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Long-term evolution of space debris under the J_2 effect, the solar radiation pressure and the solar and lunar perturbations

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Abstract The aim of this paper is the development of a model to propagate space debris in the geostationary ring considering the J_2 effect due to the Earth oblateness, the Sun and Moon perturbations, and the solar radiation pressure. We justify the importance of considering the J_2 effect when propagating space debris independently of the ratio A/m for short and long-term propagation. We study the role of the Sun and the Moon in the period and amplitude of the inclination for different values of A/m. Thanks to the Hamiltonian formulation of the problem and the use of Poincaré's variables it is possible to express the evolution of the space debris through a simplified dynamical system. We test and validate our obtained analytical solutions with the numerical ones, computed with a powerful integrator named NIMASTEP. We analyse the improvements obtained when we include the J_2 effect and the third body perturbations by a rigorous comparison with a previous model, which only considers the solar radiation pressure. Finally, we study the effect of the area-to-mass ratio on short and long-term propagation.

Keywords Space debris · solar radiation pressure · J_2 effect · third body perturbation · Poincaré's variables · Hamiltonian formulation · Long-term evolution

1 Introduction

In the last decades space debris population have dramatically increased becoming into a real problem for existing, and even for future space missions since the risk of collision between a satellite and a piece of space debris is real (Casanova et al., 2014; Bombardelli, 2013). Consequently, the study of the evolution of space debris is crucial and becomes a hot

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topic in Celestial Mechanics. The two main objectives are to avoid collisions, which create space debris systematically, and to study their long-term evolution to know the behaviour of the population in the near future. In particular, we focus our work on space debris in the geostationary ring since this area is crowded of operating satellites and space debris. Then, our model considers space debris whose semi-major axis is about $42164\,km$, and which are initially in quasi-circular and planar orbits.

Different analytical and semi-analytical investigations of space debris, with high area-to-mass ratio, orbiting in the geosynchronous region, have been developed by one of the authors of this paper (see Valk et al. (2008); Valk and Lemaître (2009); Valk et al. (2009); Lemaître et al (2009)); these studies have been completed by two other papers (see Hubaux and Lemaître (2013); Hubaux et al. (2012)) later on, focusing on the Earth shadowing effects. A semi-analytical analysis of the resonant problem (the geosynchronism between the rotation of the Earth and the orbit of the satellite) is presented in Valk et al. (2008), with the calculation of the equilibria and their stability.

Consequently, the first goal of this paper is to improve the previous models, mainly based on the solar radiation pressure effect. Indeed, for large values of the area-to-mass coefficient, it has been shown (see Fig. 1 in Valk et al. (2008)) that the solar radiation pressure is the most important perturbation of the two-body problem; the J_2 effect due to the Earth oblateness, and the effect of the Sun and the Moon as third bodies, are the next ones, in order of magnitude, although the tesseral J_{22} effect appears as a much smaller perturbation in the geosynchronous region. So in this work we neglect the tesseral terms, and concentrate our attention to the behaviour of the inclination and of the eccentricity, due to the solar radiation pressure, the J_2, the lunar and solar perturbations. The results obtained in this paper could be easily combined, if necessary, with resonant previous studies such as Valk et al. (2009) or Belyanin and Gurfil (2010).

Our second and more important goal in this paper is to provide an efficient and reliable analytical model to propagate thousands of pieces of space debris at the same time. We are not interested in the precise position of any real body, but in the statistical evolution, on long periods, of a cloud of virtual objects, characterized by high are-to-mass ratios, especially in the values of the eccentricity and inclinations reached by these objects (Casanova et al., 2015).

In this work we use the Poincaré's variables. These variables are canonical and they are non singular, when the eccentricity and inclinations are close to zero. Consequently, they are useful for studying space debris orbiting the Earth in the geostationary ring. Thanks to these variables, it is possible to obtain a complete Hamiltonian formulation of the problem, which includes the J_2 effect, the third body perturbation and the solar radiation pressure. Indeed, by means of averaging, the original Hamiltonian is replaced by the averaged one becoming into a four degrees of freedom problem since we average over the mean anomaly and consequently the semi-major axis becomes constant. This averaged Hamiltonian yields good agreement with the actual dynamical system.

The averaged dynamical system is integrated to provide an analytical solution. We compare the solution with the numerical one, that we compute with the powerful integrator NIMASTEP (Delsate and Compere, 2012) considering exactly the same perturbative effects as in our model. The goal of this comparison is to test the precession of our analytical approach, and the improve of this method with respect to the previous ones. We also study the influence of the area-to-mass ratio A/m in the propagation of space debris. This ratio plays an important role when the solar radiation pressure is considered in the propagation (See Klinkrad (2006).)

The paper is organized as follows: first, we show a numerical simulation to understand the motion of space debris. Then, we recall the Poincaré's variables and we summarize the Hamiltonian formulation of the problem. Then, we obtain the averaged Hamiltonian of the problem that maintains good agreement with the complete Hamiltonian and we solve the dynamical system obtained; finally, we present our main results.

2 Numerical simulations

We are interested in the motion of space debris, especially those orbiting in the geostationary region and for which the solar radiation pressure can not be neglected, i.e. with high values A/m. We start the analysis by numerical integrations for two specific cases : $A/m = 1\,m^2/kg$ and $A/m = 20\,m^2/kg$. We include the J_2 perturbation, the third body perturbations from the Sun and the Moon, and the solar radiation pressure.

All the numerical integrations have been performed by NIMASTEP (Delsate and Compere, 2012) with the same initials conditions : $a = 42,164.137\,km$, $e = 0.01$, $i = 0.01\,rad$, $\omega = \Omega = M = 0\,rad$ considering a fourth-order Runge-Kutta method during 200 years with a fix time step of $100\,sec$.

After many simulations, as we observe in Figures 1(a) and 1(b), we notice the following behaviours:

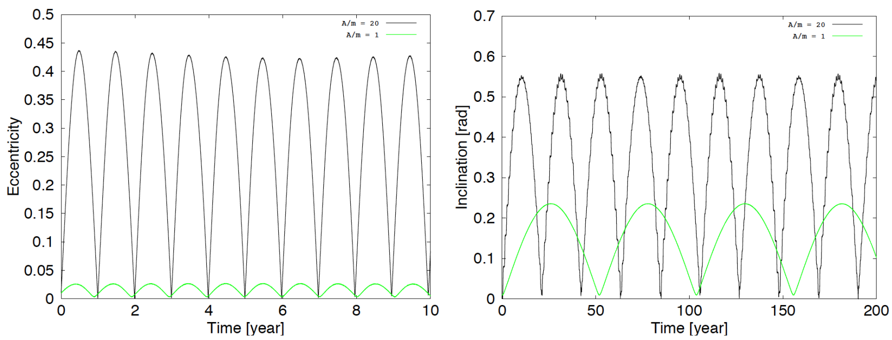


Fig. 1 Evolution of the eccentricity over 10 years (a) and evolution of the inclination over 200 years (b) of two pieces of space debris with $A/m = 1$ and $A/m = 20$, it is considered the J_2 effect, the Sun and Moon perturbations, and the solar radiation pressure

- The eccentricity has a quasi-periodic motion, dominated by a period of 1 year; a longer period appears in the simulation over 200 years, which modulates the amplitude. This amplitude is dependent on the J_2 and mainly depends on the value of A/m. Over a few years, the motion is dominated by these two effects. However over longer periods of time, as 200 years a supplementary long period appears, due to the solar and lunar perturbations, as it is shown in Figure 2
- The inclination is dominated by a long periodic motion, the period and amplitude of which are dependent on A/m, J_2 and the presence of the Sun and of the Moon. Of course, a shorter period of 1 year is also visible but its amplitude is negligible over the long periodic motion.

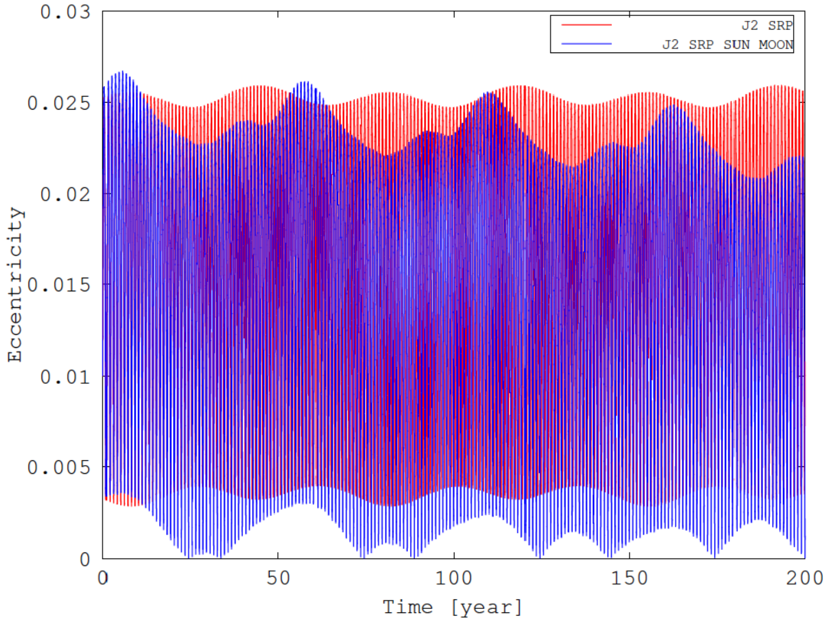


Fig. 2 Evolution of the eccentricity of a piece of space debris over 200 years with $A/m = 1$, considering the J_2 effect and the solar radiation pressure (red curve) and with the solar and lunar perturbations (blue curve)

The Moon plays a role in the motion of the inclination, as shown in Figure 3, where a comparison is performed for 200 years with or without the Moon; it is obvious than the periods and the amplitudes are modified by the presence of the Moon in the model. For large values of A/m, the error is only about a few percents, but for $A/m = 1$ it is much more important, modifying the amplitude and the period.

We propose to build a simple model, averaged over different periods, explaining the behaviour of the eccentricity over a period of 10 years and of the inclination over 200 years. Further comments about longer periods in the eccentricity and modified periods in the inclination will complete this analysis.

Our analytical model neglects the effect of the tesseral terms. This decision has been motivated by a previous analysis (see Valk et al. (2008)) in which the authors show the orders of magnitude of the radial components of the acceleration due to different perturbations (see their Figure 1). At the altitude of the geostationary orbit, it is obvious that the tesseral terms are two orders of magnitude smaller than the other perturbations. They also show the classical presence of two stable and two unstable equilibria, due to the gravitational 1:1 resonance, they calculate the libration period and show its presence on the evolution of the semi-major axis.

Nevertheless, to check this hypothesis, we have performed several simulations. In Fig. 4(a) and Fig. 4(b) we show the evolution of the eccentricity and of the inclination during 10 years and 200 years of propagation respectively. Each figure shows the evolution of two pieces of space debris ($A/m = 20$ and $A/m = 1$) considering two different models; one of them considers the J_2 effect, the solar radiation pressure, the Sun and the Moon perturbations, and the other one also includes the tesseral terms (S_{22} and C_{22}). We confirm that the tesseral

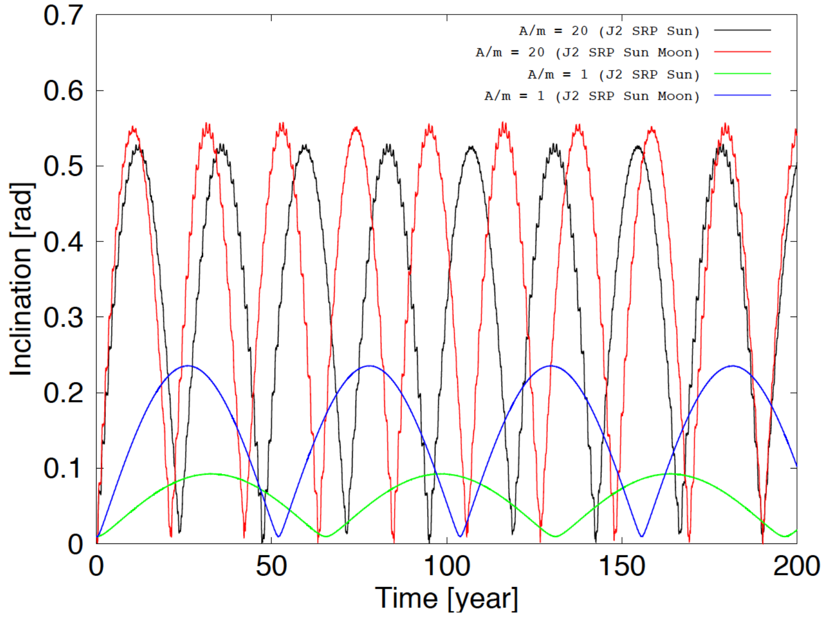
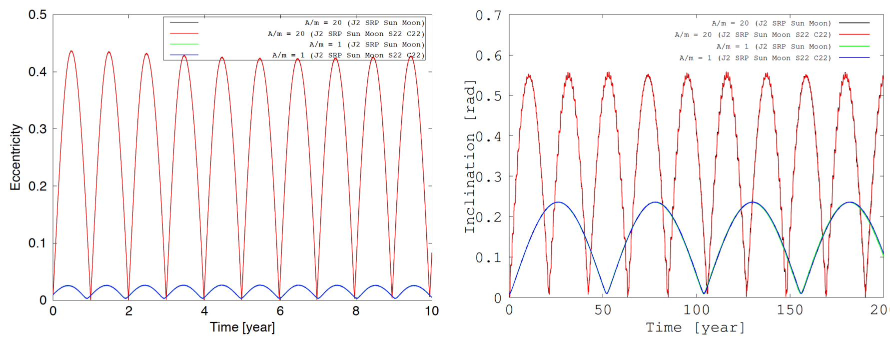


Fig. 3 Evolution of the inclination of two pieces of space debris with $A/m = 1$ and $A/m = 20$, with the J_2 effect and the Sun perturbation, with or without the lunar perturbation

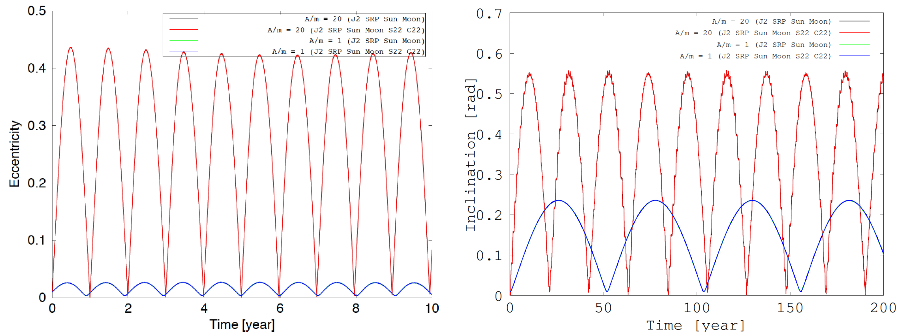
terms can be omitted. They do not play any role on the eccentricity and inclination, when the coefficients A/m are huge.



(a) (b)

Fig. 4 Evolution of the eccentricity over 10 years (a) and evolution of the inclination over 200 years (b) of two pieces of space debris with $A/m = 1$ and $A/m = 20$, with or without the S_{22} and C_{22} effect

We also study numerically the particular case of a piece of space debris located in the unstable GEO region, which presents a more complex evolution. For this purpose, Fig. 5(a) and Fig. 5(b) show the evolution of a piece of space debris without or with the C_{22} and S_{22} contribution. We observe that the tesseral terms can also be neglected in this particular case. These simulations confirm that in our goal of having a reliable and fast analytical model to propagate thousands of pieces of space debris with large values of A/m, the tesseral terms can be omitted in the eccentricity and inclination motions.



(a) (b)

Fig. 5 Evolution of the eccentricity over 10 years (a) and evolution of the inclination over 200 years (b) of a piece of space debris located in the unstable GEO region, with $A/m = 1$ and $A/m = 20$, with or without the S_{22} and C_{22} effect

3 Preliminaries

In this section we introduce the tools to be used henceforth in the paper: a specific type of non-singular elements, named Poincaé's variables, and the Hamiltonian formulation to describe the orbital motion of an object around a rigid body in the solar system.

3.1 Poincaré's variables

The classical orbital elements $(a, e, i, \Omega, \omega, M)$ are used to describe the motion of an object around a rigid body. However, when the eccentricity approaches zero, there is a large variation in ω. Actually, when $e = 0$ the argument of perigee is undefined. A similar problem happens when the inclination is zero. To overcome this problem, we use canonical and non-singular variables named Poincaré's variables. They are suitable for all eccentricities and inclinations even for null eccentricities and inclinations. These variables are especially useful for treating orbit problems with Hamiltonian dynamics (Vallado, 2001). The Poincaré's variables are given by:

$$\begin{aligned} x_1 &= \sqrt{2P}\sin p, \; y_1 = \sqrt{2P}\cos p, \\ x_2 &= \sqrt{2Q}\sin q, \; y_2 = \sqrt{2Q}\cos q, \\ &\lambda, \qquad\quad L, \end{aligned} \tag{1}$$

where λ is the mean longitude, which is the sum of the mean anomaly (M), the argument of perigee (ω), and the longitude of the ascending node (Ω). L is one of the classical Delaunay's elements given by:

$$L = \sqrt{\mu a}, \qquad G = \sqrt{\mu a(1-e^2)}, \qquad H = \sqrt{\mu a(1-e^2)}\cos i. \tag{2}$$

Finally, the modified Delaunay's elements are defined by:

$$\begin{aligned} P &= L - G, \quad p = -\omega - \Omega, \\ Q &= G - H, \quad q \quad = -\Omega. \end{aligned} \tag{3}$$

3.2 Hamiltonian formulation

Given the generalized coordinates (p, q, λ) and the generalized momenta (P, Q, L) of a piece of space debris orbiting around the Earth, it is possible to describe its motion following the Hamiltonian formulation:

$$\begin{aligned} \dot{p} &= \frac{\partial \mathscr{H}}{\partial P}, \qquad \dot{P} = -\frac{\partial \mathscr{H}}{\partial p}, \\ \dot{q} &= \frac{\partial \mathscr{H}}{\partial Q}, \qquad \dot{Q} = -\frac{\partial \mathscr{H}}{\partial q}, \\ \dot{\lambda} &= \frac{\partial \mathscr{H}}{\partial L}, \qquad \dot{L} = -\frac{\partial \mathscr{H}}{\partial \lambda}. \end{aligned} \tag{4}$$

The Hamiltonian function in this paper takes into account the attraction of the Earth as a central body, the J_2 effect due to the non-spherical shape of the Earth, the solar radiation pressure that affects the space debris, which is proportional to its area-to-mass ratio, and the solar and lunar perturbations. Then, the Hamiltonian function for this problem is:

$$\mathscr{H}(\mathbf{r}, \mathbf{v}, \mathbf{r}_\odot, \mathbf{r}_{☾}) = H_{kepler}(\mathbf{r}, \mathbf{v}) + H_{J_2}(\mathbf{r}) + H_{SRP}(\mathbf{r}, \mathbf{r}_\odot) + H_{3bS}(\mathbf{r}, \mathbf{r}_\odot) + H_{3bM}(\mathbf{r}, \mathbf{r}_{☾}), \tag{5}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the Cartesian geocentric coordinates and velocities of any piece of space debris, $\mathbf{r}_\odot$ the Cartesian geocentric coordinates of the Sun and $\mathbf{r}_{☾}$, these of the Moon. All of them are given with respect to an inertial equatorial geocentric frame. Hereinafter we denote by $r = ||\mathbf{r}||, r_\odot = ||\mathbf{r}_\odot||$, $r_{☾} = ||\mathbf{r}_{☾}||$, and $v = ||\mathbf{v}||$.

H_{kepler} represents the attraction of the Earth as a central body, H_{J_2} the potential function that affects the space debris due to the Earth oblateness. In this work we only consider the zonal harmonic J_2 which is the most representative of the potential function. H_{SRP} represents the direct solar radiation pressure potential, H_{3bS} the effect of the Sun, as third body, and H_{3bM} this of the the Moon.

More precisely,

$$H_{kepler}(\mathbf{r}, \mathbf{v}) = \frac{v^2}{2} - \frac{\mu}{r}, \tag{6}$$

where $\mu = \mathscr{G} M_\oplus$ with $\mathscr{G}$ the standard gravitational constant, $M_\oplus$ the mass of the Earth.

The expression of H_{J_2} in terms of the position is:

$$\begin{aligned} H_{J_2}(\mathbf{r}) &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 P_2(\sin\phi_{sat}) \\ &= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 \frac{1}{2} \left(3\left(\frac{z}{r}\right)^2 - 1\right), \end{aligned} \tag{7}$$

where ϕ_{sat} represents the latitude of the satellite, and consequently $\sin\phi_{sat} = z/r$.

Following Valk et al. (2008), the expression for H_{SRP} is,

$$\begin{aligned} H_{SRP}(\mathbf{r},\mathbf{r}_\odot) &= C_r P_r \frac{A}{m} a_\odot^2 \frac{1}{||\mathbf{r}-\mathbf{r}_\odot||} \\ &= C_r P_r \frac{A}{m} a_\odot \sum_{n\geq 0} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi), \end{aligned} \quad (8)$$

where C_r (fixed to 1 in this paper) is a dimension-free reflectivity coefficient, $Pr = 4.56 \times 10^{-6}\, N/m^2$ is the radiation pressure for an object located at a distance of 1 AU from the Sun, A/m is the area-to-mass ratio of the space debris given in m^2/kg and $a_\odot$ is equal to the mean distance between the Sun and the Earth (i.e. $a_\odot = 1$ AU). We consider that $r_\odot \simeq a_\odot$, and P_n is the Legendre polynomials of order n, and finally, ϕ represents the angle between the satellite and the Sun positions.

In this work we split the expression of the Hamiltonian function concerning the solar radiation pressure in three different expressions, which become very useful in the average process that we apply in the following section.

$$\begin{aligned} H_{SRP}(\mathbf{r},\mathbf{r}_\odot) &= C_r P_r \frac{A}{m} a_\odot (1 + \frac{r}{a_\odot}\cos(\phi)) + C_r P_r \frac{A}{m} a_\odot \sum_{n\geq 2} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) \\ &\simeq H_{SRP_0} + H_{SRP_1} + H_{SRP_2}, \end{aligned} \quad (9)$$

where H_{SRP_0} is only a constant term, the $H_{SRP_i}, i = 1,2$ correspond to the first and second order in $\frac{r}{a_\odot}$, neglecting terms with higher order in the distance ratio.

The solar perturbation can be expressed by:

$$\begin{aligned} H_{3bS}(\mathbf{r},\mathbf{r}_\odot) &= -\mu_\odot \frac{1}{||\mathbf{r}-\mathbf{r}_\odot||} + \mu_\odot \frac{\mathbf{r}\cdot\mathbf{r}_\odot}{||\mathbf{r}_\odot||^3} \\ &= -\frac{\mu_\odot}{a_\odot} \sum_{n\geq 0} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) + \mu_\odot \frac{r a_\odot \cos(\phi)}{a_\odot^3} \\ &= -\frac{\mu_\odot}{a_\odot}(1 + \sum_{n\geq 2} \left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi)), \end{aligned} \quad (10)$$

where $\mu_\odot = \mathscr{G} M_\odot$ with $M_\odot$ the mass of the Sun.

Similarly, the lunar perturbation writes:

$$H_{3bM}(\mathbf{r},\mathbf{r}_{☾}) = -\frac{\mu_{☾}}{a_{☾}}(1 + \sum_{n\geq 2} \left(\frac{r}{a_{☾}}\right)^n P_n(\cos\phi_M)), \quad (11)$$

where $\mu_{☾} = \mathscr{G} M_{☾}$ with $M_{☾}$ the mass of the Moon, and ϕ_M representing the angle between the satellite and the Moon positions. The Moon is also assumed to follow a circular orbit, i.e. $r_{☾} = a_{☾}$.

Thanks to Eq. (9) and Eq. (10) the Hamiltonian formulation for the solar radiation pressure plus the solar perturbation can be expressed as follows:

$$\begin{aligned} H_{SRP}(\mathbf{r},\mathbf{r}_\odot) + H_{3bS}(\mathbf{r},\mathbf{r}_\odot) &\simeq H_{SRP_1}(\mathbf{r},\mathbf{r}_\odot) + H_{SRP_2}(\mathbf{r},\mathbf{r}_\odot) + H_{3bS}(\mathbf{r},\mathbf{r}_\odot) \\ &\simeq C_r P_r \frac{A}{m} r\cos(\phi) + \left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right] \left(\frac{r}{a_\odot}\right)^2 P_2(\cos\phi), \end{aligned} \quad (12)$$

where the second term gathers terms with a quadratic dependence on $\frac{r}{a_\odot}$; we consider that the other terms with higher orders in the distance ratio can be neglected.

For the Moon, we only keep the quadratic term:

$$H_{3bM}(\mathbf{r},\mathbf{r}_\odot) \simeq -\frac{\mu_{☾}}{a_{☾}}\left(\frac{r}{a_{☾}}\right)^2 P_2(\cos\phi_M). \tag{13}$$

4 Averaged Hamiltonian

Several periods are present in the Hamiltonian, a 1-day-period (for geostationary orbits), a 1-month-period (in the Moon motion), a 1-year period (the Sun orbital motion), periods linked to the J_2 perturbation, and longer periods appear clearly in the numerical integrations. Several averaging processes are then applied to isolate those main frequencies.

4.1 Very short period averaging process : over 1 day

We average the Hamiltonian function over the mean longitude (λ) since, for long-time propagations, the short periodic oscillations caused by the mean longitude are meaningless. Hereinafter, the averaged Hamiltonians will be represented with an over-line, but we keep the same notations for the orbital elements and the canonical variables and momenta. Under the averaging assumption, our problem becomes a four degree of freedom problem in the averaged variables x_1, y_1, x_2 and y_2, since the mean longitude is not present anymore; consequently the semi-major axis (a) or the momentum associated to the mean longitude (L) will be constants. The averaged Hamiltonian function is given by:

$$\begin{aligned}\overline{\mathscr{H}}(x_1,y_1,x_2,y_2) &= \overline{H}_{kepler} + \overline{H}_{J_2}(x_1,y_1,x_2,y_2)\\ &+ \overline{H}_{SRP_1}(x_1,y_1,x_2,y_2,\mathbf{r}_\odot) + \overline{H}_{SRP_2+3bS}(x_1,y_1,x_2,y_2,\mathbf{r}_\odot)\\ &+ \overline{H}_{3bM}(x_1,y_1,x_2,y_2,\mathbf{r}_{☾}\,),\end{aligned} \tag{14}$$

where

$$\overline{H}_{kepler} = -\frac{\mu^2}{2L^2} \tag{15}$$

is now a constant term and will be omitted.

To obtain an expression for $\overline{H}_{J_2}$ we use the averaged Lagrange Planetary Equations (Abad, 2012):

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{3}{4}\sqrt{\frac{\mu}{a^3}}J_2\frac{r_\oplus^2}{a^2}\frac{4-5\sin^2 i}{(1-e^2)^2} = \frac{C_2}{2}\,\frac{4-5\sin^2 i}{(1-e^2)^2},\\ \frac{d\Omega}{dt} &= -\frac{3}{2}\sqrt{\frac{\mu}{a^3}}J_2\frac{r_\oplus^2}{a^2}\frac{\cos i}{(1-e^2)^2} = -C_2\,\frac{\cos i}{(1-e^2)^2},\end{aligned}$$

where $C_2 = \frac{3}{2}\sqrt{\frac{\mu}{a^3}}J_2\frac{r_\oplus^2}{a^2}$.

The coordinates p and q are linked to the argument of perigee (ω) and right ascension of the ascending node (Ω) through $p = -\omega - \Omega$ and $q = -\Omega$. Following the Hamiltonian formulation:

$$\begin{aligned}\dot{p} &= -\dot{\omega} - \dot{\Omega} = \frac{\partial H_{J_2}}{\partial P} = C_p,\\ \dot{q} &= -\dot{\Omega} = \frac{\partial H_{J_2}}{\partial Q} = C_q,\end{aligned}$$

and consequently, if we choose constant values for C_p and C_q,

$$\overline{H}_{J_2} = C_p\, P + C_q\, Q = \frac{C_p}{2}(x_1^2 + y_1^2) + \frac{C_q}{2}(x_2^2 + y_2^2). \tag{16}$$

In the case of $e = 0$ and $i = 0$, we obtain : $C_p = -C_2$ and $C_q = C_2$.

$\overline{H}_{SRP_1}$ can be expressed, following Hubaux and Lemaître (2013), after averaging on λ and without any truncation on e, as:

$$\overline{H}_{SRP_1} = -\frac{3}{2} C_r P_r \frac{A}{m} a\, e\, \xi, \tag{17}$$

where

$$a = \frac{L^2}{\mu}, \qquad e = \sqrt{1 - \frac{G^2}{L^2}}, \qquad \xi = \xi_1 \mathbf{r}_{\odot,1} + \xi_2 \mathbf{r}_{\odot,2} + \xi_3 \mathbf{r}_{\odot,3},$$

$$\begin{aligned}
\xi_1 &= \cos\Omega\cos\omega - \sin\Omega\cos i \sin\omega,\\
\xi_2 &= \sin\Omega\cos\omega + \cos\Omega\cos i \sin\omega,\\
\xi_3 &= \sin i \sin\omega,
\end{aligned}$$

and

$$\begin{aligned}
r_{\odot,1} &= \cos\lambda_\odot,\\
r_{\odot,2} &= \sin\lambda_\odot \cos\varepsilon,\\
r_{\odot,3} &= \sin\lambda_\odot \sin\varepsilon,
\end{aligned}$$

where a simplified circular motion for the Sun has been assumed, only depending on its mean longitude $\lambda_\odot = n_\odot t + \lambda_{\odot,0}$ (with $n_\odot = 2\pi/year$ and $\lambda_{\odot,0}$ the initial position of the Sun) and on the Earth's obliquity ε.

The averaged Hamiltonian $\overline{H}_{SRP_1}$ can be expressed in terms of Poincaré's variables, truncated at e^2 or i^2, as:

$$\overline{H}_{SRP_1} = -n_\odot k \left[r_{\odot,1}(x_1 R_2 + y_1 R_1) - r_{\odot,2}(x_1 R_3 + y_1 R_2) - r_{\odot,3}(x_1 R_5 - y_1 R_4) \right] \tag{18}$$

where

$$\begin{aligned}
R_1(x_2) &= 1 - \frac{x_2^2}{2L}, \quad R_2(x_2, y_2) = \frac{x_2 y_2}{2L}, \quad R_3(y_2) = 1 - \frac{y_2^2}{2L},\\
R_4(x_2) &= \frac{x_2}{\sqrt{L}}, \quad R_5(y_2) = \frac{y_2}{\sqrt{L}},
\end{aligned} \tag{19}$$

and

$$n_\odot k = \frac{3}{2} C_r P_r \frac{A}{m} \frac{a}{\sqrt{L}}.$$

Consequently the averaged Hamiltonian given in Eq. (14) writes now:

$$\begin{aligned}
\overline{\mathscr{H}} &= \left(\frac{x_1^2 + y_1^2}{2}\right) C_p + \left(\frac{x_2^2 + y_2^2}{2}\right) C_q\\
&- n_\odot k \left[r_{\odot,1}(x_1 R_2 + y_1 R_1) - r_{\odot,2}(x_1 R_3 + y_1 R_2) - r_{\odot,3}(x_1 R_5 - y_1 R_4) \right]\\
&+ \overline{H}_{SRP_2+3bS}(x_1, y_1, x_2, y_2, \mathbf{r}_\odot) + \overline{H}_{3bM}(x_1, y_1, x_2, y_2, \mathbf{r}_{☾}).
\end{aligned} \tag{20}$$

The second order part of the solar radiation pressure, the solar and lunar perturbations have also to be averaged over the fast variable λ. As they are second order terms in $\frac{a}{a_\odot}$, we limit their expansion to the first term, neglecting the following terms proportional to e^2. In other words, we keep the terms in $\frac{ae^2}{a_\odot}$ but not those in $(\frac{ae}{a_\odot})^2$. The immediate consequence is the dependence of the averaged perturbation $\overline{H}_{SRP_2+3bS}$ only on x_2 and y_2 and not on x_1 and y_1:

$$\overline{H}_{SRP_2+3bS} = \overline{H}_{SRP_2+3bS}(-,-,x_2,y_2,\mathbf{r}_\odot) + O(\frac{a^2e^2}{a_\odot^2}). \tag{21}$$

With this assumption, we obtain :

$$\overline{H}_{SRP_2+3bS}(x_2,y_2,\mathbf{r}_\odot) = -\left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right] \frac{3a^2}{4a_\odot^2} \upsilon_S^2 = -\beta\, \frac{3a^2}{4a_\odot^2} \upsilon_S^2 \tag{22}$$

where $\upsilon_S = \upsilon_S(x_2,y_2,\mathbf{r}_\odot) = -\sin q\, \sin i\, \mathbf{r}_{\odot,1} - \cos q\, \sin i\, \mathbf{r}_{\odot,2} + \cos i\, \mathbf{r}_{\odot,3}$, and

$$\beta = \left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right]. \tag{23}$$

With the same assumptions, the lunar perturbation is given by:

$$\overline{H}_{3bM}(x_2,y_2,\mathbf{r}_{☾}) = \frac{\mu_{☾}}{a_{☾}}\, \frac{3a^2}{4a_{☾}^2} \upsilon_M^2 \tag{24}$$

where $\upsilon_M = -\sin q\, \sin i\, \mathbf{r}_{☾,1} - \cos q\, \sin i\, \mathbf{r}_{☾,2} + \cos i\, \mathbf{r}_{☾,3}$.

The Hamiltonian formulation of the problem given in Eq. (20) allows us to express the time evolution of the Poincaré's variables through the following dynamical system:

$$\begin{aligned} \dot{x_1}(t) &= \frac{\partial \overline{\mathscr{H}}}{\partial y_1}, \quad \dot{x_2}(t) = \frac{\partial \overline{\mathscr{H}}}{\partial y_2}, \\ \dot{y_1}(t) &= -\frac{\partial \overline{\mathscr{H}}}{\partial x_1}, \quad \dot{y_2}(t) = -\frac{\partial \overline{\mathscr{H}}}{\partial x_2}. \end{aligned} \tag{25}$$

4.2 Solving the dynamical system in eccentricity : SRP_1 and J_2

Considering the system given in Eq. (25), limited to the solar radiation pressure, $\overline{H}_{SRP_1}$ and to $\overline{H}_{J_2}$ and setting $x_2 = 0 = y_2$, we have the following equations, at first order in e :

$$\begin{aligned} \dot{x_1}(t) &= -C_2\, y_1 - n_\odot\, k\, r_{\odot,1}, \\ \dot{y_1}(t) &= C_2\, x_1 - n_\odot\, k\, r_{\odot,2}, \end{aligned}$$

and the explicit solution is given by (with $\eta = \frac{C_2}{n_\odot}$):

$$\begin{aligned} x_1(t) &= \mathscr{A}\, \sin(C_2 t + \Phi) + \frac{k \sin(n_\odot t + \lambda_{\odot,0})}{1-\eta^2} \left[\eta\, \cos\varepsilon + 1\right], \\ y_1(t) &= \mathscr{A}\, \cos(C_2 t + \Phi) + \frac{k \cos(n_\odot t + \lambda_{\odot,0})}{1-\eta^2} \left[\cos\varepsilon + \eta\right], \end{aligned} \tag{26}$$

where the constants $\mathscr{A}$ and Φ are determined by the initial conditions.

Two periods are present : the 1-year periodic motion is the most important, over a few years, and the period associated to C_2 contributes to the fluctuation visible on 200 years. The amplitude of the 1-year motion is proportional to k, which means proportional to $\frac{A}{m}$.

For short periods of time, of a few years, the result is simply :

$$x_1(t) = C_x + \frac{k\sin(n_\odot t + \lambda_{\odot,0})}{1-\eta^2}\,[\eta\,\cos\varepsilon + 1]\,,$$
$$y_1(t) = C_y + \frac{k\cos(n_\odot t + \lambda_{\odot,0})}{1-\eta^2}\,[\cos\varepsilon + \eta]\,, \quad (27)$$

where C_x and C_y are fixed by the initial conditions.

We plot in Figure 6 the motion of the eccentricity over 10 years, for $A/m = 1$ and $A/m = 20$, with or without J_2. The only significant effect of J_2 is to increase the period, slightly over 10 years, more obviously over 200 years. We calculate the maximum of the eccentricity as a function of A/m and plot the result in Figure 7

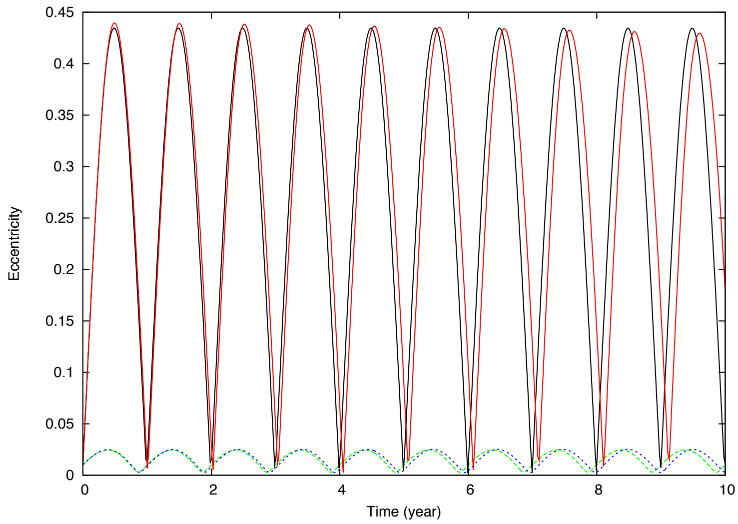


Fig. 6 Evolution of the eccentricity of two pieces of space debris over 10 years with $A/m = 1$ and $A/m = 20$, considering the J_2 effect, and the solar radiation pressure

Let us remind that our model is truncated at e^2, which explains the differences with the numerical integrations for the highest amplitudes.

Finally we represent the motion of the eccentricity on 200 years in Figure 8(b) and compare our result with a similar numerical integration in Figure 8(a) ; we clearly see the superposition of the long and short motions on both figures.

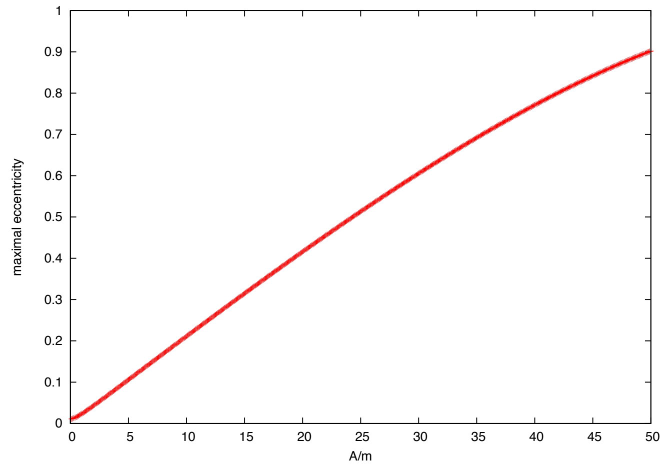


Fig. 7 The maximum of the short periodic motion of the eccentricity as a function of A/m

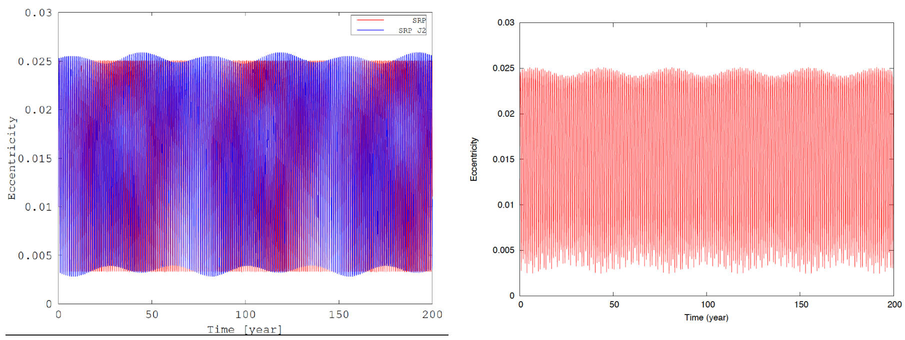


Fig. 8 Evolution of the eccentricity over 200 years, with SRP and J_2, for $A/m = 1$, obtained by numerical integration (a) and obtained by our model (b)

4.3 The second averaging process : over 1 year

Let us write the dynamical equations corresponding to the inclination, i.e. to x_2 and y_2.

$$
\begin{aligned}
\dot{x_2}(t) &= C_q\, y_2 - n_\odot k \left[r_{\odot,1}(\frac{x_1 x_2}{2L}) - r_{\odot,2}(\frac{-2x_1 y_2}{2L} + \frac{y_1 x_2}{2L}) - r_{\odot,3}(\frac{x_1}{\sqrt{L}}) \right] \\
&+ \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial y_2} + \frac{\partial \bar{H}_{3bM}}{\partial y_2}. \qquad (28) \\
\dot{y_2}(t) &= -C_q\, x_2 + n_\odot k \left[r_{\odot,1}(\frac{-2x_2 y_1}{2L} + \frac{x_1 y_2}{2L}) - r_{\odot,2}(\frac{y_1 y_2}{2L}) - r_{\odot,3}(-\frac{y_1}{\sqrt{L}}) \right] \\
&- \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial x_2} - \frac{\partial \bar{H}_{3bM}}{\partial x_2}. \qquad (29)
\end{aligned}
$$

Using the 1-year periodic expressions for $x_1(t)$ and $y_1(t)$ given by (27), we substitute them in the system of Eq. (28) and we seek the expressions for $x_2(t)$ and $y_2(t)$. We are interested in the long term evolution of the inclination, and then we average over the angle $\lambda_\odot$ to obtain a simple long period expression for $< \dot{x_2}(t) >_{\lambda_\odot}$ and $< \dot{y_2}(t) >_{\lambda_\odot}$.

To calculate the variables C_q, we use the value $i = 0$, in agreement with the truncation in i^2, and we replace e by its averaged value over $\lambda_\odot$.

The result is :

$$C_q = C_2\ (1 + 2\,\bar{e}^2) \qquad (30)$$

with

$$L\,\bar{e}^2 = \frac{1}{2}k^2 + \frac{1}{2}k^2\ \cos^2\varepsilon + C_x^2 + C_y^2. \qquad (31)$$

For the contribution $\overline{H}_{SRP_2+3bS}$, the average process over $\lambda_\odot$ is applied to υ_S^2, so to get:

$$< \overline{H}_{SRP_2+3bS} >_{\lambda_\odot} = \beta\ \frac{3a^2}{16\,L\,a_\odot^2} \left(x_2^2 + x_2^2\ \cos 2\varepsilon + 2y_2^2\ \cos 2\varepsilon - 2\ \sqrt{L}\, y_2\ \sin 2\varepsilon \right). \qquad (32)$$

For the lunar contribution, we average the Hamiltonian over the angle $\lambda_☾$ which can be considered as a faster angle that its solar equivalent, but nevertheless slower than the first period of 1-day.

$$< \overline{H}_{3bM} >_{\lambda_☾} = -\frac{\mu_☾}{a_☾}\ \frac{3a^2}{16\,L\,a_☾^2} \left(x_2^2 + x_2^2\ \cos 2\varepsilon_M + 2y_2^2\ \cos 2\varepsilon_M - 2\ \sqrt{L}\, y_2\ \sin 2\varepsilon_M \right), \qquad (33)$$

with ε_M the obliquity of the Moon.

Thus, we have averaged the equations $\dot{x_2}(t)$ and $\dot{y_2}(t)$ over the variables $\lambda_\odot$ and $\lambda_☾$, and we obtain the following simplified linear equations:

$$
\begin{aligned}
\dot{x_2}(t) &= d_1\ y_2 + d_3, \\
\dot{y_2}(t) &= -d_2\ x_2, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= n_\odot\ \frac{k^2}{4L}\ \cos\varepsilon + \frac{C_q}{2} - \delta - \delta\ \cos 2\varepsilon - \gamma - \gamma\ \cos\varepsilon_M, \\
d_2 &= n_\odot\ \frac{k^2}{4L}\ \cos\varepsilon + \frac{C_q}{2} - 2\ \delta\ \cos 2\varepsilon - 2\ \gamma\ \cos 2\varepsilon_M, \\
d_3 &= -n_\odot\ \frac{k^2}{2\sqrt{L}}\ \sin\varepsilon + 2\ \delta\ \sqrt{L}\ \sin 2\varepsilon + 2\ \gamma\ \sqrt{L}\ \sin 2\varepsilon_M,
\end{aligned}
$$

where $\delta = \beta \, \frac{3a^2}{16\,L\,a_\odot^2}$ and $\gamma = -\frac{\mu_☾}{a_☾} \, \frac{3a^2}{16\,L\,a_☾^2}$.

We write the corresponding solution for $x_2(t)$ and $y_2(t)$:

$$\begin{aligned} x_2(t) &= \mathscr{D}\,\sin(\sqrt{d_1 d_2}\,t - \psi), \\ y_2(t) &= \mathscr{D}\,\sqrt{\frac{d_2}{d_1}}\,\cos(\sqrt{d_1 d_2}\,t - \psi) - \frac{d_3}{d_1}. \end{aligned} \tag{35}$$

These equations represent an oscillatory motion, $\mathscr{D}$ is the amplitude and ψ the phase space. Both of them are calculated through the initial conditions. Eqs. (26) and Eqs. (35) are the analytical solution of the problem of space debris orbiting around the Earth in the geostationary ring.

The period in the inclination motion depends on the motion of the Moon; we represent in Fig. 9 the period (given in years) present in the inclination, as a function of A/m. The upper curve represents the period including the Moon, and the lower curve without the Moon. We clearly confirm the result of the numerical integration given in Fig. 3: for $A/m = 1$ the presence of the Moon increases the period by more than 12 years (from 52 to 65 years) while for $A/m = 20$, the increase is much smaller (from 23 to 25 years).

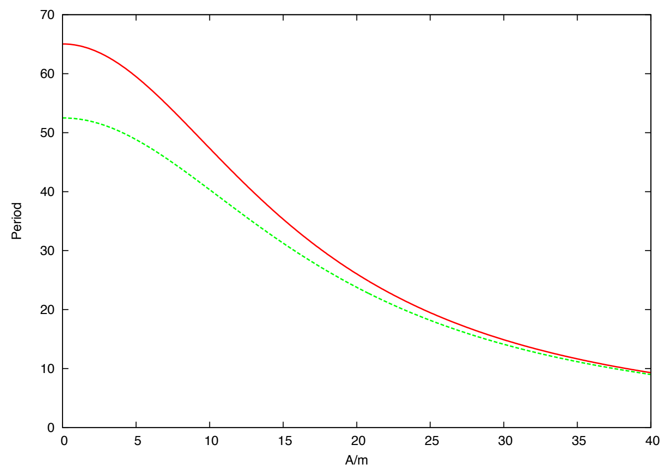


Fig. 9 The long period associated with the motion of the inclination, as a function of A/m, with the Moon (red curve) and without the Moon (green curve)

For the motion on the inclination, we can notice in Fig. 10 that each supplementary effect (J_2, Sun, Moon) reduces the main period if the value of A/m is small (a few units) while for the largest values of A/m, the solar radiation pressure is really the main perturber, the other perturbations do play a very small role.

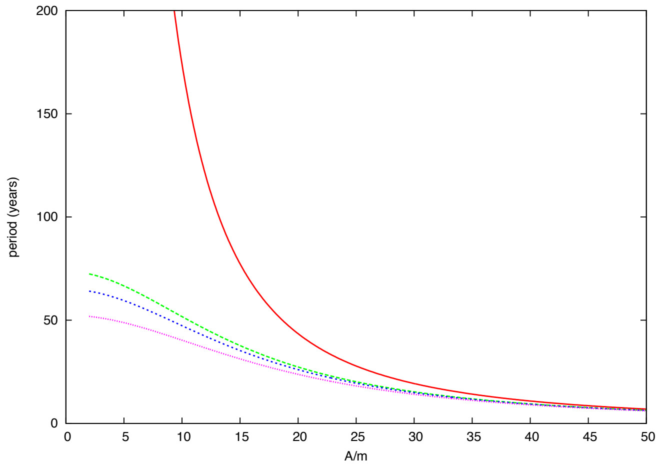
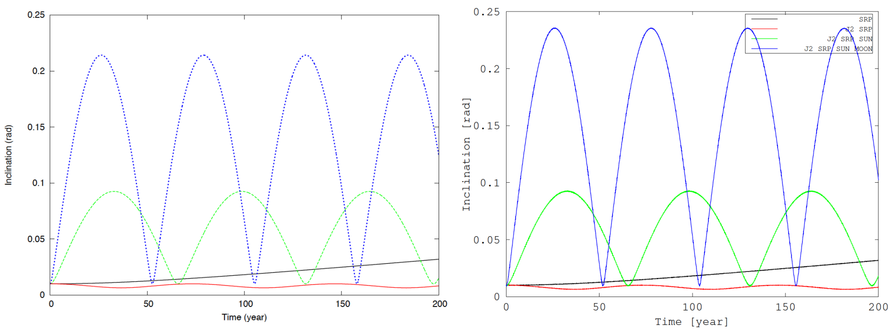


Fig. 10 Calculation of the period of the inclination motion, as a function of A/m, with only the solar radiation pressure (red curve), with SRP and J_2 (green curve), with SRP, J_2 and the Sun (blue curve), and finally with SRP, J_2, the Sun and the Moon (magenta curve); A/m is given in m^2/kg and the period in years

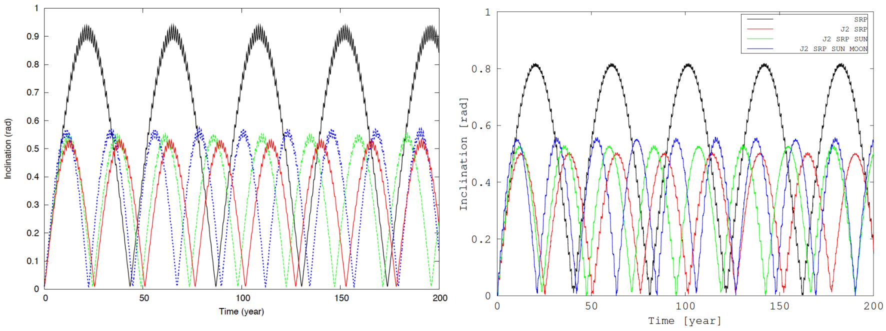
5 Results

To compare the influence of the different perturbations, we represent the motion of the inclination in 4 cases : SRP, SRP + J_2, SRP + J_2 + Sun, and SRP + J_2 + Sun + Moon, for a piece of debris with $A/m = 1$ in Fig. 11(a) and with $A/m = 20$ in Fig. 12(a). Numerical integrations have been performed with the same assumptions in Fig. 11(b) and in Fig. 12(b). The agreement is very good, except for the highest amplitudes, due to the second order truncation in i.



(a) (b)

Fig. 11 Evolution of the inclination over 200 years by using 4 different models of one piece of space debris with $A/m = 1$, obtained by our model (a) and obtained by numerical integration (b)



(a) (b)

Fig. 12 Evolution of the inclination over 200 years by using 4 different models for one piece of space debris with $A/m = 20$, obtained by our model (a) and obtained by numerical integration (b)

6 Conclusion and Future work

In this paper, we present an analytical model to propagate space debris in the geostationary ring, which includes the effect of the J_2 due to the Earth oblateness, the solar radiation pressure, and the solar and lunar perturbations. The solution provided by this model has been tested through the numerical integrator NIMASTEP proving its reliability. This model improves the previous one presented by Hubaux and Lemaître (2013), which only considers the solar radiation pressure. We justify the importance of considering the J_2 effect when propagating space debris independently of the ratio A/m for short and long-term propagation. The lunar and solar perturbations play an important role in the period and amplitude of the inclination, especially for values of A/m close to unity. For higher values of A/m the solar radiation pressure remains the dominant dynamics.

This work helps to improve the knowledge of space debris evolution for short and long-term propagation, to design future missions and also to avoid space debris collisions (Rossi and Valsecchi, 2006). However, our final goal is to use a simple but reliable analytical approach to propagate thousands of pieces of space debris at the same time, to get reliable statistical results concerning the location of this population of space debris (see Casanova et al. (2015))

Finally, this new model can also be applied to design satellite constellations, especially to improve the lattice-preserving Flower Constellations (Casanova et al., 2015; Avendaño et al., 2013). This particular satellite constellations are characterized by maintaining the initial configuration under the J_2 effect. If solar sails are included in the satellites, the analytical model presented in this paper can be easily applied to reproduce the time evolution of this kind of satellite constellation.

Acknowledgments

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