}} + 2\omega^{3\bar{2}}]\,\rangle.$$

Therefore, counting dimensions we conclude that none of the complex structures is complex-$\mathcal{C}^\infty$-full at the second stage, as the sum of $H_J^{2,0}(\mathfrak{h}_4)$, $H_J^{1,1}(\mathfrak{h}_4)$, and $H_J^{0,2}(\mathfrak{h}_4)$ never generates the whole second complex de Rham cohomology group.

What complex-$\mathcal{C}^\infty$-pureness concerns, one has the following. If $\rho = 0$, then we have

$$H_J^{1,1}(\mathfrak{h}_4) = \langle\, [\omega^{1\bar{2}}],\, [\omega^{2\bar{2}}]\,\rangle,$$

and the complex structure is complex-$\mathcal{C}^\infty$-pure at the 2nd stage (this also follows from Proposition 2.2.3 *ii)*). Finally, it is easy to see that the complex structures with $\rho = 1$ are not complex-$\mathcal{C}^\infty$-pure at the 2nd stage since, for example, the element $[\omega^{\bar{1}\bar{2}}] \in H_J^{0,2}(\mathfrak{h}_4)$ also belongs to $H_J^{1,1}(\mathfrak{h}_4)$ because $[\omega^{\bar{1}\bar{2}}] = [\omega^{1\bar{1}} + \omega^{2\bar{1}} + D\,\omega^{2\bar{2}}]$.

Similarly, one completes the calculations for the rest of the cases. As a consequence of this general study, we have found a new complex structure on a 6-dimensional nilpotent Lie algebra which is complex-$\mathcal{C}^\infty$-pure-and-full at every stage (see Table 2.5). Notice that it is abelian, as $\rho = 0$. Furthermore, it lives on the algebra $\mathfrak{h}_5$, which is precisely the one underlying the Iwasawa manifold. As a matter of notation, we use $\mathcal{N}$ for the corresponding real nilmanifold.

**Theorem 2.2.6.** *Let $X = (M, J)$ be a 6-dimensional nilmanifold, not a torus, endowed with an invariant complex structure. Then, $X$ is complex-$\mathcal{C}^\infty$-pure-and-full at every stage if and only if $X = (\mathcal{N}, \mathcal{I}^\rho)$, where $\rho \in \{0, 1\}$, and*

$$\mathcal{I}^\rho:\ \ d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \rho\,\omega^{12} + (1-\rho)\,\omega^{1\bar{2}}.$$

**Remark 2.2.7.** It is observed in [KS04] that the space of complex structures on the nilpotent Lie algebra $\mathfrak{h}_5$ has the homotopy type of the disjoint union of a point and a 2-sphere. Nevertheless, it is possible to connect these two disjoint components by means of generalized-complex structures $\mathcal{J}$ on $\mathfrak{h}_5$ [CG04]. In fact, these structures precisely connect, up to $B$-transforms and $\beta$-transforms, the two complex structures on $\mathfrak{h}_5$ that are complex-$\mathcal{C}^\infty$-pure-and-full at every stage. In [ACL15], we introduce the corresponding notions of pureness and fullness for generalized-complex structures and show that each $\mathcal{J}$ satisfies them. In addition, we provide a curve of almost-complex structures also joining the previous two complex structures which is, in contrast, not complex-$\mathcal{C}^\infty$-pure-and-full at the first stage.

By Theorem 1.2.13, it is well known that complex-$\mathcal{C}^\infty$-full at the $k$-th stage implies complex-$\mathcal{C}^\infty$-pure at the $(2n-k)$-th stage. In fact, suppose that there is a non-zero class

$$\mathbf{b} \in H_J^{p_1,q_1}(M) \cap H_J^{p_2,q_2}(M),$$

with $(p_1,q_1)\neq(p_2,q_2)$ such that $p_1+q_1=2n-k=p_2+q_2$. Since the pairing

$$\begin{array}{rrcl} p\colon & H^k_{\mathrm{dR}}(M;\mathbb{C})\times H^{2n-k}_{\mathrm{dR}}(M;\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & (\mathbf{a},\mathbf{b}) & \mapsto & p(\mathbf{a},\mathbf{b})=\int_M \alpha\wedge\beta \end{array}$$

for $\mathbf{a}=[\alpha]$ and $\mathbf{b}=[\beta]$, is non-degenerate, there is a non-zero class $\mathbf{a}\in H^k_{\mathrm{dR}}(M;\mathbb{C})$ such that $p(\mathbf{a},\mathbf{b})\neq 0$. Let us observe that we can write $\beta=\beta^{p_1,q_1}+d\eta$ and $\alpha=\sum_{r+s=k}\alpha^{r,s}+d\gamma$, with $d\alpha^{r,s}=0$. Then, $0\neq\alpha\wedge\beta=\sum_{r+s=k}\alpha^{r,s}\wedge\beta^{p_1,q_1}+d\tau$, for some complex form $\tau$. Note that only $\alpha^{r,s}$ for $r=k-(n-q_1)$ and $s=n-q_1$ remains. Similarly, one can take other representatives of the previous classes $\mathbf{a}$ and $\mathbf{b}$, $\beta=\beta^{p_2,q_2}+d\tilde\eta$ and $\alpha=\sum_{r+s=k}\tilde\alpha^{r,s}+d\tilde\gamma$, with $d\tilde\alpha^{r,s}=0$, and get $\alpha\wedge\beta=\tilde\alpha^{k-(n-q_2),n-q_2}\wedge\beta^{p_2,q_2}+d\tilde\tau$. Bearing in mind that $\beta^{p_2,q_2}=\beta^{p_1,q_1}+d(\eta-\tilde\eta)$, we see that:

$$\begin{aligned} \alpha^{k-(n-q_1),n-q_1}\wedge\beta^{p_1,q_1} &= \tilde\alpha^{k-(n-q_2),n-q_2}\wedge\beta^{p_2,q_2}+d(\tilde\tau-\tau) \\ &= \tilde\alpha^{k-(n-q_2),n-q_2}\wedge\beta^{p_1,q_1}+d\sigma, \end{aligned}$$

where $\sigma=\tilde\alpha^{k-(n-q_2),n-q_2}\wedge(\eta-\tilde\eta)+(\tilde\tau-\tau)$. As the left-hand side has degree $(n,n)$, it cannot be $d$-exact. Hence, the form $\tilde\alpha^{k-(n-q_2),n-q_2}\wedge\beta^{p_1,q_1}$ should also have degree $(n,n)$. However, this only occurs when $q_2=q_1$, and thus $(p_1,q_1)=(p_2,q_2)$, which is not possible. By induction, one finally proves that for any $N\geq 1$ the existence of a non-zero class

$$\mathbf{b}\in\left(\bigoplus_{i=1}^N H_J^{p_i,q_i}(M)\right)\cap H_J^{p,q}(M),$$

with $(p_i,q_i)\neq(p_j,q_j)$ for every $i\neq j$ and $(p_i,q_i)\neq(p,q)$ such that $p_i+q_i=2n-k=p+q$, leads to a similar contradiction. From here one gets the desired result.

However, in general it is not clear when the converse holds, that is, when pure at the $k$-th stage implies full at the $(2n-k)$-th stage. As a consequence of our study we get the following duality result (see Tables 2.5, 2.6, and 2.7):

**Proposition 2.2.8.** *Let $J$ be an invariant complex structure on a 6-dimensional nilmanifold $M$. Then, for any $1\leq k\leq 5$, $J$ is complex-$\mathcal{C}^\infty$-full at the $k$-th stage if and only if it is complex-$\mathcal{C}^\infty$-pure at the $(6-k)$-th stage.*

We still ignore if this is a consequence of the specific dimension of our nilmanifolds or maybe something general for invariant complex structures.

### 2.2.2 Different aspects of the second real stage

As we said at the beginning of Section 1.2.3, there is a particular interest in studying the (real) cohomological decomposition at the second stage. Here, we analyze the property for 6-dimensional nilmanifolds endowed with invariant complex structures and study its behaviour under holomorphic deformations.

At the sight of Tables 2.5, 2.6, and 2.7, one might wonder what happens with these manifolds when the properties of being $\mathcal{C}^\infty$-pure/full at the $k$-th stage are considered (recall Definition 1.2.10). In contrast with the complex first stage, the real first stage is trivial since $H_J^{(1,0),(0,1)}(M)_{\mathbb{R}} = H^1_{\mathrm{dR}}(M;\mathbb{R})$. In our case, we focus on the second stage, motivated by the open questions that we address below.

In the following result we prove that there exist precisely four complex structures satisfying the $\mathcal{C}^\infty$-pure-and-full property. Two of them live on the nilmanifold underlying the Iwasawa manifold (and coincide with those in Theorem 2.2.6) and the other two, on "*its* 3-*step analogue*". In more detail, the structure equations of these two nilmanifolds, which we will respectively denote by $\mathcal{N}_0$ and $\mathcal{N}_1$, are given by

$$\mathcal{N}_\epsilon: \ de^1 = de^2 = de^3 = 0, \quad de^4 = \epsilon\, e^{12}, \quad de^5 = e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23},$$

where $\epsilon \in \{0,1\}$. Notice that $\mathcal{N}_0$ corresponds to $\mathfrak{h}_5$ and $\mathcal{N}_1$ to $\mathfrak{h}_{15}$ in the notation of Theorem 1.4.20.

**Theorem 2.2.9.** *Let $X = (M, J)$ be a* 6*-dimensional nilmanifold, not a torus, endowed with an invariant complex structure. Then, $X$ is $\mathcal{C}^\infty$-pure-and-full if and only if $X = (\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$, where $\epsilon, \rho \in \{0,1\}$, and*

$$\mathcal{I}_\epsilon^\rho: \ d\omega^1 = 0, \quad d\omega^2 = \epsilon\, \omega^{1\bar{1}}, \quad d\omega^3 = \rho\, \omega^{12} + (1-\rho)\, \omega^{1\bar{2}}.$$

*Proof.* First recall that there are two complex-parallelizable nilmanifolds in dimension 6, defined by the equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\, \omega^{12},$$

where $\rho \in \{0,1\}$. One is the torus ($\rho = 0$) and the other one is the Iwasawa manifold ($\rho = 1$). The latter precisely corresponds to $(\mathcal{N}_0, \mathcal{I}_0^1)$ in the statement of the theorem. It is well known that these complex structures are $\mathcal{C}^\infty$-pure-and-full [AT11, FT10].

Next we use the description of the remaining invariant complex structures $J$ on 6-dimensional nilmanifolds obtained in [COUV16] (see Section 1.4). More concretely, we study Families I, II, and III, respectively given by equations (1.14), (1.15), and (1.16). In what follows, for the seek of simplicity we will write $H_J^\pm \equiv H_J^\pm(M)$.

- Let $J$ be a complex structure in Family I. If $\rho = 1$ then the following relation of the de Rham cohomology classes always holds:

$$[i(\omega^{12} - \omega^{\bar{1}\bar{2}})] = -2\,[i\,\omega^{1\bar{1}}] - \lambda\,[i(\omega^{1\bar{2}} + \omega^{2\bar{1}})] - 2\,\mathfrak{Re}\,D\,[i\,\omega^{2\bar{2}}].$$

Notice that the cohomology class on the left-hand side of this equality belongs to $H_J^+$ and the class on the right-hand side, to $H_J^-$. Thus, we have that $H_J^+ \cap H_J^- \neq \{\mathbf{0}\}$ and $J$ is not $\mathcal{C}^\infty$-pure. Hence, for complex structures in this family, we are led to consider the case $\rho = 0$, i.e. abelian complex structures. By Remark 2.2.4 we know that any abelian $J$ is $\mathcal{C}^\infty$-pure. Hence, we just need to see when $J$ is $\mathcal{C}^\infty$-full. Observe that the pureness of the structure implies that $H_J^+ + H_J^-$ is a direct sum, so the fullness of $J$ is equivalent to the condition $h_J^+ + h_J^- = b_2$.

According to the classification in [COUV16], for every abelian complex structure in Family I we can take $\lambda$ to be 0 or 1, and a direct computation shows that

$$\begin{aligned} H_J^+ &= \langle\, [i(\omega^{1\bar{2}}+\omega^{2\bar{1}})],\ \delta_\lambda[\omega^{1\bar{2}}-\omega^{2\bar{1}}],\ (\delta_{\lambda-1}+\delta_\lambda\delta_{\mathfrak{Im}D})[i\omega^{2\bar{2}}], \\ &\qquad \delta_D[\omega^{1\bar{3}}-\omega^{3\bar{1}}+\lambda(\omega^{2\bar{3}}-\omega^{3\bar{2}})],\ \delta_D[i(\omega^{1\bar{3}}+\omega^{3\bar{1}})+i\lambda(\omega^{2\bar{3}}+\omega^{3\bar{2}})]\,\rangle, \\ H_J^- &= \langle\, [\omega^{12}+\omega^{\bar{1}\bar{2}}],\ [i(\omega^{12}-\omega^{\bar{1}\bar{2}})],\ \delta_D[\omega^{13}+\omega^{\bar{1}\bar{3}}],\ \delta_D[i(\omega^{13}-\omega^{\bar{1}\bar{3}})]\,\rangle. \end{aligned}$$

Therefore, one has

$$h_J^+ + h_J^- = 3 + 4\,\delta_D + \delta_{\lambda-1} + \delta_\lambda(1+\delta_{\mathfrak{Im}\,D}) \leq 9.$$

Taking into account that $b_2 \geq 8$ for any of the underlying Lie algebras (see Table 2.4), it is clear that the complex structures with $D \neq 0$ are not $\mathcal{C}^\infty$-full. If $D=0$ and $\lambda=0$, the underlying algebra is $(0,0,0,0,0,12)$, but its second Betti number equals 11; thus, this complex structure is not $\mathcal{C}^\infty$-full. However, if $D=0$ and $\lambda=1$ then $J$ satisfies $h_J^+ + h_J^- = 4+4 = 8 = b_2$, and $J$ is $\mathcal{C}^\infty$-pure-and-full (notice that $J$ is equivalent to the complex structure $\mathcal{I}_0^0$ in the statement of the theorem).

- Let $J$ be a complex structure in Family II. If $\rho=1$ and $B=c=0$, then the second de Rham cohomology group is given by

$$H^2_{\rm dR}(M;\mathbb{R}) = \langle\, [\omega^{1\bar{2}}-\omega^{2\bar{1}}],\ [i(\omega^{1\bar{2}}+\omega^{2\bar{1}})],\ [i(\omega^{1\bar{3}}+\omega^{2\bar{2}}+\omega^{3\bar{1}})],\ [\omega^{13}+\omega^{\bar{1}\bar{3}}],\ [i(\omega^{13}-\omega^{\bar{1}\bar{3}})]\,\rangle.$$

Therefore, this complex structure, which is precisely $\mathcal{I}_1^1$, is clearly $\mathcal{C}^\infty$-pure-and-full.

Let us now suppose that $\rho=1$ and $(B,c)\neq(0,0)$. If $(\mathfrak{Im}\,B, \mathfrak{Re}\,B+c)\neq(0,0)$, then the following relation in the de Rham cohomology holds:

$$[i(\omega^{12}-\omega^{\bar{1}\bar{2}})] = \mathfrak{Im}\,B\,[\omega^{1\bar{2}}-\omega^{2\bar{1}}] - (\mathfrak{Re}\,B+c)\,[i(\omega^{1\bar{2}}+\omega^{2\bar{1}})],$$

whereas if $(\mathfrak{Im}\,B, \mathfrak{Re}\,B+c)=(0,0)$, then

$$[\omega^{12}+\omega^{\bar{1}\bar{2}}] = -2B\,[\omega^{1\bar{2}}-\omega^{2\bar{1}}].$$

Hence, in both cases we conclude that the structure is not $\mathcal{C}^\infty$-pure.

Hence, for complex structures in Family II, it remains to study the case $\rho=0$, i.e. the complex structure $J$ is abelian. As a consequence of Remark 2.2.4 we know that $J$ is $\mathcal{C}^\infty$-pure, so it suffices to see when the complex structure is $\mathcal{C}^\infty$-full. We proceed as in the case of Family I, now taking into account that we can suppose $B=0$ or 1 due to the choice of $\rho=0$ (see [COUV16] for more details). By direct calculation we get

$$H_J^+ = \langle\, \delta_B[\omega^{1\bar{3}}-\omega^{3\bar{1}}],\ \delta_B[i(\omega^{1\bar{3}}+\omega^{3\bar{1}})],\ \delta_{B-1}[i(\omega^{1\bar{3}}+\omega^{2\bar{2}}+\omega^{3\bar{1}})],\ \delta_{B-1}\delta_{c-1}\,[\omega^{1\bar{2}}-\omega^{2\bar{1}}]\,\rangle,$$

$$H_J^- = \langle\, [\omega^{12}+\omega^{\bar{1}\bar{2}}],\ [i(\omega^{12}-\omega^{\bar{1}\bar{2}})],\ \delta_c[\omega^{13}+\omega^{\bar{1}\bar{3}}],\ \delta_c[i(\omega^{13}-\omega^{\bar{1}\bar{3}})]\,\rangle.$$

Since the considered complex structures satisfy $h_J^+\leq 2$, $h_J^- = 2+2\delta_c$, and the underlying algebras have $b_2\geq 5$ (recall Table 2.4), it is clear that $c\neq 0$ implies non-$\mathcal{C}^\infty$-fullness.

If $c = 0$, then we must have $B = 1$. Since the underlying Lie algebra is $\mathfrak{h}_{15}$, we have $b_2 = 5 = 1 + 4 = h_J^+ + h_J^-$. Consequently, this complex structure, which is precisely $\mathcal{I}_1^0$, satisfies the $\mathcal{C}^\infty$-pure-and-full property.

- For complex structures $J$ in Family III, it is easy to see that

$$H_J^+ = \langle\, \delta_\varepsilon [i\omega^{1\bar{1}}]\,\rangle, \qquad H_J^- = \langle\, [\omega^{12} + \omega^{\bar{1}\bar{2}}],\, [i\,(\omega^{12} - \omega^{\bar{1}\bar{2}})]\,\rangle.$$

The absence of relations between the de Rham cohomology classes allows to conclude that these structures are always $\mathcal{C}^\infty$-pure. As the underlying Lie algebras have second Betti number $b_2 \geq 4$, it is clear that they are non-$\mathcal{C}^\infty$-full.

Finally, the Lie algebras underlying the complex structures $\mathcal{I}_\epsilon^\rho$ are obtained as follows. Take the real basis $\{e^1, \ldots, e^6\}$ given by

$$\omega^1 = e^1 + i\, e^2, \quad \omega^2 = -2\,(2\rho - 1)\, e^3 - 2i\, e^4, \quad \omega^3 = -2\,(2\rho - 1)\big(e^5 + i\, e^6\big)\,.$$

A direct calculation shows that

$$de^1 = de^2 = de^3 = 0, \quad de^4 = \epsilon\, e^{12}, \quad de^5 = e^{13} + e^{42}, \quad de^6 = e^{14} + e^{23},$$

so the underlying Lie algebra only depends on $\epsilon$. In fact, if $\epsilon = 0$ then the nilmanifold is the real manifold underlying the Iwasawa manifold (whose Lie algebra is $\mathfrak{h}_5 = (0, 0, 0, 0, 13 + 42, 14 + 23)$), whereas if $\epsilon = 1$ then the underlying Lie algebra is $\mathfrak{h}_{15} = (0, 0, 0, 12, 13 + 42, 14 + 23)$. □

**Remark 2.2.10.** The complex structure $\mathcal{I}_0^0$ on the manifold $\mathcal{N}_0$ coincides with the abelian complex structure in Theorem 2.2.6. The complex manifold $(\mathcal{N}_0, \mathcal{I}_0^1)$ is clearly the Iwasawa manifold.

**Remark 2.2.11.** Other stages of the real cohomological decomposition are also of interest. For instance, it can be seen in [AT12] that they serve to measure the difference between the balanced cone $\mathcal{C}_{\mathrm{b}}(M, J)$ and the strongly Gauduchon cone $\mathcal{C}_{\mathrm{sG}}(M, J)$ of a $2n$-dimensional compact complex manifold $(M, J)$. More concretely, if $\mathcal{C}_{\mathrm{b}}(M, J) \neq \emptyset$ (that is, $(M, J)$ is balanced) and $\mathbf{0} \notin \mathcal{C}_{\mathrm{sG}}(M, J)$, then

$$\mathcal{C}_{\mathrm{sG}}(M, J) \cap H_J^{(n-1,n-1)}(M)_{\mathbb{R}} = \mathcal{C}_{\mathrm{b}}(M, J),$$

$$\mathcal{C}_{\mathrm{b}}(M, J) + H_J^{(n,n-2),(n-2,n)}(M)_{\mathbb{R}} \subseteq \mathcal{C}_{\mathrm{sG}}(M, J).$$

Moreover, if $J$ is $C^\infty$-full at the $(2n-2)$-th stage, then the inequality above is an equality. Although the calculation of these cones is in general not easy, an explicit description can be found in [LUV14a] for complex nilmanifolds with underlying Lie algebra $\mathfrak{h}_{19}^-$.

Let us now move to the study of $\mathcal{C}^\infty$-pureness and $\mathcal{C}^\infty$-fullness under holomorphic deformations. We start recalling how the non-openness of the $\mathcal{C}^\infty$-pure-and-full property was obtained, in order to motivate our interest in the topic. Remember that Nakamura studied in [Nak75] the small deformations of the Iwasawa manifold, dividing them into three different classes according to their Hodge diamond. Each class was characterized as follows:

*Class (i):* $t_{11} = t_{12} = t_{21} = t_{22} = 0$,

*Class (ii):* $\mathcal{D}(\mathbf{t}) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$,

*Class (iii):* $\mathcal{D}(\mathbf{t}) \neq 0$,

where $t_{11}, t_{12}, t_{21}, t_{22}$ are parameters in the deformation space

$$\Delta = \left\{ \mathbf{t} = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in \mathbb{C}^6 \mid |\mathbf{t}| < \varepsilon \right\}$$

for a sufficiently small $\varepsilon > 0$, and $\mathcal{D}(\mathbf{t}) = t_{11}t_{22} - t_{12}t_{21}$.

In [AT11], Angella and Tomassini show that only class *(i)* is $\mathcal{C}^\infty$-pure-and-full and, as a consequence, they determine that $h^+ = h^- = 4$ for both the Iwasawa manifold and any small deformation in the class *(i)*. The remaining classes *(ii)* and *(iii)* are proved to be neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full, thus concluding that these properties are not open under holomorphic deformation.

In this setting, two questions arise. On the one hand, the dimensions $h^+$ and $h^-$ are not determined for the deformation classes *(ii)* and *(iii)*. On the other hand, the $\mathcal{C}^\infty$-pure and the $\mathcal{C}^\infty$-full properties fail simultaneously, remaining unclear how closely related their behaviours under holomorphic deformation are.

We start with the following result. It shows that $h^+$ remains constant and equal to 4 for any small deformation of the Iwasawa manifold.

**Proposition 2.2.12.** *For any sufficiently small deformation $X_{\mathbf{t}}$ of the Iwasawa manifold one has $h^+(X_{\mathbf{t}}) = 4$.*

*Proof.* The result is known for $X_{\mathbf{t}}$ in the class $(i)$. For the classes $(ii)$ and $(iii)$, one can proceed as in the proof of [AT11, Theorem 3.1] and write the complex structure equations of $X_{\mathbf{t}}$ as

$$\left\{ \begin{aligned} d\varphi^1_{\mathbf{t}} &= d\varphi^2_{\mathbf{t}} = 0, \\ d\varphi^3_{\mathbf{t}} &= \sigma_{12}\,\varphi^{12}_{\mathbf{t}} + \sigma_{1\bar{1}}\,\varphi^{1\bar{1}}_{\mathbf{t}} + \sigma_{1\bar{2}}\,\varphi^{1\bar{2}}_{\mathbf{t}} + \sigma_{2\bar{1}}\,\varphi^{2\bar{1}}_{\mathbf{t}} + \sigma_{2\bar{2}}\,\varphi^{2\bar{2}}_{\mathbf{t}}, \end{aligned} \right. \tag{2.6}$$

where the coefficients $\sigma_{12}, \sigma_{1\bar{1}}, \sigma_{1\bar{2}}, \sigma_{2\bar{1}}, \sigma_{2\bar{2}} \in \mathbb{C}$ depend only on $\mathbf{t}$, and are given by

$$\begin{aligned} \sigma_{12} &= -\gamma - \bar{\alpha}\,|t_{22}|^2 + \tfrac{1}{\bar{\gamma}}\,(\sigma_{1\bar{1}}\,\bar{\sigma}_{2\bar{2}})\,, & \sigma_{2\bar{1}} &= -\alpha\,\gamma\,(t_{11} - \bar{t}_{22}\mathcal{D}(\mathbf{t}))\,, \\ \sigma_{1\bar{1}} &= \bar{\alpha}\,\bar{\gamma}\,(t_{21} + \bar{t}_{21}\mathcal{D}(\mathbf{t}))\,, & \sigma_{2\bar{2}} &= -\alpha\,\gamma\,(t_{12} + \bar{t}_{12}\mathcal{D}(\mathbf{t}))\,, \\ \sigma_{1\bar{2}} &= \bar{\alpha}\,(t_{22} + (t_{12}\,\bar{t}_{11} + t_{22}\,\bar{t}_{12})\,\sigma_{1\bar{1}})\,, \end{aligned} \tag{2.7}$$

with $\alpha$ and $\gamma$ satisfying

$$\begin{aligned} \alpha &= \frac{1}{1 - |t_{22}|^2 - t_{21}\,\bar{t}_{12}}\,, \\ \gamma &= \frac{1}{1 - |t_{11}|^2 - t_{12}\bar{t}_{21} - \alpha\,\left(|t_{11}|^2 t_{21}\bar{t}_{12} + |t_{22}|^2 t_{12}\bar{t}_{21} + 2\mathfrak{Re}(t_{11}t_{22}\bar{t}_{12}\bar{t}_{21})\right)}\,. \end{aligned}$$

Observe that some of the coefficients have been rewritten in a different way with respect to [AT11], which will be more suitable for our purpose.

In order to compute the dimension of $H^+(X_{\mathbf{t}})$, we first determine the space of invariant closed $(1,1)$-forms on $X_{\mathbf{t}}$. It is clear that the $(1,1)$-forms $\varphi_{\mathbf{t}}^{1\bar{1}}$, $\varphi_{\mathbf{t}}^{1\bar{2}}$, $\varphi_{\mathbf{t}}^{2\bar{1}}$, and $\varphi_{\mathbf{t}}^{2\bar{2}}$ are closed. To see whether there exist any other closed $(1,1)$-forms, consider

$$\Phi = a\,\varphi_{\mathbf{t}}^{1\bar{3}} + b\,\varphi_{\mathbf{t}}^{2\bar{3}} + c\,\varphi_{\mathbf{t}}^{3\bar{1}} + e\,\varphi_{\mathbf{t}}^{3\bar{2}} + f\,\varphi_{\mathbf{t}}^{3\bar{3}},$$

where $a, b, c, e, f \in \mathbb{C}$. By (2.6), the condition $d\Phi = 0$ leads to $f = 0$ together with the system of equations:

$$\begin{cases} a\,\bar{\sigma}_{1\bar{2}} - b\,\bar{\sigma}_{1\bar{1}} + c\,\sigma_{12} = 0, \\ a\,\bar{\sigma}_{2\bar{2}} - b\,\bar{\sigma}_{2\bar{1}} + e\,\sigma_{12} = 0, \\ a\,\bar{\sigma}_{12} + c\,\sigma_{1\bar{2}} - e\,\sigma_{1\bar{1}} = 0, \\ b\,\bar{\sigma}_{12} + c\,\sigma_{2\bar{2}} - e\,\sigma_{2\bar{1}} = 0. \end{cases}$$

At the sight of (2.7), it is clear that the coefficient $\sigma_{12}$ is non-zero for any sufficiently small $\mathbf{t}$, so one has

$$c = \frac{b\,\bar{\sigma}_{1\bar{1}} - a\,\bar{\sigma}_{1\bar{2}}}{\sigma_{12}} \quad \text{and} \quad e = \frac{b\,\bar{\sigma}_{2\bar{1}} - a\,\bar{\sigma}_{2\bar{2}}}{\sigma_{12}},$$

but also

$$\begin{pmatrix} |\sigma_{12}|^2 - |\sigma_{1\bar{2}}|^2 + \sigma_{1\bar{1}}\bar{\sigma}_{2\bar{2}} & \sigma_{1\bar{2}}\bar{\sigma}_{1\bar{1}} - \sigma_{1\bar{1}}\bar{\sigma}_{2\bar{1}} \\ \sigma_{2\bar{1}}\bar{\sigma}_{2\bar{2}} - \sigma_{2\bar{2}}\bar{\sigma}_{1\bar{2}} & |\sigma_{12}|^2 - |\sigma_{2\bar{1}}|^2 + \sigma_{2\bar{2}}\bar{\sigma}_{1\bar{1}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant of the previous matrix never vanishes for any sufficiently small $\mathbf{t}$, we can conclude that $a = b = 0$ and thus $\Phi = 0$. Therefore, the space of invariant closed $(1,1)$-forms on $X_{\mathbf{t}}$ is generated by $\varphi_{\mathbf{t}}^{i\bar{k}}$, for $i, k = 1, 2$. Furthermore, any such form is never $d$-exact because the coefficient $\sigma_{12}$ does not vanish for sufficiently small $\mathbf{t}$. Then, we have

$$H^+(X_{\mathbf{t}}) = \langle\, [i\,\varphi_{\mathbf{t}}^{1\bar{1}}],\, [i\,\varphi_{\mathbf{t}}^{2\bar{2}}],\, [\varphi_{\mathbf{t}}^{1\bar{2}} - \varphi_{\mathbf{t}}^{2\bar{1}}],\, [i\,(\varphi_{\mathbf{t}}^{1\bar{2}} + \varphi_{\mathbf{t}}^{2\bar{1}})]\,\rangle,$$

for any sufficiently small deformation $X_{\mathbf{t}}$ in the classes *(ii)* and *(iii)*. This concludes the proof. □

In contrast, we have found an example that illustrates how drastically the subspace $H^-$ may change along a small deformation of the Iwasawa manifold.

**Example 2.2.13.** *There exists a small deformation $X_{\mathbf{t}}$ of the Iwasawa manifold satisfying $h^-(X_{\mathbf{t}}) = 1$ and $H^-(X_{\mathbf{t}}) \subset H^+(X_{\mathbf{t}})$ for every $\mathbf{t} \neq \mathbf{0}$.* It can be directly seen from (2.6) and (2.7) that $h^-(X_{\mathbf{t}}) \geq 1$ for any small deformation $X_{\mathbf{t}}$ of the Iwasawa manifold. We next show a particular deformation with $h^-(X_{\mathbf{t}}) = 1$ such that $H^-(X_{\mathbf{t}}) \subset H^+(X_{\mathbf{t}})$. Let us consider the small deformation given by $t_{12} = t_{21} = 0$ and $t_{11} = t_{22} = t$, with

$|t| < \varepsilon$. Observe that it belongs to class *(iii)*. The structure equations are given by (2.6), and to find their coefficients, it suffices to replace our specific values of $t_{11}, t_{12}, t_{21}$, and $t_{22}$ in (2.7):

$$\sigma_{12} = -\frac{1+|t|^2}{1-|t|^2}, \qquad \sigma_{1\bar{1}} = 0, \qquad \sigma_{1\bar{2}} = \frac{t}{1-|t|^2}, \qquad \sigma_{2\bar{1}} = -\frac{t}{1-|t|^2}, \quad \text{and} \quad \sigma_{2\bar{2}} = 0;$$

that is, the complex structure equations of $X_{\mathbf{t}}$ are

$$d\varphi^1_{\mathbf{t}} = d\varphi^2_{\mathbf{t}} = 0, \qquad d\varphi^3_{\mathbf{t}} = -\frac{1+|t|^2}{1-|t|^2}\,\varphi^{12}_{\mathbf{t}} + \frac{t}{1-|t|^2}\,\varphi^{1\bar{2}}_{\mathbf{t}} - \frac{t}{1-|t|^2}\,\varphi^{2\bar{1}}_{\mathbf{t}}.$$

For $t \neq 0$, a direct computation shows that only $\varphi^{12}_{\mathbf{t}} + \varphi^{\bar{1}\bar{2}}_{\mathbf{t}}$ and $i(\varphi^{12}_{\mathbf{t}} - \varphi^{\bar{1}\bar{2}}_{\mathbf{t}})$ define a cohomology class in $H^-(X_{\mathbf{t}})$. However, the following equality

$$(\mathfrak{Im}\,t)\,(\varphi^{12}_{\mathbf{t}} + \varphi^{\bar{1}\bar{2}}_{\mathbf{t}}) + (\mathfrak{Re}\,t)\,i(\varphi^{12}_{\mathbf{t}} - \varphi^{\bar{1}\bar{2}}_{\mathbf{t}}) = -\frac{1-|t|^2}{1+|t|^2}\,d(i\bar{t}\varphi^3_{\mathbf{t}} - it\varphi^{\bar{3}}_{\mathbf{t}})$$

implies that $h^-(X_{\mathbf{t}}) = 1$. Moreover, the space $H^-(X_{\mathbf{t}})$ is contained in $H^+(X_{\mathbf{t}})$ for $t \neq 0$ because the following relations are satisfied:

$$[\varphi^{12}_{\mathbf{t}} + \varphi^{\bar{1}\bar{2}}_{\mathbf{t}}] = \frac{2\,\mathfrak{Re}\,t}{1+|t|^2}\,[\varphi^{1\bar{2}}_{\mathbf{t}} - \varphi^{2\bar{1}}_{\mathbf{t}}] \in H^+(X_{\mathbf{t}}),$$

$$[i(\varphi^{12}_{\mathbf{t}} - \varphi^{\bar{1}\bar{2}}_{\mathbf{t}})] = -\frac{2\,\mathfrak{Im}\,t}{1+|t|^2}\,[\varphi^{1\bar{2}}_{\mathbf{t}} - \varphi^{2\bar{1}}_{\mathbf{t}}] \in H^+(X_{\mathbf{t}}).$$

◇

Drăghici, Li, and Zhang prove in [DLZ13] that for any curve of almost-complex structures $J_t$ on a 4-dimensional compact manifold $M^4$, $h^+_{J_t}(M^4)$ is a lower-semi-continuous function in $t$ whereas $h^-_{J_t}(M^4)$ is an upper-semi-continuous function in $t$, that is,

$$h^+_{J_{t_0}}(M^4) \leq h^+_{J_t}(M^4), \qquad h^-_{J_{t_0}}(M^4) \geq h^-_{J_t}(M^4),$$

for every $t$ sufficiently close to $t_0$. As we can see, every small deformation of the Iwasawa manifold shares the lower-semi-continuity of $h^+$, as a consequence of Proposition 2.2.12. In the case of $h^-$, Example 2.2.13 shows that the upper-semi-continuity of $h^-$ can also occur for higher dimensions. Nonetheless, Angella and Tomassini show in [AT12] that these properties are in general no longer true when the dimension increases. In order to do so, they construct two explicit families of curves $J_t$. Since their families $J_t$ are not $\mathcal{C}^\infty$-pure for $t \neq 0$, they pose the question of finding "*more fulfilling counterexamples*".

Next, we review the families constructed by the previous authors and give new examples satisfying similar behaviours for $h^+$ and $h^-$. Moreover, our families have a double purpose, as they serve to show that *pureness* and *fullness* do not need to fail simultaneously, in contrast with the small deformations of the Iwasawa manifold. That is, we prove that it is possible to lose $\mathcal{C}^\infty$-fullness while preserving $\mathcal{C}^\infty$-pureness, and also the converse.

In [AT12, Proposition 4.3] a curve of almost-complex structures $J_t$ on $S^3 \times \mathbb{T}^3$ is constructed, in such a way that $J_0$ is a $\mathcal{C}^\infty$-full complex structure, $J_t$ is an almost-complex structure which is not $\mathcal{C}^\infty$-pure for any $t \neq 0$, and $h^+_{J_0}(S^3 \times \mathbb{T}^3) = 3 > 1 = h^+_{J_t}(S^3 \times \mathbb{T}^3)$ for $t \neq 0$. In the following result we provide a curve of complex structures $J_t$ on $\mathcal{N}_0$, which is the nilmanifold underlying the Iwasawa manifold, constructed as an appropriate holomorphic deformation of the $\mathcal{C}^\infty$-pure-and-full abelian structure $\mathcal{I}_0^0$ found in Theorem 2.2.9. This curve $J_t$ is $\mathcal{C}^\infty$-pure for all $t$, and still satisfies $h^+_{J_0}(\mathcal{N}_0) > h^+_{J_t}(\mathcal{N}_0)$ for $t \neq 0$.

**Proposition 2.2.14.** *There exists a holomorphic family of compact complex manifolds $\{X_t = (M, J_t)\}_{t\in\Delta}$ of complex dimension 3, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that:*

*i) $X_t$ is $\mathcal{C}^\infty$-pure for every $t \in \Delta$;*

*ii) $X_0$ is $\mathcal{C}^\infty$-full, but $X_t$ is not $\mathcal{C}^\infty$-full for $t \in \Delta \setminus \{0\}$;*

*iii) $h^+(X_0) > h^+(X_t)$ for any $t \in \Delta \setminus \{0\}$.*

*Proof.* The proof is based on an appropriate deformation of the $\mathcal{C}^\infty$-pure-and-full complex nilmanifold $(\mathcal{N}_0, \mathcal{I}_0^0)$ found in Theorem 2.2.9. Nevertheless, we will next consider the more general situation $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^0)$ where $\epsilon = 0$ or 1, that is, the complex structure equations are

$$\mathcal{I}_\epsilon^0:\ \ d\omega^1 = 0, \qquad d\omega^2 = \epsilon\,\omega^{1\bar{1}}, \qquad d\omega^3 = \omega^{1\bar{2}}.$$

It is clear that the (0,1)-form $\omega^{\bar{3}}$ defines a Dolbeault cohomology class on $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^0)$. Hence, we can choose the class $[\omega^{\bar{3}}] \in H^{0,1}_{\bar{\partial}}(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^0)$ to perform an appropriate holomorphic deformation of $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^0)$. For each $t \in \mathbb{C}$ such that $|t| < 1$, define the complex structure $J_\epsilon^t$ on $\mathcal{N}_\epsilon$ given by the following basis $\{\eta_t^k\}_{k=1}^3$ of $(1,0)$-forms:

$$\eta_t^1 := \omega^1, \quad \eta_t^2 := \omega^2, \quad \eta_t^3 := \omega^3 + t\,\omega^{\bar{3}}.$$

The complex structure equations for $(\mathcal{N}_\epsilon, J_\epsilon^t)$ are

$$d\eta_t^1 = 0, \quad d\eta_t^2 = \epsilon\,\eta_t^{1\bar{1}}, \quad d\eta_t^3 = \eta_t^{1\bar{2}} - t\,\eta_t^{2\bar{1}}. \tag{2.8}$$

Observe that the initial structure $\mathcal{I}_\epsilon^0$ is recovered for $t = 0$. Since the complex structures $J_\epsilon^t$ are abelian, they are $\mathcal{C}^\infty$-pure by Remark 2.2.4.

A direct computation using (2.8) shows that for $(\mathcal{N}_\epsilon, J_\epsilon^t)$ one has

$$H^+_{J_\epsilon^t}(\mathcal{N}_\epsilon) = \begin{cases} \langle\, [\,i\eta_t^{1\bar{1}}\,],\ [\,i\eta_t^{2\bar{2}}\,],\ \delta_t\,[\,\eta_t^{2\bar{3}} - \eta_t^{3\bar{2}}\,],\ \delta_t\,[\,i\,(\eta_t^{2\bar{3}} + \eta_t^{3\bar{2}})\,]\,\rangle, & \text{if } \epsilon = 0, \\ \langle\, [\,i\,(\eta_t^{1\bar{3}} + \eta_t^{2\bar{2}} + \eta_t^{3\bar{1}})\,]\,\rangle, & \text{if } \epsilon = 1, \end{cases}$$

$$H^-_{J_\epsilon^t}(\mathcal{N}_\epsilon) = \langle\, [\,\tau_t^{12} + \tau_t^{\bar{1}\bar{2}}\,],\ [\,i\,(\tau_t^{12} - \tau_t^{\bar{1}\bar{2}})\,],\ \delta_t\,[\,\tau_t^{13} + \tau_t^{\bar{1}\bar{3}}\,],\ \delta_t\,[\,i\,(\tau_t^{13} - \tau_t^{\bar{1}\bar{3}})\,]\,\rangle,$$

together with the following relation for de Rham cohomology classes

$$[\,\tau_t^{1\bar{2}} - \tau_t^{2\bar{1}}\,] = [\,i\,(\tau_t^{1\bar{2}} + \tau_t^{2\bar{1}})\,] = \mathbf{0}, \qquad \delta_{1-\epsilon}\,[\,i\,\tau_t^{1\bar{1}}\,] = \mathbf{0}.$$

The previous description allows to conclude that, for any $t \neq 0$, the complex structure $J^t_\epsilon$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full, because $b_2(\mathcal{N}_0) = 8$ and $b_2(\mathcal{N}_1) = 5$.

Moreover, when $\epsilon = 0$ the family $X_t = (\mathcal{N}_0, J^t_0)$ satisfies $h^+_{\mathcal{I}^0_0}(\mathcal{N}_0) = 4 > 2 = h^+_{J^t_0}(\mathcal{N}_0)$ for any $t \in \Delta \setminus \{0\}$. Notice that $h^-_{\mathcal{I}^0_0}(\mathcal{N}_0) = 4 > 2 = h^-_{J^t_0}(\mathcal{N}_0)$ for $t \neq 0$. □

**Remark 2.2.15.** It follows from the previous proof that there is a holomorphic deformation $X_t$ of the $\mathcal{C}^\infty$-pure-and-full complex nilmanifold $(\mathcal{N}_1, \mathcal{I}^0_1)$ satisfying the properties *i)* and *ii)* in Proposition 2.2.14. Nonetheless, $h^+(X_t)$ remains constant along this deformation.

In [AT12, Proposition 4.1] one can find a curve of complex structures $J_t$ on a 10-dimensional nilmanifold such that $J_0$ is $\mathcal{C}^\infty$-pure-and-full, $J_t$ is not $\mathcal{C}^\infty$-pure for $t \neq 0$ and $h^-_{J_0} = 10 < 12 = h^-_{J_t}$ for $t \neq 0$. In the following result we construct a deformation in dimension 6 with a similar behaviour but also satisfying the $\mathcal{C}^\infty$-full property at every fiber of it. The proof is based on an appropriate holomorphic deformation of the $\mathcal{C}^\infty$-pure-and-full complex nilmanifold $(\mathcal{N}_1, \mathcal{I}^1_1)$ found in Theorem 2.2.9.

**Proposition 2.2.16.** *There exists a holomorphic family of compact complex manifolds* $\{X_t = (M, J_t)\}_{t \in \Delta}$ *of complex dimension* 3*, where* $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$*, such that:*

*i)* $X_t$ *is* $\mathcal{C}^\infty$*-full for every* $t \in \Delta$*;*

*ii)* $X_0$ *is* $\mathcal{C}^\infty$*-pure, but* $X_t$ *is not* $\mathcal{C}^\infty$*-pure for* $t \in \Delta \setminus \{0\}$*;*

*iii)* $h^-(X_0) < h^-(X_t)$ *for any* $t \in \Delta \setminus \{0\}$*.*

*Proof.* Let us take $J_0$ as the $\mathcal{C}^\infty$-pure-and-full complex structure $\mathcal{I}^1_1$ on the nilmanifold $\mathcal{N}_1$ found in Theorem 2.2.9. Now, as the form $\omega^{\bar 2}$ defines a non-zero Dolbeault cohomology class in $H^{0,1}_{\bar\partial}(\mathcal{N}_1, \mathcal{I}^1_1)$, it will be used to perform the following holomorphic deformation of $X_0 = (\mathcal{N}_1, \mathcal{I}^1_1)$. For each $t \in \mathbb{C}$ such that $|t| < 1$, we define the complex structure $J_t$ on $\mathcal{N}_1$ by the following basis $\{\eta^k_t\}^3_{k=1}$ of $(1,0)$-forms:

$$\eta^1_t := \omega^1, \quad \eta^2_t := \omega^2 + t\,\omega^{\bar 2}, \quad \eta^3_t := \omega^3.$$

The complex structure equations for $X_t = (\mathcal{N}_1, J_t)$ are

$$d\eta^1_t = 0, \quad d\eta^2_t = (1-t)\,\eta^{1\bar 1}_t, \quad d\eta^3_t = \frac{1}{1-|t|^2}\,\eta^{12}_t - \frac{t}{1-|t|^2}\,\eta^{1\bar 2}_t.$$

With respect to the new $(1,0)$-basis for $J_t$ given by

$$\tau^1_t = \eta^1_t, \quad \tau^2_t = \frac{1}{1-t}\,\eta^2_t, \quad \tau^3_t = \frac{1-|t|^2}{1-t}\,\eta^3_t,$$

we can rewrite the previous structure equations in a simpler way as

$$d\tau^1_t = 0, \quad d\tau^2_t = \tau^{1\bar 1}_t, \quad d\tau^3_t = \tau^{12}_t + B\,\tau^{1\bar 2}_t, \tag{2.9}$$

where $B=-\frac{t(1-\bar{t})}{1-t}$. Notice that the initial complex structure $\mathcal{I}_1^1$ is recovered for $t=0$. A direct calculation using (2.9) leads to

$$H^+_{J_t}(\mathcal{N}_1)=\langle\,[\,\tau_t^{1\bar{2}}-\tau_t^{2\bar{1}}\,],\,[\,i\,(\tau_t^{1\bar{2}}+\tau_t^{2\bar{1}})\,],\,[\,i\,\big((B-1)\,\tau_t^{1\bar{3}}+(|B|^2-1)\,\tau_t^{2\bar{2}}+(\bar{B}-1)\,\tau_t^{3\bar{1}}\big)\,]\,\rangle,$$

$$H^-_{J_t}(\mathcal{N}_1)=\langle\,[\,\tau_t^{13}+\tau_t^{\bar{1}\bar{3}}\,],\,[\,i\,(\tau_t^{13}-\tau_t^{\bar{1}\bar{3}})\,],\,(1-\delta_B)\,[\,\tau_t^{12}+\tau_t^{\bar{1}\bar{2}}\,],\,(1-\delta_B)\,[\,i\,(\tau_t^{12}-\tau_t^{\bar{1}\bar{2}})\,]\,\rangle.$$

Furthermore, the following cohomological relations hold:

$$\begin{aligned}
[\,\tau_t^{12}+\tau_t^{\bar{1}\bar{2}}\,] &= -\mathfrak{Re}B\,[\,(\tau_t^{1\bar{2}}-\tau_t^{2\bar{1}})\,]-\mathfrak{Im}B\,[\,i\,(\tau_t^{1\bar{2}}+\tau_t^{2\bar{1}})\,],\\
[\,i\,(\tau_t^{12}-\tau_t^{\bar{1}\bar{2}})\,] &= \ \ \mathfrak{Im}B\,[\,(\tau_t^{1\bar{2}}-\tau_t^{2\bar{1}})\,]-\mathfrak{Re}B\,[\,i\,(\tau_t^{1\bar{2}}+\tau_t^{2\bar{1}})\,].
\end{aligned}$$

Observe that, when $t=0$ (i.e. $B=0$), one has that $[\,\tau_t^{12}+\tau_t^{\bar{1}\bar{2}}\,]=[\,i\,(\tau_t^{12}-\tau_t^{\bar{1}\bar{2}})\,]=\mathbf{0}$. Otherwise, if $t\neq 0$ (i.e. $B\neq 0$), one can conclude that $J_t$ is not $\mathcal{C}^\infty$-pure, although it is $\mathcal{C}^\infty$-full because the second Betti number of the nilmanifold $\mathcal{N}_1$ equals 5.

Finally, counting dimensions we arrive at $h^-_{J_0}(\mathcal{N}_1)=2<4=h^-_{J_t}(\mathcal{N}_1)$ for any $t\neq 0$. Notice that $h^+_{J_t}(\mathcal{N}_1)=3$ remains constant. □

As a consequence of Propositions 2.2.14 and 2.2.16, we conclude that the properties of being $\mathcal{C}^\infty$-pure and being $\mathcal{C}^\infty$-full are not open in an independent way.

We now turn our attention to closedness and see that being $\mathcal{C}^\infty$-pure-and-full is not a closed property. In order to do so, we need to widen the class of manifolds we have been considering up to this moment.

*Solvmanifolds* $\Gamma\backslash G$ are defined in a similar way to nilmanifolds, simply letting the Lie group $G$ be solvable. Since every nilpotent Lie algebra is in particular solvable, it is clear that nilmanifolds are contained in this class. Let us notice that *invariant* complex structures $J$ can also be considered on solvmanifolds, as those complex structures arising from the ones defined on the Lie algebra $\mathfrak{g}$ of $G$. We will use the results about 6-dimensional solvmanifolds with holomorphically trivial canonical bundle obtained in [FOU15]. Concretely, let $G$ be the 6-dimensional simply-connected solvable Lie group whose Lie algebra $\mathfrak{g}$ is defined by the following structure equations:

$$\begin{cases} de^1=e^{16}-e^{25}, & de^3=-e^{36}+e^{45}, \quad de^5=0,\\ de^2=e^{15}+e^{26}, & de^4=-e^{35}-e^{46}, \quad de^6=0.\end{cases}$$

Let us consider the left-invariant almost-complex structure $J$ on the Lie group $G$ determined by

$$Je^1=e^2-e^5,\quad Je^2=-(e^1+e^6),\quad Je^3=e^4,\quad Je^4=-e^3,\quad Je^5=-e^6,\quad Je^6=e^5.$$

The forms

$$\omega^1=\frac{1}{2}\left(e^1-i(e^2-e^5)\right),\qquad \omega^2=e^3-i\,e^4,\qquad \omega^3=-\frac{1}{2}(e^5+i\,e^6)$$

have bidegree (1,0) with respect to $J$, and they satisfy

$$(2.10) \qquad d\omega^1 = 2\,i\,\omega^{13} + \omega^{3\bar{3}}, \qquad d\omega^2 = -2\,i\,\omega^{23}, \qquad d\omega^3 = 0.$$

Hence, $J$ is integrable and it defines a left-invariant complex structure on $G$.

The Lie algebra $\mathfrak{g}$ is precisely the real Lie algebra underlying the Nakamura manifold, and the Lie group $G$ admits lattices $\Gamma$, by [AK, Nak75]. Hence, $J$ defines a complex structure on any compact quotient solvmanifold $\Gamma\backslash G$.

Based on a result in [FOU15], in the following theorem we see that there is a lattice and a holomorphic deformation of the complex structure $J$ showing that being $\mathcal{C}^\infty$-pure-and-full is not a closed property.

**Theorem 2.2.17.** *There exists a holomorphic family of compact complex manifolds $\{X_t\}_{t\in\Delta}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that $X_t$ is $\mathcal{C}^\infty$-pure-and-full for every $t \in \Delta \setminus \{0\}$, but $X_0$ is neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full.*

*Proof.* Let $\Gamma\backslash G$ be any compact solvmanifold endowed with the invariant complex structure $J$ defined above. By the equations (2.10), the conjugate of the (1,0)-form $\omega^3$ defines a Dolbeault cohomology class, i.e. $[\omega^{\bar{3}}] \in H^{0,1}_{\bar\partial}(\Gamma\backslash G, J)$. Hence, for any $t \in \Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, we define the complex structure $J_t$ on $\Gamma\backslash G$ given by the following basis $\{\eta_t^k\}_{k=1}^3$ of $(1,0)$-forms:

$$\eta_t^1 := \omega^1, \quad \eta_t^2 := \omega^2, \quad \eta_t^3 := \omega^3 + t\,\omega^{\bar{3}}.$$

The complex structure equations for $(\Gamma\backslash G, J_t)$ are

$$(2.11) \qquad \begin{cases} d\eta_t^1 &= \frac{2i}{1-|t|^2}\,\eta_t^{13} - \frac{2it}{1-|t|^2}\,\eta_t^{1\bar{3}} + \frac{1}{1-|t|^2}\,\eta_t^{3\bar{3}}, \\ d\eta_t^2 &= -\frac{2i}{1-|t|^2}\,\eta_t^{23} + \frac{2it}{1-|t|^2}\,\eta_t^{2\bar{3}}, \\ d\eta_t^3 &= 0. \end{cases}$$

In [FOU15, Theorem 5.2] it is shown that every complex structure $J_t$ with $t \neq 0$ is equivalent to a complex structure obtained in [AK] as a certain deformation of the Nakamura manifold. In [AK, Proposition 4.1] it is proved that there is a lattice $\Gamma$ in $G$ such that the corresponding solvmanifold $M = \Gamma\backslash G$ endowed with the complex structure $J_t$ satisfies the $\partial\bar\partial$-lemma for any $t \neq 0$. Let us consider the holomorphic family of compact complex manifolds $X_t = (M, J_t)$, $t \in \Delta$. For every $t \in \Delta \setminus \{0\}$, it is clear that the manifold $X_t$ is $\mathcal{C}^\infty$-pure-and-full, as any compact complex manifold satisfying the $\partial\bar\partial$-lemma is.

Next we see that the central limit $X_0 = (M, J_0)$ of the holomorphic family $\{X_t\}_{t\in\Delta}$ is neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full. Since $\eta_0^k = \omega^k$, the complex structure equations (2.11) for $t = 0$ are precisely (2.10). The (real) form $i\,\omega^{3\bar{3}}$ is a closed $J_0$-invariant 2-form, and the form $\omega^{13} + \omega^{\bar{1}\bar{3}}$ is a real closed 2-form which is $J_0$-anti-invariant. Therefore, they define cohomology classes

$$(2.12) \qquad [i\,\omega^{3\bar{3}}] \in H^+_{J_0}(M), \qquad [\omega^{13} + \omega^{\bar{1}\bar{3}}] \in H^-_{J_0}(M).$$

Moreover, these classes are non-zero in $H^2_{\mathrm{dR}}(M;\mathbb{R})$. To see this, we can apply the symmetrization process described at the end of Section 1.4.1. Since the invariant forms $i\,\omega^{3\bar{3}}$ and $\omega^{13}+\omega^{\bar{1}\bar{3}}$ do not belong to the space $d(\bigwedge^1(\mathfrak{g}^*))$, we conclude that their cohomology classes in (2.12) are non-zero.

Additionally, from (2.10) we see that

$$i\,\omega^{3\bar{3}} - (\omega^{13}+\omega^{\bar{1}\bar{3}}) = \frac{1}{2}\,d(i\,\omega^1 - i\,\omega^{\bar{1}}).$$

Hence,

$$\mathbf{0} \neq [i\,\omega^{3\bar{3}}] = [\omega^{13}+\omega^{\bar{1}\bar{3}}] \in H^+_{J_0}(M) \cap H^-_{J_0}(M),$$

and $X_0 = (M, J_0)$ is not $\mathcal{C}^\infty$-pure.

Now we prove that $X_0$ is not $\mathcal{C}^\infty$-full. From the equations (2.10) we have that the real 2-form $2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}}$ is closed and defines a de Rham cohomology class

$$[2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}}] \in H^2_{\mathrm{dR}}(M;\mathbb{R}).$$

Suppose that the equality $H^2_{\mathrm{dR}}(M;\mathbb{R}) = H^+_{J_0}(M) + H^-_{J_0}(M)$ holds. Then, there exist $\mathbf{a} \in H^+_{J_0}(M)$ and $\mathbf{b} \in H^-_{J_0}(M)$ such that $[2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}}] = \mathbf{a}+\mathbf{b}$. Equivalently, there are $\alpha \in \mathcal{Z}^+_{J_0}(M)$, $\beta \in \mathcal{Z}^-_{J_0}(M)$, and $\gamma \in \Omega^1(M)$ such that

$$2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}} = \alpha+\beta+d\gamma, \tag{2.13}$$

where $\mathcal{Z}^+_{J_0}$, resp. $\mathcal{Z}^-_{J_0}$, is the space of real closed 2-forms that are $J_0$-invariant, resp. $J_0$-anti-invariant.

Applying the symmetrization process to (2.13), we get

$$2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}} = \tilde{\alpha}+\tilde{\beta}+d\tilde{\gamma},$$

for some $\tilde{\alpha} \in \mathcal{Z}^+_{J_0}(\mathfrak{g}^*)$, $\tilde{\beta} \in \mathcal{Z}^-_{J_0}(\mathfrak{g}^*)$, and $\tilde{\gamma} \in \bigwedge^1(\mathfrak{g}^*)$. However, this is not possible because a direct calculation from (2.10) yields

$$\mathcal{Z}^+_{J_0}(\mathfrak{g}^*)+\mathcal{Z}^-_{J_0}(\mathfrak{g}^*)+d\left(\bigwedge^1(\mathfrak{g}^*)\right) = \langle\, i\,\omega^{3\bar{3}},\ \omega^{13}+\omega^{\bar{1}\bar{3}},\ i(\omega^{13}-\omega^{\bar{1}\bar{3}}),\ \omega^{23}+\omega^{\bar{2}\bar{3}},\ i(\omega^{23}-\omega^{\bar{2}\bar{3}})\,\rangle.$$

In conclusion, $[2i\,\omega^{12}+\omega^{2\bar{3}}-\omega^{3\bar{2}}-2i\,\omega^{\bar{1}\bar{2}}] \notin H^+_{J_0}(M)+H^-_{J_0}(M)$, and thus $X_0$ is not $\mathcal{C}^\infty$-full. □

As a consequence of Theorem 2.2.17 we have:

**Corollary 2.2.18.** *For compact complex manifolds, the properties*

*i) "being $\mathcal{C}^\infty$-pure",*

*ii) "being $\mathcal{C}^\infty$-full", and*

*iii) "being $\mathcal{C}^\infty$-pure-and-full"*

*are not closed under holomorphic deformations.*

### 2.2.3 Relations with metric, complex, and topological properties

The idea of this section is to study $\mathcal{C}^\infty$-pureness and $\mathcal{C}^\infty$-fullness in relation with other properties. In particular, we are interested in the existence of certain types of Hermitian metrics and in some cohomological properties related to complex and topological aspects of our manifolds.

A *Hermitian-symplectic structure* on a compact complex manifold $X$ is a taming symplectic form. That is, a symplectic form $\omega$ on $X = (M, J)$ such that

$$\omega(V, JV) > 0,$$

for every non-zero $V \in \mathfrak{X}(M)$. In complex dimension 2, if $X$ has a Hermitian-symplectic structure then it admits a Kähler metric [LZ09, ST10]. Streets and Tian posed in [ST10] the problem of finding compact Hermitian-symplectic manifolds not admitting Kähler metrics. Up to date, the related results in the literature suggest that Hermitian-symplectic structures do not exist on non-Kähler manifolds. For instance, Enrietti, Fino, and Vezzoni show in [EFV12] that an invariant complex structure $J$ on a nilmanifold $M$ is tamed by a symplectic form if and only if $(M, J)$ is a complex torus (the only Kähler nilmanifold). Thus, one might think that any compact Hermitian-symplectic manifold should be $\mathcal{C}^\infty$-pure-and-full, although this has not yet been proved. In this context, it seems natural to ask to what extent the existence of a special non-Kähler Hermitian metric has an influence on the $\mathcal{C}^\infty$-pure-and-full property. Our first aim is to show that the latter property is unrelated to the existence of some special Hermitian metrics seen in Section 1.3, such as balanced, strongly Gauduchon, and SKT. We will also investigate the case of *locally conformal Kähler metrics*, defined by the condition $dF = \theta \wedge F$, where $F$ is the fundamental form and $\theta$ the (closed) Lee form. Simply recall that $\theta := Jd^*F$, being $d^*$ the co-differential induced by the Hodge star operator. Moreover, if $\theta$ is a nowhere vanishing parallel form then $F$ is called a *Vaisman metric*.

**Proposition 2.2.19.** *The compact complex nilmanifolds $(\mathcal{N}_1, \mathcal{I}_1^\rho)$, $\rho \in \{0, 1\}$, satisfy the $\mathcal{C}^\infty$-pure-and-full property, but they do not admit any SKT, locally conformal Kähler, or strongly Gauduchon metrics.*

*Proof.* By Theorem 2.2.9 we know that the complex structures $\mathcal{I}_1^\rho$ are $\mathcal{C}^\infty$-pure-and-full for $\rho = 0, 1$. The non-existence of SKT, locally conformal Kähler, or strongly Gauduchon metrics can be derived from classification results in [COUV16, EFV12, Uga07]. Nevertheless, for the sake of completeness we provide here a more direct and unified proof. First notice that by symmetrization (see Section 1.4.1) one can reduce the problem to simply studying invariant Hermitian metrics. Consider the complex equations in Theorem 2.2.9 for $\epsilon = 1$, and let $F$ be the fundamental 2-form of a generic invariant Hermitian metric, i.e.

$$2\,F = i\,(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + u\,\omega^{1\bar{2}} - \bar{u}\,\omega^{2\bar{1}} + v\,\omega^{2\bar{3}} - \bar{v}\,\omega^{3\bar{2}} + z\,\omega^{1\bar{3}} - \bar{z}\,\omega^{3\bar{1}},$$

where $r, s, t \in \mathbb{R} \setminus \{0\}$, $u, v, z \in \mathbb{C}$, satisfying the conditions that ensure the positive definiteness of the metric, namely, $r^2s^2 > |u|^2$, $s^2t^2 > |v|^2$, $r^2t^2 > |z|^2$, and $r^2s^2t^2 +$

$2\,\mathfrak{Re}\,(i\bar{u}\bar{v}z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2$. A direct calculation gives

$$(2.14)\quad 2\,\partial F = -\big(is^2 + \rho\bar{z} - (1-\rho)z\big)\,\omega^{12\bar{1}} - \rho\bar{v}\,\omega^{12\bar{2}} + i\rho t^2\,\omega^{12\bar{3}} + \bar{v}\,\omega^{13\bar{1}} - i(1-\rho)t^2\,\omega^{23\bar{1}},$$

which is never zero by the positive definiteness of the metric. Any (real) closed 1-form $\theta$ is given by $\theta = a\,\omega^1 + \bar{a}\,\omega^{\bar{1}} + b\,(\omega^2 + \omega^{\bar{2}})$, where $a \in \mathbb{C}$ and $b \in \mathbb{R}$. Since the coefficients of the 3-form $2\theta \wedge F$ in $\omega^{13\bar{3}}$ and $\omega^{23\bar{3}}$ are equal to $iat^2$ and $ibt^2$, respectively, it follows from (2.14) and the positive definiteness of the metric, that the condition $\partial F = \theta^{1,0} \wedge F$ is satisfied if and only if $a = b = 0$, i.e., $\theta = 0$. However, this contradicts $\partial F \neq 0$, so the manifold $(\mathcal{N}_1, \mathcal{I}_1^\rho)$ has no locally conformal Kähler metrics for $\rho = 0, 1$.

The nonexistence of SKT metrics follows directly from

$$2\,\partial\bar{\partial}F = -it^2\left(\rho^2 + (1-\rho)^2\right)\,\omega^{12\bar{1}\bar{2}} \neq 0.$$

Finally, it is straightforward to see that

$$4\partial F \wedge F = (1-\rho)(it^2u + \bar{v}z)\,\omega^{123\bar{1}\bar{2}} - (s^2t^2 - |v|^2)\,\omega^{123\bar{1}\bar{3}}$$

and $\bar{\partial}(\bigwedge^{3,1}) = \langle \rho\,\omega^{123\bar{1}\bar{2}} \rangle$. Therefore, $\partial F^2$ cannot be $\bar{\partial}$-exact because the positive definiteness of the metric implies $s^2t^2 > |v|^2$. We conclude that the manifold $(\mathcal{N}_1, \mathcal{I}_1^\rho)$ is not strongly Gauduchon for $\rho = 0, 1$. □

**Corollary 2.2.20.** *For compact complex manifolds, the $\mathcal{C}^\infty$-pure-and-full property is unrelated to the existence of SKT, locally conformal Kähler, Vaisman, balanced, or strongly Gauduchon metrics.*

*Proof.* In view of Theorem 2.2.9 and Proposition 2.2.19, it suffices to show the existence of SKT, Vaisman (hence locally conformal Kähler), and balanced (hence strongly Gauduchon) nilmanifolds that do not satisfy the $\mathcal{C}^\infty$-pure-and-full property, that is, that are not isomorphic to $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$, $\epsilon, \rho \in \{0, 1\}$. Again, the existence of such nilmanifolds can be derived from the classification results in [COUV16, EFV12, Uga07], but for the sake of completeness, we here provide an explicit example of each type.

First, let us consider the nilmanifold $\mathcal{N}_0$ endowed with the invariant complex structure defined by

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \frac{1}{2}\,\omega^{2\bar{2}},$$

i.e. it belongs to Family I with $\rho = 1$, $\lambda = 0$ and $D = 1/2$ (see (1.14) in Section 1.4.3). It is clear that the Hermitian metric $F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}})$ satisfies the SKT condition $\partial\bar{\partial}F = 0$, so we get a compact complex manifold that is SKT but not $\mathcal{C}^\infty$-pure-and-full.

Now, let us consider the nilmanifold $M$ whose underlying Lie algebra is $\mathfrak{h}_3$ in the notation of Theorem 1.4.20, endowed with the invariant complex structure in Family I for $\rho = \lambda = 0$ and $D = 1$, that is,

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}} + \omega^{2\bar{2}}.$$

The Hermitian metric $F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}})$ is a locally conformal Kähler metric whose Lee form $\theta = \omega^3 + \omega^{\bar{3}}$ is parallel. Hence, one has an example of a compact Vaisman manifold which does not satisfy the $\mathcal{C}^\infty$-pure-and-full property.

Finally, let us consider the nilmanifold $\mathcal{N}_0$ endowed with the invariant complex structure defined by

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} - \frac{1}{8}\,\omega^{2\bar{2}},$$

i.e. it corresponds to $\rho = 1$, $\lambda = 0$ and $D = -1/8$ in Family I. The Hermitian metric $F = 4i\,\omega^{1\bar{1}} + \frac{i}{2}(\omega^{2\bar{2}} + \omega^{3\bar{3}})$ satisfies the balanced condition, so we get a compact balanced manifold which is not $\mathcal{C}^\infty$-pure-and-full. □

Another interesting aspect is the existence of some relation between the behaviour of the Frölicher spectral sequence $\{E_r(X)\}_{r\geq 1}$ and the $\mathcal{C}^\infty$-pure-and-fullness of compact complex manifolds. As we already stated in Section 1.2.3, if the Frölicher spectral sequence of $X$ degenerates at the first step and there is a weight 2 formal Hodge decomposition, then $X$ is $\mathcal{C}^\infty$-pure-and-full. Note that by [COUV16, Theorem 4.1] we have that the $\mathcal{C}^\infty$-pure-and-full nilmanifolds $(\mathcal{N}_\epsilon, \mathcal{I}^\rho_\epsilon)$ found in Theorem 2.2.9 satisfy $E_1(\mathcal{N}_\epsilon, \mathcal{I}^\rho_\epsilon) \ncong E_2(\mathcal{N}_\epsilon, \mathcal{I}^\rho_\epsilon) \cong E_\infty(\mathcal{N}_\epsilon, \mathcal{I}^\rho_\epsilon)$ for $\epsilon, \rho \in \{0, 1\}$. Furthermore:

**Proposition 2.2.21.** *For compact complex manifolds, the $\mathcal{C}^\infty$-pure-and-full property and the degeneration of the Frölicher spectral sequence at the first step are unrelated. Moreover, there exists a compact complex manifold $X$ with $E_1(X) \cong E_\infty(X)$ whose Hodge numbers satisfy the symmetry $h^{p,q}_{\bar\partial}(X) = h^{q,p}_{\bar\partial}(X)$ for every $p, q \in \mathbb{N}$, which is neither $\mathcal{C}^\infty$-pure nor $\mathcal{C}^\infty$-full.*

*Proof.* Let us consider the complex nilmanifold $X$ defined by the complex structure equations in the Family I for $\rho = \lambda = 1$ and $D = 0$, i.e.

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}}.$$

By [COUV16, Proposition 4.3], $X$ has Frölicher spectral sequence degenerating at $E_1$ and its Hodge numbers satisfy $h^{p,q}_{\bar\partial}(X) = h^{q,p}_{\bar\partial}(X)$ for every $p, q \in \mathbb{N}$. However, $X$ is not $\mathcal{C}^\infty$-pure because $[i(\omega^{12} - \omega^{\bar{1}\bar{2}})] = -2\,[i\,\omega^{1\bar{1}}] - [i(\omega^{1\bar{2}} + \omega^{2\bar{1}})]$ is a non-zero de Rham cohomology class that belongs to $H^+(X) \cap H^-(X)$. In addition, one can check that the cohomology class $[\omega^{23} + \omega^{2\bar{3}} - \omega^{3\bar{2}} + \omega^{\bar{2}\bar{3}}]$ is a non-zero class in $H^2_{\mathrm{dR}}(X; \mathbb{R})$ which does not belong to $H^+(X) + H^-(X)$. □

**Remark 2.2.22.** It seems to be unknown whether there is some general relation between the $\mathcal{C}^\infty$-pure-and-full property and the degeneration of the Frölicher sequence at some step greater than 1. As we noticed above, the Frölicher sequence of the $\mathcal{C}^\infty$-pure-and-full complex nilmanifolds $(\mathcal{N}_\epsilon, \mathcal{I}^\rho_\epsilon)$, $\epsilon, \rho \in \{0, 1\}$, satisfies $E_1 \ncong E_2 \cong E_\infty$. This could indeed be seen as a restriction, because there are other complex nilmanifolds whose Frölicher sequences have $E_2 \ncong E_\infty$ (see [COUV16]). Also Bigalke and Rollenske provide in [BR14] a $(4n-2)$-dimensional complex nilmanifold $X_n$, for $n \geq 2$, whose Frölicher spectral sequence has $E_n \ncong E_\infty$. Following a suggestion by Angella, we studied the $\mathcal{C}^\infty$-pureness and $\mathcal{C}^\infty$-fullness of these manifolds $X_n$, observing that they are non-$\mathcal{C}^\infty$-pure for every $n \geq 3$. For $n = 2$, one has that $X_2$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

Let us now consider the complex invariants $\mathbf{f}_k(X)$ already introduced in Section 2.1.2 and defined by (2.2). Denote $\mathbf{f}(X) = \sum_{k=1}^{2n} \mathbf{f}_k(X)$. We observe that $\mathbf{f}(X) \geq 0$, since each $\mathbf{f}_k(X) \geq 0$. As a consequence of Theorem 1.2.5, one has that compact complex manifolds $X$ satisfying the $\partial\bar{\partial}$-lemma are characterized as those compact complex manifolds for which $\mathbf{f}(X) = 0$.

Since every compact 4-dimensional almost-complex manifold is $\mathcal{C}^\infty$-pure-and-full [DLZ10], we will now focus on the higher dimensional cases, i.e. real dimension greater than or equal to 6. As the $\partial\bar{\partial}$-lemma (equivalently, the vanishing of the complex invariant $\mathbf{f}$) implies $\mathcal{C}^\infty$-pure-and-fullness, one might expect to obtain low values of $\mathbf{f}$ for those compact complex manifolds satisfying the $\mathcal{C}^\infty$-pure-and-full property. Surprisingly, the propositions below suggest a possible relation between the $\mathcal{C}^\infty$-pure-and-full property and the complex structures for which $\mathbf{f}$ attains a maximal value (at least for nilmanifolds).

**Proposition 2.2.23.** *Let $M$ be a 6-dimensional nilmanifold endowed with an invariant complex structure $J$ such that the underlying Lie algebra of $M$ is not isomorphic to $\mathfrak{h}_7$. Let $\mathcal{X}$ be the set of all these pairs $(M, J)$. If $(\tilde{M}, \tilde{J}) \in \mathcal{X}$ is a pair such that $\mathbf{f}(\tilde{M}, \tilde{J}) \geq \mathbf{f}(M, J)$ for any $(M, J) \in \mathcal{X}$, then $(\tilde{M}, \tilde{J})$ is $\mathcal{C}^\infty$-pure-and-full.*

*Proof.* From Section 2.1.1 we know that the invariant $\mathbf{f}(M, J)$ can be computed for any pair $(M, J) \in \mathcal{X}$. It turns out that the maximal value of $\mathbf{f}(M, J)$ when $(M, J)$ runs the space $\mathcal{X}$ of all 6-nilmanifolds $M$ endowed with invariant complex structures $J$ is equal to 34. Furthermore, $(\tilde{M}, \tilde{J})$ satisfies that $\mathbf{f}(\tilde{M}, \tilde{J}) = 34$ if and only if $\tilde{M}$ is precisely the nilmanifold $\mathcal{N}_1$ and $\tilde{J}$ is isomorphic either to the complex structure $\mathcal{I}_1^0$ or to the complex structure $\mathcal{I}_1^1$ given in Theorem 2.2.9. Therefore, $(\tilde{M}, \tilde{J})$ is $\mathcal{C}^\infty$-pure-and-full. □

Recall that the minimal value of $\mathbf{f}(M, J)$ when $(M, J)$ runs the space of nilmanifolds $M$ of any dimension endowed with invariant complex structures $J$ is equal to 0, and it is attained at the complex tori, which are the only $\partial\bar{\partial}$-nilmanifolds.

**Proposition 2.2.24.** *Let $M$ be a nilmanifold, not a torus, of real dimension 6 admitting an invariant complex structure $\tilde{J}$ such that $(M, \tilde{J})$ is $\mathcal{C}^\infty$-pure-and-full. Then, $\mathbf{f}(M, J)$ attains a maximal value at $(M, \tilde{J})$ when $J$ runs the space of all the invariant complex structures on the nilmanifold $M$.*

*Proof.* By Theorem 2.2.9 it suffices to look at the spaces of all invariant complex structures on the nilmanifolds $\mathcal{N}_0$ and $\mathcal{N}_1$. For $M = \mathcal{N}_1$ it is clear that $\Delta$ attains a maximal value at the complex structures $\mathcal{I}_1^0$ and $\mathcal{I}_1^1$, as a consequence of Proposition 2.2.23.

Let $J$ be any invariant complex structure on $M = \mathcal{N}_0$. From [AFR15, Table 2] (see also Table 2.1 in Section 2.1.1) we have that $\mathbf{f}(\mathcal{N}_0, J)$ attains its maximal value when $J$ runs the space of all the invariant complex structures on $\mathcal{N}_0$ (which is equal to 24) if and only if $J$ is isomorphic to $\mathcal{I}_0^0$ or $\mathcal{I}_0^1$. □

These results seem to suggest the existence of some possible relationship between the $\mathcal{C}^\infty$-pure-and-full property and those pairs $(M, J)$ where the complex invariant $\mathbf{f}(M, J)$ attains maximal values.

We finish this section showing how topology might affect the preservation of *pureness* and *fullness* when the product of two $\mathcal{C}^\infty$-pure-and-full manifolds is considered. In [DLZ12, Proposition 2.7] Drăghici, Li, and Zhang prove the following result:

**Proposition 2.2.25.** [DLZ12] *Suppose $(M_1, J_1)$ and $(M_2, J_2)$ are compact almost-complex manifolds, both $\mathcal{C}^\infty$-pure-and-full, and assume that $b_1(M_1) = 0$ or $b_1(M_2) = 0$. Then $(M_1 \times M_2, J_1 + J_2)$ is $\mathcal{C}^\infty$-pure-and-full.*

They ask if the statement holds without any assumption on the first Betti number $b_1$. We next see that, even in the complex case, the previous result does not hold when both $b_1(M_1)$ and $b_1(M_2)$ are different from zero. For the construction we will consider the Kodaira-Thurston manifold $\mathbb{KT}$ (Example 1.4.6). Recall that this manifold has (real) dimension 4, so it is $\mathcal{C}^\infty$-pure-and-full by Theorem 1.2.8. A complex $m$-dimensional torus $\mathbb{T}^m$ is trivially $\mathcal{C}^\infty$-pure-and-full because it is Kähler. In the following example we show that the product $\mathbb{KT} \times \mathbb{T}^m$ is not $\mathcal{C}^\infty$-full.

**Example 2.2.26.** *For any $m \geq 1$, the compact complex manifold $X = \mathbb{KT} \times \mathbb{T}^m$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.* Writing the complex structure equations on $\mathbb{T}^m$ as $d\omega^k = 0$, for $3 \leq k \leq m+2$, we have that the structure equations for the complex nilmanifold $X = (M, J)$ are

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \cdots = d\omega^{m+2} = 0. \tag{2.15}$$

Let us first see the case $m = 1$. Since Proposition 1.4.19 also holds for $H_J^{(2,0),(0,2)}(M)$, we get

$$\begin{aligned} H_J^+(M) &= H_J^{(1,1)}(M) \cap H^2_{\mathrm{dR}}(M;\mathbb{R}) \\ &= \langle [\omega^{1\bar{2}} - \omega^{2\bar{1}}], [i\,\omega^{1\bar{2}} + i\,\omega^{2\bar{1}}], [\omega^{1\bar{3}} - \omega^{3\bar{1}}], [i\,\omega^{1\bar{3}} + i\,\omega^{3\bar{1}}], [i\,\omega^{3\bar{3}}] \rangle, \\ H_J^-(M) &= H_J^{(2,0),(0,2)}(M) \cap H^2_{\mathrm{dR}}(M;\mathbb{R}) \\ &= \langle [\omega^{12} + \omega^{\bar{1}\bar{2}}], [i\,\omega^{12} - i\,\omega^{\bar{1}\bar{2}}], [\omega^{13} + \omega^{\bar{1}\bar{3}}], [i\,\omega^{13} - i\,\omega^{\bar{1}\bar{3}}] \rangle. \end{aligned}$$

However, it is easy to see that

$$H^2_{\mathrm{dR}}(M;\mathbb{R}) = H_J^+(M) \oplus H_J^-(M) \oplus \langle [\omega^{23} + \omega^{2\bar{3}} - \omega^{3\bar{2}} + \omega^{\bar{2}\bar{3}}], [i\,\omega^{23} - i\,\omega^{2\bar{3}} - i\,\omega^{3\bar{2}} - i\,\omega^{\bar{2}\bar{3}}] \rangle,$$

so the complex product manifold $X$ is $\mathcal{C}^\infty$-pure but not $\mathcal{C}^\infty$-full.

In general, i.e. for any $m \geq 1$, let us observe that (2.15) implies that the complex structure $J$ on $M$ is abelian. Therefore, by Remark 2.2.4 we have that $X$ is $\mathcal{C}^\infty$-pure. Nevertheless, $X$ is not $\mathcal{C}^\infty$-full because the de Rham cohomology classes

$$[\omega^{2k} + \omega^{2\bar{k}} - \omega^{k\bar{2}} + \omega^{\bar{2}\bar{k}}], \quad [i\,\omega^{2k} - i\,\omega^{2\bar{k}} - i\,\omega^{k\bar{2}} - i\,\omega^{\bar{2}\bar{k}}], \quad 3 \leq k \leq m+2,$$

do not belong to the sum $H_J^+(M) \oplus H_J^-(M)$: this is a direct consequence of the fact that the invariant real exact 2-forms on $M$ belong to the space generated by $i\,\omega^{1\bar{1}}$. $\diamond$

Another example in (real) dimension 8 can be obtained by using the product of two Kodaira-Thurston manifolds.

**Example 2.2.27.** *The compact complex manifold* $X = \mathbb{KT} \times \mathbb{KT}$ *is* $\mathcal{C}^\infty$*-pure but not* $\mathcal{C}^\infty$*-full.* We write the complex structure equations for $X = (M, J)$ as

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}}, \quad d\omega^4 = \omega^{2\bar{2}}.$$

Notice that the complex structure $J$ of $M$ is again abelian, so $X$ is $\mathcal{C}^\infty$-pure (see Remark 2.2.4). However, $X$ is not $\mathcal{C}^\infty$-full because the de Rham cohomology classes

$$[\omega^{14} + \omega^{1\bar{4}} - \omega^{4\bar{1}} + \omega^{\bar{1}\bar{4}}],\ [\omega^{23} + \omega^{2\bar{3}} - \omega^{3\bar{2}} + \omega^{\bar{2}\bar{3}}],\ [\omega^{34} + \omega^{3\bar{4}} - \omega^{4\bar{3}} + \omega^{\bar{3}\bar{4}}],$$

$$[i\,\omega^{14} - i\,\omega^{1\bar{4}} - i\,\omega^{4\bar{1}} - i\,\omega^{\bar{1}\bar{4}}],\ [i\,\omega^{23} - i\,\omega^{2\bar{3}} - i\,\omega^{3\bar{2}} - i\,\omega^{\bar{2}\bar{3}}],\ [i\,\omega^{34} - i\,\omega^{3\bar{4}} - i\,\omega^{4\bar{3}} - i\,\omega^{\bar{3}\bar{4}}],$$

do not belong to the direct sum $H^+_J(M) \oplus H^-_J(M)$. This is due to the fact that the invariant real exact 2-forms on $X$ belong to the space generated by $i\,\omega^{1\bar{1}}$ and $i\,\omega^{2\bar{2}}$. ◇

Notice that in Examples 2.2.26 and 2.2.27 the first Betti numbers are far from being zero; in fact, $b_1(\mathbb{KT}) = 3$ and $b_1(\mathbb{T}^m) = 2m$. We do not know if the statement of Proposition 2.2.25 holds if $b_1(M_1) = 1$ or $b_1(M_2) = 1$.

Chapter 3

# Construction of invariant complex structures

When working in Complex Geometry, the problem of finding (integrable almost-) complex structures on a real differentiable manifold arises in a natural way. A given differentiable manifold $M$ could satisfy different properties as a complex manifold depending on the complex structure $J$ it is endowed with. As we have shown in the previous chapter, the Iwasawa manifold is complex-$\mathcal{C}^{\infty}$-pure-and-full at every stage. However, if its underlying real nilmanifold is endowed with another complex structure, this is no longer true (see Section 2.2.1). Something similar happens with respect to the existence of special Hermitian metrics, such as balanced or SKT.

Observe that determining every possible complex structure $J$ on a given manifold $M$ is not an easy task. Nevertheless, the problem can be slightly simplified for nilmanifolds when invariant complex structures are considered. In fact, the only known classifications in four and six dimensional nilmanifolds are those of invariant complex structures (recall Section 1.4.3). However, little is yet known in higher dimensions, even in this specific situation. One reason is the lack of a full list of nilpotent Lie algebras in dimensions greater than seven [Gon98], which would prevent a similar approach to the question. In fact, when dealing with the higher dimensional cases, most of the efforts have been directed to obtain algebraic constraints to the existence of invariant complex structures. See for example the work by Goze and Remm [GR02], showing the non-existence of complex structures on filiform Lie algebras, the paper by Vergnolle and Remm [VR09] proving the non-existence on quasi-filiform Lie algebras, or the recent work by Millionshchikov [Mil], bounding the nilpotency step of those algebras admitting complex structures. Although some classification results have been achieved in 8 dimensions, they generally require stronger conditions, such as the existence of hypercomplex structures [DF03], SKT metrics [EFV12], or balanced metrics with abelian complex structures [AV]. Some partial construction results have been recently obtained in [CSCO15].

In this chapter, we provide an strategy to find any complex structure $J$ on any $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ without the need of knowing the involved (real) algebras in advance. Indeed, two methods are introduced according to the degree of nilpotency of the complex structure $J$ to be constructed, in the sense of Definition 3.1.1, which is motivated by the paper [CFGU00]. Since these two methods complement each other, any possible pair $(\mathfrak{g}, J)$ could be detected. As an application, we recover the classification of invariant complex structures on four and six dimensional nilmanifolds from this new point of view. Furthermore, we parametrize all the invariant complex geometry in 8 dimensions that appears as an extension of the lower dimensional cases. This gives rise to *quasi-nilpotent* complex structures, which are those having "some nilpotency". The classification of *strongly non-nilpotent* complex structures in dimension 8 will be accomplished in the next chapter.

## 3.1 Two complementary methods

We start this section recalling some series related to nilpotent Lie algebras (NLAs for short) and complex structures. In particular, we will see that one of these series allows to measure the nilpotency of the complex structure, thus giving the starting point for our construction.

Let $\mathfrak{g}$ be an NLA of real dimension $2n$. The *descending central series* of $\mathfrak{g}$ is given by

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots, \text{ where } \begin{cases} \mathfrak{g}^0 = \mathfrak{g}, \text{ and} \\ \mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \text{ for } k \geq 1. \end{cases} \tag{3.1}$$

The *nilpotency step* of $\mathfrak{g}$ is the smallest integer $s$ for which $\mathfrak{g}^k = \{0\}$, for every $k \geq s$.

The *ascending central series* of $\mathfrak{g}$ is

$$\mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \ldots \subseteq \mathfrak{g}_k \subseteq \ldots, \text{ where } \begin{cases} \mathfrak{g}_0 = \{0\}, \text{ and} \\ \mathfrak{g}_k = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{k-1}\}, \text{ for } k \geq 1. \end{cases} \tag{3.2}$$

Remark that $\mathfrak{g}_1$ is the center of $\mathfrak{g}$. Equivalently, one can define the nilpotency step of $\mathfrak{g}$ in terms of this second series as the smallest integer $r$ for which $\mathfrak{g}_k = \mathfrak{g}$, for every $k \geq r$. Simply note that $r = s$. Thus, we will call *dimension of the ascending central series* the $s$-tuple

$$(\dim \mathfrak{g}_k)_k := (\dim \mathfrak{g}_1, \ldots, \dim \mathfrak{g}_s).$$

Since $\mathfrak{g}$ is nilpotent, it is clear that $\dim \mathfrak{g}_s = 2n$ and $\dim \mathfrak{g}_{s-1} \leq 2(n-1)$. Despite its apparent simplicity, the previous $s$-tuple encodes important information about $\mathfrak{g}$. For instance, if we have $(\dim \mathfrak{g}_k)_k = (1, 4, 6)$, then we know that $\mathfrak{g}$ is a 3-step nilpotent Lie algebra of dimension 6 whose center is 1-dimensional. In particular, if $\mathfrak{g}$ and $\mathfrak{g}'$ are two NLAs such that $(\dim \mathfrak{g}_k)_k \neq (\dim \mathfrak{g}'_k)_k$, then $\mathfrak{g}$ and $\mathfrak{g}'$ are not isomorphic.

Let us now consider a complex structure $J$ on $\mathfrak{g}$; that is, an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -id$ and the "Nijenhuis condition"

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \quad \forall X, Y \in \mathfrak{g}. \tag{3.3}$$

Observe that the previous series (3.1) and (3.2) are purely algebraic and do not take into consideration the complex framework induced by $J$. Therefore, a new series is introduced in [CFGU00], adapted to the complex structure $J$. The *ascending series of* $\mathfrak{g}$ *compatible with* $J$ is defined by

$$\mathfrak{a}_0(J) \subseteq \mathfrak{a}_1(J) \subseteq \ldots \subseteq \mathfrak{a}_k(J) \subseteq \ldots, \tag{3.4}$$

where $\begin{cases} \mathfrak{a}_0(J) = \{0\}, \text{ and} \\ \mathfrak{a}_k(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J) \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J)\}, \text{ for } k \geq 1. \end{cases}$

Note that every $\mathfrak{a}_k(J) \subseteq \mathfrak{g}_k$ is a $J$-invariant ideal of $\mathfrak{g}$, and in particular, $\mathfrak{a}_1(J)$ is the largest subspace of the center $\mathfrak{g}_1$ which is invariant under $J$. For simplicity, we will denote $\dim \mathfrak{a}_k(J) = 2n_k$ (possibly zero).

**Definition 3.1.1.** *The complex structure $J$ is said to be:*

*i)* quasi-nilpotent*, if it satisfies $\mathfrak{a}_1(J) \neq \{0\}$; moreover, $J$ will be called*

- *a)* nilpotent [CFGU00]*, if there exists an integer $t > 0$ such that $\mathfrak{a}_t(J) = \mathfrak{g}$,*
- *b)* weakly non-nilpotent*, if there is an integer $t > 0$ such that $\mathfrak{a}_t(J) = \mathfrak{a}_l(J)$, for every $l \geq t$, but $\mathfrak{a}_t(J) \neq \mathfrak{g}$;*

*ii)* strongly non-nilpotent *or* SnN [CFGU97b]*, if $\mathfrak{a}_1(J) = \{0\}$.*

Let us remark that the first division above depend on whether the ascending central series of $\mathfrak{g}$ adapted to $J$ can be constructed or not. Also notice that non-nilpotent structures are those satisfying $\mathfrak{a}_k(J) \neq \mathfrak{g}$, for every $k \geq 1$, and can be either *weakly* or *strongly* non-nilpotent. In particular, it is worth observing that the classification of complex structures might be accomplished from different points of view.

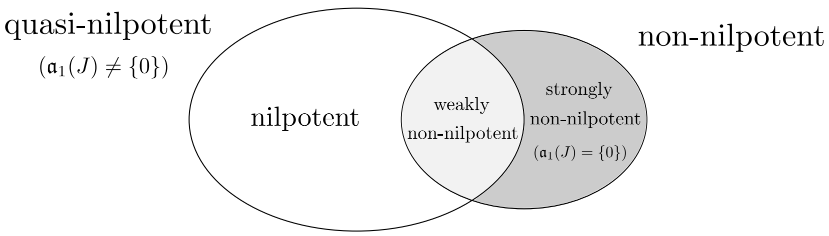



Figure 3.1: Partition of the space of complex structures.

Let us illustrate the previous definitions with some examples.

**Example 3.1.2. The Kodaira-Thurston manifold.** Recall that its underlying real nilpotent Lie algebra is

$$\mathfrak{g} = \langle X_1, X_2, X_3, X_4 \mid [X_1, X_2] = -X_4 \rangle.$$

It is easy to see that $\mathfrak{g}_1 = \langle X_3, X_4 \rangle$ and $\mathfrak{g}_2 = \mathfrak{g}$. The almost-complex structure $J$ defined in Example 1.4.2, $JX_1 = X_2$, $JX_3 = X_4$, satisfies the Nijenhuis condition (3.3), so it is a complex structure on $\mathfrak{g}$. Observe that $\mathfrak{a}_1(J) = \mathfrak{g}_1$ and $\mathfrak{a}_2(J) = \mathfrak{g}$, and thus $J$ is nilpotent. ◇

**Example 3.1.3. The Iwasawa manifold.** Consider its associated real nilpotent Lie algebra $\mathfrak{g}$, generated by $\{X_k\}_{k=1}^6$ satisfying

$$[X_1, X_3] = -[X_2, X_4] = -X_5, \quad [X_1, X_4] = [X_2, X_3] = -X_6.$$

Notice that $\mathfrak{g}_1 = \langle X_5, X_6\rangle$ and $\mathfrak{g}_2 = \mathfrak{g}$. The complex structure $J$ on $\mathfrak{g}$ is given by the almost-complex structure defined in Example 1.4.7,

$$JX_1 = X_2, \quad JX_3 = X_4, \quad JX_5 = X_6, \tag{3.5}$$

which satisfies the Nijenhuis condition. One has $\mathfrak{a}_1(J) = \mathfrak{g}_1$ and $\mathfrak{a}_2(J) = \mathfrak{g}$, so $J$ is nilpotent. ◇

**Example 3.1.4. A nilpotent $J$ with $\mathfrak{a}_1(J) \neq \mathfrak{g}_1$.** Let us consider the following 6-dimensional real nilpotent Lie algebra $\mathfrak{g}$ defined by the brackets

$$[X_1, X_3] = -X_5, \quad [X_2, X_3] = -X_6$$

endowed with the almost-complex structure defined by (3.5). Observe that $J$ satisfies the Nijenhuis condition, so it is a complex structure on $\mathfrak{g}$. One can see that $\mathfrak{g}_1 = \langle X_4, X_5, X_6\rangle$ and $\mathfrak{g}_2 = \mathfrak{g}$, whereas $\mathfrak{a}_1(J) = \langle X_5, X_6\rangle$ and $\mathfrak{a}_2(J) = \mathfrak{g}$. It can be proved that this $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_6$ in Theorem 1.4.20. ◇

**Example 3.1.5. A weakly non-nilpotent complex structure.** Let $\mathfrak{g}$ be the 8-dimensional real nilpotent Lie algebra defined by

$$[X_1, X_3] = [X_2, X_4] = X_6, \quad [X_3, X_5] = -X_1, \quad [X_4, X_5] = -X_2.$$

Let $J$ be the almost-complex structure $JX_1 = X_2,\ JX_3 = X_4,\ JX_5 = X_6,\ JX_7 = X_8$. Observe that $J$ satisfies the Nijenhuis condition, so it is a complex structure on $\mathfrak{g}$. On the one hand, $\mathfrak{g}_1 = \langle X_6, X_7, X_8\rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, X_6, X_7, X_8\rangle$, and $\mathfrak{g}_3 = \mathfrak{g}$. On the other hand, it is easy to see that $\mathfrak{a}_1(J) = \mathfrak{a}_l(J) = \langle X_7, X_8\rangle$, for every $l \geq 2$. ◇

**Example 3.1.6. A strongly non-nilpotent complex structure.** Let $\mathfrak{g}$ be the 6-dimensional real nilpotent Lie algebra given by

$$[X_1, X_3] = [X_2, X_4] = X_6, \quad [X_3, X_5] = -X_1, \quad [X_4, X_5] = -X_2.$$

Define an almost-complex structure $J$ on $\mathfrak{g}$ following (3.5). Since the Nijenhuis condition (3.3) is satisfied, $J$ is a complex structure on $\mathfrak{g}$. Note that $\mathfrak{g}_1 = \langle X_6\rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, X_6\rangle$, and $\mathfrak{g}_3 = \mathfrak{g}$. However, $\mathfrak{a}_1(J) = \{0\}$. Observe that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{19}^-$. ◇

The series (3.4) measures the nilpotency of the complex structure $J$; weakly non-nilpotent complex structures would have some nilpotency (because $\mathfrak{a}_1(J) \neq \{0\}$), whereas strongly non-nilpotent ones would not. This fact turns to be crucial in the construction of every pair $(\mathfrak{g}, J)$. Those quasi-nilpotent $J$'s can be generated in terms of already-known complex structures defined on NLAs of lower dimensions. However, the remaining ones constitute a completely new class arising in each dimension leap, and their construction requires a different approach. We develop these ideas in the next sections.

### 3.1.1 Construction of quasi-nilpotent complex structures

Let us first focus on those complex structures $J$ on $\mathfrak{g}$ such that $a_1(J) \neq \{0\}$, both nilpotent and non-nilpotent. We will see that every possible pair $(\mathfrak{g}, J)$ where $\dim \mathfrak{g} = 2n$ can be found *extending* the classification of complex structures on $2(n-1)$-dimensional nilpotent Lie algebras.

When the complex structure $J$ is quasi-nilpotent, the ascending central series of $\mathfrak{g}$ compatible with $J$ (3.4) satisfies

$$\{0\} = \mathfrak{a}_0(J) \subsetneq \mathfrak{a}_1(J) \subsetneq \ldots \subsetneq \mathfrak{a}_{t-1}(J) \subsetneq \mathfrak{a}_t(J) = \mathfrak{a}_l(J), \quad \text{for } l \geq t,$$

being $t$ the smallest integer for which the series stabilizes. It is worth noting that $t$ does not necessarily coincide with the nilpotency step $s$ of the Lie algebra $\mathfrak{g}$ (see Example 3.1.5). Nevertheless, one can ensure that $t \leq s$. However, there are some special cases in which not only $t = s$, but also both ascending central series (3.2) and (3.4) coincide.

**Proposition 3.1.7.** [CFGU00] *Let $\mathfrak{g}$ be an $s$-step nilpotent Lie algebra endowed with a complex structure $J$. If every term $\mathfrak{g}_k$ in the ascending central series of $\mathfrak{g}$ is $J$-invariant, then $\mathfrak{a}_k(J) = \mathfrak{g}_k$ for every $k \geq 0$. In particular, $\mathfrak{a}_s(J) = \mathfrak{g}_s = \mathfrak{g}$.*

For instance, this holds true when $J$ is abelian (see Example 3.1.2) or $J$ is complex-parallelizable (Example 1.4.7). These are two particular cases of nilpotent $J$'s.

Let us now consider the following sequence of quotient Lie algebras

$$\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}_1(J) \longrightarrow \ldots \longrightarrow \mathfrak{g}/\mathfrak{a}_q(J) \stackrel{\pi_{q+1}}{\longrightarrow} \mathfrak{g}/\mathfrak{a}_{q+1}(J) \longrightarrow \ldots \longrightarrow \mathfrak{g}/\mathfrak{a}_t(J), \tag{3.6}$$

where $\pi_{q+1}$ is the natural projection and $\ker \pi_{q+1} = \mathfrak{a}_{q+1}(J)/\mathfrak{a}_q(J)$. For the seek of simplicity, we will denote $\widetilde{\mathfrak{g}}_q = \mathfrak{g}/\mathfrak{a}_q(J)$, and $[\cdot\,,\cdot]_q$ will be the Lie bracket in $\widetilde{\mathfrak{g}}_q$, for each $q = 1, \ldots, t$. Observe that the algebras $\widetilde{\mathfrak{g}}_q$ are nilpotent. Moreover, one has $\widetilde{\mathfrak{g}}_t = \{0\}$ in the case of nilpotent complex structures and $\widetilde{\mathfrak{g}}_t \neq \{0\}$ otherwise.

The complex structure $J$ defined on $\mathfrak{g}$ induces a complex structure $\widetilde{J}_q$ on $\widetilde{\mathfrak{g}}_q$, for $q = 1, \ldots, t$, in the following way [CFGU97b]:

$$\widetilde{J}_q(\widetilde{X}) = \widetilde{JX}, \quad \forall \widetilde{X} \in \widetilde{\mathfrak{g}}_q,$$

being $\widetilde{X}$ and $\widetilde{JX}$ the classes of $X$ and $JX$, respectively, in the quotient $\widetilde{\mathfrak{g}}_q$.

Following the method given in [CFGU00, Theorem 12] for $J$ nilpotent and later generalized in [CFGU97b] for $J$ weakly non-nilpotent, it is possible to find a basis of $\mathfrak{g}^*$ adapted to (3.6). Let us recall this construction.

First, remember that the existence of a complex structure $J$ on $\mathfrak{g}$ allows to construct an adapted basis $\{X_1, JX_1, \ldots, X_n, JX_n\}$ of the Lie algebra $\mathfrak{g}$. Moreover, in a similar way to Section 1.1, $J$ induces a bigraduation on the complexified algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$,

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{0,1}, \text{ where } \begin{cases} \mathfrak{g}_{1,0} = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid JZ = \ i\,Z\}, \\ \mathfrak{g}_{0,1} = \{Z \in \mathfrak{g}_{\mathbb{C}} \mid JZ = -i\,Z\}, \end{cases}$$

and $\mathfrak{g}_{0,1} = \overline{\mathfrak{g}_{1,0}}$. Setting $Z_k = X_k - i\,JX_k$, for each $k = 1, \ldots, n$, one obtains a basis $\{Z_k\}_{k=1}^n$ for $\mathfrak{g}_{1,0}$. Therefore, $\mathfrak{g}_{\mathbb{C}} = \langle Z_k, \bar{Z}_k \rangle_{k=1}^n$. Since the spaces $\mathfrak{a}_q(J)$ in (3.4) are $J$-invariant, we can work with $\mathfrak{g}_{\mathbb{C}}$ and the complexification of the sequence (3.6).

Consider $\widetilde{\mathfrak{g}}_l \neq \{0\}$, where $l = t-1$ for $J$ nilpotent and $l = t$ for $J$ weakly non-nilpotent. This is an abelian Lie algebra, so any basis $\{Z_1, \bar{Z}_1, \ldots, Z_{n-n_l}, \bar{Z}_{n-n_l}\}$ of $(\widetilde{\mathfrak{g}}_l)_{\mathbb{C}}$ satisfies

$$[Z_i, Z_j]_{\,l} = [Z_i, \bar{Z}_j]_{\,l} = 0,$$

for all $i, j = 1, \ldots, n - n_l$.

Now, since $\widetilde{\mathfrak{g}}_{l-1} \cong \widetilde{\mathfrak{g}}_l \oplus \mathfrak{a}_l(J)/\mathfrak{a}_{l-1}(J)$, it is possible to complete the previous basis up to a basis of $(\widetilde{\mathfrak{g}}_{l-1})_{\mathbb{C}}$ adding some vectors $Z_{n-n_l+1},\ \bar{Z}_{n-n_l+1}, \ldots, Z_{n-n_{l-1}},\ \bar{Z}_{n-n_{l-1}}$ in $\big(\mathfrak{a}_l(J)/\mathfrak{a}_{l-1}(J)\big)_{\mathbb{C}}$. By construction, one has $[\mathfrak{a}_l(J)/\mathfrak{a}_{l-1}(J), \widetilde{\mathfrak{g}}_{l-1}]_{\,l-1} = 0$ so in fact, $\mathfrak{a}_l(J)/\mathfrak{a}_{l-1}(J)$ is in the center of $\widetilde{\mathfrak{g}}_{l-1}$. Thus,

$$[Z_i, Z_j]_{\,l-1} = [Z_i, \bar{Z}_j]_{\,l-1} = 0,$$

for all $i = n - n_l + 1, \ldots, n - n_{l-1}$, $j = 1, \ldots, n - n_l, \ldots, n - n_{l-1}$. However, when the vectors $Z_1, \bar{Z}_1, \ldots, Z_{n-n_l}, \bar{Z}_{n-n_l}$ are viewed in $(\widetilde{\mathfrak{g}}_{l-1})_{\mathbb{C}}$, they verify

$$[Z_i, Z_j]_{\,l-1},\ [Z_i, \bar{Z}_j]_{\,l-1} \in \big(\mathfrak{a}_l(J)/\mathfrak{a}_{l-1}(J)\big)_{\mathbb{C}}\,,$$

for all $i, j = 1, \ldots, n - n_l$; that is, their Lie brackets depend on those vectors which are in $\mathfrak{a}_l(J)$ but not in $\mathfrak{a}_{l-1}(J)$.

Considering the dual basis $\{\omega^1, \omega^{\bar{1}}, \ldots, \omega^{n-n_l}, \omega^{\overline{n-n_l}}, \ldots, \omega^{n-n_{l-1}}, \omega^{\overline{n-n_{l-1}}}\}$ in $(\widetilde{\mathfrak{g}}_{l-1})^*_{\mathbb{C}}$ and taking into account that $d\omega(V, W) = -\omega([V, W])$, it is easy to see

$$\begin{cases} d\omega^i = 0, \quad i = 1, \ldots, n - n_l, \\ d\omega^i \in \left(\bigwedge^{2,0} \oplus \bigwedge^{1,1}\right) \langle \omega^1, \omega^{\bar{1}}, \ldots, \omega^{n-n_l}, \omega^{\overline{n-n_l}} \rangle, \quad i = n - n_l + 1, \ldots, n - n_{l-1}. \end{cases}$$

Repeating the same process at every step in (3.6), one gets the following result:

**Theorem 3.1.8.** [CFGU97b] *Let $\mathfrak{g}$ be a $2n$-dimensional NLA endowed with a quasi-nilpotent complex structure $J$. Then, there exists an ordered basis $\{\omega^k, \omega^{\bar{k}}\}_{k=1}^n$ for $\mathfrak{g}^*_{\mathbb{C}}$ satisfying*

$$\begin{cases} d\omega^i = 0, \quad \text{for } i = 1, \ldots, n - n_l, \\ d\omega^i = \displaystyle\sum_{j<k\leq n-n_q} A_{ijk}\, \omega^{jk} + \sum_{j,k\leq n-n_q} B_{ijk}\, \omega^{j\bar{k}}, \end{cases}$$

*for $i = n - n_q + 1, \ldots, n - n_{q-1}$ and $q = l, \ldots, 1$, where $n_q = \frac{1}{2}\dim \mathfrak{a}_q(J)$.*

Now, let us note that it is possible to introduce an intermediate stage in (3.6). Since $\dim \mathfrak{a}_1(J) \geq 2$, a $J$-invariant subspace $\mathfrak{b} \subseteq \mathfrak{a}_1(J)$ can be selected. Set $\dim \mathfrak{b} = 2n_{\mathfrak{b}} \leq 2n_1$. The natural projection allows to split $\mathfrak{a}_1(J)$, obtaining $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{a}_1(J)/\mathfrak{b} \oplus \widetilde{\mathfrak{g}}_1$. Observe

$$\mathfrak{a}_1(J)/\mathfrak{b} \oplus \widetilde{\mathfrak{g}}_1 \cong \mathfrak{a}_1(J)/\mathfrak{b} \oplus (\mathfrak{g}/\mathfrak{b})\,/\,(\mathfrak{a}_1(J)/\mathfrak{b}) = \mathfrak{g}/\mathfrak{b},$$

thus $\widetilde{\mathfrak{g}}_{\mathfrak{b}} = \mathfrak{a}_1(J)/\mathfrak{b} \oplus \widetilde{\mathfrak{g}}_1$ is a nilpotent Lie algebra of dimension $2(n-n_{\mathfrak{b}})$. Moreover, since these spaces are $J$-invariant, a complex structure $\widetilde{J}_{\mathfrak{b}}$ on $\widetilde{\mathfrak{g}}_{\mathfrak{b}}$ can be defined in a similar way to above. Therefore, $(\widetilde{\mathfrak{g}}_{\mathfrak{b}}, \widetilde{J}_{\mathfrak{b}})$ is isomorphic to a pair $(\mathfrak{h}, K)$ with $\dim \mathfrak{h} = 2(n-n_{\mathfrak{b}})$. In particular, we can always choose $n_{\mathfrak{b}} = 1$.

**Definition 3.1.9.** *Let $(\mathfrak{h}, K)$ be an NLA of dimension $2(n-1)$ endowed with a complex structure. Let $(\mathfrak{g}, J)$ be a $2n$-dimensional NLA endowed with a complex structure of quasi-nilpotent type. Consider a 2-dimensional $J$-invariant subspace $\mathfrak{b} \subseteq \mathfrak{a}_1(J)$. The pair $(\mathfrak{g}, J)$ is said to be a* $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$ *if*

$$\widetilde{\mathfrak{g}}_{\mathfrak{b}} = \mathfrak{g}/\mathfrak{b} \cong \mathfrak{h} \qquad \text{and} \qquad \widetilde{J}_{\mathfrak{b}} \cong K.$$

*That is, the pair $(\widetilde{\mathfrak{g}}_{\mathfrak{b}}, \widetilde{J}_{\mathfrak{b}})$ is isomorphic to $(\mathfrak{h}, K)$.*

As a consequence of the previous lines, we can state the following result.

**Corollary 3.1.10.** *In the conditions above, every quasi-nilpotent $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of a certain $(\mathfrak{h}, K)$, for some $\mathfrak{b}$.*

Therefore, if we want to find all admissible pairs $(\mathfrak{g}, J)$, where $\dim \mathfrak{g} = 2n$ and $J$ is quasi-nilpotent, it suffices to find the $\mathfrak{b}$-extensions of each existing $(\mathfrak{h}, K)$, with $\dim \mathfrak{h} = 2(n-1)$. Let us see how.

Assume that there is a $(1,0)$-basis $\{\omega^i\}_{i=1}^{n-1}$ for each $(\mathfrak{h}, K)$ where the (complex) structure equations are totally known. By the previous reasoning, this is equivalent to saying that the pairs $(\mathfrak{g}_{\mathfrak{b}}, J_{\mathfrak{b}})$ are fully determined, whatever $\mathfrak{g}$ and $J$ might be. Hence, we will recover any $(\mathfrak{g}, J)$ just finding $\mathfrak{b}$ and "attaching" it to $(\mathfrak{h}, K)$. Note that the space $\mathfrak{b}$ remains undefined. However, it is well known that $\mathfrak{b}$ is a 2-dimensional and $J$-invariant subspace of $\mathfrak{a}_1(J)$. Thus, its dual should be generated by two conjugate elements $\omega^n$ and $\omega^{\bar{n}}$, where $d\omega^n$ follows Theorem 3.1.8 and satisfies $d^2\omega^n = 0$. The $(1,0)$-basis $\{\omega^i\}_{i=1}^n$ should parametrize every $\mathfrak{g}^{1,0}$ by means of its complex structure equations.

The freedom in the choice of the coefficients of $d\omega^n$ allows to construct non-isomorphic pairs $(\mathfrak{g}, J)$ and $(\mathfrak{g}', J')$ as $\mathfrak{b}$-extensions of a same $(\mathfrak{h}, K)$. For instance, see Proposition 3.2.1, where the complex torus $\mathbb{T}^1$ is extended. Furthermore, a pair $(\mathfrak{g}, J)$ can be seen as a $\mathfrak{b}$-extension of two different $(\mathfrak{h}, K)$ and $(\mathfrak{h}', K')$, as the next example shows.

**Example 3.1.11.** Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with a quasi-nilpotent complex structure whose (complex) structure equations are:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}}, \quad d\omega^4 = \omega^{12}.$$

Observe that one has

$$\mathfrak{g}_1 = \mathfrak{a}_1(J) = \langle Z_3, \bar{Z}_3, Z_4, \bar{Z}_4 \rangle, \quad \mathfrak{g}_2 = \mathfrak{a}_2(J) = \mathfrak{g}.$$

Choosing $\mathfrak{b}_1 = \langle Z_3, \bar{Z}_3 \rangle$, we can see that $(\mathfrak{g}/\mathfrak{b}_1, J_{\mathfrak{b}_1})$ coincides with the Iwasawa manifold. However, if we take $\mathfrak{b}_2 = \langle Z_4, \bar{Z}_4 \rangle$, then we have that $(\mathfrak{g}/\mathfrak{b}_2, J_{\mathfrak{b}_2})$ is isomorphic to the Kodaira-Thurston manifold times the complex torus $\mathbb{T}^1$. ◇

### 3.1.2 Construction of strongly non-nilpotent complex structures

Let us now turn our attention to those complex structures $J$ on $\mathfrak{g}$ satisfying $a_1(J) = \{0\}$. In this case, the construction cannot be based on lower dimensional classifications and other approach is needed. The pairs $(\mathfrak{g}, J)$, where $J$ is strongly non-nilpotent, will be found using the ascending central series and the existence of a *doubly adapted* basis.

First, note that strongly non-nilpotent structures are characterized by the absence of a $J$-invariant subspace in $\mathfrak{g}_1$. This might suggest the existence of some kind of constraint on $\mathfrak{g}_1$. Indeed, one can prove the following.

**Proposition 3.1.12.** *Let $(\mathfrak{g}, J)$ be a $2n$-dimensional nilpotent Lie algebra endowed with a strongly non-nilpotent complex structure, where $n \geq 2$. Then:*

*i)* $\mathfrak{g}_1 \cap J\mathfrak{g}_1 = \{0\}$;

*ii)* $\mathfrak{g}_2 \cap J\mathfrak{g}_1 = \{0\}$;

*iii)* $1 \leq dim\, \mathfrak{g}_1 \leq n-1$.

*Proof.* Let us begin by observing that *i)* follows directly from the definition.

In order to prove *ii)*, we assume the converse. That is, suppose that there exists $X \in \mathfrak{g}_1$ such that $JX \in \mathfrak{g}_2$. Due to the Nijenhuis condition, one has $[JX, JY] = J[JX, Y]$, for every $Y \in \mathfrak{g}$. Since any bracket involving $JX$ lies in $\mathfrak{g}_1$, from the previous equality one can indeed conclude that $[JX, JY] \in \mathfrak{g}_1 \cap J\mathfrak{g}_1 = \{0\}$, for every $Y \in \mathfrak{g}$. Therefore, $JX \in \mathfrak{g}_1$ and $J$ is nilpotent. This contradicts our initial hypothesis.

Now, let us focus on *iii)*. The nilpotency of the Lie algebra $\mathfrak{g}$ ensures $1 \leq \dim \mathfrak{g}_1$, so it suffices to prove the second inequality. Since $\mathfrak{a}_1(J) = \{0\}$ and $\dim \mathfrak{g} = 2n$, it is clear that $\dim \mathfrak{g}_1 \leq n$. Thus, we just need to discard the case $\dim \mathfrak{g}_1 = n$. If we consider $\mathfrak{g}_1 = \langle X_1, \ldots, X_n \rangle$, then one has $[JX_i, JX_j] = 0$, for all $1 \leq i, j \leq n$, as a consequence of the Nijenhuis condition. However, this implies that the Lie algebra $\mathfrak{g}$ is abelian and the complex structure $J$ is nilpotent. Therefore, $\dim \mathfrak{g}_1 \leq n-1$. □

Further results on the ascending central series in the presence of a strongly non-nilpotent complex structure can be found in Chapter 4. Here, the previous statement is enough for our purposes. Indeed, despite its simplicity, Proposition 3.1.12 allows to conclude the following.

**Corollary 3.1.13.** *Any nilpotent Lie algebra endowed with a strongly non-nilpotent complex structure is at least 3-step.*

Let us now describe the construction procedure. Our objective is to obtain nilpotent Lie algebras endowed with strongly non-nilpotent complex structures. As a starting point, we consider a $2n$-dimensional vector space $\mathfrak{g}$ with an almost-complex structure $J$. On the one hand, we want to provide $\mathfrak{g}$ with a nilpotent Lie algebra structure. On the other hand, we ask $J$ to be integrable and to satisfy $\mathfrak{a}_1(J) = \{0\}$. The idea is assembling the ascending central series (3.2) of $\mathfrak{g}$ attending to the nilpotency of $\mathfrak{g}$, the Jacobi identity

$$0 \;=\; Jac\,(X,Y,Z) \;=\; \big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big], \quad \forall\, X,Y,Z \in \mathfrak{g},$$

the Nijenhuis condition (3.3), and the strongly non-nilpotency of $J$. These four conditions need to be checked at every stage of the procedure, discarding the cases in which any of them fails. Covering all the possible combinations, the complete set of pairs $(\mathfrak{g}, J)$ will be obtained.

Let us observe that the previous method requires an appropriate basis of $\mathfrak{g}$ that allows to simplify the calculations. For this reason, we introduce the following concept.

**Definition 3.1.14.** *Let $\mathfrak{g}$ be an $s$-step nilpotent Lie algebra of dimension $2n$ endowed with an almost-complex structure $J$. A $J$-adapted basis $\mathcal{B} = \{X_k, JX_k\}_{k=1}^n$ of $\mathfrak{g}$ will be called* doubly adapted *if there is a permutation $\mathcal{B}^\sigma = \{V_1, \ldots, V_{2n}\}$ of the elements of $\mathcal{B}$ such that $\mathcal{B}^\sigma$ is adapted to the ascending central series $\{\mathfrak{g}_l\}_l$, i.e.,*

$$\mathfrak{g}_1 = \langle V_1, \ldots, V_{m_1}\rangle \subset \mathfrak{g}_2 = \langle V_1, \ldots, V_{m_1}, V_{m_1+1}, \ldots, V_{m_2}\rangle \subset \cdots \subset \mathfrak{g}_s = \mathfrak{g} = \langle V_1, \ldots, V_{2n}\rangle,$$

*being $m_l = dim\, \mathfrak{g}_l$.*

The procedure relies on finding a basis $\mathcal{B}$ of such type, starting from a basis of the center $\mathfrak{g}_1$ of the Lie algebra $\mathfrak{g}$, and successively extending it up to a basis of each $\mathfrak{g}_l$. Let us observe that the construction involves the Lie brackets of $\mathfrak{g}$, which are unknown at the beginning of the process. More precisely, the Lie brackets will be defined with respect to this doubly adapted basis, thanks to its compatibility with the ascending central series. We also need to take into account that the brackets of the generators of $\mathcal{B}$ must satisfy the Jacobi identity and the Nijenhuis condition.

Once the previous procedure is finished, all the strongly non-nilpotent pairs $(\mathfrak{g}, J)$ will have been found. Let $\mathcal{B}^* = \{e^k, Je^k\}_{k=1}^n$ denote the dual basis of $\mathcal{B}$. The brackets we have obtained together with the formula $d\alpha(X, Y) = -\alpha([X, Y])$ determine the structure equations of $\mathfrak{g}$. Since $\mathcal{B}^*$ encodes the information about the complex structure $J$, it suffices to take $\omega^k = e^k - i\, Je^k$, where $k = 1, \ldots, n$, in order to find the complex structure equations for $(\mathfrak{g}, J)$.

Although the previous description might seem too theoretical, some explicit constructions can be found in the next section (see Proposition 3.2.2 and Lemma 3.2.5) and also in Chapter 4.

## 3.2 The lower dimensional cases

In this section, we recover the already-known classifications of complex structures on nilpotent Lie algebras of dimensions four and six, using the ideas presented in Section 3.1.

### 3.2.1 Dimension four

Let $(\mathfrak{g}, J)$ be a 4-dimensional nilpotent Lie algebra endowed with a complex structure. In order to construct any possible pair $(\mathfrak{g}, J)$, two cases need to be distinguished according to the nilpotency of $J$, as we described in Section 3.1.

First, let us assume that $J$ is quasi-nilpotent. Then, we can apply the method given in Section 3.1.1. More concretely, it suffices to extend the classification of complex structures on 2-dimensional NLAs in order to find any $(\mathfrak{g}, J)$ with $\mathfrak{a}_1(J) \neq \{0\}$. Recall that the abelian algebra is the only nilpotent Lie algebra in dimension 2 admitting complex structures. In fact, there is only one complex structure on it, which gives rise to the complex torus $\mathbb{T}^1$. Therefore, one concludes the following.

**Proposition 3.2.1.** *Let $\mathfrak{h}$ be the 2-dimensional abelian Lie algebra endowed with the trivial complex structure $K$. Then, the $\mathfrak{b}$-extensions of $(\mathfrak{h}, K)$ are parametrized by*

$$d\omega^1 = 0, \qquad d\omega^2 = A\,\omega^{1\bar{1}}.$$

*Moreover, up to equivalence of complex structures, $A \in \{0, 1\}$.*

We clearly recover the complex torus $\mathbb{T}^2$ for $A = 0$ and the Kodaira-Thurston manifold for $A \neq 0$ (it suffices to apply a change of basis to obtain $A = 1$).

Let us now turn our attention to the SnN case, i.e., $\mathfrak{a}_1(J) = \{0\}$. It is well known that this type of complex stuctures do not appear in dimension 4. However, here we would like to reach the result using the construction method presented in Section 3.1.2.

**Proposition 3.2.2.** *There are no strongly non-nilpotent complex structures on nilpotent Lie algebras of dimension four.*

*Proof.* As we already mentioned, we follow the ideas contained in Section 3.1.2. Let us consider a 4-dimensional NLA $\mathfrak{g}$ endowed with an SnN complex structure $J$, both initially undefined. Our aim is to construct the ascending central series of every possible $\mathfrak{g}$ admitting $J$, in terms of a doubly adapted basis $\mathcal{B}$ (see Definition 3.1.14).

Let us start observing that $\dim \mathfrak{g}_1 = 1$, as a consequence of Proposition 3.1.12 *iii)*. Hence, we can set $\mathfrak{g}_1 = \langle X_1 \rangle$ and choose $X_1$ as the first element in the basis $\mathcal{B}$. Since the algebra is nilpotent, the ascending series (3.2) should contain a new vector $Y$ in $\mathfrak{g}_2$ which is not in $\mathfrak{g}_1$. There are two possibilities, depending on how the (generic) complex structure $J$ is defined: either this vector is linearly dependent with $JX_1$, or it is linearly independent with both $X_1$ and $JX_1$. Due to Proposition 3.1.12 *ii)*, the former case is not valid. Therefore, one can assume that the latter case holds and take $X_2 = Y$ as a new element in $\mathcal{B}$. Moreover, since $JX_1 \notin \mathfrak{g}_2$ and $\mathfrak{g}$ is nilpotent, we necessarily have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

Notice that $[JX_1, X_2] \in \mathfrak{g}_1$. By the Nijenhuis condition,

$$[JX_1, JX_2] = J[JX_1, X_2] \in J\mathfrak{g}_1 = \langle JX_1 \rangle.$$

The nilpotency of $\mathfrak{g}$ leads to $[JX_1, JX_2] = [JX_1, X_2] = 0$ and thus, $JX_1 \in \mathfrak{g}_1$. This is a contradiction with the strongly non-nilpotency of the complex structure. □

Hence, the only existing complex structures on 4-dimensional NLAs are those parametrized by Proposition 3.2.1. As already stated, they coincide with the complex torus and the Kodaira-Thurston manifold, recovering in this way the usual classification.

### 3.2.2 Dimension six

Let us now consider a 6-dimensional nilpotent Lie algebra $\mathfrak{g}$ endowed with a complex structure $J$. As in the previous case, the nilpotency degree of $J$ determines the approach needed to find any possible $(\mathfrak{g}, J)$.

First, consider those pairs $(\mathfrak{g}, J)$ with $\mathfrak{a}_1(J) \neq \{0\}$. One can then follow Section 3.1.1 and extend the classification of complex structures on 4-dimensional NLAs.

**Proposition 3.2.3.** *Let $(\mathfrak{h}, K)$ be a 4-dimensional NLA endowed with a complex structure. Then, any quasi-nilpotent pair $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$ and has the following structure equations:*

$$\tag{3.7} \begin{cases} d\omega^1 = 0, \\ d\omega^2 = \varepsilon\, \omega^{1\bar{1}}, \\ d\omega^3 = \rho\, \omega^{12} + (1-\varepsilon)\, A\, \omega^{1\bar{1}} + B\, \omega^{1\bar{2}} + C\, \omega^{2\bar{1}} + (1-\varepsilon)\, D\, \omega^{2\bar{2}}, \end{cases}$$

*where $\varepsilon = 0$ if $(\mathfrak{h}, K)$ is isomorphic to $\mathbb{T}^2$, and $\varepsilon = 1$ if it is isomorphic to the Kodaira-Thurston manifold. Furthermore, $\rho \in \{0,1\}$ and $A, B, C, D \in \mathbb{C}$.*

*Proof.* Consider the classification of complex structures on 4-dimensional NLAs,

$$d\eta^1 = 0, \quad d\eta^2 = \varepsilon\, \eta^{1\bar{1}},$$

where $\varepsilon = 0$ for the complex torus, and $\varepsilon = 1$ for the Kodaira-Thurston manifold. Add a third element $\eta^3$ to the basis $\{\eta^1, \eta^2\}$ satisfying Theorem 3.1.8, namely,

$$d\eta^3 = A'\, \eta^{12} + B'\, \eta^{1\bar{1}} + C'\, \eta^{1\bar{2}} + D'\, \eta^{2\bar{1}} + E'\, \eta^{2\bar{2}},$$

where $A', B', C', D', E' \in \mathbb{C}$. Imposing $d^2\eta^3 = 0$, one obtains

$$\begin{cases} d\eta^1 = 0, \\ d\eta^2 = \varepsilon\, \eta^{1\bar{1}}, \\ d\eta^3 = A'\, \eta^{12} + B'\, \eta^{1\bar{1}} + C'\, \eta^{1\bar{2}} + D'\, \eta^{2\bar{1}} + (1-\varepsilon)\, E'\, \eta^{2\bar{2}}. \end{cases}$$

Now, we observe the following.

If $A' = 0$, apply the change of basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3 - B'\eta^2,$$

to obtain (3.7) with $\rho = 0$, $A = B'$, $B = C'$, $C = D'$, and $D = E'$.

If $A' \neq 0$, take

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \frac{1}{A'}\,(\eta^3 - B'\eta^2).$$

We get (3.7) with $\rho = 1$, $A = B'/A'$, $B = C'/A'$, $C = D'/A'$, and $D = E'/A'$. □

**Remark 3.2.4.** Note that all the previous $J$'s are nilpotent. In fact, Proposition 3.2.3 recovers the parametrization given in [Uga07, Theorem 2.2] for this type of complex structures using the new point of view.

Let us now move to study those $J$'s such that $\mathfrak{a}_1(J) = 0$. We will apply Section 3.1.2 and follow a similar argument to the 4-dimensional case, this time leading to a positive answer. As a first stage, we compute the valid ascending central series in terms of a doubly adapted basis. In the second part (Proposition 3.2.6), the complex structure equations are obtained.

**Lemma 3.2.5.** *Let $(\mathfrak{g}, J)$ be a 6-dimensional NLA endowed with a strongly non-nilpotent complex structure. Then, its ascending central series is given in terms of a doubly adapted basis $\mathcal{B} = \{X_k, JX_k\}_{k=1}^3$ by:*

*i)* $\mathfrak{g}_1 = \langle X_1 \rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle$, *and* $\mathfrak{g}_3 = \mathfrak{g}$; *or*

*ii)* $\mathfrak{g}_1 = \langle X_1 \rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle$, $\mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle$, *and* $\mathfrak{g}_4 = \mathfrak{g}$.

*Proof.* Let us note that, by Corollary 3.1.13, the Lie algebra $\mathfrak{g}$ is at least 3-step. The idea is exactly the same as in the proof of Proposition 3.2.2. According to the described procedure, we want to construct the doubly adapted basis $\mathcal{B}$.

Following Proposition 3.1.12 *iii)*, two cases can be distinguished depending on the dimension of the center $\mathfrak{g}_1$, namely, $\dim \mathfrak{g}_1 = 1$ or $\dim \mathfrak{g}_1 = 2$. We first see that the latter case is not valid.

Let us suppose $\dim \mathfrak{g}_1 = 2$, and consider $\mathfrak{g}_1 = \langle X_1, X_2 \rangle$. Take $X_1$ and $X_2$ as two vectors of $\mathcal{B}$. Since $\mathfrak{g}$ is nilpotent, there is an element $Y$ in $\mathfrak{g}_2$ which is not in $\mathfrak{g}_1$. Moreover, we can assume that $Y$ satisfies $\pi_{\mathfrak{g}_1}(Y) = 0$, being $\pi_{\mathfrak{g}_1}$ the natural projection of $\mathfrak{g}$ on $\mathfrak{g}_1$. Otherwise, it suffices to replace $Y$ by $Y - \pi_{\mathfrak{g}_1}(Y) \in \mathfrak{g}_2$. Now, two possibilities arise: either $Y = JX$, for some $X \in \mathfrak{g}_1$, or $Y$ is linearly independent with $X_1$, $JX_1$, $X_2$, and $JX_2$. According to Proposition 3.1.12 *ii)* the first option is not valid, so the second one necessarily holds. Let us then set $X_3 = Y$ as a new element in $\mathcal{B}$. One has

$$\mathfrak{g}_1 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_2 \supseteq \langle X_1, X_2, X_3 \rangle.$$

Although we still do not know where $JX_1$ and $JX_2$ enter the ascending central series, it is clear that we have

$$[X_3, JX_1] = \alpha_1 X_1 + \alpha_2 X_2, \quad [X_3, JX_2] = \beta_1 X_1 + \beta_2 X_2,$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Furthermore, the Nijenhuis condition (3.3) implies

$$[JX_1, JX_2] = 0, \quad [JX_k, JX_3] = J[JX_k, X_3], \text{ for } k = 1, 2.$$

In particular, it is worth observing that

$$[JX_3, JX_1] = \alpha_1 JX_1 + \alpha_2 JX_2, \quad [JX_3, JX_2] = \beta_1 JX_1 + \beta_2 JX_2. \tag{3.8}$$

Notice that both $(\alpha_1, \alpha_2) \neq (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$, or otherwise either $JX_1 \in \mathfrak{g}_1$ or $JX_2 \in \mathfrak{g}_1$. As a consequence, we can ensure that $JX_3 \notin \mathfrak{g}_2$. Hence,

$$\mathfrak{g}_1 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3 \rangle.$$

Observe that $\mathcal{B} = \{X_k, JX_k\}_{k=1}^3$ is a $J$-adapted basis of $\mathfrak{g}$ which is also adapted to its ascending central series up to the term $\mathfrak{g}_2$.

Let us move to study $\mathfrak{g}_3$. The nilpotency of $\mathfrak{g}$ leads to the existence of an element $Y \in \mathfrak{g}_3$ such that $Y \notin \mathfrak{g}_2$ and linearly independent with $X_1$, $X_2$, and $X_3$. Therefore, $Y$ is a generator of $\mathfrak{g}_3$, but we do not know if $Y$ belongs to $\mathcal{B}$. Changing $Y$ by $Y - \pi_{\mathfrak{g}_2}(Y)$ if necessary, it is possible to assume $Y = JX$, for some $X \in \mathfrak{g}_2$. Notice that $X$ can be written as $X = \mu\, X_1 + \lambda\, X_2 + \nu\, X_3$, with $(\mu, \lambda, \nu) \neq (0,0,0)$. In order to include $Y$ as a new element of the doubly adapted basis we are searching for, we perform a change of generators. More precisely, we replace $X_1, X_2$, and $X_3$ by three new vectors $X_1', X_2'$, and $X_3'$ defined as follows:

- if $\nu \neq 0$, take $X_1' = X_1,\ X_2' = X_2,$ and $X_3' = X,$
- if $\nu = 0$ and $\lambda \neq 0$, choose $X_1' = X_1,\ X_2' = X,$ and $X_3' = X_3,$
- if $\nu = \lambda = 0$ and $\mu \neq 0$, consider $X_1' = X,\ X_2' = X_2,$ and $X_3' = X_3.$

Consequently, we also change $JX_k$ by $JX_k'$. In this way, $Y = JX_k' \in \mathfrak{g}_3$ for some $k = 1, 2, 3$, and $\mathfrak{g}_1 = \langle X_1', X_2' \rangle$, $\mathfrak{g}_2 = \langle X_1', X_2', X_3' \rangle$. Since the ascending central series is preserved, the brackets involving the new elements coincide with the old ones, maybe varying the parameters (which were anyway free). For the seek of clarity, we can rename $X_k' \equiv X_k$, for each $k = 1, 2, 3$, and simply claim that $JX_k \in \mathfrak{g}_3$, for some $k$, *up to an arrangement of generators*:

$$\mathfrak{g}_1 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3 \rangle \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_k \rangle.$$

For this reason, at least one of the brackets in (3.8) should lay in $\mathfrak{g}_2$. However, we can then conclude that either $JX_1$ or $JX_2$ belongs to $\mathfrak{g}_1$, which is a contradiction. Therefore, we are led to assume $\dim \mathfrak{g}_1 = 1$.

Let $\mathfrak{g}_1 = \langle X_1 \rangle$ and consider $X_1$ as the first element in $\mathcal{B}$. Due to Proposition 3.1.12 *ii)*, one has $JX_1 \notin \mathfrak{g}_2$. Necessarily, there is $X_2 \in \mathfrak{g}_2$ linearly independent with $X_1$ and $JX_1$ which can be thought as an element in $\mathcal{B}$,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 \supseteq \langle X_1, X_2 \rangle.$$

We ignore where $JX_1$ and $JX_2$ enter the ascending central series. Nonetheless, it is clear that $[X_2, JX_1] = \mu\, X_1$, for some $\mu \in \mathbb{R}$. Due to the Nijenhuis condition, we can also establish $[JX_1, JX_2] = J[JX_1, X_2] = -\mu\, JX_1$. The nilpotency of $\mathfrak{g}$ requires $\mu = 0$. Hence,

$$[X_1, Y] = 0, \forall\, Y \in \mathfrak{g}, \quad [X_2, JX_1] = 0, \quad [X_2, JX_2] = \beta\, X_1, \quad [JX_1, JX_2] = 0,$$

where $\beta \in \mathbb{R}$. Furthermore, a new element $Y$ linearly independent with $X_1$, $JX_1$, $X_2$, and $JX_2$ cannot be in $\mathfrak{g}_2$. Otherwise, one would have $[Y, JX_1] = a\,X_1$ and $[JY, JX_1] = a\,JX_1$, with $a \in \mathbb{R}$. Since $\mathfrak{g}$ is nilpotent we would get $a = 0$, but this implies $JX_1 \in \mathfrak{g}_1$ which is a contradiction. Hence, only two possibilities arise: either $\mathfrak{g}_2 = \langle X_1, X_2\rangle$ or $\mathfrak{g}_2 = \langle X_1, X_2, JX_2\rangle$.

<u>Case 1</u>: Let us first suppose $JX_2 \notin \mathfrak{g}_2$; that is,

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2\rangle.$$

We should now focus on $\mathfrak{g}_3$. If $JX_1 \in \mathfrak{g}_3$, then $[JX_1, JY] = J[JX_1, Y] \in \mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$ for every $Y \in \mathfrak{g}$. However, this implies $\mathfrak{a}_1(J) \neq \{0\}$ (because $JX_1 \in \mathfrak{g}_1$) and hence, we can conclude $JX_1 \notin \mathfrak{g}_3$.

If $JX_2 \in \mathfrak{g}_3$ then

$$[JX_2, JY] - [X_2, Y] = J[JX_2, Y] + J[X_2, JY] \in \mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\},$$

for every $Y \in \mathfrak{g}$. Consequently, $[JX_2, JY] = [X_2, Y] \in \mathfrak{g}_1$ for every $Y \in \mathfrak{g}$, which contradicts the initial assumption of this case. Thus, we can also ensure $JX_2 \notin \mathfrak{g}_3$.

Since the ascending central series should continue until it reaches the whole algebra $\mathfrak{g}$, there is a vector $Y \in \mathfrak{g}_3$ linearly independent with $X_1$, $JX_1$, $X_2$, and $JX_2$. We set $X_3 = Y \in \mathcal{B}$ and have

$$[X_2, X_3] = \alpha\, X_1, \quad [X_2, JX_3] = \gamma\, X_1,$$

$$[X_3, JX_1] = a_1\, X_1 + a_2\, X_2, \quad [X_3, JX_2] = b_1\, X_1 + b_2\, X_2, \quad [X_3, JX_3] = c_1\, X_1 + c_2\, X_2,$$

where $\alpha, \gamma, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Therefore, the only yet undetermined brackets are

$$[JX_1, JX_3] = -a_1\, JX_1 - a_2\, JX_2, \quad [JX_2, JX_3] = \alpha\, X_1 + (\gamma - b_1)\, JX_1 - b_2\, JX_2.$$

If $JX_3 \in \mathfrak{g}_3$, then $a_1 = a_2 = 0$ and $JX_1 \in \mathfrak{g}_1$. Thus, $JX_3 \notin \mathfrak{g}_3$ and we set

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2\rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3\rangle.$$

One necessarily has $JX_k \in \mathfrak{g}_4$ for some $k = 1, 2, 3$, up to an arrangement of generators. Then, for every $Y \in \mathfrak{g}$

$$[JX_k, JY] - [X_k, Y] = J[JX_k, Y] + J[X_k, JY] \in \mathfrak{g}_3 \cap J\mathfrak{g}_3 = \{0\}.$$

This leads to $[JX_k, JY] = [X_k, Y] \in \mathfrak{g}_2$ and hence, $JX_k \in \mathfrak{g}_3$. As we have seen, this is not possible.

<u>Case 2</u>: Let us suppose $JX_2 \in \mathfrak{g}_2$ and

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2\rangle.$$

Observe that we can take

$$[X_2, X_3] = \alpha\, X_1, \quad [X_2, JX_3] = \gamma\, X_1, \quad [X_3, JX_2] = \delta\, X_1,$$

for some $\alpha, \gamma, \delta \in \mathbb{R}$. Following Nijenhuis, $[JX_2, JX_3] = \alpha\, X_1 + (\gamma - \delta) JX_1$. Since the last bracket should belong to $\mathfrak{g}_1$, one needs $\delta = \gamma$. In the end, we conclude

$$[X_3, JX_2] = \gamma\, X_1, \quad [JX_2, JX_3] = \alpha\, X_1.$$

In order to study $\mathfrak{g}_3$, we separate two possibilities. In particular, observe that $\dim \mathfrak{g}_3 \geq 4$.

If we assume that there exists $Y \in \mathfrak{g}_3$ linearly independent with $X_1$, $JX_1$, $X_2$, and $JX_2$, then one can take $X_3 = Y$ in $\mathcal{B}$. Therefore, $[X_3, JX_1] = a_1\, X_1 + a_2\, X_2 + A_2\, JX_2$, where $a_1, a_2, A_2 \in \mathbb{R}$, so $[JX_1, JX_3] = -a_1\, JX_1 - a_2\, JX_2 + A_2\, X_2$. In order to ensure $JX_1 \notin \mathfrak{g}_3$, one would need $a_1 \neq 0$. However, this contradicts the nilpotency of $\mathfrak{g}$.

We now consider the opposite situation, namely, $JX_1 \in \mathfrak{g}_3$ (after an eventual arrangement of generators). Let us set $[X_3, JX_1] = a_1\, X_1 + a_2\, X_2 + A_2\, JX_2$, for some $a_1, a_2, A_2 \in \mathbb{R}$. We have $[JX_1, JX_3] = -a_1\, JX_1 - a_2\, JX_2 + A_2\, X_2 \in \mathfrak{g}_2$ if and only if $a_1 = 0$. Moreover, the condition $(a_2, A_2) \neq (0,0)$ is required in order to ensure $JX_1 \in \mathfrak{g}_3 \smallsetminus \mathfrak{g}_2$. By the Jacobi identity,

$$\begin{cases} 0 = Jac\,(X_3, JX_1, X_2) = -A_2\, \beta\, X_1, \\ 0 = Jac\,(JX_1, JX_3, X_2) = a_2\, \beta\, X_1, \end{cases}$$

and necessarily, $\beta = 0$. To summarize, at this stage of the construction the ascending central series fulfills

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, JX_1, JX_2 \rangle,$$

and the already fixed brackets are

$$\begin{aligned} &[X_1, Y] = 0, \ \forall Y \in \mathfrak{g}, \\ &[X_2, X_3] = \alpha\, X_1, \quad [X_2, JX_1] = [X_2, JX_2] = 0, \quad [X_2, JX_3] = \gamma\, X_1, \\ &[X_3, JX_1] = a_2\, X_2 + A_2\, JX_2, \quad [X_3, JX_2] = \gamma\, X_1, \\ &[JX_1, JX_2] = 0, \quad [JX_1, JX_3] = A_2\, X_2 - a_2\, JX_2, \quad [JX_2, JX_3] = \alpha\, X_1. \end{aligned} \tag{3.9}$$

Observe that the parameters must be chosen appropriately in order to preserve the dimension of the ascending central series. In particular, one needs $(\alpha, \gamma) \neq (0,0)$ and $(a_2, A_2) \neq (0,0)$ to guarantee $X_2 \in \mathfrak{g}_2 \smallsetminus \mathfrak{g}_1$ and $JX_1 \in \mathfrak{g}_3 \smallsetminus \mathfrak{g}_2$. Notice that just the bracket $[X_3, JX_3]$ remains undefined.

Case 2.1: Suppose $\dim \mathfrak{g}_3 \geq 5$. Due to the nilpotency of $\mathfrak{g}$, $\dim \mathfrak{g}_3 = 6$ and thus

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

In particular, one has

$$[X_3, JX_3] = b_1\, X_1 + b_2\, X_2 + B_2\, JX_2,$$

where $b_1, b_2, B_2 \in \mathbb{R}$ and $(b_2, B_2) \neq (0,0)$. At this point, just the Jacobi identity needs to be checked. The only non-trivial choice of elements turns to be

$$0 = Jac\,(X_3, JX_3, JX_1) = -2\,(\alpha\, A_2 + \gamma\, a_2)\, X_1.$$

In consequence, we need to impose the condition $\alpha\, A_2 + \gamma\, a_2 = 0$. For example, it suffices to take $\alpha = a_2 = b_2 = 0$ and $\gamma = A_2 = B_2 = 1$ to find a suitable solution. One can then conclude that our assumptions are valid, thus getting part *i)* of the lemma.

Case 2.2: Now assume the converse to Case 2.1, i.e., $\dim \mathfrak{g}_3 = 4$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g},$$

where the last term comes as a consequence of the nilpotency of $\mathfrak{g}$. As $X_3, JX_3 \notin \mathfrak{g}_3$, we have

$$[X_3, JX_3] = b_1\, X_1 + b_2\, X_2 + B_1\, JX_1 + B_2\, JX_2,$$

where $b_1, b_2, B_1, B_2 \in \mathbb{R}$ and $B_1 \neq 0$. The only non-trivial Jacobi identity coincides with the one above, so the extra condition $\alpha\, A_2 + \gamma\, a_2 = 0$ should be imposed. This time, a suitable solution could be $\alpha = a_2 = b_2 = 0$ and $\gamma = B_1 = A_2 = B_2 = 1$. We have obtained another valid case, which corresponds to part *ii)*.

Our decision tree concludes here. Observe it only leaves two admissible pairs $(\mathfrak{g}, J)$: those given by Cases 2.1 and 2.2. Therefore, $\mathfrak{g}$ is defined by the brackets (3.9) and

$$[X_3, JX_3] = b_1\, X_1 + b_2\, X_2 + B_1\, JX_1 + B_2\, JX_2, \tag{3.10}$$

where $B_1 = 0$ and $(b_2, B_2) \neq (0,0)$ for Case 2.1 ($\mathfrak{g}$ is 3-step), and $B_1 \neq 0$ for Case 2.2 ($\mathfrak{g}$ is 4-step). Moreover, in both cases we need $(\alpha, \gamma) \neq (0,0)$, $(a_2, A_2) \neq (0,0)$, and $\alpha\, A_2 + \gamma\, a_2 = \mathfrak{Im}\big((\alpha + i\gamma)(a_2 + iA_2)\big) = 0$. □

Using the previous lemma, we are able to provide the complex structure equations of the corresponding pairs $(\mathfrak{g}, J)$.

**Proposition 3.2.6.** *Let $J$ be a strongly non-nilpotent structure on a 6-dimensional nilpotent Lie algebra $\mathfrak{g}$. Then, there is a basis $\{\omega^i\}_{i=1}^3$ for $\mathfrak{g}^{1,0}$ satisfying*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 = i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \end{cases} \tag{3.11}$$

*where $\varepsilon = 0$ if $\mathfrak{g}$ is 3-step, and $\varepsilon = 1$ if $\mathfrak{g}$ is 4-step.*

*Proof.* Let $\{e^1, \ldots, e^6\}$ be the dual basis to $\{X_1, X_2, X_3, JX_1, JX_2, JX_3\}$ (note we have rearranged the doubly adapted basis $\mathcal{B}$). Due to the formula $de(V, W) = -e([V, W])$, where $e \in \mathfrak{g}^*$ and $V, W \in \mathfrak{g}$, and the lemma above (in particular, see expressions (3.9) and (3.10)), we obtain the (real) structure equations

$$\begin{cases} de^1 = -\alpha\, e^{23} - \gamma\, e^{26} - \gamma\, e^{35} - b_1\, e^{36} - \alpha\, e^{56}, \\ de^2 = -a_2\, e^{34} - b_2\, e^{36} - A_2\, e^{46}, \\ de^4 = -B_1\, e^{36}, \\ de^5 = -A_2\, e^{34} - B_2\, e^{36} + a_2\, e^{46}, \\ de^3 = de^6 = 0. \end{cases}$$

Some necessary (but not sufficient) conditions to preserve $(\dim \mathfrak{g}_l)_l$ are

$$(\alpha, \gamma) \neq (0,0), \quad (a_2, A_2) \neq (0,0), \quad \text{and} \quad \alpha\, A_2 + \gamma\, a_2 = \mathfrak{Im}\left((\alpha + i\gamma)(a_2 + iA_2)\right) = 0.$$

When $B_1 = 0$, also $(b_2, B_2) \neq (0,0)$. Furthermore, it is worth recalling that $\mathfrak{g}$ is 3-step if $B_1 = 0$ and 4-step if $B_1 \neq 0$.

We now define a basis for $\mathfrak{g}^{1,0}$ as follows

$$\omega^1 = e^3 - ie^6, \quad \omega^2 = e^2 - ie^5, \quad \omega^3 = e^1 - ie^4.$$

The (complex) structure equations are

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= -\frac{A_2+ia_2}{2}\,\omega^{13} + \frac{B_2+ib_2}{2}\,\omega^{1\bar{1}} + \frac{A_2+ia_2}{2}\,\omega^{1\bar{3}}, \\ d\omega^3 &= \frac{B_1+ib_1}{2}\,\omega^{1\bar{1}} + \frac{\alpha+i\gamma}{2}\,\omega^{1\bar{2}} - \frac{\alpha-i\gamma}{2}\,\omega^{2\bar{1}}. \end{cases}$$

Since $\mathfrak{Re}\big((\alpha + i\gamma)(a_2 + iA_2)\big) = a_2\,\alpha - A_2\,\gamma \neq 0$, it is possible to apply the following change of basis:

$$\tau^1 = \frac{\sqrt{|a_2\,\alpha - A_2\,\gamma|}}{2}\,\omega^1, \quad \tau^2 = \frac{\alpha - i\,\gamma}{2}\,\omega^2 - i\,\frac{b_1}{4}\,\omega^1,$$

$$\tau^3 = -i\,\delta\,\frac{\sqrt{|a_2\,\alpha - A_2\,\gamma|}}{2}\left(\omega^3 + i\,\frac{(B_2 - i\,b_2)(\alpha + i\,\gamma)}{a_2\,\alpha - A_2\,\gamma}\,\omega^1\right),$$

where $\delta = 1$ if $a_2\,\alpha - A_2\,\gamma > 0$, and $\delta = -1$ if $a_2\,\alpha - A_2\,\gamma < 0$. Then,

$$\begin{cases} d\tau^1 &= 0, \\ d\tau^2 &= \tau^{13} + \tau^{1\bar{3}}, \\ d\tau^3 &= -i\,\delta\,\frac{B_1}{\sqrt{|a_2\,\alpha - A_2\,\gamma|}}\,\tau^{1\bar{1}} - i\,\delta\,(\tau^{1\bar{2}} - \tau^{2\bar{1}}). \end{cases}$$

For $B_1 = 0$, we clearly recover (3.11) with $\varepsilon = 0$. For $B_1 \neq 0$, it suffices to apply a second change of basis

$$\eta^1 = \tau^1, \quad \eta^2 = -\delta\,\frac{\sqrt{|a_2\,\alpha - A_2\,\gamma|}}{B_1}\,\tau^2, \quad \eta^3 = -\delta\,\frac{\sqrt{|a_2\,\alpha - A_2\,\gamma|}}{B_1}\,\tau^3$$

in order to get (3.11) with $\varepsilon = 1$. □

**Remark 3.2.7.** We have independently recovered the classification of non-nilpotent complex structures on 6-dimensional NLAs given in [UV14].

The key point about Propositions 3.2.3 and 3.2.6 is the fact that they parametrize invariant complex geometry on 6-dimensional nilmanifolds without depending on the NLA classification. This entails a difference with the approaches previously followed, and opens a new path for classifying invariant complex structures on higher dimensional nilmanifolds.

### 3.2.3 Classifications up to equivalence

Since nilpotent Lie algebras of dimensions 4 and 6 are completely classified, it could be interesting to study the corresponce between our structure equations and these algebras. Furthermore, one would also like to study when two complex structures defined on a fixed NLA determine the same complex framework; that is, when the complex structures are equivalent. We will try to answer these questions in this section.

In dimension 4, there are 3 non-isomorphic nilpotent Lie algebras.

| | Structure | $b_1$ | $(\dim \mathfrak{g}_k)_k$ |
|---|---|---|---|
| $\mathfrak{g}_1$ | $(0, 0, 0, 0)$ | 4 | (4) |
| $\mathfrak{g}_2$ | $(0, 0, 0, 12)$ | 3 | (2,4) |
| $\mathfrak{g}_3$ | $(0, 0, 12, 13)$ | 2 | (1,2,4) |

Here, we follow Notation 1.4.21. The column $b_1$ shows the first Betti number of the corresponding Lie algebra and the last column, the dimension of the ascending central series $\{\mathfrak{g}_k\}_{k\geq 0}$.

We would like to see which of these algebras admit a complex structure. Recall that every complex structure $J$ on a 4-dimensional NLA $\mathfrak{g}$ is nilpotent. Therefore, the existence of $J$ implies that the center of $\mathfrak{g}$ should be at least 2-dimensional. This leaves aside the algebra $\mathfrak{g}_3$. Now, consider the structure equations in Proposition 3.2.1 with $A$ already normalized, i.e., $A \in \{0, 1\}$. If we split $\omega^1 = e^1 - i\, e^2$ and $\omega^2 = e^3 - i\, e^4$ into their real and imaginary parts, one can clearly see that the complex torus lives on $\mathfrak{g}_1$ and the Kodaira-Thurston manifold, on $\mathfrak{g}_2$.

When one moves to six dimensions, the problem becomes not only richer, but also more difficult. First, one should note that there are, up to isomorphism, 34 nilpotent Lie algebras.

| | Structure | $b_1$ | $b_2$ | $(\dim \mathfrak{g}_k)_k$ |
|---|---|---|---|---|
| $\mathfrak{h}_1$ | $(0, 0, 0, 0, 0, 0)$ | 6 | 15 | (6) |
| $\mathfrak{h}_2$ | $(0, 0, 0, 0, 12, 34)$ | 4 | 8 | $(2, 6)$ |
| $\mathfrak{h}_3$ | $(0, 0, 0, 0, 0, 12 + 34)$ | 5 | 9 | $(2, 6)$ |
| $\mathfrak{h}_4$ | $(0, 0, 0, 0, 12, 14 + 23)$ | 4 | 8 | $(2, 6)$ |
| $\mathfrak{h}_5$ | $(0, 0, 0, 0, 13 + 42, 14 + 23)$ | 4 | 8 | $(2, 6)$ |

| | Structure | $b_1$ | $b_2$ | $(\dim \mathfrak{g}_k)_k$ |
|---|---|---|---|---|
| $\mathfrak{h}_6$ | $(0,0,0,0,12,13)$ | 4 | 9 | $(3,6)$ |
| $\mathfrak{h}_7$ | $(0,0,0,12,13,23)$ | 3 | 8 | $(3,6)$ |
| $\mathfrak{h}_8$ | $(0,0,0,0,0,12)$ | 5 | 11 | $(4,6)$ |
| $\mathfrak{h}_9$ | $(0,0,0,0,12,14+25)$ | 4 | 7 | $(2,4,6)$ |
| $\mathfrak{h}_{10}$ | $(0,0,0,12,13,14)$ | 3 | 6 | $(2,4,6)$ |
| $\mathfrak{h}_{11}$ | $(0,0,0,12,13,14+23)$ | 3 | 6 | $(2,4,6)$ |
| $\mathfrak{h}_{12}$ | $(0,0,0,12,13,24)$ | 3 | 6 | $(2,4,6)$ |
| $\mathfrak{h}_{13}$ | $(0,0,0,12,13+14,24)$ | 3 | 5 | $(2,4,6)$ |
| $\mathfrak{h}_{14}$ | $(0,0,0,12,14,13+42)$ | 3 | 5 | $(2,4,6)$ |
| $\mathfrak{h}_{15}$ | $(0,0,0,12,13+42,14+23)$ | 3 | 5 | $(2,4,6)$ |
| $\mathfrak{h}_{16}$ | $(0,0,0,12,14,24)$ | 3 | 5 | $(3,4,6)$ |
| $\mathfrak{h}_{17}$ | $(0,0,0,0,12,15)$ | 4 | 7 | $(3,4,6)$ |
| $\mathfrak{h}_{18}$ | $(0,0,0,12,13,14+35)$ | 3 | 5 | $(1,3,6)$ |
| $\mathfrak{h}_{19}^+$ | $(0,0,0,12,23,14+35)$ | 3 | 5 | $(1,3,6)$ |
| $\mathfrak{h}_{19}^-$ | $(0,0,0,12,23,14-35)$ | 3 | 5 | $(1,3,6)$ |
| $\mathfrak{h}_{20}$ | $(0,0,0,0,12,15+34)$ | 4 | 6 | $(1,4,6)$ |
| $\mathfrak{h}_{21}$ | $(0,0,0,12,14,15)$ | 3 | 5 | $(2,3,4,6)$ |
| $\mathfrak{h}_{22}$ | $(0,0,0,12,14,15+24)$ | 3 | 5 | $(2,3,4,6)$ |
| $\mathfrak{h}_{23}$ | $(0,0,12,13,23,14)$ | 2 | 4 | $(2,3,4,6)$ |
| $\mathfrak{h}_{24}$ | $(0,0,0,12,14,15+23+24)$ | 3 | 5 | $(1,3,4,6)$ |
| $\mathfrak{h}_{25}$ | $(0,0,0,12,14,15+23)$ | 3 | 5 | $(1,3,4,6)$ |
| $\mathfrak{h}_{26}^+$ | $(0,0,12,13,23,14+25)$ | 2 | 4 | $(1,3,4,6)$ |
| $\mathfrak{h}_{26}^-$ | $(0,0,12,13,23,14-25)$ | 2 | 4 | $(1,3,4,6)$ |
| $\mathfrak{h}_{27}$ | $(0,0,0,12,14-23,15+34)$ | 3 | 4 | $(1,2,4,6)$ |
| $\mathfrak{h}_{28}$ | $(0,0,12,13,14,15)$ | 2 | 3 | $(1,2,3,4,6)$ |
| $\mathfrak{h}_{29}$ | $(0,0,12,13,14,23+15)$ | 2 | 3 | $(1,2,3,4,6)$ |
| $\mathfrak{h}_{30}$ | $(0,0,12,13,14+23,24+15)$ | 2 | 3 | $(1,2,3,4,6)$ |
| $\mathfrak{h}_{31}$ | $(0,0,12,13,14,34+52)$ | 2 | 2 | $(1,2,3,4,6)$ |
| $\mathfrak{h}_{32}$ | $(0,0,12,13,14+23,34+52)$ | 2 | 2 | $(1,2,3,4,6)$ |

In the case of strongly non-nilpotent structures, the real structure equations given in the proof of Proposition 3.2.6 should completely determine the underlying algebras. However, one would need to find the appropriate changes of basis in order to reduce the number of parameters. Since we already have a very reduced version of the complex structure equations (3.11), a similar argument to that in the 4-dimensional case can be applied. Denoting $\omega^1 = \alpha^1 - i\,\alpha^2$, $\omega^2 = \alpha^3 - i\,\alpha^4$, and $\omega^3 = \alpha^5 - i\,\alpha^6$, we obtain

$$d\alpha^1 = d\alpha^2 = 0, \quad d\alpha^3 = 2\,\alpha^{15}, \quad d\alpha^4 = 2\,\alpha^{25}, \quad d\alpha^5 = 2\,\varepsilon\,\alpha^{12}, \quad d\alpha^6 = \pm 2\,(\alpha^{13} + \alpha^{24}).$$

If $\varepsilon = 0$, considering the following change of basis

$$e^1 = \sqrt{2}\,\alpha^1, \quad e^2 = \alpha^5, \quad e^3 = -\sqrt{2}\,\alpha^2, \quad e^4 = \frac{1}{\sqrt{2}}\,\alpha^3, \quad e^5 = \frac{1}{\sqrt{2}}\,\alpha^4, \quad e^6 = \pm\frac{1}{2}\,\alpha^6,$$

one exactly gets the structure equations for the Lie algebra $\mathfrak{h}_{19}^-$. If $\varepsilon = 1$, take

$$e^1 = \sqrt{2}\,\alpha^1, \quad e^2 = \sqrt{2}\,\alpha^2, \quad e^3 = \alpha^5, \quad e^4 = \frac{1}{\sqrt{2}}\,\alpha^3, \quad e^5 = \frac{1}{\sqrt{2}}\,\alpha^4, \quad e^6 = \pm\frac{1}{2}\,\alpha^6,$$

and it is immediate to see that the underlying algebra corresponds to $\mathfrak{h}_{26}^+$.

Let us now move to the case of extensions. Most of the parameters in equations (3.7) are completely free. Therefore, it seems difficult to obtain direct results using the same idea as before. Nevertheless, it is possible to identify which of the previous algebras admit a nilpotent structure following a similar argument to that in [CFGU97c].

Let $\mathfrak{g}$ be a 6-dimensional NLA admitting a nilpotent complex structure $J$. Taking into account the nilpotency of $J$, the center of $\mathfrak{g}$ should be at least 2-dimensional. This discards $\mathfrak{h}_{18}, \mathfrak{h}_{19}^{\pm}, \mathfrak{h}_{20}, \mathfrak{h}_{24}, \ldots, \mathfrak{h}_{32}$. Furthermore, since the ascending central series of $\mathfrak{g}$ adapted to $J$ stabilizes at $\mathfrak{g}$, the longest admissible sequence has dimension $(2, 4, 6)$. This implies that the Lie algebra is at most 3-step, thus eliminating $\mathfrak{h}_{21}, \mathfrak{h}_{22}, \mathfrak{h}_{23}$. Now, we should focus on $\mathfrak{h}_1, \ldots, \mathfrak{h}_{17}$. The structure equations (3.7) come into play.

Let us consider $\mathfrak{h}_{17}$. If we choose $\varepsilon = 0$ in (3.7), the nilpotency step of the underlying algebra turns to be $s = 2$. However, $\mathfrak{h}_{17}$ is 3-step, so we can fix $\varepsilon = 1$. Now, take $\rho = 1$ and focus on the first Betti number. It is clear that neither $\omega^3$ nor $\omega^{\bar{3}}$ can be used to generate a closed 1-form. Therefore, the case $(\varepsilon, \rho) = (1, 1)$ implies $b_1 = 3 < 4 = b_1(\mathfrak{h}_{17})$. Then, the only remaining possibility is $(\varepsilon, \rho) = (1, 0)$. If we denote $\{Z_k, \bar{Z}_k\}_{k=1}^3$ the dual basis of $\{\omega^k, \omega^{\bar{k}}\}_{k=1}^3$, it is clear that $Z_3 + \bar{Z}_3$ and $i\,(Z_3 + \bar{Z}_3)$ are in the center of the algebra. Since the center of $\mathfrak{h}_{17}$ has dimension 3, another real element should be found. Take $0 \neq T = \mu\,Z_1 + \bar{\mu}\,\bar{Z}_1 + \tau\,Z_2 + \bar{\tau}\,\bar{Z}_2$, and suppose $[T, Y] = 0$ for every $Y \in \mathfrak{h}_{17}$. Observe

$$0 \;=\; [T, Z_1] \;=\; \bar{\mu}\,(Z_2 - \bar{Z}_2) + \bar{\tau}\,(B\,Z_3 - \bar{C}\,\bar{Z}_3),$$

thus getting $\mu = 0$ and $\tau\,B = \tau\,C = 0$. Considering that $\tau \neq 0$ (otherwise, $T = 0$), we can conclude $B = C = 0$. However, this leads to $b_1 = 5 > 4 = b_1(\mathfrak{h}_{17})$ and consequently, the Lie algebra $\mathfrak{h}_{17}$ does not admit any complex structure.

For $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, it is possible to find appropriate values of the parameters in (3.7) ensuring the existence of complex structures.

**Proposition 3.2.8.** *Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra endowed with a complex structure $J$. Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, or $\mathfrak{h}_{26}^+$. Furthermore, $J$ is nilpotent for $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$ and non-nilpotent for $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$.*

This proposition recovers the results previously obtained in [CFGU97c, Sal01]. In particular, note that non-nilpotent and nilpotent structures cannot coexist on the same 6-dimensional NLA.

Observe that in the case of strongly non-nilpotent structures we have easily found all the possible values of the parameters in (3.11) depending on the underlying algebra. However, when working in the nilpotent case two problems arise: the great number of free parameters in (3.7) and the big amount of admissible algebras. Although the solution is far from being trivial, it was finally obtained in [Uga07] after carrying further reductions of the complex structure equations.

Concerning the equivalence of complex structures on a fixed algebra, a similar situation occurs. For the strongly non-nilpotent case, applying the arguments in [UV14] one can directly conclude that the two complex structures given by the signs $+$ and $-$ in (3.11) are non-equivalent, both on $\mathfrak{h}_{19}^{-}$ ($\varepsilon = 0$) and $\mathfrak{h}_{26}^{+}$ ($\varepsilon = 1$). For the nilpotent case, the situation is more complicated due to the same two reasons as above. Nonetheless, the problem was recently solved in [COUV16], where it is possible to find the appropiate changes of basis linking equations (3.7) to their complete classification without repetitions (see Section 1.4.3).

**Remark 3.2.9.** For the seek of simplicity, we will parametrize complex structures on 6-dimensional NLAs using the classification up to equivalence provided by the complex-parallelizable family (1.13), Family I (1.14), Family II (1.15), and Family III (1.16) described in Section 1.4.3.

As we can see, the question we proposed at the beginning of this section does not have a straight answer and becomes more complicated as the dimension increases. Thus, when dealing with the 8-dimensional case, we will forget about this point and just focus on the more general parametrization.

## 3.3 The eight-dimensional case

The lack of a nilpotent Lie algebra classification in 8 dimensions may have prevented, up to date, from parametrizing invariant complex geometry on 8-dimensional nilmanifolds. The approach previously described could allow to overcome this obstacle.

Let $\mathfrak{g}$ be an 8-dimensional NLA endowed with a complex structure $J$. As in the lower dimensional cases, two paths need to be distinguished according to the nilpotency of $J$. Considering that the difficulty of our methods seems to increase with the dimension of the underlying algebra, here we will simply focus on those pairs such that $\mathfrak{a}_1(J) \neq \{0\}$. The study of strongly non-nilpotent structures will be fulfilled in the next chapter.

Let us then suppose that our complex structure $J$ is quasi-nilpotent. According to Section 3.1.1, it suffices to extend the classification (up to equivalence) of complex structures on 6-dimensional NLAs in order to parametrize every possible pair $(\mathfrak{g}, J)$.

**Lemma 3.3.1.** *Let $\mathfrak{h}$ be a 6-dimensional NLA endowed with a complex structure $K$. Then, the $\mathfrak{b}$-extensions of $(\mathfrak{h}, K)$ are of the following types:*

*i) if* $(\mathfrak{h}, K)$ *is determined by the complex-parallelizable family* (1.13)*, then*

$$\begin{cases} d\omega^1 &=& d\omega^2 \;=\; 0, \\ d\omega^3 &=& \rho\,\omega^{12}, \\ d\omega^4 &=& A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} \\ && +(1-\rho)\left(B_{1\bar{3}}\,\omega^{1\bar{3}} + C_{2\bar{3}}\,\omega^{2\bar{3}} + D_{3\bar{1}}\,\omega^{3\bar{1}} + D_{3\bar{2}}\,\omega^{3\bar{2}} + D_{3\bar{3}}\,\omega^{3\bar{3}}\right), \end{cases}$$

*where* $\rho = 0$ *for the complex torus and* $\rho = 1$ *for the Iwasawa manifold;*

*ii) if* $(\mathfrak{h}, K)$ *is isomorphic to a structure in Family I* (1.14)*, then*

$$\begin{cases} d\omega^1 &=& d\omega^2 \;=\; 0, \\ d\omega^3 &=& \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \\ d\omega^4 &=& A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} \\ && +C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} + C_{2\bar{3}}\,\omega^{2\bar{3}} + D_{3\bar{1}}\,\omega^{3\bar{1}} + D_{3\bar{2}}\,\omega^{3\bar{2}}, \end{cases}$$

*where the parameters satisfy* $\rho \in \{0,1\}$*,* $\lambda \in \mathbb{R}^{\geq 0}$*,* $D \in \mathbb{C}$ *with* $\mathfrak{Im}\, D \geq 0$*, and*

$$(3.12)\qquad \begin{cases} A_{23} + \lambda\, B_{1\bar{3}} - C_{2\bar{3}} + \rho\, D_{3\bar{1}} = 0, \\ D\, A_{13} - \lambda\, A_{23} - \bar{D}\, B_{1\bar{3}} - \rho\, D_{3\bar{2}} = 0, \end{cases} \qquad \begin{cases} \rho\, B_{1\bar{3}} + \lambda\, D_{3\bar{1}} - D_{3\bar{2}} = 0, \\ \rho\, C_{2\bar{3}} + D\, D_{3\bar{1}} = 0. \end{cases}$$

*iii) if* $(\mathfrak{h}, K)$ *is defined by Family II* (1.15)*, then*

$$\begin{cases} d\omega^1 &=& 0, \\ d\omega^2 &=& \omega^{1\bar{1}}, \\ d\omega^3 &=& \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \\ d\omega^4 &=& A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} \\ && +C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} + D_{3\bar{1}}\,\omega^{3\bar{1}} - A_{23}\,\omega^{3\bar{2}}, \end{cases}$$

*where* $\rho \in \{0,1\}$*,* $B \in \mathbb{C}$*,* $c \in \mathbb{R}^{\geq 0}$ *such that* $(\rho, B, c) \neq (0,0,0)$*, and*

$$(3.13)\qquad \begin{cases} c\, A_{13} - \bar{B}\, B_{1\bar{3}} + C_{2\bar{2}} - \rho\, D_{3\bar{1}} = 0, \\ \rho\, B_{1\bar{3}} - C_{2\bar{2}} + B\, D_{3\bar{1}} = 0, \end{cases} \qquad \begin{cases} (B-\rho)\, A_{23} = 0, \\ c\, A_{23} = 0. \end{cases}$$

*iv) if* $(\mathfrak{h}, K)$ *is parametrized by Family III* (1.16)*, then*

$$\begin{cases} d\omega^1 &=& 0, \\ d\omega^2 &=& \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 &=& i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \\ d\omega^4 &=& A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} \\ && +(A_{13} \pm 2\,\varepsilon\, A_{23})\,\omega^{1\bar{3}} - B_{1\bar{2}}\,\omega^{2\bar{1}} + A_{23}\,\omega^{2\bar{3}}, \end{cases}$$

*where* $\varepsilon = 0$ *for* $\mathfrak{h}_{19}^-$ *and* $\varepsilon = 1$ *for* $\mathfrak{h}_{26}^+$.

*Proof.* Let us remark that the complex structure equations of $(\mathfrak{h}, K)$ are given by (1.13), (1.14), (1.15), and (1.16). Consider a basis $\{\eta^i\}_{i=1}^3$ for $\mathfrak{h}^{1,0}$ satisfying such equations. Add a 4th element $\eta^4$ following Theorem 3.1.8, i.e.,

$$\begin{aligned} d\eta^4 \;\; = \;\; & A_{12}\,\eta^{12} + A_{13}\,\eta^{13} + A_{23}\,\eta^{23} + B_{1\bar{1}}\,\eta^{1\bar{1}} + B_{1\bar{2}}\,\eta^{1\bar{2}} + B_{1\bar{3}}\,\eta^{1\bar{3}} \\ & + C_{2\bar{1}}\,\eta^{2\bar{1}} + C_{2\bar{2}}\,\eta^{2\bar{2}} + C_{2\bar{3}}\,\eta^{2\bar{3}} + D_{3\bar{1}}\,\eta^{3\bar{1}} + D_{3\bar{2}}\,\eta^{3\bar{2}} + D_{3\bar{3}}\,\eta^{3\bar{3}}, \end{aligned}$$

and impose the condition $d^2\eta^4 = 0$. □

We now proceed to find further reductions of the previous 8-dimensional families of complex structure equations.

### 3.3.1 Extensions of the complex-parallelizable family

We recall that there are, up to equivalence, two complex-parallelizable structures on 6-dimensional NLAs, which correspond to equations (1.13).

**Proposition 3.3.2.** *Let* $\mathfrak{h}$ *be a* 6*-dimensional NLA endowed with a complex-parallelizable structure* $K$*. The quasi-nilpotent pair* $(\mathfrak{g}, J)$ *is a* $\mathfrak{b}$*-extension of* $(\mathfrak{h}, K)$ *if it can be parametrized by*

$$(3.14)\qquad \begin{cases} d\omega^1 \;=\; d\omega^2 \;=\; 0, \\ d\omega^3 \;=\; \rho\,\omega^{12}, \\ d\omega^4 \;=\; \nu\,\omega^{13} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} \\ \qquad\quad + (1-\rho)\left(B_{1\bar{3}}\,\omega^{1\bar{3}} + C_{2\bar{3}}\,\omega^{2\bar{3}} + D_{3\bar{1}}\,\omega^{3\bar{1}} + D_{3\bar{2}}\,\omega^{3\bar{2}} + D_{3\bar{3}}\,\omega^{3\bar{3}}\right), \end{cases}$$

*where* $\rho, \nu \in \{0, 1\}$, $B_{1\bar{1}}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, C_{2\bar{2}}, C_{2\bar{3}}, D_{3\bar{1}}, D_{3\bar{2}}, D_{3\bar{3}} \in \mathbb{C}$.

*Proof.* Consider a basis $\{\eta^i\}_{i=1}^4$ satisfying Lemma 3.3.1 *i)* with coefficients $A'_{ij}$, $B'_{i\bar{j}}$, $C'_{i\bar{j}}$, and $D'_{i\bar{j}}$. Consider a new $(1,0)$-basis

$$\sigma^1 = \eta^1, \quad \sigma^2 = \eta^2, \quad \sigma^3 = \eta^3, \quad \sigma^4 = \eta^4 - A'_{12}\eta^3.$$

The structure equations become $d\sigma^1 = d\sigma^2 = 0$, $d\sigma^3 = \rho\,\sigma^{12}$, and

$$\begin{aligned} d\sigma^4 \;\; = \;\; & A'_{13}\,\sigma^{13} + A'_{23}\,\sigma^{23} + B'_{1\bar{1}}\,\sigma^{1\bar{1}} + B'_{1\bar{2}}\,\sigma^{1\bar{2}} + C'_{2\bar{1}}\,\sigma^{2\bar{1}} + C'_{2\bar{2}}\,\sigma^{2\bar{2}} \\ & + (1-\rho)\left(A'_{12}\,\sigma^{12} + B'_{1\bar{3}}\,\sigma^{1\bar{3}} + C'_{2\bar{3}}\,\sigma^{2\bar{3}} + D'_{3\bar{1}}\,\sigma^{3\bar{1}} + D'_{3\bar{2}}\,\sigma^{3\bar{2}} + D'_{3\bar{3}}\,\sigma^{3\bar{3}}\right). \end{aligned}$$

Observe the following:

- If $((1-\rho)\,A'_{12}, A'_{13}, A'_{23}) = (0, 0, 0)$, it suffices to rename the coefficients and basis in order to get (3.14) with $\nu = 0$.

- If $((1-\rho)\,A'_{12}, A'_{13}) = (0,0)$ and $A'_{23} \neq 0$, take a new basis

$$\omega^1 = \sigma^2, \quad \omega^2 = \sigma^1, \quad \omega^3 = -\sigma^3, \quad \omega^4 = -\frac{1}{A'_{23}}\,\sigma^4.$$

Then we have (3.14) with $\nu = 1$ and

$$B_{1\bar 1} = -\frac{C'_{2\bar 2}}{A'_{23}}, \quad B_{1\bar 2} = -\frac{C'_{2\bar 1}}{A'_{23}}, \quad B_{1\bar 3} = \frac{C'_{2\bar 3}}{A'_{23}}, \quad C_{2\bar 1} = -\frac{B'_{1\bar 2}}{A'_{23}},$$

$$C_{2\bar 2} = -\frac{B'_{1\bar 1}}{A'_{23}}, \quad C_{2\bar 3} = \frac{B'_{1\bar 3}}{A'_{23}}, \quad D_{3\bar 1} = \frac{D'_{3\bar 2}}{A'_{23}}, \quad D_{3\bar 2} = \frac{D'_{3\bar 1}}{A'_{23}}, \quad D_{3\bar 3} = -\frac{D'_{3\bar 3}}{A'_{23}}.$$

- If $(1-\rho)\,A'_{12} = 0$ and $A'_{13} \neq 0$, the $(1,0)$-basis

$$\omega^1 = A'_{13}\,\sigma^1 + A'_{23}\,\sigma^2, \quad \omega^2 = \sigma^2, \quad \omega^3 = A'_{13}\,\sigma^3, \quad \omega^4 = A'_{13}\,\sigma^4$$

satisfies (3.14) with $\nu = 1$ and

$$B_{1\bar 1} = \frac{B'_{1\bar 1}}{\bar A'_{13}}, \quad B_{1\bar 2} = \frac{B'_{1\bar 2}\bar A'_{13} - B'_{1\bar 1}\bar A'_{23}}{\bar A'_{13}}, \quad B_{1\bar 3} = \frac{B'_{1\bar 3}}{\bar A'_{13}},$$

$$C_{2\bar 1} = \frac{A'_{13}C'_{2\bar 1} - A'_{23}B'_{1\bar 1}}{\bar A'_{13}}, \quad C_{2\bar 2} = \frac{|A'_{23}|^2 B'_{1\bar 1} - A'_{23}B'_{1\bar 2}\bar A'_{13} - A'_{13}C'_{2\bar 1}\bar A'_{23} + |A'_{13}|^2 C'_{2\bar 2}}{\bar A'_{13}},$$

$$C_{2\bar 3} = \frac{A'_{13}C'_{2\bar 3} - A'_{23}B'_{1\bar 3}}{\bar A'_{13}}, \quad D_{3\bar 1} = \frac{D'_{3\bar 1}}{\bar A'_{13}}, \quad D_{3\bar 2} = \frac{D'_{3\bar 2}\bar A'_{13} - D'_{3\bar 1}\bar A'_{23}}{\bar A'_{13}}, \quad D_{3\bar 3} = \frac{D'_{3\bar 3}}{\bar A'_{13}}.$$

- If $(1-\rho)\,A'_{12} \neq 0$, then $\rho = 0$ and one can consider a new basis

$$\omega^1 = \sigma^1 - \frac{A'_{23}}{A'_{12}}\,\sigma^3, \quad \omega^2 = \sigma^3, \quad \omega^3 = A'_{12}\sigma^2 + A'_{13}\sigma^3, \quad \omega^4 = \sigma^4.$$

It satisfies (3.14) with $\nu = 1$, $B_{1\bar 1} = B'_{1\bar 1}$ and

$$B_{1\bar 2} = \frac{B'_{1\bar 1}\bar A'_{23} - B'_{1\bar 2}\bar A'_{13} + B'_{1\bar 3}\bar A'_{12}}{\bar A'_{12}}, \quad B_{1\bar 3} = \frac{B'_{1\bar 2}}{\bar A'_{12}}, \quad C_{2\bar 1} = \frac{A'_{23}B'_{1\bar 1} - A'_{13}C'_{2\bar 1} + A'_{12}D'_{3\bar 1}}{A'_{12}},$$

$$C_{2\bar 2} = \frac{A'_{12}(D'_{3\bar 1}\bar A'_{23} - D'_{3\bar 2}\bar A'_{13} + D'_{3\bar 3}\bar A'_{12})}{|A'_{12}|^2} - \frac{A'_{13}(C'_{2\bar 1}\bar A'_{23} - C'_{2\bar 2}\bar A'_{13} + C'_{2\bar 3}\bar A'_{12})}{|A'_{12}|^2} +$$
$$\frac{A'_{23}(B'_{1\bar 1}\bar A'_{23} - B'_{1\bar 2}\bar A'_{13} + B'_{1\bar 3}\bar A'_{12})}{|A'_{12}|^2}, \quad C_{2\bar 3} = \frac{A'_{23}B'_{1\bar 2} - A'_{13}C'_{2\bar 2} + A'_{12}D'_{3\bar 2}}{|A'_{12}|^2},$$

$$D_{3\bar 1} = \frac{C'_{2\bar 1}}{A'_{12}}, \quad D_{3\bar 2} = \frac{C'_{2\bar 1}\bar A'_{23} - C'_{2\bar 2}\bar A'_{13} + C'_{2\bar 3}\bar A'_{12}}{|A'_{12}|^2}, \quad D_{3\bar 3} = \frac{C'_{2\bar 2}}{|A'_{12}|^2}.$$

□

Although we are not able to identify the non-isomorphic real NLAs underlying the previous extensions, we can at least compute their nilpotency steps.

**Proposition 3.3.3.** *The Lie algebra* $\mathfrak{g}$ *associated to the* $\mathfrak{b}$*-extension* (3.14) *of the* 6*-dimensional complex-parallelizable family can be:*

*a)* 1*-step, if* $(\rho, \nu, B_{1\bar{1}}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, C_{2\bar{2}}, C_{2\bar{3}}, D_{3\bar{1}}, D_{3\bar{2}}, D_{3\bar{3}}) = (0, \ldots, 0)$ *(torus),*

*b)* 2*-step, if* $\rho\,\nu = 0$*, and* $(\rho, \nu, B_{1\bar{1}}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, C_{2\bar{2}}, C_{2\bar{3}}, D_{3\bar{1}}, D_{3\bar{2}}, D_{3\bar{3}}) \neq (0, \ldots, 0)$,

*c)* 3*-step, if* $(\rho, \nu) = (1, 1)$.

*Proof.* The nilpotency step can be calculated via the descending central series (3.1) of $\mathfrak{g}$ using the Lie brackets determined by (3.14). Let us denote $\{Z_i\}_{i=1}^4$ the dual basis of $\{\omega^i\}_{i=1}^4$. Observe that the elements $Z_4$ and $\bar{Z}_4$ belong to the center of $\mathfrak{g}$, so one immediately has $[Z_4, \cdot\,] = [\bar{Z}_4, \cdot\,] = 0$. Moreover, the space $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is generated by the following brackets and their conjugates

$$
\begin{aligned}
[Z_1, Z_2] &= -\rho\, Z_3, & [Z_1, \bar{Z}_3] &= (1-\rho)(-B_{1\bar{3}}\, Z_4 + \bar{D}_{3\bar{1}}\, \bar{Z}_4),\\
[Z_1, Z_3] &= -\nu\, Z_4, & [Z_2, \bar{Z}_2] &= -C_{2\bar{2}}\, Z_4 + \bar{C}_{2\bar{2}}\, \bar{Z}_4,\\
[Z_1, \bar{Z}_1] &= -B_{1\bar{1}}\, Z_4 + \bar{B}_{1\bar{1}}\, \bar{Z}_4, & [Z_2, \bar{Z}_3] &= (1-\rho)(-C_{2\bar{3}}\, Z_4 + \bar{D}_{3\bar{2}}\, \bar{Z}_4),\\
[Z_1, \bar{Z}_2] &= -B_{1\bar{2}}\, Z_4 + \bar{C}_{2\bar{1}}\, \bar{Z}_4, & [Z_3, \bar{Z}_3] &= (1-\rho)(-D_{3\bar{3}}\, Z_4 + \bar{D}_{3\bar{3}}\, \bar{Z}_4).
\end{aligned}
$$

From these expressions one can easily see part *a)* of the statement. To calculate $\mathfrak{g}^2$, it suffices to focus on those brackets which do not completely lay on $\langle\, Z_4,\, \bar{Z}_4\,\rangle \subset \mathfrak{g}_1$. In this way, one can conclude that the space $\mathfrak{g}^2$ is generated by the following element and its conjugate:

$$\big[[Z_1, Z_2], Z_1\big] = -\rho\,\nu\, Z_4.$$

This yields to the parts *b)* and *c)*. □

**Remark 3.3.4.** It is always possible to find an extension of the complex torus or the Iwasawa manifold rising the nilpotency step of the original 6-dimensional algebra by 1.

### 3.3.2 Extensions of Family I

Nilpotent complex structures on NLAs $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$, and $\mathfrak{h}_8$ are given by equations (1.14) (with the exception of the Iwasawa manifold). Note that each algebra has its own particular values of the parameters $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}^{\geq 0}$, and $D \in \mathbb{C}$, which can be found in Table A (see Section 1.4.3). The relations among them will determine the construction of the extension.

**Proposition 3.3.5.** *Let* $\mathfrak{h}$ *be a* 6*-dimensional NLA endowed with a nilpotent complex structure* $K$ *parametrized by Family I. The quasi-nilpotent pair* $(\mathfrak{g}, J)$ *is a* $\mathfrak{b}$*-extension of* $(\mathfrak{h}, K)$ *when it can be parametrized by one of the following sets of structure equations:*

*i) if* $(\mathfrak{h}, K)$ *satisfies* $\rho = \lambda$ *and* $D = 0$*, then*

$$(3.15)\quad \begin{cases} d\omega^1 &= \; d\omega^2 \; = \; 0, \\ d\omega^3 &= \; \rho\,(\omega^{12} + \omega^{1\bar{2}}) + (1-\rho)\,\omega^{1\bar{1}}, \\ d\omega^4 &= \; A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + \rho\,B_{1\bar{1}}\,\omega^{1\bar{1}} \\ & \quad +(1-\rho)\,B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} + C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} \\ & \quad +(1-\rho)\,A_{23}\,\omega^{2\bar{3}} + ((1-\rho)\,D_{3\bar{1}} - \rho\,B_{1\bar{3}})\,\omega^{3\bar{1}} - \rho\,A_{23}\,\omega^{3\bar{2}}, \end{cases}$$

*where* $A_{12}, A_{13}, A_{23}, B_{1\bar{1}}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, C_{2\bar{2}}, D_{3\bar{1}} \in \mathbb{C}$*;*

*ii) if* $(\mathfrak{h}, K)$ *fulfills the conditions* $\rho \neq \lambda$ *and* $D = 0$*, then*

$$(3.16)\quad \begin{cases} d\omega^1 &= \; d\omega^2 \; = \; 0, \\ d\omega^3 &= \; \rho\,\omega^{12} + \nu\,\omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}}, \\ d\omega^4 &= \; A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + \rho\,A_{23}\,\omega^{23} + (1-\nu)\,B_{1\bar{1}}\,\omega^{1\bar{1}} + \nu\,B_{1\bar{2}}\,\omega^{1\bar{2}} \\ & \quad +C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} + (1-\rho)\,C_{2\bar{3}}\,\omega^{2\bar{3}} - \nu\,A_{23}\,\omega^{3\bar{1}} - \lambda\,A_{23}\,\omega^{3\bar{2}}, \end{cases}$$

*where* $\nu \in \{0, 1\}$*,* $A_{12}, A_{13}, A_{23}, B_{1\bar{1}}, B_{1\bar{2}}, C_{2\bar{1}}, C_{2\bar{2}}, C_{2\bar{3}} \in \mathbb{C}$*. Furthermore, the case* $\nu = 1$ *is only valid for* $(\rho, \lambda) = (1, 0)$*;*

*iii) if* $(\mathfrak{h}, K)$ *has* $D \neq 0$*, then*

$$(3.17)\quad \begin{cases} d\omega^1 &= \; d\omega^2 \; = \; 0, \\ d\omega^3 &= \; \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \\ d\omega^4 &= \; A_{12}\,\omega^{12} + \big((\bar{D} - \lambda^2 + \rho)\,B_{1\bar{3}} + \lambda\,C_{2\bar{3}}\big)\;\omega^{13} \\ & \quad + ((D+\rho)\,C_{2\bar{3}} - \lambda\,D\,B_{1\bar{3}})\;\omega^{23} + B_{1\bar{2}}\,\omega^{1\bar{2}} + D\,B_{1\bar{3}}\,\omega^{1\bar{3}} + C_{2\bar{1}}\,\omega^{2\bar{1}} \\ & \quad +C_{2\bar{2}}\,\omega^{2\bar{2}} + D\,C_{2\bar{3}}\,\omega^{2\bar{3}} - \rho\,C_{2\bar{3}}\,\omega^{3\bar{1}} + \rho\,(D\,B_{1\bar{3}} - \lambda\,C_{2\bar{3}})\,\omega^{3\bar{2}}, \end{cases}$$

*where* $A_{12}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, C_{2\bar{2}}, C_{2\bar{3}} \in \mathbb{C}$*.*

*Proof.* Let us consider a basis $\{\eta^i\}_{i=1}^4$ satisfying Lemma 3.3.1 *ii)* with coefficients $A'_{ij}$, $B'_{i\bar{j}}$, $C'_{i\bar{j}}$, and $D'_{i\bar{j}}$. Let us discuss how to attain our result depending on the solutions of the system of equations given by (3.12):

- If $\rho = \lambda$ and $D = 0$, one has

$$\begin{cases} \text{for } \rho = 0: \; A'_{23} = C'_{2\bar{3}}, \; D'_{3\bar{2}} = 0, \;\; \text{and} \\ \text{for } \rho = 1: \; A'_{23} = -\,(B'_{1\bar{3}} + D'_{3\bar{1}}), \; C'_{2\bar{3}} = 0, \; D'_{3\bar{2}} = B'_{1\bar{3}} + D'_{3\bar{1}}. \end{cases}$$

In both cases, consider a new $(1,0)$-basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2 + \rho\,\eta^1, \quad \omega^3 = \eta^3, \quad \omega^4 = \eta^4 - \big((1-\rho)\,B'_{1\bar{1}} + \rho\,(B'_{1\bar{2}} - C'_{2\bar{2}})\big)\,\eta^3.$$

For $\rho = 0$, it suffices to rename the coefficients to obtain equations (3.15). For $\rho = 1$, we still get (3.15) but this time with

$$A_{12} = A'_{12} - B'_{1\bar{2}} + C'_{2\bar{2}}, \quad A_{13} = A'_{13} + B'_{1\bar{3}} + D'_{3\bar{1}}, \quad A_{23} = -(B'_{1\bar{3}} + D'_{3\bar{1}}),$$

$$B_{1\bar{1}} = B'_{1\bar{1}} - B'_{1\bar{2}} - C'_{2\bar{1}} + C'_{2\bar{2}}, \quad B_{1\bar{3}} = B'_{1\bar{3}}, \quad C_{2\bar{1}} = C'_{2\bar{1}} - C'_{2\bar{2}}, \quad C_{2\bar{2}} = C'_{2\bar{2}}.$$

- If $\rho \neq \lambda$ and $D = 0$, one has $D'_{3\bar{2}} = \lambda\,D'_{3\bar{1}}$ together with

$$\begin{cases} \text{for } \rho = 0: \ A'_{23} = 0, \ C'_{2\bar{3}} = \lambda\,B'_{1\bar{3}}, \ \text{ and} \\ \text{for } \rho = 1: \ A'_{23} = -D'_{3\bar{1}}, \ B'_{1\bar{3}} = C'_{2\bar{3}} = 0. \end{cases}$$

If $\lambda = 0$ (thus, $\rho = 1$), defining a new basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = \eta^4 - B'_{1\bar{1}}\,\eta^3,$$

equations (3.16) are recovered for $\nu = 1$. One then just needs to rename the coefficients. If $\lambda \neq 0$, consider

$$\omega^1 = \eta^1, \quad \omega^2 = \lambda\,\eta^2 + \eta^1, \quad \omega^3 = \lambda\,\eta^3, \quad \omega^4 = \eta^4 - \frac{\lambda\,B'_{1\bar{2}} - C'_{2\bar{2}}}{\lambda^2}\,\eta^3.$$

The structure equations become (3.16) with $\nu = 0$, for any $\rho \in \{0,1\}$. More concretely,

$$A_{12} = \frac{A'_{12}}{\lambda}, \quad A_{13} = \frac{A'_{13}}{\lambda}, \quad A_{23} = -\frac{D'_{3\bar{1}}}{\lambda^2}, \quad B_{1\bar{1}} = B'_{1\bar{1}} + \frac{C'_{2\bar{2}} - \lambda\,(B'_{1\bar{2}} + C'_{2\bar{1}})}{\lambda^2},$$

$$C_{2\bar{1}} = \frac{\lambda\,C'_{2\bar{1}} - C'_{2\bar{2}}}{\lambda^2}, \quad C_{2\bar{2}} = \frac{C'_{2\bar{2}}}{\lambda^2}, \quad C_{2\bar{3}} = \frac{B'_{1\bar{3}}}{\lambda},$$

for $\rho = 0$, and

$$A_{12} = \frac{\lambda^2\,A'_{12} - \lambda\,B'_{1\bar{2}} + C'_{2\bar{2}}}{\lambda^3}, \quad A_{13} = \frac{\lambda\,A'_{13} + D'_{3\bar{1}}}{\lambda^2}, \quad A_{23} = -\frac{D'_{3\bar{1}}}{\lambda^2},$$

$$B_{1\bar{1}} = B'_{1\bar{1}} + \frac{C'_{2\bar{2}} - \lambda\,(B'_{1\bar{2}} + C'_{2\bar{1}})}{\lambda^2}, \quad C_{2\bar{1}} = \frac{\lambda\,C'_{2\bar{1}} - C'_{2\bar{2}}}{\lambda^2}, \quad C_{2\bar{2}} = \frac{C'_{2\bar{2}}}{\lambda^2},$$

for $\rho = 1$.

- If $D \neq 0$, then $A'_{13} = \frac{\bar{D} - \lambda^2 + \rho}{D}\,B'_{1\bar{3}} + \frac{\lambda}{D}\,C'_{2\bar{3}}$, $A'_{23} = -\lambda\,B'_{1\bar{3}} + \frac{D+\rho}{D}\,C'_{2\bar{3}}$, and

$$\begin{cases} \text{for } \rho = 0: \ D'_{3\bar{1}} = D'_{3\bar{2}} = 0, \\ \text{for } \rho = 1: \ D'_{3\bar{1}} = -\frac{1}{D}\,C'_{2\bar{3}}, \ D'_{3\bar{2}} = B'_{1\bar{3}} - \frac{\lambda}{D}\,C'_{2\bar{3}}. \end{cases}$$

Take a new basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = D\,(\eta^4 - B'_{1\bar{1}}\eta^3),$$

and the structure equations become (3.17) with

$$A_{12} = D\,A'_{12}, \quad B_{1\bar{2}} = D\,(B'_{1\bar{2}} - \lambda\,B'_{1\bar{1}}), \quad B_{1\bar{3}} = B'_{1\bar{3}},$$
$$C_{2\bar{1}} = D\,C'_{2\bar{1}}, \quad C_{2\bar{2}} = D\,(C'_{2\bar{2}} - D\,B'_{1\bar{1}}), \quad C_{2\bar{3}} = C'_{2\bar{3}}.$$

□

**Notation 3.3.6.** For the seek of simplity, we will refer to (3.15) as *Family I/i*, to (3.16) as *Family I/ii*, and to (3.17) as *Family I/iii.*

We now proceed to compute the nilpotency step of the previous 8-dimensional nilpotent Lie algebras.

**Proposition 3.3.7.** *Let* $(\mathfrak{g}, J)$ *be a* $\mathfrak{b}$*-extension of Family I. Then,*

*i) if* $(\mathfrak{g}, J)$ *belongs to Family I/i, the nilpotent Lie algebra* $\mathfrak{g}$ *is*

*i.a) 2-step, if either* $(\rho, B_{1\bar{3}} - A_{13}, D_{3\bar{1}}) = (0,0,0)$ *or* $(\rho, A_{13}, A_{23}, B_{1\bar{3}}) = (1,0,0,0)$,

*i.b) 3-step, in other case;*

*ii) if it belongs to Family I/ii, then* $\mathfrak{g}$ *is*

*ii.a) 2-step, if* $(A_{13}, A_{23}, (1-\rho)\,C_{2\bar{3}}) = (0,0,0)$,

*ii.b) 3-step, in other case;*

*iii) if it belongs to Family I/iii, the Lie algebra is*

*iii.a) 2-step, if* $(B_{1\bar{3}}, C_{2\bar{3}}) = (0,0)$ *or* $(B_{1\bar{3}}, C_{2\bar{3}}) \neq (0,0)$ *with* $(\rho, \lambda, \mathfrak{Im} D) = (0,0,0)$,

*iii.b) 3-step, in other case.*

*Proof.* The nilpotency step will be calculated using the descending central series (3.1) of $\mathfrak{g}$, by means of the Lie brackets determined by (3.15), (3.16) and (3.17). Denote $\{Z_i\}_{i=1}^4$ the dual basis of $\{\omega^i\}_{i=1}^4$.

In the case of the $\mathfrak{b}$-extensions $(\mathfrak{g}, J)$ defined by equations (3.15), the space $\mathfrak{g}^1$ is generated by the following brackets and their conjugates

$$[Z_1, Z_2] = -\rho\,Z_3 - A_{12}\,Z_4, \qquad [Z_1, Z_3] = -A_{13}\,Z_4, \qquad [Z_2, Z_3] = -A_{23}\,Z_4,$$
$$[Z_1, \bar{Z}_1] = -(1-\rho)\,(Z_3 - \bar{Z}_3) - \rho\,(B_{1\bar{1}}\,Z_4 - \bar{B}_{1\bar{1}}\,\bar{Z}_4),$$
$$[Z_1, \bar{Z}_2] = -\rho\,Z_3 - (1-\rho)\,B_{1\bar{2}}\,Z_4 + \bar{C}_{2\bar{1}}\,\bar{Z}_4,$$
$$[Z_1, \bar{Z}_3] = -B_{1\bar{3}}\,Z_4 + ((1-\rho)\bar{D}_{3\bar{1}} - \rho\,\bar{B}_{1\bar{3}})\,\bar{Z}_4,$$
$$[Z_2, \bar{Z}_2] = -C_{2\bar{2}}\,Z_4 + \bar{C}_{2\bar{2}}\,\bar{Z}_4, \qquad [Z_2, \bar{Z}_3] = -(1-\rho)A_{23}\,Z_4 - \rho\,\bar{A}_{23}\,\bar{Z}_4.$$

Observing the brackets $[Z_1, \bar{Z}_1]$ for $\rho = 0$ and $[Z_1, \bar{Z}_2]$ for $\rho = 1$, it is clear that the algebras $\mathfrak{g}$ are at least 2-step. Focusing on those brackets not lying on $\langle Z_4, \bar{Z}_4 \rangle \subset \mathfrak{g}_1$, one can see that the space $\mathfrak{g}^2$ is generated by

$$\begin{aligned}
\big[[Z_1, Z_2], Z_1\big] &= \big[[Z_1, \bar{Z}_2], Z_1\big] = -\rho\, A_{13}\, Z_4,\\
\big[[Z_1, Z_2], Z_2\big] &= \big[[Z_1, \bar{Z}_2], Z_2\big] = \big[[Z_1, Z_2], \bar{Z}_2\big] = \big[[Z_1, \bar{Z}_2], \bar{Z}_2\big] = -\rho\, A_{23}\, Z_4,\\
\big[[Z_1, Z_2], \bar{Z}_1\big] &= \big[[Z_1, \bar{Z}_2], \bar{Z}_1\big] = -\rho\, (B_{1\bar{3}}\, Z_4 + \bar{B}_{1\bar{3}}\, \bar{Z}_4),\\
\big[[Z_1, \bar{Z}_1], Z_1\big] &= (1-\rho)\, ((B_{1\bar{3}} - A_{13})\, Z_4 - \bar{D}_{3\bar{1}}\, \bar{Z}_4)
\end{aligned}$$

and their conjugates. From here we obtain i.*a)* and i.*b)*.

Let us now move to the case of those structures defined by equations (3.16). Now the space $\mathfrak{g}^1$ is given by the brackets below and their conjugates

$$\begin{aligned}
[Z_1, Z_2] &= -\rho\, Z_3 - A_{12}\, Z_4, \qquad [Z_1, Z_3] = -A_{13}\, Z_4, \qquad [Z_2, Z_3] = -\rho\, A_{23}\, Z_4,\\
[Z_1, \bar{Z}_1] &= -\nu\, (Z_3 - \bar{Z}_3) - (1-\nu)\, (B_{1\bar{1}}\, Z_4 - \bar{B}_{1\bar{1}}\, \bar{Z}_4),\\
[Z_1, \bar{Z}_2] &= -\lambda\, Z_3 - \nu\, B_{1\bar{2}}\, Z_4 + \bar{C}_{2\bar{1}}\, \bar{Z}_4, \qquad [Z_1, \bar{Z}_3] = -\nu\, \bar{A}_{23}\, \bar{Z}_4,\\
[Z_2, \bar{Z}_2] &= -C_{2\bar{2}}\, Z_4 + \bar{C}_{2\bar{2}}\, \bar{Z}_4, \qquad [Z_2, \bar{Z}_3] = -(1-\rho)\, C_{2\bar{3}}\, Z_4 - \lambda\, \bar{A}_{23}\, \bar{Z}_4.
\end{aligned}$$

Due to $[Z_1, \bar{Z}_1]$ for $\nu = 1$ and $[Z_1, \bar{Z}_2]$ for $\nu = 0$ (thus, $\lambda \neq 0$), we can at least ensure the 2-step nilpotency of our underlying algebras. Notice that $\mathfrak{g}^2$ is generated by the following brackets and their conjugates

$$\begin{aligned}
\big[[Z_1, Z_2], Z_1\big] &= -\rho\, A_{13}\, Z_4, \qquad \big[[Z_1, \bar{Z}_1], Z_2\big] = -\nu\, (\rho\, A_{23}\, Z_4 - (1-\rho)\, C_{2\bar{3}}\, \bar{Z}_4),\\
\big[[Z_1, Z_2], Z_2\big] &= -\rho\, A_{23}\, Z_4, \qquad \big[[Z_1, \bar{Z}_2], Z_1\big] = -\lambda\, A_{13}\, Z_4,\\
\big[[Z_1, Z_2], \bar{Z}_1\big] &= -\nu\, \rho\, A_{23}\, Z_4, \qquad \big[[Z_1, \bar{Z}_2], \bar{Z}_2\big] = -\lambda\, (\lambda\, A_{23}\, Z_4 + (1-\rho)\, \bar{C}_{2\bar{3}}\, \bar{Z}_4),\\
\big[[Z_1, Z_2], \bar{Z}_2\big] &= \big[[Z_1, \bar{Z}_2], Z_2\big] = -\rho\, \lambda\, A_{23}\, Z_4,\\
\big[[Z_1, \bar{Z}_1], Z_1\big] &= -\nu\, (A_{13}\, Z_4 - \bar{A}_{23}\, \bar{Z}_4).
\end{aligned}$$

From these expressions we are able to conclude ii.*a)* and ii.*b)*.

Finally, let us consider the case of those extensions $(\mathfrak{g}, J)$ given by equations (3.17). The only non-zero brackets in $\mathfrak{g}^1$ are the following ones, together with their conjugates:

$$\begin{aligned}
[Z_1, Z_2] &= -\rho\, Z_3 - A_{12}\, Z_4, \qquad [Z_1, Z_3] = -((\bar{D} - \lambda^2 + \rho)\, B_{1\bar{3}} + \lambda\, C_{2\bar{3}})\, Z_4,\\
[Z_2, Z_3] &= -((D + \rho)\, C_{2\bar{3}} - \lambda\, D\, B_{1\bar{3}})\, Z_4,\\
[Z_1, \bar{Z}_1] &= -Z_3 + \bar{Z}_3, \qquad [Z_1, \bar{Z}_2] = -\lambda\, Z_3 - B_{1\bar{2}}\, Z_4 + \bar{C}_{2\bar{1}}\, \bar{Z}_4,\\
[Z_1, \bar{Z}_3] &= -D\, B_{1\bar{3}}\, Z_4 - \rho\, \bar{C}_{2\bar{3}}\, \bar{Z}_4,
\end{aligned}$$

$$[Z_2,\bar Z_2] = -D\,Z_3 - C_{2\bar 2}\,Z_4 + \bar D\,\bar Z_3 + \bar C_{2\bar 2}\,\bar Z_4,$$
$$[Z_2,\bar Z_3] = -D\,C_{2\bar 3}\,Z_4 + \rho\,(\bar D\,\bar B_{1\bar 3} - \lambda\,\bar C_{2\bar 3})\,\bar Z_4.$$

From the bracket $[Z_1,\bar Z_1]$, one can clearly see that the algebras $\mathfrak{g}$ are at least 2-step. The space $\mathfrak{g}^2$ is now generated by the following brackets and their conjugates:

$$\begin{aligned}
\big[[Z_1,Z_2],Z_1\big] &= -\rho\,((\bar D-\lambda^2+\rho)\,B_{1\bar 3}+\lambda\,C_{2\bar 3})\,Z_4,\\
\big[[Z_1,Z_2],Z_2\big] &= -\rho\,((D+\rho)\,C_{2\bar 3}-\lambda\,D\,B_{1\bar 3})\,Z_4,\\
\big[[Z_1,Z_2],\bar Z_1\big] &= -\rho\,(C_{2\bar 3}\,Z_4+\bar D\,\bar B_{1\bar 3}\,\bar Z_4),\\
\big[[Z_1,Z_2],\bar Z_2\big] &= \rho\,((D\,B_{1\bar 3}-\lambda\,C_{2\bar 3})\,Z_4-\bar D\,\bar C_{2\bar 3}\,\bar Z_4),\\
\big[[Z_1,\bar Z_1],Z_1\big] &= ((\lambda^2-\rho+2\,i\,\mathfrak{Im}D)\,B_{1\bar 3}+\lambda\,C_{2\bar 3})\,Z_4+\rho\,\bar C_{2\bar 3}\,\bar Z_4,\\
\big[[Z_1,\bar Z_1],Z_2\big] &= -(\rho\,C_{2\bar 3}-\lambda\,D\,B_{1\bar 3})\,Z_4-\rho\,(\bar D\,\bar B_{1\bar 3}-\lambda\,\bar C_{2\bar 3})\,\bar Z_4,\\
\big[[Z_1,\bar Z_2],Z_1\big] &= -\lambda\,((\bar D-\lambda^2+\rho)\,B_{1\bar 3}+\lambda\,C_{2\bar 3})\,Z_4,\\
\big[[Z_1,\bar Z_2],Z_2\big] &= -\lambda\,((D+\rho)\,C_{2\bar 3}-\lambda\,D\,B_{1\bar 3})\,Z_4,\\
\big[[Z_1,\bar Z_2],\bar Z_1\big] &= -\rho\,\lambda\,C_{2\bar 3}\,Z_4-\lambda\,\bar D\,\bar B_{1\bar 3}\,\bar Z_4,\\
\big[[Z_1,\bar Z_2],\bar Z_2\big] &= -\rho\,\lambda\,(D\,B_{1\bar 3}-\lambda\,C_{2\bar 3})\,Z_4-\lambda\,\bar D\,\bar C_{2\bar 3}\,\bar Z_4,\\
\big[[Z_2,\bar Z_2],Z_1\big] &= D((\lambda^2-\rho)\,B_{1\bar 3}-\lambda\,C_{2\bar 3})\,Z_4+\rho\,\bar D\,\bar C_{2\bar 3}\,Z_4,\\
\big[[Z_2,\bar Z_2],Z_2\big] &= -D((2\,i\,\mathfrak{Im}D+\rho)\,C_{2\bar 3}-\lambda\,D\,B_{1\bar 3})\,Z_4-\rho\,\bar D\,(\bar D\,B_{1\bar 3}-\lambda\,\bar C_{2\bar 3})\,\bar Z_4.
\end{aligned}$$

Thus, we can conclude iii.*a)* and iii.*b)*. □

**Remark 3.3.8.** The 8-dimensional algebras arising as $\mathfrak{b}$-extensions of $\mathfrak{h}_3$ correspond to Family I/iii with $\rho=\lambda=0$ and $D=\pm 1$. Observe that they all have the same nilpotency step as this 6-dimensional Lie algebra. For the rest of the algebras underlying Family I, it is always possible to find a $\mathfrak{b}$-extension rising the nilpotency step.

### 3.3.3 Extensions of Family II

Remember that this family parametrizes nilpotent complex structures on $\mathfrak{h}_7, \mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$. The concrete values of the parameters $\rho \in \{0,1\}$, $B \in \mathbb{C}$ and $c \in \mathbb{R}^{c\geq 0}$ can be found in Table B (see Section 1.4.3) according to the underlying Lie algebra. In particular, we should remark that $(\rho, B, c) \neq (0,0,0)$.

**Proposition 3.3.9.** *Let $\mathfrak{h}$ be a 6-dimensional NLA endowed with a nilpotent complex structure $K$ parametrized by Family II. The quasi-nilpotent pair $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$ if its structure equations can be given by one of the following sets:*

*i) if* $(\mathfrak{h}, K)$ *has* $\rho = B = 1$ *and* $c = 0$*, then*

$$
(3.18) \qquad \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{1\bar{1}}, \\ d\omega^3 &= \omega^{12} + \omega^{1\bar{2}}, \\ d\omega^4 &= A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} \\ & \quad + C_{2\bar{1}}\,\omega^{2\bar{1}} + (B_{1\bar{3}} + D_{3\bar{1}})\,\omega^{2\bar{2}} + D_{3\bar{1}}\,\omega^{3\bar{1}} - A_{23}\,\omega^{3\bar{2}}, \end{cases}
$$

*where* $A_{13}, A_{23}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, D_{3\bar{1}} \in \mathbb{C}$.

*ii) if* $(\mathfrak{h}, K)$ *satisfies* $\rho \neq B$ *and* $c = 0$*, then*

$$
(3.19) \qquad \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{1\bar{1}}, \\ d\omega^3 &= \rho\,\omega^{12} + B\,\omega^{1\bar{2}}, \\ d\omega^4 &= (1-\rho)\,A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + \rho\,B_{1\bar{2}}\,\omega^{1\bar{2}} + (B-\rho)\,B_{1\bar{3}}\,\omega^{1\bar{3}} \\ & \quad + C_{2\bar{1}}\,\omega^{2\bar{1}} + (|B|^2 - \rho)\,B_{1\bar{3}}\,\omega^{2\bar{2}} + (\bar{B} - \rho)\,B_{1\bar{3}}\,\omega^{3\bar{1}}, \end{cases}
$$

*where* $A_{12}, A_{13}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}} \in \mathbb{C}$.

*iii) if* $(\mathfrak{h}, K)$ *admits* $c \neq 0$*, then*

$$
(3.20) \qquad \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{1\bar{1}}, \\ d\omega^3 &= \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \\ d\omega^4 &= A_{12}\,\omega^{12} + \big((\bar{B}-\rho)\,B_{1\bar{3}} - (B-\rho)\,D_{3\bar{1}}\big)\,\omega^{13} + B_{1\bar{2}}\,\omega^{1\bar{2}} \\ & \quad + c\,B_{1\bar{3}}\,\omega^{1\bar{3}} + c\,(\rho\,B_{1\bar{3}} + B\,D_{3\bar{1}})\,\omega^{2\bar{2}} + c\,D_{3\bar{1}}\,\omega^{3\bar{1}}, \end{cases}
$$

*where* $A_{12}, B_{1\bar{2}}, B_{1\bar{3}}, D_{3\bar{1}} \in \mathbb{C}$.

*Proof.* Let us consider a basis $\{\eta^i\}_{i=1}^4$ satisfying Lemma 3.3.1 *ii)* with coefficients $A'_{ij}$, $B'_{i\bar{j}}$, $C'_{i\bar{j}}$, and $D'_{i\bar{j}}$. Let us see how to reach our result depending on the solutions of the system (3.13).

- If $B = \rho$ and $c = 0$, first observe that $B = \rho = 1$ because $(\rho, B, c) \neq (0, 0, 0)$. Then, one has $C'_{2\bar{2}} = B'_{1\bar{3}} + D'_{3\bar{1}}$ and $D'_{3\bar{2}} = -A'_{23}$. In terms of a new basis

$$
\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = \eta^4 - B'_{1\bar{1}}\,\eta^2 - A'_{12}\,\eta^3,
$$

the structure equations become (3.18) with

$$A_{13} = A'_{13}, \quad A_{23} = A'_{23}, \quad B_{1\bar{2}} = B'_{1\bar{2}} - A'_{12}, \quad B_{1\bar{3}} = B'_{1\bar{3}}, \quad C_{2\bar{1}} = C'_{2\bar{1}}, \quad D_{3\bar{1}} = D'_{3\bar{1}}.$$

• If $B \neq \rho$ and $c = 0$, then $A'_{23} = 0$, $C'_{2\bar{2}} = \frac{|B|^2-\rho}{B-\rho} B'_{1\bar{3}}$, $D'_{3\bar{1}} = \frac{\bar{B}-\rho}{B-\rho} B'_{1\bar{3}}$ and $D'_{3\bar{2}} = 0$. If $\rho = 0$, one can define a new basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = B\,\eta^4 - BB'_{1\bar{1}}\,\eta^2 - B'_{1\bar{2}}\,\eta^3.$$

The structure equations are now given by (3.19) with

$$A_{12} = B\,A'_{12}, \quad A_{13} = B\,A'_{13}, \quad B_{1\bar{3}} = B'_{1\bar{3}}, \quad C_{2\bar{1}} = B\,C'_{2\bar{1}}.$$

If $\rho = 1$, consider

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = (B-1)(\eta^4 - B'_{1\bar{1}}\,\eta^2 - A'_{12}\,\eta^3).$$

The structure equations still follow (3.19) but now

$$A_{13} = (B-1)\,A'_{13}, \quad B_{1\bar{2}} = (B-1)(B'_{1\bar{2}} - B\,A'_{12}), \quad B_{1\bar{3}} = B'_{1\bar{3}}, \quad C_{2\bar{1}} = (B-1)\,C'_{2\bar{1}}.$$

• If $c \neq 0$, then $A'_{13} = \frac{\bar{B}-\rho}{c} B'_{1\bar{3}} - \frac{B-\rho}{c} D'_{3\bar{1}}$, $A'_{23} = 0$, $C'_{2\bar{2}} = \rho\,B'_{1\bar{3}} + B\,D'_{3\bar{1}}$ and $D'_{3\bar{2}} = 0$. In terms of a new basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = c\,\eta^4 - c\,B'_{1\bar{1}}\,\eta^2 - C'_{2\bar{1}}\,\eta^3,$$

the structure equations become (3.20) with

$$A_{12} = c\,A'_{12} - \rho\,C'_{2\bar{1}}, \quad B_{1\bar{2}} = c\,B'_{1\bar{2}} - B\,C'_{2\bar{1}}, \quad B_{1\bar{3}} = B'_{1\bar{3}}, \quad D_{3\bar{1}} = D'_{3\bar{1}}.$$

□

**Notation 3.3.10.** Following the previous section, we will refer to (3.18) as *Family II/i*, to (3.19) as *Family II/ii*, and to (3.20) as *Family II/iii.*

**Proposition 3.3.11.** *Let* $(\mathfrak{g}, J)$ *be some* $\mathfrak{b}$*-extension of Family II. Then,*

*i) if* $(\mathfrak{g}, J)$ *belongs to Family II/i, the nilpotent Lie algebra* $\mathfrak{g}$ *is*

*i.a)* 2*-step, if* $(A_{13}, A_{23}, B_{1\bar{2}}, B_{1\bar{3}}, C_{2\bar{1}}, D_{3\bar{1}}) = (0,0,0,0,0,0)$ *(product by a torus),*

*i.b)* 3*-step, in other case;*

*ii) if it belongs to Family II/ii, then* $\mathfrak{g}$ *is*

*ii.a)* 3*-step, if* $(A_{13}, B_{1\bar{3}}) = (0,0)$*,*

*ii.b)* 4*-step, in other case;*

*iii) if it belongs to Family II/iii, the Lie algebra is* *iii.a)* 3*-step, if* $D_{3\bar{1}} = 0$ *and either* $c = |B - \rho|$ *or* $c \neq |B - \rho|$ *with* $B_{1\bar{3}} = 0$*,*

*iii.b)* 4*-step, in other case.*

*Proof.* The nilpotency step can be calculated in terms of the Lie brackets obtained from (3.18), (3.19), and (3.20), by means of the descending central series (3.1). Let us denote $\{Z_i\}_{i=1}^4$ the dual basis of $\{\omega^i\}_{i=1}^4$.

For the extensions $(\mathfrak{g}, J)$ defined by (3.18), the space $\mathfrak{g}^1$ is generated by the subsequent brackets and their conjugates

$$\begin{aligned}
[Z_1, Z_2] &= -Z_3, \qquad [Z_1, Z_3] = -A_{13}\, Z_4, \qquad [Z_2, Z_3] = -A_{23}\, Z_4,\\
[Z_1, \bar{Z}_1] &= -Z_2 + \bar{Z}_2, \qquad [Z_1, \bar{Z}_2] = -Z_3 - B_{1\bar{2}}\, Z_4 + \bar{C}_{2\bar{1}}\, \bar{Z}_4,\\
[Z_1, \bar{Z}_3] &= -B_{1\bar{3}}\, Z_4 + \bar{D}_{3\bar{1}}\, \bar{Z}_4, \qquad [Z_2, \bar{Z}_2] = -(B_{1\bar{3}} + D_{3\bar{1}})\, Z_4 + (\bar{B}_{1\bar{3}} + \bar{D}_{3\bar{1}})\, \bar{Z}_4,\\
[Z_2, \bar{Z}_3] &= -\bar{A}_{23}\, \bar{Z}_4.
\end{aligned}$$

Due to the bracket $[Z_1, \bar{Z}_1]$ it is clear that the extended algebras will be at least 2-step nilpotent. Now the space $\mathfrak{g}^2$ is generated by these brackets and their conjugates:

$$\begin{aligned}
\big[[Z_1, Z_2], Z_1\big] &= \big[[Z_1, \bar{Z}_2], Z_1\big] = -A_{13}\, Z_4,\\
\big[[Z_1, Z_2], Z_2\big] &= \big[[Z_1, Z_2], \bar{Z}_2\big] = \big[[Z_1, \bar{Z}_2], Z_2\big] = \big[[Z_1, \bar{Z}_2], \bar{Z}_2\big] = -A_{23}\, Z_4,\\
\big[[Z_1, Z_2], \bar{Z}_1\big] &= \big[[Z_1, \bar{Z}_2], \bar{Z}_1\big] = D_{3\bar{1}}\, Z_4 - \bar{B}_{1\bar{3}}\, \bar{Z}_4,\\
\big[[Z_1, \bar{Z}_1], Z_1\big] &= B_{1\bar{2}}\, Z_4 - \bar{C}_{2\bar{1}}\, \bar{Z}_4,\\
\big[[Z_1, \bar{Z}_1], Z_2\big] &= (B_{1\bar{3}} + D_{3\bar{1}})\, Z_4 - (\bar{B}_{1\bar{3}} + \bar{D}_{3\bar{1}})\, \bar{Z}_4.
\end{aligned}$$

From here we get i.*a)* and i.*b)*.

We now consider the case of those structures defined by equations (3.19). Observe that the space $\mathfrak{g}^1$ is generated by the following brackets and their conjugates

$$\begin{aligned}
[Z_1, Z_2] &= -\rho\, Z_3 - (1 - \rho)\, A_{12}\, Z_4, \qquad [Z_1, Z_3] = -A_{13}\, Z_4,\\
[Z_1, \bar{Z}_1] &= -Z_2 + \bar{Z}_2, \qquad [Z_1, \bar{Z}_2] = -B\, Z_3 - \rho\, B_{1\bar{2}}\, Z_4 + \bar{C}_{2\bar{1}} \bar{Z}_4,\\
[Z_1, \bar{Z}_3] &= (\rho - B)\, (B_{1\bar{3}}\, Z_4 - \bar{B}_{1\bar{3}}\, \bar{Z}_4), \qquad [Z_2, \bar{Z}_2] = (\rho - |B|^2)\, (B_{1\bar{3}}\, Z_4 - \bar{B}_{1\bar{3}}\, \bar{Z}_4).
\end{aligned}$$

Once again, the bracket $[Z_1, \bar{Z}_1]$ allows to conclude that all our underlying Lie algebras are at least 2-step. We now compute $\mathfrak{g}^2$, which will be generated by the brackets below and their conjugates

$$\begin{aligned}
\big[[Z_1, Z_2], Z_1\big] &= -\rho\, A_{13}\, Z_4,\\
\big[[Z_1, Z_2], \bar{Z}_1\big] &= \rho\, (\bar{B} - \rho)\, (B_{1\bar{3}}\, Z_4 - \bar{B}_{1\bar{3}}\, \bar{Z}_4),\\
\big[[Z_1, \bar{Z}_1], Z_1\big] &= (B - \rho)\, Z_3 + (\rho\, B_{1\bar{2}} - (1 - \rho)\, A_{12})\, Z_4 - \bar{C}_{2\bar{1}}\, \bar{Z}_4,
\end{aligned}$$

$$\begin{aligned}
\big[[Z_1,\bar Z_1],Z_2\big] &= (|B|^2-\rho)\,(B_{1\bar 3}\,Z_4-\bar B_{1\bar 3}\,\bar Z_4),\\
\big[[Z_1,\bar Z_2],Z_1\big] &= -B\,A_{13}\,Z_4,\\
\big[[Z_1,\bar Z_2],\bar Z_1\big] &= B\,(\bar B-\rho)\,(B_{1\bar 3}\,Z_4-\bar B_{1\bar 3}\,\bar Z_4).
\end{aligned}$$

As a consequence of the 6-dimensional classification one has $B \neq \rho$, so the bracket $\big[[Z_1,\bar Z_1],Z_1\big]$ ensures that the nilpotency step of our extended algebras will be at least 3. Next observe that $\mathfrak{g}^3$ is generated, up to conjugation, by the brackets:

$$\begin{aligned}
\Big[\big[[Z_1,\bar Z_1],Z_1\big],Z_1\Big] &= (B-\rho)\,A_{13}\,Z_4,\\
\Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_1\Big] &= -|B-\rho|^2\,(B_{1\bar 3}\,Z_4-\bar B_{1\bar 3}\,\bar Z_4).
\end{aligned}$$

We can now conclude ii.*a)* and ii.*b)*.

Finally, equations (3.20) are considered. The space $\mathfrak{g}^1$ is generated by the following brackets and their conjugates

$$\begin{aligned}
[Z_1,Z_2] &= -\rho\,Z_3-A_{12}\,Z_4, & [Z_1,Z_3] &= -\big((\bar B-\rho)\,B_{1\bar 3}-(B-\rho)\,D_{3\bar 1}\big)\,Z_4,\\
[Z_1,\bar Z_1] &= -Z_2+\bar Z_2, & [Z_1,\bar Z_2] &= -B\,Z_3-\,B_{1\bar 2}\,Z_4+c\,\bar Z_3,\\
[Z_1,\bar Z_3] &= -c\,B_{1\bar 3}\,Z_4+c\,\bar D_{3\bar 1}\,\bar Z_4, & &\\
[Z_2,\bar Z_2] &= -c\,(\rho\,B_{1\bar 3}+B\,D_{3\bar 1})\,Z_4+c\,(\rho\,\bar B_{1\bar 3}+\bar B\,\bar D_{3\bar 1})\,\bar Z_4. & &
\end{aligned}$$

Note that the nilpotency step of our underlying algebras must be equal or greater than 2. The space $\mathfrak{g}^2$ is generated by

$$\begin{aligned}
\big[[Z_1,Z_2],Z_1\big] &= -\rho\,\big((\bar B-\rho)\,B_{1\bar 3}-(B-\rho)\,D_{3\bar 1}\big)\,Z_4,\\
\big[[Z_1,Z_2],\bar Z_1\big] &= \rho\,c\,(D_{3\bar 1}\,Z_4-\bar B_{1\bar 3}\,\bar Z_4),\\
\big[[Z_1,\bar Z_1],Z_1\big] &= (B-\rho)\,Z_3+(B_{1\bar 2}-A_{12})\,Z_4-c\,\bar Z_3,\\
\big[[Z_1,\bar Z_1],Z_2\big] &= c\,(\rho\,B_{1\bar 3}+B\,D_{3\bar 1})\,Z_4-c\,(\rho\,\bar B_{1\bar 3}+\bar B\,\bar D_{3\bar 1})\,\bar Z_4,\\
\big[[Z_1,\bar Z_2],Z_1\big] &= \big((c^2-B\,(\bar B-\rho))B_{1\bar 3}+B\,(B-\rho)\,D_{3\bar 1}\big)Z_4-c^2\,\bar D_{3\bar 1}\,\bar Z_4,\\
\big[[Z_1,\bar Z_2],\bar Z_1\big] &= c\,\big(B\,D_{3\bar 1}\,Z_4-(\rho\,\bar B_{1\bar 3}+(\bar B-\rho)\,\bar D_{3\bar 1})\,\bar Z_4\big),
\end{aligned}$$

and their conjugates. Considering that $c \neq 0$, from the bracket $\big[[Z_1,\bar Z_1],Z_1\big]$ we can ensure that the underlying algebras are at least 3-step. The generators of $\mathfrak{g}^3$ are

$$\begin{aligned}
\Big[\big[[Z_1,\bar Z_1],Z_1\big],Z_1\Big] &= \big((|B-\rho|^2-c^2)\,B_{1\bar 3}-(B-\rho)^2\,D_{3\bar 1}\big)\,Z_4+c^2\,\bar D_{3\bar 1}\,\bar Z_4,\\
\Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_1\Big] &= -c\,(B-\rho)\,D_{3\bar 1}\,Z_4+c\,(\bar B-\rho)\,\bar D_{3\bar 1}\,\bar Z_4,
\end{aligned}$$

together with their conjugates. From these expressions we obtain iii.*a)* and iii.*b)*. □

**Remark 3.3.12.** One can always find an extension of the underlying algebras of Family II rising their nilpotency step.

### 3.3.4 Extensions of Family III

Now, our starting point is the strongly non-nilpotent family in 6 dimensions. Recall it is given by equations (1.16) and the parameter $\varepsilon = 0$, for $\mathfrak{h}_{19}^-$, and $\varepsilon = 1$, for $\mathfrak{h}_{26}^+$.

**Proposition 3.3.13.** *Let $\mathfrak{h}$ be a 6-dimensional NLA endowed with a non-nilpotent complex structure $K$. The quasi-nilpotent pair $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$ if its structure equations are given by:*

$$(3.21)\qquad \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 &= i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \\ d\omega^4 &= A\,\omega^{12} + B\,\omega^{1\bar{1}} + \nu\,(\omega^{23} \pm 2\,\varepsilon\,\omega^{1\bar{3}} + \omega^{2\bar{3}}), \end{cases}$$

*where $\nu \in \{0, 1\}$, $A, B \in \mathbb{C}$.*

*Proof.* Let us consider a basis $\{\eta^i\}_{i=1}^4$ satisfying Lemma 3.3.1 *iv)* with coefficients $A'_{ij}$, $B'_{i\bar{j}}$, $C'_{i\bar{j}}$, and $D'_{i\bar{j}}$. Let us discuss how to attain our result depending on the values of some of the previous structure constants.

If $A'_{23} = 0$, a new $(1,0)$-basis

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = \eta^4 - A'_{13}\,\eta^2 \pm i\,B'_{1\bar{2}}\,\eta^3,$$

leads to (3.21) with: $\nu = 0, \quad A = A'_{12}, \quad B = B'_{1\bar{1}} \mp \varepsilon B'_{1\bar{2}}$.

Otherwise, we can define

$$\omega^1 = \eta^1, \quad \omega^2 = \eta^2, \quad \omega^3 = \eta^3, \quad \omega^4 = \frac{1}{A'_{23}}\,(\eta^4 - A'_{13}\,\eta^2 \pm i\,B'_{1\bar{2}}\,\eta^3)$$

and obtain (3.21) with: $\nu = 1, \quad A = \frac{A'_{12}}{A'_{23}}, \quad B = \frac{1}{A'_{23}}\,(B'_{1\bar{1}} \mp \varepsilon B'_{1\bar{2}})$. □

**Remark 3.3.14.** The complex structures obtained in Proposition 3.3.13 are weakly non-nilpotent. This contrasts with the 6-dimensional case, where all the non-nilpotent structures are strongly non-nilpotent.

**Proposition 3.3.15.** *The Lie algebra $\mathfrak{g}$ associated to a $\mathfrak{b}$-extension* (3.21) *of the* 6-*dimensional strongly non-nilpotent family is:*

*a)* 3-*step, if $\varepsilon = 0$,*

*b)* 4-*step, if $\varepsilon = 1$ and $\nu = 0$,*

*c)* 5-*step, in other case.*

*Proof.* Let us now compute the nilpotency step of the extended algebras $\mathfrak{g}$. Denote $\{Z_k\}_{k=1}^4$ the dual basis of $\{\omega^k\}_{k=1}^4$. Then, due to the equations (3.21) the space $\mathfrak{g}^1$ is generated by the following brackets and their conjugates:

$$[Z_1,Z_2] = -A\,Z_4, \qquad [Z_1,Z_3] = -Z_2, \qquad [Z_2,Z_3] = -\nu\,Z_4,$$
$$[Z_1,\bar Z_1] = -i\,\varepsilon\,Z_3 - B\,Z_4 - i\,\varepsilon\,\bar Z_3 + \bar B\,\bar Z_4, \qquad [Z_1,\bar Z_2] = \mp i\,(Z_3-\bar Z_3),$$
$$[Z_1,\bar Z_3] = -Z_2 \mp 2\,\varepsilon\,\nu\,Z_4, \qquad [Z_2,\bar Z_3] = -\nu\,Z_4.$$

Note $\mathfrak{g}^1 \neq \{0\}$. If we now compute $\mathfrak{g}^2$, we see that this space is generated by the following brackets and their conjugates

$$\big[[Z_1,Z_3],Z_1\big] = \big[[Z_1,\bar Z_3],Z_1\big] = -A\,Z_4,$$
$$\big[[Z_1,Z_3],Z_3\big] = \big[[Z_1,Z_3],\bar Z_3\big] = \big[[Z_1,\bar Z_3],Z_3\big] = \big[[Z_1,\bar Z_3],\bar Z_3\big] = \nu\,Z_4,$$
$$\big[[Z_1,Z_3],\bar Z_1\big] = \big[[Z_1,\bar Z_3],\bar Z_1\big] = \mp i\,(Z_3-\bar Z_3),$$
$$\big[[Z_1,\bar Z_1],Z_1\big] = -2\,i\,\varepsilon\,(Z_2 \pm \nu\,Z_4),$$
$$\big[[Z_1,\bar Z_1],Z_2\big] = -\big[[Z_1,\bar Z_2],Z_1\big] = -2\,i\,\varepsilon\,\nu\,Z_4,$$
$$\big[[Z_1,\bar Z_2],\bar Z_1\big] = -2\,i\,\varepsilon\,\nu\,\bar Z_4.$$

In particular, $\mathfrak{g}^2 \neq \{0\}$. Next observe that $\mathfrak{g}^3$ is generated, up to conjugation, by

$$-\Big[\big[[Z_1,Z_3],\bar Z_1\big],Z_1\Big] = \Big[\big[[Z_1,\bar Z_1],Z_1\big],Z_3\Big] = \Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_3\Big] = 2\,i\,\varepsilon\,\nu\,Z_4,$$
$$\Big[\big[[Z_1,Z_3],\bar Z_1\big],\bar Z_1\Big] = 2\,i\,\varepsilon\,\nu\,\bar Z_4, \qquad \Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_1\Big] = \pm 2\,\varepsilon\,(Z_3-\bar Z_3),$$
$$\Big[\big[[Z_1,\bar Z_1],Z_1\big],Z_1\Big] = -2\,i\,\varepsilon\,A\,Z_4.$$

From here, it is easy to conclude *a)*. Therefore, one can focus on the case $\varepsilon = 1$. For this value, the space $\mathfrak{g}^4$ is given by

$$\bigg[\Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_1\Big],Z_1\bigg] = -4\,\nu\,Z_4, \qquad \bigg[\Big[\big[[Z_1,\bar Z_1],Z_1\big],\bar Z_1\Big],\bar Z_1\bigg] = 4\,\nu\,\bar Z_4,$$

so we finally obtain *b)* and *c)*. □

**Remark 3.3.16.** Any extended algebra of $\mathfrak{h}_{19}^-$ has the same nilpotency step as $\mathfrak{h}_{19}^-$. However, for $\mathfrak{h}_{26}^+$ it is always possible to find an extension rising its nilpotency step.

Chapter 4

# Strongly non-nilpotent complex structures

Let $(M, J)$ be a $2n$-dimensional nilmanifold endowed with an invariant complex structure. As seen in the previous chapter, quasi-nilpotent pairs can be found as $\mathfrak{b}$-extensions of $2(n-1)$-dimensional NLAs admitting complex structures. However, those complex nilmanifolds whose underlying $(\mathfrak{g}, J)$ satisfies $\mathfrak{a}_1(J) = \{0\}$ are missed in this approach, and they should be explicitly constructed from zero. Therefore, strongly non-nilpotent pairs (or SnN for short) might be seen as a completely new family which arises in each dimension leap.

The situation in the lower dimensional cases comes in the following way (for clarity, bear in mind Figure 3.1). When one moves from $n = 1$ to $n = 2$, $\mathfrak{b}$-extensions succeed in parametrizing every pair $(\mathfrak{g}, J)$, due to the non-existence of SnN structures (see Proposition 3.2.2). In fact, note that every complex structure is nilpotent in this dimension. Nevertheless, this is no longer true when we consider $n = 3$. On the one hand, nilpotent structures still exist. They can be generated as $\mathfrak{b}$-extensions and completely classified, up to equivalence, by the complex-parallelizable family, Family I, and Family II (see Section 1.4.3). On the other hand, we have Family III, which cannot be found by the same procedure because it exactly parametrizes those $J$'s satisfying $\mathfrak{a}_1(J) = \{0\}$. Hence, $n = 3$ is the first dimension where strongly non-nilpotent structures appear. In fact, it turns out that every non-nilpotent structure belongs to this special class for $n = 3$. This entails a difference with $n = 4$, where weakly non-nilpotent structures have been found in Proposition 3.3.13 (see Remark 3.3.14). It is worth noting that they precisely arised when the 6-dimensional SnN family was extended.

Apart from some concrete examples in [CFGU00], little is yet known about strongly non-nilpotent structures for $n \geq 4$. Nevertheless, they turn to be the remaining piece to completely understand the complex geometry on any nilpotent Lie algebra $\mathfrak{g}$. In this chapter, we will try to cast some light on the topic. In particular, we parametrize every SnN complex structure on NLAs of dimension eight.

The first section contains some technical lemmas that allow to study the ascending central series $\{\mathfrak{g}_k\}_k$ of any $2n$-dimensional NLA $\mathfrak{g}$ admitting an SnN complex structure $J$. This part concludes with a bound for the dimension of the center $\mathfrak{g}_1$. In the next sections, we focus on $n = 4$ and give a description of all the terms in the series $\{\mathfrak{g}_k\}_k$, using a *doubly* *adapted* basis. By the result by Vergnolle and Remm [VR09] asserting that quasi-filiform Lie algebras do not admit complex structures, we have that $\mathfrak{g}$ is at most 5-step nilpotent. Moreover, since $\dim \mathfrak{g}_1 = 1$ for $n = 4$, we only need to describe $\mathfrak{g}_2$ (see Section 4.2), $\mathfrak{g}_3$, and $\mathfrak{g}_4$ (see Section 4.3). At the same time, we get an explicit description of the Lie brackets of $\mathfrak{g}$. As a final result, in Section 4.4 we obtain the corresponding structure equations for each SnN pair $(\mathfrak{g}, J)$.

## 4.1 Restrictions on the ascending central series

We introduce here a collection of technical lemmas which will allow to simplify the construction procedure accomplished in Section 4.2. In particular, we show that the arrangement of the ascending central series of an NLA $\mathfrak{g}$ determines the way in which a complex structure $J$ on $\mathfrak{g}$ can be defined. In the case of $J$ being strongly non-nilpotent, the dimension of the center of $\mathfrak{g}$ is bounded.

Let us start with some results where $J$ can be a complex structure of any type.

**Lemma 4.1.1.** *Let $\mathfrak{g}$ be a $2n$-dimensional nilpotent Lie algebra endowed with a complex structure $J$. Suppose that $l > 1$ is an integer for which there exists a subspace $V \subset \mathfrak{g}_l$ such that $\dim V = n$ and $V \cap JV = \{0\}$. It holds:*

*i) If $\mathfrak{g}_{l-1} = J\mathfrak{g}_{l-1}$, then $\mathfrak{g}_l = \mathfrak{g}$.*

*ii) If there is $X \in V$ such that $JX \in \mathfrak{g}_{l+1} \smallsetminus \mathfrak{g}_l$, then it exists $Y \in \mathfrak{g}_{l-1}$ such that $JY \in \mathfrak{g}_l \smallsetminus \mathfrak{g}_{l-1}$.*

*Proof.* Let $\{X_i\}_{i=1}^n$ be a basis for $V$, where $X_i \neq JX_j$ for every $1 \leq i, j \leq n$. First observe that $\{X_i, JX_i\}_{i=1}^n$ gives a basis of $\mathfrak{g}$, and for every $i = 1, \ldots, n$ one has

$$[X_i, X_j],\ [X_i, JX_j] \in \mathfrak{g}_{l-1},\ \forall j = 1, \ldots, n.$$

Therefore, the brackets $[JX_i, JX_j]$ should be the ones to study.

The Nijenhuis condition together with the hypothesis in *i)* yields to

$$[JX_i, JX_j] = [X_i, X_j] + J[JX_i, X_j] + J[X_i, JX_j] \ \in \ \mathfrak{g}_{l-1} + J\mathfrak{g}_{l-1} = \mathfrak{g}_{l-1},$$

for all $i, j = 1, \ldots, n$. Thus, we can easily conclude that $JX_1, \ldots, JX_n \in \mathfrak{g}_l$ and $\mathfrak{g}_l = \mathfrak{g}$.

For the second part, let us suppose that there exists $X \in V \subset \mathfrak{g}_l$ such that $JX \in \mathfrak{g}_{l+1}$ but $JX \notin \mathfrak{g}_l$. Then, it is possible to find $Z \in V$ such that $0 \neq [JX, JZ] \in \mathfrak{g}_l \smallsetminus \mathfrak{g}_{l-1}$. Due to the Nijenhuis condition one has

$$[JX, JZ] - [X, Z] = J\left([JX, Z] + [X, JZ]\right).$$

Observe that the left-hand side belongs to $\mathfrak{g}_l$, but it is not completely contained in $\mathfrak{g}_{l-1}$. Hence, the element $0 \neq Y = [JX, Z] + [X, JZ] \in \mathfrak{g}_{l-1}$ and $JY \in \mathfrak{g}_l \smallsetminus \mathfrak{g}_{l-1}$. $\square$

**Lemma 4.1.2.** *Let* $(\mathfrak{g}, J)$ *be an NLA of dimension* $2n$ *endowed with a complex structure. Suppose that there exist a* 2*-dimensional* $J$*-invariant subspace* $W \subset \mathfrak{g}$ *and an element* $X \in \mathfrak{g}_1$ *such that* $\mathfrak{g} = \mathfrak{g}_k \oplus W \oplus \langle JX \rangle$*, for some* $k \geq 1$*. One has:*

*i) if* $\mathfrak{g}_{k+1} \cap W \neq \{0\}$*, then* $\mathfrak{g}_{k+1} = \mathfrak{g}$*; and*

*ii) if* $\mathfrak{g}_{k+1} \cap W = \{0\}$*, then* $\mathfrak{g}_{k+1} = \mathfrak{g}_k \oplus \langle JX \rangle$ *and* $\mathfrak{g}_{k+2} = \mathfrak{g}$*.*

*Proof.* Let $\{X_i, JX_i\}_{i=1}^n$ be a $J$-adapted basis of $\mathfrak{g}$. We can suppose, without loss of generality, that $X_1 = X$ and $W = \langle X_n, JX_n \rangle$. In this way, it is clear that

$$\mathfrak{g}_k = \langle X_1, \ldots, X_{n-1}, JX_2, \ldots, JX_{n-1} \rangle.$$

We first prove part *i)*. If $\mathfrak{g}_{k+1} \cap W \neq \{0\}$, then we can assume that $X_n \in \mathfrak{g}_{k+1}$, interchanging the roles of $X_n$ and $JX_n$ if necessary. We would like to see that also $JX_1, JX_n \in \mathfrak{g}_{k+1}$. In particular, it suffices to check that $[JX_1, JX_n] \in \mathfrak{g}_k$. We observe that one has

$$[X_n, JX_1] = a_1\, X_1 + \cdots + a_{n-1}\, X_{n-1} + b_2\, JX_2 + \cdots + b_{n-1}\, JX_{n-1},$$

where $a_i, b_i \in \mathbb{R}$. Applying the Nijenhuis condition (3.3), we obtain

$$[JX_n, JX_1] = J[X_n, JX_1] = -b_2\, X_2 - \cdots - b_{n-1}\, X_{n-1} + a_1\, JX_1 + \cdots + a_{n-1}\, JX_{n-1}.$$

Due to the nilpotency of $\mathfrak{g}$ one necessarily has $a_1 = 0$, and thus we get the result.

For part *ii)*, we note that the condition $\mathfrak{g}_{k+1} \cap W = \{0\}$ implies $\mathfrak{g}_{k+1} = \mathfrak{g}_k \oplus \langle JX \rangle$. Since $\mathfrak{g}$ is nilpotent, this yields $\mathfrak{g}_{k+2} = \mathfrak{g}$. □

If we now impose further conditions on $J$, some other restrictions arise.

**Lemma 4.1.3.** *Let* $\mathfrak{g}$ *be an NLA endowed with a complex structure* $J$*. Consider its ascending central series, and assume that* $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$*, for some integer* $k \geq 1$ *(in particular,* $J$ *is SnN). Then, one has:*

*i)* $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$*.*

*ii) If* $r > 1$ *is the smallest integer such that* $\mathfrak{g}_{k+r} \cap J\mathfrak{g}_k \neq \{0\}$*, then*

$$\mathfrak{g}_{k+r-1} \cap J\mathfrak{g}_{k+r-1} \neq \{0\}.$$

*Proof.* Let $r \geq 1$ and suppose that for every non-zero $Z \in \mathfrak{g}_{k+r-1}$ one has $JZ \notin \mathfrak{g}_{k+r-1}$ (notice that it holds trivially for $r = 1$, by hypothesis). We assume that there exists $X \in \mathfrak{g}_k$ such that $JX \in \mathfrak{g}_{k+r}$ but $JX \notin \mathfrak{g}_{k+r-1}$. Observe that we are denying the claim given in *i)* for $r = 1$ and assuming the hypothesis in *ii)* for $r > 1$. Then, it is possible to find $Y \in \mathfrak{g}$ satisfying $0 \neq [JX, Y] \in \mathfrak{g}_{k+r-1}$ and $[JX, Y] \notin \mathfrak{g}_{k+r-2}$. Let us denote by $T$ the non-zero element given by $T = [JX, Y] + [X, JY] \in \mathfrak{g}_{k+r-1}$. By the Nijenhuis condition, we have

$$JT = J\left([JX, Y] + [X, JY]\right) = [JX, JY] - [X, Y].$$

Therefore, both the element $T$ and its image by $J$ belong to $\mathfrak{g}_{k+r-1}$. This is a contradiction for $r = 1$ and proves the statement for $r > 1$. □

**Corollary 4.1.4.** *Let $\mathfrak{g}$ be an NLA endowed with a complex structure $J$. If one has $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$, for some integer $k \geq 1$, then the nilpotency step of $\mathfrak{g}$ satisfies $s \geq k+2$. Moreover, if $s = k+2$ then $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} \neq \{0\}$.*

*Proof.* By Lemma 4.1.3 *i)*, it is clear that $\mathfrak{g}_{k+1} \neq \mathfrak{g}$. Hence, $s \geq k+2$. Furthermore, if $s = k+2$, then one has $\mathfrak{g}_{k+2} = \mathfrak{g}$ and $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$ by Lemma 4.1.3 *i)*. We are precisely in the conditions of Lemma 4.1.3 *ii)* for $r = 2$, and the result is clear. □

**Corollary 4.1.5.** *Let $(\mathfrak{g}, J)$ be a $2n$-dimensional NLA endowed with a complex structure. If $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$, for some integer $k \geq 1$, then $\mathfrak{g}_{k+1}$ cannot contain a subspace $V$ such that $\dim V = n$ and $V \cap JV = \{0\}$.*

*Proof.* Let us assume the opposite, i.e., suppose that there exists $V \subset \mathfrak{g}_{k+1}$ such that $\dim V = n$ and $V \cap JV = \{0\}$. By Corollary 4.1.4 it is clear that $\mathfrak{g}_{k+1} \neq \mathfrak{g}$. Necessarily, there is $Z \in \mathfrak{g}_{k+2}$ such that $Z \notin \mathfrak{g}_{k+1}$. If $\dim \mathfrak{g}_{k+1} = n+p$, we can choose a basis $\{X_1, \ldots, X_n\}$ for $V$ in such a way that $\mathfrak{g}_{k+1} = \langle X_1, \ldots, X_n, JX_1, \ldots, JX_p \rangle$. As $\{X_i, JX_i\}_{i=1}^n$ is a basis for $\mathfrak{g}$, we can write

$$Z = \sum_{i=1}^{n} \lambda_i X_i + \sum_{i=1}^{p} \mu_i JX_i + \sum_{j=p+1}^{n} \mu_j JX_j,$$

where $\lambda_i, \mu_i \in \mathbb{R}$, for every $i = 1, \ldots, n$. Since $\sum_{i=1}^n \lambda_i X_i + \sum_{i=1}^p \mu_i JX_i \in \mathfrak{g}_{k+1}$ but $Z \notin \mathfrak{g}_{k+1}$, we can ensure that $0 \neq \sum_{j=p+1}^n \mu_j JX_j = J\left(\sum_{j=p+1}^n \mu_j X_j\right)$. Therefore, there is $0 \neq X = \sum_{j=p+1}^n \mu_j X_j \in V$ such that $JX \in \mathfrak{g}_{k+2} \smallsetminus \mathfrak{g}_{k+1}$. Applying Lemma 4.1.1 *ii)* for $l = k+1$, it would then be possible to find $Y \in \mathfrak{g}_k$ such that $JY \in \mathfrak{g}_{k+1} \smallsetminus \mathfrak{g}_k$. However, this contradicts Lemma 4.1.3 *i)*. □

**Corollary 4.1.6.** *Let $J$ be a complex structure on a $2n$-dimensional NLA $\mathfrak{g}$, with $n \geq 3$. If $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$, for some integer $k \geq 1$, then $k \leq \dim \mathfrak{g}_k \leq n-2$.*

*Proof.* The lower bound is clear, because $\mathfrak{g}$ is a nilpotent Lie algebra and its ascending central series increases strictly. For the upper bound, first note that the hypothesis $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ leads to $\dim \mathfrak{g}_k \leq n$. Thus, we just need to discard the cases $\dim \mathfrak{g}_k = n$ and $\dim \mathfrak{g}_k = n-1$.

Let us first suppose that $\dim \mathfrak{g}_k = n$, and set $\mathfrak{g}_k = \langle X_1, \ldots, X_n \rangle$. We apply the same idea as in the proof of the previous corollary, with $V = \mathfrak{g}_k$. Since the Lie algebra is nilpotent, the ascending central series should reach $\mathfrak{g}$, and we can find some element in $\mathfrak{g}_{k+1}$ which is not in $\mathfrak{g}_k$. Indeed, the dimension of $\mathfrak{g}$ implies that this new element can be chosen to be $JX$, for some $X \in V$. This contradicts Lemma 4.1.3 *i)*, so we have $\dim \mathfrak{g}_k \leq n-1$.

Let us now consider $\dim \mathfrak{g}_k = n-1$, and take $\mathfrak{g}_k = \langle X_1, \ldots, X_{n-1} \rangle$. The nilpotency of $\mathfrak{g}$ guarantees the existence of some $Y \in \mathfrak{g}_{k+1} \smallsetminus \mathfrak{g}_k$. Furthermore, as a consequence of Lemma 4.1.3 *i)* we have that $Y \neq JX$, for every $X \in \mathfrak{g}_k$. Hence, we can set $X_n = Y$ and find $V = \langle X_1, \ldots, X_n \rangle \subseteq \mathfrak{g}_{k+1}$ such that $V \cap JV = \{0\}$. This contradicts Corollary 4.1.5. Therefore, we can conclude $1 \leq \dim \mathfrak{g}_k \leq n-2$. □

**Lemma 4.1.7.** *Let* $\mathfrak{g}$ *be a* $2n$*-dimensional NLA with* $n \geq 3$ *endowed with a complex structure* $J$*. If* $\mathfrak{g}_k \cap J\mathfrak{g}_k = \{0\}$ *and* $\dim \mathfrak{g}_{k+1} = \dim \mathfrak{g}_k + 1$*, for some integer* $k \geq 1$*, then* $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1} = \{0\}$*.*

*Proof.* Let $\mathfrak{g}_k = \langle X_1, \ldots, X_l \rangle$, with $1 \leq l \leq n-2$ (see Corollary 4.1.6). By part *i)* of Lemma 4.1.3, we have that $\mathfrak{g}_{k+1} \cap J\mathfrak{g}_k = \{0\}$, so there is an element $X_{l+1}$ linearly independent with $X_1, \ldots, X_l, JX_1, \ldots, JX_l$ such that $\mathfrak{g}_{k+1} = \langle X_1, \ldots, X_{l+1} \rangle$. Let us take an element $Z \in \mathfrak{g}_{k+1} \cap J\mathfrak{g}_{k+1}$. Observe that $Z$ can be written as

$$Z = a_1 X_1 + \ldots + a_{l+1} X_{l+1} = J\big(b_1 X_1 + \ldots + b_{l+1} X_{l+1}\big),$$

where $a_i, b_i \in \mathbb{R}$, for $i = 1, \ldots, l+1$. In particular, the previous expression gives

$$a_1 X_1 + \ldots + a_{l+1} X_{l+1} - b_1 JX_1 - \ldots - b_{l+1} JX_{l+1} = 0,$$

which is a linear combination of elements which are linearly independent. Therefore, $a_1 = \ldots = a_{l+1} = b_1 = \ldots = b_{l+1} = 0$, and we can conclude $Z = 0$. □

**Corollary 4.1.8.** *Let* $\mathfrak{g}$ *be a* $2n$*-dimensional NLA. If* $\dim \mathfrak{g}_{n-1} = n-1$*, then* $\mathfrak{g}$ *does not admit a complex structure.*

*Proof.* Since the Lie algebra is nilpotent, we have

$$1 \leq \dim \mathfrak{g}_1 \lneq \dim \mathfrak{g}_2 \lneq \ldots \lneq \dim \mathfrak{g}_{n-1} = n-1.$$

Therefore, $\dim \mathfrak{g}_k = k$ for each $1 \leq k \leq n-1$. In particular, $\mathfrak{g}_1 \cap J\mathfrak{g}_1 = \{0\}$ and $\mathfrak{g}$ can only admit SnN complex structures.

For $n = 2$, the result comes directly from Proposition 3.2.2. For $n \geq 3$, let us set $\mathfrak{g}_1 = \langle X_1 \rangle$. By Lemma 4.1.3 *i)* we have that $JX_1 \notin \mathfrak{g}_2$, so there is an element $X_2$ linearly independent with $X_1, JX_1$ such that $\mathfrak{g}_2 = \langle X_1, X_2 \rangle$. Moreover, we have that $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$ by Lemma 4.1.7. Repeating the process, one has $\mathfrak{g}_{n-1} = \langle X_1, \ldots, X_{n-1} \rangle$ and $\mathfrak{g}_{n-1} \cap J\mathfrak{g}_{n-1} = \{0\}$. This is contradicts Corollary 4.1.6. □

As a consequence, we recover the following result proven by Goze and Remm in [GR02]. Let us simply note that we make use of the ascending central series $\{\mathfrak{g}_k\}_{k\geq 1}$ instead of the descending central series $\{\mathfrak{g}^k\}_{k\geq 1}$ (see Section 3.1 for their definitions).

**Corollary 4.1.9.** *Filiform Lie algebras do not admit complex structures.*

*Proof.* Recall that an $m$-dimensional filiform Lie algebra is an $(m-1)$-step nilpotent Lie algebra. If we consider a $2n$-dimensional filiform Lie algebra $\mathfrak{g}$, then

$$(\dim \mathfrak{g}_k)_k = (1, 2, \ldots, n-1, \ldots, 2n-2, 2n).$$

In particular, $\dim \mathfrak{g}_{n-1} = n-1$ and applying Corollary 4.1.9 one gets the result. □

**Remark 4.1.10.** An $m$-dimensional quasi-filiform Lie algebra $\mathfrak{g}$ is an $(m-2)$-step nilpotent Lie algebra. Observe that one can partially recover the result about the non-existence of complex structures on quasi-filiform Lie algebras given by Vergnolle and Remm in [VR09]. For those quasi-filiform Lie algebras whose ascending central series satisfies $(\dim \mathfrak{g}_k)_{k=1}^{n-1} = (1, 2, \ldots, n-1)$, it suffices to apply Corollary 4.1.9. For those $\mathfrak{g}$ such that $\dim \mathfrak{g}_1 = 2$, the result comes by contradiction, simply distinguishing two possibilities. If we assume that $\mathfrak{g}$ admits a nilpotent complex structure, then the quotient $\mathfrak{h} = \mathfrak{g}/\mathfrak{g}_1 = \mathfrak{g}/\mathfrak{a}_1(J)$ must admit a complex structure. However, $\mathfrak{h}$ is filiform so this cannot happen. If $\mathfrak{g}$ admits a non-nilpotent complex structure, then it must be SnN. We can then combine Lemma 4.1.3 *i)* and Corollary 4.1.5 in order to reach a contradiction.

Let us observe that Corollary 4.1.6 improves the bound given in Proposition 3.1.12 for the dimension of $\mathfrak{g}_1$ when $(\mathfrak{g}, J)$ is a $2n$-dimensional NLA endowed with a strongly non-nilpotent complex structure, and one has $n \geq 3$. Moreover, when $n \geq 4$ we prove the following.

**Theorem 4.1.11.** *Let $\mathfrak{g}$ be a $2n$-dimensional nilpotent Lie algebra with $n \geq 4$ endowed with a strongly non-nilpotent complex structure $J$. Then, $1 \leq \dim \mathfrak{g}_1 \leq n-3$.*

*Proof.* For the seek of clarity, we include an outline of the proof in Appendix A.

In view of Corollary 4.1.6, we have $1 \leq \dim \mathfrak{g}_1 \leq n-2$. Therefore, it suffices to discard the case $\dim \mathfrak{g}_1 = n-2$. We will proceed by contradiction.

The idea is using the construction procedure described in Section 3.1.2, in a similar way to the proof of Lemma 3.2.5. At the initial point, we simply consider a vector space $\mathfrak{g}$ endowed with a complex structure $J$. We want to construct a doubly adapted basis $\mathcal{B}$ in the sense of Definition 3.1.14. If necessary, we will perform arrangements of generators along the process, as in the proof of Lemma 3.2.5 (see p. 81). Starting from an $(n-2)$-dimensional center and covering all the possible combinations, we will define the Lie brackets of $\mathfrak{g}$ in terms of the elements of $\mathcal{B}$, always attending to the nilpotency of $\mathfrak{g}$, the Jacobi identity, the Nijenhuis condition, and the strongly non-nilpotency of $J$. These four conditions are checked at every stage of the method, discarding the cases in which any of them fails.

Let us then suppose that

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle$$

and choose $X_1, \ldots, X_{n-2}$ as generators in $\mathcal{B}$. It is clear that for each $1 \leq k \leq n-2$,

$$[X_k, Y] = 0, \quad \forall Y \in \mathfrak{g}.$$

Furthermore, as a consequence of the Nijenhuis condition:

$$[JX_k, JX_l] = 0, \quad 1 \leq k < l \leq n-2,$$

even though we ignore where the elements $JX_k$, $k = 1, \ldots, n-2$, enter the ascending central series. We want to complete $\{X_k, JX_k\}_{k=1}^{n-2}$ up to a doubly adapted basis $\mathcal{B}$. In particular, we need to find two vectors, that will be called $X_{n-1}$ and $X_n$, that are generators of some $\mathfrak{g}_l$, for $l \geq 2$. In fact, note that the yet completely undetermined brackets for $\mathfrak{g}$ are exactly those involving at least one of the elements $X_{n-1}$, $X_n$, $JX_{n-1}$, $JX_n \in \mathcal{B}$.

We should now focus on $\mathfrak{g}_2$. The nilpotency of $\mathfrak{g}$ implies that $\dim \mathfrak{g}_2 > n-2$. Since $J$ is SnN, $\mathfrak{g}_1 \cap J\mathfrak{g}_1 = \{0\}$ and by Lemma 4.1.3 it is clear that $JX \notin \mathfrak{g}_2$, for every $X \in \mathfrak{g}_1$. Therefore, there is a vector $Y$ linearly independent with $X_i$, $JX_i$, $i = 1, \ldots, n-2$, such that $Y \in \mathfrak{g}_2 \smallsetminus \mathfrak{g}_1$. We can take $X_{n-1} = Y$ as a new element in $\mathcal{B}$, and

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \quad \mathfrak{g}_2 \supseteq \langle X_1, \ldots, X_{n-1} \rangle.$$

By Corollary 4.1.5, a new vector linearly independent with $X_i$, $JX_i$, $i = 1, \ldots, n-1$, cannot belong to $\mathfrak{g}_2$. Furthermore, Corollary 4.1.6 implies that also $JX_{n-1}$ should lie in $\mathfrak{g}_2$. Hence we can conclude

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \quad \mathfrak{g}_2 = \langle X_1, \ldots, X_{n-1}, JX_{n-1} \rangle.$$

Although we do not know where $X_n$ enters the ascending central series, we can set

$$\begin{aligned} [X_{n-1}, X_n] &= \textstyle\sum_{i=1}^{n-2} \alpha_i X_i, & [X_{n-1}, JX_k] &= \textstyle\sum_{i=1}^{n-2} \mu_i^k X_i, \ k = 1, \ldots, n-2, \\ [X_{n-1}, JX_n] &= \textstyle\sum_{i=1}^{n-2} \gamma_i X_i, & [X_n, JX_{n-1}] &= \textstyle\sum_{i=1}^{n-2} a_i X_i, \end{aligned}$$

where $\alpha_i$, $\mu_i^k$, $\gamma_i$, $a_i \in \mathbb{R}$, for $i, k = 1, \ldots, n-2$. If we now apply the Nijenhuis condition,

$$\begin{aligned} [JX_k, JX_{n-1}] &= -\textstyle\sum_{i=1}^{n-2} \mu_i^k JX_i, \ k = 1, \ldots, n-2, \\ [JX_{n-1}, JX_n] &= \textstyle\sum_{i=1}^{n-2} \alpha_i X_i + \sum_{i=1}^{n-2} (\gamma_i - a_i) JX_i. \end{aligned}$$

These two brackets belong to $\mathfrak{g}_1$ (because $JX_{n-1} \in \mathfrak{g}_2$), so necessarily $\mu_i^k = 0$ and $a_i = \gamma_i$, for all $i, k = 1, \ldots, n-2$. Taking into account all the previous lines, we have:

$$\begin{aligned} &[X_k, Y] = 0, \quad \forall Y \in \mathfrak{g}, \ k = 1 \ldots n-2, \\ &[X_{n-1}, X_n] = \textstyle\sum_{i=1}^{n-2} \alpha_i X_i, \qquad [X_{n-1}, JX_k] = 0, \ k = 1, \ldots, n-2, \\ &[X_{n-1}, JX_{n-1}] = \textstyle\sum_{i=1}^{n-2} \beta_i X_i, \quad [X_{n-1}, JX_n] = \sum_{i=1}^{n-2} \gamma_i X_i, \\ &[X_n, JX_{n-1}] = \textstyle\sum_{i=1}^{n-2} \gamma_i X_i, \qquad [X_n, JX_k] \text{ unknown}, \ k = 1, \ldots, n-2, n, \\ &[JX_k, JX_l] = 0, \ 1 \leq k < l \leq n-1, \\ &[JX_k, JX_n] = -J[X_n, JX_k], \ k = 1, \ldots, n-2, \\ &[JX_{n-1}, JX_n] = \textstyle\sum_{i=1}^{n-2} \alpha_i X_i, \end{aligned} \tag{4.1}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, for $i = 1, \ldots, n-2$. Notice that not all the possible values for these parameters are valid: the choice must preserve the dimension of the ascending central series we have fixed above.

We proceed to study $\mathfrak{g}_3$. First, let us note that $\dim \mathfrak{g}_3 > n = \dim \mathfrak{g}_2$. If we assume $JX \notin \mathfrak{g}_3$, for any $X \in \mathfrak{g}_1$, then there should be a vector in $\mathfrak{g}_3$ linearly independent with $X_i$, $JX_i$, for $i = 1, \ldots, n-1$. Therefore, this vector can belong to $\mathcal{B}$ and we denote it by $X_n$. Moreover, due to the nilpotency of $\mathfrak{g}$ one could find $Z \in V = \langle X_1, \ldots, X_n \rangle \subset \mathfrak{g}_3$ such that $JZ \in \mathfrak{g}_4 \setminus \mathfrak{g}_3$. Applying Lemma 4.1.1 *ii)* with $l = 3$ there is an element $Y \in \mathfrak{g}_2$ such that $JY \in \mathfrak{g}_3 \setminus \mathfrak{g}_2$, but this is not possible by construction.

We are forced to suppose that $JX \in \mathfrak{g}_3$, for some $X \in \mathfrak{g}_1$. We can write $X = \sum_{i=1}^{n-2} s_i X_i$, with $(s_1, \ldots, s_{n-2}) \neq (0, \ldots, 0)$. Since $JX$ is a generator of $\mathfrak{g}_3$, we would like to include it in $\mathcal{B}$. As we want $\mathcal{B}$ to be $J$-adapted, we must arrange generators as follows:

- if $s_1 \neq 0$, consider $X'_1 = X$ and $X'_k = X_k$, for every $k = 2, \ldots, n-1$;
- if $s_1 = \ldots = s_l = 0$, for some $1 \leq l \leq n-3$, and $s_{l+1} \neq 0$, then choose $X'_1 = X$, $X'_{l+1} = X_1$, and $X'_k = X_k$, for every $k = 2, \ldots, n-1$ such that $k \neq l+1$.

Let us notice that the previous change does not affect the structure of the ascending central series that has been adjusted up to this moment. Therefore, the brackets of the new elements still follow (4.1), maybe modifying the coefficients $\alpha_i$, $\beta_i$, and $\gamma_i$ if necessary (which were anyway free). Furthermore, we get $JX'_1 \in \mathfrak{g}_3$.

This fact allows to conclude that our assumption is equivalent to $JX_1 \in \mathfrak{g}_3$, *up to arrangement of generators*. In this way, we have

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \;\; \mathfrak{g}_2 = \langle X_1, \ldots, X_{n-1}, JX_{n-1} \rangle, \;\; \mathfrak{g}_3 \supseteq \langle X_1, \ldots, X_{n-1}, JX_1, JX_{n-1} \rangle.$$

Necessarily, one can find $Y \in \mathfrak{g}$ such that $[JX_1, Y] \in \mathfrak{g}_2 \setminus \mathfrak{g}_1$. In view of the brackets (4.1), this element $Y$ should be linearly independent with $X_i$, $JX_i$, for $i = 1, \ldots, n-1$. Hence, we can choose $X_n = Y$ as a new generator in $\mathcal{B}$, and then $X_n \in \mathfrak{g}_k \setminus \mathfrak{g}_2$ for some $k \geq 3$. Therefore, one can set

$$[X_n, JX_1] = \sum_{i=1}^{n-2} b_i^1 \, X_i + b_{n-1}^1 \, X_{n-1} + B_{n-1}^1 \, JX_{n-1},$$

where $b_i^1, B_{n-1}^1 \in \mathbb{R}$, for $i = 1, \ldots, n-1$. Observe that we need $(b_{n-1}^1, B_{n-1}^1) \neq (0,0)$ in order to ensure $X_n \notin \mathfrak{g}_2$. Furthermore, Nijenhuis condition yields

$$[JX_1, JX_n] = -\sum_{i=1}^{n-2} b_i^1 \, JX_i - b_{n-1}^1 \, JX_{n-1} + B_{n-1}^1 \, X_{n-1} \;\in\; \mathfrak{g}_2,$$

and thus $b_i^1 = 0$, for every $i = 1, \ldots, n-2$. Applying the Jacobi identity,

$$\begin{array}{rcccccc} 0 & = & Jac\,(X_n, JX_1, X_{n-1}) & = & B_{n-1}^1 \, [JX_{n-1}, X_{n-1}] & = & -B_{n-1}^1 \sum_{i=1}^{n-2} \beta_i \, X_i, \\ 0 & = & Jac\,(X_n, JX_1, JX_{n-1}) & = & b_{n-1}^1 \, [X_{n-1}, JX_{n-1}] & = & b_{n-1}^1 \sum_{i=1}^{n-2} \beta_i \, X_i, \end{array}$$

because $X_{n-1}$, $JX_{n-1} \in \mathfrak{g}_2$. Therefore, the following system of equations is obtained

$$\left\{ \begin{array}{rcl} B_{n-1}^1 \, \beta_i & = & 0, \;\; i = 1, \ldots, n-2, \\ b_{n-1}^1 \, \beta_i & = & 0, \;\; i = 1, \ldots, n-2. \end{array} \right.$$

Since $(b^1_{n-1}, B^1_{n-1}) \neq (0,0)$, we can conclude $\beta_i = 0$ for every $i = 1,\ldots,n-2$. The brackets (4.1) now become

$$
\begin{aligned}
&[X_k, Y] \ = \ 0, \quad \forall Y \in \mathfrak{g}, \ \ k = 1\ldots n-2,\\
&[X_{n-1}, X_n] \ = \ \textstyle\sum_{i=1}^{n-2} \alpha_i\, X_i, \quad [X_{n-1}, JX_k] \ = \ 0, \ \ k = 1,\ldots,n-1,\\
&[X_{n-1}, JX_n] \ = \ \textstyle\sum_{i=1}^{n-2} \gamma_i\, X_i,\\
&[X_n, JX_1] \ = \ b^1_{n-1}\, X_{n-1} + B^1_{n-1}\, JX_{n-1},\\
&[X_n, JX_{n-1}] = \textstyle\sum_{i=1}^{n-2} \gamma_i\, X_i, \quad [X_n, JX_k] \text{ unknown}, \ \ k = 2,\ldots,n-2,\, n,\\
&[JX_1, JX_k] \ = \ 0, \ \ k = 2,\ldots,n-1,\\
&[JX_1, JX_n] \ = \ B^1_{n-1}\, X_{n-1} - b^1_{n-1}\, JX_{n-1},\\
&[JX_k, JX_l] \ = \ 0, \ \ 2 \leq k < l \leq n-1,\\
&[JX_k, JX_n] \ = \ -J[X_n, JX_k], \ \ k = 2,\ldots,n-2,\\
&[JX_{n-1}, JX_n] \ = \ \textstyle\sum_{i=1}^{n-2} \alpha_i\, X_i,
\end{aligned}
\tag{4.2}
$$

with $\alpha_i, \gamma_i, b^1_{n-1}, B^1_{n-1} \in \mathbb{R}$, for $i = 1,\ldots,n-2$, such that the dimension of the ascending central series is preserved. Solving from the Jacobi identity $Jac\,(X_n, JX_n, JX_1)$ and using (4.2), we obtain

$$\big[[X_n, JX_n], JX_1\big] \ = \ 2\sum_{i=1}^{n-2} \big(b^1_{n-1}\,\gamma_i + B^1_{n-1}\,\alpha_i\big)\ X_i.$$

At this point, we ignore the value of $[X_n, JX_n]$. However, it is clear that this bracket will depend, at most, on all the other elements of the basis, $X_1, \ldots, X_{n-1}, JX_1, \ldots, JX_{n-1}$. Since the bracket of each one of these elements with $JX_1$ equals zero, we can conclude $\big[[X_n, JX_n], JX_1\big] = 0$. Hence,

$$b^1_{n-1}\,\gamma_i + B^1_{n-1}\,\alpha_i = 0, \ \forall i = 1,\ldots,n-2. \tag{4.3}$$

At this point, two cases need to be distinguished.

<u>Case 1</u>: Assume there is another element in $\mathfrak{g}_1$ whose image by $J$ belongs to $\mathfrak{g}_3$, i.e. $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) \geq 2$. Arranging generators if necessary, we can suppose $JX_2 \in \mathfrak{g}_3$:

$$\mathfrak{g}_1 = \langle X_1,\ldots,X_{n-2}\rangle, \ \ \mathfrak{g}_2 = \langle X_1,\ldots,X_{n-1}, JX_{n-1}\rangle,$$
$$\mathfrak{g}_3 \supseteq \langle X_1,\ldots,X_{n-1}, JX_1, JX_2, JX_{n-1}\rangle.$$

Then, it should be possible to find an element $Y \in \mathfrak{g}$ such that $[JX_2, Y] \in \mathfrak{g}_2 \smallsetminus \mathfrak{g}_1$. From the brackets (4.2), one can see that $Y$ depends on $X_n$ or $JX_n$. Therefore, repeating a similar argument as for $JX_1$ we have

$$
\begin{aligned}
[X_n, JX_2] \ &= \ b^2_{n-1}\, X_{n-1} + B^2_{n-1}\, JX_{n-1},\\
[JX_2, JX_n] \ &= \ B^2_{n-1}\, X_{n-1} - b^2_{n-1}\, JX_{n-1},
\end{aligned}
$$

where $b^2_{n-1}, B^2_{n-1} \in \mathbb{R}$. In particular $(b^2_{n-1}, B^2_{n-1}) \neq (0,0)$, or otherwise $JX_2$ would belong to $\mathfrak{g}_2$ which is not possible. The Jacobi identity together with (4.2) leads to

$$\big[[X_n, JX_n], JX_2\big] \;=\; 2 \sum\nolimits_{i=1}^{n-2} \left(b^2_{n-1}\,\gamma_i + B^2_{n-1}\,\alpha_i\right)\, X_i.$$

By a similar argument to the one used for (4.3), we get

$$b^2_{n-1}\,\gamma_i + B^2_{n-1}\,\alpha_i = 0,\ \forall i = 1, \ldots, n-2. \tag{4.4}$$

Let us solve the system of equations generated by (4.3) and (4.4).

*a)* If $b^1_{n-1} = 0$ then $B^1_{n-1} \neq 0$, and one obtains $\alpha_1 = \ldots = \alpha_{n-2} = 0$ from (4.3). Observe that $(\gamma_1, \ldots, \gamma_{n-2}) \neq (0, \ldots, 0)$, or otherwise $X_{n-1}$, $JX_{n-1} \in \mathfrak{g}_1$, which is a contradiction. Therefore, from (4.4) we conclude $b^2_{n-1} = 0$. Take $0 \neq X = B^2_{n-1}\,JX_1 - B^1_{n-1}\,JX_2 \;\in\; J\mathfrak{g}_1$. Note that the bracket of $X$ with every element of the basis vanishes. Thus $X \in \mathfrak{g}_1 \cap J\mathfrak{g}_1$, but this is not possible.

*b)* If $b^1_{n-1} \neq 0$, then we can solve $\gamma_i$ from (4.3) and get

$$\gamma_i = -\frac{B^1_{n-1}}{b^1_{n-1}}\,\alpha_i, \quad i = 1, \ldots, n-2.$$

Replacing these values in (4.4),

$$\alpha_i \left(B^2_{n-1} - \frac{B^1_{n-1}\,b^2_{n-1}}{b^1_{n-1}}\right) = 0.$$

If $\alpha_i = 0$, for every $i = 1, \ldots, n-2$, then $\gamma_i = 0$, for every $i = 1, \ldots, n-2$, and $X_{n-1}$, $JX_{n-1} \in \mathfrak{g}_1$. This is a contradiction. Hence, we should have

$$B^2_{n-1} = \frac{B^1_{n-1}\,b^2_{n-1}}{b^1_{n-1}}.$$

Take $0 \neq X = b^2_{n-1}\,JX_1 - b^1_{n-1}\,JX_2 \;\in\; J\mathfrak{g}_1$. Computing the brackets of $X$ with each element of the basis, we can see that $X \in \mathfrak{g}_1 \cap J\mathfrak{g}_1$. This contradicts the hypothesis $\mathfrak{a}_1(J) = \{0\}$.

Since Case 1 is not possible, let us move to the opposite situation.

<u>CASE 2</u>: Let us now suppose that $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) = 1$, that is, $JX_k \notin \mathfrak{g}_3$ for every $k = 2, \ldots, n-2$. Two new possibilities should be studied.

<u>Case 2.1</u>: Assume that $X_n \in \mathfrak{g}_3$, being $X_n$ a vector linearly independent with $X_1, \ldots, X_{n-1}$ and their images by $J$. Then, it is possible to fix the unknown brackets in (4.2) as follows

$$\begin{aligned}
[X_n, JX_k] \;&=\; \sum\nolimits_{i=1}^{n-2} c^k_i\, X_i + c^k_{n-1}\, X_{n-1} + C^k_{n-1}\, JX_{n-1}, \quad k = 2, \ldots, n-2,\, n,\\
[JX_k, JX_n] \;&=\; -\sum\nolimits_{i=1}^{n-2} c^k_i\, JX_i + C^k_{n-1}\, X_{n-1} - c^k_{n-1}\, JX_{n-1}, \quad k = 2, \ldots, n-2,
\end{aligned}$$

where $c_i^k, C_{n-1}^k \in \mathbb{R}$, for $i = 1, \ldots, n-1$ and $k = 2, \ldots, n-2, n$. If there is some $k \in \{2, \ldots, n-2\}$ such that $c_i^k = 0$, for every $i = 1, \ldots, n-2$, then $JX_k \in \mathfrak{g}_3$. However, this is not possible by the conditions imposed at the beginning of Case 2. Therefore, we can ensure $(c_1^k, \ldots, c_{n-2}^k) \neq (0, \ldots, 0)$, for every $k = 2, \ldots, n-2$. Let us note that this implies $JX_n \notin \mathfrak{g}_3$. Hence,

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \quad \mathfrak{g}_2 = \langle X_1, \ldots, X_{n-1}, JX_{n-1} \rangle, \quad \mathfrak{g}_3 = \langle X_1, \ldots, X_n, JX_1, JX_{n-1} \rangle.$$

Due to the nilpotency of the Lie algebra, either $JX_2$ (up to arrangement of generators) or $JX_n$ should enter in $\mathfrak{g}_4$. Let us study these two paths.

*a)* Assume $JX_2 \in \mathfrak{g}_4$. In particular we have $[JX_2, JX_n] \in \mathfrak{g}_3$, so one needs $c_i^2 = 0$, for every $i = 2, \ldots, n-2$. By the Jacobi identity $Jac\,(X_n, JX_n, JX_2)$, one has

$$0 \;=\; -c_1^2\, b_{n-1}^1\, X_{n-1} - c_1^2\, B_{n-1}^1\, JX_{n-1} - 2 \sum_{i=1}^{n-2} \left( c_{n-1}^2\, \gamma_i + C_{n-1}^2\, \alpha_i \right) X_i.$$

Since $(b_{n-1}^1, B_{n-1}^1) \neq (0,0)$, we get $c_1^2 = 0$. However, this implies $JX_2 \in \mathfrak{g}_3$, which is not allowed.

*b)* Suppose $JX_k \notin \mathfrak{g}_4$, for every $k = 2, \ldots, n-2$. The nilpotency of the algebra leads to $JX_n \in \mathfrak{g}_4$. Then $[JX_k, JX_n] \in \mathfrak{g}_3$, so $c_i^k = 0$, for every $i, k = 2, \ldots, n-2$. From here, we conclude $JX_k \in \mathfrak{g}_4$, for $k = 2, \ldots, n-2$, which is a contradiction by hypothesis.

<u>Case 2.2</u>: Let us suppose the opposite to Case 2.1. Then,

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \; \mathfrak{g}_2 = \langle X_1, \ldots, X_{n-1}, JX_{n-1} \rangle, \; \mathfrak{g}_3 = \langle X_1, \ldots, X_{n-1}, JX_1, JX_{n-1} \rangle.$$

We turn our attention to $\mathfrak{g}_4$, with the aim of determining the corresponding brackets in (4.2) for this case.

<u>Case 2.2.1</u>: Let us assume that $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) \geq 2$. Up to arrangement of generators, one can suppose that $JX_2 \in \mathfrak{g}_4$. Then, there should exist an element $Y \in \mathfrak{g}$ such that $[JX_2, Y] \in \mathfrak{g}_3 \setminus \mathfrak{g}_2$. Necessarily, $Y$ depends on $X_n$ or $JX_n$ (see the brackets in (4.2) involving $JX_2$), and it is possible to set

$$\begin{aligned} [X_n, JX_2] &= \textstyle\sum_{i=1}^{n-2} c_i^2\, X_i + c_{n-1}^2\, X_{n-1} + C_{n-1}^2\, JX_{n-1} + C_1^2\, JX_1, \\ [JX_2, JX_n] &= -\textstyle\sum_{i=1}^{n-2} c_i^2\, JX_i - c_{n-1}^2\, JX_{n-1} + C_{n-1}^2\, X_{n-1} + C_1^2\, X_1, \end{aligned}$$

where $c_i^2, C_1^2, C_{n-1}^2 \in \mathbb{R}$, for $i = 1, \ldots, n-1$. Observe that $c_i^2 = 0$, for every $i = 2, \ldots, n-2$, because both brackets belong to $\mathfrak{g}_3$. Furthermore, they cannot lie in $\mathfrak{g}_2$, so one has $(c_1^2, C_1^2) \neq (0,0)$. Now, applying the Jacobi identity and a similar argument to that used to prove (4.3) we get:

$$\begin{aligned} Jac\,(X_n, JX_n, JX_2) \;=\; & -\left( C_1^2\, B_{n-1}^1 + c_1^2\, b_{n-1}^1 \right) X_{n-1} + \left( C_1^2\, b_{n-1}^1 - c_1^2\, B_{n-1}^1 \right) JX_{n-1} \\ & -2 \textstyle\sum_{i=1}^{n-2} \left( c_{n-1}^2\, \gamma_i + C_{n-1}^2\, \alpha_i \right) X_i. \end{aligned}$$

The following system of equations needs to be solved:

$$\begin{cases} 0 &= C_1^2\, B_{n-1}^1 + c_1^2\, b_{n-1}^1, \\ 0 &= C_1^2\, b_{n-1}^1 - c_1^2\, B_{n-1}^1. \end{cases}$$

In order to obtain a solution $(c_1^2, C_1^2) \neq (0,0)$, one needs $\left(B_{n-1}^1\right)^2 + \left(b_{n-1}^1\right)^2 = 0$. The only real solution for this last equation is $b_{n-1}^1 = B_{n-1}^1 = 0$, but it implies $JX_1 \in \mathfrak{g}_2$. This contradicts the arrangement of our ascending central series.

Case 2.2.2: Assume $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) = 1$, i.e., $JX_k \notin \mathfrak{g}_4$, for every $k = 2, \ldots, n-2$. Then, either $X_n$ or $JX_n$ should belong to $\mathfrak{g}_4$. Observe that we can suppose $X_n \in \mathfrak{g}_4$ because the role of the elements $X_n$ and $JX_n$ in (4.2) is interchangeable. In this case,

$$\begin{aligned} [X_n, JX_k] &= \textstyle\sum_{i=1}^{n-2} c_i^k\, X_i + c_{n-1}^k\, X_{n-1} + C_{n-1}^k\, JX_{n-1} + C_1^k\, JX_1, \;\; k = 2, \ldots, n-2,\, n, \\ [JX_k, JX_n] &= -\textstyle\sum_{i=1}^{n-2} c_i^k\, JX_i - c_{n-1}^k\, JX_{n-1} + C_{n-1}^k\, X_{n-1} + C_1^k\, X_1, \;\; k = 2, \ldots, n-2, \end{aligned}$$

where $c_i^k, C_1^k, C_{n-1}^k \in \mathbb{R}$, for $i = 1, \ldots, n-1$ and $k = 2, \ldots, n-2, n$. Note that $JX_n \in \mathfrak{g}_4$ would imply $JX_k \in \mathfrak{g}_4$, for every $k = 2, \ldots, n-2$, but this contradicts our assumption. Hence, $JX_n \notin \mathfrak{g}_4$ and

$$\mathfrak{g}_1 = \langle X_1, \ldots, X_{n-2} \rangle, \quad \mathfrak{g}_2 = \langle X_1, \ldots, X_{n-1}, JX_{n-1} \rangle,$$

$$\mathfrak{g}_3 = \langle X_1, \ldots, X_{n-1}, JX_1, JX_{n-1} \rangle, \quad \mathfrak{g}_4 = \langle X_1, \ldots, X_n, JX_1, JX_{n-1} \rangle.$$

Recall that the nilpotency of the algebra implies that the ascending central series finishes in $\mathfrak{g}$, so there is $Y \in \mathfrak{g}_5$ such that $Y \notin \mathfrak{g}_4$. Necessarily, $Y$ is linearly dependent with some $JX_2, \ldots,\, JX_{n-2},\, JX_n$. In fact, we can assume that, up to a change of basis, $Y = JX$ for some $X \in V = \langle X_1, \ldots, X_n \rangle \subset \mathfrak{g}_4$. Let us note that we are in the conditions of Lemma 4.1.1 *ii)* for $l = 4$. Thus, there is $Z \in \mathfrak{g}_3$ such that $JZ \in \mathfrak{g}_4 \smallsetminus \mathfrak{g}_3$. However, if we take an element $Z = \sum_{i=1}^{n-1} s_i X_i + S_1 JX_1 + S_{n-1} JX_{n-1} \in \mathfrak{g}_3$, then we can see that $JZ \in \mathfrak{g}_4$ if and only if $JZ \in \mathfrak{g}_3$, which is a contradiction.

Notice that the previous lines cover all the possible arrangements for the ascending central series when the center of the Lie algebra $\mathfrak{g}$ has dimension $n-2$ and $\mathfrak{a}_1(J) = \{0\}$. Since only contradictions have been obtained, we can derive our result. □

As a consequence of the the previous theorem, we obtain the following.

**Corollary 4.1.12.** *Nilpotent Lie algebras of dimensions six and eight admitting strongly non-nilpotent complex structures have* 1*-dimensional centers.*

Although the 6-dimensional case had already been proven [Uga07], this is the first result concerning dimension 8, as far as we know.

At this point, it is worth recalling that the existence of a nilpotent complex structure requires a center of dimension greater than or equal to 2. This observation together with Corollary 4.1.12 allows to conclude that neither 6- nor 8-dimensional nilpotent Lie algebras simultaneously admit SnN and nilpotent complex structures. Once again, the observation for the 6-dimensional case is not new, as it comes from the combination of [CFGU97c] and [Sal01] (see also Section 3.2). Indeed, in 6 dimensions it is not possible to have both a nilpotent and a non-nilpotent complex structure on the same NLA. Things are different in dimension 8, due to the presence of weakly non-nilpotent complex structures.

**Example 4.1.13.** Consider the 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ given in Example 3.1.5. Let us remember that $\mathfrak{g}$ was there endowed with a weakly non-nilpotent complex structure $J$. Now, we define an almost-complex structure $\tilde{J}$ on $\mathfrak{g}$ as follows

$$\tilde{J}X_1 = X_2, \quad \tilde{J}X_3 = X_4, \quad \tilde{J}X_5 = X_8, \quad \tilde{J}X_6 = -X_7.$$

One can check that $\tilde{J}$ is indeed a complex structure on $\mathfrak{g}$, because it satisfies the Nijenhuis condition (3.3). Furthermore, it is nilpotent and has $\mathfrak{a}_1(\tilde{J}) = \langle X_6, X_7\rangle$, $\mathfrak{a}_2(\tilde{J}) = \langle X_1, X_2, X_6, X_7\rangle$, and $\mathfrak{a}_3(\tilde{J}) = \mathfrak{g}$. ◇

In sight of the previous lines, it is natural to ask what happens in dimension greater than or equal to 10 (that is, $n \geq 5$), where there could be more flexibility for $\dim \mathfrak{g}_1$. The next example shows that, in the 10-dimensional case, the upper bound given in Theorem 4.1.11 can indeed be attained.

**Example 4.1.14.** [CFGU00] Let $\mathfrak{g}$ be the nilpotent Lie algebra of dimension 10 defined by the following non-zero brackets

$$[X_3, X_9] = [X_4, X_{10}] = X_1, \quad [X_5, X_9] = [X_6, X_{10}] = X_2,$$
$$[X_7, X_9] = X_3, \quad [X_7, X_{10}] = X_4, \quad [X_8, X_9] = X_5, \quad [X_8, X_{10}] = X_6.$$

It is easy to see that $\mathfrak{g}_1 = \langle X_1, X_2\rangle$, $\mathfrak{g}_2 = \langle X_1, X_2, X_3, X_4, X_5, X_6\rangle$, and $\mathfrak{g}_3 = \mathfrak{g}$. In particular, the dimension of the center is 2.

Let us define an almost-complex structure $J$ on $\mathfrak{g}$ by

$$JX_1 = -X_7, \quad JX_2 = -X_8, \quad JX_3 = X_4, \quad JX_5 = X_6, \quad JX_9 = X_{10}.$$

One can check that the Nijenhuis condition holds, so $J$ turns to be a complex structure. Furthermore, it satisfies $\mathfrak{a}_1(J) = \{0\}$, i.e., it is strongly non-nilpotent. ◇

This fact opens the possibility of finding coexistent strongly non-nilpotent and nilpotent complex structures in this dimension. In fact, the previous $\mathfrak{g}$ in Example 4.1.14 appears in [CFGU00] as a nilpotent Lie algebra precisely admitting both types of complex structures.

**Example 4.1.15.** [CFGU00] The almost-complex structure $\hat{J}$ on the Lie algebra $\mathfrak{g}$ of Example 4.1.14 given by

$$\hat{J}X_1 = X_2, \quad \hat{J}X_3 = X_4, \quad \hat{J}X_5 = X_6, \quad \hat{J}X_7 = X_8, \quad \hat{J}X_9 = X_{10}$$

verifies the Nijenhuis condition (3.3) and has $\mathfrak{a}_1(\hat{J}) = \mathfrak{g}_1$, $\mathfrak{a}_2(\hat{J}) = \mathfrak{g}_2$, and $\mathfrak{a}_3(\hat{J}) = \mathfrak{g}_3$. In particular, $\hat{J}$ is nilpotent.

We ignore whether Theorem 4.1.11 can be improved for $n \geq 6$. Concerning other stages of the ascending central series, we have the following results.

**Proposition 4.1.16.** *Let $\mathfrak{g}$ be a $2n$-dimensional NLA with $n \geq 3$ endowed with an SnN complex structure $J$. Then, $2 \leq \dim \mathfrak{g}_2 \leq 2n-3$. In addition, when $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$ one indeed has $3 \leq \dim \mathfrak{g}_2 \leq 2n-3$.*

*Proof.* Since $1 \leq \dim \mathfrak{g}_1$, it is clear that $2 \leq \dim \mathfrak{g}_2$ because of the nilpotency of $\mathfrak{g}$. In particular, if $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$ there exist at least two linearly independent elements in $\mathfrak{g}_2$ which do not belong to $\mathfrak{g}_1$.

For the upper bound, let us assume that $\mathfrak{g}_1 = \langle X_1, \ldots, X_l \rangle$, for some $l \geq 1$. Using Lemma 4.1.3 *i)*, it is clear that $\mathfrak{g}_2 \cap J\mathfrak{g}_1 = \{0\}$. Furthermore, by Corollary 4.1.5 we know that $\mathfrak{g}_2$ cannot contain a subspace $V$ of dimension $n$ such that $V \cap JV = \{0\}$. These two observations lead to $\dim \mathfrak{g}_2 \leq l + 2(n-l-1) = 2n-l-2 \leq 2n-3$. □

**Proposition 4.1.17.** *Let $(\mathfrak{g}, J)$ be a $2n$-dimensional NLA, where $n \geq 4$, endowed with a strongly non-nilpotent complex structure. If $\dim \mathfrak{g}_1 = n-3$, then $\mathfrak{g}_3 \cap J\mathfrak{g}_3 \neq \{0\}$.*

*Proof.* We first observe that if $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$, then the result is trivial. Hence, let us focus on $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$. By Corollary 4.1.6, one has $\dim \mathfrak{g}_2 \leq n-2$. Bearing in mind that $n-3 = \dim \mathfrak{g}_1 < \dim \mathfrak{g}_2$, we obtain $\dim \mathfrak{g}_2 = n-2$. Hence, $\dim \mathfrak{g}_3 \geq n-1$. At the sight of Corollary 4.1.6, the result holds. □

The rest of this chapter is devoted to parametrize strongly non-nilpotent complex structures on 8-dimensional nilpotent Lie algebras $\mathfrak{g}$. Following the ideas contained in Section 3.1.2, we will carry a similar procedure to that accomplished in dimensions 4 and 6 (see Section 3.2). This will allow us to find the corresponding structure equations and complete the classification of invariant complex geometry on 8-dimensional nilmanifolds.

Let us recall that the first step in our construction consists on finding the ascending central series of the 8-dimensional nilpotent Lie algebras $\mathfrak{g}$ admitting SnN complex structures. The idea is motivated by the 6-dimensional case, which was developed in Lemma 3.2.5. However, as the increase in the dimension makes the argument much longer, we have divided the initial decision tree into smaller parts, for the seek of clarity.

## 4.2 Initial terms in dimension eight

In this section, we concentrate on calculating the initial terms of the ascending central series for those NLAs of dimension eight endowed with SnN complex structures. In particular, we make use of Corollary 4.1.12 for the first term $\mathfrak{g}_1$ and then characterize the second term $\mathfrak{g}_2$.

Let $\mathfrak{g}$ be an 8-dimensional NLA endowed with an SnN complex structure $J$. In order to study the ascending central series of $\mathfrak{g}$, we consider a doubly adapted basis $\mathcal{B}$. We apply the constructive procedure described in Section 3.1.2 by means of which the elements $X_k, JX_k$, $1 \leq k \leq 4$, of $\mathcal{B}$ are found along the discussion, giving rise to every $(\mathfrak{g}, J)$. At the end of this section, we characterize the term $\mathfrak{g}_2$ for each $\mathfrak{g}$ admitting an SnN complex structure. In the next section, we obtain a structure theorem for $(\mathfrak{g}, J)$.

Let us start noting that one can take $\mathfrak{g}_1 = \langle X_1 \rangle$, as a consequence of Corollary 4.1.12. In this way, a first element in $\mathcal{B}$ can be fixed. Moreover, it is clear that

$$[X_1, Y] = 0, \quad \forall\, Y \in \mathfrak{g}.$$

Furthermore, from the Nijenhuis condition (3.3) one also has

$$[JX_1, JY] = J[JX_1, Y], \quad \forall\, Y \in \mathfrak{g}. \tag{4.5}$$

We now focus on $\mathfrak{g}_2$. Applying Lemma 4.1.3, we conclude that $JX_1 \notin \mathfrak{g}_2$. By the nilpotency of $\mathfrak{g}$, one can find $Y \in \mathfrak{g}_2$ linearly independent with $X_1$ and $JX_1$. Observe that $X_2 = Y$ can be taken as another element of $\mathcal{B}$. Since $[X_2, Z] \in \mathfrak{g}_1$, for every $Z \in \mathfrak{g}$, one in particular has $[X_2, JX_1] = \lambda\, X_1$, for some $\lambda \in \mathbb{R}$. From equation (4.5), one obtains $[JX_1, JX_2] = -\lambda\, JX_1$. The nilpotency of the Lie algebra requires $\lambda = 0$. Hence,

$$\begin{aligned} &[X_1, Z] = 0, \quad \forall\, Z \in \mathfrak{g}, && \\ &[X_2, X_k] = a^1_{2k}\, X_1, \;\; k = 3, 4, && [X_2, JX_1] = 0, \\ &[X_2, JX_k] = b^1_{2k}\, X_1, \;\; k = 2, 3, 4, && [JX_1, JX_2] = 0, \end{aligned} \tag{4.6}$$

where $a^1_{2k}, b^1_{22}, b^1_{2k} \in \mathbb{R}$, for $k = 3, 4$. We use these relations as the starting point to prove the following result, which improves Proposition 4.1.17:

**Proposition 4.2.1.** *Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with a strongly non-nilpotent complex structure. Then, $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$.*

*Proof.* We proceed by contradiction. Since several cases arise along the proof, we provide a scheme of the decision tree in Appendix A.

Let us suppose that $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$. From Corollary 4.1.6 we have that $\dim \mathfrak{g}_2 = 2$, thus

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle.$$

By Lemma 4.1.3 *i)*, it is clear that $JX_1, JX_2 \notin \mathfrak{g}_3$. However, the ascending central series should increase until it reaches $\mathfrak{g}$, so there is $Y \in \mathfrak{g}_3$ linearly independent with $X_1, X_2, JX_1$, and $JX_2$. We can take $X_3 = Y$ as an element in $\mathcal{B}$. Moreover, Proposition 4.1.17 indicates that $\mathfrak{g}_3 \cap J\mathfrak{g}_3 \neq \{0\}$. Hence, one can assume that $JX_3 \in \mathfrak{g}_3$, up to an arrangement of generators. Furthermore, as a consequence of Corollary 4.1.5 we know that it is not possible to find $X \in \mathfrak{g}_3$ such that $X$ is linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Therefore,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle.$$

Together with (4.6), we have the following brackets:

$$[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2, \quad [X_3, JX_k] = b^1_{3k}\, X_1 + b^2_{3k}\, X_2,\ k = 1, \ldots, 4,$$
$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2,$$

where $a^i_{34}, b^i_{3k}, b^i_{43} \in \mathbb{R}$, for $i = 1, 2$ and $k = 1, \ldots, 4$. Applying (4.5) for $Y = X_3$, we obtain:

$$[JX_1, JX_3] = -b^1_{31}\, JX_1 - b^2_{31}\, JX_2.$$

In view of $[JX_1, JX_3] \in \mathfrak{g}_2$, we can conclude $b^1_{31} = b^2_{31} = 0$. Now, for $k = 2, 3$ we use the Nijenhuis condition (3.3), obtaining

$$[JX_2, JX_3] = a^1_{23}\, X_1 + (b^1_{23} - b^1_{32})\, JX_1 - b^2_{32}\, JX_2,$$
$$[JX_3, JX_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 + (b^2_{34} - b^2_{43})\, JX_2.$$

Once again $[JX_2, JX_3], [JX_3, JX_4] \in \mathfrak{g}_2$, which implies $b^2_{32} = 0$, $b^1_{32} = b^1_{23}$, $b^1_{43} = b^1_{34}$, and $b^2_{43} = b^2_{34}$. In this way, our Lie algebra $\mathfrak{g}$ is defined by the brackets:

$$
\begin{aligned}
&[X_1, Z] = 0, \quad \forall\, Z \in \mathfrak{g},\\
&[X_2, X_k] = a^1_{2k}\, X_1,\ \ k = 3, 4, \quad [X_2, JX_1] = 0,\\
&[X_2, JX_k] = b^1_{2k}\, X_1,\ \ k = 2, 3, 4,\\
&[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2, \quad [X_3, JX_1] = 0, \quad [X_3, JX_2] = b^1_{23}\, X_1\\
&[X_3, JX_k] = b^1_{3k}\, X_1 + b^2_{3k}\, X_2,\ k = 3, 4,\\
&[X_4, JX_3] = b^1_{34}\, X_1 + b^2_{34}\, X_2,\\
&[JX_1, JX_2] = [JX_1, JX_3] = 0,\\
&[JX_2, JX_3] = a^1_{23}\, X_1, \quad [JX_3, JX_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2,
\end{aligned}
\tag{4.7}
$$

where $a^1_{2k}, a^i_{34}, b^1_{22}, b^1_{2k}, b^i_{3k} \in \mathbb{R}$, for $i = 1, 2$ and $k = 3, 4$. We move to the study of $\mathfrak{g}_4$. Different possibilities are distinguished:

<u>Case 1</u>: Let us suppose that there is $Y \in \mathfrak{g}_4$ linearly independent with $X_1$, $X_2$, $X_3$, $JX_1$, $JX_2$, and $JX_3$. Set $X_4 = Y$ as an element in $\mathcal{B}$, and we have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle, \quad \mathfrak{g}_4 \supseteq \langle X_1, X_2, X_3, X_4, JX_3 \rangle.$$

We can then fix the brackets

$$[X_4, JX_k] = b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + b^3_{4k}\, X_3 + c^3_{4k}\, JX_3,\ \ k = 1, 2, 4,$$

where $b^i_{4k}, c^3_{4k} \in \mathbb{R}$, for $i = 1, 2, 3$. Applying (4.5) to $Y = X_4$ and (3.3) to $X_2, X_4$, we get:

$$[JX_1, JX_4] = c^3_{41}\, X_3 - b^1_{41}\, JX_1 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3,$$
$$[JX_2, JX_4] = a^1_{24}\, X_1 + c^3_{42}\, X_3 + (b^1_{24} - b^1_{42})\, JX_1 - b^2_{42}\, JX_2 - b^3_{42}\, JX_3.$$

To preserve the nilpotency of $\mathfrak{g}$, one needs $b_{41}^1 = b_{42}^2 = 0$. In this way, the Lie algebra is defined by (4.7) and

$$
\begin{aligned}
&[X_4, JX_1] = b_{41}^2\, X_2 + b_{41}^3\, X_3 + c_{41}^3\, JX_3,\\
&[X_4, JX_2] = b_{42}^1\, X_1 + b_{42}^3\, X_3 + c_{42}^3\, JX_3,\\
&[X_4, JX_4] = b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^3\, JX_3,\\
&[JX_1, JX_4] = c_{41}^3\, X_3 - b_{41}^2\, JX_2 - b_{41}^3\, JX_3,\\
&[JX_2, JX_4] = a_{24}^1\, X_1 + c_{42}^3\, X_3 + (b_{24}^1 - b_{42}^1)\, JX_1 - b_{42}^3\, JX_3,
\end{aligned}
\tag{4.8}
$$

with $b_{41}^2, b_{42}^1, b_{4k}^3, b_{44}^i, c_{4k}^3, c_{44}^3 \in \mathbb{R}$, for $k = 1, 2$ and $i = 1, 2, 3$. Note that all the brackets are now settled, even though we still do not know where $JX_1, JX_2$, and $JX_4$ enter the ascending central series. The position of these elements will determine part of the previous parameters. Let us distinguish different cases.

Case 1.1: Consider that $JX_4 \in \mathfrak{g}_4$. Taking into account that the brackets $[JX_1, JX_4]$ and $[JX_2, JX_4]$ in (4.8) should lie in $\mathfrak{g}_3$, we get $b_{41}^2 = 0$ and $b_{42}^1 = b_{24}^1$. Hence, $\mathfrak{g}$ is defined by (4.7) and (4.8) with $b_{41}^2 = 0$ and $b_{42}^1 = b_{24}^1$. Its ascending central series is

$$
\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.
$$

Let us study the Jacobi identity to contradict this case. First, we calculate

$$
\begin{aligned}
Jac\,(X_3, JX_1, X_4) &= b_{33}^1\, c_{41}^3\, X_1 + b_{33}^2\, c_{41}^3\, X_2,\\
Jac\,(X_3, JX_1, JX_4) &= b_{33}^1\, b_{41}^3\, X_1 + b_{33}^2\, b_{41}^3\, X_2.
\end{aligned}
$$

If $b_{41}^3 = c_{41}^3 = 0$ then $JX_1 \in \mathfrak{g}_1$, which contradicts the strongly non-nilpotency of our complex structure. Therefore, $(b_{41}^3, c_{41}^3) \neq (0, 0)$ and $b_{33}^1 = b_{33}^2 = 0$.

Moreover, from $Jac\,(X_2, JX_1, X_4)$ and $Jac\,(X_2, JX_1, JX_4)$ we obtain the system of equations:

$$
\begin{pmatrix} b_{41}^3 & c_{41}^3 \\ c_{41}^3 & -b_{41}^3 \end{pmatrix} \begin{pmatrix} a_{23}^1 \\ b_{23}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Since $(b_{41}^3, c_{41}^3) \neq (0, 0)$, the determinant of the previous matrix cannot be zero, and we conclude that $a_{23}^1 = b_{23}^1 = 0$.

Next, we calculate $Jac\,(X_3, JX_2, X_4)$ and $Jac\,(X_3, JX_2, JX_4)$. Bearing in mind that $a_{23}^1 = b_{23}^1 = b_{33}^1 = b_{33}^2 = 0$, we obtain the equations:

$$
\begin{cases} b_{22}^1\, a_{34}^2 = 0,\\ b_{22}^1\, b_{34}^2 = 0. \end{cases}
\tag{4.9}
$$

If $a_{34}^2 = b_{34}^2 = 0$, then $X_3 \in \mathfrak{g}_2$. As this is not possible (recall the initial construction), we conclude that $(a_{34}^2, b_{34}^2) \neq (0, 0)$ and $b_{22}^1 = 0$.

Therefore, the Lie algebra is defined by (4.7) and (4.8) with $a^1_{23} = b^1_{22} = b^1_{23} = b^1_{33} = b^2_{33} = b^2_{41} = 0$ and $b^1_{42} = b^1_{24}$. We now compute the following:

$$
\begin{aligned}
Jac\,(X_4, JX_4, X_3) &= -\big(a^1_{24}\, b^2_{34} - b^1_{24}\, a^2_{34}\big)\, X_1, \\
Jac\,(X_4, JX_4, JX_3) &= -\big(a^1_{24}\, a^2_{34} + b^1_{24}\, b^2_{34}\big)\, X_1.
\end{aligned} \tag{4.10}
$$

In matrix form, we have

$$
\begin{pmatrix} b^2_{34} & -a^2_{34} \\ a^2_{34} & b^2_{34} \end{pmatrix} \begin{pmatrix} a^1_{24} \\ b^1_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Since $(a^2_{34}, b^2_{34}) \neq (0,0)$, the determinant is not zero. This yields $a^1_{24} = b^1_{24} = 0$. As we already had $a^1_{23} = b^1_{22} = b^1_{23} = 0$, it is clear that we get $X_2 \in \mathfrak{g}_1$ from (4.7). This is a contradiction.

<u>Case 1.2</u>: Let us assume that $JX_4 \notin \mathfrak{g}_4$. From Lemma 4.1.1 *ii)* for $l = 4$, we know that either $JX_1$ or $JX_2$ belong to $\mathfrak{g}_4$ (up to a change of basis). Observe that the former leads to $b^2_{41} = 0$ in (4.8) and the latter, to $b^1_{42} = b^1_{24}$. Simply note that both conditions cannot hold at the same time (otherwise $JX_4$ would belong to $\mathfrak{g}_4$). Calculating the Jacobi identity directly from (4.7) and (4.8), one observes that

$$
\begin{aligned}
Jac\,(X_4, JX_1, JX_4) &= \big(2\,(a^1_{34}\, c^3_{41} + b^1_{34}\, b^3_{41}) + b^2_{41}\,(b^1_{24} + b^1_{42})\big)\, X_1 \\
&\quad + \;2\,\big(a^2_{34}\, c^3_{41} + b^3_{41}\, b^2_{34}\big)\, X_2 \;+\; b^2_{41}\, b^3_{42}\, X_3 \;+\; b^2_{41}\, c^3_{42}\, JX_3, \\
Jac\,(X_4, JX_2, JX_4) &= \big(2\,(a^1_{34}\, c^3_{42} + b^1_{34}\, b^3_{42}) + a^1_{23}\, c^3_{44} - b^2_{44}\, b^1_{22} - b^3_{44}\, b^1_{23}\big)\, X_1 \\
&\quad + \;\big(2\,(a^2_{34}\, c^3_{42} + b^2_{34}\, b^3_{42}) + b^2_{41}\,(b^1_{42} - b^1_{24})\big)\, X_2 \\
&\quad + \;b^3_{41}\,(b^1_{42} - b^1_{24})\, X_3 \;+\; c^3_{41}\,(b^1_{42} - b^1_{24})\, JX_3.
\end{aligned}
$$

If we consider the system of equations generated by the coefficients of $X_3$ and $JX_3$ in the previous expressions, we get

$$
\begin{cases} b^2_{41}\, b^3_{42} = 0, \\ b^2_{41}\, c^3_{42} = 0, \end{cases} \qquad \begin{cases} b^3_{41}\,(b^1_{42} - b^1_{24}) = 0, \\ c^3_{41}\,(b^1_{42} - b^1_{24}) = 0. \end{cases}
$$

If $JX_1 \in \mathfrak{g}_4$, then $b^2_{41} = 0$ and $b^1_{42} \neq b^1_{24}$. From the previous system, we conclude $b^3_{41} = c^3_{41} = 0$. Nonetheless, replacing these values in (4.7) and (4.8), we get $JX_1 \in \mathfrak{g}_1$, which is not possible.

If $JX_2 \in \mathfrak{g}_4$, then $b^1_{42} = b^1_{24}$ and $b^2_{41} \neq 0$. Therefore, we have $b^3_{42} = c^3_{42} = 0$. However, this makes $JX_2 \in \mathfrak{g}_2$, which contradicts the hypothesis $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$.

All the possibilities in Case 1 lead to contradiction.

<u>Case 2</u>: Suppose that there are no elements linearly independent with $X_1$, $X_2$, $X_3$, $JX_1$, $JX_2$, and $JX_3$ in the space $\mathfrak{g}_4$. This means that $\mathfrak{g}$ is at least 5-step nilpotent.

However, by [VR09] (see also Remark 4.1.10) we know that quasi-filiform Lie algebras do not admit complex structures, which implies that our ascending central series should stabilize before the nilpotency step equals 6. Therefore, $\mathfrak{g} = \mathfrak{g}_5$ and there exists $Y \in \mathfrak{g}$ such that $Y, JY \notin \mathfrak{g}_4$. Take $X_4 = Y$ as an element of $\mathcal{B}$.

Let us study $\mathfrak{g}_4$. Since its dimension should be bigger than $\dim \mathfrak{g}_3$, we have that, up to an arrangement of generators, $JX_1 \in \mathfrak{g}_4$ or $JX_2 \in \mathfrak{g}_4$ (even both). Let us study all the possible situations.

Case 2.1: First, we assume that $JX_1 \in \mathfrak{g}_4$, that is,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle, \quad \mathfrak{g}_4 \supseteq \langle X_1, X_2, X_3, JX_1, JX_3 \rangle.$$

We consider the bracket

$$[X_4, JX_1] = b_{41}^1\, X_1 + b_{41}^2\, X_2 + b_{41}^3\, X_3 + c_{41}^3\, JX_3,$$

where $b_{41}^i, c_{41}^3 \in \mathbb{R}$ for $i = 1, 2, 3$, and from (4.5) we obtain

$$[JX_1, JX_4] = c_{41}^3\, X_3 - b_{41}^1\, JX_1 - b_{41}^2\, JX_2 - b_{41}^3\, JX_3.$$

Recall that $[JX_1, JX_4] \in \mathfrak{g}_3$, so $b_{41}^1 = b_{41}^2 = 0$. Therefore, our Lie algebra is defined, up to this point, by the real brackets (4.7) and

$$[X_4, JX_1] = b_{41}^3\, X_3 + c_{41}^3\, JX_3, \qquad [JX_1, JX_4] = c_{41}^3\, X_3 - b_{41}^3\, JX_3. \tag{4.11}$$

Let us know separate two different cases depending on the position of $JX_2$.

Case 2.1.1: Suppose that $JX_2 \in \mathfrak{g}_4$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle,$$

$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

We can define the bracket $[X_4, JX_2]$ as in Case 1 and repeat the same argument. Then, the expressions for $[X_4, JX_2]$ and $[JX_2, JX_4]$ coincide with those in (4.8). In fact, from (4.7), (4.11), and the brackets $[X_4, JX_2]$, $[JX_2, JX_4]$, we observe that the situation is quite similar to that of Case 1.1. In fact, the only difference between the two cases come from

$$[X_4, JX_4] = b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^1\, JX_1 + c_{44}^2\, JX_2 + c_{44}^3\, JX_3,$$

where $b_{44}^i, c_{44}^i \in \mathbb{R}$, for $i = 1, 2, 3$. Hence, a similar study can be done in order to check the Jacobi identity. Indeed, the only Jacobi triples that might change are those involving both $X_4$ and $JX_4$. Therefore, following the same reasoning as in Case 1.1 we have $a_{23}^1 = b_{22}^1 = b_{23}^1 = b_{33}^1 = b_{33}^2 = 0$. In addition, if we recalculate $Jac\,(X_4, JX_4, X_3)$ and $Jac\,(X_4, JX_4, JX_3)$ the same formulas as in (4.10) are found. A similar argument can be applied, and a contradiction is obtained.

Case 2.1.2: Consider that $JX_2 \notin \mathfrak{g}_4$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle,$$

$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

We can set

$$[X_4, JX_k] = b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + b^3_{4k}\, X_3 + c^1_{4k}\, JX_1 + c^3_{4k}\, JX_3, \ k = 2, 4,$$

where $b^i_{4k}, c^1_{4k}, c^2_{4k} \in \mathbb{R}$, for $i = 1, 2, 3$ and $k = 2, 4$. Take $k = 2$ and apply the Nijenhuis condition. We get

$$[JX_2, JX_4] = (a^1_{24} + c^1_{42})\, X_1 + c^3_{42}\, X_3 + (b^1_{24} - b^1_{42})\, JX_1 - b^2_{42}\, JX_2 - b^3_{42}\, JX_3.$$

Since $[JX_2, JX_4] \in \mathfrak{g}_4$, one needs to take $b^2_{42} = 0$. In this way, our Lie algebra is defined by the brackets (4.7), (4.11), and

$$[X_4, JX_2] = b^1_{42}\, X_1 + b^3_{42}\, X_3 + c^1_{42}\, JX_1 + c^3_{42}\, JX_3,$$
$$[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^3_{44}\, JX_3,$$
$$[JX_2, JX_4] = (a^1_{24} + c^1_{42})\, X_1 + c^3_{42}\, X_3 + (b^1_{24} - b^1_{42})\, JX_1 - b^3_{42}\, JX_3.$$

We now study the Jacobi identity. Note that the only difference between these brackets and those defined in Case 1.1 are precisely $[X_4, JX_2]$, $[X_4, JX_4]$, and $[JX_2, JX_4]$. In other words, only those Jacobi identities involving two of the three elements $X_4, JX_2, JX_4$ will change. In particular, $Jac\,(X_3, JX_1, X_4)$, $Jac\,(X_3, JX_1, JX_4)$, $Jac\,(X_2, JX_1, X_4)$, and $Jac\,(X_2, JX_1, JX_4)$ remain the same, so we can use a similar argument and conclude $a^1_{23} = b^1_{23} = b^1_{33} = b^2_{33} = 0$.

Now, if we recalculate $Jac\,(X_3, JX_2, X_4)$ and $Jac\,(X_3, JX_2, JX_4)$, we see that we still obtain (4.9), as in Case 1.1. Therefore, repeating the argument we also have $b^1_{22} = 0$.

The same happens when we compute $Jac\,(X_4, JX_4, X_3)$ and $Jac\,(X_4, JX_4, JX_3)$: we recover (4.10). Applying the same ideas, we reach a contradiction.

Hence, we conclude that the Case 2.1 is not valid. We next study the opposite.

Case 2.2: Let us now suppose that $JX_1 \notin \mathfrak{g}_4$. Then necessarily $JX_2 \in \mathfrak{g}_4$, and

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle,$$
$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

We can take

$$[X_4, JX_2] = b^1_{42}\, X_1 + b^2_{42}\, X_2 + b^3_{42}\, X_3 + c^3_{42}\, JX_3,$$

where $b^i_{42}, c^3_{42} \in \mathbb{R}$, for $i = 1, 2, 3$, and obtain

$$[JX_2, JX_4] = a^1_{24}\, X_1 + c^3_{42}\, X_3 + (b^1_{24} - b^1_{42})\, JX_1 - b^2_{42}\, JX_2 - b^3_{42}\, JX_3$$

by the Nijenhuis condition (3.3). As $[JX_2, JX_4] \in \mathfrak{g}_3$, one needs $b^2_{42} = 0$ and $b^1_{42} = b^1_{24}$. In the same way, one can fix

$$[X_4, JX_1] = b^1_{41}\, X_1 + b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3,$$

where $b_{41}^1, b_{41}^i, c_{41}^i \in \mathbb{R}$, for $i = 2,3$. Using (4.5) and the fact that $[X_4, JX_1] \in \mathfrak{g}_4$, one concludes $b_{41}^1 = 0$.

Taking into account the previous lines, one sees that our Lie algebra $\mathfrak{g}$ is defined by the real brackets (4.7) and

$$\begin{aligned}
[X_4, JX_1] &= b_{41}^2\, X_2 + b_{41}^3\, X_3 + c_{41}^2\, JX_2 + c_{41}^3\, JX_3, \\
[X_4, JX_2] &= b_{24}^1\, X_1 + b_{42}^3\, X_3 + c_{42}^3\, JX_3, \\
[X_4, JX_4] &= b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^2\, JX_2 + c_{44}^3\, JX_3, \\
[JX_1, JX_4] &= c_{41}^2\, X_2 + c_{41}^3\, X_3 - b_{41}^2\, JX_2 - b_{41}^3\, JX_3 \\
[JX_2, JX_4] &= a_{24}^1\, X_1 + c_{42}^3\, X_3 - b_{42}^3\, JX_3.
\end{aligned}$$

We now study the Jacobi condition. We start calculating the following identities:

$$\begin{aligned}
Jac\,(X_2, JX_2, X_4) &= \left(a_{23}^1\, b_{42}^3 + b_{23}^1\, c_{42}^3\right) X_1, \\
Jac\,(X_2, JX_2, JX_4) &= -\left(a_{23}^1\, c_{42}^3 - b_{23}^1\, b_{42}^3\right) X_1.
\end{aligned}$$

In matrix form, we get the system

$$\begin{pmatrix} b_{42}^3 & c_{42}^3 \\ c_{42}^3 & -b_{42}^3 \end{pmatrix} \begin{pmatrix} a_{23}^1 \\ b_{23}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant equals zero, then we have $b_{42}^3 = c_{42}^3 = 0$. However, this implies that $JX_2 \in \mathfrak{g}_2$, which is not possible due to the initial hypothesis $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$. Therefore, $(b_{42}^3, c_{42}^3) \neq (0,0)$ and necessarily, $a_{23}^1 = b_{23}^1 = 0$.

Let us now consider

$$\begin{aligned}
Jac\,(X_3, JX_2, X_4) &= \left(b_{33}^1\, c_{42}^3 - b_{22}^1\, a_{34}^2\right) X_1 + b_{33}^2\, c_{42}^3\, X_2, \\
Jac\,(X_3, JX_2, JX_4) &= \left(b_{33}^1\, b_{42}^3 - b_{22}^1\, a_{34}^2\right) X_1 + b_{33}^2\, b_{42}^3\, X_2.
\end{aligned}$$

Since $(b_{42}^3, c_{42}^3) \neq (0,0)$, from the expressions accompanying $X_2$ we solve $b_{33}^2 = 0$.

We next compute

$$\begin{aligned}
Jac\,(X_4, JX_1, JX_4) &= 2\left(a_{24}^1\, c_{41}^2 + a_{34}^1\, c_{41}^3 + b_{24}^1\, b_{41}^2 + b_{34}^1\, b_{41}^3\right) X_1 \\
&\quad + \; 2\left(a_{34}^2\, c_{41}^3 + b_{34}^2\, b_{41}^3\right) X_2 \; + \; \left(b_{41}^2\, b_{42}^3 + c_{41}^2\, c_{42}^3\right) X_3 \\
&\quad + \; \left(b_{41}^2\, c_{42}^3 - c_{41}^2\, b_{42}^3\right) JX_3.
\end{aligned}$$

Equalling the coefficients of $X_3$ and $JX_3$ in the previous formula to zero, this system of equations arise:

$$\begin{pmatrix} b_{42}^3 & c_{42}^3 \\ -c_{42}^3 & b_{42}^3 \end{pmatrix} \begin{pmatrix} b_{41}^2 \\ c_{41}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have seen before that the determinant cannot be zero, so we get $b_{41}^2 = c_{41}^2 = 0$. This leads to $JX_1 \in \mathfrak{g}_4$, which supposes a contradiction with the hypothesis of Case 2.2.

In conclusion, we always obtain a contradiction when we try to continue the ascending central series of an 8-dimensional NLA $\mathfrak{g}$ endowed with a strongly non-nilpotent complex structure $J$ such that $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$. This makes that such a pair $(\mathfrak{g}, J)$ cannot exist. Thus, the only option is having $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$. □

As a consequence of Proposition 4.2.1, we arrive at the following result which gives the structure of the second term $\mathfrak{g}_2$ in the ascending central series.

**Corollary 4.2.2.** *Let $\mathfrak{g}$ be an 8-dimensional NLA admitting a strongly non-nilpotent complex structure $J$. Then,*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \text{ and } \mathfrak{g}_2 = \begin{cases} \langle X_1, X_2, JX_2 \rangle, \\ \langle X_1, X_2, X_3, JX_2 \rangle, \text{ or} \\ \langle X_1, X_2, X_3, JX_2, JX_3 \rangle. \end{cases}$$

*Proof.* By Corollary 4.1.12, it is clear that $\mathfrak{g}_1 = \langle X_1 \rangle$. Now, as a consequence of Lemma 4.1.3 *i)* and Proposition 4.2.1 we conclude that $\mathfrak{g}_2 \supseteq \langle X_1, X_2, JX_2 \rangle$, up to a change of basis. In fact, thanks to Proposition 4.1.16 one has that $3 \leq \dim \mathfrak{g}_2 \leq 5$. Taking into account Corollary 4.1.5, the only possibilities for the space $\mathfrak{g}_2$ are those in the statement above. □

Once the second term is determined, we need to calculate the rest of the ascending central series in order to obtain the complete description of our algebras.

## 4.3 Final terms and structure of the admissible algebras

We now study the rest of the terms in the ascending central series of our 8-dimensional NLAs $\mathfrak{g}$ admitting SnN complex structures $J$. Our starting point is the description of $\mathfrak{g}_2$ obtained in Corollary 4.2.2. The final result is a structure theorem given at the end of this section (see p. 153).

Recall that we are constructing a doubly adapted basis $\mathcal{B}$. From the beginning of Section 4.2, two of its elements, namely $X_1$ and $X_2$, belong to $\mathfrak{g}_2$. Indeed, their Lie brackets are given by (4.6). Moreover, $X_1$ actually belongs to $\mathfrak{g}_1$, so one has the expression

$$[JX_1, JY] = J[JX_1, Y], \quad \forall\, Y \in \mathfrak{g}, \tag{4.12}$$

as a direct consequence of the Nijenhuis condition (3.3). At the sight of Corollary 4.2.2, we know that also $JX_2 \in \mathfrak{g}_2$. Hence,

$$[X_k, JX_2] = b_{k2}^1\, X_1, \quad k = 3, 4,$$

with $b^1_{k2} \in \mathbb{R}$, for $k = 3, 4$, and applying the Nijenhuis condition we obtain

$$[JX_2, JX_k] = a^1_{2k}\, X_1 + (b^1_{2k} - b^1_{k2})\, JX_1.$$

To be sure that $[JX_2, JX_k] \in \mathfrak{g}_1$, we need to set $b^1_{k2} = b^1_{2k}$, for $k = 3, 4$. Together with (4.6), we then have the brackets

$$\begin{aligned}
&[X_1, Y] = 0, \ \ \forall\, Y \in \mathfrak{g}, \\
&[X_2, X_k] = a^1_{2k}\, X_1, \ \ k = 3, 4, \\
&[X_2, JX_1] = 0, \quad [X_2, JX_k] = b^1_{2k}\, X_1, \ \ k = 2, 3, 4, \\
&[X_k, JX_2] = b^1_{2k}\, X_1, \ \ k = 3, 4 \\
&[JX_1, JX_2] = 0, \quad [JX_2, JX_k] = a^1_{2k}\, X_1, \ \ k = 3, 4,
\end{aligned} \tag{4.13}$$

where $a^1_{2k}, b^1_{22}, b^1_{2k} \in \mathbb{R}$, for $k = 3, 4$. This is the starting point of our next results. For each description of $\mathfrak{g}_2$ given in Corollary 4.2.2, we study $\mathfrak{g}_3$ and $\mathfrak{g}_4$. Let us note that this is enough for our purposes, because neither quasi-filiform nor filiform NLAs admit complex structures (see [VR09] and [GR02]).

We start with a collection of lemmas which allow us to investigate the case $\dim \mathfrak{g}_2 = 3$ given in the statement of Corollary 4.2.2.

**Lemma 4.3.1.** *Let us suppose that* $(\mathfrak{g}, J)$ *is an* 8*-dimensional NLA endowed with an SnN complex structure. If* $\dim \mathfrak{g}_2 = 3$*, then* $\mathfrak{g}_3 \neq \mathfrak{g}_2 \oplus J\mathfrak{g}_1$ *(as vector spaces).*

*Proof.* A sketch of this proof is provided in Appendix A. We argue by contradiction. Let us suppose that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle.$$

Then, there is an element $Y \in \mathfrak{g}_4$ linearly independent with $X_i, JX_i$, $i = 1, 2$. We can take $X_3 = Y$ as an element of $\mathcal{B}$, in such a way that

$$\mathfrak{g}_4 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

Thanks to $JX_1 \in \mathfrak{g}_3$, we can set

$$[X_k, JX_1] = b^1_{k1}\, X_1 + b^2_{k1}\, X_2 + c^2_{k1}\, JX_2, \ \ k = 3, 4,$$

where $b^1_{k1}, b^2_{k1}, c^2_{k1} \in \mathbb{R}$, for $k = 3, 4$. Applying (4.12) for $k = 3, 4$, one gets $b^1_{k1} = 0$ in order to ensure the nilpotency of $\mathfrak{g}$. Hence, our Lie algebra is now defined by the brackets (4.13) and

$$\begin{aligned}
&[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2 + \alpha^1_{34}\, JX_1 + \alpha^2_{34}\, JX_2, \\
&[X_3, JX_k] = b^1_{3k}\, X_1 + b^2_{3k}\, X_2 + c^1_{3k}\, JX_1 + c^2_{3k}\, JX_2, \ \ k = 3, 4, \\
&[X_k, JX_1] = b^2_{k1}\, X_2 + c^2_{k1}\, JX_2, \ \ k = 3, 4, \\
&[JX_1, JX_k] = c^2_{k1}\, X_2 - b^2_{k1}\, JX_2, \ \ k = 3, 4,
\end{aligned} \tag{4.14}$$

where $a^i_{34}, \alpha^i_{34}, b^i_{3k}, c^i_{3k}, b^2_{k1}, c^2_{k1} \in \mathbb{R}$, for $i = 1, 2$ and $k = 3, 4$. Two options should be studied, depending on how the ascending central series continues.

Case 1: Let us consider that there is an element $Y \in \mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Take $X_4 = Y \in \mathcal{B}$. Observe that we are in the conditions of Lemma 4.1.1 *i)* with $l = 4$. Therefore,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

The Lie algebra $\mathfrak{g}$ is then defined via the brackets (4.13), (4.14), and

$$\begin{aligned} [X_4, JX_k] &= b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + c^1_{4k}\, JX_1 + c^2_{4k}\, JX_2, \quad k = 3, 4, \\ [JX_3, JX_4] &= (a^1_{34} - c^1_{34} + c^1_{43})\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 \\ &\quad + (\alpha^1_{34} + b^1_{34} - b^1_{43})\, JX_1 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2, \end{aligned} \tag{4.15}$$

where $b^i_{4k}, c^i_{4k} \in \mathbb{R}$, for $i = 1, 2$ and $k = 3, 4$. Simply note that the expression of $[JX_3, JX_4]$ comes from the Nijenhuis condition (3.3). Moreover, it is worth recalling that the previous parameters should preserve the ascending central series above.

Let us now study the Jacobi identity. We first consider $Jac\,(X_3, JX_3, X_4)$ and $Jac\,(X_3, JX_3, JX_4)$. Since the coefficients of $X_2$ and $JX_2$ must be zero, we need:

$$\left\{ \begin{aligned} c^1_{43} b^2_{31} - \alpha^1_{34} c^2_{31} &= c^1_{33} b^2_{41}, \\ \alpha^1_{34} b^2_{31} + c^1_{43} c^2_{31} &= c^1_{33} c^2_{41}, \\ (\alpha^1_{34} + b^1_{34} - b^1_{43}) b^2_{31} + c^1_{34} c^2_{31} &= c^1_{33} c^2_{41}, \\ c^1_{34} b^2_{31} - (\alpha^1_{34} + b^1_{34} - b^1_{43}) c^2_{31} &= c^1_{33} b^2_{41}. \end{aligned} \right. \tag{4.16}$$

Equalling the expressions in (4.16) whose right-hand side coincides, we get the homogeneous system:

$$\begin{pmatrix} c^1_{43} - c^1_{34} & b^1_{34} - b^1_{43} \\ -(b^1_{34} - b^1_{43}) & c^1_{43} - c^1_{34} \end{pmatrix} \begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let us denote this $2 \times 2$ matrix by $A$. We proceed analogously for $Jac\,(X_4, JX_4, X_3)$ and $Jac\,(X_4, JX_4, JX_3)$, obtaining

$$\left\{ \begin{aligned} c^1_{34} b^2_{41} + \alpha^1_{34} c^2_{41} &= c^1_{44} b^2_{31}, \\ -\alpha^1_{34} b^2_{41} + c^1_{34} c^2_{41} &= c^1_{44} c^2_{31}, \\ -(\alpha^1_{34} + b^1_{34} - b^1_{43}) b^2_{41} + c^1_{43} c^2_{41} &= c^1_{44} c^2_{31}, \\ c^1_{43} b^2_{41} + (\alpha^1_{34} + b^1_{34} - b^1_{43}) c^2_{41} &= c^1_{44} b^2_{31}. \end{aligned} \right. \tag{4.17}$$

Using the same idea as before, we find another homogeneous system:

$$A \begin{pmatrix} b^2_{41} \\ c^2_{41} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In this way, if the determinant of $A$ is different from zero, then $b^2_{31} = c^2_{31} = b^2_{41} = c^2_{41} = 0$. However, replacing these values in (4.13), (4.14), and (4.15), one observes that $JX_1 \in \mathfrak{g}_1$, which is not possible. Therefore, the determinant of $A$ equals to zero. This makes $c^1_{43} = c^1_{34}$ and $b^1_{43} = b^1_{34}$. Replacing these values in (4.16) and (4.17), we obtain the following systems of equations:

$$B\begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix} = c^1_{33}\begin{pmatrix} c^2_{41} \\ b^2_{41} \end{pmatrix}, \quad B\begin{pmatrix} c^2_{41} \\ b^2_{41} \end{pmatrix} = c^1_{44}\begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} \alpha^1_{34} & c^1_{34} \\ c^1_{34} & -\alpha^1_{34} \end{pmatrix}.$$

Their solutions depend on $B$:

- If the determinant of $B$ is zero, then $\alpha^1_{34} = c^1_{34} = 0$ (thus, $c^1_{43} = 0$). The equations to solve become

$$\begin{cases} c^1_{33}c^2_{41} = 0, \\ c^1_{33}b^2_{41} = 0, \end{cases} \qquad \begin{cases} c^1_{44}c^2_{31} = 0, \\ c^1_{44}b^2_{31} = 0. \end{cases}$$

  If $c^1_{33} = 0$, the fact that also $\alpha^1_{34} = c^1_{34} = 0$ implies that $X_3 \in \mathfrak{g}_3$, due to the brackets (4.13), (4.14), and (4.15). This is a contradiction.

  If $c^1_{33} \neq 0$, then $b^2_{41} = c^2_{41} = 0$ and we focus on the last two equations. If one assumes $c^1_{44} = 0$, then we can replace this value and $\alpha^1_{34} = c^1_{43} = 0$ in (4.13), (4.14), and (4.15), and one concludes that $X_4 \in \mathfrak{g}_3$. As this is not possible by hypothesis, we need $c^1_{44} \neq 0$. However, this implies $c^2_{31} = b^2_{31} = 0$, which joined to $b^2_{41} = c^2_{41} = 0$ yields $JX_1 \in \mathfrak{g}_1$. A new contradiction arises.

  This case can never hold.

- Let us assume that the determinant of $B$ is different from zero. Observe that this makes $(\alpha^1_{34}, c^1_{34}) \neq (0, 0)$. Furthermore, since $B$ is symmetric one can check that $B^{-1} = -\frac{1}{\det B}\, B$. From our two systems of equations we conclude

$$\begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix} = -\frac{c^1_{33}}{\det B}\begin{pmatrix} \alpha^1_{34} & c^1_{34} \\ c^1_{34} & -\alpha^1_{34} \end{pmatrix}\begin{pmatrix} c^2_{41} \\ b^2_{41} \end{pmatrix} = -\frac{c^1_{33}c^1_{44}}{\det B}\begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix}.$$

  Hence, $\det B = -\big((\alpha^1_{34})^2 + (c^1_{34})^2\big) = -c^1_{33}c^1_{44}$ and, in particular, $c^1_{33}c^1_{44} \neq 0$. Bearing in mind this relation among $\alpha^1_{34}$, $c^1_{34}$, $c^1_{33}$, and $c^1_{44}$, together with the equalities $b^1_{43} = b^1_{34}$ and $c^1_{43} = c^1_{34}$, one can see that the element $\alpha^1_{34}\, X_4 - c^1_{44}\, JX_3 + c^1_{34}\, JX_4 \neq 0$ belongs to $\mathfrak{g}_3$. This contradicts the assumption made at the beginning of the proof.

The study of Case 1 is now finished. Notice that it does not provide any valid case, as only contradictions have been obtained.

<u>Case 2</u>: Suppose that the elements in $\mathfrak{g}_4$ are linearly dependent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Two options can be found.

Case 2.1: First, let us assume that $JX_3 \in \mathfrak{g}_4$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle,$$
$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

Note that the brackets in this case coincide with those given in Case 1, with the only exception of

$$[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3,$$

with $b^i_{44}, c^i_{44} \in \mathbb{R}$, for $i = 1, 2, 3$. In particular, $(b^3_{44}, c^3_{44}) \neq (0, 0)$ is needed in order to avoid lying in Case 1.

Remark that only those Jacobi identities involving both $X_4$ and $JX_4$ need to be recalculated with respect to the previous case. We focus on

$$\begin{aligned} Jac\,(X_2, JX_1, X_3) &= b^1_{22}\, c^2_{31}\, X_1, & Jac\,(X_2, JX_1, X_4) &= b^1_{22}\, c^2_{41}\, X_1,\\ Jac\,(X_2, JX_1, JX_3) &= b^1_{22}\, b^2_{31}\, X_1, & Jac\,(X_2, JX_1, JX_4) &= b^1_{22}\, b^2_{41}\, X_1. \end{aligned}$$

If $b^1_{22} \neq 0$, then $c^2_{31} = b^2_{31} = c^2_{41} = b^2_{41} = 0$ and $JX_1 \in \mathfrak{g}_1$, but this is a contradiction. Therefore, $b^1_{22} = 0$. Now, let us consider:

$$\begin{aligned} Jac\,(X_4, JX_4, X_2) &= -\big(a^1_{23}\, b^3_{44} + b^1_{23}\, c^3_{44}\big)\, X_1,\\ Jac\,(X_4, JX_4, JX_2) &= -\big(a^1_{23}\, c^3_{44} - b^1_{23}\, b^3_{44}\big)\, X_1,\\ Jac\,(X_4, JX_4, JX_1) &= -\big(a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41}\big)\, X_1 \;+\; \big(b^2_{31}\, b^3_{44} - c^2_{31}\, c^3_{44}\big)\, X_2\\ &\quad +\; \big(b^2_{31}\, c^3_{44} + c^2_{31}\, b^3_{44}\big)\, JX_2. \end{aligned}$$

In particular, we have

$$\begin{pmatrix} b^3_{44} & c^3_{44}\\ c^3_{44} & -b^3_{44} \end{pmatrix} \begin{pmatrix} a^1_{23}\\ b^1_{23} \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}, \qquad \begin{pmatrix} b^3_{44} & -c^3_{44}\\ c^3_{44} & b^3_{44} \end{pmatrix} \begin{pmatrix} b^2_{31}\\ c^2_{31} \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}.$$

Since the determinant of these matrices is different from zero, we get $a^1_{23} = b^1_{23} = b^2_{31} = c^2_{31} = 0$. Considering these values, we obtain

$$\begin{aligned} Jac\,(X_3, JX_3, X_4)_{X_1} &= a^1_{24}\, b^2_{33} - b^1_{24}\, c^2_{33},\\ Jac\,(X_3, JX_3, JX_4)_{X_1} &= a^1_{24}\, c^2_{33} + b^1_{24}\, b^2_{33}, \end{aligned}$$

where the notation $Jac\,(X, Y, Z)_W$ means that we simply focus on the coefficient of $W$ in the expression of $Jac\,(X, Y, Z)$. If we equal to zero the expressions above, we obtain the following system of equations:

$$\begin{pmatrix} a^1_{24} & -b^1_{24}\\ b^1_{24} & a^1_{24} \end{pmatrix} \begin{pmatrix} b^2_{33}\\ c^2_{33} \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix}.$$

If the determinant of the matrix is zero, that means $a_{24}^1 = b_{24}^1 = 0$. As we already had $a_{23}^1 = b_{23}^1 = 0$, we could then conclude $X_2 \in \mathfrak{g}_1$, which is not possible. Hence, the determinant is non-zero and $b_{33}^2 = c_{33}^2 = 0$.

Let us now take

$$\begin{aligned} Jac\,(X_4, JX_4, X_3) &= \left(a_{24}^1\,(\alpha_{34}^2 - b_{34}^2) + b_{24}^1\,(a_{34}^2 + c_{34}^2) - b_{33}^1\,c_{44}^3\right) X_1 \\ &+ \left(b_{41}^2\,c_{34}^1 + c_{41}^2\,\alpha_{34}^1\right) X_2 \;-\; c_{33}^1\,c_{44}^3\,JX_1 \\ &- \left(b_{41}^2\,\alpha_{34}^1 - c_{41}^2\,c_{34}^1\right) JX_2, \end{aligned}$$

and we can consider the following system of equations:

$$\begin{pmatrix} c_{34}^1 & \alpha_{34}^1 \\ -\alpha_{34}^1 & c_{34}^1 \end{pmatrix} \begin{pmatrix} b_{41}^2 \\ c_{41}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are two options. If the determinant of the matrix is different from zero, then $b_{41}^2 = c_{41}^2 = 0$, but we have previously seen that this yields $JX_1 \in \mathfrak{g}_1$, which is a contradiction. Thus, we need $c_{34}^1 = \alpha_{34}^1 = 0$.

Furthermore, using $Jac\,(X_4, JX_4, X_3)_{JX_1}$ and $Jac\,(X_4, JX_4, JX_3)_{JX_1}$ we have the system of equations

$$\begin{cases} c_{33}^1\,c_{44}^3 = 0, \\ c_{33}^1\,b_{44}^3 = 0. \end{cases}$$

Taking into account that $(b_{44}^3, c_{44}^3) \neq (0,0)$, one concludes that $c_{33}^1 = 0$. However, this solution together with the previous ones leads to $X_3 \in \mathfrak{g}_3$. This contradicts our initial assumption $\mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2\rangle$.

<u>Case 2.2</u>: We now suppose that $JX_3 \notin \mathfrak{g}_4$. This implies that

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2\rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2\rangle,$$
$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2\rangle, \quad \mathfrak{g}_5 = \mathfrak{g},$$

as neither quasi-filiform nor filiform NLAs can admit complex structures [GR02, VR09], so our highest possible nilpotency step is 5.

Furthermore, we can use the brackets (4.13) and (4.14). Now, consider $X_4 \in \mathfrak{g}_5$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Then,

$$[X_4, JX_3] = b_{43}^1\,X_1 + b_{43}^2\,X_2 + b_{43}^3\,X_3 + c_{43}^1\,JX_1 + c_{43}^2\,JX_2,$$

where $b_{43}^i, b_{43}^3, c_{43}^i \in \mathbb{R}$, for $i = 1, 2$. Applying the Nijenhuis condition, one obtains

$$\begin{aligned} [JX_3, JX_4] &= (a_{34}^1 - c_{34}^1 + c_{43}^1)\,X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2)\,X_2 \\ &\quad + (\alpha_{34}^1 + b_{34}^1 - b_{43}^1)\,JX_1 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2)\,JX_2 - b_{43}^3\,JX_3. \end{aligned}$$

Since our Lie algebra $\mathfrak{g}$ is nilpotent, we necessarily have $b_{43}^3 = 0$. However, this choice makes that $JX_3 \in \mathfrak{g}_4$, which is not possible.

This completes the proof of the lemma. □

In the following lemmas we analyze the relative position of $J\mathfrak{g}_1$ with respect to $\mathfrak{g}_3$.

**Lemma 4.3.2.** *Let* $(\mathfrak{g}, J)$ *be an 8-dimensional NLA endowed with an SnN complex structure. If* $\dim \mathfrak{g}_2 = 3$ *and* $\mathfrak{g}_3 \cap J\mathfrak{g}_1 \neq \{0\}$*, then the ascending central series of* $\mathfrak{g}$ *is given by*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle,$$

*and one of the following cases:*

*i)* $\mathfrak{g}_3 = \mathfrak{g}$*;*

*ii)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \ \mathfrak{g}_4 = \mathfrak{g}$*;*

*Proof.* For the seek of clarity, see Appendix A for an outline of the proof. We here provide the complete discussion. Let us first observe that the brackets involving the elements $X_1$, $X_2$, and $JX_2$ of the doubly adapted basis $\mathcal{B} = \{X_k, JX_k\}_{k=1}^4$ we want to construct are given by (4.13). By hypothesis we have that $JX_1 \in \mathfrak{g}_3$, so by Lemma 4.3.1 there is also some element $Y \in \mathfrak{g}_3$ which is linearly independent with $X_i, JX_i$ for $i = 1, 2$. Take $X_3 = Y$, and

$$\mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

We note that one has

$$[X_k, JX_1] = b_{k1}^1\, X_1 + b_{k1}^2\, X_2 + c_{k1}^2\, JX_2, \ \ k = 3, 4,$$

where $b_{k1}^1, b_{k1}^2, c_{k1}^2 \in \mathbb{R}$, for $k = 3, 4$. Using (4.12) for each $k = 3, 4$ and the nilpotency of $\mathfrak{g}$, we obtain $b_{k1}^1 = 0$. Hence, our Lie algebra is determined by the brackets (4.13) and

$$\begin{aligned}
&[X_3, X_4] = a_{34}^1\, X_1 + a_{34}^2\, X_2 + \alpha_{34}^2\, JX_2, \quad [X_3, JX_1] = b_{31}^2\, X_2 + c_{31}^2\, JX_2,\\
&[X_3, JX_k] = b_{3k}^1\, X_1 + b_{3k}^2\, X_2 + c_{3k}^2\, JX_2, \ \ k = 3, 4\\
&[X_4, JX_1] = b_{41}^2\, X_2 + c_{41}^2\, JX_2,\\
&[JX_1, JX_3] = c_{31}^2\, X_2 - b_{31}^2\, JX_2, \quad [JX_1, JX_4] = c_{41}^2\, X_2 - b_{41}^2\, JX_2,
\end{aligned} \tag{4.18}$$

where all the coefficients are real numbers. If we keep on studying $\mathfrak{g}_3$, then two possibilities arise.

<u>Case 1</u>: Suppose that there is $Y \in \mathfrak{g}_3$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Take $X_4 = Y \in \mathcal{B}$. We have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, \ldots, X_4, JX_1, JX_2 \rangle.$$

The brackets are given by (4.13), (4.18), and

$$\begin{aligned}
&[X_4, JX_k] = b_{4k}^1\, X_1 + b_{4k}^2\, X_2 + c_{4k}^2\, JX_2, \ \ k = 3, 4,\\
&[JX_3, JX_4] = a_{34}^1\, X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2)\, X_2 + (b_{34}^1 - b_{43}^1)\, JX_1 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2)\, JX_2,
\end{aligned}$$

where $b^1_{4k}, b^2_{4k}, c^2_{4k} \in \mathbb{R}$, for $k = 3, 4$, Let us remark that the bracket $[JX_3, JX_4]$ comes from the Nijenhuis condition (3.3). Furthermore, it will determine where $JX_3$ and $JX_4$ enter the ascending central series: if $b^1_{43} = b^1_{34}$ then $JX_3, JX_4 \in \mathfrak{g}_3$ and $\mathfrak{g}_3 = \mathfrak{g}$; otherwise, $\mathfrak{g}_4 = \mathfrak{g}$. We will shortly see that only the first one of these two situations is valid.

If one computes the Jacobi identities

$$\begin{aligned}
Jac\,(X_3, JX_3, JX_4) &= \big(a^1_{23}(a^2_{34} - 2\,c^2_{34} + c^2_{43}) + a^1_{24}\,c^1_{33} - b^1_{23}(\alpha^2_{34} + 2\,b^2_{34} - b^2_{43}) \\
&\quad + b^1_{24}\,b^2_{33}\big)\,X_1 \; - \; b^2_{31}\,(b^1_{34} - b^1_{43})X_2 \; - \; c^2_{31}\,(b^1_{34} - b^1_{43})\,JX_2, \\
Jac\,(X_4, JX_3, JX_4) &= \big(-a^1_{23}\,c^2_{44} + a^1_{24}(a^2_{34} - c^2_{34} + 2\,c^2_{43}) - b^1_{23}\,b^2_{44} \\
&\quad - b^1_{24}(\alpha^2_{34} + b^2_{34} - 2\,b^2_{43})\big)\,X_1 \; - \; b^2_{41}\,(b^1_{34} - b^1_{43})\,X_2 \\
&\quad - \; c^2_{41}\,(b^1_{34} - b^1_{43})\,JX_2,
\end{aligned}$$

then one sees that the following equations must hold:

$$\begin{cases} b^2_{31}\,\big(b^1_{34} - b^1_{43}\big) = 0, \\ c^2_{31}\,\big(b^1_{34} - b^1_{43}\big) = 0, \end{cases} \qquad \begin{cases} b^2_{41}\,\big(b^1_{34} - b^1_{43}\big) = 0, \\ c^2_{41}\,\big(b^1_{34} - b^1_{43}\big) = 0. \end{cases}$$

If we take $b^2_{31} = b^2_{41} = c^2_{31} = c^2_{41} = 0$ and replace these values in the brackets, then we observe that indeed $JX_1 \in \mathfrak{g}_1$. This is a contradiction. Therefore, $b^1_{43} = b^1_{34}$ and

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

The brackets are now defined by (4.13), (4.18), and

$$\begin{aligned}
&[X_4, JX_3] = b^1_{34}\,X_1 + b^2_{43}\,X_2 + c^2_{43}\,JX_2, \\
&[X_4, JX_4] = b^1_{44}\,X_1 + b^2_{44}\,X_2 + c^2_{44}\,JX_2, \qquad (4.19) \\
&[JX_3, JX_4] = a^1_{34}\,X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\,X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\,JX_2,
\end{aligned}$$

where $b^1_{44}, b^2_{4k}, c^2_{4k} \in \mathbb{R}$, for $k = 1, 3, 4$. Let us also recall that the choice of parameters in (4.13), (4.18), and (4.19) should preserve the arrangement of the ascending central series. Moreover, the Jacobi identity still needs to be checked. Let us start remarking that

$$\begin{aligned}
Jac\,(X_3, JX_1, X_2) &= -b^1_{22}\,c^2_{31}\,X_1, & Jac\,(JX_1, JX_3, X_2) &= b^1_{22}\,b^2_{31}\,X_1, \\
Jac\,(X_4, JX_1, X_2) &= -b^1_{22}\,c^2_{41}\,X_1, & Jac\,(JX_1, JX_4, X_2) &= b^1_{22}\,b^2_{41}\,X_1.
\end{aligned}$$

As we have seen, $(b^2_{31}, c^2_{31}, b^2_{41}, c^2_{41}) \neq (0,0,0,0)$ so one should take $b^1_{22} = 0$. The remaining Jacobi identities give a quadratic system with 8 equations and 22 unknowns (see Lemma B.0.1 in Appendix B). One observes that it is possible to find a particular solution preserving the fixed dimension of the ascending central series. This gives the case *i)* in the statement of the lemma.

<u>Case 2</u>: Let us now suppose that we cannot find a new element in $\mathfrak{g}_3$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. New possibilities arise.

<u>Case 2.1</u>: Let us suppose that $JX_3 \in \mathfrak{g}_3$. Then, applying the nilpotency of $\mathfrak{g}$ and Lemma 4.1.1 *i)* for $l = 3$, we see that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Let us check that this is a valid case. We can set the bracket

$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2,$$

with $b^1_{43}, b^2_{43}, c^2_{43} \in \mathbb{R}$, and applying the Nijenhuis condition

$$[JX_3, JX_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2.$$

Since this last bracket must lie in $\mathfrak{g}_2$, necessarily $b^1_{43} = b^1_{34}$. In this way, the Lie algebra $\mathfrak{g}$ is defined by the brackets (4.13), (4.18), and

$$\begin{aligned}
&[X_4, JX_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2,\\
&[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3,\\
&[JX_3, JX_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2,
\end{aligned} \tag{4.20}$$

where all the coefficients are real numbers preserving the ascending central series above. Next, we proceed to verify the Jacobi identity.

First, let us observe that the only difference between the previous expressions and (4.19) is the bracket $[X_4, JX_4]$. Thus, we can repeat the same argument as before in order to conclude that $b^1_{22} = 0$. In fact, only those identities involving both $X_4$ and $JX_4$ need to be recalculated. Let us start with the following ones:

$$\begin{aligned}
Jac\,(X_4, JX_4, X_2) &= -\big(a^1_{23}\, b^3_{44} + b^1_{23}\, c^3_{44}\big)\, X_1,\\
Jac\,(X_4, JX_4, JX_1) &= -\,2\,\big(a^1_{24}\, c^1_{41} + b^1_{24}\, b^2_{41}\big)\, X_1 \;+\; \big(b^2_{31}\, b^3_{44} - c^2_{31}\, c^3_{44}\big)\, X_2\\
&\quad +\; \big(b^2_{31}\, c^3_{44} + c^2_{31}\, b^3_{44}\big)\, JX_2,\\
Jac\,(X_4, JX_4, JX_2) &= -\big(a^1_{23}\, c^3_{44} - b^1_{23}\, b^3_{44}\big) X_1.
\end{aligned}$$

From here, we can built two systems of equations:

$$\begin{pmatrix} b^3_{44} & c^3_{44} \\ c^3_{44} & -b^3_{44} \end{pmatrix} \begin{pmatrix} a^1_{23} \\ b^1_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} b^3_{44} & c^3_{44} \\ c^3_{44} & -b^3_{44} \end{pmatrix} \begin{pmatrix} c^2_{31} \\ b^2_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will discuss their solutions according to the determinant of the matrix and see that they lead to the same conditions on the parameters of the brackets.

- First, let us suppose that the determinant is non-zero. That means that $(b^3_{44}, c^3_{44}) \neq (0,0)$. Then, we have
$$a^1_{23} = b^1_{23} = b^2_{31} = c^2_{31} = 0.$$
If we substitute these values in the brackets and compute $Jac\,(X_4, JX_4, X_3)$ and $Jac\,(X_4, JX_4, JX_3)$, a new system of equations is obtained:
$$\begin{cases} c^3_{44}\, b^2_{33} = 0, \\ c^3_{44}\, c^2_{33} = 0, \end{cases} \qquad \begin{cases} b^3_{44}\, b^2_{33} = 0, \\ b^3_{44}\, c^2_{33} = 0. \end{cases}$$
Due to the initial hypothesis, we conclude $b^2_{33} = c^2_{33} = 0$.

- Let us now assume that the determinant equals zero, which implies that $b^3_{44} = c^3_{44} = 0$. In order to avoid lying in Case 1, we need $c^1_{44} \neq 0$. Replacing these values in the brackets and considering $Jac\,(X_4, JX_4, X_3)$, a new system of equations can be found:
$$\begin{cases} c^1_{44}\, b^2_{31} = 0, \\ c^1_{44}\, c^2_{31} = 0. \end{cases}$$
The condition $c^1_{44} \neq 0$ yields $b^2_{31} = c^2_{31} = 0$.

  If we now substitute these values in the expressions of $Jac\,(X_3, X_4, JX_1)$ and $Jac\,(X_3, JX_1, JX_4)$, then we get the system
$$\begin{pmatrix} c^2_{41} & b^2_{41} \\ b^2_{41} & -c^2_{41} \end{pmatrix} \begin{pmatrix} a^1_{23} \\ b^1_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
If the determinant of the previous matrix equals zero, then $b^2_{41} = c^2_{41} = 0$. However, this implies that $JX_1 \in \mathfrak{g}_1$ which is a contradiction. Therefore, the determinant cannot be zero and hence $a^1_{23} = b^1_{23} = 0$.

  Replacing this in $Jac\,(X_3, JX_3, X_4)$ and $Jac\,(X_3, JX_3, JX_4)$, we obtain a new system of equations:
$$\begin{pmatrix} a^1_{24} & -b^1_{24} \\ b^1_{24} & a^1_{24} \end{pmatrix} \begin{pmatrix} b^2_{33} \\ c^2_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
If the determinant equals zero, then $a^1_{24} = b^1_{24} = 0$ and $X_2, JX_2 \in \mathfrak{g}_1$, which is again a contradiction. Thus, $(a^1_{24}, b^1_{24}) \neq (0,0)$ and $b^2_{33} = c^2_{33} = 0$.

Therefore, we are led to the vanishing of the following parameters in the brackets (4.13), (4.18), and (4.20):
$$a^1_{23} = b^1_{22} = b^1_{23} = b^2_{31} = b^2_{33} = c^2_{31} = c^2_{33} = 0.$$
The remaining 19 coefficients should preserve the ascending central series fixed at the beginning of this case and satisfy the 3 equations given by the Jacobi identities (see

Lemma B.0.2 in Appendix B). Here, we simply note that it is possible to give a particular solution fulfilling the two previous requirements. This leads to case *ii)* in the statement.

Case 2.2: Finally, let us assume that $JX_3 \notin \mathfrak{g}_3$, i.e.,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle,$$

with brackets defined by (4.13) and (4.18). Let us see that this case is not possible.

Case 2.2.1: Assume that we can find an element $Y \in \mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. We can choose $X_4 = Y$ as an element of $\mathcal{B}$. Setting

$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + b^3_{43}\, X_3 + c^1_{43}\, JX_1 + c^2_{43}\, JX_2,$$

where $b^i_{43}, b^3_{43}, c^i_{43} \in \mathbb{R}$, for $i = 1, 2$, and applying the Nijenhuis condition (3.3), we obtain

$$\begin{aligned}[JX_3, JX_4] = {} & (a^1_{34} + c^1_{43})\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 \\ & + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2 - b^3_{43}\, JX_3.\end{aligned}$$

Necessarily $b^3_{43} = 0$, in order to ensure that the Lie algebra is nilpotent. In this way, we have Lie brackets (4.13), (4.18), and

$$\begin{aligned}
[X_4, JX_3] &= b^1_{43}\, X_1 + b^2_{43}\, X_2 + c^1_{43}\, JX_1 + c^2_{43}\, JX_2, \\
[X_4, JX_4] &= b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2, \\
[JX_3, JX_4] &= (a^1_{34} + c^1_{43})\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 \\
&\quad + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2,
\end{aligned}$$

with $b^i_{4k}, b^3_{44}, c^i_{4k} \in \mathbb{R}$, for $i = 1, 2$ and $k = 3, 4$. In particular, observe that also $JX_3, JX_4 \in \mathfrak{g}_4$, so we conclude

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Simply recall that the parameters in the brackets must preserve this arrangement for the ascending central series.

Let us study the Jacobi identity in order to get a contradiction. We first note that $Jac\,(X_2, JX_1, X_3)$, $Jac\,(X_2, JX_1, JX_3)$, $Jac\,(X_2, JX_1, X_4)$, and $Jac\,(X_2, JX_1, JX_4)$ lead to the following system of equations:

$$\left\{ \begin{array}{l} b^1_{22}\, c^2_{31} = 0, \\ b^1_{22}\, b^2_{31} = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} b^1_{22}\, c^2_{41} = 0, \\ b^1_{22}\, b^2_{41} = 0. \end{array} \right.$$

If $b^2_{31} = c^2_{31} = b^2_{41} = c^2_{41} = 0$ then $JX_1 \in \mathfrak{g}_1$, which is not possible. Therefore, we need $b^1_{22} = 0$. Next, we compute:

$$\begin{array}{llll}
Jac\,(X_3, JX_3, X_4)_{X_2} & = b^2_{31}\, c^1_{43}, & Jac\,(X_3, JX_3, X_4)_{JX_2} & = c^2_{31}\, c^1_{43}, \\
Jac\,(X_3, JX_3, JX_4)_{X_2} & = -b^2_{31}(b^1_{34} - b^1_{43}), & Jac\,(X_3, JX_3, JX_4)_{JX_2} & = -c^2_{31}(b^1_{34} - b^1_{43}), \\
Jac\,(X_4, JX_4, X_3)_{X_2} & = -b^2_{31}\, c^1_{44}, & Jac\,(X_4, JX_4, X_3)_{JX_2} & = -c^2_{31}\, c^1_{44}, \\
Jac\,(X_4, JX_4, JX_1)_{X_2} & = b^2_{31}\, b^3_{44}, & Jac\,(X_4, JX_4, JX_1)_{JX_2} & = c^2_{31}\, b^3_{44}.
\end{array}$$

If we assume that $(b^2_{31}, c^2_{31}) \neq (0,0)$, then we get $c^1_{43} = c^1_{44} = b^3_{44} = 0$ and $b^1_{43} = b^1_{34}$. These choices imply that $X_4, JX_4 \in \mathfrak{g}_3$, which is not possible. Hence, $b^2_{31} = c^2_{31} = 0$.

Taking into account the previous values, we have:

$$\begin{aligned} Jac\,(X_3, JX_1, X_4) &= -\big(a^1_{23}\, b^2_{41} - b^1_{23}\, c^2_{41}\big)\, X_1, \\ Jac\,(X_3, JX_1, JX_4) &= \big(a^1_{23}\, c^2_{41} + b^1_{23}\, b^2_{41}\big)\, X_1. \end{aligned}$$

A new homogeneous system of equations needs to be solved, namely,

$$\begin{pmatrix} b^2_{41} & -c^2_{41} \\ c^2_{41} & b^2_{41} \end{pmatrix} \begin{pmatrix} a^1_{23} \\ b^1_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant of the previous matrix equals zero, it means that $b^2_{41} = c^2_{41} = 0$. Nevertheless, this implies $JX_1 \in \mathfrak{g}_1$ which is not possible by hypothesis. Therefore, the determinant cannot be zero and $a^1_{23} = b^1_{23} = 0$.

Replacing these two new values in $Jac\,(X_3, JX_3, X_4)_{X_1}$ and $Jac\,(X_3, JX_3, JX_4)_{X_1}$, one reaches the system

$$\begin{pmatrix} b^2_{33} & -c^2_{33} \\ c^2_{33} & b^2_{33} \end{pmatrix} \begin{pmatrix} a^1_{24} \\ b^1_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant is different from zero, then $a^1_{24} = b^1_{24} = 0$ and $X_2 \in \mathfrak{g}_1$. Since this contradicts our construction, we necessarily have $b^2_{33} = c^2_{33} = 0$.

We now calculate

$$\begin{aligned} Jac\,(X_4, JX_3, JX_4) = &\ \big(a^1_{24}\,(a^2_{34} - c^2_{34} + 2c^2_{43}) - b^1_{24}\,(\alpha^2_{34} + b^2_{34} - 2b^2_{43}) - b^1_{33}\, b^3_{44}\big)\, X_1 \\ &- \big(b^2_{41}\,(b^1_{34} - b^1_{43}) - c^2_{41}\, c^1_{43}\big)\, X_2 \\ &- \big(b^2_{41}\, c^1_{43} + c^2_{41}\,(b^1_{34} - b^1_{43})\big)\, JX_2, \end{aligned}$$

using the brackets that define our algebra and taking into consideration the choices of the parameters that we have already made. Consider the system of equations obtained when we equal to zero the expressions accompanying $X_2$ and $JX_2$. We have seen that $b^2_{41} = c^2_{41} = 0$ leads to $JX_1 \in \mathfrak{g}_1$, which is a contradiction. Therefore, the only option is having $c^1_{43} = 0$ and $b^1_{43} = b^1_{34}$. If we look again at the brackets, this choice makes $JX_3 \in \mathfrak{g}_3$. Another contradiction is obtained, so this case is not valid.

<u>Case 2.2.2</u>: Let us now suppose the converse, that is, there is no element in $\mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Then, the algebra must be 5-step nilpotent, and

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle,$$

$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

In fact, we can follow the same ideas as in Case 2.2.1 and conclude that the only different bracket in these two situations is

$$[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3,$$

where $b^i_{44}, c^i_{44} \in \mathbb{R}$, for $i = 1, 2, 3$. Indeed, we need $c^3_{44} \neq 0$ so as not to recover the previous case.

Next, we study the Jacobi identities. Let us remark that only those involving both $X_4$ and $JX_4$ will change with respect to Case 2.2.1. Therefore, we will follow the same stages, just modifying the necessary expressions.

Since none of the first Jacobi identities contains both $X_4$ and $JX_4$, we can immediately conclude $b^1_{22} = 0$. Regarding the second set of expressions, only two of them need to be recalculated:

$$\begin{aligned}
Jac\,(X_4, JX_4, X_3) \quad &= \quad \big(a^1_{23}\, b^2_{44} + a^1_{24}(\alpha^2_{34} - b^2_{34}) - b^1_{23}\, c^2_{44} + b^1_{24}(a^2_{34} + c^2_{34}) \\
&\quad - b^1_{33}\, c^3_{44}\big)\, X_1 \;-\; \big(b^2_{31}\, c^1_{44} + b^2_{33}\, c^3_{44}\big)\, X_2 \\
&\quad - \;\big(c^2_{31}\, c^1_{44} + c^2_{33}\, c^3_{44}\big)\, JX_2, \\
Jac\,(X_4, JX_4, JX_1) \quad &= \quad -2\big(a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41}\big)\, X_1 \;+\; \big(b^2_{31}\, b^3_{44} - c^2_{31}\, c^3_{44}\big)\, X_2 \\
&\quad + \;\big(c^2_{31}\, b^3_{44} + b^2_{31}\, c^3_{44}\big)\, JX_2.
\end{aligned}$$

Let us start focusing on the equations

$$\begin{pmatrix} b^3_{44} & -c^3_{44} \\ c^3_{44} & b^3_{44} \end{pmatrix} \begin{pmatrix} b^2_{31} \\ c^2_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which come from equalling $Jac\,(X_4, JX_4, JX_1)$ to zero. As $c^3_{44} \neq 0$ the determinant of the previous matrix never vanishes, and we get $b^2_{31} = c^2_{31} = 0$. Notice that these solutions coincide with those obtained in Case 2.2.1. Furthermore, we additionally conclude that $b^2_{33} = c^2_{33} = 0$, simply replacing the previous values in $Jac\,(X_4, JX_4, X_3)$.

Let us observe that $Jac\,(X_3, JX_1, X_4)$ and $Jac\,(X_3, JX_1, JX_4)$ remain the same with respect to Case 2.2.1. Therefore, the same argument can be applied, and we conclude that $a^1_{23} = b^1_{23} = 0$.

If we finally recalculate the Jacobi identity $Jac\,(X_4, JX_3, JX_4)$, we realize that it coincides with that given in Case 2.2.1 (the only modification comes from the triple $\big[[X_4, JX_4], JX_3\big]$ and more precisely, from the Lie bracket involving the term $JX_3$ in $[X_4, JX_4]$ with $JX_3$, which is zero). Using the same ideas as before, we reach another contradiction.

This concludes the proof of the lemma. □

**Lemma 4.3.3.** *Let* $(\mathfrak{g}, J)$ *be an* 8*-dimensional NLA endowed with an SnN complex structure. If* $\dim \mathfrak{g}_2 = 3$ *and* $\mathfrak{g}_3 \cap J\mathfrak{g}_1 = \{0\}$*, then the ascending central series of* $\mathfrak{g}$ *is given by*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle,$$

*and one of the following cases:*

*i)* $\mathfrak{g}_4 = \mathfrak{g}$*;*

*ii)* $\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3\rangle, \quad \mathfrak{g}_5 = \mathfrak{g}$*;*

*Proof.* See Appendix A for a plan of the following lines. We recall that the brackets involving the elements of the doubly adapted basis $\mathcal{B} = \{X_k, JX_k\}_{k=1}^4$ which belong to $\mathfrak{g}_2$ follow (4.13). Since $\mathfrak{g}$ is nilpotent and $JX_1 \notin \mathfrak{g}_3$, there exists an element $Y \in \mathfrak{g}_3$ which is linearly independent with $X_i, JX_i$ for $i = 1, 2$. Take $X_3 = Y \in \mathcal{B}$, and we set

$$[X_3, JX_1] = b_{31}^1\, X_1 + b_{31}^2\, X_2 + c_{31}^2\, JX_2,$$

where $b_{31}^1, b_{31}^2, c_{31}^2 \in \mathbb{R}$. Applying (4.12), we get

$$[JX_1, JX_3] = c_{31}^2\, X_2 - b_{31}^1\, JX_1 - b_{31}^2\, JX_2.$$

The nilpotency of $\mathfrak{g}$ yields $b_{31}^1 = 0$. Therefore, we have

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2\rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_2\rangle,$$

with brackets given by (4.13) and

$$\begin{aligned}
&[X_3, X_4] = a_{34}^1\, X_1 + a_{34}^2\, X_2 + \alpha_{34}^2\, JX_2,\\
&[X_3, JX_1] = b_{31}^2\, X_2 + c_{31}^2\, JX_2,\\
&[X_3, JX_k] = b_{3k}^1\, X_1 + b_{3k}^2\, X_2 + c_{3k}^2\, JX_2, \quad k = 3, 4,\\
&[JX_1, JX_3] = c_{31}^2\, X_2 - b_{31}^2\, JX_2,
\end{aligned} \tag{4.21}$$

where all the coefficients are real numbers. We next observe the following.

If there exists $Y \in \mathfrak{g}_3$ linearly independent with $X_i, JX_i$ for $i = 1, 2, 3$, then we take $X_4 = Y \in \mathcal{B}$ and set

$$[X_4, JX_1] = b_{41}^1\, X_1 + b_{41}^2\, X_2 + c_{41}^2\, JX_2,$$

where $b_{41}^1, b_{41}^2, c_{41}^2 \in \mathbb{R}$. By equation (4.12), we get

$$[JX_1, JX_4] = c_{41}^2\, X_2 - b_{41}^1\, JX_1 - b_{41}^2\, JX_2,$$

and the nilpotency of the Lie algebra requires $b_{41}^1 = 0$. In view of these two brackets together with (4.13) and (4.21), one concludes that $JX_1 \in \mathfrak{g}_3$. However, this is not possible by the hypothesis of the lemma. Hence, we may assume that we cannot find a new element in $\mathfrak{g}_3$ being linearly independent with $X_i, JX_i$ for $i = 1, 2, 3$.

It suffices to study whether $JX_3$ can belong to $\mathfrak{g}_3$ or not.

<u>Case 1</u>: Let us assume that $JX_3 \notin \mathfrak{g}_3$. We have

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2\rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2\rangle,$$

and two possibilities arise. Let us see that both of them give invalid cases.

Case 1.1: Suppose that one can find a new generator $Y \in \mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Then, we take $X_4 = Y \in \mathcal{B}$ and have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_4 \supseteq \langle X_1, \ldots, X_4, JX_2 \rangle.$$

Let us fix the brackets

$$[X_4, JX_k] = b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + b^3_{4k}\, X_3 + c^2_{4k}\, JX_2, \;\; k = 1, 3, 4,$$

where $b^i_{4k}, c^2_{4k} \in \mathbb{R}$, for $i = 1, 2, 3$ and $k = 1, 3, 4$. If we apply the Nijenhuis condition for $k = 1, 3$ using the previous brackets and those defined by (4.13) and (4.21), we obtain

$$\begin{aligned} [JX_1, JX_4] &= c^2_{41}\, X_2 - b^1_{41}\, JX_1 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3, \\ [JX_3, JX_4] &= a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 \\ &\qquad + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2 - b^3_{43}\, JX_3. \end{aligned}$$

Since the Lie algebra is nilpotent, we should take $b^1_{41} = b^3_{43} = 0$. In this way, $\mathfrak{g}$ is defined by the real brackets (4.13), (4.21), and

$$\begin{aligned} &[X_4, JX_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2, \\ &[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2, \\ &[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2, \\ &[JX_1, JX_4] = c^2_{41}\, X_2 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3, \\ &[JX_3, JX_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, JX_2. \end{aligned}$$

Although all the brackets of $\mathfrak{g}$ are now defined, we have not fixed yet where $JX_1, JX_3$, and $JX_4$ enter the series. Nonetheless, since the values of the parameters depend on the arrangement of the series, this is an important task to be done.

First, if we suppose that $JX_4 \in \mathfrak{g}_4$ then $b^3_{41} = 0$ and $b^1_{43} = b^1_{34}$. Notice that this implies $JX_1, JX_3 \in \mathfrak{g}_3$, which is not possible by construction. Therefore, $JX_4 \notin \mathfrak{g}_4$. This fact implies that $\mathfrak{g}_5 = \mathfrak{g}$. Applying Lemma 4.1.1 *ii)*, either $JX_1$ or $JX_3$ belong to $\mathfrak{g}_4$ (up to an arrangement of generators).

If $JX_1 \in \mathfrak{g}_4$ (and $JX_3, JX_4 \in \mathfrak{g}_5$), then we can conclude that $b^3_{41} = 0$ from the fact that $[JX_1, JX_4] \in \mathfrak{g}_3$. However, this makes that $JX_1$ actually belongs to $\mathfrak{g}_3$. This is a contradiction with the hypothesis given in the statement of the lemma.

If $JX_3 \in \mathfrak{g}_4$ (and $JX_1, JX_4 \in \mathfrak{g}_5$), then we get $b^1_{43} = b^1_{34}$ from $[JX_3, JX_4] \in \mathfrak{g}_3$. Nonetheless, this causes that $JX_3$ descends to $\mathfrak{g}_3$, which contradicts the assumption at the beginning of Case 1.

Case 1.2: Let us assume that there is no element in $\mathfrak{g}_4$ linearly independent with $X_i, JX_i$, for $i = 1, 2, 3$. Then, up to an arrangement of generators, we have $JX_1 \in \mathfrak{g}_4$

or $JX_3 \in \mathfrak{g}_4$ (even both) as the ascending central series should strictly increase until it reaches the algebra $\mathfrak{g}$. Observe that if both belong to $\mathfrak{g}_4$, then $\mathfrak{g}_4 = \mathfrak{g}$.

If we assume that $JX_3 \in \mathfrak{g}_4$, then we can take the same bracket $[X_4, JX_3]$ as at the beginning of Case 1.1 and apply Nijenhuis. The expression for $[JX_3, JX_4]$ coincides with the first one given above. Since in this new case $[JX_3, JX_4]$ lies in $\mathfrak{g}_3$, we need $b_{43}^3 = 0$ and $b_{43}^1 = b_{34}^1$. Let us note that this choice makes $JX_3 \in \mathfrak{g}_3$, which turns to contradict the assumption of Case 1. Thus $JX_3 \notin \mathfrak{g}_4$, and necessarily $JX_1 \in \mathfrak{g}_4$. However, a similar argument can be applied, leading to $JX_1 \in \mathfrak{g}_3$. This again entails a contradiction (due to the hypothesis in the statement of the lemma).

We have finished the study of all the possibilities contained in Case 1. We now analyze the opposite situation.

<u>Case 2</u>: Let us now suppose that $JX_3 \in \mathfrak{g}_3$. This makes

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

Then, we can set the bracket

$$[X_4, JX_3] = b_{43}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2,$$

with $b_{43}^1, b_{43}^2, c_{43}^2 \in \mathbb{R}$, and apply the Nijenhuis condition

$$[JX_3, JX_4] = a_{34}^1 X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (b_{34}^1 - b_{43}^1) JX_1 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2.$$

Since this last bracket must lie in $\mathfrak{g}_2$, necessarily $b_{43}^1 = b_{34}^1$. In this way, the Lie algebra $\mathfrak{g}$ is given by the brackets (4.13), (4.21), and

$$\begin{aligned} &[X_4, JX_3] = b_{34}^1 X_1 + b_{43}^2 X_2 + c_{43}^2 JX_2, \\ &[JX_3, JX_4] = a_{34}^1 X_1 + (a_{34}^2 - c_{34}^2 + c_{43}^2) X_2 + (\alpha_{34}^2 + b_{34}^2 - b_{43}^2) JX_2, \end{aligned} \tag{4.22}$$

where $b_{43}^2, c_{43}^2 \in \mathbb{R}$.

We should now focus on $\mathfrak{g}_4$. Let us note that we are in the conditions of Lemma 4.1.2. Two new options arise, according to parts *i)* and *ii)* of the cited result.

<u>Case 2.1</u>: Assume that there is an element $Y \in \mathfrak{g}_4$ linearly independent with $X_1$, $X_2$, $X_3$, $JX_1$, $JX_2$, and $JX_3$. We take $X_4 = Y$ in the doubly adapted basis $\mathcal{B}$. As a consequence of Lemma 4.1.2 *i)*, we immediately have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Let us see that this is a valid case. We can take

$$[X_4, JX_1] = b_{41}^1 X_1 + b_{41}^2 X_2 + b_{41}^3 X_3 + c_{41}^2 JX_2 + c_{41}^3 JX_3,$$

with $b_{41}^1, b_{41}^i, c_{41}^i \in \mathbb{R}$, for $i = 2, 3$. As a consequence of (4.12) we will also have

$$[JX_1, JX_4] = c_{41}^2 X_2 + c_{41}^3 X_3 - b_{41}^1 JX_1 - b_{41}^2 JX_2 - b_{41}^3 JX_3.$$

The nilpotency of $\mathfrak{g}$ makes $b^1_{41} = 0$. In particular, we notice that $\mathfrak{g}$ is defined by (4.13), (4.21), (4.22), and

$$
\begin{aligned}
[X_4, JX_1] &= b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3,\\
[X_4, JX_4] &= b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3,\\
[JX_1, JX_4] &= c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3,
\end{aligned}
\tag{4.23}
$$

where $b^1_{44}, b^i_{4k}, c^i_{4k} \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. Furthermore, the parameters in these expressions should preserve the arrangement of the ascending central series. We turn our attention to the Jacobi identity.

First, let us consider

$$
\begin{aligned}
Jac\,(X_3, X_4, X_2) &= -\alpha^2_{34}\, b^1_{22}\, X_1, & Jac\,(X_3, JX_3, X_2) &= -c^2_{33}\, b^1_{22}\, X_1,\\
Jac\,(X_3, X_4, JX_2) &= a^2_{34}\, b^1_{22}\, X_1, & Jac\,(X_3, JX_3, JX_2) &= b^2_{33}\, b^1_{22}\, X_1,\\
Jac\,(X_2, JX_1, X_3) &= c^2_{31}\, b^1_{22}\, X_1, & Jac\,(X_3, JX_4, X_2) &= -c^2_{34}\, b^1_{22}\, X_1,\\
Jac\,(X_2, JX_1, JX_3) &= b^2_{31}\, b^1_{22}\, X_1, & Jac\,(X_3, JX_4, JX_2) &= b^2_{34}\, b^1_{22}\, X_1.
\end{aligned}
$$

If $\alpha^2_{34} = a^2_{34} = c^2_{31} = b^2_{31} = c^2_{33} = b^2_{33} = c^2_{34} = b^2_{34} = 0$, then one observes that $X_3 \in \mathfrak{g}_2$. This contradicts our assumption, so we necessarily have $b^1_{22} = 0$. Now, we calculate:

$$
\begin{aligned}
Jac\,(X_3, JX_1, X_4)_{X_2} &= b^2_{33}\, c^3_{41}, & Jac\,(X_3, JX_1, X_4)_{JX_2} &= c^2_{33}\, c^3_{41},\\
Jac\,(X_3, JX_1, JX_4)_{X_2} &= b^2_{33}\, b^3_{41}, & Jac\,(X_3, JX_1, JX_4)_{JX_2} &= c^2_{33}\, b^3_{41},\\
Jac\,(X_4, JX_4, X_3)_{X_2} &= -b^2_{33}\, c^3_{44}, & Jac\,(X_4, JX_4, X_3)_{JX_2} &= -c^2_{33}\, c^3_{44},\\
Jac\,(X_4, JX_4, JX_3)_{X_2} &= b^2_{33}\, b^3_{44}, & Jac\,(X_4, JX_4, JX_3)_{JX_2} &= c^2_{33}\, b^3_{44}.
\end{aligned}
$$

If we take $(b^2_{33}, c^2_{33}) \neq (0, 0)$, then $b^3_{41} = c^3_{41} = b^3_{44} = c^3_{44} = 0$. Nonetheless, this would imply that $X_4, JX_1, JX_4 \in \mathfrak{g}_3$, which contradicts our hypothesis. Therefore, we must have $b^2_{33} = c^2_{33} = 0$.

Next, using that $b^1_{22} = b^2_{33} = c^2_{33} = 0$, let us consider

$$
\begin{aligned}
Jac\,(X_2, JX_1, X_4) &= \left(a^1_{23}\, b^3_{41} + b^1_{23}\, c^3_{41}\right) X_1,\\
Jac\,(X_2, JX_1, JX_4) &= -\left(a^1_{23}\, c^3_{41} - b^1_{23}\, b^3_{41}\right) X_1,\\
Jac\,(X_4, JX_4, X_2) &= -\left(a^1_{23}\, b^3_{44} + b^1_{23}\, c^3_{44}\right) X_1,\\
Jac\,(X_4, JX_4, JX_2) &= -\left(a^1_{23}\, c^3_{44} - b^1_{23}\, b^3_{44}\right) X_1.
\end{aligned}
$$

Two systems of equations can be built:

$$
\begin{pmatrix} a^1_{23} & b^1_{23} \\ -b^1_{23} & a^1_{23} \end{pmatrix} \begin{pmatrix} b^3_{41} \\ c^3_{41} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a^1_{23} & b^1_{23} \\ -b^1_{23} & a^1_{23} \end{pmatrix} \begin{pmatrix} b^3_{44} \\ c^3_{44} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

If the determinant of the matrix is different from zero, then $b_{41}^3 = c_{41}^3 = b_{44}^3 = c_{44}^3 = 0$, but we have seen that this is not a valid choice. Hence, we need $a_{23}^1 = b_{23}^1 = 0$.

Bearing in mind the previous discussion, the coefficients of the brackets are reduced. In fact, only 22 unknowns are left, and they should satisfy the system of equations derived from the remaining Jacobi identities (see Lemma B.0.3 in Appendix B). We remark that one can find particular solutions satisfying the system and preserving the fixed dimension of the ascending central series. This leads to case *i)*.

Case 2.2: Suppose that there is no element $Y \in \mathfrak{g}_4$ linearly independent with $X_i, JX_i$, for $i = 1, 2, 3$. By Lemma 4.1.2 *ii)*, the ascending central series has the form

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle,$$
$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

Observe that we can take the same bracket $[X_4, JX_1]$ as in Case 2.1 and repeat the reasoning. In this way, we conclude that $\mathfrak{g}$ is defined by the brackets (4.13), (4.21), (4.22), and

$$\begin{aligned}
&[X_4, JX_1] = b_{41}^2\, X_2 + b_{41}^3\, X_3 + c_{41}^2\, JX_2 + c_{41}^3\, JX_3,\\
&[X_4, JX_4] = b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^1\, JX_1 + c_{44}^2\, JX_2 + c_{44}^3\, JX_3,\\
&[JX_1, JX_4] = c_{41}^2\, X_2 + c_{41}^3\, X_3 - b_{41}^2\, JX_2 - b_{41}^3\, JX_3,
\end{aligned}$$

where $b_{44}^1, b_{4k}^i, c_{44}^1, c_{4k}^i \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. Moreover, we need $c_{44}^1 \neq 0$ so as not to recover Case 2.1.

We next study the Jacobi identity. First, let us remark that only those expressions that involve both $X_4$ and $JX_4$ change with respect to Case 1.1. Therefore, the system of equations obtained immediately after (4.23) still holds, and we can conclude $b_{22}^1 = 0$. Now, recalculating the necessary expressions in the second set of Jacobi identities after (4.23), we get:

$$\begin{cases}
b_{33}^2\, c_{41}^3 = 0,\\
b_{33}^2\, b_{41}^3 = 0,\\
b_{33}^2\, c_{44}^3 + b_{31}^2\, c_{44}^1 = 0,\\
b_{33}^2\, b_{44}^3 + c_{31}^2\, c_{44}^1 = 0,
\end{cases}
\qquad
\begin{cases}
c_{33}^2\, c_{41}^3 = 0,\\
c_{33}^2\, b_{41}^3 = 0,\\
c_{33}^2\, c_{44}^3 + c_{31}^2\, c_{44}^1 = 0,\\
c_{33}^2\, b_{44}^3 - b_{31}^2\, c_{44}^1 = 0.
\end{cases}$$

If $(b_{33}^2, c_{33}^2) \neq (0, 0)$, then $b_{41}^3 = c_{41}^3 = 0$. However, this implies $JX_1 \in \mathfrak{g}_3$ which is a contradiction. Hence, we must have $b_{33}^2 = c_{33}^2 = 0$. Replacing these two values in the last four equations and taking into account that $c_{44}^1 \neq 0$, we conclude $b_{31}^2 = c_{31}^2 = 0$.

We keep on following the same ideas as in Case 2.1. Observe that in the third set of Jacobi identities after (4.23), two of them include both $X_4$ and $JX_4$. Nonetheless, the fact that $[X_2, JX_1] = [JX_1, JX_2] = 0$ indicates that the new expressions should coincide with the given ones. Hence, the same arguments can be applied and we get $a_{23}^1 = b_{23}^1 = 0$.

Bearing in mind all the previous choices, we go back to $Jac\,(X_3, JX_1, X_4)$ and $Jac\,(X_3, JX_1, JX_4)$ in Case 2.1. The coefficients of $X_1$ lead to:

$$\begin{cases} b^1_{33}\, c^3_{41} = 0, \\ b^1_{33}\, b^3_{41} = 0. \end{cases}$$

If $b^3_{41} = c^3_{41} = 0$ then $JX_1 \in \mathfrak{g}_3$, and we obtain a contradiction. Therefore, one should have $(b^3_{41}, c^3_{41}) \neq (0, 0)$ and $b^1_{33} = 0$. In the end, we have 20 unknowns that must satisfy the system of equations generated by the remaining Jacobi identities (see Lemma B.0.4 in Appendix B). We simply note that it admits a particular solution preserving the arrangement of the ascending central series that we have fixed. We then obtain part *ii)*.

This concludes the study of every possibility, and thus our result is obtained. □

As a consequence of the previous three lemmas, namely Lemmas 4.3.1, 4.3.2, and 4.3.3, we obtain the following structure result.

**Proposition 4.3.4.** *Let* $(\mathfrak{g}, J)$ *be an* 8*-dimensional NLA endowed with an SnN complex structure. If* $\dim \mathfrak{g}_2 = 3$*, then the ascending central series of* $\mathfrak{g}$ *is given by*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle,$$

*and one of the following cases:*

*i)* $\mathfrak{g}_3 = \mathfrak{g}$*;*

*ii)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \;\; \mathfrak{g}_4 = \mathfrak{g}$*;*

*iii)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \;\; \mathfrak{g}_4 = \mathfrak{g}$*;*

*iv)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \;\; \mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \;\; \mathfrak{g}_5 = \mathfrak{g}$.

After the analysis of the case $\dim \mathfrak{g}_2 = 3$, let us move to study the other two cases considered in Corollary 4.2.2. One should notice that, in both of them, there is an element $X_3$ of the doubly adapted basis $\mathcal{B}$ such that $X_3 \in \mathfrak{g}_2$. Hence, we have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 \supseteq \langle X_1, X_2, X_3, JX_2 \rangle,$$

and by a similar argument to that used for $X_2 \in \mathfrak{g}_2$ we get $[X_3, JX_1] = [JX_1, JX_3] = 0$. Therefore, we have the brackets:

$$\begin{aligned}
&[X_1, Y] = 0, \;\; \forall\, Y \in \mathfrak{g}, \\
&[X_2, X_k] = a^1_{2k}\, X_1, \;\; k = 3, 4, \\
&[X_2, JX_1] = 0, \quad [X_2, JX_k] = b^1_{2k}\, X_1, \quad k = 2, 3, 4, \\
&[X_3, X_4] = a^1_{34}\, X_1, \\
&[X_3, JX_1] = 0, \quad [X_3, JX_2] = b^1_{23}\, X_1, \quad [X_3, JX_k] = b^1_{3k}\, X_1, \;\; k = 3, 4, \\
&[X_4, JX_2] = b^1_{24}\, X_1, \\
&[JX_1, JX_k] = 0, \;\; k = 2, 3, \quad [JX_2, JX_k] = a^1_{2k}\, X_1, \;\; k = 3, 4,
\end{aligned} \tag{4.24}$$

where $a^1_{2k}, a^1_{34}, b^1_{22}, b^1_{2k}, b^1_{32}, b^1_{3k} \in \mathbb{R}$, for $k = 3, 4$. In the next propositions, we continue the study of the ascending central series for the last two cases in Corollary 4.2.2.

**Proposition 4.3.5.** *Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with an SnN complex structure. If $\dim \mathfrak{g}_2 = 4$, then the ascending central series of $\mathfrak{g}$ is given by*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle,$$

*and one of the following cases:*

*i)* $\mathfrak{g}_3 = \mathfrak{g}$;

*ii)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \ \mathfrak{g}_4 = \mathfrak{g}$.

*Proof.* We first remark that a sketch of the decision tree constructed along these lines can be found in Appendix A. We here provide its complete discussion.

From the previous results, it is clear that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle.$$

We need to study $\mathfrak{g}_3$. Two possible paths are opened.

Case 1: Suppose there is $Y \in \mathfrak{g}_3$ linearly independent with $X_1$, $X_2$, $X_3$, $JX_1$, $JX_2$, and $JX_3$. Take $X_4 = Y$ as an element in the doubly adapted basis $\mathcal{B}$. Let us set

$$[X_4, JX_k] = b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + b^3_{4k}\, X_3 + c^2_{4k}\, JX_2, \ \ k = 1, 3, 4,$$

with $b^i_{4k}, c^2_{4k} \in \mathbb{R}$, for $i = 1, 2, 3$ and $k = 1, 3, 4$. If we now compute $[JX_1, JX_4]$ from (4.12) and $[JX_3, JX_4]$ from Nijenhuis (3.3), we can see that $b^1_{41} = b^3_{43} = 0$, in order to ensure the nilpotency of $\mathfrak{g}$. In this way, the ascending central series has the form

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 \supseteq \langle X_1, \ldots, X_4, JX_2 \rangle,$$

and we can add the following brackets to (4.24)

$$\begin{aligned}
&[X_4, JX_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2,\\
&[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2,\\
&[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2,\\
&[JX_1, JX_4] = c^2_{41}\, X_2 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3,\\
&[JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 - b^2_{43}\, JX_2,
\end{aligned} \tag{4.25}$$

where $b^3_{41}, b^1_{43}, b^1_{44}, b^3_{44}, b^2_{4k}, c^2_{4k} \in \mathbb{R}$, for $k = 1, 3, 4$. Let us observe that the last two brackets depend on some elements for which we still do not know the term of the series in which they enter (namely, $JX_1$ and $JX_3$). It is also worth remarking that we are in the conditions of Lemma 4.1.1 for $l = 3$. We distinguish two possibilities.

Case 1.1: Let us consider that $JX_4 \in \mathfrak{g}_3$. From the brackets in (4.25) involving this element, we see that both $b^3_{41} = 0$ and $b^1_{43} = b^1_{34}$ are needed. In particular, also $JX_1$ and $JX_3$ will enter in $\mathfrak{g}_3$, so we get

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 = \mathfrak{g}.$$

Therefore, the brackets defining our Lie algebra are (4.24) and

$$\begin{aligned} & [X_4, JX_1] = b^2_{41}\, X_2 + c^2_{41}\, JX_2, \quad [X_4, JX_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2, \\ & [X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2, \\ & [JX_1, JX_4] = c^2_{41}\, X_2 - b^2_{41}\, JX_2, \quad [JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 - b^2_{43}\, JX_2, \end{aligned} \tag{4.26}$$

where $b^1_{44}, b^3_{44}, b^2_{4k}, c^2_{4k} \in \mathbb{R}$, for $k = 1, 3, 4$. Recall that the parameters in (4.24) and (4.26) should preserve the fixed ascending central series.

We now move to study the Jacobi identity. First, let us observe that

$$Jac\,(X_4, JX_3, X_2) = -b^1_{22}\, c^2_{43}\, X_1, \qquad Jac\,(JX_3, JX_4, X_2) = b^1_{22}\, b^2_{43}\, X_1.$$

If $b^2_{43} = c^2_{43} = 0$ then $JX_3 \in \mathfrak{g}_2$, which is not possible. Therefore, $(b^2_{43}, c^2_{43}) \neq (0,0)$ and one necessarily has $b^1_{22} = 0$ in (4.24). Furthermore,

$$\begin{aligned} Jac\,(X_4, JX_1, X_3) &= \left(a^1_{23}\, b^2_{41} - b^1_{23}\, c^2_{41}\right) X_1, \\ Jac\,(JX_1, JX_4, X_3) &= \left(a^1_{23}\, c^2_{41} + b^1_{23}\, b^2_{41}\right) X_1. \end{aligned}$$

Writing the previous equations as a system, we obtain

$$\begin{pmatrix} b^2_{41} & -c^2_{41} \\ c^2_{41} & b^2_{41} \end{pmatrix} \begin{pmatrix} a^1_{23} \\ b^1_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant of the previous matrix equals zero, then $b^2_{41} = c^2_{41} = 0$ and $JX_1 \in \mathfrak{g}_2$. As this is not possible, we need $(b^2_{41}, c^2_{41}) \neq (0,0)$. Hence, $a^1_{23} = b^1_{23} = 0$ in (4.24).

In the end, we have 13 unknown parameters which satisfy 2 equations coming from the remaining Jacobi identities (see Lemma B.0.5 in Appendix B). It is possible to find a particular solution preserving the fixed dimension of the ascending central series. We obtain statement *i)* in the proposition.

Case 1.2: Let us suppose that $JX_4 \notin \mathfrak{g}_3$. The nilpotency of the Lie algebra implies that there is an element $Y \in \mathfrak{g}_4 \smallsetminus \mathfrak{g}_3$. In particular, one can assume that $Y = JX$, for some $X \in V = \langle X_1, \ldots, X_4 \rangle \subset \mathfrak{g}_3$, replacing $Y$ by $Y - \pi_{\mathfrak{g}_3}(Y)$ if necessary. Applying Lemma 4.1.1 *ii)*, it is possible to find $Z \in \mathfrak{g}_2$ such that $JZ \in \mathfrak{g}_3 \smallsetminus \mathfrak{g}_2$. Clearly, this element $JZ$ can be chosen in such a way that it is linearly dependent with $JX_1$ or $JX_3$.

Case 1.2.1: Suppose that $JZ$ is a multiple of $JX_1$, namely $JZ = \lambda\, JX_1$, for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then, we can arrange the generators and assume that $Z = X_1$. One has:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 \supseteq \langle X_1, \ldots, X_4, JX_2, JX_1 \rangle.$$

At the sight of (4.25) and bearing in mind that $[JX_1, Y] \in \mathfrak{g}_2$, for every $Y \in \mathfrak{g}$, one concludes that $b_{41}^3 = 0$. Furthermore, one needs $b_{43}^1 \neq b_{34}^1$ in order to ensure $JX_3 \notin \mathfrak{g}_3$. In fact, this condition leads to $JX_3, JX_4 \in \mathfrak{g}_4$, and $\mathfrak{g}_4 = \mathfrak{g}$. Considering these choices in (4.24) and (4.25) one obtains:

$$\begin{aligned} Jac\,(X_4, JX_3, JX_4) \;&=\; \left(2\,a_{24}^1\,c_{43}^2 + 2\,b_{24}^1\,b_{43}^2 - b_{23}^1\,b_{44}^2 - b_{33}^1\,b_{44}^3 - a_{23}^1\,c_{44}^2\right) X_1 \\ &\quad - \; b_{41}^2\,(b_{34}^1 - b_{43}^1)\,X_2 \;-\; c_{41}^2\,(b_{34}^1 - b_{43}^1)\,JX_2. \end{aligned}$$

Since $b_{43}^1 \neq b_{34}^1$, we get $b_{41}^2 = c_{41}^2 = 0$. Replacing these values and $b_{41}^3 = 0$ in (4.24) and (4.25), we conclude that, indeed, $JX_1 \in \mathfrak{g}_1$. However, this is a contradiction.

Case 1.2.2: If $JZ$ is not a multiple of $JX_1$, then it depends on $JX_3$. We can then apply a change of basis in $\mathfrak{g}_2$ and assume that, in fact, $JX_3 \in \mathfrak{g}_3$. Hence,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 \supseteq \langle X_1, \ldots, X_4, JX_2, JX_3 \rangle.$$

Since $[JX_3, Y] \in \mathfrak{g}_2$, from (4.25) we conclude that $b_{43}^1 = b_{34}^1$. Moreover, in order to ensure $JX_4 \notin \mathfrak{g}_3$ we need $b_{41}^3 \neq 0$, which implies $JX_1, JX_4 \in \mathfrak{g}_4$. In particular, observe that $\mathfrak{g}_4 = \mathfrak{g}$. Considering (4.24) and (4.25) with the previous conditions, we see

$$Jac\,(X_4, JX_4, JX_1) \;=\; -2\,(a_{24}^1\,c_{41}^2 + b_{24}^1\,b_{41}^2 + b_{34}^1\,b_{41}^3)\,X_1 \;-\; b_{41}^3\,b_{43}^2\,X_2 \;-\; b_{41}^3\,c_{43}^2\,JX_2.$$

As $b_{41}^3 \neq 0$, we will need $b_{43}^2 = c_{43}^2 = 0$ in order to have $Jac\,(X_4, JX_4, JX_1) = 0$. However, this choice makes $JX_3 \in \mathfrak{g}_2$, which contradicts $\dim \mathfrak{g}_2 = 4$.

This finishes the discussion of Case 1. We now need to study what happens when the contrary holds.

Case 2: Let us assume that there is no element in $\mathfrak{g}_3$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. The nilpotency of the Lie algebra requires the existence of at least one element in $\mathfrak{g}_3$ which is not in $\mathfrak{g}_2$. Necessarily, this element is linearly dependent with $JX_1$ and/or $JX_3$. In fact, one can consider that it is exactly either $JX_1$ or $JX_3$, up to an arrangement of generators. Therefore, two new cases arise.

Case 2.1: Assume that $JX_1 \in \mathfrak{g}_3$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

As a consequence, it is possible to set

$$[X_4, JX_1] = b_{41}^1\,X_1 + b_{41}^2\,X_2 + b_{41}^3\,X_3 + c_{41}^2\,JX_2,$$

where $b_{41}^1, b_{41}^2, b_{41}^3, c_{41}^2 \in \mathbb{R}$, and from (4.12),

$$[JX_1, JX_4] = c_{41}^2\,X_2 - b_{41}^1\,JX_1 - b_{41}^2\,JX_2 - b_{41}^3\,JX_3.$$

Since $JX_1 \in \mathfrak{g}_3$, we have $[JX_1, JX_4] \in \mathfrak{g}_2$ and we should take $b_{41}^1 = b_{41}^3 = 0$. The brackets defining the Lie algebra are (4.24) and

$$(4.27) \qquad [X_4, JX_1] = b_{41}^2\,X_2 + c_{41}^2\,JX_2, \quad [JX_1, JX_4] = c_{41}^2\,X_2 - b_{41}^2\,JX_2,$$

where $b^2_{41}, c^2_{41} \in \mathbb{R}$. New options appear.

Case 2.1.1: Let us suppose that also $JX_3 \in \mathfrak{g}_3$. Then, we can apply Lemma 4.1.1 *i)* for $l = 3$ and conclude

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Now, we can set

$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + b^3_{43}\, X_3 + c^2_{43}\, JX_2,$$

where $b^1_{43}, b^2_{43}, b^3_{43}, c^2_{43} \in \mathbb{R}$. Applying the Nijenhuis condition and (4.24)

$$[JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 - b^2_{43}\, JX_2 - b^3_{43}\, JX_3.$$

This bracket should belong to $\mathfrak{g}_2$, so we need $b^3_{43} = 0$ and $b^1_{43} = b^1_{34}$. Therefore, $\mathfrak{g}$ is defined by the brackets (4.24), (4.27), and

$$\begin{aligned} &[X_4, JX_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2, \\ &[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3, \\ &[JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 - b^2_{43}\, JX_2, \end{aligned} \tag{4.28}$$

where $b^1_{44}, b^3_{44}, b^2_{4k}, c^1_{44}, c^3_{44}, c^2_{4k} \in \mathbb{R}$, for $k = 3, 4$. Furthermore, the choice of parameters in (4.24), (4.27), and (4.28) must preserve the layout of the ascending central series that has been considered. In particular, $(c^1_{44}, c^3_{44}) \neq (0, 0)$ to avoid recovering Case 1.1. We now move to study the Jacobi identity.

First, let us note that the bracket $[X_4, JX_4]$ is the only difference between (4.26) and (4.27)-(4.28). Therefore, many of the arguments used in Case 1.1 can also be applied here. Indeed, just those Jacobi identities involving both $X_4$ and $JX_4$ change in this new case.

In this way, one definitely has $b^1_{22} = 0$ and $a^1_{23} = b^1_{23} = 0$. Nonetheless, still 15 parameters remain unknown, which additionally satisfy 3 equations coming from the remaining Jacobi identities (see Lemma B.0.6 in Appendix B). A particular solution can be attained in such a way that the fixed ascending central series is preserved. This gives part *ii)* of the proposition.

Let us now study the other possibility under Case 2.1.

Case 2.1.2: Assume $JX_3 \notin \mathfrak{g}_3$. Then, we can conclude

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

We turn out attention to $\mathfrak{g}_4$. Two possibilities arise.

*Case 2.1.2.1*: Let $Y \in \mathfrak{g}_4$ be a linearly independent element with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Take $X_4 = Y$ as an element of $\mathcal{B}$. Then, we can fix

$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + b^3_{43}\, X_3 + c^1_{43}\, JX_1 + c^2_{43}\, JX_2,$$

where $b_{43}^1, b_{43}^2, b_{43}^3, c_{43}^1, c_{43}^2 \in \mathbb{R}$, and from the Nijenhuis condition and the brackets (4.24), (4.27), we get

$$[JX_3, JX_4] = (a_{34}^1 + c_{43}^1)\, X_1 + c_{43}^2\, X_2 + (b_{34}^1 - b_{43}^1)\, JX_1 - b_{43}^2\, JX_2 - b_{43}^3\, JX_3.$$

Necessarily $b_{43}^3 = 0$ in order to ensure that $[JX_3, JX_4] \in \mathfrak{g}_3$, but then also $JX_3, JX_4 \in \mathfrak{g}_4$. Indeed, $\mathfrak{g}_4 = \mathfrak{g}$ and the Lie algebra can be completely defined by (4.24), (4.27), and

$$\begin{aligned}
[X_4, JX_3] &= b_{43}^1\, X_1 + b_{43}^2\, X_2 + c_{43}^1\, JX_1 + c_{43}^2\, JX_2,\\
[X_4, JX_4] &= b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^1\, JX_1 + c_{44}^2\, JX_2,\\
[JX_3, JX_4] &= (a_{34}^1 + c_{43}^1)\, X_1 + c_{43}^2\, X_2 + (b_{34}^1 - b_{43}^1)\, JX_1 - b_{43}^2\, JX_2,
\end{aligned}$$

with $b_{44}^3, b_{4k}^i, c_{4k}^i \in \mathbb{R}$, for $i = 1, 2$ and $k = 3, 4$. From these brackets, one can compute

$$\begin{aligned}
Jac\,(X_4, JX_3, JX_4) \;=\; & -\big(a_{23}^1\, c_{44}^2 - 2\, a_{24}^1\, c_{43}^2 + b_{23}^1\, b_{44}^2 - 2\, b_{24}^1\, b_{43}^2 + b_{33}^1\, b_{44}^3\big)\, X_1\\
& + \big(c_{41}^2\, c_{43}^1 - (b_{34}^1 - b_{43}^1)\, b_{41}^2\big)\, X_2\\
& - \big(b_{41}^2\, c_{43}^1 + (b_{34}^1 - b_{43}^1)\, c_{41}^2\big)\, JX_2.
\end{aligned}$$

This expression equals zero, so in particular we get

$$\begin{pmatrix} c_{43}^1 & -(b_{34}^1 - b_{34}^1) \\ b_{34}^1 - b_{34}^1 & c_{43}^1 \end{pmatrix} \begin{pmatrix} c_{41}^2 \\ b_{41}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant of the previous matrix is non-zero, then $c_{41}^2 = b_{41}^2 = 0$. However, if we replace these values in (4.24) and (4.27) we can see that $JX_1 \in \mathfrak{g}_1$, which is not possible. Therefore, the determinant should be zero. Notice that this yields $(b_{34}^1 - b_{43}^3, c_{43}^1) = (0, 0)$, which in turn implies $JX_3 \in \mathfrak{g}_3$. This contradicts the assumption on the ascending central series.

<u>*Case 2.1.2.2*</u>: Suppose that one cannot find an element in $\mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. The nilpotency of $\mathfrak{g}$ implies that there is an element in $\mathfrak{g}_4$ which is not in $\mathfrak{g}_3$. Necessarily, this element is linearly dependent with $JX_3$. Hence, we can assume that, up to an arrangement of generators, $JX_3 \in \mathfrak{g}_4$. This leaves $X_4, JX_4 \in \mathfrak{g}_5$, and thus $\mathfrak{g}_5 = \mathfrak{g}$. Observe that we can reproduce the same argument as in the previous case in order to find $[X_4, JX_3]$ and $[JX_3, JX_4]$. Indeed, even the same conditions on the parameters of these two brackets hold, and the only difference with respect to Case 2.1.2.1 turns to be

$$[X_4, JX_4] = b_{44}^1\, X_1 + b_{44}^2\, X_2 + b_{44}^3\, X_3 + c_{44}^1\, JX_1 + c_{44}^2\, JX_2 + c_{44}^3\, JX_3,$$

with $b_{44}^1$, $b_{44}^2$, $b_{44}^3$, $c_{44}^1$, $c_{44}^2$, $c_{44}^3 \in \mathbb{R}$. Observe that one should take $c_{44}^3 \neq 0$ in order to ensure $X_4, JX_4 \in \mathfrak{g}_5 \setminus \mathfrak{g}_4$. If one now computes $Jac\,(X_4, JX_3, JX_4)$ in this new case, it is easy to see that the new expression coincides with that given in Case 2.1.2.1. The same argument can be applied and reach a contradiction.

This finishes Case 2.1. Let us move to study the other possibility.

Case 2.2: Let us assume that $JX_1 \notin \mathfrak{g}_3$. The nilpotency of $\mathfrak{g}$ and the hypothesis in Case 2 allow us to set $JX_3 \in \mathfrak{g}_3$, up to a change of basis in $\mathfrak{g}_2$. Then,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \qquad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

Notice that one has

$$[X_4, JX_3] = b^1_{43}\, X_1 + b^2_{43}\, X_2 + b^3_{43}\, X_3 + c^2_{43}\, JX_2,$$

where $b^1_{43}$, $b^2_{43}$, $b^3_{43}$, $c^2_{43} \in \mathbb{R}$. By the Nijenhuis condition and the brackets (4.24), we get

$$[JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 + (b^1_{34} - b^1_{43})\, JX_1 - b^2_{43}\, JX_2 - b^3_{43}\, JX_3.$$

In order to preserve the arrangement of the ascending central series, we need $b^3_{43} = 0$ and $b^1_{43} = b^1_{34}$. Hence, the brackets defining our Lie algebra are (4.24) and

$$\begin{aligned} &[X_4, JX_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, JX_2, \\ &[JX_3, JX_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 - b^2_{43}\, JX_2, \end{aligned} \tag{4.29}$$

where $b^2_{43}, c^2_{43} \in \mathbb{R}$. We now need to see where $X_4, JX_1$, and $JX_4$ enter the ascending central series in order to have the complete description of $(\mathfrak{g}, J)$. Simply note that we are in the conditions of Lemma 4.1.2 for $k = 3$.

Case 2.2.1: Let us suppose that there is $Y \in \mathfrak{g}_4$ linearly independent with $X_1, X_2, X_3, JX_1, JX_2$, and $JX_3$. Then, we can set $X_4 = Y$ as an element of the doubly adapted basis $\mathcal{B}$. Applying Lemma 4.1.2 *i)* for $k = 3$, we get $\mathfrak{g}_4 = \mathfrak{g}$. We can then take

$$[X_4, JX_1] = b^1_{41}\, X_1 + b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3,$$

where $b^1_{41}$, $b^2_{41}$, $b^3_{41}$, $c^2_{41}$, $c^3_{41} \in \mathbb{R}$. From equation (4.12),

$$[JX_1, JX_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^1_{41}\, JX_1 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3,$$

so one should choose $b^1_{41} = 0$. Indeed, the Lie algebra is defined by the brackets (4.24), (4.29), and

$$\begin{aligned} &[X_4, JX_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3, \\ &[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3, \\ &[JX_1, JX_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3, \end{aligned}$$

where $b^1_{44}, b^i_{4k}, c^i_{4k} \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. We should now verify the Jacobi identity. Let us consider

$$\begin{aligned} Jac\,(X_4, JX_1, JX_4) \;&=\; 2\big(a^1_{24}\, c^2_{41} + a^1_{34}\, c^3_{41} + b^1_{24}\, b^2_{41} + b^1_{34}\, b^3_{41}\big) X_1 \\ &\quad + \; \big(b^3_{41}\, b^2_{43} + c^3_{41}\, c^2_{43}\big)\, X_2 \;+\; \big(b^3_{41}\, c^2_{43} - c^3_{41}\, b^2_{43}\big)\, JX_2. \end{aligned}$$

Observe that we have the following system of equations:

$$\begin{pmatrix} b_{41}^3 & c_{41}^3 \\ -c_{41}^3 & b_{41}^3 \end{pmatrix} \begin{pmatrix} b_{43}^2 \\ c_{43}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the determinant of the matrix is different from zero, then $b_{43}^2 = c_{43}^2 = 0$. Replacing these values in the brackets we can see that $JX_3 \in \mathfrak{g}_2$, which contradicts our assumption. Therefore, the determinant is zero and $(b_{41}^3, c_{41}^3) = (0, 0)$. However, this makes that $JX_1$ descends to $\mathfrak{g}_3$ which is not possible due to the hypothesis of Case 2.2.

Case 2.2.2: We now suppose the converse to Case 2.2.1. Then, by Lemma 4.1.2 *ii)* for $k = 3$, we directly conclude that $JX_1 \in \mathfrak{g}_4$ and $\mathfrak{g}_5 = \mathfrak{g}$. Hence, we have the same expression for the bracket $[X_4, JX_1]$ as in Case 2.2.1. In fact, we can repeat the argument and obtain analogous $[X_4, JX_1]$ and $[JX_1, JX_4]$. Now, since the hypothesis of this case implies $X_4, JX_4 \in \mathfrak{g}_5$, we can set

$$[X_4, JX_4] = b_{44}^1 X_1 + b_{44}^2 X_2 + b_{44}^3 X_3 + c_{44}^1 JX_1 + c_{44}^2 JX_2 + c_{44}^3 JX_3,$$

with $c_{44}^1 \neq 0$. If we compute $Jac\,(X_4, JX_1, JX_4)$ using these brackets together with (4.24) and (4.29), we obtain exactly the same expression as before. Hence, we can again construct the system and reach a contradiction.

We have already studied all the possibilities, so the proposition is proved. □

Finally, we study the last case considered in Corollary 4.2.2, namely, $\dim \mathfrak{g}_2 = 5$.

**Proposition 4.3.6.** *Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with an SnN complex structure. If $\dim \mathfrak{g}_2 = 5$, then the ascending central series of $\mathfrak{g}$ is given by*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle,$$

*and one of the following cases:*

*i)* $\mathfrak{g}_3 = \mathfrak{g}$;

*ii)* $\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \;\; \mathfrak{g}_4 = \mathfrak{g}$.

*Proof.* As a consequence of previous results, one clearly has

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

Since $JX_3 \in \mathfrak{g}_2$, one can take $[X_4, JX_3] = b_{43}^1 X_1$, for some $b_{43}^1 \in \mathbb{R}$, and then $[JX_3, JX_4] = a_{34}^1 X_1 + (b_{34}^1 - b_{43}^1) JX_1$ from Nijenhuis and (4.24). As $[JX_3, JX_4] \in \mathfrak{g}_1$, we conclude $b_{43}^1 = b_{34}^1$. Hence, we add the following brackets to (4.24)

$$[X_4, JX_3] = b_{34}^1 X_1, \qquad [JX_3, JX_4] = a_{34}^1 X_1, \tag{4.30}$$

where $a_{34}^1, b_{34}^1 \in \mathbb{R}$. We now need to study the different possibilities for $\mathfrak{g}_3$. In order to do so, bear in mind that we are in the conditions of Lemma 4.1.2 for $k = 2$.

<u>CASE 1</u>: Let us assume that there is $Y \in \mathfrak{g}_3$ linearly independent with $X_1, X_2, X_3$, $JX_1$, $JX_2$, and $JX_3$. Then, we take $X_4 = Y \in \mathcal{B}$ and applying Lemma 4.1.2 *i)* for $k = 2$, we directly have that

$$\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \qquad \mathfrak{g}_3 = \mathfrak{g}.$$

We take

$$[X_4, JX_k] = b^1_{4k}\, X_1 + b^2_{4k}\, X_2 + b^3_{4k}\, X_3 + c^2_{4k}\, JX_2 + c^3_{4k}\, JX_3, \ \ k = 1, 4,$$

where $b^1_{4k}, b^i_{4k}, c^i_{4k} \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. Moreover, from equation (4.12) one can see that

$$[JX_1, JX_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^1_{41}\, JX_1 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3.$$

The nilpotency of $\mathfrak{g}$ requires $b^1_{41} = 0$. Thus, we have the brackets (4.24), (4.30), and

$$\begin{aligned} &[X_4, JX_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3, \\ &[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3, \\ &[JX_1, JX_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3, \end{aligned} \tag{4.31}$$

where $b^1_{44}, b^i_{4k}, c^i_{4k} \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. It is worth remarking that all the brackets are now completely defined. Simply observe that the coefficients of the brackets (4.24), (4.30), and (4.31) are not completely free: they should be chosen according to the previous arrangement of the ascending central series. For instance, $a^1_{23} = a^1_{34} = b^1_{32} = b^1_{33} = b^1_{34} = 0$ would yield $X_3 \in \mathfrak{g}_1$ and $a^1_{23} = b^1_{22} = b^1_{23} = b^1_{24} = 0$ would imply $JX_2 \in \mathfrak{g}_1$, so these are not admissible values.

The construction finishes here. Nevertheless, we still need to verify the Jacobi identity, which gives some relations among the parameters. In fact, using the brackets (4.24), (4.30), and (4.31), one obtains a quadratic system with 9 equations and 17 (real) unknowns (see Lemma B.0.7 in Appendix B). A particular solution preserving the layout of the desired ascending central series can be found. Hence, the Case 1 leads to a first positive result, contained in part *i)* of the proposition.

<u>CASE 2</u>: Let us assume that there is no element in $\mathfrak{g}_3$ linearly independent with $X_1, X_2, X_3$, $JX_1, JX_2$, and $JX_3$. The nilpotency of the Lie algebra implies that, up to an arrangement of generators, $JX_1 \in \mathfrak{g}_3$. Indeed, by Lemma (4.1.2) *ii)* for $k = 2$, the ascending central series is

$$\begin{aligned} &\mathfrak{g}_1 = \langle X_1 \rangle, \qquad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \\ &\mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \qquad \mathfrak{g}_4 = \mathfrak{g}. \end{aligned}$$

Observe that the expressions for $[X_4, JX_1]$ and $[JX_1, JX_4]$ coincide with those in (4.31). More concretely, $\mathfrak{g}$ is defined by the brackets (4.24), (4.30), and

$$\begin{aligned} &[X_4, JX_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, JX_2 + c^3_{41}\, JX_3, \\ &[X_4, JX_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, JX_1 + c^2_{44}\, JX_2 + c^3_{44}\, JX_3, \\ &[JX_1, JX_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, JX_2 - b^3_{41}\, JX_3, \end{aligned}$$

where $b^1_{44}, b^i_{4k}, c^1_{44}, c^i_{4k} \in \mathbb{R}$, for $i = 2, 3$ and $k = 1, 4$. In particular, observe that $c^1_{44} \neq 0$ or otherwise, we would go back to Case 1. The construction is completed.

Now, the Jacobi identity needs to be checked. It turns out that one exactly obtains the same equations as in Case 1, since only the bracket $[X_4, JX_4]$ has changed (see Lemma B.0.8 in Appendix B). A particular solution preserving $(\dim \mathfrak{g}_k) = (1, 5, 6, 8)$ can be attained. Hence, we obtain another valid case, corresponding to part *ii)* of the statement above.

This finishes the proof of the proposition. □

We have found 8 admissible cases for the dimension of the ascending central series of an 8-dimensional NLA $\mathfrak{g}$ admitting SnN complex structures $J$. The precise descriptions of these $(\mathfrak{g}, J)$ can be found in Appendix B. Furthermore, as a consequence of Propositions 4.3.4, 4.3.5, and 4.3.6 (see p. 144, p. 145, and p. 151, respectively), we obtain the following structure theorem.

**Theorem 4.3.7.** *Let $\mathfrak{g}$ be an NLA of dimension 8. If $J$ admits an SnN complex structure, then the dimension of its ascending central series $\{\mathfrak{g}_k\}_k$ is*

$$\begin{aligned}(\dim \mathfrak{g}_k)_k \;=\;& (1,3,8),\ (1,3,6,8),\ (1,3,5,8),\ (1,3,5,6,8),\\ & (1,4,8),\ (1,4,6,8),\ (1,5,8),\ \textit{or}\ (1,5,6,8).\end{aligned}$$

## 4.4 Complexification and 8-dimensional classification

In the previous section, we have completely determine those NLAs $\mathfrak{g}$ of dimension eight admitting an SnN complex structure $J$. Here, we describe the complex structure equations for each pair $(\mathfrak{g}, J)$. In this way, we are able to complete the study of invariant complex structures on 8-dimensional nilmanifolds that we initiated in Chapter 3 with the study of quasi-nilpotent complex structures.

Let $\mathfrak{g}$ be an NLA of dimension 8 endowed with a strongly non-nilpotent complex structure. Let $\mathcal{B} = \{X_1, \ldots, X_4, Y_1, \ldots, Y_4\}$ be a doubly adapted basis of $(\mathfrak{g}, J)$, with $Y_k = JX_k$ for $k = 1, \ldots, 4$. Consider its dual basis $\mathcal{B}^* = \{e^1, \ldots, e^8\}$. First, we recall that the differential of any element $e \in \mathfrak{g}^*$ can be computed using the Lie brackets of $\mathfrak{g}$ by means of the formula

$$de(X, Y) = -e([X, Y]),$$

where $X, Y \in \mathfrak{g}$. Since $J$ is a complex structure, one can then construct the $(1,0)$-basis

$$\eta^1 = e^4 - ie^8, \quad \eta^2 = e^3 - ie^7, \quad \eta^3 = e^2 - ie^6, \quad \eta^4 = e^1 - ie^5, \tag{4.32}$$

and calculate the (complex) structure equations of $(\mathfrak{g}, J)$. A reduced version of them can be obtained performing appropriate changes of basis, depending on each admissible pair $(\mathfrak{g}, J)$ found in Propositions 4.3.4, 4.3.5, and 4.3.6. We devote the following lines to develop these ideas.

**Lemma 4.4.1.** *Let $J$ be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that $(\dim \mathfrak{g}_k)_k = (1,3,8)$ or $(1,3,6,8)$. The pair $(\mathfrak{g}, J)$ is parametrized by the structure equations*

$$\left\{\begin{aligned} d\omega^1 &= 0,\\ d\omega^2 &= \varepsilon\,\omega^{1\bar{1}},\\ d\omega^3 &= \omega^{14} \pm i\,\varepsilon\, b\,\omega^{1\bar{2}} + \omega^{1\bar{4}} + A\,\omega^{2\bar{1}},\\ d\omega^4 &= i\,\nu\,\omega^{1\bar{1}} \pm i\,\omega^{1\bar{3}} + b\,\omega^{2\bar{2}} \mp i\,\omega^{3\bar{1}}, \end{aligned}\right. \tag{4.33}$$

*where $\varepsilon, \nu \in \{0,1\}$, $A \in \mathbb{C}$, and $b \in \mathbb{R}$. Furthermore,*

- *if $(\dim \mathfrak{g}_k)_k = (1,3,8)$, then one has $(\varepsilon,\nu) = (0,0)$; and*
- *if $(\dim \mathfrak{g}_k)_k = (1,3,6,8)$, then $(\varepsilon,\nu) \neq (0,0)$.*

*Proof.* From the Lie brackets obtained in the proof of Lemma 4.3.2 (see also Lemmas B.0.1 and B.0.2 in Appendix B) one has real structure equations

$$\left\{\begin{array}{lcl} de^1 &=& -a^1_{23}\,e^{23} - a^1_{24}\,e^{24} - b^1_{23}\,e^{27} - b^1_{24}\,e^{28} - a^1_{34}\,e^{34} - b^1_{23}\,e^{36} - b^1_{33}\,e^{37}\\ && -b^1_{34}\,e^{38} - b^1_{24}\,e^{46} - b^1_{34}\,e^{47} - b^1_{44}\,e^{48} - a^1_{23}\,e^{67} - a^1_{24}\,e^{68} - a^1_{34}\,e^{78},\\ de^2 &=& -a^2_{34}\,e^{34} - b^2_{31}\,e^{35} - b^2_{33}\,e^{37} - b^2_{34}\,e^{38} - b^2_{41}\,e^{45} - b^2_{43}\,e^{47}\\ && -b^2_{44}\,e^{48} - c^2_{31}\,e^{57} - c^2_{41}\,e^{58} - (a^2_{34} - c^2_{34} + c^2_{43})\,e^{78},\\ de^3 &=& -b^3_{44}\,e^{48},\\ de^4 &=& 0,\\ de^5 &=& -c^1_{44}\,e^{48},\\ de^6 &=& -\alpha^2_{34}\,e^{34} - c^2_{31}\,e^{35} - c^2_{33}\,e^{37} - c^2_{34}\,e^{38} - c^2_{41}\,e^{45} - c^2_{43}\,e^{47}\\ && -c^2_{44}\,e^{48} + b^2_{31}\,e^{57} + b^2_{41}\,e^{58} - (\alpha^2_{34} + b^2_{34} - b^2_{43})\,e^{78},\\ de^7 &=& -c^3_{44}\,e^{48},\\ de^8 &=& 0, \end{array}\right.$$

where the structure constants satisfy $b^3_{44} = c^1_{44} = c^3_{44} = 0$ for $(\dim \mathfrak{g}_k)_k = (1,3,8)$ and $a^1_{23} = b^1_{23} = b^2_{31} = b^2_{33} = c^2_{31} = c^2_{33} = 0$ for $(\dim \mathfrak{g}_k)_k = (1,3,6,8)$. If we construct the basis $\{\eta^k\}_{k=1}^4$ described by (4.32), then we obtain

$$\left\{\begin{array}{lcl} d\eta^1 &=& 0,\\ d\eta^2 &=& M\,\eta^{1\bar{1}},\\ d\eta^3 &=& (A' - B)\,\eta^{12} - C\,\eta^{14} + D\,\eta^{1\bar{1}} + (E + B)\,\eta^{1\bar{2}} + C\,\eta^{1\bar{4}}\\ && -F\,\eta^{24} + (A' - E)\,\eta^{2\bar{1}} + G\,\eta^{2\bar{2}} + F\,\eta^{2\bar{4}},\\ d\eta^4 &=& N\,\eta^{1\bar{1}} + H\,\eta^{1\bar{2}} + K\,\eta^{1\bar{3}} - \bar{H}\,\eta^{2\bar{1}} + \frac{i\,b^1_{33}}{2}\,\eta^{2\bar{2}} + L\,\eta^{2\bar{3}} - \bar{K}\,\eta^{3\bar{1}} - \bar{L}\,\eta^{3\bar{2}}, \end{array}\right.$$

where

$$A' = \frac{c^2_{34} + i\, b^2_{34}}{2}, \quad B = \frac{c^2_{43} + i\, b^2_{43}}{2}, \quad C = \frac{c^2_{41} + i\, b^2_{41}}{2}, \quad D = \frac{c^2_{44} + i\, b^2_{44}}{2},$$
$$E = \frac{a^2_{34} - i\, \alpha^2_{34}}{2}, \quad F = \frac{c^2_{31} + i\, b^2_{31}}{2}, \quad G = \frac{c^2_{33} + i\, b^2_{33}}{2}, \quad H = \frac{a^1_{34} + i\, b^1_{34}}{2},$$
$$K = \frac{a^1_{24} + i\, b^1_{24}}{2}, \quad L = \frac{a^1_{23} + i\, b^1_{23}}{2}, \quad M = \frac{c^3_{44} + i\, b^3_{44}}{2}, \quad N = \frac{c^1_{44} + i\, b^1_{44}}{2}.$$

Simply observe that $M = 0$, $N = \frac{i\, b^1_{44}}{2}$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$ and $F = G = L = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 3, 6, 8)$. Furthermore, the condition $d^2\eta^k = 0$, for $k = 1, 2, 3, 4$, which is equivalent to the Jacobi identity, gives the system of equations:

$$\text{(4.34)} \qquad \begin{cases} K(\bar{B} + \bar{E}) + (B - A')\bar{K} - L\bar{D} - \frac{i\, b^1_{33}}{2}\, \bar{M} = 0, \\ L(\bar{E} - \bar{A}') + (B - A')\bar{L} + K\bar{G} = 0, \\ \mathfrak{Re}\,(K\bar{C}) = 0, \\ F\bar{K} + L\bar{C} = 0, \\ \mathfrak{Re}\,(L\bar{F}) = 0. \end{cases}$$

We now perform different changes of basis, according to the values of these parameters.

- Let us consider the case $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$ with $C = 0$, i.e., we have

$$M = C = 0, \quad N = \frac{i\, b^1_{44}}{2}.$$

First, we observe that if $F = 0$, then $b^2_{31} = b^2_{41} = c^2_{31} = c^2_{41} = 0$ and we have $Y_1 \in \mathfrak{g}_1$ (see the brackets in Lemma B.0.1), which contradicts $\dim \mathfrak{g}_1 = 1$. Hence, $F \neq 0$ and from the fourth equation in (4.34) we can conclude $K = 0$.

We next notice that $K = 0$ implies $L \neq 0$, or otherwise one would have $a^1_{23} = a^1_{24} = b^1_{23} = b^1_{24} = 0$ and $X_2, Y_2 \in \mathfrak{g}_1$, which is not possible. As a consequence, the first equation in (4.34) leads to $D = 0$.

From the last equation in (4.34), we know that $\mathfrak{Re}\,(L\bar{F}) = 0$. Since $F, L \neq 0$, one necessarily has $\mathfrak{Im}\,(L\bar{F}) \neq 0$. We define a new basis for $\mathfrak{g}^{1,0}$ as follows:

$$\omega^1 = \sqrt{|\mathfrak{Im}\,(L\bar{F})|}\, \eta^2, \qquad \omega^3 = \sqrt{|\mathfrak{Im}\,(L\bar{F})|}\left(-\frac{i}{F}\,\eta^3 + \frac{b^1_{33}}{4\,L\bar{F}}\,\eta^2 - \frac{i\,H}{L\bar{F}}\,\eta^1\right),$$
$$\omega^2 = \eta^1, \qquad \omega^4 = i\left(\eta^4 + \frac{\bar{G}}{\bar{F}}\,\eta^2 + \frac{A' - B}{F}\,\eta^1\right).$$

Then, using equations (4.34) with $C = K = D = 0$, one concludes that the structure equations follow (4.33) with $\varepsilon = \nu = 0$, and

$$A = -i\,\frac{B + E}{F}, \quad b = -\frac{b^1_{44}}{2}.$$

- We now study the case $(\dim \mathfrak{g}_k)_k = (1,3,8)$ with $C \neq 0$, i.e.,

$$M = 0, \quad N = \frac{i\, b^1_{44}}{2}, \quad C \neq 0.$$

Let us first note the following: if $K = 0$, then the fourth equation in (4.34) gives $L = 0$, but $a^1_{23} = a^1_{24} = b^1_{23} = b^1_{24} = 0$ implies $X_2, Y_2 \in \mathfrak{g}_1$, which is a contradiction. Therefore, $K \neq 0$.

Since $\mathfrak{Re}\,(K\bar{C}) = 0$ but $C, K \neq 0$, it is clear that $\mathfrak{Im}\,(K\bar{C}) \neq 0$, and one can take the following change of basis:

$$\begin{aligned}
\omega^1 &= \sqrt{|\mathfrak{Im}\,(K\bar{C})|}\left(\eta^1 + \frac{F}{C}\,\eta^2\right),\\
\omega^2 &= \eta^2,\\
\omega^3 &= \sqrt{|\mathfrak{Im}\,(K\bar{C})|}\left(-\frac{i}{C}\,\eta^3 + \frac{4\,i\bar{H}C - b^1_{44}\,F}{4\,|C|^2 K}\,\eta^2 + \frac{b^1_{44}}{4\,K\bar{C}}\,\eta^1\right),\\
\omega^4 &= i\left(\eta^4 + \frac{F\bar{D} + (B - A')\bar{C}}{|C|^2}\,\eta^2 + \frac{\bar{D}}{\bar{C}}\,\eta^1\right).
\end{aligned}$$

Taking into account (4.34), the structure equations in terms of the new basis correspond to (4.33) with $\varepsilon = \nu = 0$, and

$$A = -i\,\frac{(A' - E)\,C - DF}{C^2}, \quad b = \frac{2\,\mathfrak{Im}\,(FH\bar{C})}{|C|^2} - \frac{b^1_{44}\,|F|^2}{2\,|C|^2} - \frac{b^1_{33}}{2}.$$

- Let $(\dim \mathfrak{g}_k)_k = (1,3,6,8)$, that is,

$$F = G = L = 0.$$

We first note that if $K = 0$ then $X_2, Y_2 \in \mathfrak{g}_1$, but this is not possible as $\dim \mathfrak{g}_1 = 1$. Therefore, we have $K \neq 0$. Similarly, we can check that $C \neq 0$, because otherwise we would have $Y_1 \in \mathfrak{g}_1$.

Since $K, C \neq 0$, one necessarily has $\mathfrak{Im}\,(K\bar{C}) \neq 0$ due to the equation $\mathfrak{Re}\,(K\bar{C}) = 0$ in (4.34). Bearing the previous conditions in mind, the following basis can be considered:

$$\begin{aligned}
\tau^1 &= \sqrt{|\mathfrak{Im}\,(K\bar{C})|}\,\eta^1,\\
\tau^2 &= \eta^2,\\
\tau^3 &= \sqrt{|\mathfrak{Im}\,(K\bar{C})|}\left(-\frac{i}{C}\,\eta^3 - \frac{i\,\bar{H}}{C\bar{K}}\,\eta^2 + \frac{1}{2\,K\bar{C}}\,\mathfrak{Im}\left(\frac{NC + (B - A')M}{C}\right)\eta^1\right),\\
\tau^4 &= i\left(\eta^4 - \frac{A' - B}{C}\,\eta^2 + \frac{H\bar{M} + K\bar{D}}{K\bar{C}}\,\eta^1\right).
\end{aligned}$$

The structure equations become:

$$\begin{cases} d\tau^1 = 0, \\ d\tau^2 = \frac{M}{|\mathfrak{Im}\,(K\bar{C})|}\,\tau^{1\bar{1}}, \\ d\tau^3 = \tau^{14} - i\big(\frac{\bar{A}'-\bar{B}}{\bar{C}} + \frac{B+E}{C}\big)\,\tau^{1\bar{2}} + \tau^{1\bar{4}} - i\,\frac{A'-E}{C}\,\tau^{2\bar{1}}, \\ d\tau^4 = \frac{i}{|\mathfrak{Im}\,(K\bar{C})|}\,\mathfrak{Re}\left(\frac{CN+(B-A')M}{C}\right)\tau^{1\bar{1}} \pm i\,\tau^{1\bar{3}} - \frac{b^1_{33}}{2}\,\tau^{2\bar{2}} \mp i\,\tau^{3\bar{1}}, \end{cases}$$

where the upper signs are taken when $\mathfrak{Im}\,(K\bar{C}) > 0$ and the lower ones, when $\mathfrak{Im}\,(K\bar{C}) < 0$. Moreover, as a consequence of equations (4.34) with $F = G = L = 0$, one has

$$\frac{\bar{A}' - \bar{B}}{\bar{C}} + \frac{B + E}{C} = \frac{i\,b^1_{33}\,\bar{M}}{2\,K\bar{C}}.$$

We next distinguish different cases.

– Let $M = 0$. When this conditon holds, then we can ensure that $\mathfrak{Re}\,N \neq 0$. Otherwise we would have $b^3_{44} = c^1_{44} = c^3_{44} = 0$ and $X_4, Y_4 \in \mathfrak{g}_3$, i.e., $\mathfrak{g}_3 = \mathfrak{g}$, which contradicts the nilpotency step of $\mathfrak{g}$. This allows us to consider a basis

$$\omega^1 = \tau^1, \quad \omega^2 = \tau^2, \quad \omega^3 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\,N}\,\tau^3, \quad \omega^4 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\,N}\,\tau^4,$$

in terms of which the structure equations follow (4.33) with $\varepsilon = 0$, $\nu = 1$, and

$$A = \frac{i\,(E - A')\,|\mathfrak{Im}\,(K\bar{C})|}{C\,\mathfrak{Re}\,N}, \quad b = -\frac{b^1_{33}\,|\mathfrak{Im}\,(K\bar{C})|}{2\,\mathfrak{Re}\,N}.$$

– Let $M \neq 0$ and $\mathfrak{Re}\left(\frac{CN+(B-A')M}{C}\right) = 0$. If we consider the basis

$$\omega^1 = \tau^1, \quad \omega^2 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{M}\,\tau^2, \quad \omega^3 = \tau^3, \quad \omega^4 = \tau^4,$$

then we obtain (4.33) with $\varepsilon = 1$, $\nu = 0$, and

$$A = \frac{i\,M\,(E - A')}{C\,|\mathfrak{Im}\,(K\bar{C})|}, \quad b = -\frac{b^1_{33}\,|M|^2}{2\,\big(\mathfrak{Im}\,(K\bar{C})\big)^2}.$$

– We finally study the case $M \neq 0$ and $\mathfrak{Re}\left(\frac{CN+(B-A')M}{C}\right) \neq 0$. We take the $(1,0)$-basis defined by

$$\omega^1 = \tau^1, \qquad \omega^3 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\left(\frac{CN+(B-A')M}{C}\right)}\,\tau^3,$$

$$\omega^2 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{M}\,\tau^2, \qquad \omega^4 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\left(\frac{CN+(B-A')M}{C}\right)}\,\tau^4.$$

The structure equations are (4.33) with $\varepsilon = \nu = 1$, and

$$A = \frac{i\, M\,(E - A')}{C\,\mathfrak{Re}\,\left(\frac{CN+(B-A')M}{C}\right)}, \qquad b = -\frac{b^1_{33}\,|M|^2}{2\,|\mathfrak{Im}\,(K\bar{C})|\,\mathfrak{Re}\,\left(\frac{CN+(B-A')M}{C}\right)}.$$

This concludes the proof of the Lemma. □

**Lemma 4.4.2.** *Let $J$ be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ or $(1,3,5,6,8)$. The pair $(\mathfrak{g}, J)$ is parametrized by*

$$(4.35)\qquad \begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = A\,\omega^{1\bar{1}} + \varepsilon\,(\omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,\mu\,(\omega^{24} + \omega^{2\bar{4}}), \\ d\omega^4 = i\,\nu\,\omega^{1\bar{1}} + i\,b\,\omega^{1\bar{2}} + i\,\omega^{1\bar{3}} - i\,b\,\omega^{2\bar{1}} - \mu\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}, \end{cases}$$

*where $\varepsilon, \nu, \mu \in \{0,1\}$, $A \in \mathbb{C}$ such that $\mathfrak{Im}\, A = 0$ for $\varepsilon = 1$, and $b \in \mathbb{R}$. In addition,*

- *if* $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ *then* $\nu = 0$, $(\varepsilon, \mu) \neq (0,0)$; *and*
- *if* $(\dim \mathfrak{g}_k)_k = (1,3,5,6,8)$ *then* $\nu = 1$, $(\varepsilon, \mu) = (1,0)$.

*Proof.* As a consequence of the Lie brackets obtained in the proof of Lemma 4.3.3 (see Lemmas B.0.3 and B.0.4 in Appendix B for a summarized version) we have real structure equations:

$$\begin{cases} de^1 = -a^1_{24}\,e^{24} - b^1_{24}\,e^{28} - a^1_{34}\,e^{34} - b^1_{33}\,e^{37} - b^1_{34}\,e^{38} \\ \qquad\quad -b^1_{24}\,e^{46} - b^1_{34}\,e^{47} - b^1_{44}\,e^{48} - a^1_{24}\,e^{68} - a^1_{34}\,e^{78}, \\ de^2 = -a^2_{34}\,e^{34} - b^2_{31}\,e^{35} - b^2_{34}\,e^{38} - b^2_{41}\,e^{45} - b^2_{43}\,e^{47} \\ \qquad\quad -b^2_{44}\,e^{48} - c^2_{31}\,e^{57} - c^2_{41}\,e^{58} - (a^2_{34} - c^2_{34} + c^2_{43})\,e^{78}, \\ de^3 = -b^3_{41}\,e^{45} - b^3_{44}\,e^{48} - c^3_{41}\,e^{58}, \\ de^4 = 0, \\ de^5 = -c^1_{44}\,e^{48}, \\ de^6 = -\alpha^2_{34}\,e^{34} - c^2_{31}\,e^{35} - c^2_{34}\,e^{38} - c^2_{41}\,e^{45} - c^2_{43}\,e^{47} \\ \qquad\quad -c^2_{44}\,e^{48} + b^2_{31}\,e^{57} + b^2_{41}\,e^{58} - (\alpha^2_{34} + b^2_{34} - b^2_{43})\,e^{78}, \\ de^7 = -c^3_{41}\,e^{45} - c^3_{44}\,e^{48} + b^3_{41}\,e^{58}, \\ de^8 = 0, \end{cases}$$

where the structure constants satisfy $c^1_{44} = 0$ for $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$, and $b^2_{31} = c^2_{31} = b^1_{33} = 0$ for $(\dim \mathfrak{g}_k)_k = (1,3,5,6,8)$. Considering the $(1,0)$-basis $\{\eta^k\}_{k=1}^4$ given by (4.32), we obtain

$$\begin{cases} d\eta^1 &= 0, \\ d\eta^2 &= -P\,\eta^{14} + M\,\eta^{1\bar{1}} + P\,\eta^{1\bar{4}}, \\ d\eta^3 &= (A'-B)\,\eta^{12} - C\,\eta^{14} + D\,\eta^{1\bar{1}} + (E+B)\,\eta^{1\bar{2}} + C\,\eta^{1\bar{4}} \\ & \quad -F\,\eta^{24} + (A'-E)\,\eta^{2\bar{1}} + F\,\eta^{2\bar{4}}, \\ d\eta^4 &= N\,\eta^{1\bar{1}} + H\,\eta^{1\bar{2}} + K\,\eta^{1\bar{3}} - \bar{H}\,\eta^{2\bar{1}} + \frac{i\,b^1_{33}}{2}\,\eta^{2\bar{2}} - \bar{K}\,\eta^{3\bar{1}}, \end{cases}$$

where

$$A' = \frac{c^2_{34} + i\,b^2_{34}}{2}, \quad B = \frac{c^2_{43} + i\,b^2_{43}}{2}, \quad C = \frac{c^2_{41} + i\,b^2_{41}}{2}, \quad D = \frac{c^2_{44} + i\,b^2_{44}}{2},$$

$$E = \frac{a^2_{34} - i\,\alpha^2_{34}}{2}, \quad F = \frac{c^2_{31} + i\,b^2_{31}}{2}, \quad H = \frac{a^1_{34} + i\,b^1_{34}}{2}, \quad K = \frac{a^1_{24} + i\,b^1_{24}}{2},$$

$$M = \frac{c^3_{44} + i\,b^3_{44}}{2}, \quad N = \frac{c^1_{44} + i\,b^1_{44}}{2} \quad P = \frac{c^3_{41} + i\,b^3_{41}}{2}.$$

Simply observe that $N = \frac{i\,b^1_{44}}{2}$ for $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ and $F = b^1_{33} = 0$ and $\mathfrak{Re}\,N \neq 0$ for $(\dim \mathfrak{g}_k)_k = (1,3,5,6,8)$. The Jacobi condition $d^2\eta^k = 0$, where $k = 1,2,3,4$, gives the system of equations:

$$\begin{cases} P(E-A') + (B+E)\bar{P} + FM = 0, \\ K(\bar{B}+\bar{E}) + (B-A')\bar{K} - \frac{i\,b^1_{33}}{2}\,\bar{M} = 0, \\ \mathfrak{Re}\,\big(K\bar{C} + H\bar{P}\big) = 0, \\ K\bar{F} - \frac{i\,b^1_{33}}{2}\,P = 0. \end{cases} \tag{4.36}$$

Furthermore, it is important to note that one always has $K, P \neq 0$, because $K = 0$ implies $X_2, Y_2 \in \mathfrak{g}_1$, and $P = 0$ leads to $Y_1 \in \mathfrak{g}_1$, both contradicting $\dim \mathfrak{g}_1 = 1$. In addition, the third expression in (4.36) implies that $K\bar{C} + H\bar{P}$ is a purely imaginary number. We separate different cases, according to the values of the structure constants.

- We consider $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ with $F = 0$, that is,

  $$N = \frac{i\,b^1_{44}}{2}, \quad F = 0.$$

  From the fourth equation in (4.36), it is clear that $b^1_{33} = 0$ (recall $P \neq 0$).

  In addition, if we take $E = -B$, then the equations (4.36) give $A' = B = E = 0$. However, these values joined to $F = 0$ lead to $X_3, Y_3 \in \mathfrak{g}_2$, which is a contradiction. Hence, $B + E \neq 0$.

At this point, we can define a new $(1,0)$-basis as follows:

$$\begin{aligned}
\omega^1 &= \frac{i\,PK(\bar{B}+\bar{E})}{|KP\,(B+E)|^{2/3}}\,\eta^1,\\
\omega^2 &= \frac{1}{(B+E)\,\bar{K}}\left(\eta^2 + \frac{1}{2\,\bar{P}}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\eta^1\right),\\
\omega^3 &= \frac{1}{|KP\,(B+E)|^{2/3}}\left(\frac{i}{B+E}\,(P\,\eta^3 - C\,\eta^2) + \right.\\
&\qquad \left.\frac{P}{4\,\bar{K}\,(B+E)}\left(b^1_{44} + 2\,i\,\frac{K\bar{C}+H\bar{P}}{|P|^2}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\right)\eta^1\right),\\
\omega^4 &= \frac{i\,|P|^{2/3}}{|K\,(B+E)|^{4/3}}\left(\eta^4 + \frac{\bar{M}}{\bar{P}}\,\eta^1\right).
\end{aligned}$$

Bearing in mind (4.36) with $F = b^1_{33} = 0$, one obtains the structure equations given by (4.35) with $\nu = \mu = 0$, $\varepsilon = 1$, and

$$A = \frac{1}{|KP\,(B+E)|^{4/3}}\,\mathfrak{Im}\left(\frac{CM-DP}{B+E}\right), \quad b = -\frac{i\,(K\bar{C}+H\bar{P})}{|KP\,(B+E)|^{2/3}}.$$

- Let $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ with $F \neq 0$ and $B+E=0$, i.e.,

$$N = \frac{i\,b^1_{44}}{2}, \quad F \neq 0, \quad E = -B.$$

We first observe that, as a consequence of the last equation in (4.36), one has $b^1_{33} \neq 0$. The following change of basis is considered:

$$\begin{aligned}
\omega^1 &= \sqrt{\frac{b^1_{33}}{2}}\,P\,\eta^1, & \omega^3 &= -i\sqrt{\frac{b^1_{33}}{2}}\left(\frac{1}{F}\,\eta^3 - \frac{C}{FP}\,\eta^2 + \frac{b^1_{44}}{2\,b^1_{33}\bar{P}}\,\eta^1\right),\\
\omega^2 &= -i\sqrt{\frac{b^1_{33}}{2}}\,\eta^2, & \omega^4 &= i\left(\eta^4 + \frac{\bar{M}}{\bar{P}}\,\eta^1\right).
\end{aligned}$$

By simply applying equations (4.36) with $E=-B$, we obtain (4.35) with structure constants $\nu = 0$, $\mu = 1$, $\varepsilon = 0$, and

$$A = \frac{i\sqrt{2}\,(CM-DP)}{\sqrt{b^1_{33}}\,|P|^2\,FP}, \quad b = -\frac{2\,i\,(K\bar{C}+H\bar{P})}{b^1_{33}\,|P|^2}.$$

- We take $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ with $F \neq 0$ and $B+E \neq 0$, i.e.,

$$N = \frac{i\,b^1_{44}}{2}, \quad F \neq 0, \quad E \neq -B.$$

By the last equation in (4.36), it is important to note that $K = i\,\frac{b^1_{33}P}{2\,\bar{F}}$. Since $K \neq 0$, it is clear that also $b^1_{33} \neq 0$. One can take the following change of basis:

$$\omega^1 = -\left(\frac{b^1_{33}\,|F\,P|^2}{2\,|B+E|^2}\right)^{1/3} \frac{P\,(\bar{B}+\bar{E})}{\bar{F}\bar{P}}\,\eta^1,$$

$$\omega^2 = i\left(\frac{b^1_{33}\,|F\,P|^2}{2\,|B+E|^2}\right)^{2/3} \left(\frac{\bar{B}+\bar{E}}{\bar{F}\bar{P}}\right)\left(\eta^2 + \frac{1}{2\,\bar{P}}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\eta^1\right),$$

$$\omega^3 = \frac{i\,b^1_{33}}{2\,(B+E)}\left[P\,\eta^3 + \left(\frac{F}{2\,\bar{P}}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right) - C\right)\eta^2 + \frac{FP}{8\,\bar{P}}\left(\frac{4\,b^1_{44}}{b^1_{33}} + \frac{1}{|P|^2}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\left(3\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right) + \frac{8\,i\,(K\bar{C}+H\bar{P})}{b^1_{33}}\right)\right)\eta^1\right],$$

$$\omega^4 = i\left(\frac{b^1_{33}\,|F\,P|^2}{2\,|B+E|^2}\right)^{1/3}\left(\eta^4 + \frac{\bar{M}}{\bar{P}}\,\eta^1\right).$$

Bearing in mind (4.36) and the relation $\bar{F}\left(F\bar{M} + \bar{P}\,(B-A')\right) = FP\,(\bar{B}+\bar{E})$, which comes from the second and fourth equations in (4.36), one reaches (4.35) with $\nu = 0$, $\mu = 1$, $\varepsilon = 1$,

$$A = \frac{1}{|P|^2}\left(\frac{b^1_{33}\,|F\,P|^2}{2\,|B+E|^2}\right)^{1/3}\mathfrak{Im}\left(\frac{CM-DP}{B+E}\right),$$

$$b = \frac{1}{b^1_{33}|P|^2}\left(\frac{b^1_{33}\,|F\,P|^2}{2\,|B+E|^2}\right)^{1/3}\left(b^1_{33}\,\mathfrak{Re}\left(\frac{CM-DP}{B+E}\right) - 2\,i\,(K\bar{C}+H\bar{P})\right).$$

- Let us now study the case $(\dim \mathfrak{g}_k)_k = (1,3,5,6,8)$, that is,

$$F = b^1_{33} = 0, \quad \mathfrak{Re}\,N \neq 0.$$

If we assume $B+E=0$, then the first equation in (4.36) becomes $P\,(E-A') = 0$. Since $P \neq 0$, we can conclude $E = A$ and $B = -A'$. Substituting these values in the second expression of (4.36), we obtain $A'\,\bar{K} = 0$, and as $K \neq 0$, one necessarily has $A' = B = E = 0$. However, this implies that $\alpha^2_{34} = a^2_{34} = b^2_{34} = b^2_{43} = c^2_{34} = c^2_{43} = 0$, so $X_3, Y_3 \in \mathfrak{g}_2$, which is a contradiction. Hence, $B+E \neq 0$.

We define the $(1,0)$-basis:

$$\omega^1 = \frac{i\,KP\,(\bar{B}+\bar{E})}{|KP\,(B+E)|^{2/3}}\,\eta^1,$$

$$\omega^2 = \frac{K\,(\bar{B}+\bar{E})}{\mathfrak{Re}\,N}\left(\eta^2 + \frac{1}{2\,\bar{P}}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\eta^1\right),$$

$$\omega^3 = \frac{i\,|K|^2(\bar B+\bar E)}{\mathfrak{Re}\,N\,|KP\,(B+E)|^{2/3}}\Bigg(P\,\eta^3 - C\,\eta^2 - \frac{P}{2\,\bar K}\left(i\,\mathfrak{Im}\,N - \frac{K\bar C+H\bar P}{|P|^2}\,\mathfrak{Re}\left(\frac{DP-CM}{B+E}\right)\right)\eta^1\Bigg),$$

$$\omega^4 = \frac{i\,|KP\,(B+E)|^{2/3}}{\mathfrak{Re}\,N}\left(\eta^4 + \frac{\bar M}{\bar P}\,\eta^1\right).$$

Using equations (4.36), one obtains (4.35) with $\nu=1$, $\mu=0$, $\varepsilon=1$, and

$$A = \frac{|K\,(B+E)|^{2/3}}{|P|^{4/3}\,\mathfrak{Re}\,N}\,\mathfrak{Im}\left(\frac{CM-DP}{B+E}\right),\qquad b = -\frac{i\,(K\bar C+H\bar P)}{|KP\,(B+E)|^{2/3}}.$$

This finishes our proof. □

**Lemma 4.4.3.** *Let $J$ be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that $\dim\mathfrak{g}_1 = 1$ and $\dim\mathfrak{g}_2 = 4$. The pair $(\mathfrak{g}, J)$ is parametrized by the structure equations*

$$(4.37)\qquad \begin{cases} d\omega^1 = 0,\\ d\omega^2 = \varepsilon\,\omega^{1\bar 1},\\ d\omega^3 = \omega^{14} \pm i\,\mu\,b\,\omega^{1\bar 2} + \omega^{1\bar 4},\\ d\omega^4 = i\,\nu\,\omega^{1\bar 1} \pm i\,\omega^{1\bar 3} + b\,\omega^{2\bar 2} \mp i\,\omega^{3\bar 1},\end{cases}$$

*where $\varepsilon,\mu,\nu\in\{0,1\}$ and $b\in\mathbb{R}$. Furthermore,*

- *if $(\dim\mathfrak{g}_k)_k = (1,4,8)$, then $\mu=\nu$ and $(\varepsilon,\nu)=(0,0)$ or $(\varepsilon,\nu,b)=(1,0,0)$, $(1,1,\pm2)$; whereas*
- *if $(\dim\mathfrak{g}_k)_k = (1,4,6,8)$, then $\mu=\varepsilon$ and $(\varepsilon,\nu)\neq(0,0)$.*

*Proof.* Using the Lie brackets calculated in the proof of Proposition 4.3.4 (see their description in Lemmas B.0.5 and B.0.6 of the Appendix B), one obtains real structure equations

$$\begin{cases} de^1 = -a^1_{24}\,e^{24} - b^1_{24}\,e^{28} - a^1_{34}\,e^{34} - b^1_{33}\,e^{37} - b^1_{34}\,e^{38}\\ \qquad\quad -b^1_{24}\,e^{46} - b^1_{34}\,e^{47} - b^1_{44}\,e^{48} - a^1_{24}\,e^{68} - a^1_{34}\,e^{78},\\ de^2 = -b^2_{41}\,e^{45} - b^2_{43}\,e^{47} - b^2_{44}\,e^{48} - c^2_{41}\,e^{58} - c^2_{43}\,e^{78},\\ de^3 = -b^3_{44}\,e^{48},\\ de^4 = 0,\\ de^5 = -c^1_{44}\,e^{48},\\ de^6 = -c^2_{41}\,e^{45} - c^2_{43}\,e^{47} - c^2_{44}\,e^{48} + b^2_{41}\,e^{58} + b^2_{43}\,e^{78},\\ de^7 = -c^3_{44}\,e^{48},\\ de^8 = 0,\end{cases}$$

where $c^1_{44} = c^3_{44} = 0$ when one has $(\dim \mathfrak{g}_k)_k = (1,4,8)$ and $(c^1_{44}, c^3_{44}) \neq (0,0)$ when $(\dim \mathfrak{g}_k)_k = (1,4,6,8)$. If we construct the basis $\{\eta^k\}_{k=1}^4$ given by (4.32), we get

$$\begin{cases} d\eta^1 &= 0, \\ d\eta^2 &= M\,\eta^{1\bar{1}}, \\ d\eta^3 &= -B\,\eta^{12} - C\,\eta^{14} + D\,\eta^{1\bar{1}} + B\,\eta^{1\bar{2}} + C\,\eta^{1\bar{4}}, \\ d\eta^4 &= N\,\eta^{1\bar{1}} + H\,\eta^{1\bar{2}} + K\,\eta^{1\bar{3}} - \bar{H}\,\eta^{2\bar{1}} + \frac{i\,b^1_{33}}{2}\,\eta^{2\bar{2}} - \bar{K}\,\eta^{3\bar{1}}, \end{cases}$$

where

$$B = \frac{c^2_{43} + i\,b^2_{43}}{2}, \quad C = \frac{c^2_{41} + i\,b^2_{41}}{2}, \quad D = \frac{c^2_{44} + i\,b^2_{44}}{2}, \quad H = \frac{a^1_{34} + i\,b^1_{34}}{2},$$

$$K = \frac{a^1_{24} + i\,b^1_{24}}{2}, \quad M = \frac{c^3_{44} + i\,b^3_{44}}{2}, \quad N = \frac{c^1_{44} + i\,b^1_{44}}{2}.$$

Let us note that $M = \frac{i\,b^3_{44}}{2}$, $N = \frac{i\,b^1_{44}}{2}$ for $(\dim \mathfrak{g}_k)_k = (1,4,8)$, whereas $(\mathfrak{Re}\,M, \mathfrak{Re}\,N) \neq (0,0)$ for $(\dim \mathfrak{g}_k)_k = (1,4,6,8)$. Applying the condition $d^2\eta^k = 0$, for $k = 1,2,3,4$, the following relations hold:

$$\text{(4.38)} \qquad \begin{cases} \mathfrak{Re}\,(K\bar{C}) = 0, \\ K\bar{B} + B\bar{K} + \frac{i\,b^1_{33}}{2}\,M = 0. \end{cases}$$

It is important to note the following. If $K = 0$ then $a^1_{24} = b^1_{24} = 0$, which at the sight of the Lie brackets implies that $X_2, Y_2 \in \mathfrak{g}_1$, which is a contradiction. Similarly, if $C = 0$ then we get $Y_1 \in \mathfrak{g}_1$, again contradicting $\dim \mathfrak{g}_1 = 1$. Therefore, $C, K \neq 0$. Applying this fact, equations (4.38) yield

$$\text{(4.39)} \qquad B\bar{C} - C\bar{B} = \frac{i\,b^1_{33} C M}{2K}.$$

Moreover, if we suppose that $B = 0$, then one can see that $Y_3 \in \mathfrak{g}_2$, which is not possible by hypothesis. Hence, we also have that $B \neq 0$. In addition, from the first equation in (4.38) one necessarily has $\mathfrak{Im}\,(K\bar{C}) \neq 0$. We define the following $(1,0)$-basis:

$$\tau^1 = \sqrt{|\mathfrak{Im}\,(K\bar{C})|}\,\eta^1, \quad \tau^3 = -i\,\sqrt{|\mathfrak{Im}\,(K\bar{C})|}\left(\frac{1}{C}\,\eta^3 + \frac{\bar{H}}{C\bar{K}}\,\eta^2 + \frac{\mathfrak{Im}\,\left(\frac{BM+CN}{C}\right)}{2\,\mathfrak{Im}\,(K\bar{C})}\,\eta^1\right),$$

$$\tau^2 = \eta^2, \qquad \tau^4 = i\left(\eta^4 + \frac{B}{C}\,\eta^2 + \frac{H\bar{M} + K\bar{D}}{K\bar{C}}\,\eta^1\right).$$

We next distinguish different possibilities according to the values of the parameters.

- Let $M = 0$ and $\mathfrak{Re}\,N = 0$. Notice that these two conditions can only hold for the case $(\dim \mathfrak{g}_k)_k = (1,4,8)$. It suffices to rename the basis $\{\tau^k\}_{k=1}^4$ in order to obtain (4.37) with

$$\varepsilon = \mu = \nu = 0, \quad b = -\frac{b^1_{33}}{2}.$$

- We now consider $M = 0$ and $\mathfrak{Re}\, N \neq 0$. Then, we are in the case for which $(\dim \mathfrak{g}_k)_k = (1, 4, 6, 8)$. Considering

$$\omega^1 = \tau^1, \quad \omega^2 = \tau^2, \quad \omega^3 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\, N}\, \tau^3, \quad \omega^4 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\, N}\, \tau^4,$$

we reach equations (4.37) with structure constants

$$\varepsilon = \mu = 0, \quad \nu = 1, \quad b = -\frac{b^1_{33}\, |\mathfrak{Im}\,(K\bar{C})|}{2\, \mathfrak{Re}\, N}.$$

- Take $M \neq 0$ and $\mathfrak{Re}\, \left(\frac{BM+CN}{C}\right) = 0$. Note that these two conditions can hold for both $(\dim \mathfrak{g}_k)_k = (1, 4, 8)$ and $(1, 4, 6, 8)$. If we consider the change of basis

$$\omega^1 = \tau^1, \quad \omega^2 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{M}\, \tau^2, \quad \omega^3 = \tau^3, \quad \omega^4 = \tau^4,$$

then the structure equations become (4.37). Concerning the values of the structure constants, we observe two different things depending on the dimension of the ascending central series.

  - If $(\dim \mathfrak{g}_k)_k = (1, 4, 8)$, then $M = \frac{i\, b^3_{44}}{2} \neq 0$ and $N = \frac{i\, b^1_{44}}{2}$. As a consequence,

$$\mathfrak{Re}\, \left(\frac{BM+CN}{C}\right) = -\frac{b^3_{44}}{2\, |C|^2}\, \mathfrak{Im}\,(B\bar{C}). \tag{4.40}$$

    Since (4.40) annihilates and $b^3_{44} \neq 0$, one necessarily has $\mathfrak{Im}\,(B\bar{C}) = 0$. From (4.39), we conclude $b^1_{33} = 0$. Hence, we get

$$\varepsilon = 1, \quad \mu = \nu = 0, \quad b = 0.$$

  - If $(\dim \mathfrak{g}_k)_k = (1, 4, 6, 8)$, then the previous change gives

$$\varepsilon = \mu = 1, \quad \nu = 0, \quad b = -\frac{b^1_{33}\, |M|^2}{2\, \left(\mathfrak{Im}\,(K\bar{C})\right)^2}.$$

- We finally study the case $M \neq 0$ and $\mathfrak{Re}\, \left(\frac{BM+CN}{C}\right) \neq 0$, valid for the two possible ascending central series. We define

$$\omega^1 = \tau^1, \;\; \omega^2 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{M}\, \tau^2, \;\; \omega^3 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\, \left(\frac{BM+CN}{C}\right)}\, \tau^3, \;\; \omega^4 = \frac{|\mathfrak{Im}\,(K\bar{C})|}{\mathfrak{Re}\, \left(\frac{BM+CN}{C}\right)}\, \tau^4.$$

After this change of basis, the structure equations become (4.37) with

$$\varepsilon = \mu = \nu = 1, \quad b = -\frac{b^1_{33}\, |M|^2}{2\, |\mathfrak{Im}\,(K\bar{C})|\, \mathfrak{Re}\, \left(\frac{BM+CN}{C}\right)}.$$

In particular, one can see that $b = \pm 2$ for $(\dim \mathfrak{g}_k)_k = (1, 4, 8)$, as a consequence of (4.38) and (4.40).

As we have covered all the admissible cases, this finishes the proof of the lemma. □

**Lemma 4.4.4.** *Let $J$ be an SnN complex structure on an 8-dimensional NLA $\mathfrak{g}$ such that $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g}_2 = 5$. The pair $(\mathfrak{g}, J)$ is parametrized by the structure equations*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \varepsilon\, \omega^{1\bar{1}}, \\ d\omega^3 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^4 = i\,\nu\, \omega^{1\bar{1}} \pm i\, \omega^{1\bar{3}} + (1-\varepsilon)\, b\, \omega^{2\bar{2}} \mp i\, \omega^{3\bar{1}}, \end{cases} \tag{4.41}$$

*where $\varepsilon, \nu \in \{0, 1\}$ and $b \in \mathbb{R}$. Furthermore,*

- *if $(\dim \mathfrak{g}_k)_k = (1, 5, 8)$, then $\nu = 0$; and*
- *if $(\dim \mathfrak{g}_k)_k = (1, 5, 6, 8)$, then $\nu = 1$.*

*Proof.* If we consider Lie brackets calculated in the proof of Proposition 4.3.4 (see also Lemmas B.0.7 and B.0.8 of the Appendix B), then we obtain real structure equations

$$\begin{cases} de^1 &=& -a^1_{23}\, e^{23} - a^1_{24}\, e^{24} - b^1_{22}\, e^{26} - b^1_{23}\, e^{27} - b^1_{24}\, e^{28} - a^1_{34}\, e^{34} - b^1_{23}\, e^{36} - b^1_{33}\, e^{37} \\ & & -b^1_{34}\, e^{38} - b^1_{24}\, e^{46} - b^1_{34}\, e^{47} - b^1_{44}\, e^{48} - a^1_{23}\, e^{67} - a^1_{24}\, e^{68} - a^1_{34}\, e^{78}, \\ de^2 &=& -b^2_{41}\, e^{45} - b^2_{44}\, e^{48} - c^2_{41}\, e^{58}, \\ de^3 &=& -b^3_{41}\, e^{45} - b^3_{44}\, e^{48} - c^3_{41}\, e^{58}, \\ de^4 &=& 0, \\ de^5 &=& -c^1_{44}\, e^{48}, \\ de^6 &=& -c^2_{41}\, e^{45} - c^2_{44}\, e^{48} + b^2_{41}\, e^{58}, \\ de^7 &=& -c^3_{41}\, e^{45} - c^3_{44}\, e^{48} + b^3_{41}\, e^{58}, \\ de^8 &=& 0, \end{cases}$$

where $c^1_{44} = 0$ for $(\dim \mathfrak{g}_k)_k = (1, 5, 8)$ and $c^1_{44} \neq 0$ for $(\dim \mathfrak{g}_k)_k = (1, 4, 6, 8)$. Constructing the $(1, 0)$-basis $\{\eta^k\}_{k=1}^4$ described in (4.32), we get

$$\begin{cases} d\eta^1 &=& 0, \\ d\eta^2 &=& -P\, \eta^{14} + M\, \eta^{1\bar{1}} + P\, \eta^{1\bar{4}}, \\ d\eta^3 &=& -C\, \eta^{14} + D\, \eta^{1\bar{1}} + C\, \eta^{1\bar{4}}, \\ d\eta^4 &=& N\, \eta^{1\bar{1}} + H\, \eta^{1\bar{2}} + K\, \eta^{1\bar{3}} - \bar{H}\, \eta^{2\bar{1}} + \frac{i\, b^1_{33}}{2}\, \eta^{2\bar{2}} + L\, \eta^{2\bar{3}} - \bar{K}\, \eta^{3\bar{1}} - \bar{L}\, \eta^{3\bar{2}} + \frac{i\, b^1_{22}}{2}\, \eta^{3\bar{3}}, \end{cases}$$

where

$$C = \frac{c^2_{41} + i\, b^2_{41}}{2}, \quad D = \frac{c^2_{44} + i\, b^2_{44}}{2}, \quad H = \frac{a^1_{34} + i\, b^1_{34}}{2}, \quad K = \frac{a^1_{24} + i\, b^1_{24}}{2},$$

$$L=\frac{a_{23}^1+i\,b_{23}^1}{2},\quad M=\frac{c_{44}^3+i\,b_{44}^3}{2},\quad N=\frac{c_{44}^1+i\,b_{44}^1}{2},\quad P=\frac{c_{41}^3+i\,b_{41}^3}{2}.$$

Moreover, $N=\frac{i\,b_{44}^1}{2}$ for $(\dim\mathfrak{g}_k)_k=(1,5,8)$, and $\mathfrak{Re}\,N\neq 0$ for $(\dim\mathfrak{g}_k)_k=(1,5,6,8)$. As a consequence of $d^2\eta^k=0$, for $k=1,2,3,4$, the following relations among the parameters are found:

$$\begin{cases} 2\,L\bar{D}+i\,b_{33}^1\,\bar{M}=0,\\ 2\,LM+i\,b_{22}^1\,D=0,\\ \mathfrak{Re}\,\left(K\bar{C}+P\bar{H}\right)=0,\\ 2\,L\bar{C}+i\,b_{33}^1\,\bar{P}=0,\\ 2\,LP+i\,b_{22}^1\,C=0.\end{cases}\tag{4.42}$$

First, we are going to see that there exists a basis $\{\tau^k\}_{k=1}^4$ for $\mathfrak{g}^{1,0}$ in terms of which the structure equations are:

$$\begin{cases} d\tau^1 &= 0,\\ d\tau^2 &= A_{1\bar{1}}\,\tau^{1\bar{1}},\\ d\tau^3 &= \tau^{14}+\tau^{1\bar{4}},\\ d\tau^4 &= i\nu\,\tau^{1\bar{1}}\pm i\,\eta^{1\bar{3}}+B_{2\bar{2}}\,\eta^{2\bar{2}}\mp i\,\eta^{2\bar{3}},\end{cases}\tag{4.43}$$

where $\nu\in\{0,1\}$, $A_{1\bar{1}}\in\mathbb{C}$, and $B_{2\bar{2}}\in\mathbb{R}$. We distinguish two different cases.

- Let us take $C=0$. We observe the following. If we also have $P=0$, then $b_{41}^2=b_{41}^3=c_{41}^2=c_{41}^3=0$ and $Y_1\in\mathfrak{g}_1$, which is a contradiction. Hence, one can assume $P\neq 0$, and from the last two equations in (4.42) we conclude that $b_{33}^1=L=0$. Moreover, let us note that this implies $H\neq 0$, because otherwise we would get $a_{23}^1=a_{34}^1=b_{23}^1=b_{33}^1=b_{34}^1=0$ and $X_3,Y_3\in\mathfrak{g}_1$. Hence, we have

  $$C=L=b_{33}^1=0,\quad H,P\neq 0.$$

  Furthermore, from the third equation in (4.42) and the fact that $H,P\neq 0$, it is clear that $\mathfrak{Im}\,(P\bar{H})\neq 0$. We start defining the $(1,0)$-forms

  $$\sigma^1=\sqrt{|\mathfrak{Im}\,(H\bar{P})|}\,\eta^1,\quad \sigma^3=-i\sqrt{|\mathfrak{Im}\,(H\bar{P})|}\left(\frac{1}{P}\,\eta^2+\frac{\bar{K}}{P\bar{H}}\,\eta^3+\frac{\mathfrak{Im}\,N}{2\,\mathfrak{Im}\,(H\bar{P})}\,\eta^1\right),$$
  $$\sigma^2=\eta^3,\qquad \sigma^4=i\left(\eta^4+\frac{H\bar{M}+K\bar{D}}{H\bar{P}}\,\eta^1\right),$$

  and then distinguish two cases according to the nilpotency step of $\mathfrak{g}$.

  - Let $\mathfrak{Re}\,N=0$, that is, $(\dim\mathfrak{g}_k)_k=(1,5,8)$. It suffices to rename $\sigma^k=\tau^k$, for each $k=1,2,3,4$, in order to obtain equations (4.43) with

    $$\nu=0,\quad A_{1\bar{1}}=\frac{D}{|\mathfrak{Im}\,(H\bar{P})|},\quad B_{2\bar{2}}=-\frac{b_{22}^1}{2}.$$

– Consider $\mathfrak{Re}\, N \neq 0$, i.e., $(\dim \mathfrak{g}_k)_k = (1, 5, 6, 8)$. Then, we take

$$\tau^1 = \sigma^1, \quad \tau^2 = \sigma^2, \quad \tau^3 = \frac{|\mathfrak{Im}\,(H\bar{P})|}{\mathfrak{Re}\, N}\, \sigma^3, \quad \tau^4 = \frac{|\mathfrak{Im}\,(H\bar{P})|}{\mathfrak{Re}\, N}\, \sigma^4,$$

and we reach the structure equations (4.43) with

$$\nu = 1, \quad A_{1\bar{1}} = \frac{D}{|\mathfrak{Im}\,(H\bar{P})|}, \quad B_{2\bar{2}} = -\frac{b^1_{22}\, |\mathfrak{Im}\,(H\bar{P})|}{2\, \mathfrak{Re}\, N}.$$

- Let us suppose that $C \neq 0$. We first apply the change of basis

$$\varphi^1 = \eta^1, \quad \varphi^2 = \eta^2 - \frac{P}{C}\, \eta^3, \quad \varphi^3 = \eta^3, \quad \varphi^4 = \eta^4.$$

Using the fourth and fifth equations in (4.42) in order to simplify the appropriate expressions, one obtains the following structure equations:

$$\begin{cases} d\varphi^1 &= 0, \\ d\varphi^2 &= \frac{CM-DP}{C}\, \varphi^{1\bar{1}}, \\ d\varphi^3 &= -C\, \varphi^{14} + D\, \varphi^{1\bar{1}} + C\, \varphi^{1\bar{4}}, \\ d\varphi^4 &= N\, \varphi^{1\bar{1}} + H\, \varphi^{1\bar{2}} + \frac{K\bar{C}+H\bar{P}}{\bar{C}}\, \varphi^{1\bar{3}} - \bar{H}\, \varphi^{2\bar{1}} + \frac{i\, b^1_{33}}{2}\, \varphi^{2\bar{2}} - \frac{P\bar{H}+C\bar{K}}{C}\, \varphi^{3\bar{1}}. \end{cases}$$

At this point, it is important to note the following. If $K\bar{C} + H\bar{P} = 0$, then we can define a new basis $\omega^1 = \varphi^1$, $\omega^2 = \varphi^2$, $\omega^3 = \varphi^4$, $\omega^4 = \varphi^3$ in terms of which the structure equations follow Theorem 3.1.8. However, this implies that $J$ is nilpotent, which is not possible by hypothesis. Necessarily, we have $K\bar{C}+H\bar{P} \neq 0$. Moreover, from the third expression in (4.42) one necessarily has $\mathfrak{Im}\,(K\bar{C} + H\bar{P}) \neq 0$.

We take the $(1,0)$-basis given by

$$\begin{aligned} \sigma^1 &= \sqrt{|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}\, \varphi^1, \\ \sigma^2 &= \varphi^2, \\ \sigma^3 &= -i\, \sqrt{|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|} \left( \frac{1}{C}\, \varphi^3 + \frac{\bar{H}}{C\bar{K} + P\bar{H}}\, \varphi^2 + \frac{\mathfrak{Im}\, N}{2\, \mathfrak{Im}\,(K\bar{C} + H\bar{P})}\, \varphi^1 \right), \\ \sigma^4 &= i \left( \varphi^4 + \frac{(K\bar{C} + H\bar{P})\, \bar{D} + H\, (\bar{C}\bar{M} - \bar{D}\bar{P})}{\bar{C}\, (K\bar{C} + H\bar{P})}\, \varphi^1 \right), \end{aligned}$$

and then observe the following.

– Assume $\mathfrak{Re}\, N = 0$, and thus $(\dim \mathfrak{g}_k)_k = (1, 5, 8)$. Renaming $\sigma^k = \tau^k$, for each $k = 1, 2, 3, 4$, and using the relations (4.42), we directly obtain (4.43) with parameters

$$\nu = 0, \quad A_{1\bar{1}} = \frac{CM - DP}{C\, |\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}, \quad B_{2\bar{2}} = -\frac{b^1_{33}}{2}.$$

– Let $\mathfrak{Re}\, N \neq 0$, i.e., $(\dim \mathfrak{g}_k)_k = (1,5,6,8)$. We apply another change of basis, given by

$$\tau^1 = \sigma^1, \quad \tau^2 = \sigma^2, \quad \tau^3 = \frac{|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}{\mathfrak{Re}\, N}\,\sigma^3, \quad \tau^4 = \frac{|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}{\mathfrak{Re}\, N}\,\sigma^4,$$

and we reach the structure equations (4.43) with

$$\nu = 1, \quad A_{1\bar{1}} = \frac{CM - DP}{C\,|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}, \quad B_{2\bar{2}} = -\frac{b^1_{33}\,|\mathfrak{Im}\,(K\bar{C} + H\bar{P})|}{2\,\mathfrak{Re}\, N}.$$

The next step consists on normalizing the coefficient $A_{1\bar{1}}$ in (4.43). Two cases should be distinguished according to the value of $A_{1\bar{1}}$.

- Let $A_{1\bar{1}} = 0$. Taking $\omega^k = \tau^k$, for $k = 1,2,3,4$, we directly obtain (4.41) with

$$\varepsilon = 0, \quad b = B_{2\bar{2}}, \quad \nu = \begin{cases} 0, & \text{for } (\dim \mathfrak{g}_k)_k = (1,5,8), \\ 1, & \text{for } (\dim \mathfrak{g}_k)_k = (1,5,6,8). \end{cases}$$

- We now consider $A_{1\bar{1}} \neq 0$ and make the following observations.

  For the case $C = 0$, the condition $A_{1\bar{1}} \neq 0$ implies that $D \neq 0$. Bearing in mind that $L = 0$, we can use the second equation in (4.42) in order to obtain $b^1_{22} = 0$. Thus, we get $B_{2\bar{2}} = 0$.

  For $C \neq 0$, the fact that $A_{1\bar{1}} \neq 0$ leads to $CM - DP \neq 0$. Solving $L$ from the fourth equation in (4.42) and replacing its value in the first expression, we can see that $b^1_{33} = 0$. As a consequence, $B_{2\bar{2}} = 0$.

  Therefore, in any case we have $B_{2\bar{2}} = 0$, and it suffices to define the basis

$$\omega^1 = \tau^1, \quad \omega^2 = \frac{1}{A_{1\bar{1}}}\,\tau^2, \quad \omega^3 = \tau^3, \quad \omega^4 = \tau^4,$$

  in order to get (4.41) with

$$\varepsilon = 1, \quad \nu = \begin{cases} 0, & \text{for } (\dim \mathfrak{g}_k)_k = (1,5,8), \\ 1, & \text{for } (\dim \mathfrak{g}_k)_k = (1,5,6,8). \end{cases}$$

This finishes the proof of our result. □

**Remark 4.4.5.** Although we have greatly reduced the number of parameters in the complex structure equations, their range of choice is not totally satisfactory. For instance, if we choose $\varepsilon = \nu = A = b = 0$ in (4.33), then the dual elements of $\omega^2 + \omega^{\bar{2}}$ and $i\,(\omega^2 - \omega^{\bar{2}})$ belong to the center of $\mathfrak{g}$, obtaining a complex structure $J$ of nilpotent type. Another illustrative example arises when we take $\varepsilon = \nu = A = 0$, $b = 1$ in (4.33) and $\varepsilon = \nu = 0$, $b = 1$ in (4.41): the structure equations coincide, but the first case should correspond to $(\dim \mathfrak{g}_k)_k = (1,3,8)$ and the second one, to $(\dim \mathfrak{g}_k)_k = (1,5,8)$. The reason is that the values of the structure constants found in the previous lemmas give necessary but not sufficient conditions to ensure the dimension of the ascending central series. We can refine these values by simply recalculating the series from the simplified complex structure equations.

**Theorem 4.4.6.** *Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with an SnN complex structure. We have:*

*i) If $(\dim \mathfrak{g}_k)_k = (1,3,8), (1,3,6,8), (1,4,8), (1,4,6,8), (1,5,8),$ or $(1,5,6,8)$, then the pair $(\mathfrak{g}, J)$ is defined by*

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \varepsilon\,\omega^{1\bar{1}}, \\ d\omega^3 &= \omega^{14} + \omega^{1\bar{4}} + A\,\omega^{2\bar{1}} \pm i\,\varepsilon\, b\,\omega^{1\bar{2}}, \\ d\omega^4 &= i\,\nu\,\omega^{1\bar{1}} \pm i\,\omega^{1\bar{3}} + b\,\omega^{2\bar{2}} \mp i\,\omega^{3\bar{1}}, \end{cases}$$

*where $\varepsilon, \nu \in \{0,1\}$, $A \in \mathbb{C}$, and $b \in \mathbb{R}$. Furthermore:*

- $(\varepsilon,\nu) = (0,0)$, $A \neq 0$, *for* $(\dim \mathfrak{g}_k)_k = (1,3,8)$,
- $(\varepsilon,\nu) \neq (0,0)$, $A \neq 0$, *for* $(\dim \mathfrak{g}_k)_k = (1,3,6,8)$,
- $(\varepsilon,\nu) = (1,1)$, $A = 0$, $b = \pm 2$, *for* $(\dim \mathfrak{g}_k)_k = (1,4,8)$,
- $(\varepsilon,\nu) = (1,\nu)$, $A = 0$, $b \neq 0, \pm 2$, *for* $(\dim \mathfrak{g}_k)_k = (1,4,6,8)$,
- $(\varepsilon,\nu) = (0,0)$, $A = 0$, $b \neq 0$, *for* $(\dim \mathfrak{g}_k)_k = (1,5,8)$,
- $(\varepsilon,\nu) = (0,1)$, $A = 0$, $b \neq 0$, *for* $(\dim \mathfrak{g}_k)_k = (1,5,6,8)$.

*ii) If $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ or $(1,3,5,6,8)$, then the pair $(\mathfrak{g}, J)$ is defined by the structure equations*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = A\,\omega^{1\bar{1}} + \varepsilon\,(\omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,\mu\,(\omega^{24} + \omega^{2\bar{4}}), \\ d\omega^4 = i\,\nu\,\omega^{1\bar{1}} + i\,b\,\omega^{1\bar{2}} + i\,\omega^{1\bar{3}} - i\,b\,\omega^{2\bar{1}} - \mu\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}, \end{cases}$$

*where $\varepsilon, \nu, \mu \in \{0,1\}$, $A \in \mathbb{C}$ such that $\mathfrak{Im}\, A = 0$ for $\varepsilon = 1$, and $b \in \mathbb{R}$. Moreover,*

- $\nu = 0$, $(\varepsilon,\mu) \neq (0,0)$, *for* $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$,
- $\nu = 1$, $(\varepsilon,\mu) = (1,0)$, *for* $(\dim \mathfrak{g}_k)_k = (1,3,5,6,8)$.

*Proof.* We start with part *i)*. In order to do so, we first observe that it is possible to collect the structure equations found in Lemmas 4.4.1, 4.4.3, and 4.4.4 (see p. 154, p.

162, and p. 165) by simply considering

$$(4.44)\qquad \begin{cases} d\omega^1 &=& 0, \\ d\omega^2 &=& \varepsilon\,\omega^{1\bar 1}, \\ d\omega^3 &=& \omega^{14}+\omega^{1\bar 4}+A\,\omega^{2\bar 1}\pm i\,\mu\,b\,\omega^{1\bar 2}, \\ d\omega^4 &=& i\,\nu\,\omega^{1\bar 1}\pm i\,\omega^{1\bar 3}+b\,\omega^{2\bar 2}\mp i\,\omega^{3\bar 1}, \end{cases}$$

where $\varepsilon,\mu,\nu\in\{0,1\}$, $A\in\mathbb{C}$, and $b\in\mathbb{R}$. We can now recover the real structure equations of our 8-dimensional NLAs taking the real basis

$$(4.45)\qquad e^k=\frac{1}{2}\,(\omega^k+\omega^{\bar k}),\quad e^{4+k}=-\frac{i}{2}\,(\omega^k-\omega^{\bar k}),$$

for $k=1,2,3,4$. If we denote $a_1=\mathfrak{Re}\,A$ and $a_2=\mathfrak{Im}\,A$, then one has

$$\begin{cases} de^1 &=& de^2 \;=\; 0, \\ de^3 &=& -a_1\,e^{12}+2\,e^{14}+(a_2\pm\mu\,b)\,e^{16}+(a_2\pm\mu\,b)\,e^{25}-a_1\,e^{56}, \\ de^4 &=& 2\,\nu\,e^{15}, \\ de^5 &=& 0, \\ de^6 &=& -2\,\varepsilon\,e^{15}, \\ de^7 &=& (-a_2\pm\mu\,b)\,e^{12}-a_1\,e^{16}-a_1\,e^{25}-2\,e^{45}+(-a_2\pm\mu\,b)\,e^{56}, \\ de^8 &=& \pm 2\,e^{13}-2\,b\,e^{26}\pm 2\,e^{57}. \end{cases}$$

We recall that these expressions completely define $\mathfrak{g}$. Nonetheless, we are now interested in the Lie brackets. Let $\{X_k\}_{k=1}^8$ be the dual basis of $\{e^k\}_{k=1}^8$. Using the well-known formula

$$d\alpha(A,B)=-\alpha([A,B]),$$

for every $\alpha\in\mathfrak{g}^*$ and every $A,B\in\mathfrak{g}$, it turns out that the only possibly non-zero Lie brackets of $\mathfrak{g}$ are:

$$\begin{array}{lll@{\qquad}lll} [X_1,X_2] &=& a_1\,X_3+(a_2\mp\mu\,b)\,X_7, & [X_2,X_5] &=& -(a_2\pm\mu\,b)\,X_3+a_1\,X_7, \\ [X_1,X_3] &=& \mp 2\,X_8, & [X_2,X_6] &=& 2\,b\,X_8, \\ [X_1,X_4] &=& -2\,X_3, & [X_4,X_5] &=& 2\,X_7, \\ [X_1,X_5] &=& -2\,\nu\,X_4+2\,\varepsilon\,X_6, & [X_5,X_6] &=& a_1\,X_3+(a_2\mp\mu\,b)\,X_7, \\ [X_1,X_6] &=& -(a_2\pm\mu\,b)\,X_3+a_1\,X_7, & [X_5,X_7] &=& \mp 2\,X_8. \end{array}$$

We next observe that any element $Y\in\mathfrak{g}$ can be written in terms of the given basis $\{X_k\}_{k=1}^8$ as follows:

$$(4.46)\qquad Y=\alpha\,X_1+\beta\,X_2+\gamma\,X_3+\delta\,X_4+\varphi\,X_5+\psi\,X_6+\zeta\,X_7+\tau\,X_8,$$

where $\alpha,\beta,\gamma,\delta,\varphi,\psi,\zeta,\tau\in\mathbb{R}$. The brackets between this generic element and each $X_k$, for $k=1,2,3,4$, will allow us to calculate the ascending central series. We will shortly give the details.

First, by direct calculation, we have:

$$
(4.47)\quad
\begin{aligned}
[Y,X_1] &= (-a_1\,\beta+2\,\delta+\psi\,(a_2\pm\mu\,b))\,X_3+2\,\nu\,\varphi\,X_4-2\,\varepsilon\,\varphi\,X_6\\
&\quad -(a_1\,\psi+\beta\,(a_2\mp\mu\,b))\,X_7\pm 2\,\gamma\,X_8,\\
[Y,X_2] &= (a_1\,\alpha+\varphi\,(a_2\pm\mu\,b))\,X_3+(-a_1\,\varphi+\alpha\,(a_2\mp\mu\,b))\,X_7-2\,b\,\psi\,X_8,\\
[Y,X_3] &= \mp 2\,\alpha\,X_8,\\
[Y,X_4] &= -2\,\alpha\,X_3-2\,\varphi\,X_7,\\
[Y,X_5] &= -(a_1\,\psi+\beta\,(a_2\pm\mu\,b))\,X_3-2\,\nu\,\alpha\,X_4+2\,\varepsilon\,\alpha\,X_6\\
&\quad +(a_1\,\beta+2\,\delta-\psi\,(a_2\mp\mu\,b))\,X_7\pm 2\,\zeta\,X_8,\\
[Y,X_6] &= (a_1\,\varphi-\alpha(a_2\pm\mu\,b))\,X_3+(a_1\,\alpha+\varphi\,(a_2\mp\mu\,b))\,X_7+2\,b\,\beta\,X_8,\\
[Y,X_7] &= \mp 2\,\varphi\,X_8,\\
[Y,X_8] &= 0.
\end{aligned}
$$

Now, let us start computing $\mathfrak{g}_1$. In order to do so, we simply suppose that $Y\in\mathfrak{g}_1$. Then, one must have $[Y,X_k]=0$ for every $k=1,\dots,8$. At the sight of the brackets $[Y,X_3]$ and $[Y,X_7]$ in (4.47), one clearly needs $\alpha=\varphi=0$. Moreover, equalling to zero the coefficients of $X_8$ in the brackets $[Y,X_1]$ and $[Y,X_5]$, one can also conclude that $\gamma=\zeta=0$. The rest of the parameters satisfy the following system of equations:

$$
\begin{cases}
b\,\psi=0,\\
b\,\beta=0,\\
a_1\,\beta-2\,\delta-\psi\,(a_2\pm\mu\,b)=0 \quad\Leftrightarrow\quad a_1\,\beta-2\,\delta-a_2\,\psi=0,\\
a_1\,\beta+2\,\delta-\psi\,(a_2\mp\mu\,b)=0 \quad\Leftrightarrow\quad a_1\,\beta+2\,\delta-a_2\,\psi=0,\\
a_1\,\psi+\beta\,(a_2\mp\mu\,b)=0 \quad\Leftrightarrow\quad a_1\,\psi+a_2\,\beta=0,\\
a_1\,\psi+\beta\,(a_2\pm\mu\,b)=0 \quad\Leftrightarrow\quad a_1\,\psi+a_2\,\beta=0.
\end{cases}
$$

We observe that some of the equations have been reduced using $b\,\psi=b\,\beta=0$. Now, from the third expression above we have $a_1\,\beta=2\,\delta+a_2\,\psi$. Replacing this value of $a_1\,\beta$ in the fourth equation, we conclude $\delta=0$. Hence, we have $\alpha=\gamma=\delta=\varphi=\zeta=0$, whereas $\beta,\psi,\tau$ fulfill the equations:

$$
\begin{cases}
b\,\psi=0,\\
b\,\beta=0,
\end{cases}
\qquad
\begin{cases}
a_1\,\beta-a_2\,\psi=0,\\
a_2\,\beta+a_1\,\psi=0.
\end{cases}
$$

In particular, let us observe that $\tau$ is free, so one always has $X_8 \in \mathfrak{g}_1$. If we assume $A = b = 0$, then also $\beta$ and $\psi$ are free, so $\dim \mathfrak{g}_1 > 1$. This is a contradiction with Corollary 4.1.12. Hence, $(A, b) \neq (0, 0)$ and $\beta = \psi = 0$, making $\mathfrak{g}_1 = \langle X_8 \rangle$.

We now move to study $\mathfrak{g}_2$, bearing in mind that $(A, b) \neq (0, 0)$ in order to ensure that $\mathfrak{g}_1$ is 1-dimensional. We focus on the brackets (4.47) and look for the conditions that make $Y \in \mathfrak{g}_2$, i.e., $[Y, X_k] \in \mathfrak{g}_1 = \langle X_8 \rangle$ for every $k = 1, \ldots, 8$. From $[Y, X_4]$, it is clear that one needs $\alpha = \varphi = 0$. The other parameters satisfy:

$$(4.48) \qquad \begin{cases} \mu\, b\, \beta = 0, \\ 2\,\delta \pm \mu\, b\, \psi = 0 \end{cases} \qquad \text{and} \qquad \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} \beta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We distinguish two possible cases according to the value of $A$.

- Let $A \neq 0$. Then, the determinant of the matrix in (4.48) equals $|A|^2$, which is non-zero, so we have $\beta = \psi = 0$. Replacing these values in the other two equations, we also get $\delta = 0$. Recall that we had already taken $\alpha = \varphi = 0$, so just $\gamma, \zeta, \tau$ remain free. Hence, we can conclude that $\mathfrak{g}_2 = \langle X_3, X_7, X_8 \rangle$ and $\dim \mathfrak{g}_2 = 3$.
- Let $A = 0$, and thus $b \neq 0$. The system in (4.48) is satisfied trivially, so we just need to focus on other two equations. Two cases can be distinguished according to the value of $\mu \in \{0, 1\}$.
  - Consider $\mu = 1$. As $b \neq 0$, using (4.48), we clearly obtain $\beta = 0$ and $\delta = \mp \frac{b}{2}\, \psi$. We also have $\alpha = \varphi = 0$, so simply $\gamma, \psi, \zeta, \tau$ remain free. In this way, we get $\mathfrak{g}_2 = \langle X_3, X_6 \mp \frac{b}{2}\, X_4, X_7, X_8 \rangle$ and $\dim \mathfrak{g}_2 = 4$.
  - Take $\mu = 0$. From (4.48), it is clear that $\delta = 0$. Recall that we also had $\alpha = \varphi = 0$. Therefore, the parameters $\beta, \gamma, \psi, \zeta, \tau$ in (4.46) are free, and we can conclude $\mathfrak{g}_2 = \langle X_2, X_3, X_6, X_7, X_8 \rangle$ and $\dim \mathfrak{g}_2 = 5$.

This finishes our study of the space $\mathfrak{g}_2$. Let us focus on $\mathfrak{g}_3$. Three cases are distinguished depending on the dimension of $\mathfrak{g}_2$.

- Let $\dim \mathfrak{g}_2 = 3$. According to the previous analysis, one has $A \neq 0$,

  $$\mathfrak{g}_1 = \langle X_8 \rangle, \quad \mathfrak{g}_2 = \langle X_3, X_7, X_8 \rangle.$$

  If an element $Y \in \mathfrak{g}$ belongs to $\mathfrak{g}_3$, then the brackets (4.47) should lie in $\mathfrak{g}_2$. This can only happen when the following equations holds:

  $$\begin{cases} \nu\, \varphi = 0, \\ \varepsilon\, \varphi = 0, \end{cases} \qquad \begin{cases} \nu\, \alpha = 0, \\ \varepsilon\, \alpha = 0. \end{cases}$$

  Two possibilities arise:

  - If $(\varepsilon, \nu) = (0, 0)$, then $\mathfrak{g}_3 = \mathfrak{g}$ and thus $(\dim \mathfrak{g}_k) = (1, 3, 8)$.

– If $(\varepsilon, \nu) \neq (0,0)$, then one needs $\alpha = \varphi = 0$, and we can conclude that

$$\mathfrak{g}_3 = \langle X_2, X_3, X_4, X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

In particular, we get $(\dim \mathfrak{g}_k) = (1,3,6,8)$.

- Suppose $\dim \mathfrak{g}_2 = 4$. In this case, we have $\mu = 1$, $A = 0$, $b \neq 0$, and

$$\mathfrak{g}_1 = \langle X_8 \rangle, \quad \mathfrak{g}_2 = \langle X_3, X_6 \mp \frac{b}{2} X_4, X_7, X_8 \rangle.$$

At the sight of (4.47), we first observe that the parameters $\beta, \gamma, \delta, \psi, \zeta, \tau$ are always free. However, the value of $\alpha$ and $\varphi$ depend on whether the vectors

$$\begin{aligned} 2\,\nu\,\varphi\,X_4 - 2\,\varepsilon\,\varphi\,X_6 &= -2\,\varphi\,(\varepsilon\,X_6 - \nu\,X_4), \\ -2\,\nu\alpha\,X_4 + 2\,\varepsilon\,\alpha\,X_6 &= 2\,\alpha\,(\varepsilon\,X_6 - \nu\,X_4) \end{aligned}$$

(see brackets $[Y, X_1]$ and $[Y, X_5]$) are linearly independent with $X_6 \mp \frac{b}{2} X_4$. In particular, it suffices to study the linear dependence between

$$Y_1 = \varepsilon\,X_6 - \nu\,X_4 \quad \text{and} \quad Y_2 = X_6 \mp \frac{b}{2} X_4.$$

Simply recall that $b \neq 0$. Four cases are considered according to the value of $(\varepsilon, \nu)$.

– Take $(\varepsilon, \nu) = (0,0)$. Then $Y_1 = 0$, so $\alpha$, $\varphi$ are free and we get $\mathfrak{g}_3 = \mathfrak{g}$. In this way, $(\dim \mathfrak{g}_k) = (1,4,8)$.

– Assume $\varepsilon \neq \nu$. We have that the vectors $Y_1$ and $Y_2$ are linearly independent. Hence, we need $\alpha = \varphi = 0$ to ensure $Y \in \mathfrak{g}_3$ and thus,

$$\mathfrak{g}_3 = \langle X_2, X_3, X_4, X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

Therefore, the ascending central series satisfies $(\dim \mathfrak{g}_k) = (1,4,6,8)$.

– Finally, we consider $(\varepsilon, \nu) = (1,1)$. Since we have $Y_1 = X_6 - X_4$, everything depends on the value of $b$ in $Y_2$.

  * When $b = \pm 2$, it is clear that $Y_1 = Y_2$. Thus, $\alpha$ and $\varphi$ are free, and this fact makes that $\mathfrak{g}_3 = \mathfrak{g}$. Then, $(\dim \mathfrak{g}_k) = (1,4,8)$.
  * If $b \neq \pm 2$, then $Y_1$ and $Y_2$ are linearly independent, so we need $\alpha = \varphi = 0$. Hence,

$$\mathfrak{g}_3 = \langle X_2, X_3, X_4, X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

In this case, we have $(\dim \mathfrak{g}_k) = (1,4,6,8)$.

- Let $\dim \mathfrak{g}_2 = 5$. Recall that we have $\mu = 0$, $A = 0$, $b \neq 0$, and

$$\mathfrak{g}_1 = \langle X_8 \rangle, \quad \mathfrak{g}_2 = \langle X_2, X_3, X_6, X_7, X_8 \rangle.$$

The element $Y \in \mathfrak{g}$ belongs to $\mathfrak{g}_3$ if the brackets (4.47) satisfy

$$\begin{cases} \nu\,\varphi = 0, \\ \nu\,\alpha = 0. \end{cases}$$

Hence, it is clear that two cases appear.

- If $\nu = 0$, then $\mathfrak{g}_3 = \mathfrak{g}$ and $(\dim \mathfrak{g}_k) = (1, 5, 8)$.
- If $\nu = 1$, then $\alpha = \varphi = 0$ and one has

$$\mathfrak{g}_3 = \langle X_2, X_3, X_4, X_6, X_7, X_8 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

  Observe that $(\dim \mathfrak{g}_k) = (1, 5, 6, 8)$.

Finally, to obtain part *i)* of the Lemma, one simply needs to overlap the conditions found in the previous study with the possible values of the parameters $\varepsilon, \mu, \nu, A, b$ in (4.44), according to Lemmas 4.4.1 , 4.4.3, and 4.4.4. Let us see it in detail.

- Let $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$. According to Lemma 4.4.1 (see p. 154), SnN complex structures on $\mathfrak{g}$ are given by (4.44) with $\mu = \varepsilon$ and $(\varepsilon, \nu) = (0, 0)$. As a consequence of the calculations above, we know that the only values among them that preserve the dimension of the ascending central series are those satisfying $(\varepsilon, \nu) = (0, 0)$ and $A \neq 0$. Hence, we conclude that

$$(\dim \mathfrak{g}_k)_k = (1, 3, 8) \;\Leftrightarrow\; \mu = \varepsilon = \nu = 0,\; A \neq 0.$$

- Consider $(\dim \mathfrak{g}_k)_k = (1, 3, 6, 8)$. From Lemma 4.4.1 we know that the allowed values for the initial equations (4.44) are $\mu = \varepsilon$ and $(\varepsilon, \nu) \neq (0, 0)$. As we have seen above, the dimension of the ascending central series is preserved when $(\varepsilon, \nu) \neq (0, 0)$ and $A \neq 0$. Hence, we can then conclude that

$$(\dim \mathfrak{g}_k)_k = (1, 3, 6, 8) \;\Leftrightarrow\; \mu = \varepsilon,\; (\varepsilon, \nu) \neq (0, 0),\; A \neq 0.$$

- Take $(\dim \mathfrak{g}_k)_k = (1, 4, 8)$. Following Lemma 4.4.3 (p. 162), we have that the admissible values for the parameters in (4.44) are $\mu = \nu$, $A = 0$, and $(\varepsilon, \nu) = (0, 0)$ or $(\varepsilon, \nu, b) = (1, 0, 0), (1, 1, \pm 2)$. According to our calculations, the values that give the appropriate dimension of the ascending central series satisfy $\mu = 1$, $A = 0$, $b \neq 0$, and $(\varepsilon, \nu) = (0, 0)$ or $(\varepsilon, \nu, b) = (1, 1, \pm 2)$. However, we observe that according to the lemma $\mu = \nu = 1$, so the case $(\varepsilon, \nu) = (0, 0)$ is not valid. In this way, we can conclude that

$$(\dim \mathfrak{g}_k)_k = (1, 4, 8) \;\Leftrightarrow\; \mu = \varepsilon = \nu = 1,\; A = 0,\; b = \pm 2.$$

- Suppose $(\dim \mathfrak{g}_k)_k = (1,4,6,8)$. From Lemma 4.4.3 we have initial values $\mu = \varepsilon$, $A = 0$, and $(\varepsilon, \nu) \neq (0,0)$ in (4.44). By the analysis above, only those satisfying $\mu = 1$, $A = 0$, $b \neq 0$, and $(\varepsilon,\nu) = (0,1),(1,0)$ or $(\varepsilon,\nu) = (1,1)$ with $b \neq \pm 2$ truly preserve the dimension of $\{\mathfrak{g}_k\}_k$. In this way, it is easy to see that

$$(\dim \mathfrak{g}_k)_k = (1,4,6,8) \;\Leftrightarrow\; \mu = \varepsilon = 1,\ A = 0,\ \begin{cases} \nu = 0,\ b \neq 0, \\ \nu = 1,\ b \neq 0, \pm 2. \end{cases}$$

- Let $(\dim \mathfrak{g}_k)_k = (1,5,8)$. As a consequence of Lemma 4.4.4, the structure equations (4.44) fulfill $\nu = \mu = A = 0$, and $\varepsilon\, b = 0$. Among them, the ones preserving the dimension of the ascending central series are those satisfying $\nu = \mu = A = 0$ and $b \neq 0$, as a consequence of the calculations above. Hence, we conclude that

$$(\dim \mathfrak{g}_k)_k = (1,5,8) \;\Leftrightarrow\; \mu = \varepsilon = \nu = 0,\ A = 0,\ b \neq 0.$$

- We finally study the case $(\dim \mathfrak{g}_k)_k = (1,5,6,8)$. By Lemma 4.4.4, the admissible values in (4.44) are $\nu = 1$, $\mu = A = 0$, and $\varepsilon\, b = 0$. By the previous study of the ascending central series, only those such that $\nu = 1$, $\mu = A = 0$, and $b \neq 0$ preserve $(\dim \mathfrak{g}_k)_k$. Therefore,

$$(\dim \mathfrak{g}_k)_k = (1,5,6,8) \;\Leftrightarrow\; \mu = \varepsilon = 0,\ \nu = 1,\ A = 0,\ b \neq 0.$$

This concludes part *i)* of the theorem. We now move to the proof of part *ii)*. The idea is applying the same method as before. Let us consider the (1,0)-basis $\{\omega^k\}_{k=1}^4$ satisfying the structure equations (4.35) in Lemma 4.4.2 (p. 158). Defining the real basis (4.45) and denoting $a_1 = \mathfrak{Re}\, A$, $a_2 = \mathfrak{Im}\, A$, one obtains

$$\begin{cases} de^1 &= 0, \\ de^2 &= 2\, e^{14}, \\ de^3 &= 3\,\varepsilon\, e^{12} + 2\, a_2\, e^{15} + 2\,\mu\, e^{46} + \varepsilon\, e^{56}, \\ de^4 &= 2\,\nu\, e^{15}, \\ de^5 &= 0, \\ de^6 &= -2\, e^{45}, \\ de^7 &= -2\, a_1\, e^{15} + \varepsilon\, e^{16} + 2\,\mu\, e^{24} - \varepsilon\, e^{25}, \\ de^8 &= 2\, b\, e^{12} + 2\, e^{13} + 2\,\mu\, e^{26} + 2\, b\, e^{56} + 2\, e^{57}. \end{cases}$$

When we consider the dual basis $\{X_k\}_{k=1}^8$ of $\{e^k\}_{k=1}^8$ and apply the same formula as above, the possibly non-zero Lie brackets of $\mathfrak{g}$ turn to be:

$$\begin{aligned} [X_1,X_2] &= -3\,\varepsilon\, X_3 - 2\, b\, X_8, & [X_1,X_5] &= -2\, a_2\, X_3 - 2\,\nu\, X_4 + 2\, a_1\, X_7, \\ [X_1,X_3] &= -2\, X_8, & [X_1,X_6] &= -\varepsilon\, X_7, \\ [X_1,X_4] &= -2\, X_2 & [X_2,X_4] &= -2\,\mu\, X_7, \end{aligned}$$

$$
\begin{aligned}
[X_2,X_5] &= \varepsilon\, X_7, & [X_4,X_6] &= -2\,\mu\, X_3,\\
[X_2,X_6] &= -2\,\mu\, X_8, & [X_5,X_6] &= -\varepsilon\, X_3 - 2\,b\, X_8,\\
[X_4,X_5] &= 2\, X_6, & [X_5,X_7] &= -2\, X_8.
\end{aligned}
$$

Any element $Y \in \mathfrak{g}$ is given by (4.46), in such a way that its brackets with the elements of the previous basis are given as follows:

$$
\begin{aligned}
[Y,X_1] &= 2\,\delta\, X_2 + (3\,\varepsilon\,\beta + 2\,a_2\,\varphi)\, X_3 + 2\,\nu\,\varphi\, X_4\\
&\quad -(2\,a_1\,\varphi - \varepsilon\,\psi)\, X_7 + 2\,(b\,\beta + \gamma)\, X_8,\\
[Y,X_2] &= -3\,\varepsilon\,\alpha\, X_3 + (2\,\mu\,\delta - \varepsilon\,\varphi)\, X_7 - 2\,(b\,\alpha - \mu\,\psi)\, X_8,\\
[Y,X_3] &= -2\,\alpha\, X_8,\\
[Y,X_4] &= -2\,\alpha\, X_2 + 2\,\mu\,\psi\, X_3 - 2\,\varphi\, X_6 - 2\,\mu\,\beta\, X_7,\\
[Y,X_5] &= -(2\,a_2\,\alpha - \varepsilon\,\psi)\, X_3 - 2\,\nu\,\alpha\, X_4 + 2\,\delta\, X_6\\
&\quad +(2\,a_1\,\alpha + \varepsilon\,\beta)\, X_7 + 2\,(\zeta + b\,\psi)\, X_8,\\
[Y,X_6] &= -(2\,\mu\,\delta + \varepsilon\,\varphi)\, X_3 - \varepsilon\,\alpha\, X_7 - 2\,(\mu\,\beta + b\,\varphi)\, X_8,\\
[Y,X_7] &= -2\,\varphi\, X_8,\\
[Y,X_8] &= 0.
\end{aligned}
\tag{4.49}
$$

Let us start calculating $\mathfrak{g}_1$. Simply recall that $Y \in \mathfrak{g}_1$ when $[Y, X_k] = 0$, for every $k = 1,\ldots,8$. At the sight of the brackets $[Y, X_3]$ and $[Y, X_7]$ in (4.49), one clearly needs $\alpha = \varphi = 0$. Furthermore, the coefficient of $X_2$ in $[Y, Z_1]$ implies the condition $\delta = 0$. In this way, the equations to solve are reduced to:

$$
\begin{cases} \varepsilon\,\beta = 0,\\ \varepsilon\,\psi = 0, \end{cases}
\qquad
\begin{cases} \mu\,\beta = 0,\\ \mu\,\psi = 0, \end{cases}
\qquad
\begin{cases} \gamma + b\,\beta = 0,\\ \zeta + b\,\psi = 0. \end{cases}
$$

Observe that $\tau$ is always free. If $(\varepsilon,\mu) = (0,0)$, then the parameters $\beta, \psi$ and $\tau$ are free, so $\dim \mathfrak{g}_1 > 1$. This contradicts Corollary 4.1.12. Therefore, $(\varepsilon,\mu) \neq (0,0)$ and thus we can ensure $\mathfrak{g}_1 = \langle X_8\rangle$.

We now calculate $\mathfrak{g}_2$, bearing in mind that $(\varepsilon,\mu) \neq (0,0)$. Observe that if $Y \in \mathfrak{g}_2$, then the brackets (4.49) should lie in $\mathfrak{g}_1 = \langle X_8\rangle$. From the coefficient of $X_2$ in the expression of $[Y, X_1]$, one clearly has $\delta = 0$. Similarly, from the coefficients of $X_2$ and $X_6$ in $[Y, X_4]$, we get $\alpha = \varphi = 0$. The rest of the parameters in (4.46) satisfy the four first equations above, namely,

$$
\begin{cases} \varepsilon\,\beta = 0,\\ \varepsilon\,\psi = 0, \end{cases}
\qquad
\begin{cases} \mu\,\beta = 0,\\ \mu\,\psi = 0. \end{cases}
$$

As $(\varepsilon, \mu) \neq (0,0)$, we conclude that $\beta = \psi = 0$ and thus, $\mathfrak{g}_2 = \langle X_3, X_7, X_8 \rangle$. Hence, we have that $\dim \mathfrak{g}_2 = 3$.

To compute $\mathfrak{g}_3$ one can apply the same ideas as above to the brackets (4.49) (which this time must belong to $\mathfrak{g}_2$), and then get $\alpha = \delta = \varphi = 0$. This choice makes that $\mathfrak{g}_3 = \langle X_2, X_3, X_6, X_7, X_8 \rangle$ and $\dim \mathfrak{g}_3 = 5$.

Finally, we focus on $\mathfrak{g}_4$. Notice that $Y \in \mathfrak{g}_4$ whenever its coefficients satisfy

$$\begin{cases} \nu\, \varphi = 0, \\ \nu\, \alpha = 0. \end{cases}$$

Two cases arise, depending on the value of $\nu$:

- if $\nu = 0$, then $\mathfrak{g}_4 = \mathfrak{g}$;
- if $\nu = 1$, then $\alpha = \varphi = 0$ and one obtains $\mathfrak{g}_4 = \langle X_2, X_3, X_4, X_6, X_7, X_8 \rangle$, $\mathfrak{g}_5 = \mathfrak{g}$.

Simply observe that all the values of the parameters considered in the statement of Lemma 4.4.2 fulfill the conditions. Hence, they all preserve the appropriate $(\dim \mathfrak{g}_k)_k$.

□

Chapter 5

# Geometric aspects of eight-dimensional nilmanifolds

Let $M = \Gamma\backslash G$ be a $2n$-dimensional nilmanifold endowed with an invariant complex structure $J$. In the last years, many geometrical structures have been investigated for $n = 3$, as SKT/astheno-Kähler metrics [FPS04], 1-st Gauduchon [FU13], pseudo-Kähler structures [CFU04], balanced Hermitian metrics [UV15] and their role in heterotic string theory [FIUV09, UV14],... Moreover, many results concerning the behaviour under holomorphic deformations of these and other metric properties are obtained in this dimension (see Section 1.4.3). Nonetheless, $n = 3$ is still a low complex dimension and it makes difficult to distinguish among some of the Hermitian metrics we presented in Section 1.3. For example, the SKT and astheno-Kähler conditions are exactly the same, and also invariant 1-st Gauduchon metrics turn to coincide with invariant SKT ones (see [FU13]). In the case of holomorphic symplectic structures, they do not even exist in complex dimension 3, since an even complex dimension is required. The parametrization of invariant complex structures on 8-dimensional nilmanifolds accomplished in Chapters 3 and 4 allows to analyze the previous geometrical structures. This is the aim of our last chapter.

In Section 5.1, we focus on Hermitian metrics. We first show that the expression of the fundamental 2-form of any invariant Hermitian metric on a $2n$-dimensional nilmanifold $(M, J)$ can be simplified when $J$ is quasi-nilpotent. As a consequence, we see that $\mathfrak{b}$-extensions can help in the characterization of certain types of special Hermitian metrics. Making use of these results, we then study the existence of invariant astheno-Kähler and generalized Gauduchon metrics on every $(M, J)$ with $n = 4$. In particular, we show that neither astheno-Kähler nor generalized Gauduchon metrics on nilmanifolds require a nilpotency step equal to 2, unlike SKT ones [EFV12]. This constitutes an important difference among these types of invariant Hermitian metrics, despite being coincident in dimension six.

We then move to the study of other special geometry. More concretely, in Section 5.2 we concentrate on holomorphic symplectic structures. We first show that their existence in dimension eight implies a strong restriction on the complex structure, namely, $J$ must be of nilpotent type. Afterwards, we study their behaviour under holomorphic deformations. In [Gua95b], Guan proved that the property of existence of holomorphic symplectic structures is preserved under additional hypothesis on the second de Rham cohomology group. However, the property is not open in general. Here, we prove that it is neither closed, i.e., we construct an analytical family of holomorphic symplectic manifolds $\{X_t\}$ such that its central limit $X_0$ does not admit any holomorphic symplectic structure.

Bearing in mind that holomorphic symplectic structures define non-trivial cohomology classes in $H_J^-(M)$, one also considers the structures corresponding to their counterparts in $H_J^+(M)$, namely, pseudo-Kähler structures. We check that, even on nilmanifolds with abelian complex structures, these two special geometries are no related to each other (see also [Yam05]). We also prove that pseudo-Kähler structures do exist on 8-dimensional nilmanifolds endowed with non-nilpotent complex structures, providing a counterexample to a conjecture in [CFU04]. Furthermore, we show that there exists only one SnN complex structure admitting pseudo-Kähler metrics.

## 5.1 Special Hermitian metrics

In order to study metric properties of 8-dimensional nilmanifolds endowed with complex structures, we first focus on special Hermitian structures (see Section 1.3). We concentrate on *invariant* Hermitian metrics, in such a way that the discussion can be reduced to the Lie algebra level.

Let $\mathfrak{g}$ be a $2n$-dimensional NLA endowed with a complex structure $J$. We start showing that the fundamental form of any Hermitian metric defined on $(\mathfrak{g}, J)$ can be simplified in terms of a special basis when $J$ is quasi-nilpotent.

As we saw in Chapter 3, the fact that $J$ is quasi-nilpotent implies that one can obtain $(\mathfrak{g}, J)$ as a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$, a $2(n-1)$-dimensional NLA endowed with an arbitrary complex structure. In particular, there is an appropriate basis for $\mathfrak{g}^{1,0}$ in terms of which the first $n-1$ structure equations coincide with those of $(\mathfrak{h}, K)$. The next result shows that, under these assumptions, also any Hermitian metric on $(\mathfrak{g}, J)$ with fundamental form $\Omega$ can be written in terms of the fundamental form $F$ of a Hermitian metric on $(\mathfrak{h}, K)$.

**Lemma 5.1.1.** *Let $(\mathfrak{g}, J)$ be a $\mathfrak{b}$-extension of a NLA $\mathfrak{h}$ of dimension $2(n-1)$ endowed with a complex structure $K$. Let $\Omega$ be the fundamental 2-form associated to a Hermitian metric on $(\mathfrak{g}, J)$. If $\mathcal{B}$ is a basis for $\mathfrak{h}^{1,0}$, then there is a basis $\mathcal{B}_{\mathfrak{g}} = \mathcal{B} \cup \{\omega^n\}$ for $\mathfrak{g}^{1,0}$ such that $\Omega = F + \frac{i}{2}\,\omega^{n\bar{n}}$, where $F$ is the fundamental form of a Hermitian metric on $(\mathfrak{h}, K)$.*

*Proof.* Let $\mathcal{B} = \{\eta^k\}_{k=1}^{n-1}$ be a basis for $\mathfrak{h}^{1,0}$. Since $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$, we can add an element $\eta^n$ to $\mathcal{B}$ in order to obtain a basis $\{\eta^k\}_{k=1}^{n}$ for $\mathfrak{g}^{1,0}$. Recall that the first $n-1$ structure equations of $(\mathfrak{g}, J)$ coincide with those of $(\mathfrak{h}, K)$ and $d\eta^n$ satisfies Theorem 3.1.8 and $d^2\eta^n = 0$ (see Section 3.1.1). Moreover, any Hermitian metric on $(\mathfrak{g}, J)$ with fundamental 2-form $\Omega$ is determined by

$$\Omega \;=\; \sum_{k=1}^{n} i\, x'_{k\bar{k}}\, \eta^{k\bar{k}} \;+\; \sum_{1\leq k<l\leq n} \big(x'_{k\bar{l}}\, \eta^{k\bar{l}} - \bar{x}'_{k\bar{l}}\, \eta^{l\bar{k}}\big),$$

where $x'_{k\bar{k}} \in \mathbb{R}^{>0}$, $x'_{k\bar{l}} \in \mathbb{C}$, for $1 \leq k < l \leq n$, and satisfying certain conditions that ensure the positive definiteness of the metric. The idea is applying an appropriate change of basis which reduces the metric and "*keeps the model*" of a generic extension.

Take the $(1,0)$-basis given by

$$\omega^k = \eta^k,\ k = 1, \ldots, n-1, \qquad \omega^n = \sqrt{2\, x'_{n\bar{n}}} \left( \eta^n - i \sum_{k=1}^{n-1} \frac{x'_{k\bar{n}}}{x'_{n\bar{n}}}\, \eta^k \right).$$

We first observe that $\mathcal{B}$ remains unchanged. In addition, it is clear that $\omega^n$ is defined using elements that already satisfy Theorem 3.1.8 and $d^2\eta^k = 0$ themselves. One just needs to rewrite $\Omega$ in terms of the new basis $\mathcal{B}_{\mathfrak{g}} = \mathcal{B} \oplus \langle \omega^n \rangle$. After some calculations, it is not difficult to see that

$$\Omega \;=\; \sum_{k=1}^{n-1} i\, x_{k\bar{k}}\, \omega^{k\bar{k}} \;+\; \frac{i}{2}\, \omega^{n\bar{n}} \;+ \sum_{1\leq k<l\leq n-1} \left( x_{k\bar{l}}\, \omega^{k\bar{l}} - \bar{x}_{k\bar{l}}\, \omega^{l\bar{k}} \right),$$

where for every $k, l$ one has

$$x_{k\bar{k}} = \frac{x'_{k\bar{k}}\, x'_{n\bar{n}} - |x_{k\bar{n}}|^2}{x'_{n\bar{n}}}, \qquad x_{k\bar{l}} = \frac{x'_{k\bar{l}}\, x'_{n\bar{n}} - i\, x'_{k\bar{n}}\, \bar{x}'_{l\bar{n}}}{x'_{n\bar{n}}}.$$

Now, simply observe that the positive definiteness of $\Omega$ implies that

$$F \;=\; \sum_{k=1}^{n-1} i\, x_{k\bar{k}}\, \omega^{k\bar{k}} \;+ \sum_{1\leq k<l\leq n-1} \left( x_{k\bar{l}}\, \omega^{k\bar{l}} - \bar{x}_{k\bar{l}}\, \omega^{l\bar{k}} \right)$$

is a Hermitian metric on $(\mathfrak{h}, K)$. Furthermore, $\Omega = F + \frac{i}{2}\, \omega^{n\bar{n}}$. □

Thanks to this lemma, we can find obstructions on $(\mathfrak{h}, K, F)$ and on the structure constants of the $\mathfrak{b}$-extension in order to obtain certain types of Hermitian metrics with fundamental form $\Omega$ on $(\mathfrak{g}, J)$. In the next result, we concentrate on balanced, astheno-Kähler, and 1-st Gauduchon metrics (see Section 1.3).

**Proposition 5.1.2.** *In the conditions of Lemma 5.1.1 with $n \geq 4$, one has that $\Omega$ is:*

*i) balanced if and only if $(\mathfrak{h}, K, F)$ is balanced and $F^{n-2} \wedge d\omega^n = 0$;*

*ii) astheno-Kähler if and only if $(\mathfrak{h}, K, F)$ is astheno-Kähler and*

$$F^{n-3} \wedge \partial\bar{\partial}\omega^n + \partial F^{n-3} \wedge \bar{\partial}\omega^n - \bar{\partial} F^{n-3} \wedge \partial\omega^n = 0,$$

$$F^{n-3} \wedge \left( \bar{\partial}\omega^n \wedge \partial\omega^{\bar{n}} - \partial\omega^n \wedge \bar{\partial}\omega^{\bar{n}} \right) = 0;$$

*iii) 1-st Gauduchon if and only if*

$$F^{n-3} \wedge \left( \bar{\partial}\omega^n \wedge \partial\omega^{\bar{n}} - \partial\omega^n \wedge \bar{\partial}\omega^{\bar{n}} \right) - 4\, \tilde{\gamma}_1(F)\, F^{n-1} = 0,$$

*where $\tilde{\gamma}_1(F)$ is the constant given by $\frac{i}{2}\, \partial\bar{\partial} F \wedge F^{n-3} = \tilde{\gamma}_1(F)\, F^{n-1}$.*

*Proof.* As a consequence of Lemma 5.1.1, we can write $\Omega = F + \frac{i}{2}\omega^{n\bar{n}}$. Then, by induction on $2 \leq k \leq n$, it is easy to see that

$$\Omega^k = F^k + \frac{i\,k}{2}\,F^{k-1} \wedge \omega^{n\bar{n}}. \tag{5.1}$$

Let us start with part *i)*, which involves the balanced condition. Using (5.1) for $k = n-1$ and applying the differential $d$, we see that

$$d\Omega^{n-1} = dF^{n-1} + \tfrac{i\,(n-1)}{2}\Big(dF^{n-2} \wedge \omega^{n\bar{n}} + F^{n-2} \wedge d\omega^n \wedge \omega^{\bar{n}} - F^{n-2} \wedge d\omega^{\bar{n}} \wedge \omega^n\Big).$$

Since $F^{n-1}$ is a volume form on $(\mathfrak{h}, K)$, it is clear that $dF^{n-1} = 0$. Now, we focus on the summands inside the parenthesis. Notice that the first one depends on both $\omega^n$ and $\omega^{\bar{n}}$, whereas the second one only depends on $\omega^{\bar{n}}$, and the third one, on $\omega^n$. Hence, they should annihilate independently. In particular, one needs $dF^{n-2} = 0$, that is, $F$ is a balanced metric on $(\mathfrak{h}, K)$, and also $F^{n-2} \wedge d\omega^n = 0$, which automatically implies $F^{n-2} \wedge d\omega^{\bar{n}} = 0$ by conjugation (recall that $\bar{F} = F$). This concludes the proof of *i)*.

In order to see *ii)*, we apply the astheno-Kähler condition to $\Omega$. Using (5.1), we obtain after some calculations:

$$\begin{aligned}
\partial\bar\partial\Omega^{n-2} \;=\;& \partial\bar\partial F^{n-2} + \tfrac{i\,(n-2)}{2}\Big(F^{n-3} \wedge \bar\partial\omega^n \wedge \partial\omega^{\bar{n}} - F^{n-3} \wedge \partial\omega^n \wedge \bar\partial\omega^{\bar{n}}\\
&+ \big(\bar\partial F^{n-3} \wedge \partial\omega^{\bar{n}} - \partial F^{n-3} \wedge \bar\partial\omega^{\bar{n}} - F^{n-3} \wedge \partial\bar\partial\omega^{\bar{n}}\big) \wedge \omega^n\\
&+ \big(\partial F^{n-3} \wedge \bar\partial\omega^n - \bar\partial F^{n-3} \wedge \partial\omega^n + F^{n-3} \wedge \partial\bar\partial\omega^n\big) \wedge \omega^{\bar{n}} + \partial\bar\partial F^{n-3} \wedge \omega^{n\bar{n}}\Big).
\end{aligned}$$

Since $F$ is a Hermitian metric on $(\mathfrak{h}, K)$, it turns out that the form $\partial\bar\partial F^{n-2}$ attains maximal degree. This allows to conclude that $\partial\bar\partial F^{n-2} = 0$, because volume forms cannot be exact. For the rest of the summands above, one observes the following. The first two expressions inside the parenthesis depend neither on $\omega^n$ nor on $\omega^{\bar{n}}$. The third one depends on $\omega^n$ but not on $\omega^{\bar{n}}$, whereas the fourth one depends on $\omega^{\bar{n}}$ but not on $\omega^n$. In fact, these two expressions are conjugate to each other. Finally, we note that the last summand inside the parenthesis depends both on $\omega^n$ and on $\omega^{\bar{n}}$. Hence, it suffices to annihilate the previous expressions separately in order to annihilate $\partial\bar\partial\Omega^{n-2}$. This gives the result.

In order to check *iii)*, we proceed in a similar way. Applying the 1-st Gauduchon condition to $\Omega$ and taking into account (5.1) and the formula $\frac{i}{2}\,\partial\bar\partial F \wedge F^{n-3} = \tilde\gamma_1(F)\,F^{n-1}$, one has

$$\begin{aligned}
\partial\bar\partial\Omega \wedge \Omega^{n-2} \;=\;& \tfrac{i}{2}\big(F^{n-2} \wedge \bar\partial\omega^n \wedge \partial\omega^{\bar{n}} - F^{n-2} \wedge \partial\omega^n \wedge \bar\partial\omega^{\bar{n}}\big)\\
&+ \tfrac{i}{2}\,F^{n-2} \wedge \partial\bar\partial\omega^n \wedge \omega^{\bar{n}} + \tfrac{i}{2}\,F^{n-2} \wedge \partial\bar\partial\omega^{\bar{n}} \wedge \omega^n\\
&+ \tfrac{n-2}{4}\big(4\tilde\gamma_1(F)\,F^{n-1} - F^{n-3} \wedge \bar\partial\omega^n \wedge \partial\omega^{\bar{n}} + F^{n-3} \wedge \partial\omega^n \wedge \bar\partial\omega^{\bar{n}}\big) \wedge \omega^{n\bar{n}}.
\end{aligned}$$

We now focus on the right hand side of this equality. The expression inside the first parenthesis equals zero, as it is a $(n,n)$-form given in terms of $n-1$ elements $\omega^1, \ldots, \omega^{n-1}$ (recall Lemma 5.1.1). Similarly, we note that $F^{n-2} \wedge \partial\bar\partial\omega^n$ is a $(n, n-1)$-form obtained in terms of $\omega^1, \ldots, \omega^{n-1}$. Hence, it must be zero, and also its conjugate. In the end, just the last expression remains, which leads to *iii)*. □

### 5.1.1 The astheno-Kähler condition

Here, we study astheno-Kähler metrics on 8-dimensional NLAs endowed with complex structures. In particular, we show that the nilpotency step of the underlying algebras can be equal to 3, in contrast with the case of NLAs admitting SKT metrics.

In order to give a short background on astheno-Kähler metrics, we first consider 6-dimensional NLAs $\mathfrak{h}$ endowed with complex structures $K$. In this dimension, recall that the astheno-Kähler condition coincides with the SKT condition. Therefore, the classification of invariant astheno-Kähler metrics on 6-dimensional nilmanifolds with invariant complex structure come from the following result:

**Proposition 5.1.3.** [FPS04, Uga07] *Let $\mathfrak{h}$ be a 6-dimensional NLA endowed with a complex structure $K$. Then, $(\mathfrak{h}, K)$ admits SKT metrics if and only if it is a torus or it is given by Family I, i.e.,*

$$\begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \end{cases}$$

*where $\rho \in \{0, 1\}$, $\lambda \in \mathbb{R}^{\geq 0}$, $D \in \mathbb{C}$, and structure constants satisfying $\rho + \lambda^2 = 2\,\mathfrak{Re}D$. Moreover, in this case any Hermitian metric is SKT.*

It is not casual that all the underlying algebras in Proposition 5.1.3 are 2-step, as the next result shows:

**Theorem 5.1.4.** [EFV12] *The existence of an SKT structure on a $2n$-dimensional NLA $\mathfrak{g}$ endowed with a complex structure implies that $\mathfrak{g}$ is 2-step.*

Hence, Theorem 5.1.4 gives a strong obstruction on the nilpotency step of the algebra for the existence of SKT metrics. Indeed, it plays an important role in the description of SKT metrics on 8-dimensional NLAs $\mathfrak{g}$ with complex structures $J$ attained in [EFV12]. As a consequence of the results in this paper, one observes that not all the metrics on a fixed 8-dimensional $(\mathfrak{g}, J)$ are either SKT or non-SKT, in contrast with dimension 6. In addition, our study of strongly non-nilpotent complex structures (see Corollary 3.1.13) allows to conclude the following.

**Corollary 5.1.5.** *SKT metrics do not exist on nilmanifolds endowed with SnN complex structures.*

It is worth remarking that in dimension eight the SKT and the astheno-Kähler conditions are no longer the same. In this sense, it was shown in [RT12] that a fixed metric $\Omega$ on a certain $(\mathfrak{g}, J)$ can be SKT but not astheno-Kähler, and also the other way round. This definitely shows the independence of these two types of metrics in higher dimensions. Nonetheless, it is true that all the examples in [RT12] are 2-step. Hence, it seems natural to wonder which similarities and differences these two types of metrics have.

The description of 8-dimensional $(\mathfrak{g}, J)$ accomplished in Chapters 3 and 4 opens a new setting in which studying the astheno-Kähler condition more generally. In the next result, we obtain a complex obstruction to the existence of this type of metrics on $(\mathfrak{g}, J)$.

**Proposition 5.1.6.** *Let* $(\mathfrak{g}, J)$ *be an* 8*-dimensional NLA endowed with a complex structure. If* $(\mathfrak{g}, J)$ *admits an astheno-Kähler metric* $\Omega$*, then* $J$ *is quasi-nilpotent.*

*Proof.* Let $\Omega$ be an astheno-Kähler metric on $(\mathfrak{g}, J)$. Suppose that the complex structure $J$ is SnN. By Theorem 4.4.6, we know that $(\mathfrak{g}, J)$ can be parametrized by two different families depending on the ascending central series of $\mathfrak{g}$. Take a generic fundamental $(1,1)$-form

$$\Omega \;=\; \sum_{k=1}^{4} i\, x_{k\bar{k}}\, \omega^{k\bar{k}} \;+\; \sum_{1\leq k<l\leq 4} \big(x_{k\bar{l}}\, \omega^{k\bar{l}} - \bar{x}_{k\bar{l}}\, \omega^{l\bar{k}}\big),$$

with $x_{k\bar{k}} \in \mathbb{R}^{>0}$ and $x_{k\bar{l}} \in \mathbb{C}$, for $1 \leq k, l \leq 4$. We want to study the astheno-Kähler condition for $\Omega$ on each of these families.

If $(\mathfrak{g}, J)$ is given by Theorem 4.4.6 *i)*, then the structure equations are

$$\begin{cases} d\omega^1 &=& 0, \\ d\omega^2 &=& \varepsilon\, \omega^{1\bar{1}}, \\ d\omega^3 &=& \omega^{14} + \omega^{1\bar{4}} + A\, \omega^{2\bar{1}} \pm i\, \varepsilon\, b\, \omega^{1\bar{2}}, \\ d\omega^4 &=& i\, \nu\, \omega^{1\bar{1}} \pm i\, \omega^{1\bar{3}} + b\, \omega^{2\bar{2}} \mp i\, \omega^{3\bar{1}}, \end{cases}$$

with $\varepsilon, \nu \in \{0, 1\}$, $A \in \mathbb{C}$, and $b \in \mathbb{R}$. From these equations we compute $\partial\bar{\partial}\Omega^2$, and we need to annihilate the resulting expression. In order to do so, the coefficient accompanying each term of the form $\omega^{rst\bar{u}\bar{v}\bar{w}}$ in $\partial\bar{\partial}\Omega^2$ must vanish. If we focus on $\omega^{234\bar{1}\bar{2}\bar{4}}$, then we must have $2\, b\, (x_{3\bar{3}} x_{4\bar{4}} - |x_{3\bar{4}}|^2) = 0$. The positive definiteness of the metric yields $x_{3\bar{3}} x_{4\bar{4}} - |x_{3\bar{4}}|^2 > 0$, so we get $b = 0$. Thanks to this observation, the astheno-Kähler condition reduces to

$$\begin{aligned} \tfrac{1}{2}\, \partial\bar{\partial}\Omega^2 \;&=\; 2\, (x_{2\bar{2}} x_{4\bar{4}} - |x_{2\bar{4}}|^2)\, \omega^{123\bar{1}\bar{2}\bar{3}} \\ &+\; \big(2\, x_{2\bar{2}} x_{3\bar{3}} + |A|^2 (x_{3\bar{3}} x_{4\bar{4}} - |x_{3\bar{4}}|^2) - 2\, |x_{2\bar{3}}|^2\big)\, \omega^{124\bar{1}\bar{2}\bar{4}} \\ &\pm\; i\, \bar{A}\, (x_{2\bar{4}} \bar{x}_{3\bar{4}} + i\, x_{4\bar{4}} x_{2\bar{3}})\, \omega^{123\bar{1}\bar{2}\bar{4}} \;\mp\; i\, A\, (\bar{x}_{2\bar{4}} x_{3\bar{4}} - i\, x_{4\bar{4}} \bar{x}_{2\bar{3}})\, \omega^{124\bar{1}\bar{2}\bar{3}} \\ &\pm\; (x_{4\bar{4}} x_{2\bar{3}} - i\, x_{2\bar{4}} \bar{x}_{3\bar{4}})\, \omega^{124\bar{1}\bar{3}\bar{4}} \;\mp\; (x_{4\bar{4}} \bar{x}_{2\bar{3}} + i\, \bar{x}_{2\bar{4}} x_{3\bar{4}})\, \omega^{134\bar{1}\bar{2}\bar{4}}. \end{aligned}$$

By the positive definiteness of $\Omega$ one also has $x_{2\bar{2}} x_{4\bar{4}} - |x_{2\bar{4}}|^2 > 0$, so it is clear that the metric cannot be astheno-Kähler.

Now, when $(\mathfrak{g}, J)$ is parametrized by Theorem 4.4.6 *ii)*, one has

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = A\,\omega^{1\bar{1}} + \varepsilon\,(\omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,\mu\,(\omega^{24} + \omega^{2\bar{4}}), \\ d\omega^4 = i\,\nu\,\omega^{1\bar{1}} + i\,b\,\omega^{1\bar{2}} + i\,\omega^{1\bar{3}} - i\,b\,\omega^{2\bar{1}} - \mu\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}, \end{cases}$$

with $\varepsilon, \nu, \mu \in \{0,1\}$, $A \in \mathbb{C}$ such that $\mathfrak{Im}\,A = 0$ for $\varepsilon = 1$, and $b \in \mathbb{R}$. Moreover, $(\varepsilon, \mu) \neq (0,0)$. If we calculate $\partial\bar{\partial}\Omega^2$ and concentrate on the coefficient of $\omega^{234\bar{1}\bar{3}\bar{4}}$, then we see that $\mu = 0$, because the positive definiteness of the metric yields $x_{3\bar{3}}x_{4\bar{4}} - |x_{3\bar{4}}|^2 > 0$. With the choice $\mu = 0$, we have

$$\begin{aligned} \tfrac{1}{2}\partial\bar{\partial}\Omega^2 \;=\; & 2\,\big(x_{2\bar{2}}x_{3\bar{3}} - |x_{2\bar{3}}|^2 - x_{4\bar{4}}\,\mathfrak{Re}\,x_{2\bar{3}} - \mathfrak{Im}\,(x_{2\bar{4}}\bar{x}_{3\bar{4}})\,\big)\,\omega^{134\bar{1}\bar{3}\bar{4}} \\ & + 3\,\varepsilon\,(x_{3\bar{3}}x_{4\bar{4}} - |x_{3\bar{4}}|^2)\,\omega^{124\bar{1}\bar{2}\bar{4}} \\ & + 2\,\big(x_{2\bar{2}}x_{4\bar{4}} - |x_{2\bar{4}}|^2 + b^2\,(x_{3\bar{3}}x_{4\bar{4}} - |x_{3\bar{4}}|^2) + 2\,b\,\big(\,\mathfrak{Re}\,(x_{2\bar{4}}\bar{x}_{3\bar{4}}) - x_{4\bar{4}}\mathfrak{Im}\,x_{2\bar{3}}\,\big)\big)\,\omega^{123\bar{1}\bar{2}\bar{3}} \\ & + 2\,i\,\varepsilon\,\big(\,x_{4\bar{4}}\,\mathfrak{Im}\,x_{2\bar{3}} - b\,(x_{3\bar{3}}x_{4\bar{4}} - |x_{3\bar{4}}|^2) - \mathfrak{Re}\,(x_{2\bar{4}}\bar{x}_{3\bar{4}})\,\big)\,(\omega^{123\bar{1}\bar{2}\bar{4}} + \omega^{124\bar{1}\bar{2}\bar{3}}) \\ & + \big(2\,\varepsilon\big(\mathfrak{Re}\,(x_{2\bar{3}}x_{3\bar{4}}) + x_{3\bar{3}}\mathfrak{Im}\,x_{2\bar{4}}\big) + i\,(x_{2\bar{2}}x_{4\bar{4}} - |x_{2\bar{4}}|^2) - b\,(x_{4\bar{4}}x_{2\bar{3}} - i\,x_{2\bar{4}}\bar{x}_{3\bar{4}})\big)\,\omega^{124\bar{1}\bar{3}\bar{4}} \\ & + \big(2\,\varepsilon\big(\mathfrak{Re}\,(x_{2\bar{3}}x_{3\bar{4}}) + x_{3\bar{3}}\mathfrak{Im}\,x_{2\bar{4}}\big) - i\,(x_{2\bar{2}}x_{4\bar{4}} - |x_{2\bar{4}}|^2) - b\,(x_{4\bar{4}}\bar{x}_{2\bar{3}} + i\,\bar{x}_{2\bar{4}}x_{3\bar{4}})\big)\,\omega^{134\bar{1}\bar{2}\bar{4}}. \end{aligned}$$

At the sight of the coefficient $\omega^{124\bar{1}\bar{2}\bar{4}}$, we have $\varepsilon = 0$, as $x_{3\bar{3}}x_{4\bar{4}} - |x_{3\bar{4}}|^2 > 0$. However, according to Theorem 4.4.6 *ii)* the case $(\varepsilon, \mu) = (0,0)$ is not valid.

Therefore, there are no astheno-Kähler metrics when $J$ is SnN. $\square$

As a consequence of Proposition 5.1.6, it suffices to study the existence of astheno-Kähler metrics on those $(\mathfrak{g}, J)$ where $J$ is quasi-nilpotent. From Corollary 3.1.10, we know that each quasi-nilpotent $(\mathfrak{g}, J)$ is a $\mathfrak{b}$-extension of a 6-dimensional NLA $\mathfrak{h}$ endowed with a complex structure $K$. We consider a basis $\mathcal{B} = \{\omega^k\}_{k=1}^3$ of $\mathfrak{h}^{1,0}$. Applying Lemma 5.1.1, we can extend $\mathcal{B}$ up to a basis $\{\omega^k\}_{k=1}^4$ of $\mathfrak{g}^{1,0}$, in such a way that one can write the fundamental form of any Hermitian metric on $(\mathfrak{g}, J)$ as $\Omega = F + \frac{1}{2}\,\omega^{4\bar{4}}$, being

$$F \;=\; \sum_{k=1}^{3} i\,x_{k\bar{k}}\,\omega^{k\bar{k}} \;+ \sum_{1\leq k<l\leq 3} \big(x_{k\bar{l}}\,\omega^{k\bar{l}} - \bar{x}_{k\bar{l}}\,\omega^{l\bar{k}}\big) \tag{5.2}$$

the fundamental form of a Hermitian metric on $(\mathfrak{h}, K)$. In particular, $x_{k\bar{k}} \in \mathbb{R}^{>0}$ and $x_{k\bar{l}} \in \mathbb{C}$. By Proposition 5.1.2 *ii)*, if $\Omega$ is astheno-Kähler then $F$ must be astheno-Kähler (equivalently SKT, as these two metrics coincide in complex dimension $n = 3$) and fulfill the equations

$$\begin{gathered} F \wedge \partial\bar{\partial}\omega^4 + \partial F \wedge \bar{\partial}\omega^4 - \bar{\partial}F \wedge \partial\omega^4 = 0, \\ F \wedge \big(\bar{\partial}\omega^4 \wedge \partial\omega^{\bar{4}} - \partial\omega^4 \wedge \bar{\partial}\omega^{\bar{4}}\big) = 0. \end{gathered} \tag{5.3}$$

Let us note that they relate $F$ and the structure constants of the $\mathfrak{b}$-extension. In addition, the basis $\mathcal{B}$ can be chosen according to Proposition 5.1.3.

In the following result we prove that if $\Omega$ is an astheno-Kähler metric on a nilmanifold of dimension 8, then the complex structure $J$ must be nilpotent. Moreover, we describe the complex geometry underlying astheno-Kähler nilmanifolds.

**Theorem 5.1.7.** *Let $M$ be an 8-dimensional nilmanifold endowed with an invariant complex structure $J$. Suppose that there exists an invariant astheno-Kähler metric $\Omega$ on $(M, J)$. Then, $J$ is of nilpotent type. Moreover, $(M, J)$ is the trivial product of an SKT 6-nilmanifold and a complex torus, or it admits a $(1,0)$-basis $\{\omega^k\}_{k=1}^4$ for its underlying Lie algebra in terms of which $\Omega = F + \frac{1}{2}\omega^4$ and $(M, J)$ has structure equations:*

$$\text{i)}\quad \begin{cases} d\omega^1 = d\omega^2 = d\omega^3 = 0, \\ d\omega^4 = A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} + \\ \qquad C_{2\bar{1}}\,\omega^{2\bar{1}} + C_{2\bar{2}}\,\omega^{2\bar{2}} + C_{2\bar{3}}\,\omega^{2\bar{3}} + D_{3\bar{1}}\,\omega^{3\bar{1}} + D_{3\bar{2}}\,\omega^{3\bar{2}} + D_{3\bar{3}}\,\omega^{3\bar{3}}, \end{cases}$$

*with (complex) structure constants and metric $F$ related by*

$$(5.4)\quad \begin{aligned} 0 \;=\;& x_{1\bar{1}}\left(2\,\mathfrak{Re}\left(C_{2\bar{2}}\bar{D}_{3\bar{3}}\right) - |A_{23}|^2 - |C_{2\bar{3}}|^2 - |D_{3\bar{2}}|^2\right) \\ +\;& x_{2\bar{2}}\left(2\,\mathfrak{Re}\left(B_{1\bar{1}}\bar{D}_{3\bar{3}}\right) - |A_{13}|^2 - |B_{1\bar{3}}|^2 - |D_{3\bar{1}}|^2\right) \\ +\;& x_{3\bar{3}}\left(2\,\mathfrak{Re}\left(B_{1\bar{1}}\bar{C}_{2\bar{2}}\right) - |A_{12}|^2 - |B_{1\bar{2}}|^2 - |C_{2\bar{1}}|^2\right) \\ +\;& 2\,\mathfrak{Im}\left(x_{1\bar{2}}\left(A_{23}\bar{A}_{13} - C_{2\bar{1}}\bar{D}_{3\bar{3}} + C_{2\bar{3}}\bar{B}_{1\bar{3}} + D_{3\bar{1}}\bar{D}_{3\bar{2}} - D_{3\bar{3}}\bar{B}_{1\bar{2}}\right)\right) \\ -\;& 2\,\mathfrak{Im}\left(x_{1\bar{3}}\left(A_{23}\bar{A}_{12} - C_{2\bar{1}}\bar{C}_{2\bar{3}} + C_{2\bar{2}}\bar{B}_{1\bar{3}} + D_{3\bar{1}}\bar{C}_{2\bar{2}} - D_{3\bar{2}}\bar{B}_{1\bar{2}}\right)\right) \\ +\;& 2\,\mathfrak{Im}\left(x_{2\bar{3}}\left(A_{13}\bar{A}_{12} - B_{1\bar{1}}\bar{C}_{2\bar{3}} + B_{1\bar{2}}\bar{B}_{1\bar{3}} + D_{3\bar{1}}\bar{C}_{2\bar{1}} - D_{3\bar{2}}\bar{B}_{1\bar{1}}\right)\right); \end{aligned}$$

$$\text{ii)}\quad \begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \omega^{1\bar{1}}, \\ d\omega^4 = A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} + B_{1\bar{3}}\,\omega^{1\bar{3}} + C_{2\bar{1}}\,\omega^{2\bar{1}} + D_{3\bar{1}}\,\omega^{3\bar{1}}, \end{cases}$$

*where the (complex) structure constants and the metric $F$ satisfy*

$$(5.5)\quad \begin{aligned} 0 \;=\;& i\,x_{2\bar{2}}(|A_{13}|^2 + |B_{1\bar{3}}|^2 + |D_{3\bar{1}}|^2) + i\,x_{3\bar{3}}(|A_{12}|^2 + |B_{2\bar{2}}|^2 + |C_{2\bar{1}}|^2) \\ +\;& 2\,\mathfrak{Im}\left(x_{2\bar{3}}(A_{13}\bar{A}_{12} + B_{1\bar{2}}\bar{B}_{1\bar{3}} + D_{3\bar{1}}\bar{C}_{2\bar{1}})\right). \end{aligned}$$

*Proof.* Let $\mathfrak{g}$ be the Lie algebra underlying $M$. We observe that $\Omega$ is an astheno-Kähler metric on $(\mathfrak{g}, J)$. Due to Proposition 5.1.6, one has that $J$ must be quasi-nilpotent. Hence, we can apply the ideas above and study the $\mathfrak{b}$-extensions of those 6-dimensional NLAs $\mathfrak{h}$ endowed with complex structures $K$ that admit SKT metrics $F$.

As a consequence of Proposition 5.1.3, there is a basis $\mathcal{B} = \{\omega^k\}_{k=1}^3$ for $\mathfrak{h}^{1,0}$ in terms of which the pairs $(\mathfrak{h}, K)$ are defined by the complex-parallelizable family (1.13) with $\rho = 0$ (torus) and Family I (1.14) with underlying algebras (see Table A in Section 1.4.3):

- $\mathfrak{h}_2$, for which $(\rho, \lambda, D) = (0, 0, i)$ or $(\rho, \lambda, D) = (1, 1, 1 + i\,\mathfrak{Im}\,D)$ and $\mathfrak{Im}\,D > 0$,
- $\mathfrak{h}_4$, but only with the complex structure parametrized by $(\rho, \lambda, D) = (1, 1, 1)$,
- $\mathfrak{h}_5$, only for $(\rho, \lambda, D) = (1, 0, \frac{1}{2} + i\,\mathfrak{Im}\,D)$,
- $\mathfrak{h}_8$, where $\rho = \lambda = D = 0$.

In addition, any $F$ on the previous $(\mathfrak{h}, K)$ satisfies the SKT condition.

We start with the $\mathfrak{b}$-extensions of the torus, which are described in Lemma 3.3.1 *i)* for $\rho = 0$. Notice that the first expression in (5.3) holds trivially, as $d\omega^1 = d\omega^2 = d\omega^3 = 0$. For the second one, a direct calculation gives (5.4).

Next, we consider the $\mathfrak{b}$-extensions of Family I, which can be found in Lemma 3.3.1 *ii)*. Let us remark that the structure constants must satisfy (3.12). Calculating the first condition in (5.3), we obtain:

$$(5.6)\qquad \begin{cases} i\,x_{3\bar{3}}\,(\rho\,A_{12} - \bar{D}B_{1\bar{1}} + \lambda\,B_{1\bar{2}} - C_{2\bar{2}}) + x_{1\bar{3}}\,(\rho\,A_{23} + \bar{D}D_{3\bar{1}} - \lambda\,D_{3\bar{2}}) - \\ \quad x_{2\bar{3}}\,\rho\,A_{13} + \bar{x}_{1\bar{3}}\,\rho\,D_{3\bar{2}} - \bar{x}_{2\bar{3}}\,\rho\,D_{3\bar{1}} = 0, \\ \lambda\,B_{1\bar{3}} - C_{2\bar{3}} + \rho\,D_{3\bar{1}} = 0, \\ \bar{D}\,B_{1\bar{3}} + \rho\,D_{3\bar{2}} = 0. \end{cases}$$

The last two equations above give the following simplification of (3.12):

$$(5.7)\qquad \begin{cases} A_{23} = 0, \\ D\,A_{13} = 0, \\ \rho\,B_{1\bar{3}} + \lambda\,D_{3\bar{1}} - D_{3\bar{2}} = 0, \\ \rho\,C_{2\bar{3}} + D\,D_{3\bar{1}} = 0. \end{cases}$$

Moreover, if we consider the system of equations generated by the last two equations in (5.6) and the last two in (5.7), we get:

$$\begin{pmatrix} \rho & 0 & \lambda & -1 \\ 0 & \rho & D & 0 \\ -\lambda & 1 & -\rho & 0 \\ \bar{D} & 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} B_{1\bar{3}} \\ C_{2\bar{3}} \\ D_{3\bar{1}} \\ D_{3\bar{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Two cases can be distinguished according to the determinant of the previous matrix, which is precisely

$$Det = -\left(\rho^4 + \rho^2(2\mathfrak{Re}\,D - \lambda^2) + |D|^2\right).$$

When the 6-dimensional NLA is $\mathfrak{h}_2$, $\mathfrak{h}_4$, or $\mathfrak{h}_5$, it is easy to see that $Det$ never vanishes, due to the values of $\rho$, $\lambda$, and $D$ given above. Moreover, one observes that $D \neq 0$ for all these cases. Hence, from the previous system and the first two equations in (5.7), we conclude that:

$$A_{13} = A_{23} = B_{1\bar{3}} = C_{2\bar{3}} = D_{3\bar{1}} = D_{3\bar{2}} = 0.$$

In this way, equations (5.6), (5.7), and the second condition in (5.3) (which have not yet been studied) simply become:

$$\begin{cases} \rho\, A_{12} + \lambda B_{1\bar{2}} - C_{2\bar{2}} - \bar{D} B_{1\bar{1}} = 0, \\ |A_{12}|^2 + |B_{1\bar{2}}|^2 + |C_{2\bar{1}}|^2 - 2\,\mathfrak{Re}\,(B_{1\bar{1}}\bar{C}_{2\bar{2}}) = 0. \end{cases}$$

From the first equation, we obtain $C_{2\bar{2}} = \rho\, A_{12} + \lambda B_{1\bar{2}} - \bar{D} B_{1\bar{1}}$. Replacing this value in the second one and solving $|C_{2\bar{1}}|^2$, it can be seen that the existence of solution requires

$$|C_{2\bar{1}}|^2 = 2\,\mathfrak{Re}\left(B_{1\bar{1}}\,(\rho\,\bar{A}_{12} + \lambda \bar{B}_{1\bar{2}})\right) - 2\,\mathfrak{Re}\,D\,|B_{1\bar{1}}|^2 - |A_{12}|^2 - |B_{1\bar{2}}|^2 \geq 0. \tag{5.8}$$

For $\mathfrak{h}_2$ with $\rho = 0,\ \lambda = 0$ and $D = i$, the only possibility is $A_{12} = B_{1\bar{2}} = 0$. Thus, $C_{2\bar{1}} = 0$ and $C_{2\bar{2}} = i\, B_{1\bar{1}}$. We get $d\omega^4 \equiv d\omega^3$, i.e. a product by a complex torus.

For $\mathfrak{h}_2$ with $\rho = \lambda = 1$, $\mathfrak{Re}\, D = 1$ and $\mathfrak{Im}\, D > 0$, we consider the real and imaginary parts of the structure constants and see that (5.8) is equivalent to

$$(\mathfrak{Re}\, A_{12} - \mathfrak{Re}\, B_{1\bar{1}})^2 + (\mathfrak{Im}\, A_{12} - \mathfrak{Im}\, B_{1\bar{1}})^2 + (\mathfrak{Re}\, B_{1\bar{2}} - \mathfrak{Re}\, B_{1\bar{1}})^2 + (\mathfrak{Im}\, B_{1\bar{2}} - \mathfrak{Im}\, B_{1\bar{1}})^2 \leq 0.$$

Therefore, we must have $A_{12} = B_{1\bar{1}} = B_{1\bar{2}}$, which gives $C_{2\bar{1}} = 0$ and $C_{2\bar{2}} = (2 - \bar{D})B_{1\bar{1}}$. Taking into account that $\mathfrak{Re}\, D = 1$ and $\mathfrak{Im}\, D > 0$, we conclude that $d\omega^4 \equiv d\omega^3$. Again, this is a product by a complex torus.

By a similar argument, we get $d\omega^4 \equiv d\omega^3$ for both $\mathfrak{h}_4$ and $\mathfrak{h}_5$.

Let us finally study the NLA $\mathfrak{h}_8$, which is the only one for which $Det = 0$. Recall that in this case we have $\rho = \lambda = D = 0$. Replacing these values in (5.6) and (5.7), it is easy to see that

$$A_{23} = C_{2\bar{2}} = C_{2\bar{3}} = D_{3\bar{2}} = 0.$$

Now, just the second condition in (5.3) remains to be solved. A direct calculation shows that it is precisely (5.5) in the statement of the theorem. □

**Remark 5.1.8.** For any extension of the 6-dimensional torus one can indeed suppose that $\Omega$ is a diagonal metric. Under this assumption, Theorem 5.1.7 *i)* recovers the result obtained by Fino and Tomassini in [FT11, Theorem 2.7] for this particular case.

By Theorem 5.1.4, we know that the underlying algebras of nilmanifolds admitting SKT metrics must be 2-step. In contrast, Theorem 5.1.7 allows to conclude the following:

**Corollary 5.1.9.** *There exist* 3*-step nilmanifolds of dimension* 8 *admitting (invariant) astheno-Kähler metrics.*

*Proof.* Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with a quasi-nilpotent complex structure defined by Theorem 5.1.7 *ii)*, with $A_{12} = A_{13}$, $B_{1\bar{2}} = B_{1\bar{3}}$, and $C_{2\bar{1}} = D_{3\bar{1}}$; that is,

$$\begin{cases} d\omega^1 = d\omega^2 = 0, \\ d\omega^3 = \omega^{1\bar{1}}, \\ d\omega^4 = A_{13}(\omega^{12} + \omega^{13}) + B_{1\bar{1}}\omega^{1\bar{1}} + B_{1\bar{3}}(\omega^{1\bar{2}} + \omega^{1\bar{3}}) + D_{3\bar{1}}(\omega^{2\bar{1}} + \omega^{3\bar{1}}). \end{cases}$$

If we choose $F$ satisfying $i\,x_{2\bar{2}} + i\,x_{3\bar{3}} + 2\,\mathfrak{Im}\,x_{2\bar{3}} = 0$, then the condition (5.5) holds trivially. Hence, $\Omega = F + \frac{i}{2}\omega^{4\bar{4}}$ is an astheno-Kähler metric on $(\mathfrak{g}, J)$. It suffices to take $A_{13}, B_{1\bar{3}}$, or $D_{3\bar{1}} \neq 0$ in order to obtain a 3-step NLA $\mathfrak{g}$. In addition, if we choose such coefficients in $\mathbb{Q}[i]$, then we can apply Malcev Theorem 1.4.9 and there is a lattice $\Gamma$ on the connected and simply connected Lie group $G$ associated to $\mathfrak{g}$. This gives the desired nilmanifolds. □

As far as we know, these are the first examples of 3-step nilmanifolds endowed with a complex structure that admit an astheno-Kähler metric. This definitely provides an important difference between invariant SKT and astheno-Kähler metrics on nilmanifolds. In fact, one has the following result.

**Corollary 5.1.10.** *Let $(M, J)$ an 8-dimensional nilmanifold endowed with an invariant complex structure. If there exists an invariant astheno-Kähler metric on $(M, J)$, then $M$ is at most 3-step.*

### 5.1.2 Generalized Gauduchon metrics

Since both SKT and astheno-Kähler metrics belong to the larger class of generalized Gauduchon metrics, we devote these lines to it. We mainly concentrate on 8-dimensional nilmanifolds and make some observations related to their nilpotency steps. In particular, we find generalized Gauduchon metrics on a 4-step nilmanifold endowed with an SnN complex structure. This entails a difference with respect to SKT and astheno-Kähler metrics.

Let $(M, J, F)$ be a Hermitian manifold of complex dimension $n$. Then, one has that $\frac{i}{2}\,\partial\bar{\partial}F^k \wedge F^{n-k-1} = \tilde{\gamma}_k(F)\,F^n$, where $\tilde{\gamma}_k(F)$ is a real constant whose sign only depends on the conformal class of $F$ (see Section 1.3 for more details). In particular, we recall that $F$ is said to be $k$-th Gauduchon if $\frac{i}{2}\,\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$, that is, $\tilde{\gamma}_k(F) = 0$.

It was shown in [FU13] that an invariant metric on a $2n$-dimensional nilmanifold endowed with an invariant complex structure is $k$-th Gauduchon if and only if it is $(n-k-1)$-th Gauduchon, for $1 \leq k \leq \left[\frac{n}{2}\right] - 1$. However, our next result shows that the connection among generalized Gauduchon metrics on nilmanifolds is still stronger.

**Proposition 5.1.11.** *Let $(M, J)$ be a nilmanifold of real dimension $2n$, with $n \geq 3$, endowed with an invariant complex structure. Consider an invariant Hermitian metric on $(M, J)$ whose fundamental 2-form is given by $\Omega$. Then, $\Omega$ is either $k$-th Gauduchon for every $1 \leq k \leq n-1$ or simply Gauduchon in the usual sense.*

*Proof.* Let us start recalling that every invariant Hermitian metric $\Omega$ on a nilmanifold is standard or Gauduchon in the usual sense, that is, $(n-1)$-th Gauduchon. Indeed, if $0 \neq \partial\bar\partial\Omega^{n-1} = d(\bar\partial\Omega^{n-1})$, then one would have an exact volume form, but this is not possible. Now, let $2 \leq k \leq n-2$. The idea is comparing the $k$-th Gauduchon condition with the 1-st Gauduchon one. By induction, one can see that $\partial\Omega^k = k\,\partial\Omega\wedge\Omega^{k-1}$ and, respectively, $\bar\partial\Omega^k = k\,\bar\partial\Omega\wedge\Omega^{k-1}$. Therefore, we have

$$\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} = k\left(\partial\bar\partial\Omega\wedge\Omega^{n-2} - (k-1)\,\bar\partial\Omega\wedge\partial\Omega\wedge\Omega^{n-3}\right). \tag{5.9}$$

In addition, for every $1 \leq k \leq n-2$

$$\begin{aligned}\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} &= d\left(\bar\partial\Omega^k\wedge\Omega^{n-k-1}\right) + \bar\partial\Omega^k\wedge\partial\Omega^{n-k-1}\\ &= d\left(\bar\partial\Omega^k\wedge\Omega^{n-k-1}\right) + k\,(n-k-1)\bar\partial\Omega\wedge\partial\Omega\wedge\Omega^{n-3},\end{aligned}$$

so we get

$$\bar\partial\Omega\wedge\partial\Omega\wedge\Omega^{n-3} = \frac{1}{k(n-k-1)}\left(\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} - d\left(\bar\partial\Omega^k\wedge\Omega^{n-k-1}\right)\right).$$

Replacing this expression in (5.9) we have

$$\begin{aligned}\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} &= k\,\partial\bar\partial\Omega\wedge\Omega^{n-2} - \frac{k-1}{n-k-1}\,\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1}\\ &+ \frac{k-1}{n-k-1}\,d\left(\bar\partial\Omega^k\wedge\Omega^{n-k-1}\right),\end{aligned}$$

which in turn leads to

$$\frac{n-2}{k-1}\,\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} = \frac{k(n-k-1)}{k-1}\,\partial\bar\partial\Omega\wedge\Omega^{n-2} + d\left(\bar\partial\Omega^k\wedge\Omega^{n-k-1}\right).$$

By Stokes' Theorem

$$\frac{n-2}{k-1}\,\frac{i}{2}\int_M \partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} = \frac{k(n-k-1)}{k-1}\,\frac{i}{2}\int_M \partial\bar\partial\Omega\wedge\Omega^{n-2}.$$

Finally, taking into account that $\frac{i}{2}\,\partial\bar\partial\Omega^k\wedge\Omega^{n-k-1} = \tilde\gamma_k(\Omega)\,\Omega^n$, we conclude

$$\frac{n-2}{k-1}\,\tilde\gamma_k(\Omega)\int_M \Omega^n = \frac{k(n-k-1)}{k-1}\,\tilde\gamma_1(\Omega)\int_M \Omega^n.$$

Therefore,

$$\tilde\gamma_k(\Omega) = \frac{k(n-k-1)}{n-2}\,\tilde\gamma_1(\Omega),$$

for every $2 \leq k \leq n-2$. Hence, it is clear that $\tilde\gamma_1(\Omega) = 0$ if and only if $\tilde\gamma_k(\Omega) = 0$. From here we get the result. □

As a consequence of Proposition 5.1.11, the study of invariant generalized Gauduchon metrics on $(M, J)$ can be reduced to the study of invariant 1-st Gauduchon metrics. For the 6-dimensional case, i.e., $n = 3$, something special happens:

**Proposition 5.1.12.** [FU13] *Let $(M, J)$ be a 6-dimensional nilmanifold endowed with an invariant complex structure. An invariant Hermitian metric on $(M, J)$ is 1-st Gauduchon if and only if it is SKT.*

It is worth noting that there are non-invariant 1-st Gauduchon metrics that are not SKT, as shown in [FU13].

If we now consider the case $n = 4$, then invariant SKT, astheno-Kähler, and generalized Gauduchon metrics on 8-dimensional $(M, J)$ are related according to Figure 5.1.

$$\begin{array}{ccc} \{\text{1-st Gauduchon}\} & = & \{\text{2-nd Gauduchon}\} \\ \cup\!\shortmid & & \cup\!\shortmid \\ \{\text{SKT}\} & \neq & \{\text{astheno-Kähler}\} \end{array}$$

Figure 5.1: Invariant generalized Gauduchon metrics in complex dimension 4.

We first observe that the equality between invariant SKT and 1-st Gauduchon metrics does no longer hold in this dimension. Since every invariant astheno-Kähler metric is 2-nd Gauduchon, it is also 1-st Gauduchon (see Proposition 5.1.11). In particular, the invariant metrics in Corollary 5.1.9 must satisfy the 1-st Gauduchon condition. As the underlying algebra is 3-step, it is clear that the nilmanifold cannot admit any SKT metric by Theorem 5.1.4.

We next show that 1-st Gauduchon metrics also exist on 8-dimensional NLAs with nilpotency steps higher than 3. As a consequence of Corollary 5.1.10, the inclusion of astheno-Kähler metrics in the class of 2-nd Gauduchon metrics is also strict.

**Proposition 5.1.13.** *There exist 1-st Gauduchon metrics on 8-dimensional nilmanifolds with nilpotency steps 4 and 5. In particular, there exist non-nilpotent complex structures admitting generalized Gauduchon metrics.*

*Proof.* Let $(\mathfrak{h}, K)$ be a 6-dimensional NLA endowed with a complex structure parametrized by Family III. Let $(\mathfrak{g}, J)$ be a $\mathfrak{b}$-extension of $(\mathfrak{h}, K)$. By Lemma 3.3.1 *iii)*, there is a basis $\{\omega^k\}_{k=1}^4$ for $\mathfrak{g}^{1,0}$ in terms of which the structure equations are

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 &= i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \\ d\omega^4 &= A_{12}\,\omega^{12} + A_{13}\,\omega^{13} + A_{23}\,\omega^{23} + B_{1\bar{1}}\,\omega^{1\bar{1}} + B_{1\bar{2}}\,\omega^{1\bar{2}} \\ & \quad +(A_{13} \pm 2\,\varepsilon\,A_{23})\,\omega^{1\bar{3}} - B_{1\bar{2}}\,\omega^{2\bar{1}} + A_{23}\,\omega^{2\bar{3}}, \end{cases}$$

with $\varepsilon = 0$ for $\mathfrak{h} \cong \mathfrak{h}_{19}^{-}$ and $\varepsilon = 1$ for $\mathfrak{h} \cong \mathfrak{h}_{26}^{+}$. Furthermore, applying Lemma 5.1.1, one can assume that any Hermitian metric on $(\mathfrak{g}, J)$ is given by $\Omega = F + \frac{i}{2}\omega^{4\bar{4}}$, where $F$ is a Hermitian metric on $(\mathfrak{h}, K)$ described by (5.2). In particular, recall $x_{k\bar{k}} \in \mathbb{R}^{>0}$.

By Propositon 5.1.2, we have that $\Omega$ is 1-st Gauduchon if and only if

$$F \wedge \left(\bar{\partial}\omega^4 \wedge \partial\omega^{\bar{4}} - \partial\omega^4 \wedge \bar{\partial}\omega^{\bar{4}}\right) - 4\,\tilde{\gamma}_1(F)\,F^3 = 0, \tag{5.10}$$

where $\tilde{\gamma}_1(F)$ is the constant given by $\frac{i}{2}\,\partial\bar{\partial}F \wedge F = \tilde{\gamma}_1(F)\,F^3$. Observe that $\tilde{\gamma}_1(F)$ has been calculated in [FU13]. In the case of $(\mathfrak{h}, K)$ parametrized by Family III, its value for any $F$ is precisely

$$\tilde{\gamma}_1(F) = \frac{i\,(x_{2\bar{2}}^2 + x_{3\bar{3}}^2)}{6\,det(x_{k\bar{l}})},$$

being $(x_{k\bar{l}})$ the matrix associated to $F$. A direct calculation shows that condition (5.10) is equivalent to

$$\begin{aligned} 0 \;=\; & 2\,x_{1\bar{1}}|A_{23}|^2 \;+\; x_{2\bar{2}}\left(|A_{13}|^2 + |A_{13} \pm 2\,\varepsilon\,A_{23}|^2\right) \;+\; x_{3\bar{3}}\left(|A_{12}|^2 + 2\,|B_{1\bar{2}}|^2\right) \\ & - \; 4\,\mathfrak{Im}\left(x_{1\bar{2}}\,A_{23}\,(\bar{A}_{13} \pm \varepsilon\,\bar{A}_{23})\right) \;+\; 2\,\mathfrak{Im}\left(x_{1\bar{3}}(A_{23}\bar{A}_{12} + B_{1\bar{2}}\bar{A}_{23})\right) \\ & - \; 2\,\mathfrak{Im}\left(x_{2\bar{3}}(A_{13}\bar{A}_{12} - B_{1\bar{1}}\bar{A}_{23} + B_{1\bar{2}}(\bar{A}_{13} \pm 2\,\varepsilon\,\bar{A}_{23}))\right) \;+\; 4\left(x_{2\bar{2}}^2 + x_{3\bar{3}}^2\right). \end{aligned}$$

In particular, if we choose $\mathfrak{Im}\,(x_{2\bar{3}}) \neq 0$, and

$$A_{12} = B_{1\bar{2}} = 0, \quad A_{13} = \mp\varepsilon, \quad A_{23} = 1, \quad B_{1\bar{1}} = -\frac{x_{1\bar{1}} + \varepsilon\,x_{2\bar{2}} + 2\,(x_{2\bar{2}}^2 + x_{3\bar{3}}^2)}{\mathfrak{Im}\,(x_{2\bar{3}})},$$

then the previous equation holds. Hence, $F$ gives rise to a 1-st Gauduchon metric $\Omega$ on the corresponding extension $(\mathfrak{g}, J)$. Recall that if $\varepsilon = 0$ then $\mathfrak{g}$ is 4-step, whereas if $\varepsilon = 1$ then it is 5-step. Furthermore, $J$ is weakly non-nilpotent. It suffices to choose appropriate structure constants (see Malcev's Theorem 1.4.9) in order to find the desired nilmanifolds. □

**Corollary 5.1.14.** *There exist* 8*-dimensional nilmanifolds endowed with complex structures that admit a Hermitian metric which is both* 1*-st and* 2*-nd Gauduchon but it is neither SKT nor astheno-Kähler.*

Although the complex structures considered in the proof of Proposition 5.1.13 are weakly non-nilpotent, we note that there also exist strongly non-nilpotent $J$'s admitting generalized Gauduchon metrics.

**Example 5.1.15.** Let $(\mathfrak{g}, J)$ be an 8-dimensional NLA endowed with an SnN complex structure defined by the structure equations

$$\begin{cases} d\omega^1 \;=\; d\omega^2 \;=\; 0, \\ d\omega^3 \;=\; \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^4 \;=\; i\,\nu\,\omega^{1\bar{1}} + i\,\omega^{1\bar{3}} + 2\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}, \end{cases}$$

with $\nu = 0$ for $(\dim\mathfrak{g}_k)_k = (1,5,8)$, and $\nu = 1$ for $(\dim\mathfrak{g}_k)_k = (1,5,6,8)$. A direct calculation shows that the Hermitian metric

$$\Omega = i\,(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}} + \omega^{4\bar{4}}) + \frac{1}{2}\,(\omega^{1\bar{3}} - \omega^{3\bar{1}})$$

satisfies $\partial\bar{\partial}\Omega \wedge \Omega^2 = 0$. Therefore, it is 1-st Gauduchon. ◇

The 1-st Gauduchon condition turns to be much weaker than the SKT or the astheno-Kähler ones: not only the underlying algebras admit large nilpotency steps, but also the complex structure can be of non-nilpotent type.

## 5.2 Holomorphic symplectic structures

In this section, we focus on holomorphic symplectic structures on 8-dimensional nilmanifolds endowed with invariant complex structures. In the first part, we provide some obstructions to their existence and study holomorphic deformations. In the second part, we are mainly concern about their relation with pseudo-Kähler structures.

### 5.2.1 Obstructions to their existence and deformations

Here, we show that the existence of holomorphic symplectic structures on 8-nilmanifolds requires a complex structure of nilpotent type. Moreover, we show that their existence along holomorphic deformations is not a closed property.

Let $(M,J)$ be a compact complex manifold. As we already observed in Section 1.3, holomorphic symplectic structures are related with the subspace $H^-_J(M)$ of the de Rham cohomology group $H^2_{\rm dR}(M;\mathbb{R})$. By analogy with the symplectic case, one can introduce the following notion that helps to study this type of structures.

**Definition 5.2.1.** *A compact complex manifold $(M,J)$ of complex dimension $n$ is said to be* cohomologically holomorphic symplectic *if there exists $\mathbf{a} \in H^-_J(M)$ such that the cup product $\mathbf{a}^n \neq \mathbf{0}$.*

**Remark 5.2.2.** The definition implies that the complex dimension of the manifold must be even, i.e., $n = 2p$ for some integer $p \geq 1$.

**Remark 5.2.3.** Let $(M,J,\Omega)$ be a holomorphic symplectic manifold. Then, $\omega = \frac{1}{2}(\Omega + \bar{\Omega})$ is a symplectic form on $(M,J)$ which defines a class in $H^-_J(M) \subseteq H^2_{\rm dR}(M;\mathbb{R})$. It is well known that every symplectic manifold is cohomologically symplectic. Therefore, $[\omega]^n \neq \mathbf{0}$ and $(M,J)$ is cohomologically holomorphic symplectic, where $\omega$ is the symplectic form.

Let us note that not every cohomologically symplectic manifold is symplectic (see [Kas11] and the references therein). Therefore, we cannot ensure that in general every cohomologically holomorphic symplectic manifold is holomorphic symplectic. Nonetheless, we next show that such result holds for the case of nilmanifolds endowed with invariant complex structures.

**Lemma 5.2.4.** *Let $M$ be a $4p$-dimensional nilmanifold endowed with an invariant complex structure $J$. If $(M, J)$ is cohomologically holomorphic symplectic, then it admits an invariant holomorphic symplectic structure.*

*Proof.* By hypothesis, there exists $\mathbf{a} \in H_J^-(M)$ such that $\mathbf{a}^{2p} \neq \mathbf{0}$. Let $\mathfrak{g}$ denote the Lie algebra underlying $M$. According to Proposition 1.4.19, one has $H_J^-(M) \cong H_J^-(\mathfrak{g})$. Therefore, there is $\Omega \in \bigwedge^{2,0}(\mathfrak{g}^*)$ such that $\mathbf{a} = [\Omega + \bar{\Omega}]$. Then,

$$\mathbf{0} \neq \mathbf{a}^{2p} = \left[(\Omega + \bar{\Omega})^{2p}\right] = \left[\sum_{k=0}^{2p} \binom{2p}{k} \Omega^{2p-k} \wedge \bar{\Omega}^k\right] = \left[\binom{2p}{p} \Omega^p \wedge \bar{\Omega}^p\right].$$

Notice that $\Omega^p \wedge \bar{\Omega}^p$ defines a volume form. In particular, we have $d\Omega = 0$ and $\Omega^p \neq 0$; that is, $\Omega$ is an invariant holomorphic symplectic structure. □

**Corollary 5.2.5.** *A nilmanifold endowed with an invariant complex structure is holomorphic symplectic if and only if it is cohomologically holomorphic symplectic.*

In fact, the combination of Lemma 5.2.4 and Corollary 5.2.5 leads to the following result, which will be very useful for our purposes.

**Corollary 5.2.6.** *Let $(M, J)$ be a nilmanifold endowed with an invariant complex structure. The existence of a holomorphic symplectic structure on $(M, J)$ implies the existence of an invariant one.*

From now on, we focus on the study of holomorphic symplectic structures on nilmanifolds of dimension eight.

**Proposition 5.2.7.** *Let $M$ be an 8-dimensional nilmanifold endowed with an invariant complex structure $J$. If $(M, J)$ admits a holomorphic symplectic structure, then $J$ cannot be non-nilpotent.*

*Proof.* By Corollary 5.2.6, we can reduce the problem to the Lie algebra level. Therefore, it suffices to prove that an 8-dimensional NLA $\mathfrak{g}$ endowed with a non-nilpotent complex structure $J$ cannot admit a holomorphic symplectic structure.

Recall that non-nilpotent complex structures on $\mathfrak{g}$ can be of two different types, namely, weakly or strongly non-nilpotent. The former arise as $\mathfrak{b}$-extensions of the 6-dimensional Family III, and their structure equations can be found in Lemma 3.3.1 *iv)*, together with a reduction of them in Proposition 3.3.13 (p. 103). The latter were directly computed in Chapter 4, and their structure equations appear in Theorem 4.4.6 (p. 169).

Any holomorphic symplectic structure $\Omega$ on $(\mathfrak{g}, J)$ is given by a $(2,0)$-form

$$\Omega = \alpha\, \omega^{12} + \beta\, \omega^{13} + \gamma\, \omega^{14} + \tau\, \omega^{23} + \theta\, \omega^{24} + \zeta\, \omega^{34},$$

where $\alpha, \beta, \gamma, \tau, \theta, \zeta \in \mathbb{C}$, satisfying $d\Omega = 0$ and $\Omega \wedge \Omega \neq 0$. We simply observe that

$$\Omega \wedge \Omega \neq 0 \quad \Leftrightarrow \quad \alpha\, \zeta - \beta\, \theta + \tau\, \gamma \neq 0. \tag{5.11}$$

We now study the condition $d\Omega = 0$.

First, let $J$ be weakly non-nilpotent. A direct calculation from Proposition 3.3.13 shows that the coefficients of $\omega^{134}$ and $\omega^{14\bar{2}}$ in $d\Omega$ lead to $\theta = \zeta = 0$, in order to ensure $d\Omega = 0$. With this choice, one has

$$d\Omega \;=\; -\nu\,\gamma\,\omega^{12\bar{3}} + i\,(\varepsilon\,\tau \pm \beta)\,\omega^{12\bar{1}} \pm i\,\tau\,\omega^{12\bar{2}} - \nu\,\gamma\,\omega^{12\bar{3}} - \tau\,\omega^{13\bar{3}}.$$

Clearly, one also needs $\tau = 0$, but this ruins the condition (5.11). Hence, holomorphic symplectic structures do not exist for $(\mathfrak{g}, J)$ when $J$ is weakly non-nilpotent.

Next, consider a strongly non-nilpotent complex structure $J$. Let us first study the existence of $\Omega$ on those $(\mathfrak{g}, J)$ parametrized by part *i)* of Theorem 4.4.6. Using those structure equations, one has

$$\begin{aligned} d\Omega \;=\;& \tau\,\omega^{124} - (A\,\beta - i\,\nu\,\theta)\,\omega^{12\bar{1}} - b\,(\gamma \mp i\,\varepsilon\,\tau)\,\omega^{12\bar{2}} \pm i\,\theta\,\omega^{12\bar{3}} + \tau\,\omega^{12\bar{4}} \\ & + i\,(\nu\,\zeta + i\,\varepsilon\,\tau \pm \gamma)\,\omega^{13\bar{1}} \pm i\,\zeta\,\omega^{13\bar{3}} - \varepsilon\,\theta\,\omega^{14\bar{1}} \mp i\,\varepsilon\,b\,\zeta\,\omega^{14\bar{2}} \\ & - \zeta\,\omega^{14\bar{4}} \pm i\,\theta\,\omega^{23\bar{1}} + b\,\zeta\,\omega^{23\bar{2}} - A\,\zeta\,\omega^{24\bar{1}}. \end{aligned}$$

Observe that one needs $\tau = \theta = \zeta = 0$ in order to get $d\Omega = 0$. However, this contradicts the non-degeneration condition (5.11). Next, we focus on $(\mathfrak{g}, J)$ given by part *ii)* of Theorem 4.4.6. If we calculate $d\Omega$, then we see that $\tau = \theta = 0$, by simply equalling to zero the coefficients of $\omega^{134}$ and $\omega^{12\bar{3}}$ in the resulting expression. In this way, we get

$$\begin{aligned} d\Omega \;=\;& (\varepsilon\,\zeta - i\,\mu\beta)\,\omega^{124} + (\varepsilon\,\beta + i\,b\,\gamma)\,\omega^{12\bar{1}} + \mu\,\gamma\,\omega^{12\bar{2}} - i\,\mu\,\beta\,\omega^{12\bar{4}} \\ & + i\,(\gamma + \nu\,\zeta)\,\omega^{13\bar{1}} + i\,b\,\zeta\,\omega^{13\bar{2}} + i\,\zeta\,\omega^{13\bar{3}} - A\,\zeta\,\omega^{14\bar{1}} - \varepsilon\,\zeta\,\omega^{14\bar{2}} \\ & - i\,b\,\zeta\,\omega^{23\bar{1}} - \mu\,\zeta\,\omega^{23\bar{2}} + \varepsilon\,\zeta\,\omega^{24\bar{1}} - i\,\mu\,\zeta\,\omega^{24\bar{4}}. \end{aligned}$$

Let us recall that $\mu \in \{0, 1\}$. If $\mu = 1$, then we directly obtain $\zeta = 0$ from the previous expression, and the condition (5.11) cannot hold. When $\mu = 0$, we first observe that $\varepsilon \neq 0$ as a consequence of Theorem 4.4.6 *ii)*. At the sight of $d\Omega$, we again reach $\zeta = 0$ and thus (5.11) is not satisfied. As we have covered any possible SnN complex structure $J$ on an 8-dimensional NLA $\mathfrak{g}$, it is clear that such $(\mathfrak{g}, J)$ cannot admit holomorphic symplectic structures. □

Hence, we are led to focus on those 8-dimensional nilmanifolds $M$ endowed with nilpotent complex structures $J$. Recall that they are parametrized by Lemma 3.3.1 in parts *i)*, *ii)*, and *iii)*. From there, one can perform a general study of invariant holomorphic symplectic structures on $(M, J)$, obtaining many new non-Kähler examples. Observe that these examples are compact but not simply connected, in the line of [Gua94] and [Yam05] (for simply connected ones, we refer to [Gua95a, Gua95b, Bog96]). Nonetheless, in this work we are interested in the property of existence of holomorphic symplectic structures under holomorphic deformations of the complex structure. Let us start recalling the following theorem, which provides a condition under which the property is stable.

**Theorem 5.2.8.** [Gua95b] *Let $X$ be a compact holomorphic symplectic manifold such that $H^2_{\rm dR}(X;\mathbb{C}) = H^{2,0}_{\bar\partial}(X) \oplus H^{1,1}_{\bar\partial}(X) \oplus H^{0,2}_{\bar\partial}(X)$. Then, every small deformation of $X$ also admits a holomorphic symplectic structure.*

In the previous result, the hypothesis on the second de Rham cohomology group is essential to ensure the stability. In fact, the existence of holomorphic symplectic structures is in general not open under holomorphic deformations. We illustrate this fact in the example below. We also show that in our example it is possible to preserve the property along particular deformations when the hypothesis on $H^2_{\rm dR}(X;\mathbb{C})$ given in Theorem 5.2.8 does not hold.

**Example 5.2.9.** In Section 2.2.2, we presented the small deformations $X_{\mathbf{t}}$ of the Iwasawa manifold $X_{\mathbf{0}}$. We now want to focus on those directions given by $t_{12} = t_{21} = 0$, with $t_{11}, t_{22}$ free sufficiently small parameters. More precisely, we consider $Y_{\mathbf{t}} = X_{\mathbf{t}} \times \mathbb{T}$, where $\mathbb{T}$ is a complex torus. For sufficiently small $\mathbf{t}$, this is a deformation of the complex manifold $Y_{\mathbf{0}}$, which corresponds to the product of the Iwasawa manifold $X_{\mathbf{0}}$ by $\mathbb{T}$. Let $\varphi^4$ be an invariant $(1,0)$-form on the torus $\mathbb{T}$. By (2.6) and (2.7), the structure equations in terms of the basis $\{\varphi^k_{\mathbf{t}}\}_{k=1}^4$, where $\varphi^4_{\mathbf{t}} = \varphi^4$, are:

$$\begin{cases} d\varphi^1_{\mathbf{t}} &= d\varphi^2_{\mathbf{t}} = 0, \\ d\varphi^3_{\mathbf{t}} &= -\frac{1-|t_{11}t_{22}|^2}{(1-|t_{11}|^2)(1-|t_{22}|^2)}\,\varphi^{12}_{\mathbf{t}} + \frac{t_{22}}{1-|t_{22}|^2}\,\varphi^{1\bar2}_{\mathbf{t}} - \frac{t_{11}}{1-|t_{11}|^2}\,\varphi^{2\bar1}_{\mathbf{t}}, \\ d\varphi^4_{\mathbf{t}} &= 0. \end{cases}$$

Thanks to Corollary 5.2.6, we can study the existence of homolomorphic symplectic structures on $Y_{\mathbf{t}}$ at the Lie algebra level. Let us then consider an invariant $(2,0)$-form

$$\Omega = \alpha\,\varphi^{12}_{\mathbf{t}} + \beta\,\varphi^{13}_{\mathbf{t}} + \gamma\,\varphi^{14}_{\mathbf{t}} + \tau\,\varphi^{23}_{\mathbf{t}} + \theta\,\varphi^{24}_{\mathbf{t}} + \zeta\,\varphi^{34}_{\mathbf{t}},$$

where $\alpha, \beta, \gamma, \tau, \theta, \zeta \in \mathbb{C}$, satisfying $d\Omega = 0$ and $\Omega\wedge\Omega \neq 0$. A direct calculation from the previous structure equations shows that

$$\begin{aligned} d\Omega &= -\frac{\zeta\,(1-|t_{11}t_{22}|^2)}{(1-|t_{11}|^2)(1-|t_{22}|^2)}\,\varphi^{124}_{\mathbf{t}} + \frac{\beta\,t_{11}}{1-|t_{11}|^2}\,\varphi^{12\bar1}_{\mathbf{t}} + \frac{\tau\,t_{22}}{1-|t_{22}|^2}\,\varphi^{12\bar2}_{\mathbf{t}} \\ &\quad -\frac{\zeta\,t_{22}}{1-|t_{22}|^2}\,\varphi^{14\bar2}_{\mathbf{t}} + \frac{\zeta\,t_{11}}{1-|t_{11}|^2}\,\varphi^{24\bar1}_{\mathbf{t}}. \end{aligned}$$

From here, it is clear that $\zeta = 0$, as $|t_{11}t_{22}|^2$ is sufficiently small. Moreover, we have the following conditions:

$$\begin{cases} \beta\,t_{11} = 0, \\ \tau\,t_{22} = 0, \\ \gamma\,\tau - \beta\,\theta \neq 0. \end{cases}$$

Simply note that the last one comes from the non-degeneration condition $\Omega\wedge\Omega \neq 0$ with $\zeta = 0$. Several cases arise:

- If $t_{11} = t_{22} = 0$, then it suffices to choose $\alpha, \beta, \gamma, \tau, \theta$ such that $\gamma\,\tau - \beta\,\theta \neq 0$ in order to find a holomorphic symplectic structure on $Y_{\mathbf{0}}$ (see also [Gua94]).
- If $t_{11} = 0$ but $t_{22} \neq 0$, then we can take $\tau = 0$ and $\alpha, \beta, \gamma, \theta$ such that $\beta\,\theta \neq 0$ in order to obtain a holomorphic symplectic structure.
- If $t_{11} \neq 0$ and $t_{22} = 0$, then $\beta = 0$ and $\alpha, \gamma, \tau, \theta$ are free parameters which must satisfy $\gamma\,\tau \neq 0$ in order to provide an appropriate $\Omega$ on the corresponding $Y_{\mathbf{t}}$.
- If $t_{11}, t_{22} \neq 0$, then one necessarily has $\beta = \tau = 0$. However, this implies $\Omega \wedge \Omega = 0$, and there are no holomorphic symplectic structures on these $Y_{\mathbf{t}}$.

As a consequence, one can clearly see that if we deform $Y_{\mathbf{0}}$ along either $t_{11} = 0$, $t_{22} \neq 0$ or $t_{11} \neq 0$, $t_{22} = 0$, then the existence of holomorphic symplectic structure is preserved. However, when we deform this complex manifold along $t_{11}, t_{22} \neq 0$, holomorphic symplectic structures do not longer exist. This observation is consistent with Theorem 5.2.8, because the required hypothesis does not hold: simply note that the non-zero Dolbeault cohomology class $[\varphi_{\mathbf{0}}^{12}] \in H^{2,0}_{\bar\partial}(Y_{\mathbf{0}})$ vanishes in $H^2_{\rm dR}(Y_{\mathbf{0}}; \mathbb{C})$.

Let us remark that the non-existence of holomorphic symplectic structures for the case $t_{11}, t_{22} \neq 0$ can be also deduced from $H^-(Y_{\mathbf{t}})$, accordingly to Corollary 5.2.5. In fact, this can be directly seen considering the space $\mathcal{Z}^-_{\mathbf{t}}$ of closed $J$-anti-invariant 2-forms on the Lie algebra, which is

$$\begin{aligned}\mathcal{Z}^-_{\mathbf{t}} \;=\; & \langle\, \varphi_{\mathbf{t}}^{12} + \varphi_{\mathbf{t}}^{\bar1\bar2},\, i\,(\varphi_{\mathbf{t}}^{12} - \varphi_{\mathbf{t}}^{\bar1\bar2}),\, \varphi_{\mathbf{t}}^{14} + \varphi_{\mathbf{t}}^{\bar1\bar4},\, i\,(\varphi_{\mathbf{t}}^{14} - \varphi_{\mathbf{t}}^{\bar1\bar4}),\, \varphi_{\mathbf{t}}^{24} + \varphi_{\mathbf{t}}^{\bar2\bar4},\, i\,(\varphi_{\mathbf{t}}^{24} - \varphi_{\mathbf{t}}^{\bar2\bar4}),\\ & \delta_{t_{11}}(\varphi_{\mathbf{t}}^{13} + \varphi_{\mathbf{t}}^{\bar1\bar3}),\, \delta_{t_{11}} i\,(\varphi_{\mathbf{t}}^{13} - \varphi_{\mathbf{t}}^{\bar1\bar3}),\, \delta_{t_{22}}(\varphi_{\mathbf{t}}^{23} + \varphi_{\mathbf{t}}^{\bar2\bar3}),\, \delta_{t_{22}} i\,(\varphi_{\mathbf{t}}^{23} - \varphi_{\mathbf{t}}^{\bar2\bar3})\,\rangle,\end{aligned}$$

where we use the notation $\delta_t = 1$, if $t = 0$, and $\delta_t = 0$, if $t \neq 0$. Notice that there are no forms involving $\varphi_{\mathbf{t}}^{3}$ and $\varphi_{\mathbf{t}}^{\bar3}$ when one takes $t_{11}, t_{22} \neq 0$. Hence, one cannot achieve an element in $H^-(Y_{\mathbf{t}})$ which makes the manifold cohomologically holomorphic symplectic. As a consequence, the manifold cannot be holomorphic symplectic. ◇

Our next result reveals that the property of admitting a holomorphic symplectic structure is not closed under holomorphic deformations.

**Theorem 5.2.10.** *There is a holomorphic family of compact complex manifolds* $\{X_t\}_{t\in\Delta}$ *of complex dimension* 4*, where* $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$*, such that* $X_t$ *admits a holomorphic symplectic structure for every* $t \in \Delta \setminus \{0\}$*, but* $X_0$ *is not a holomorphic symplectic manifold.*

*Proof.* Let $X_0 = (M, J_0)$ be an 8-dimensional nilmanifold endowed with an invariant complex structure, defined by the structure equations:

$$\begin{cases} d\omega^1 \;=\; d\omega^2 \;=\; 0,\\ d\omega^3 \;=\; \omega^{12},\\ d\omega^4 \;=\; i\,\omega^{1\bar1} + \omega^{1\bar2} + \omega^{2\bar1}.\end{cases}$$

Observe that $J_0$ is a $\mathfrak{b}$-extension of the Iwasawa manifold. Note that the $(0,1)$-forms $\omega^{\bar 1}$ and $\omega^{\bar 2}$ define non-zero Dolbeault cohomology classes on $X_0$. Therefore, they provide appropriate directions along which performing holomorphic deformations. For each $t \in \mathbb{C}$ such that $|t| < 1$, define an invariant complex structure $J_t$ on $M$ given by the following basis of $(1,0)$-forms:

$$\eta_t^1 = \omega^1 + t\,\omega^{\bar 1} - i\,t\,\omega^{\bar 2}, \quad \eta_t^2 = \omega^2, \quad \eta_t^3 = \omega^3, \quad \eta_t^4 = \omega^4.$$

The complex structure equations for $X_t = (M, J_t)$ are

$$\text{(5.12)} \qquad \begin{cases} d\eta_t^1 \;=\; d\eta_t^2 \;=\; 0, \\ d\eta_t^3 \;=\; \frac{1}{1-|t|^2}\left(\eta_t^{12} + t\,\eta_t^{2\bar 1} - i\,t\,\eta_t^{2\bar 2}\right), \\ d\eta_t^4 \;=\; \frac{1}{1-|t|^2}\left(2\,\bar t\,\eta_t^{12} + i\,\eta_t^{1\bar 1} + \eta_t^{1\bar 2} + \eta_t^{2\bar 1} - i\,t\,\eta_t^{2\bar 2}\right). \end{cases}$$

Let us remark that we recover $X_0$, for $t = 0$. By Corollary 5.2.6, the existence of holomorphic symplectic structures can be studied at the Lie algebra level. Hence, we can take an invariant $(2,0)$-form

$$\Omega = \alpha\,\eta_t^{12} + \beta\,\eta_t^{13} + \gamma\,\eta_t^{14} + \tau\,\eta_t^{23} + \theta\,\eta_t^{24} + \zeta\,\eta_t^{34},$$

where $\alpha, \beta, \gamma, \tau, \theta, \zeta \in \mathbb{C}$, and impose the condition $d\Omega = 0$, where

$$\begin{aligned} d\Omega \;=\; & -\frac{1}{1-|t|^2}\Big(2\,\bar t\,\zeta\,\eta_t^{123} - \zeta\,\eta_t^{124} + (t\,\beta - i\,\theta + \gamma)\,\eta_t^{12\bar 1} - i\,(t\,\beta - i\,\theta + \gamma\,|t|^2)\,\eta_t^{12\bar 2} \\ & - i\,\zeta\eta_t^{13\bar 1} - \zeta\,\eta_t^{13\bar 2} - \zeta\,\eta_t^{23\bar 1} + i\,\zeta\,|t|^2\,\eta_t^{23\bar 2} + t\,\zeta\,\eta_t^{24\bar 1} - i\,t\,\zeta\,\eta_t^{24\bar 2}\Big). \end{aligned}$$

It is straightforward to see that $\zeta = 0$. Therefore, one simply needs to solve the system of equations

$$\begin{cases} t\,\beta - i\,\theta + \gamma = 0, \\ (1 - |t|^2)\,\gamma = 0, \end{cases}$$

bearing in mind the non-degeneration of $\Omega$, which is equivalent to $\gamma\,\tau - \beta\,\theta \neq 0$. Since $|t| < 1$, it is clear that $\gamma = 0$. The problem is reduced to a single equation

$$t\,\beta - i\,\theta = 0, \quad \text{where} \;\; \beta\,\theta \neq 0.$$

If $t = 0$, then $\theta = 0$ and the non-degeneration condition does not hold. As a consequence, there are no holomorphic symplectic structures on $X_0 = (M, J_0)$. However, the complex manifold $X_t = (M, J_t)$ with $t \neq 0$ admits holomorphic symplectic structures

$$\Omega = \alpha\,\eta_t^{12} + \frac{i}{t}\,\theta\,\eta_t^{13} + \tau\,\eta_t^{23} + \theta\,\eta_t^{24},$$

where $\alpha,\ \tau,\ \theta \in \mathbb{C}$ and $\theta \neq 0$. □

**Remark 5.2.11.** The previous result can also be proved using the notion of cohomologically holomorphic symplectic manifold. Let us recall that the space $H^-_{J_t}(M)$ can be calculated at the Lie algebra level, by Proposition 1.4.19. Hence, from (5.12) we get:

$$\begin{aligned} H^-_{J_t}(M) \;=\; & \langle\, [\eta_t^{23}+\eta_t^{\bar{2}\bar{3}}],\, [i(\eta_t^{23}-\eta_t^{\bar{2}\bar{3}})],\, [\eta_t^{13}+\eta_t^{\bar{1}\bar{3}}-i\,(t\,\eta_t^{24}-\bar{t}\,\eta_t^{\bar{2}\bar{4}})], \\ & [i(\eta_t^{13}-\eta_t^{\bar{1}\bar{3}})+t\,\eta_t^{24}+\bar{t}\,\eta_t^{\bar{2}\bar{4}}],\, \delta_{\mathfrak{Im}\,t}[\eta_t^{12}+\eta_t^{\bar{1}\bar{2}}],\, \delta_{\mathfrak{Re}\,t}[i(\eta_t^{12}-\eta_t^{\bar{1}\bar{2}})]\,\rangle. \end{aligned}$$

It is clear that for $t=0$ there are no cohomology classes involving $\eta_t^4$ and $\eta_t^{\bar{4}}$, so one cannot find $\mathbf{a}\in H^-_{J_0}(M)$ such that $\mathbf{a}^4\neq\mathbf{0}$. In contrast, for $t\neq 0$ it suffices to take $\mathbf{a}=[\eta_t^{13}+\eta_t^{\bar{1}\bar{3}}-i\,(t\,\eta_t^{24}-\bar{t}\,\eta_t^{\bar{2}\bar{4}})]$.

### 5.2.2 Relation with pseudo-Kähler structures

Next, we study pseudo-Kähler structures on 8-nilmanifolds endowed with invariant complex structures. More concretely, we show that their existence is in general non-related to the existence of holomorphic symplectic structures. Furthermore, we find pseudo-Kähler structures on complex structures of non-nilpotent type, which contrasts with the 6-dimensional case.

In the literature, the study of holomorphic symplectic structures is frequently combined with the study of pseudo-Kähler structures. One reason is that they give rise to two special and complementary types of symplectic structures $\omega$ with respect to the complex structure $J$, namely, those satisfying $\omega(J\cdot,J\cdot)=-\omega(\cdot,\cdot)$ and those fulfilling $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$. In addition, let us observe that whereas holomorphic symplectic structures define non-zero cohomology classes in $H^-_J(M)$, pseudo-Kähler structures behave similarly for $H^+_J(M)$. As we saw in Section 1.2.3, these are two subspaces of the second de Rham cohomology group $H^2_{\mathrm{dR}}(M)$.

Although it is possible to associate the existence of pseudo-Kähler and holomorphic symplectic structures under certain conditions (see [Gua10, Theorem 10]), there is in general no relation. This was shown in [Yam05] using a certain type of solvmanifolds with invariant complex structures. Here, we reinforce this idea using the class of nilmanifolds with abelian complex structures.

**Proposition 5.2.12.** *There are* 8*-dimensional nilmanifolds endowed with abelian complex structures admitting*

*i) both holomorphic symplectic and pseudo-Kähler structures,*

*ii) holomorphic symplectic structures but no pseudo-Kähler structures,*

*iii) pseudo-Kähler structures but no holomorphic symplectic structures,*

*iv) neither holomorphic symplectic nor pseudo-Kähler structures.*

*Proof.* Let us consider the product of the Kodaira-Thurston manifold $\mathbb{KT}$ by a complex torus $\mathbb{T}$. It is well-known that $\mathbb{KT}\times\mathbb{T}$ is a 6-dimensional nilmanifold endowed with an invariant complex structure $K$ whose underlying Lie algebra is $\mathfrak{h}_8$ (recall Theorem 1.4.20).

The structure equations for $(\mathfrak{h}_8, K)$ belong to Family I with $\rho = \lambda = D = 0$ (see Table A in Section 1.4.3), so its $\mathfrak{b}$-extensions $(\mathfrak{g}, J)$ are given in Proposition 3.3.5 *i)*. Take those of abelian type, i.e. $A_{12} = A_{13} = A_{23} = 0$. For the seek of simplicity, we choose $B_{1\bar{3}} = C_{2\bar{2}} = 0$. These $(\mathfrak{g}, J)$ are parametrized by:

$$\begin{cases} d\omega^1 \;=\; d\omega^2 \;=\; 0, \\ d\omega^3 \;=\; \omega^{1\bar{1}}, \\ d\omega^4 \;=\; B_{1\bar{2}}\,\omega^{1\bar{2}} + C_{2\bar{1}}\,\omega^{2\bar{1}} + D_{3\bar{1}}\,\omega^{3\bar{1}}, \end{cases} \tag{5.13}$$

where $B_{1\bar{2}}, C_{2\bar{1}}, D_{3\bar{1}} \in \mathbb{C}$. By Corollary 5.2.6, the existence of holomorphic symplectic structures on any 8-dimensional nilmanifold with underlying Lie algebra $\mathfrak{g}$ and invariant complex structure $J$ implies the existence of invariant ones. A similar argument using $H_J^+$ instead of $H_J^-$ shows that the same happens for pseudo-Kähler metrics. Hence, it suffices to study the existence of such structures on $(\mathfrak{g}, J)$.

A pseudo-Kähler structure $F$ on $(\mathfrak{g}, J)$ can be written in terms of a $(1,1)$-form

$$F \;=\; \sum_{k=1}^{4} i\,x_{k\bar{k}}\,\omega^{k\bar{k}} \;+\; \sum_{1\leq k<l\leq 4} \big(x_{k\bar{l}}\,\omega^{k\bar{l}} - \bar{x}_{k\bar{l}}\,\omega^{l\bar{k}}\big), \quad \text{such that} \begin{cases} dF = 0, \\ F^4 \neq 0, \end{cases} \tag{5.14}$$

where $x_{k\bar{k}} \in \mathbb{R}$ and $x_{k\bar{l}} \in \mathbb{C}$, for $1 \leq k, l \leq 4$.

Any holomorphic symplectic structure $\Omega$ on $(\mathfrak{g}, J)$ is given by a $(2,0)$-form

$$\Omega = \alpha\,\omega^{12} + \beta\,\omega^{13} + \gamma\,\omega^{14} + \tau\,\omega^{23} + \theta\,\omega^{24} + \zeta\,\omega^{34}, \quad \text{such that} \begin{cases} d\Omega = 0, \\ \Omega^2 \neq 0, \end{cases}$$

where $\alpha, \beta, \gamma, \tau, \theta, \zeta \in \mathbb{C}$. If we compute $d\Omega$ from (5.13), then we clearly see that $\zeta = 0$ and $\tau = C_{2\bar{1}}\,\gamma$. In this way, the conditions $d\Omega = 0$ and $\Omega \wedge \Omega \neq 0$ that ensure the existence of a holomorphic symplectic structure become

$$\begin{cases} B_{1\bar{2}}\,\theta\,\omega^{12\bar{2}} - D_{3\bar{1}}\,\gamma\,\omega^{13\bar{1}} - D_{3\bar{1}}\,\theta\,\omega^{23\bar{1}} \;=\; 0, \\ \beta\,\theta - C_{2\bar{1}}\,\gamma^2 \neq 0. \end{cases} \tag{5.15}$$

We distinguish different cases according to the values of the structure constants in (5.13).

- Let us suppose that $B_{1\bar{2}} = D_{3\bar{1}} = 0$. From (5.15), it is clear that it suffices to choose $\beta\,\theta \neq C_{2\bar{1}}\,\gamma^2$ in order to find a holomorphic symplectic structure on $(\mathfrak{g}, J)$.

  We now study the existence of pseudo-Kähler structures, i.e., $dF = 0$.

  From (5.13) and (5.14), one has

$$\begin{aligned} dF \;=\; & -x_{2\bar{3}}\,\omega^{12\bar{1}} - x_{2\bar{4}}\,\bar{C}_{2\bar{1}}\,\omega^{12\bar{2}} - i\,x_{3\bar{3}}\,\omega^{13\bar{1}} - x_{3\bar{4}}\,\bar{C}_{2\bar{1}}\omega^{13\bar{2}} + \bar{x}_{3\bar{4}}\,\omega^{14\bar{1}} \\ & -i\,x_{4\bar{4}}\,\bar{C}_{2\bar{1}}\,\omega^{14\bar{2}} - \bar{x}_{2\bar{3}}\,\omega^{1\bar{1}\bar{2}} + i\,x_{3\bar{3}}\,\omega^{1\bar{1}\bar{3}} + x_{3\bar{4}}\,\omega^{1\bar{1}\bar{4}} - \bar{x}_{2\bar{4}}\,C_{2\bar{1}}\,\omega^{2\bar{1}\bar{2}} \\ & -\bar{x}_{3\bar{4}}\,C_{2\bar{1}}\,\omega^{2\bar{1}\bar{3}} + i\,x_{4\bar{4}}\,C_{2\bar{1}}\,\omega^{2\bar{1}\bar{4}}. \end{aligned}$$

It is straightforward to see that $x_{2\bar{3}} = x_{3\bar{3}} = x_{3\bar{4}} = 0$. In this way, we have

$$dF = -x_{2\bar{4}}\,\bar{C}_{2\bar{1}}\,\omega^{12\bar{2}} - i\,x_{4\bar{4}}\,\bar{C}_{2\bar{1}}\,\omega^{14\bar{2}} - \bar{x}_{2\bar{4}}\,C_{2\bar{1}}\,\omega^{2\bar{1}\bar{2}} + i\,x_{4\bar{4}}\,C_{2\bar{1}}\,\omega^{2\bar{1}\bar{4}}.$$

Furthermore, the non-degeneration condition for $F$ becomes

$$0 \neq F^4 = 24\,|x_{1\bar{3}}|^2(|x_{2\bar{4}}|^2 - x_{2\bar{2}}\,x_{4\bar{4}})\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}.$$

We then note the following:

– If $C_{2\bar{1}} = 0$, then $dF = 0$ and it suffices to choose $x_{1\bar{3}} \neq 0$, $|x_{2\bar{4}}|^2 - x_{2\bar{2}}\,x_{4\bar{4}} \neq 0$ in order to obtain a pseudo-Kähler metric $F$. Therefore, $(\mathfrak{g}, J)$ given by (5.13) with structure constants $B_{1\bar{3}} = C_{2\bar{1}} = D_{3\bar{1}} = 0$ satisfies *i)* in the statement of the proposition. Observe that this manifold is simply the product of $\mathbb{KT}$ and a 2-dimensional complex torus.

– If $C_{2\bar{1}} \neq 0$, then one takes $x_{2\bar{4}} = x_{4\bar{4}} = 0$, but this choice makes $F^4 = 0$. As a consequence, there are no pseudo-Kähler structures. Hence, the complex nilmanifold corresponding to $(\mathfrak{g}, J)$ with $B_{1\bar{3}} = D_{3\bar{1}} = 0$ and $C_{2\bar{1}} \neq 0$ gives part *ii)* of the proposition.

- Let us assume $B_{1\bar{2}} = 0$ and $D_{3\bar{1}} \neq 0$. By (5.15), one gets $\gamma = \theta = 0$. Nonetheless, this choice yields $\Omega \wedge \Omega = 0$, and there are no holomorphic symplectic structures. We now study the pseudo-Kähler condition on these $(\mathfrak{g}, J)$. A direct calculation from (5.13) and (5.14) shows that we need $x_{2\bar{3}} = x_{2\bar{4}} = x_{3\bar{3}} = x_{3\bar{4}} = x_{4\bar{4}} = 0$, in order to have $dF = 0$. Observe that these values lead to $F^4 = 0$, contradicting the non-degeneration of $F$. Therefore, $(\mathfrak{g}, J)$ with $B_{1\bar{2}} = 0$ and $D_{3\bar{1}} \neq 0$ gives part *iv)* of the proposition.

- Consider $B_{1\bar{2}} \neq 0$ and $D_{3\bar{1}} = 0$. From (5.15), it is clear that $\theta = 0$ and the non-degeneration condition of $\Omega$ becomes $C_{2\bar{1}}\,\gamma^2 \neq 0$. In particular, it directly depends on the value of $C_{2\bar{1}}$.

– If $C_{2\bar{1}} = 0$, then there are no holomorphic symplectic structures on $(\mathfrak{g}, J)$. We now study the pseudo-Kähler condition. A direct calculation from (5.13) and (5.14) shows that $dF = 0$ implies $x_{3\bar{3}} = x_{3\bar{4}} = x_{4\bar{4}} = 0$ and $x_{2\bar{3}} = \bar{B}_{1\bar{2}}\,x_{1\bar{4}}$. As a consequence,

$$F^4 = 24\,|x_{1\bar{3}}\,x_{2\bar{4}} - \bar{B}_{1\bar{2}}\,x_{1\bar{4}}^2|^2\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}.$$

It is easy to see that one can take values in $F$ satisfying $x_{1\bar{3}}\,x_{2\bar{4}} \neq \bar{B}_{1\bar{2}}\,x_{1\bar{4}}^2$, thus preserving the non-degeneration condition $F^4 \neq 0$. We conclude that $(\mathfrak{g}, J)$ given by (5.13) with $D_{3\bar{1}} = C_{2\bar{1}} = 0$ and $B_{1\bar{2}} \neq 0$ leads to part *iii)* in the statement of the proposition.

– If we now consider the case $C_{2\bar{1}} \neq 0$, then it suffices to take $\gamma \neq 0$ in order to find a holomorphic symplectic structure $\Omega$ on $(\mathfrak{g}, J)$. For pseudo-Kähler structures, we apply (5.13) and (5.14), obtaining the values $x_{2\bar{4}} = x_{3\bar{3}} = x_{3\bar{4}} = x_{4\bar{4}} = 0$ and $x_{2\bar{3}} = \bar{B}_{1\bar{2}}\, x_{1\bar{4}}$. Then,

$$F^4 \;=\; 24\,|B_{1\bar{2}}|^2\,|x_{1\bar{4}}|^4\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}.$$

Since $B_{1\bar{2}} \neq 0$, it suffices to choose $x_{1\bar{4}} \neq 0$. We then observe that $(\mathfrak{g}, J)$ with $D_{3\bar{1}} = 0$ and $B_{1\bar{2}}, C_{2\bar{1}} \neq 0$ admits both holomorphic symplectic and pseudo-Kähler structures. Once again, we are in case *i)* of the proposition. Note that this time $(\mathfrak{g}, J)$ does not correspond to $\mathbb{KT} \times \mathbb{T}^2$.

- Finally, we suppose $B_{1\bar{2}}, D_{3\bar{1}} \neq 0$. At the sight of (5.15), we get $\gamma = \theta = 0$. However, these values make $\Omega \wedge \Omega = 0$, so there are no holomorphic symplectic structures on $(\mathfrak{g}, J)$. For the pseudo-Kähler condition, we calculate $dF = 0$ from (5.13) and see that $x_{2\bar{4}} = x_{3\bar{3}} = x_{3\bar{4}} = x_{4\bar{4}} = 0$ and $x_{2\bar{3}} = \bar{B}_{1\bar{2}}\, x_{1\bar{4}}$. It suffices to choose $x_{1\bar{4}} \neq 0$ in order to ensure $F^4 \neq 0$. Hence, we have found another $(\mathfrak{g}, J)$ satisfying part *iii)* of the proposition, this time with structure constants $B_{1\bar{2}}, D_{3\bar{1}} \neq 0$.

In all the previous cases, the structure constants $B_{1\bar{2}}, C_{2\bar{1}}, D_{3\bar{1}}$ can be chosen in $\mathbb{Q}[i]$. By Malcev Theorem 1.4.9, this choice ensures the existence of a lattice $\Gamma$ on the corresponding Lie group $G$ with associated Lie algebra $\mathfrak{g}$. Then, we have a compact nilmanifold $\Gamma\backslash G$ satisfying the desired conditions. For the seek of clarity, we summarize our results in the following table according to the values of the structure constants in (5.13):

| $B_{1\bar{2}}$ | $C_{2\bar{1}}$ | $D_{3\bar{1}}$ | holomorphic symplectic | pseudo-Kähler |
|---|---|---|---|---|
| 0 | 0 | 0 | ✓ | ✓ |
| | $\neq 0$ | | ✓ | – |
| | *free* | $\neq 0$ | – | – |
| $\neq 0$ | 0 | 0 | – | ✓ |
| | $\neq 0$ | | ✓ | ✓ |
| | *free* | $\neq 0$ | – | ✓ |

□

In [CFU04], it is conjectured that the existence of a pseudo-Kähler metric on a $2n$-dimensional nilmanifold endowed with an invariant complex structure $J$ implies the nilpotency of $J$. The authors prove that this holds up to dimension $n \leq 3$. However, we next see that this is not the case in higher dimensions. We start showing which 8-dimensional NLAs endowed with non-nilpotent complex structures admit pseudo-Kähler structures.

**Theorem 5.2.13.** *Let $M$ be an 8-dimensional nilmanifold endowed with a non-nilpotent complex structure $J$. There exists a pseudo-Kähler structure on $(M, J)$ if and only if there is a $(1,0)$-basis $\{\omega^k\}_{k=1}^4$ such that the structure equations are one of the following:*

$$i)\ \begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = \omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}, \\ d\omega^4 = i\,(\omega^{1\bar{3}} - \omega^{3\bar{1}}), \end{cases} \qquad ii)\ \begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 &= \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \\ d\omega^4 &= A\,\omega^{12} + \omega^{23} + B\,\omega^{1\bar{1}} + \omega^{2\bar{3}}, \end{cases}$$

*where $A, B \in \mathbb{C}$. Moreover, the complex structure given by i) is strongly non-nilpotent, and that in ii) is weakly non-nilpotent.*

*Proof.* Let $\mathfrak{g}$ be the NLA underlying $M$. The idea is to compute $dF = 0$ and the non-degeneration condition $F^4 \neq 0$ using the structure equations found in Chapters 3 and 4 for each $(\mathfrak{g}, J)$, where $J$ is non-nilpotent.

We start with those $J$'s of SnN type, which are determined by Theorem 4.4.6. Recall that if $\{\omega^k\}_{k=1}^4$ is a basis for $\mathfrak{g}^{1,0}$, then any pseudo-Kähler structure $F$ on $(\mathfrak{g}, J)$ can be written as in (5.14). Two cases are distinguished according to the dimension of the ascending central series of $\mathfrak{g}$.

If $(\dim \mathfrak{g}_k)_k = (1,3,8), (1,3,6,8), (1,4,8), (1,4,6,8), (1,5,8)$, or $(1,5,6,8)$, then $(\mathfrak{g}, J)$ is parametrized by

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \varepsilon\,\omega^{1\bar{1}}, \\ d\omega^3 &= \omega^{14} + \omega^{1\bar{4}} + A\,\omega^{2\bar{1}} \pm i\,\varepsilon\,b\,\omega^{1\bar{2}}, \\ d\omega^4 &= i\,\nu\,\omega^{1\bar{1}} \pm i\,\omega^{1\bar{3}} + b\,\omega^{2\bar{2}} \mp i\,\omega^{3\bar{1}}, \end{cases}$$

where $\varepsilon, \nu \in \{0,1\}$, $A \in \mathbb{C}$, and $b \in \mathbb{R}$. A direct calculation shows that $dF = 0$ implies $x_{1\bar{4}} = x_{2\bar{3}} = x_{2\bar{4}} = x_{3\bar{4}} = 0$. With this choice, we obtain:

$$\begin{aligned} dF \;=\; & -i\,\varepsilon\,(x_{2\bar{2}} \pm b\,x_{1\bar{3}})\,\omega^{12\bar{1}} - i\,x_{3\bar{3}}\,\bar{A}\,\omega^{13\bar{2}} + (x_{1\bar{3}} - \bar{x}_{1\bar{3}} - \nu\,x_{4\bar{4}})\,\omega^{14\bar{1}} \\ & + (i\,x_{3\bar{3}} \pm x_{4\bar{4}})\,\omega^{14\bar{3}} + i\,\varepsilon\,(x_{2\bar{2}} \pm b\,\bar{x}_{1\bar{3}})\,\omega^{1\bar{1}\bar{2}} - (x_{1\bar{3}} - \bar{x}_{1\bar{3}} + \nu\,x_{4\bar{4}})\,\omega^{1\bar{1}\bar{4}} \\ & \mp \varepsilon\,b\,x_{3\bar{3}}\,\omega^{1\bar{2}\bar{3}} - (i\,x_{3\bar{3}} \pm x_{4\bar{4}})\,\omega^{1\bar{3}\bar{4}} \mp \varepsilon\,b\,x_{3\bar{3}}\,\omega^{23\bar{1}} - i\,x_{4\bar{4}}\,b\,\omega^{24\bar{2}} + i\,x_{3\bar{3}}\,A\,\omega^{2\bar{1}\bar{3}} \\ & + i\,x_{4\bar{4}}\,b\,\omega^{2\bar{2}\bar{4}} + (i\,x_{3\bar{3}} \mp x_{4\bar{4}})\,\omega^{34\bar{1}} - (i\,x_{3\bar{3}} \mp x_{4\bar{4}})\,\omega^{3\bar{1}\bar{4}}, \\ F^4 \;=\; & 24\,x_{4\bar{4}}\,(x_{1\bar{1}}\,x_{2\bar{2}}\,x_{3\bar{3}} - x_{3\bar{3}}\,|x_{1\bar{2}}|^2 - x_{2\bar{2}}|x_{1\bar{3}}|^2)\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}. \end{aligned}$$

We recall that $x_{3\bar{3}}$ and $x_{4\bar{4}}$ are real numbers. Hence, one has $i\,x_{3\bar{3}} \pm x_{4\bar{4}} = 0$ if and only if $x_{3\bar{3}} = x_{4\bar{4}} = 0$. However, this yields $F^4 = 0$, contradicting the non-degeneration condition. Therefore, there are no pseudo-Kähler structures on these $(\mathfrak{g}, J)$.

If $(\dim \mathfrak{g}_k)_k = (1,3,5,8)$ or $(1,3,5,6,8)$, then $(\mathfrak{g}, J)$ is defined by

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = A\,\omega^{1\bar{1}} + \varepsilon\,(\omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,\mu\,(\omega^{24} + \omega^{2\bar{4}}), \\ d\omega^4 = i\,\nu\,\omega^{1\bar{1}} + i\,b\,\omega^{1\bar{2}} + i\,\omega^{1\bar{3}} - i\,b\,\omega^{2\bar{1}} - \mu\,\omega^{2\bar{2}} - i\,\omega^{3\bar{1}}, \end{cases}$$

where $\varepsilon, \nu, \mu \in \{0,1\}$, $A \in \mathbb{C}$ such that $\mathfrak{Im}\, A = 0$ for $\varepsilon = 1$, and $b \in \mathbb{R}$. Moreover, $(\varepsilon, \mu) \neq (0,0)$. Calculating the condition $dF = 0$ from the previous structure equations, we directly get $x_{2\bar{4}} = x_{3\bar{4}} = 0$. Now, since $(\varepsilon, \mu) \neq (0,0)$ we also have $x_{3\bar{3}} = 0$, which in turn leads to $x_{1\bar{4}} = 0$ and $x_{2\bar{3}} = -x_{4\bar{4}}$. In this way,

$$\begin{aligned} dF \;=\;& \big(\varepsilon\,(x_{1\bar{3}} - \bar{x}_{1\bar{3}}) + \bar{A}\,x_{4\bar{4}}\big)\,\omega^{12\bar{1}} + (x_{1\bar{2}} - \bar{x}_{1\bar{2}} - \nu\,x_{4\bar{4}})\,\omega^{14\bar{1}} \\ & + i\,(x_{2\bar{2}} - i\,b\,x_{4\bar{4}} - \mu\,x_{1\bar{3}})\,\omega^{14\bar{2}} - \big(\varepsilon\,(x_{1\bar{3}} - \bar{x}_{1\bar{3}}) - A\,x_{4\bar{4}}\big)\,\omega^{1\bar{1}\bar{2}} \\ & -(x_{1\bar{2}} - \bar{x}_{1\bar{2}} + \nu\,x_{4\bar{4}})\,\omega^{1\bar{1}\bar{4}} - i\,(x_{2\bar{2}} - i\,b\,x_{4\bar{4}} - \mu\,x_{1\bar{3}})\,\omega^{1\bar{2}\bar{4}} \\ & + i\,(x_{2\bar{2}} + i\,b\,x_{4\bar{4}} - \mu\,\bar{x}_{1\bar{3}})\,\omega^{24\bar{1}} + 3\,i\,\mu\,x_{4\bar{4}}\,\omega^{24\bar{2}} \\ & - i\,(x_{2\bar{2}} + i\,b\,x_{4\bar{4}} - \mu\,\bar{x}_{1\bar{3}})\,\omega^{2\bar{1}\bar{4}} - 3\,i\,\mu\,x_{4\bar{4}}\,\omega^{2\bar{2}\bar{4}}, \\ F^4 \;=\;& -24\,x_{4\bar{4}}\,\big(x_{1\bar{1}}\,x_{4\bar{4}}^2 + x_{2\bar{2}}\,|x_{1\bar{3}}|^2 - i\,x_{4\bar{4}}\,(x_{1\bar{2}}\bar{x}_{1\bar{3}} - \bar{x}_{1\bar{2}}\,x_{1\bar{3}})\big)\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}. \end{aligned}$$

It is clear that if $\mu = 1$, then $x_{4\bar{4}} = 0$ and the non-degeneration condition fails. Hence, we concentrate on the case $\mu = 0$, which implies $\varepsilon = 1$ and $A \in \mathbb{R}$ by Theorem 4.4.6 *ii)*. Observe that one must have $x_{2\bar{2}} + i\,b\,x_{4\bar{4}} = 0$ in order to get $dF = 0$. However, this is only possible if $x_{2\bar{2}} = 0$ and $b\,x_{4\bar{4}} = 0$. If $b \neq 0$, then $x_{4\bar{4}} = 0$ and we again arrive to a contradiction, namely $F^4 = 0$. Thus, we focus on $\mu = b = 0$, $\varepsilon = 1$, $A \in \mathbb{R}$. We then have

$$\begin{aligned} dF \;=\;& \big((x_{1\bar{3}} - \bar{x}_{1\bar{3}}) + A\,x_{4\bar{4}}\big)\,\omega^{12\bar{1}} + (x_{1\bar{2}} - \bar{x}_{1\bar{2}} - \nu\,x_{4\bar{4}})\,\omega^{14\bar{1}} \\ & - \big((x_{1\bar{3}} - \bar{x}_{1\bar{3}}) - A\,x_{4\bar{4}}\big)\,\omega^{1\bar{1}\bar{2}} - (x_{1\bar{2}} - \bar{x}_{1\bar{2}} + \nu\,x_{4\bar{4}})\,\omega^{1\bar{1}\bar{4}} \\ F^4 \;=\;& -24\,x_{4\bar{4}}^2\,\big(x_{1\bar{1}}\,x_{4\bar{4}} - i\,(x_{1\bar{2}}\bar{x}_{1\bar{3}} - \bar{x}_{1\bar{2}}\,x_{1\bar{3}})\big)\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}. \end{aligned}$$

Notice that the condition $dF = 0$ implies $\mathfrak{Im}\,x_{1\bar{3}} = \mathfrak{Im}\,x_{1\bar{2}} = A\,x_{4\bar{4}} = \nu\,x_{4\bar{4}} = 0$. In particular, it turns out that $x_{1\bar{3}}, x_{1\bar{2}} \in \mathbb{R}$, so

$$F^4 \;=\; -24\,x_{1\bar{1}}\,x_{4\bar{4}}^3\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}.$$

If $(A, \nu) \neq (0,0)$, then $x_{4\bar{4}} = 0$ and the non-degeneration condition fails. Nonetheless, if we choose $(A,\nu) = (0,0)$, then it suffices to take $x_{1\bar{1}}, x_{4\bar{4}} \neq 0$ so that $(\mathfrak{g}, J)$ admits a pseudo-Kähler structure. This leads to part *i)* of the theorem.

Let us now concentrate on weakly non-nilpotent $J$'s. These $(\mathfrak{g}, J)$ are precisely parametrized by Lemma 3.3.1 *iv)*. With the aim of simplifying the discussion, we consider the reduced version of the structure equations given in Proposition 3.3.13. Hence, we have that $(\mathfrak{g}, J)$ is determined by

$$\begin{cases} d\omega^1 &= 0, \\ d\omega^2 &= \omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 &= i\,\varepsilon\,\omega^{1\bar{1}} \pm i\,(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \\ d\omega^4 &= A\,\omega^{12} + B\,\omega^{1\bar{1}} + \nu\,(\omega^{23} \pm 2\,\varepsilon\,\omega^{1\bar{3}} + \omega^{2\bar{3}}), \end{cases}$$

where $\varepsilon, \nu \in \{0,1\}$ and $A, B \in \mathbb{C}$. If we consider $dF = 0$, then it is straightforward to see that $x_{2\bar{3}} = x_{2\bar{4}} = x_{3\bar{4}} = 0$. Moreover, one has the equations

$$\nu\,x_{4\bar{4}} = 0, \qquad A\,x_{4\bar{4}} = 0, \qquad B\,x_{4\bar{4}} = 0.$$

Two cases can be distinguished.

- Let $(\nu, A, B) = (0,0,0)$. Then, we observe that $(\mathfrak{g}, J)$ is the product of a 6-dimensional NLA endowed with a SnN complex structure and a complex torus. We have

$$\begin{aligned} dF &= \mp i\,x_{1\bar{3}}\,\omega^{12\bar{1}} + (x_{1\bar{2}} - \bar{x}_{1\bar{2}} - \varepsilon\,x_{3\bar{3}})\,\omega^{13\bar{1}} + (i\,x_{2\bar{2}} \pm x_{3\bar{3}})\,\omega^{13\bar{2}} \\ &\quad \pm i\,\bar{x}_{1\bar{3}}\,\omega^{1\bar{1}\bar{2}} - (x_{1\bar{2}} - \bar{x}_{1\bar{2}} + \varepsilon\,x_{3\bar{3}})\,\omega^{1\bar{1}\bar{3}} - (i\,x_{2\bar{2}} \pm x_{3\bar{3}})\,\omega^{1\bar{2}\bar{3}} \\ &\quad +(i\,x_{2\bar{2}} \mp x_{3\bar{3}})\,\omega^{23\bar{1}} - (i\,x_{2\bar{2}} \mp x_{3\bar{3}})\,\omega^{2\bar{1}\bar{3}}. \end{aligned}$$

  The condition $dF = 0$ implies that $x_{1\bar{3}} = x_{2\bar{2}} = x_{3\bar{3}} = 0$, but this makes $F^4 = 0$. Hence, there are no pseudo-Kähler structures on $(\mathfrak{g}, J)$.

- If $(\nu, A, B) \neq (0,0,0)$, then $x_{4\bar{4}} = 0$ and one gets

$$\begin{aligned} dF &= -(A\,\bar{x}_{1\bar{4}} \pm i\,x_{1\bar{3}})\,\omega^{12\bar{1}} + \big(x_{1\bar{2}} - \bar{x}_{1\bar{2}} - \varepsilon\,(x_{3\bar{3}} \mp 2\,\nu\,x_{1\bar{4}})\big)\,\omega^{13\bar{1}} \\ &\quad +(i\,x_{2\bar{2}} + \nu\,x_{1\bar{4}} \pm x_{3\bar{3}})\,\omega^{13\bar{2}} - (\bar{A}\,x_{1\bar{4}} \mp i\,\bar{x}_{1\bar{3}})\,\omega^{1\bar{1}\bar{2}} \\ &\quad -\big(x_{1\bar{2}} - \bar{x}_{1\bar{2}} + \varepsilon\,(x_{3\bar{3}} \mp 2\,\nu\,\bar{x}_{1\bar{4}})\big)\,\omega^{1\bar{1}\bar{3}} - (i\,x_{2\bar{2}} + \nu\,x_{1\bar{4}} \pm x_{3\bar{3}})\,\omega^{1\bar{2}\bar{3}} \\ &\quad +(i\,x_{2\bar{2}} - \nu\,\bar{x}_{1\bar{4}} \mp x_{3\bar{3}})\,\omega^{23\bar{1}} - (i\,x_{2\bar{2}} - \nu\,\bar{x}_{1\bar{4}} \mp x_{3\bar{3}})\,\omega^{2\bar{1}\bar{3}}, \\ F^4 &= -24\,x_{2\bar{2}}\,x_{3\bar{3}}\,|x_{1\bar{4}}|^2\,\omega^{1234\bar{1}\bar{2}\bar{3}\bar{4}}. \end{aligned}$$

  In order to obtain $dF = 0$, we take $x_{1\bar{3}} = \pm i\,A\,\bar{x}_{1\bar{4}}$. Moreover, as $x_{2\bar{2}}, x_{3\bar{3}} \in \mathbb{R}$, we also fix $x_{2\bar{2}} = -\nu\,\mathfrak{Im}\,x_{1\bar{4}}$, and $x_{3\bar{3}} = \mp\,\nu\,\mathfrak{Re}\,x_{1\bar{4}}$. We note the following. If $\nu = 0$, then $x_{2\bar{2}} = x_{3\bar{3}} = 0$ and $F^4 = 0$, which is not possible. Therefore, we can assume $\nu = 1$ and choose $\mathfrak{Re}\,x_{1\bar{4}}, \mathfrak{Im}\,x_{1\bar{4}} \neq 0$ in order to have $x_{2\bar{2}}, x_{3\bar{3}} \neq 0$. Furthermore, to ensure the condition $dF = 0$ one also needs

$$\begin{aligned} 0 &= 2\,i\,\mathfrak{Im}\,x_{1\bar{2}} - \varepsilon\,(x_{3\bar{3}} \mp 2\,x_{1\bar{4}}) \\ &= 2\,i\,\big(\mathfrak{Im}\,x_{1\bar{2}} \pm \varepsilon\,\mathfrak{Im}\,x_{1\bar{4}}\big) \pm 3\,\varepsilon\,\mathfrak{Re}\,x_{1\bar{4}}. \end{aligned}$$

Since we have chosen $\mathfrak{Re}\,(x_{1\bar{4}}) \neq 0$, the only possible pseudo-Kähler structures arise when $\varepsilon = 0$. With this choice, it suffices to take $\mathfrak{Im}\, x_{1\bar{2}} = 0$. We obtain part *ii)* of the statement above.

This finishes the study of pseudo-Kähler geometry on those 8-dimensional NLAs endowed with non-nilpotent complex structures. □

A consequence of the previous study is that there is a restriction on the first Betti number of those nilmanifolds of dimension less than or equal to eight endowed with invariant pseudo-Kähler structures.

**Corollary 5.2.14.** *Let $M$ be a $2n$-dimensional nilmanifold with $2 \leq n \leq 4$ endowed with an invariant complex structure $J$. If $(M, J)$ admits a pseudo-Kähler structure, then $b_1(M) \geq 3$.*

*Proof.* By Nomizu's Theorem, we recall that the de Rham cohomology of any $2n$-dimensional nilmanifold $M = \Gamma\backslash G$ can be computed at the level of the Lie algebra $\mathfrak{g}$ of $G$. We now discuss according to the type of the invariant complex structure $J$. If $J$ is nilpotent, then it is straightforward to see that $b_1(M) \geq 3$ (see Theorem 3.1.8). For a non-nilpotent $J$, we observe the following. Remember that every invariant complex structure on a 4-dimensional nilmanifold is nilpotent, so there is nothing to study for $n = 2$. When $n = 3$, it is already known [CFU04] that pseudo-Kähler structures do not exist on such $(M, J)$. We now focus on $n = 4$. The only pseudo-Kähler $(M, J)$ are parametrized in Theorem 5.2.13 by structure equations *i)* and *ii)*. A direct calculation shows that

$$H^1_{\mathrm{dR}}(M) = \langle\, [\omega^1 + \omega^{\bar{1}}],\, [i\,(\omega^1 - \omega^{\bar{1}})],\, \begin{cases} [\omega^4 + \omega^{\bar{4}}]\,\rangle, & \text{for } i), \\ [\omega^3 + \omega^{\bar{3}}]\,\rangle, & \text{for } ii). \end{cases}$$

This concludes the proof. □

**Remark 5.2.15.** There exist 8-dimensional nilmanifolds $M$ endowed with SnN complex structures whose first Betti number is $b_1(M) = 2$. More concretely, those with underlying Lie algebra $\mathfrak{g}$ such that $(\dim \mathfrak{g}_k)_k = (1, 3, 5, 6, 8)$, given in Chapter 4.

# Appendices

# Appendix A
# Outline of some technical proofs

We provide here the outline of some proofs provided in Chapter 4, with the aim of clarifying the procedure.

**Sketch for Theorem 4.1.11**
It suffices to discard the case $\dim \mathfrak{g}_1 = n-2$. We proceed by contradiction. Using previous results, our starting point is:

$$\mathfrak{g}_1 = \langle X_1, \dots, X_{n-2} \rangle, \ \ \mathfrak{g}_2 = \langle X_1, \dots, X_{n-1}, JX_{n-1} \rangle, \ \ \mathfrak{g}_3 \supseteq \langle X_1, \dots, X_{n-1}, JX_1, JX_{n-1} \rangle.$$

We need to complete the study of $\mathfrak{g}_3$.

- $\boxed{1}$ $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) \geq 2 \ \Rightarrow \ JX_2 \in \mathfrak{g}_3$ Contradiction
- $\boxed{2}$ $\dim(\mathfrak{g}_3 \cap J\mathfrak{g}_1) = 1 \ \Rightarrow \ JX_k \notin \mathfrak{g}_3, \ \forall\, k = 2, \dots, n-2$
  - $\boxed{2.1}$ $X_n \in \mathfrak{g}_3 \ \Rightarrow \ \mathfrak{g}_3 = \langle X_1, \dots, X_n, JX_1, JX_{n-1} \rangle$ Contradiction
  - $\boxed{2.2}$ $X_n \notin \mathfrak{g}_3 \ \Rightarrow \ \mathfrak{g}_3 = \langle X_1, \dots, X_{n-1}, JX_1, JX_{n-1} \rangle$
    - $\boxed{2.2.1}$ $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) \geq 2 \ \Rightarrow \ JX_2 \in \mathfrak{g}_4$ Contradiction
    - $\boxed{2.2.2}$ $\dim(\mathfrak{g}_4 \cap J\mathfrak{g}_1) = 1 \ \Rightarrow \ \mathfrak{g}_4 = \langle X_1, \dots, X_n, JX_1, JX_{n-1} \rangle$ Contradiction

**Sketch for Proposition 4.2.1**
Supposing $\mathfrak{g}_2 \cap J\mathfrak{g}_2 = \{0\}$ and applying some results contained in Section 4.1 one gets:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_3 \rangle.$$

The study of $\mathfrak{g}_4$ opens two paths, contained in the following decision tree:

- $\boxed{1}$ $X_4 \in \mathfrak{g}_4$
  - $\boxed{1.1}$ $JX_4 \in \mathfrak{g}_4 \ \Rightarrow \ JX_1, JX_2 \in \mathfrak{g}_4$ Contradiction
  - $\boxed{1.2}$ $JX_4 \notin \mathfrak{g}_4$ Contradiction
- $\boxed{2}$ $X_4 \notin \mathfrak{g}_4$
  - $\boxed{2.1}$ $JX_1 \in \mathfrak{g}_4$

$\boxed{2.1.1}$ $JX_2 \in \mathfrak{g}_4 \quad \Rightarrow \quad X_4, JX_4 \in \mathfrak{g}_5$ Contradiction

$\boxed{2.1.2}$ $JX_2 \notin \mathfrak{g}_4 \quad \Rightarrow \quad X_4, JX_2, JX_4 \in \mathfrak{g}_5$ Contradiction

$\boxed{2.2}$ $JX_1 \notin \mathfrak{g}_4 \quad \Rightarrow \quad JX_2 \in \mathfrak{g}_4$ and $X_4, JX_1, JX_4 \in \mathfrak{g}_5$ Contradiction

**Sketch for Lemma 4.3.1**

Using previous results and arguing by contradiction, we have:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, JX_1, JX_2 \rangle,$$

$$\mathfrak{g}_4 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

Different possibilities arise depending on the elements which enter in $\mathfrak{g}_4$.

$\boxed{1}$ $X_4 \in \mathfrak{g}_4 \quad \Rightarrow \quad \mathfrak{g}_4 = \mathfrak{g}$ Contradiction

$\boxed{2}$ $X_4 \notin \mathfrak{g}_4$.

$\boxed{2.1}$ $JX_3 \in \mathfrak{g}_4 \quad \Rightarrow \quad X_4, JX_4 \in \mathfrak{g}_5$ Contradiction

$\boxed{2.2}$ $JX_3 \notin \mathfrak{g}_4 \quad \Rightarrow \quad \mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle$ Contradiction

**Sketch for Lemma 4.3.2**

By hypothesis, $JX_1 \in \mathfrak{g}_3$. Using previous results, one has

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_1, JX_2 \rangle.$$

We study whether more elements can belong to $\mathfrak{g}_3$, distinguishing two cases.

$\boxed{1}$ $X_4 \in \mathfrak{g}_3 \quad \Rightarrow \quad JX_3, JX_4 \in \mathfrak{g}_3$. Then:

$$\boxed{\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}} \qquad (1,3,8)$$

$\boxed{2}$ $X_4 \notin \mathfrak{g}_3$

$\boxed{2.1}$ $JX_3 \in \mathfrak{g}_3 \quad \Rightarrow \quad X_4, JX_4 \in \mathfrak{g}_4$. Then:

$$\boxed{\begin{array}{l} \mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \\ \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g} \end{array}} \qquad (1,3,6,8)$$

$\boxed{2.2}$ $JX_3 \notin \mathfrak{g}_3 \quad \Rightarrow \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle$

$\boxed{2.2.1}$ $X_4 \in \mathfrak{g}_4 \quad \Rightarrow \quad JX_3, JX_4 \in \mathfrak{g}_4$ Contradiction

$\boxed{2.2.2}$ $X_4 \notin \mathfrak{g}_4 \quad \Rightarrow \quad JX_3 \in \mathfrak{g}_4$ and $X_4, JX_4 \in \mathfrak{g}_5$ Contradiction

**Sketch for Lemma 4.3.3**

By hypothesis, $JX_1 \notin \mathfrak{g}_3$. As a consequence of previous results, we have

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \quad \mathfrak{g}_3 \supseteq \langle X_1, X_2, X_3, JX_2 \rangle.$$

Moreover, we see that $X_4 \notin \mathfrak{g}_3$. Two options to complete $\mathfrak{g}_3$: either $JX_3 \in \mathfrak{g}_3$ or $\mathfrak{g}_3$ only contains the elements above.

$\boxed{1}$ $JX_3 \notin \mathfrak{g}_3 \quad \Rightarrow \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2 \rangle$

$\boxed{1.1}$ $X_4 \in \mathfrak{g}_4$ Contradiction

$\boxed{1.2}$ $X_4 \notin \mathfrak{g}_4$ Contradiction

$\boxed{2}$ $JX_3 \in \mathfrak{g}_3 \quad \Rightarrow \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle$

$\boxed{2.1}$ $X_4 \in \mathfrak{g}_4 \quad \Rightarrow \quad JX_1, JX_4 \in \mathfrak{g}_4$. Then:

$$\boxed{\begin{array}{l} \mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \\ \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g} \end{array}} \qquad (1,3,5,8)$$

$\boxed{2.2}$ $X_4 \notin \mathfrak{g}_4 \quad \Rightarrow \quad JX_1 \in \mathfrak{g}_4$ and $X_4, JX_4 \in \mathfrak{g}_5$. Then:

$$\boxed{\begin{array}{l} \mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, JX_2 \rangle, \\ \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \\ \mathfrak{g}_4 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_5 = \mathfrak{g} \end{array}} \qquad (1,3,5,6,8)$$

**Sketch for Proposition 4.3.5**

From previous results, we get:

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle.$$

Therefore, the attention is focused on $\mathfrak{g}_3$. The different cases are summarized in the next scheme.

$\boxed{1}$ $X_4 \in \mathfrak{g}_3$

$\boxed{1.1}$ $JX_4 \in \mathfrak{g}_3$. Then:

$$\boxed{\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}} \qquad (1,4,8)$$

$\boxed{1.2}$ $JX_4 \notin \mathfrak{g}_3$.

$\boxed{1.2.1}$ $JX_1 \in \mathfrak{g}_3$ Contradiction

$\boxed{1.2.2}$ $JX_3 \in \mathfrak{g}_3$ Contradiction

$\boxed{2}$ $X_4 \notin \mathfrak{g}_3$

$\boxed{2.1}$ $JX_1 \in \mathfrak{g}_3$

$\boxed{2.1.1}$ $JX_3 \in \mathfrak{g}_3$. Then:

$$\boxed{\begin{array}{l} \mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2 \rangle, \\ \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g} \end{array}} \qquad (1,4,6,8)$$

$\boxed{2.1.2}$ $JX_3 \notin \mathfrak{g}_3 \quad \Rightarrow \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2 \rangle$

$\boxed{2.1.2.1}$ $X_4 \in \mathfrak{g}_4 \Rightarrow JX_3, JX_4 \in \mathfrak{g}_4$ Contradiction

$\boxed{2.1.2.2}$ $X_4 \notin \mathfrak{g}_4 \Rightarrow JX_3 \in \mathfrak{g}_4, \quad X_4, JX_4 \in \mathfrak{g}_5$ Contradiction

$\boxed{2.2}$ $JX_1 \notin \mathfrak{g}_3 \Rightarrow JX_3 \in \mathfrak{g}_3 \Rightarrow \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle$

$\boxed{2.2.1}$ $X_4 \in \mathfrak{g}_4 \Rightarrow JX_1, JX_4 \in \mathfrak{g}_4$ Contradiction

$\boxed{2.2.2}$ $X_4 \notin \mathfrak{g}_4 \Rightarrow JX_1 \in \mathfrak{g}_4, \quad X_4, JX_4 \in \mathfrak{g}_5$ Contradiction

**Sketch for Proposition 4.3.6**

From previous results,

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle.$$

Concerning $\mathfrak{g}_3$, the following two possibilities are studied:

$\boxed{1}$ $X_4 \in \mathfrak{g}_3$. Then:

$$\boxed{\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}} \qquad (1,5,8)$$

$\boxed{2}$ $X_4 \notin \mathfrak{g}_3$. Then:

$$\boxed{\begin{array}{l} \mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, JX_2, JX_3 \rangle, \\ \mathfrak{g}_3 = \langle X_1, X_2, X_3, JX_1, JX_2, JX_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g} \end{array}} \qquad (1,5,6,8)$$

# Appendix B
# Structural lemmas

**Lemma B.0.1.** *Let* $\mathfrak{g}_{(1,3,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,8)}$ *admits a basis* $\langle X_k, Y_k\rangle_{k=1}^4$ *in terms of which the possibly non-zero brackets are*

$$\begin{aligned}
&[X_2, X_k] = a^1_{2k}\, X_1, \;\; k = 3, 4, \quad [X_2, Y_k] = b^1_{2k}\, X_1, \;\; k = 3, 4,\\
&[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2 + \alpha^2_{34}\, Y_2, \quad [X_3, Y_1] = b^2_{31}\, X_2 + c^2_{31}\, Y_2,\\
&[X_3, Y_2] = b^1_{23}\, X_1, \quad [X_3, Y_k] = b^1_{3k}\, X_1 + b^2_{3k}\, X_2 + c^2_{3k}\, Y_2, \;\; k = 3, 4\\
&[X_4, Y_1] = b^2_{41}\, X_2 + c^2_{41}\, Y_2, \quad [X_4, Y_2] = b^1_{24}\, X_1,\\
&[X_4, Y_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, Y_2, \quad [X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + c^2_{44}\, Y_2,\\
&[Y_1, Y_3] = c^2_{31}\, X_2 - b^2_{31}\, Y_2, \quad [Y_1, Y_4] = c^2_{41}\, X_2 - b^2_{41}\, Y_2,\\
&[Y_2, Y_k] = a^1_{2k}\, X_1, \;\; k = 3, 4,\\
&[Y_3, Y_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, Y_2,
\end{aligned}$$

*where the coefficients fulfill equations*

$$\begin{cases}
a^1_{23}\, c^2_{31} + b^2_{31}\, b^1_{23} = 0,\\
a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41} = 0,\\
a^1_{23}\, b^2_{41} - a^1_{24}\, b^2_{31} - b^1_{23}\, c^2_{41} + b^1_{24}\, c^2_{31} = 0,\\
a^1_{23}\, c^2_{41} + a^1_{24}\, c^2_{31} + b^1_{23}\, b^2_{41} + b^1_{24}\, b^2_{31} = 0,\\
a^1_{23}\, (\alpha^2_{34} + b^2_{43}) - a^1_{24}\, b^2_{33} + b^1_{23}\, (a^2_{34} - c^2_{43}) + b^1_{24}\, c^2_{33} = 0,\\
a^1_{23}\, b^2_{44} + a^1_{24}\, (\alpha^2_{34} - b^2_{34}) - b^1_{23}\, c^2_{44} + b^1_{24}\, (a^2_{34} + c^2_{34}) = 0,\\
a^1_{23}\, (a^2_{34} - 2c^2_{34} + c^2_{43}) + a^1_{24}\, c^2_{33} - b^1_{23}\, (\alpha^2_{34} + 2b^2_{34} - b^2_{43}) + b^1_{24}\, b^2_{33} = 0,\\
a^1_{23}\, c^2_{44} - a^1_{24}\, (a^2_{34} - c^2_{34} + 2c^2_{43}) + b^1_{23}\, b^2_{44} + b^1_{24}\, (\alpha^2_{34} + b^2_{34} - 2b^2_{43}) = 0.
\end{cases}$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, Y_2\rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$JX_k = Y_k, \ \textit{for } k = 1, \ldots, 4.$$

**Lemma B.0.2.** *Let* $\mathfrak{g}_{(1,3,6,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,6,8)}$ *admits a basis* $\langle X_k,\, Y_k \rangle_{k=1}^4$ *in which the possibly non-zero brackets follow*

$$\begin{aligned}
&[X_2, X_4] = a^1_{24}\, X_1, \quad [X_2, Y_4] = b^1_{24}\, X_1, \\
&[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2 + \alpha^2_{34}\, Y_2, \quad [X_3, Y_3] = b^1_{33}\, X_1, \\
&[X_3, Y_4] = b^1_{34}\, X_1 + b^2_{34}\, X_2 + c^2_{34}\, Y_2, \\
&[X_4, Y_1] = b^2_{41}\, X_2 + c^2_{41}\, Y_2, \quad [X_4, Y_2] = b^1_{24}\, X_1, \\
&[X_4, Y_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, Y_2, \\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, Y_1 + c^2_{44}\, Y_2 + c^3_{44}\, Y_3, \\
&[Y_1, Y_4] = c^2_{41}\, X_2 - b^2_{41}\, Y_2, \\
&[Y_2, Y_4] = a^1_{24}\, X_1, \\
&[Y_3, Y_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, Y_2,
\end{aligned}$$

*where the coefficients fulfill equations*

$$\begin{cases}
a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41} = 0, \\
a^1_{24}\left(\alpha^2_{34} - b^2_{34}\right) + b^1_{24}\left(a^2_{34} + c^2_{34}\right) - b^1_{33}\, c^3_{44} = 0, \\
a^1_{24}\left(c^2_{34} - a^2_{34} - 2\, c^2_{43}\right) + b^1_{24}\left(b^2_{34} + \alpha^2_{34} - 2\, b^2_{43}\right) + b^1_{33}\, b^3_{44} = 0.
\end{cases}$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, Y_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, Y_1, Y_2, Y_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,6,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$JX_k = Y_k, \ \textit{for } k = 1, \ldots, 4.$$

**Lemma B.0.3.** *Let* $\mathfrak{g}_{(1,3,5,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,5,8)}$ *admits a basis* $\langle X_k,\, Y_k \rangle_{k=1}^4$ *in terms of which the possibly* *non-zero brackets are*

$$
\begin{aligned}
&[X_2, X_4] = a^1_{24}\, X_1, \quad [X_2, Y_4] = b^1_{24}\, X_1,\\
&[X_3, X_4] = a^1_{34}\, X_1 + a^2_{34}\, X_2 + \alpha^2_{34}\, Y_2, \quad [X_3, Y_1] = b^2_{31}\, X_2 + c^2_{31}\, Y_2,\\
&[X_3, Y_3] = b^1_{33}\, X_1, \quad [X_3, Y_4] = b^1_{34}\, X_1 + b^2_{34}\, X_2 + c^2_{34}\, Y_2,\\
&[X_4, Y_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, Y_2 + c^3_{41}\, Y_3, \quad [X_4, Y_2] = b^1_{24}\, X_1,\\
&[X_4, Y_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, Y_2,\\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, Y_2 + c^3_{44}\, Y_3,\\
&[Y_1, Y_3] = c^2_{31}\, X_2 - b^2_{31}\, Y_2, \quad [Y_1, Y_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, Y_2 - b^3_{41}\, Y_3,\\
&[Y_2, Y_4] = a^1_{24}\, X_1,\\
&[Y_3, Y_4] = a^1_{34}\, X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\, X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\, Y_2,
\end{aligned}
$$

*where the coefficients fulfill equations*

$$
\begin{cases}
a^1_{24}\, b^2_{31} - b^1_{24}\, c^2_{31} + b^1_{33}\, c^3_{41} = 0,\\
a^1_{24}\, c^2_{31} + b^1_{24}\, b^2_{31} + b^1_{33}\, b^3_{41} = 0,\\
a^1_{24}\, c^2_{41} + a^1_{34}\, c^3_{41} + b^1_{24}\, b^2_{41} + b^1_{34}\, b^3_{41} = 0,\\
a^1_{24}\left(\alpha^2_{34} - b^2_{34}\right) + b^1_{24}\left(a^2_{34} + c^2_{34}\right) - b^1_{33}\, c^3_{44} = 0,\\
a^1_{24}\left(c^2_{34} - a^2_{34} - 2\, c^2_{43}\right) + b^1_{24}\left(\alpha^2_{34} + b^2_{34} - 2\, b^2_{43}\right) + b^1_{33}\, b^3_{44} = 0,\\
b^3_{41}\left(b^2_{34} + b^2_{43}\right) + c^3_{41}\left(c^2_{43} + 2\, a^2_{34} - c^2_{34}\right) - b^2_{31}\, b^3_{44} + c^2_{31}\, c^3_{44} = 0,\\
b^3_{41}\left(c^2_{34} + c^2_{43}\right) + c^3_{41}\left(b^2_{34} + 2\, \alpha^2_{34} - b^2_{43}\right) - b^2_{31}\, c^3_{44} + c^2_{31}\, b^3_{44} = 0.
\end{cases}
$$

*and preserve the ascending central series*

$$
\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, Y_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, Y_2, Y_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.
$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,5,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$
JX_k = Y_k, \ \textit{for } k = 1, \ldots, 4.
$$

**Lemma B.0.4.** *Let* $\mathfrak{g}_{(1,3,5,6,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,3,5,6,8)}$ *admits a basis* $\langle X_k,\, Y_k \rangle_{k=1}^4$ *in which the possibly non-zero* *brackets have the form*

$$
\begin{aligned}
&[X_2,X_4] = a^1_{24}\,X_1, \quad [X_2,Y_4] = b^1_{24}\,X_1,\\
&[X_3,X_4] = a^1_{34}\,X_1 + a^2_{34}\,X_2 + \alpha^2_{34}\,Y_2, \quad [X_3,Y_4] = b^1_{34}\,X_1 + b^2_{34}\,X_2 + c^2_{34}\,Y_2,\\
&[X_4,Y_1] = b^2_{41}\,X_2 + b^3_{41}\,X_3 + c^2_{41}\,Y_2 + c^3_{41}\,Y_3, \quad [X_4,Y_2] = b^1_{24}\,X_1,\\
&[X_4,Y_3] = b^1_{34}\,X_1 + b^2_{43}\,X_2 + c^2_{43}\,Y_2,\\
&[X_4,Y_4] = b^1_{44}\,X_1 + b^2_{44}\,X_2 + b^3_{44}\,X_3 + c^1_{44}\,Y_1 + c^2_{44}\,Y_2 + c^3_{44}\,Y_3,\\
&[Y_1,Y_4] = c^2_{41}\,X_2 + c^3_{41}\,X_3 - b^2_{41}\,Y_2 - b^3_{41}\,Y_3,\\
&[Y_2,Y_4] = a^1_{24}\,X_1,\\
&[Y_3,Y_4] = a^1_{34}\,X_1 + (a^2_{34} - c^2_{34} + c^2_{43})\,X_2 + (\alpha^2_{34} + b^2_{34} - b^2_{43})\,Y_2,
\end{aligned}
$$

*where the coefficients fulfill equations*

$$
\begin{cases}
a^1_{24}\,(\alpha^2_{34} - b^2_{34}) + b^1_{24}\,(a^2_{34} + c^2_{34}) = 0,\\
a^1_{24}\,(c^2_{34} - a^2_{34} - 2\,c^2_{43}) + b^1_{24}\,(\alpha^2_{34} + b^2_{34} - 2\,b^2_{43}) = 0,\\
b^3_{41}\,(b^2_{34} + b^2_{43}) + c^3_{41}\,(c^2_{43} + 2\,a^2_{34} - c^2_{34}) = 0,\\
b^3_{41}\,(c^2_{34} + c^2_{43}) + c^3_{41}\,(b^2_{34} + 2\,\alpha^2_{34} - b^2_{43}) = 0,\\
a^1_{24}\,c^2_{41} + a^1_{34}\,c^3_{41} + b^1_{24}\,b^2_{41} + b^1_{34}\,b^3_{41} = 0.
\end{cases}
$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, Y_2\rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, Y_2, Y_3\rangle,$$

$$\mathfrak{g}_4 = \langle X_1, X_2, X_3, Y_1, Y_2, Y_3\rangle, \quad \mathfrak{g}_5 = \mathfrak{g}.$$

*Then, every $\mathfrak{g} \in \mathfrak{g}_{(1,3,5,6,8)}$ admits a strongly non-nilpotent complex structure $J$ given by*

$$J X_k = Y_k, \text{ for } k = 1, \ldots, 4.$$

**Lemma B.0.5.** *Let $\mathfrak{g}_{(1,4,8)}$ be the family of 8-dimensional real nilpotent Lie algebras such that every $\mathfrak{g} \in \mathfrak{g}_{(1,4,8)}$ admits a basis $\langle X_k,\, Y_k\rangle_{k=1}^4$ in which the only possibly non-*

*zero brackets have the form*

$$\begin{aligned}
&[X_2, X_4] = a^1_{24}\, X_1, \quad [X_2, Y_4] = b^1_{24}\, X_1,\\
&[X_3, X_4] = a^1_{34}\, X_1, \quad [X_3, Y_k] = b^1_{3k}\, X_1, \;\; k = 3, 4,\\
&[X_4, Y_1] = b^2_{41}\, X_2 + c^2_{41}\, Y_2, \quad [X_4, Y_2] = b^1_{24}\, X_1,\\
&[X_4, Y_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, Y_2,\\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, Y_2,\\
&[Y_1, Y_4] = c^2_{41}\, X_2 - b^2_{41}\, Y_2,\\
&[Y_2, Y_4] = a^1_{24}\, X_1,\\
&[Y_3, Y_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 - b^2_{43}\, Y_2,
\end{aligned}$$

*where the coefficients satisfy equations*

$$\begin{cases} a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41} = 0,\\ 2\,(a^1_{24}\, c^2_{43} + b^1_{24}\, b^2_{43}) - b^1_{33}\, b^3_{44} = 0, \end{cases}$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1\rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, Y_2\rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,4,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$JX_k = Y_k, \text{ for } k = 1, \ldots, 4.$$

**Lemma B.0.6.** *Let* $\mathfrak{g}_{(1,4,6,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,4,6,8)}$ *admits a basis* $\langle X_k, Y_k\rangle_{k=1}^4$ *in terms of which the possibly non-zero brackets are given by*

$$\begin{aligned}
&[X_2, X_4] = a^1_{24}\, X_1, \quad [X_2, Y_4] = b^1_{24}\, X_1,\\
&[X_3, X_4] = a^1_{34}\, X_1, \quad [X_3, Y_k] = b^1_{3k}\, X_1, \;\; k = 3, 4,\\
&[X_4, Y_1] = b^2_{41}\, X_2 + c^2_{41}\, Y_2, \quad [X_4, Y_2] = b^1_{24}\, X_1,\\
&[X_4, Y_3] = b^1_{34}\, X_1 + b^2_{43}\, X_2 + c^2_{43}\, Y_2,\\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, Y_1 + c^2_{44}\, Y_2 + c^3_{44}\, Y_3,\\
&[Y_1, Y_4] = c^2_{41}\, X_2 - b^2_{41}\, Y_2,\\
&[Y_2, Y_4] = a^1_{24}\, X_1,\\
&[Y_3, Y_4] = a^1_{34}\, X_1 + c^2_{43}\, X_2 - b^2_{43}\, Y_2,
\end{aligned}$$

*where the coefficients fulfill equations*

$$\begin{cases} b^1_{33}\, c^3_{44} = 0, \\ a^1_{24}\, c^2_{41} + b^1_{24}\, b^2_{41} = 0, \\ 2\,(a^1_{24}\, c^2_{43} + b^1_{24}\, b^2_{43}) - b^1_{33}\, b^3_{44} = 0. \end{cases}$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, Y_2 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, Y_1, Y_2, Y_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,4,6,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$J X_k = Y_k, \ \text{for } k = 1, \ldots, 4.$$

**Lemma B.0.7.** *Let* $\mathfrak{g}_{(1,5,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every element* $\mathfrak{g} \in \mathfrak{g}_{(1,5,8)}$ *admits a basis* $\langle X_k,\, Y_k \rangle_{k=1}^4$ *in terms of which the only possibly non-zero brackets have the form*

$$\begin{aligned}
&[X_2, X_k] = a^1_{2k}\, X_1, \ \ k = 3, 4, \quad [X_2, Y_k] = b^1_{2k}\, X_1, k = 2, 3, 4, \\
&[X_3, X_4] = a^1_{34}\, X_1, \quad [X_3, Y_2] = b^1_{23}\, X_1, \quad [X_3, Y_k] = b^1_{3k}\, X_1, \ \ k = 3, 4, \\
&[X_4, Y_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, Y_2 + c^3_{41}\, Y_3, \quad [X_4, Y_k] = b^1_{k4}\, X_1, \ \ k = 2, 3, \\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^2_{44}\, Y_2 + c^3_{44}\, Y_3, \\
&[Y_1, Y_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, Y_2 - b^3_{41}\, Y_3, \\
&[Y_2, Y_k] = a^1_{2k}\, X_1, \ \ k = 3, 4, \\
&[Y_3, Y_4] = a^1_{34}\, X_1,
\end{aligned}$$

*where the coefficients fulfill equations*

$$\begin{cases} a^1_{23}\, b^3_{41} + b^1_{22}\, c^2_{41} + b^1_{23}\, c^3_{41} = 0, \\ a^1_{23}\, b^3_{44} + b^1_{22}\, c^2_{44} + b^1_{23}\, c^3_{44} = 0, \\ a^1_{23}\, c^3_{41} - b^1_{22}\, b^2_{41} - b^1_{23}\, b^3_{41} = 0, \\ a^1_{23}\, b^2_{41} - b^1_{23}\, c^2_{41} - b^1_{33}\, c^3_{41} = 0, \\ a^1_{23}\, b^2_{44} - b^1_{23}\, c^2_{44} - b^1_{33}\, c^3_{44} = 0, \\ a^1_{23}\, c^2_{41} + b^1_{23}\, b^2_{41} + b^1_{33}\, b^3_{41} = 0, \\ a^1_{23}\, c^3_{44} - b^1_{22}\, b^2_{44} - b^1_{23}\, b^3_{44} = 0, \\ a^1_{23}\, c^2_{44} + b^1_{23}\, b^2_{44} + b^1_{33}\, b^3_{44} = 0, \\ a^1_{24}\, c^2_{41} + a^1_{34}\, c^3_{41} + b^1_{24}\, b^2_{41} + b^1_{34}\, b^3_{41} = 0, \end{cases}$$

*and preserve the ascending central series*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, Y_2, Y_3 \rangle, \quad \mathfrak{g}_3 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,5,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$J X_k = Y_k, \text{ for } k = 1, \ldots, 4.$$

**Lemma B.0.8.** *Let* $\mathfrak{g}_{(1,5,6,8)}$ *be the family of 8-dimensional real nilpotent Lie algebras such that every* $\mathfrak{g} \in \mathfrak{g}_{(1,5,6,8)}$ *admits a basis* $\langle X_k,\, Y_k \rangle_{k=1}^4$ *in which the possible non-zero brackets are given by*

$$\begin{aligned}
&[X_2, X_k] = a^1_{2k}\, X_1, \;\; k = 3, 4, \quad [X_2, Y_k] = b^1_{2k}\, X_1, k = 2, 3, 4, \\
&[X_3, X_4] = a^1_{34}\, X_1, \quad [X_3, Y_2] = b^1_{23}\, X_1, \quad [X_3, Y_k] = b^1_{3k}\, X_1, \;\; k = 3, 4, \\
&[X_4, Y_1] = b^2_{41}\, X_2 + b^3_{41}\, X_3 + c^2_{41}\, Y_2 + c^3_{41}\, Y_3, \quad [X_4, Y_k] = b^1_{k4}\, X_1, \;\; k = 2, 3, \\
&[X_4, Y_4] = b^1_{44}\, X_1 + b^2_{44}\, X_2 + b^3_{44}\, X_3 + c^1_{44}\, Y_1 + c^2_{44}\, Y_2 + c^3_{44}\, Y_3, \\
&[Y_1, Y_4] = c^2_{41}\, X_2 + c^3_{41}\, X_3 - b^2_{41}\, Y_2 - b^3_{41}\, Y_3, \\
&[Y_2, Y_k] = a^1_{2k}\, X_1, \;\; k = 3, 4, \\
&[Y_3, Y_4] = a^1_{34}\, X_1,
\end{aligned}$$

*where the coefficients fulfill equations*

$$\begin{cases}
a^1_{23}\, b^3_{41} + b^1_{22}\, c^2_{41} + b^1_{23}\, c^3_{41} = 0, \\
a^1_{23}\, b^3_{44} + b^1_{22}\, c^2_{44} + b^1_{23}\, c^3_{44} = 0, \\
a^1_{23}\, c^3_{41} - b^1_{22}\, b^2_{41} - b^1_{23}\, b^3_{41} = 0, \\
a^1_{23}\, b^2_{41} - b^1_{23}\, c^2_{41} - b^1_{33}\, c^3_{41} = 0, \\
a^1_{23}\, b^2_{44} - b^1_{23}\, c^2_{44} - b^1_{33}\, c^3_{44} = 0, \\
a^1_{23}\, c^2_{41} + b^1_{23}\, b^2_{41} + b^1_{33}\, b^3_{41} = 0, \\
a^1_{23}\, c^3_{44} - b^1_{22}\, b^2_{44} - b^1_{23}\, b^3_{44} = 0, \\
a^1_{23}\, c^2_{44} + b^1_{23}\, b^2_{44} + b^1_{33}\, b^3_{44} = 0, \\
a^1_{24}\, c^2_{41} + a^1_{34}\, c^3_{41} + b^1_{24}\, b^2_{41} + b^1_{34}\, b^3_{41} = 0,
\end{cases}$$

*and preserve the arrangement of the series*

$$\mathfrak{g}_1 = \langle X_1 \rangle, \quad \mathfrak{g}_2 = \langle X_1, X_2, X_3, Y_2, Y_3 \rangle, \quad \mathfrak{g}_3 = \langle X_1, X_2, X_3, Y_1, Y_2, Y_3 \rangle, \quad \mathfrak{g}_4 = \mathfrak{g}.$$

*Then, every* $\mathfrak{g} \in \mathfrak{g}_{(1,5,6,8)}$ *admits a strongly non-nilpotent complex structure* $J$ *defined by*

$$J X_k = Y_k, \text{ for } k = 1, \ldots, 4.$$

# Conclusions

In this section we summarize the most important results obtained in this thesis, devoted to the study of nilmanifolds endowed with invariant complex structure.

The aim of Chapter 1 is to introduce the main concepts that will be used along this work, such as complex manifolds and the theory of holomorphic deformations. Furthermore, we set the basis for the investigation of two large areas in the field of Complex Geometry: cohomological invariants and special geometric structures. With respect to the former, we describe those cohomology groups associated to complex manifolds (de Rham, Dolbeault, Aeppli, Bott-Chern), as well as the relations among them and the problem of cohomological decomposition. Concerning the latter, we recall the main ways of weakening the *Kähler* condition, from both the point of view of Hermitian metrics (strongly Gauduchon, SKT, astheno-Kähler,...) and that of Symplectic Geometry (pseudo-Kähler and holomorphic symplectic structures). We also introduce the notion of nilmanifolds endowed with invariant complex structures, together with their most relevant aspects.

In Chapter 2 we investigate several cohomological aspects of 6-dimensional nilmanifolds $M$ endowed with invariant complex structure $J$. Using Angella's Theorem [Ang13], the results by Rollenske [Rol09a], and the classification attained in [COUV16], we calculate the Bott-Chern numbers $h^{p,q}_{\mathrm{BC}}(X)$ of the complex nilmanifolds $X=(M,J)$ at the level of the underlying real Lie algebra $\mathfrak{g}$, whenever $\mathfrak{g}\not\cong\mathfrak{h}_7$. Moreover, we relate the Bott-Chern cohomology groups to the existence of certain metrics and the behaviour of some properties under holomorphic deformation of the complex structure. More concretely, given a compact complex manifold $X$ of complex dimension $n$, we consider the following invariants of cohomological type, which are related to the $\partial\bar\partial$-lemma condition:

$$\mathbf{f}_k(X)=\sum_{p+q=k}\left(h^{p,q}_{\mathrm{BC}}(X)+h^{n-p,n-q}_{\mathrm{BC}}(X)\right)-2b_k(X),$$

$$\mathbf{k}_r(X)=h^{1,1}_{\mathrm{BC}}(X)+2\dim E_r^{0,2}(X)-b_2(X),$$

where $0\leq k\leq n$ and $r\geq 1$. Let us recall that $b_k(X)$ is the $k$-th Betti number and $E_r^{0,2}(X)$ are the terms of bidegree $(0,2)$ in the Frölicher spectral sequence. By [AT13] we know that $\mathbf{f}_k(X)$ are non-negative integer numbers, all of them equal to zero for compact $\partial\bar\partial$-manifolds (indeed, their vanishing provides a characterization of such manifolds). The invariants $\mathbf{k}_r(X)$ are inspired by [Sch], and they are also non-negative integer numbers that vanish when the manifold $X$ satisfies the $\partial\bar{\partial}$-lemma condition. In addition, they satisfy $\mathbf{k}_1(X) \geq \mathbf{k}_2(X) \geq \mathbf{k}_r(X)$ for every $r \geq 3$. Bearing in mind these observations, we consider the following properties

$$\mathcal{F}_k = \{X \text{ satisfies } \mathbf{f}_k(X) = 0\}, \qquad \mathcal{K} = \{X \text{ satisfies } \mathbf{k}_1(X) = 0\}.$$

The most important result in this part of the thesis is the following one, where we show that the properties considered above are in general not closed under holomorphic deformation:

**Result 1.** *Let $(M, J_0)$ be a compact nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with an abelian complex structure. Then, there is a holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1/3\}$, such that for each $t \in \Delta \setminus \{0\}$ every Gauduchon metric is strongly Gauduchon, $(M, J_t)$ admits balanced metric, satisfies the properties $\mathcal{F}_2$ and $\mathcal{K}$, and has degenerate Frölicher spectral sequence. However, the central limit $(M, J_0)$ does not admit strongly Gauduchon metrics, the properties $\mathcal{F}_2$ and $\mathcal{K}$ fail, and the Frölicher spectral sequence does not degenerate at the first step.* (Theorem 2.1.16, p. 44.)

Next, the problem of cohomological decomposition is considered. One of the first purposes is finding non-Kählerian manifolds that satisfy an analogous result to the Hodge Decomposition Theorem, that is, whose complex structure is *complex-$\mathcal{C}^\infty$-pure-and-full.* Apart from recovering the case of the Iwasawa manifold (see [AT11]), we obtain a new example in real dimension 6 that allows us to complete a classification theorem. In fact, we prove the following:

**Result 2.** *Let $X = (M, J)$ be a nilmanifold of dimension 6, not a torus, endowed with an invariant complex structure. The study of the complex-$\mathcal{C}^\infty$-pure and complex-$\mathcal{C}^\infty$-full properties can be found in Tables 2.5, 2.6, and 2.7 (p. 48 and 49). In particular, one has that $X$ is complex-$\mathcal{C}^\infty$-pure-and-full at every stage if and only if $X$ is the complex nilmanifold $(\mathcal{N}_0, \mathcal{I}_0^\rho)$ determined by the structure equations*

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \rho\,\omega^{12} + (1-\rho)\omega^{1\bar{2}},$$

*where $\rho \in \{0, 1\}$ and $\mathcal{N}_0$ is the real nilmanifold underlying the Iwasawa manifold $(\mathcal{N}_0, \mathcal{I}_0^1)$. Moreover, the following duality result holds: $J$ is complex-$\mathcal{C}^\infty$-full at the $k$-th stage if and only if $J$ is complex-$\mathcal{C}^\infty$-pure at the $(6-k)$-th stage.* (Theorem 2.2.6, p. 50, Proposition 2.2.8, p. 51.)

Later, we concentrate on the corresponding real notions at the second stage and investigate their behaviour under holomorphic deformations. We recall that in [AT11] the properties of "being $\mathcal{C}^\infty$-pure" and "being $\mathcal{C}^\infty$-full" are studied for the small deformations of the Iwasawa manifold, showing that they are lost simultaneously.

**Result 3.** *Let $X = (M, J)$ be a nilmanifold of dimension 6, not a torus, endowed with an invariant complex structure. Then, $X$ is $\mathcal{C}^\infty$-pure-and-full if and only if $X$ is the complex nilmanifold $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$ defined by the structure equations*

$$d\omega^1 = 0, \qquad d\omega^2 = \epsilon\,\omega^{1\bar{1}}, \qquad d\omega^3 = \rho\,\omega^{12} + (1-\rho)\omega^{1\bar{2}},$$

*where* $\epsilon, \rho \in \{0, 1\}$*. Furthermore, there exist analytic families of compact complex manifolds* $\{X_t\}$*, constructed as appropriate holomorphic deformations of* $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$*, such that* $X_0$ *is* $\mathcal{C}^\infty$*-pure and* $\mathcal{C}^\infty$*-full, but for* $X_t$ *with* $t \neq 0$ *one of the properties is lost while the other one is preserved.* (Theorem 2.2.9, p. 52, Proposition 2.2.14, p. 58, Proposition 2.2.16, p. 59.)

With the aim of studying the deformation limits of the previous properties, we leave nilmanifolds aside for a moment, and we consider the larger class of solvmanifolds. Then, we are able to prove the following result.

**Result 4.** *For compact complex manifolds, the properties of "being* $\mathcal{C}^\infty$*-pure", "being* $\mathcal{C}^\infty$*-full" and "being* $\mathcal{C}^\infty$*-pure-and-full" are not closed under holomorphic deformation of the complex structure.* (Theorem 2.2.17, p. 61, Corollary 2.2.18, p. 62.)

We also examine the relation between "being $\mathcal{C}^\infty$-pure-and-full" and the existence of some special geometric structures on compact complex manifolds.

**Result 5.** *The property of "being* $\mathcal{C}^\infty$*-pure-and-full" is unrelated to the existence of SKT, locally conformal Kähler, balanced, or strongly Gauduchon Hermitian metrics. It is neither related to the degeneration of the Frölicher spectral sequence at the first step.* (Corollary 2.2.20, p. 64, Proposition 2.2.21, p. 65.)

Motivated by certain aspects of special geometric structures that cannot be clarified by only studying dimension 6, we are led to consider nilmanifolds of higher dimensions. In this way, we face the problem of how to parametrize invariant complex structures on any nilmanifold of arbitrary dimension $2n$. We observe that the lack of a classification of nilpotent Lie algebras for $n \geq 4$ makes difficult to directly generalize the method applied for $n \leq 3$. For this reason, in Chapter 3 we introduce a different approach based on the type of the complex structure. In order to do so, we split the space of invariant complex structures into two classes: that of quasi-nilpotent complex structures and that of strongly non-nilpotent complex structures.

Let $\mathfrak{g}$ a $2n$-dimensional nilpotent Lie algebra. A complex structure $J$ on $\mathfrak{g}$ is called *quasi-nilpotent* when

$$\{0\} \neq \mathfrak{a}_1(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = [JX, \mathfrak{g}] = 0\}\,.$$

In such case, we can take a subspace $\mathfrak{b} \subseteq \mathfrak{a}_1(J)$ that is $J$-invariant and 2-dimensional. It turns out that $\tilde{\mathfrak{g}}_\mathfrak{b} = \mathfrak{g}/\mathfrak{b}$ is a nilpotent Lie algebra of dimension $2(n-1)$ endowed with a complex structure $\tilde{J}_\mathfrak{b}$ given by:

$$\tilde{J}_\mathfrak{b}(\tilde{X}) = \widetilde{JX}, \quad \forall\, \tilde{X} \in \tilde{\mathfrak{g}}_\mathfrak{b},$$

being $\tilde{X}$ and $\widetilde{JX}$ the classes of $X$ and $JX$, respectively, in the quotient $\tilde{\mathfrak{g}}_\mathfrak{b}$. This observation allows to conclude the following:

**Result 6.** *Every pair* $(\mathfrak{g}, J)$*, with* $\dim \mathfrak{g} = 2n$ *and* $J$ *of quasi-nilpotent type, can be found extending with an appropriate space* $\mathfrak{b}$ *those nilpotent Lie algebras of dimension* $2(n-1)$

*that admit complex structures. In particular, we obtain a parametrization of all such (invariant) $J$'s on* 8*-dimensional nilmanifolds.* (Corollary 3.1.10, p. 75, Lemma 3.3.1, p. 89.)

A complex structure $J$ on $\mathfrak{g}$ is said to be *strongly non-nilpotent* when it satisfies $\mathfrak{a}_1(J) = \{0\}$. Let us note that this is the essentially new geometry that arises in each dimension. It is well known that for $n = 1, 2$, there are no complex structures of this type. For $n = 3$, the classification can be found in [UV14]. Here, we develop a constructive procedure that allows to recover these results from a new point of view. Its main advantage is the applicability to the search of strongly non-nilpotent complex structures in higher dimensions than 6.

Chapter 4 is completely devoted to strongly non-nilpotent complex structures $J$. After a collection of technical lemmas, we show that the existence of such $J$ implies a bound in the dimension of the center of the Lie algebra. More concretely:

**Result 7.** *Let $\mathfrak{g}$ be a $2n$-dimensional nilpotent Lie algebra, with $n \geq 4$, endowed with a strongly non-nilpotent complex structure. Then, $1 \leq \dim \mathfrak{g}_1 \leq n - 3$.* (Theorem 4.1.11, p. 110.)

We observe that the upper bound can be attained for $n = 5$. In our case, we focus on $n = 4$ and use the previous result as the starting point to find every pair $(\mathfrak{g}, J)$. By means of the constructive procedure mentioned above, we obtain a basis of $\mathfrak{g}$ adapted to the complex structure $J$ that provides the ascending central series $\{\mathfrak{g}_k\}_k$ of the Lie algebra. More precisely, we first see that in dimension 8 one has $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$, and then we find the rest of the terms. We make use of the results in [GR02, VR09], which imply that any 8-dimensional $\mathfrak{g}$ endowed with a complex structure must be at most 5-step nilpotent.

**Result 8.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension* 8*. If $\mathfrak{g}$ admits a strongly non-nilpotent complex structure, then the dimension of its ascending central series $\{\mathfrak{g}_k\}_k$ is*

$$\begin{aligned}(\dim \mathfrak{g}_k)_k \;=\; & (1,3,8),\ (1,3,6,8),\ (1,3,5,8),\ (1,3,5,6,8),\\ & (1,4,8),\ (1,4,6,8),\ (1,5,8),\ \textit{or}\ (1,5,6,8).\end{aligned}$$

*This allows us to find the complex structure equations of each pair $(\mathfrak{g}, J)$.* (Theorem 4.3.7, p. 153, Theorem 4.4.6, p. 169.)

In this way, we complete the parametrization of all invariant complex structures on 8-dimensional nilmanifolds that we initiated in Chapter 3.

In Chapter 5 we make us of the previous classification to study several special geometric structures and answer some related questions. In the first part, we focus on special Hermitian metrics, more precisely, astheno-Kähler [JY93] and generalized Gauduchon (in the sense of [FWW13]). Let $M$ be a $2n$-dimensional nilmanifold endowed with an invariant complex structure $J$. It is well known [EFV12] that the existence of an SKT metric on $(M, J)$ implies that the Lie algebra $\mathfrak{g}$ underlying $M$ is 2-step nilpotent. Since the astheno-Kähler condition coincides with the SKT one for $n = 3$, we wonder whether the previous restriction on the nilpotency step also holds for this other type of metrics in higher dimensions. In order to do so, we concentrate on $n = 4$ and prove the following:

**Result 9.** *Let $M$ be an $s$-step nilmanifold of dimension 8 endowed with an invariant complex structure $J$. If $(M, J)$ admits invariant astheno-Kähler metrics, then $J$ is nilpotent and $s \leq 3$. Moreover, there exist 8-dimensional astheno-Kähler nilmanifolds with nilpotency step $s = 3$.* (Theorem 5.1.7, p. 186, Corollary 5.1.9, p. 188, Corollary 5.1.10, p. 189.)

Both SKT and astheno-Kähler metrics are particular cases of generalized Gauduchon metrics. For $n = 3$, it turns out that invariant 1-st Gauduchon metrics on $(M, J)$ exactly coincide with SKT ones [FU13], that is:

$$\big\{\text{1-st Gauduchon}\big\} \quad = \quad \big\{\text{SKT/astheno-Kähler}\big\}.$$

In fact, a non-invariant metric is needed in order to distinguish these two types of structures in dimension $2n = 6$. For $n = 4$, we show that invariant 1-st and 2-nd Gauduchon metrics coincide, but not the other classes:

$$\begin{array}{ccc} \big\{\text{1-st Gauduchon}\big\} & = & \big\{\text{2-nd Gauduchon}\big\} \\ \cup\!\!\!\!\!\nmid & & \cup\!\!\!\!\!\nmid \\ \big\{\text{SKT}\big\} & \neq & \big\{\text{astheno-Kähler}\big\}. \end{array}$$

More concretely, we obtain the following result.

**Result 10.** *There exist 1-st Gauduchon metrics on 8-dimensional nilmanifolds with nilpotency steps 4 and 5 where the complex structure is non-nilpotent.* (Proposition 5.1.13, p. 191.)

The second part of Chapter 5 is devoted to the study of other special geometric structures. In particular, we focus our attention on holomorphic symplectic structures and pseudo-Kähler ones. Notice that these two structures give rise to symplectic forms which are anti-compatible and compatible, respectively, with the complex structure. Although it is possible to relate their existence on certain complex manifolds [Gua10], there is in general no relation. This fact was proved by Yamada on solvmanifolds [Yam05], and we recover the result making use of abelian complex structures on nilmanifolds.

For holomorphic symplectic structures, we analyze their behaviour under holomorphic deformation. Despite it is possible to find conditions under which the property of existence of holomorphic symplectic structures is stable [Gua95b], it is known that this is in general not true. For this reason, we mainly focus on the deformation limits of analytic families of holomorphic symplectic manifolds, showing that the existence property is not closed. First, we prove that if an 8-dimensional nilmanifold endowed with an invariant complex structure $J$ admits a holomorphic symplectic structure, then $J$ must be of nilpotent type. This reduces the space where searching for an appropriate deformation.

**Result 11.** *There exists an analytic family of compact complex manifolds* $\{X_t\}_{t\in\Delta}$ *of complex dimension* 4*, where* $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$*, such that* $X_t$ *admits holomorphic symplectic structures for each* $t \in \Delta \setminus \{0\}$*, but* $X_0$ *is not a holomorphic symplectic manifold.* (Theorem 5.2.10, p. 197.)

Concerning pseudo-Kähler structures, we observe that in dimension 6 they can only exist on those nilmanifolds whose complex structure is nilpotent [CFU04]. Surprisingly, this is no longer true in dimension 8.

**Result 12.** *There exist* 8*-dimensional nilmanifolds with non-nilpotent complex structures that admit pseudo-Kähler structures. Furthermore, these complex structures can be either of strongly non-nilpotent type or of quasi-nilpotent type. This provides counterexamples to a conjecture in* [CFU04]. (Theorem 5.2.13, p. 203.)

# Conclusiones

En esta sección resumimos los resultados más importantes obtenidos en esta tesis, dedicada al estudio de la geometría de nilvariedades dotadas de estructura compleja invariante.

El objetivo del Capítulo 1 es introducir los conceptos básicos que se utilizan en este trabajo, como son las variedades complejas y la teoría de deformaciones holomorfas. Además, sentamos las bases para la investigación de dos grandes áreas dentro de la Geometría Compleja: invariantes cohomológicos y estructuras geométricas especiales. En relación con la primera, se describen diversos grupos de cohomología que están asociados a las variedades complejas (de Rham, Dolbeault, Aeppli, Bott-Chern), así como las relaciones entre ellos y el problema de la descomposición cohomológica. Respecto a la segunda, recordamos las principales formas de debilitar la condición *Kähler*, tanto desde el punto de vista de métricas Hermíticas (fuertemente Gauduchon, SKT, astheno-Kähler,...) como desde la geometría simpléctica (estructuras pseudo-Kähler y simplécticas holomorfas). También presentamos la noción de nilvariedad dotada de estructura compleja invariante, junto con sus aspectos más relevantes.

En el Capítulo 2 se investigan ciertos aspectos cohomológicos de las nilvariedades 6-dimensionales $M$ dotadas de estructura compleja invariante $J$. Usando el teorema de Angella [Ang13], los resultados de Rollenske [Rol09a] y la clasificación obtenida en [COUV16], calculamos los números de Bott-Chern $h^{p,q}_{\mathrm{BC}}(X)$ de las nilvariedades complejas $X = (M, J)$ a nivel del álgebra de Lie real subyacente $\mathfrak{g}$, siempre y cuando $\mathfrak{g} \ncong \mathfrak{h}_7$. Además, relacionamos los grupos de cohomología de Bott-Chern con la existencia de ciertas métricas y el comportamiento de algunas propiedades por deformación holomorfa de la estructura compleja. En concreto, dada una variedad compleja compacta $X$ de dimensión compleja $n$, se definen los siguientes invariantes de tipo cohomológico relacionados con la condición del $\partial\bar{\partial}$-lema:

$$\mathbf{f}_k(X) = \sum_{p+q=k} \left(h^{p,q}_{\mathrm{BC}}(X) + h^{n-p,n-q}_{\mathrm{BC}}(X)\right) - 2b_k(X),$$

$$\mathbf{k}_r(X) = h^{1,1}_{\mathrm{BC}}(X) + 2\dim E^{0,2}_r(X) - b_2(X),$$

donde $0 \leq k \leq n$ y $r \geq 1$. Recordamos que $b_k(X)$ es el $k$-ésimo número de Betti y $E^{0,2}_r(X)$ son términos de bigrado $(0,2)$ de la sucesión espectral de Frölicher. Por [AT13] sabemos que los $\mathbf{f}_k(X)$ son números enteros no negativos, y que todos son iguales a cero para $\partial\bar{\partial}$-variedades compactas (de hecho, su anulación proporciona una caracterización de dichas variedades). Los invariantes $\mathbf{k}_r(X)$ están inspirados en [Sch] y son también números enteros no negativos que se anulan cuando la variedad $X$ cumple la condición del $\partial\bar{\partial}$-lema. Además, verifican $\mathbf{k}_1(X) \geq \mathbf{k}_2(X) \geq \mathbf{k}_r(X)$ para todo $r \geq 3$. En base a estas observaciones, consideramos las propiedades

$$\mathcal{F}_k = \big\{X \text{ cumple } \mathbf{f}_k(X) = 0\big\}, \qquad \mathcal{K} = \big\{X \text{ cumple } \mathbf{k}_1(X) = 0\big\}.$$

El resultado más relevante de esta parte es el siguiente, en el que se muestra que muchas de las propiedades consideradas no son cerradas por deformación holomorfa:

**Resultado 1.** *Sea $(M, J_0)$ una nilvariedad compacta con álgebra de Lie subyacente $\mathfrak{h}_4$ dotada de una estructura compleja abeliana. Entonces, existe una familia holomorfa de variedades complejas compactas $(M, J_t)_{t\in\Delta}$, con $\Delta = \{t \in \mathbb{C} \mid |t| \leq 1/3\}$, tal que para cada $t \in \Delta \setminus \{0\}$ toda métrica Gauduchon es fuertemente Gauduchon, $(M, J_t)$ admite métrica balanced, cumple las propiedades $\mathcal{F}_2$ y $\mathcal{K}$, y su sucesión espectral de Frölicher es degenerada. Sin embargo, el límite central $(M, J_0)$ no admite métricas fuertemente Gauduchon, no satisface la propiedad $\mathcal{F}_2$ ni la propiedad $\mathcal{K}$, y su sucesión espectral de Frölicher no degenera en primer paso.* (Teorema 2.1.16, p. 44.)

A continuación, se analiza el problema de la descomposición cohomológica. Un primer objetivo es encontrar nuevas variedades no Kählerianas que cumplan un análogo al Teorema de descomposición de Hodge, esto es, cuya estructura compleja sea *complex-$\mathcal{C}^\infty$-pure-and-full*. Además de recuperar el caso de la variedad de Iwasawa (véase [AT11]), obtenemos un nuevo ejemplo que nos permite completar un teorema de clasificación. De hecho, demostramos lo siguiente:

**Resultado 2.** *Sea $X = (M, J)$ una nilvariedad de dimensión $6$, distinta del toro, dotada de una estructura compleja invariante. El estudio de las propiedades complex-$\mathcal{C}^\infty$-pure y complex-$\mathcal{C}^\infty$-full para cada $X$ viene dado en las Tablas 2.5, 2.6 y 2.7 (p. 48 y 49). En particular, se tiene que $X$ es complex-$\mathcal{C}^\infty$-pure-and-full en todo paso si y solo si $X$ es la nilvariedad compleja $(\mathcal{N}_0, \mathcal{I}_0^\rho)$ determinada por las ecuaciones de estructura*

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \rho\,\omega^{12} + (1-\rho)\omega^{1\bar{2}},$$

*donde $\rho \in \{0, 1\}$ y $\mathcal{N}_0$ es la nilvariedad real subyacente a la variedad de Iwasawa $(\mathcal{N}_0, \mathcal{I}_0^1)$. Además, se obtiene la siguiente dualidad: $J$ es complex-$\mathcal{C}^\infty$-full en paso $k$ si y solo si es complex-$\mathcal{C}^\infty$-pure en paso $6-k$.* (Teorema 2.2.6, p. 50, Proposición 2.2.8, p. 51.)

Posteriormente, nos centramos en los conceptos análogos reales en paso 2 e investigamos su comportamiento por deformación holomorfa. Recordemos que en [AT11] se estudian las propiedades “ser $\mathcal{C}^\infty$-pure” y “ser $\mathcal{C}^\infty$-full” para las pequeñas deformaciones de la variedad de Iwasawa, mostrando que se pierden simultáneamente.

**Resultado 3.** *Sea $X = (M, J)$ una nilvariedad $6$-dimensional $M$, distinta del toro, dotada de una estructura compleja invariante $J$. Entonces, $X$ es $\mathcal{C}^\infty$-pure-and-full si y solo si $X$ es la nilvariedad compleja $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$ determinada por las ecuaciones de estructura*

$$d\omega^1 = 0, \qquad d\omega^2 = \epsilon\,\omega^{1\bar{1}}, \qquad d\omega^3 = \rho\,\omega^{12} + (1-\rho)\omega^{1\bar{2}},$$

*donde $\epsilon, \rho \in \{0,1\}$. Además, existen familias analíticas de variedades complejas compactas $\{X_t\}$, construidas como deformaciones holomorfas apropiadas de $(\mathcal{N}_\epsilon, \mathcal{I}_\epsilon^\rho)$, tales que $X_0$ es $\mathcal{C}^\infty$-pure y $\mathcal{C}^\infty$-full pero para $X_t$ con $t \neq 0$ una de las propiedades se pierde mientras la otra se preserva.* (Teorema 2.2.9, p. 52, Proposición 2.2.14, p. 58, Proposición 2.2.16, p. 59.)

Con el objetivo de estudiar los límites de deformación de las propiedades anteriores, es preciso salir del ámbito de las nilvariedades y considerar la clase más amplia de las solvariedades. Logramos así demostrar el siguiente resultado.

**Resultado 4.** *Para variedades complejas compactas, las propiedades "ser $\mathcal{C}^\infty$-pure", "ser $\mathcal{C}^\infty$-full" y "ser $\mathcal{C}^\infty$-pure-and-full" no son cerradas por deformación holomorfa de la estructura compleja.* (Teorema 2.2.17, p. 61, Corolario 2.2.18, p. 62.)

También se analiza la relación entre "ser $\mathcal{C}^\infty$-pure-and-full" y la existencia de algunas estructuras geométricas especiales sobre variedades complejas compactas.

**Resultado 5.** *La propiedad "ser $\mathcal{C}^\infty$-pure-and-full" no está relacionada con la existencia de métricas Hermíticas de tipo SKT, localmente conforme Kähler, balanced o fuertemente Gauduchon. Tampoco está relacionada con la degeneración de la sucesión espectral de Frölicher en primer paso.* (Corolario 2.2.20, p. 64, Proposición 2.2.21, p. 65.)

Motivados por ciertas cuestiones sobre estructuras geométricas especiales que la dimensión 6 no logra responder, pasamos a analizar nilvariedades de dimensiones más altas. Surge así el problema de cómo parametrizar las estructuras complejas invariantes sobre cualquier nilvariedad de dimensión arbitraria $2n$. Observamos que la ausencia de una clasificación de álgebras de Lie nilpotentes para $n \geq 4$ hace difícil generalizar de manera directa el método llevado a cabo para $n \leq 3$. Por este motivo, en el Capítulo 3 proponemos un nuevo enfoque basado en el tipo de estructura compleja. Para ello, dividimos el espacio de estructuras complejas invariantes en dos grandes clases: cuasi-nilpotentes y fuertemente no-nilpotentes.

Sea $\mathfrak{g}$ un álgebra de Lie nilpotente $2n$-dimensional. Una estructura compleja $J$ sobre $\mathfrak{g}$ se dice *cuasi-nilpotente* cuando

$$\{0\} \neq \mathfrak{a}_1(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = [JX, \mathfrak{g}] = 0\}.$$

En tal caso, podemos tomar un subespacio $\mathfrak{b} \subseteq \mathfrak{a}_1(J)$ que sea $J$-invariante y de dimensión dos. Resulta que $\tilde{\mathfrak{g}}_\mathfrak{b} = \mathfrak{g}/\mathfrak{b}$ es un álgebra de Lie nilpotente de dimensión $2(n-1)$ dotada de una estructura compleja $\tilde{J}_\mathfrak{b}$ dada por:

$$\tilde{J}_\mathfrak{b}(\tilde{X}) = \widetilde{JX}, \quad \forall\, \tilde{X} \in \tilde{\mathfrak{g}}_\mathfrak{b},$$

siendo $\tilde{X}$ y $\widetilde{JX}$ las clases de $X$ y $JX$, respectivamente, en el cociente $\tilde{\mathfrak{g}}_\mathfrak{b}$. Esta observación permite concluir lo siguiente:

**Resultado 6.** *Todos los pares $(\mathfrak{g}, J)$, con $\dim \mathfrak{g} = 2n$ y $J$ cuasi-nilpotente, se pueden obtener adjuntando adecuadamente un subespacio $\mathfrak{b}$ a aquellas álgebras de Lie nilpotentes de dimensión $2(n-1)$ que poseen estructuras complejas. En particular, obtenemos* *una parametrización de todas estas $J$ (invariantes) en nilvariedades de dimensión* 8. (Corolario 3.1.10, p. 75, Lema 3.3.1, p. 89.)

Una estructura compleja $J$ sobre $\mathfrak{g}$ se dice *fuertemente no-nilpotente* cuando verifica que $\mathfrak{a}_1(J) = \{0\}$. Notemos que esta es la geometría esencialmente nueva que surge al pasar de una dimensión a otra superior. Es bien conocido que para $n = 1, 2$, no existen estructuras complejas de este tipo. Para $n = 3$, su clasificación puede encontrarse en [UV14]. Nosotros desarrollamos un procedimiento constructivo que nos permite recuperar estos resultados desde un nuevo punto de vista. Su principal ventaja es que abre las puertas a la búsqueda de este tipo de estructuras en dimensiones más altas.

El Capítulo 4 está íntegramente dedicado a las estructuras complejas $J$ fuertemente no-nilpotentes. Tras una serie de lemas técnicos, vemos que la existencia de una tal $J$ conlleva una cota sobre la dimensión del centro del álgebra de Lie. Más concretamente:

**Resultado 7.** *Sea $\mathfrak{g}$ un álgebra de Lie nilpotente $2n$-dimensional, con $n \geq 4$, dotada de una estructura compleja fuertemente no-nilpotente. Entonces,* $1 \leq \dim \mathfrak{g}_1 \leq n - 3$. (Teorema 4.1.11, p. 110.)

Observamos que para $n = 5$ se puede alcanzar la cota superior. Nosotros nos centramos en el caso $n = 4$ y usamos el resultado anterior como punto de partida para calcular todos los pares $(\mathfrak{g}, J)$. A través del proceso constructivo mencionado anteriormente, conseguimos una base de $\mathfrak{g}$ adaptada a la estructura compleja $J$ que nos proporciona la serie central ascendente $\{\mathfrak{g}_k\}_k$ del álgebra. En concreto, primero mostramos que en dimensión 8 se tiene $\mathfrak{g}_2 \cap J\mathfrak{g}_2 \neq \{0\}$ y luego hallamos el resto de términos. Para ello usamos los resultados de [GR02, VR09], que implican que cualquier $\mathfrak{g}$ de dimensión 8 dotada de estructura compleja debe ser a lo sumo 5-step.

**Resultado 8.** *Sea $\mathfrak{g}$ un álgebra de Lie nilpotente 8-dimensional. Si $\mathfrak{g}$ admite una estructura compleja fuertemente no-nilpotente, entonces la dimensión de su serie central ascendente $\{\mathfrak{g}_k\}_k$ es*

$$
\begin{aligned}
(\dim \mathfrak{g}_k)_k \;=\; & (1,3,8),\ (1,3,6,8),\ (1,3,5,8),\ (1,3,5,6,8),\\
& (1,4,8),\ (1,4,6,8),\ (1,5,8),\ \textit{o bien}\ (1,5,6,8).
\end{aligned}
$$

*Esto nos permite hallar las ecuaciones de estructura complejas de todos los pares $(\mathfrak{g}, J)$.* (Teorema 4.3.7, p. 153, Teorema 4.4.6, p. 169.)

De esta manera, completamos la parametrización de todas las estructuras complejas invariantes sobre nilvariedades de dimensión 8 ya iniciada en el Capítulo 3.

En el Capítulo 5 hacemos uso de la clasificación anterior para estudiar varios tipos de estructuras geométricas especiales y responder algunas cuestiones relacionadas. En una primera parte nos centramos en métricas Hermíticas especiales, concretamente, astheno-Kähler [JY93] y Gauduchon generalizadas (en el sentido de [FWW13]). Sea $M$ una nilvariedad de dimensión $2n$ dotada de una estructura compleja invariante $J$. Es bien conocido [EFV12] que la existencia de una métrica SKT sobre $(M, J)$ implica que el álgebra de Lie $\mathfrak{g}$ subyacente a $M$ es nilpotente en paso 2. Puesto que para $n = 3$ la condición astheno-Kähler coincide con la SKT, nos planteamos si la restricción anterior en el paso de nilpotencia es extensible a este otro tipo de métricas en dimensiones superiores. Para ello, nos centramos en $n = 4$ y demostramos lo siguiente:

**Resultado 9.** *Sea $M$ una nilvariedad s-step de dimensión 8 dotada de estructura compleja invariante $J$. Si $(M, J)$ admite métricas astheno-Kähler invariantes, entonces $J$ es nilpotente y $s \leq 3$. Además, existen nilvariedades astheno-Kähler 8-dimensionales que son nilpotentes en paso $s = 3$.* (Teorema 5.1.7, p. 186, Corolario 5.1.9, p. 188, Corolario 5.1.10, p. 189.)

Las métricas SKT y astheno-Kähler son casos particulares de métricas Gauduchon generalizadas. Para $n = 3$ resulta que las métricas 1-Gauduchon invariantes sobre $(M, J)$ coinciden exactamente con las SKT [FU13]. Esto es:

$$\big\{\text{1-Gauduchon}\big\} \quad = \quad \big\{\text{SKT/astheno-Kähler}\big\}$$

De hecho, una métrica de tipo no invariante es necesaria para poder distinguir estas estructuras. Para $n = 4$, no solo demostramos que las métricas invariantes 1-Gauduchon y 2-Gauduchon coinciden, sino que tenemos:

$$\begin{array}{ccc} \big\{\text{1-Gauduchon}\big\} & = & \big\{\text{2-Gauduchon}\big\} \\ \cup\!\!\!\!\!\not\;\; & & \cup\!\!\!\!\!\not\;\; \\ \big\{\text{SKT}\big\} & \neq & \big\{\text{astheno-Kähler}\big\} \end{array}$$

De hecho, obtenemos el siguiente resultado.

**Resultado 10.** *Existen métricas 1-Gauduchon sobre nilvariedades de dimensión 8 con pasos de nilpotencia 4 y 5 donde la estructura compleja es no-nilpotente.* (Proposición 5.1.13, p. 191.)

La segunda parte del Capítulo 5 está dedicada al estudio de otras estructuras geométricas especiales. Más en concreto, centramos nuestra atención en las estructuras simplécticas holomorfas y pseudo-Kähler. Conviene notar que ambas dan lugar a formas simplécticas anti-compatibles y compatibles, respectivamente, con la estructura compleja asociada. Aunque es posible vincular la existencia de unas y otras sobre determinadas variedades complejas [Gua10], se sabe que en general no hay relación. Este hecho fue probado por Yamada en solvariedades [Yam05], y nosotros recuperamos el resultado haciendo uso de estructuras complejas abelianas sobre nilvariedades.

Para las estructuras simplécticas holomorfas analizamos su comportamiento por deformación holomorfa. A pesar de que se pueden encontrar condiciones bajo las cuales la propiedad de existencia de estructuras simplécticas holomorfas es estable [Gua95b], se sabe que generalmente no lo es. Nos centramos así en los límites de deformación de familias analíticas de variedades simplécticas holomorfas, demostrando que se trata de una propiedad no cerrada. Para ello, en primer lugar se prueba que si una nilvariedad 8-dimensional dotada de estructura compleja invariante $J$ admite tales estructuras, entonces $J$ debe ser de tipo nilpotente. Esto reduce el espacio de búsqueda de una deformación adecuada.

**Resultado 11.** *Existe una familia analítica de variedades complejas compactas* $\{X_t\}_{t\in\Delta}$ *de dimensión compleja* 4*, donde* $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$*, tal que* $X_t$ *admite estructura simpléctica holomorfa para todo* $t \in \Delta \setminus \{0\}$*, pero* $X_0$ *no es una variedad simpléctica holomorfa.* (Teorema 5.2.10, p. 197.)

Respecto a las estructuras pseudo-Kähler, observamos que en dimensión 6 solo existen sobre aquellas nilvariedades cuya estructura compleja es nilpotente [CFU04]. Sorprendentemente, esto no ocurre en dimensión mayor.

**Resultado 12.** *Existen nilvariedades* 8*-dimensionales con estructuras complejas no-nilpotentes que admiten estructuras pseudo-Kähler. Además, estas estructuras complejas pueden ser tanto de tipo fuertemente no-nilpotente como de tipo cuasi-nilpotente.* (Teorema 5.2.13, p. 203.)

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