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On asymptotic behaviour of one-parameter families of bounded operators on Banach spaces

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ON ASYMPTOTIC BEHAVIOUR OF ONE-PARAMETER FAMILIES OF BOUNDED OPERATORS ON BANACH SPACES

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Memoria de Tesis Doctoral realizada bajo la dirección de los doctores
D. José Esteban Galé Gimeno y D. Pedro José Miana Sanz

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Chapter 0

Introduction

The study of the asymptotic behaviour of one-parameter families (in particular C_0-semigroups) of operators on a Banach space has received a lot of attention in recent years. In particular the convergence of orbits of a given family to zero is a milestone in operator theory and differential equations. Under assumptions of very different nature, and motivated by applications to partial differential equations, a number of results about stability of C_0-semigroups and other families have been obtained. A fairly complete overview on the techniques used and the results obtained on this topic can be found in [B, CT, EN, N].

In this introduction we do not pretend to give an exhaustive presentation about the history and main results of asymptotic analysis of one-parameter families of operators. Our purpose is just to collect a set of results which allows us to understand this memoir properly.

Stability of semigroups

Let X be a complex Banach space and let A be a closed operator on X with domain $D(A)$ and range $R(A)$. Let $u\colon [0,\infty) \to X$ be a (vector-valued) continuous function which is differentiable on $(0,\infty)$ and such that $u(t) \in D(A)$ for all $t > 0$.

The so-called *Cauchy problem* for A consists on finding a function u as above such that it satisfies the equation

$$\begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = x, & x \in X, \end{cases} \tag{ACP}$$

where $x \in X$ is a given initial value. The problem is said to be *well-posed* if A is the infinitesimal generator of a strongly continuous C_0-semigroup $T(t) = e^{tA}$ of bounded operators on X. Then the solution u is given by $u(t) = e^{tA}x$, $t \geq 0$.

An important question about the behaviour of the above solution u is whether or not it is *stable*, which is to say by definition that $\lim_{t\to\infty} u(t) = 0$. Thus one says that for a given $x \in X$ the orbit $\{T(t)x : t \geq 0\}$ is stable when $\lim_{t\to\infty} T(t)x = 0$, and that the semigroup $(T(t))_{t\geq 0}$ is stable if all its orbits are stable. Next, we recall well established facts about stability of one-parameter semigroups.

• Liapunov Theorem

We start with the simplest case. Let $\mathscr{B}(X)$ denote the Banach algebra of bounded operators on a Banach space X. Then A is the generator of the exponential semigroup $(e^{tA})_{t\geq 0}$ given by the (convergent in $\mathscr{B}(X)$) series

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \qquad (t \in \mathbb{R}).$$

In particular we can consider the full algebra $\mathrm{M}_n(\mathbb{C})$ of $n \times n$ matrices on the complex field $\mathbb{C}$ -which corresponds to $\mathscr{B}(X)$ when $X = \mathbb{C}^n$- and $A \in \mathrm{M}_n(\mathbb{C})$.

The classical Liapunov stability theorem goes back to 1892, see [Li] and also [EN, Theorem I.2.10]. For a given $A \in \mathrm{M}_n(\mathbb{C})$, it characterizes the stability of $(e^{tA})_{t\geq 0}$ in terms of the location of the eigenvalues of A:

Theorem 1. *Let $(e^{tA})_{t\geq 0}$ be the one-parameter semigroup generated by $A \in M_n(\mathbb{C})$. Then the following assertions are equivalent:*

(a) The semigroup is stable, i.e., $\lim_{t\to\infty} \|e^{tA}\| = 0$.

(b) All eigenvalues of A have negative real part, i.e., $\Re\lambda < 0$ for all $\lambda \in \sigma(A)$.

Naturally, mathematicians have tried to extend this finite-dimensional theorem to infinite dimensions as much as possible. A research line in this direction is the following.

Let A be the (closed) infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then the (uniform) exponential growth bound of A -or $T(t)$ alternatively- is the (possibly infinite) number

$$\omega(A) := \inf\{\omega \in \mathbb{R} \;:\; \|\exp(tA)\| \leq Me^{\omega t} \text{ for some } M > 0 \text{ and all } t \geq 0\}.$$

Let $\sigma(A)$ denote the spectrum of A. The spectral bound $s(A)$ of A is defined by

$$s(A) := \sup\{\Re\lambda \;:\lambda \in \sigma(A)\}.$$

One has, for *bounded* A, the following extension of Theorem 1, which is also known as Liapunov Theorem.

Theorem 2. *Let A be a bounded operator on the Banach space X. Then,*

$$s(A) = \omega(A).$$

Thus the Liapunov theorem shows that the spectrum $\sigma(A)$ of A is responsible for the asymptotic behavior of the solution u of the equation (ACP). Note that the relation between the two versions of the theorem -namely, Theorem 1 and Theorem 2- is given by the fact that if $\sigma(A)$ is contained in the left-hand half-plane, then

$s(A) < 0$, so $\omega(A) < 0$ and the solution is uniformly asymptotically stable, that is, $\|e^{tA}\| \to 0$ as $t \to \infty$, see [EN, Theorem I.3.14].

The proof of Theorem 2 -see for instance [DK, Theorem I.4.1] or [EN, Corollary IV.2.4]- relies upon the *spectral mapping* theorem

$$e^{t\sigma(A)} = \sigma(e^{tA}), \qquad t \geq 0.$$

See [EN, Theorem I.3.13].

Neither the spectral mapping theorem nor the Liapunov theorem are longer true for general unbounded operators A. Even in the case of Hilbert spaces, there exist examples of C_0-semigroups whose uniform growth bound $\omega(A)$ is strictly larger than the spectral bound $s(A)$, see [EN, Counterexamples IV.2.7 and IV.3.4]. These “patologies" are the starting point for the modern asymptotic theory of semigroups.

Remark. As in the bounded case, the spectral mapping theorem implies the equality $\omega(A) = s(A)$ for general operators A. So it is important to find conditions on a C_0-semigroup (or on its generator) which allow us to prove the spectral mapping theorem. In this sense, a well-known assumption is the eventual continuity with respect to the uniform operator topology; this holds for compact semigroups and holomorphic semigroups, for instance. Let us also mention some other new spectral mapping theorems, like for example the weak spectral mapping theorem for non-quasianalytic groups ([EN, Section IV.3. c]) and the spectral mapping theorem of Latushkin and Montgomery-Smith ([N, Section 2.5]).

Nevertheless, the equality $\omega(A) = s(A)$ can be directly proved in some particular cases. For example, this holds when A generates a positive semigroup on spaces $L^p(\mu)$ or $C_0(\Omega)$; see [N, Section 3.3.5] and references therein.

In the general case, one has to look for additional spectral conditions to determine the stability of some (or all) orbits $(T(t)x)_{t\geq 0}$ with $x \in X$. In this direction we must quote the classical stability theorem proved by Arendt-Batty and Lyubich-Vũ ([AB, LV]), and also the semigroup version of the Katznelson-Tzafriri theorem established by Esterle-Strouse-Zouakia and Vũ ([ESZ, V]). It is in this setting where we focus our research. We devote the next paragraph to explain these items in some more detail.

• Stability of orbits

Let A be a closed operator and let $\sigma_P(A^*)$ denote the point spectrum of the adjoint operator A^* of A. Arendt-Batty in [AB] and Lyubich-Vũ in [LV] showed independently and with different proofs from each other the following result.

Theorem 3. *Let $(T(t))_{t\geq 0}$ be a uniformly bounded C_0-semigroup on a Banach space X, with generator A. If*

(i) $\sigma(A) \cap i\mathbb{R}$ *is countable, and*

(ii) $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$

then $T(t)$ is asymptotically stable, i.e. $\lim_{t\to\infty} T(t)x = 0$ *for all $x \in X$.*

We refer to this stability result as the Arendt-Batty-Lyubich-Vũ theorem. None of the above spectral requirements (i) and (ii) in the above theorem is superfluous, see [AB]. This result was subsequently extended by Vũ to semigroups with non-quasianalytic growth in [V1].

A positive measurable locally bounded function $\omega(t)$ with domain $\mathbb{R}$ or $[0,\infty)$ is said to be a weight if $\omega(t) \geq 1$ and $\omega(s+t) \leq \omega(s)\omega(t)$ for all t, s in its domain. A weight ω on $[0,\infty)$ is called nonquasianalytic if

$$\int_0^\infty \frac{\log \omega(t)}{t^2+1}\, dt < \infty.$$

Assume that $\liminf_{t\to\infty} \omega(t)^{-1}\omega(s+t) \geq 1$ for all $s > 0$. Then one can define the associated weight function $\widetilde{\omega}$ on $\mathbb{R}$ given by

$$\widetilde{\omega}(s) := \limsup_{t\to\infty} \frac{\omega(t+s)}{\omega(t)}, \text{if s} \geq 0 \text{ , and } \widetilde{\omega}(\text{s}) := 1, \text{if s} < 0.$$

Clearly, $\widetilde{\omega}(t) \leq \omega(t)$ for every $t \geq 0$.

The Vũ's result is as follows.

Theorem 4. *Let A be the generator of a C_0-semigroup $T(t)$ such that*

$$\sup_{t\geq 1} \omega(t)^{-1}\|T(t)\| < +\infty,$$

for some nonquasianalytic weight ω on $[0,\infty)$ for which $\widetilde{\omega}(t) = O(t^k)$ as $t \to \infty$, for some $k \geq 0$. Assume also that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$. Then

$$\lim_{t\to\infty} \omega(t)^{-1}T(t)x = 0 \quad \textit{for all } x \in X.$$

The stability of the semigroup obtained in the above theorems involves the strong operator topology. It is possible to get results on *uniform* asymptotic convergence (to 0) of the semigroup, that is, convergence in the operator norm, when the semigroup acts on suitable operator-valued weights. These results are semigroup extensions of the Katznelson-Tzafriri theorem:

Let $A(\mathbb{T})$ be the convolution Wiener algebra formed by all continuous periodic functions $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$, $t \in [-\pi,\pi]$, such that $\|f\|_{A(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |a_n| < \infty$, endowed with the norm $\|\cdot\|_{A(\mathbb{T})}$. This algebra is regular. Let $A_+(\mathbb{T})$ be the convolution closed subalgebra of $A(\mathbb{T})$ formed by the functions f with $a_n = 0$ for all $n < 0$. Assume that $T \in \mathscr{B}(X)$ is a power bounded operator, that is, $\sup_{n\geq 0} \|T^n\| < \infty$. Clearly, the operator sum

$$f(T) := \sum_{n=0}^{\infty} a_n T^n$$

is well defined for every $f \in A_+(\mathbb{T})$ with $f(t) = \sum_{n=0}^{\infty} a_n e^{int}, t \in [-\pi, \pi]$. In [KT], Katznelson and Tzafriri proved that *if $f \in A_+(\mathbb{T})$ is of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$ then*

$$\lim_{n\to\infty} \|T^n f(T)\| = 0.$$

What spectral synthesis does mean in regular Banach algebras is explained in Chapter 1 (Section 1.4).

The continuous version of this theorem for uniformly bounded C_0-semigroups was given in [ESZ, Théorème 3.4] and [V, Theorem 3.2]), independently one paper of each other, and with different proofs. Recall that the convolution algebra $L^1(\mathbb{R})$ is a regular Banach algebra and that the Banach space $L^1(\mathbb{R}^+)$ can be seen as a subalgebra of $L^1(\mathbb{R})$ in a similar way as $A_+(\mathbb{T})$ is of $A(\mathbb{T})$ (convolution in $L^1(\mathbb{R})$ corresponds to convolution of coefficients in $A(\mathbb{T})$). This version is as follows.

Theorem 5. *Let A be the infinitesimal generator of a uniformly bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X. If $f \in L^1(\mathbb{R}^+)$ is of spectral synthesis in $L^1(\mathbb{R})$ with respect to $i\sigma(A) \cap \mathbb{R}$ then*

$$\lim_{t\to\infty} \|T(t)\pi_0(f)\| = 0,$$

where $\pi_0 \colon L^1(\mathbb{R}^+) \to \mathscr{B}(X)$ is the bounded Banach algebra homomorphism defined by

$$\pi_0(f)x := \int_0^{\infty} f(t)\, T(t)x\, dt, \quad x \in X,\, f \in L^1(\mathbb{R}^+).$$

As an obvious corollary we obtain that, under the conditions of the above theorem, the orbits $\{T(t)y : t \geq 0\}$ are stable for all $y = \pi_0(f)x$, $x \in X$. That is,

$$\lim_{t\to\infty} T(t)y = 0, \quad \text{for every } y = \pi_0(f)x,\ x \in X.$$

Let $\mathfrak{S}_0$ denote the set of all functions in $L^1(\mathbb{R}^+)$ which are of spectral synthesis in $L^1(\mathbb{R})$ with respect to $i\sigma(A)\cap\mathbb{R}$. From the above, we have that whenever the subspace $Y := \{\pi_0(f)x : f \in \mathfrak{S}_0, x \in X\}$ is dense in X then the semigroup $T(t)$ is stable. The density of Y on X was proved in [ESZ] provided A satisfies

$$\sigma(A)\cap i\mathbb{R} \text{ is countable and } \sigma_P(A^*)\cap i\mathbb{R} = \emptyset,$$

so it gives in this way another different proof of the Arendt-Batty-Lyubisch-Vũ theorem.

• Decay rate of stable orbits

In the last decades, many authors have approached the issue about searching for estimates of the decay rate of stable orbits. Motivated by applications to damped wave equations and many other hyperbolic problems, great progress in the problem of getting decay estimates has been achieved in the case that the infinitesimal generator of the C_0-semigroup has empty boundary spectrum; that is, when $\sigma(A)\cap i\mathbb{R} = \emptyset$ (see [BEPS], [BD], [Bu], [Le], [LR], for instance). It turns out from these results that the growth of the resolvent on the imaginary axis determines the rate of decay of smooth orbits of the semigroup. A unified statement to this subject was given by Batty and Duyckaerts (see [BD, Theorem 1.5]):

Theorem 6. *Let $(T(t))_{t\geq 0}$ be a uniformly bounded C_0-semigroup on a Banach space X. Let A be its generator and assume that $\sigma(A)\cap i\mathbb{R} = \emptyset$. Let $k\in\mathbb{N}$. Then, there are constants $C_k, T_k > 0$ such that*

$$\left\|T(t)(1-A)^{-k}\right\| \leq \frac{C_k}{\left(M_{log}^{-1}(t/C_k)\right)^k} \qquad \forall t \geq T_k\,,$$

where

$$\begin{aligned} M(\xi) &:= \sup_{1\leq|\tau|\leq\xi} \|(i\tau - A)^{-1}\|\,,\ \xi \geq 1\,,\\ M_{log}(\xi) &:= M(\xi)\log((1+M(\xi))(1+\xi))\,,\ \xi \geq 1\,. \end{aligned}$$

Remark. In [BD], it is also conjectured that the logarithmic correction considered in $M_{\log}$ is necessary when X is a general Banach space but it can be dropped if X is a Hilbert space. In the recent paper [BT], A. Borichev and Y. Tomilov confirm this conjecture for polynomially growing M. In addition, they show that Theorem 6 is sharp.

The proof of Theorem 6 is based on a classical contour integral method initiated by Newman and Korevaar in [Ne] and [K], respectively. Using this technique, C. J. K. Batty and T. Duyckaerts also estimate the decay of some Cesaro means of a bounded vector-valued function whose Laplace transform extends to a suitable region containing the imaginary axis ([BD, Theorem 4.1]). They conclude the work by considering the case of semigroups whose associated boundary spectrum is at most finite ([BD, Proposition 4.3]). Such method and results are further considered in the present memoir (see Chapter 2 below).

Vector-valued Laplace theorems and asymptotics

As noticed at the end of the preceding section, the study of asymptotics of orbits is related with the study of vector-valued functions through the analytic properties of their Laplace transforms. In the present section, we follow on this idea.

For a Banach space X, let $\mathbb{R}^+ := [0,\infty)$ and let $L^1_{loc}(\mathbb{R}^+;X)$ denote the vector space of functions $f : \mathbb{R}^+ \to X$ which are Bochner integrable on $[0,R]$ for all $R > 0$.

For a function $f \in L^1_{loc}(\mathbb{R}^+;X)$, the Laplace transform $\mathscr{L}f$ of f is given by

$$\mathscr{L}f(\lambda) = \int_0^\infty f(t)e^{-\lambda t}dt$$

for those complex values λ for which the integral exists. If such a set of numbers λ is non-empty one says that the function f is *Laplace transformable*. In the particular case that $f(t) = T(t)x$ for $t \geq 0$, $x \in X$ and $(T(t))_{t\geq 0}$ a uniformly bounded C_0-semigroup generated by A, we get that $\mathbb{C}^+ \subset \rho(A)$ and

$$(\lambda - A)^{-1}x = \int_0^\infty T(t)xe^{-\lambda t}dt, \qquad \Re\lambda > 0.$$

Then the Laplace transform is the link between Cauchy problems and spectral properties of operators, that is, between solutions and resolvents. We will serve from this fruitful relationship (or other variants of it) to obtain some of the main results of this memoir. We shall not be concerned here with representation theorems for Laplace transforms, corresponding to well-posedness of the Cauchy problem; see [ABHN, Section 3.1] for a wide range of results on this matter. On the other hand, we will concentrate on results of tauberian nature relating Laplace transforms and their applications to the asymptotic behaviour of orbits of semigroups -that is, asymptotics of solutions to (ACP)- and of other one-parameter families of bounded operators.

• Post-Widder inversion formula

Recall that $t > 0$ is said to be a Lebesgue point of a function $f \in L^1_{loc}(\mathbb{R}^+;X)$ if

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|f(s)-f(t)\|ds = 0.$$

Every point of continuity is a Lebesgue point of f and almost all points are Lebesgue points of f (see [ABHN, p. 16]).

It is a well known fact that any Laplace transformable function $f \in L^1_{loc}(\mathbb{R}^+;X)$ is (uniquely) determined by its Laplace transform, as the following (vector-valued) Post-Widder formula shows (see [ABHN, Theorem 1.7.7]).

Theorem 7. *Let $f \in L^1_{loc}([0,\infty);X)$ be such that $\mathscr{L}f(\lambda)$ converges for some $\lambda \in \mathbb{C}$. Let $t > 0$ be a Lebesgue point of f. Then*

$$f(t) = \lim_{n\to\infty}(-1)^n \frac{1}{n!}\left(\frac{n}{t}\right)^{n+1}(\mathscr{L}f)^{(n)}\left(\frac{n}{t}\right).$$

The above theorem provides us with the vector-valued version of the classical Post-Widder inversion formula for the Laplace transform; see [P, W]. Such a limit is known as a real inversion formula since only properties of $\mathscr{L}f(\lambda)$ for large real λ are used. In recent years, the Post-Widder formula has been fruitfully applied to numerical problems; see for instance [MCPS, SB].

In Theorem 7, if one takes $f(t) = T(t)x$ for some $x \in X$, where $T(t) = e^{tA}$ is a C_0-semigroup, and applies the resolvent equation

$$\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}x = (\lambda - A)^{-(n+1)}x, \qquad n \in \mathbb{N},$$

then one gets the *Euler formula* for semigroups

$$T(t)x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}x, \qquad t > 0;$$

see [ABHN, Corollary 3.3.6]. Note that this formula can be regarded as that the orbit of the semigroup is an asymptotic limit of orbits of the resolvent function of its generator. We give an integrated version of this property in Chapter 3 below.

Ill-posed Cauchy problems and asymptotics of solutions

There are (closed) interesting operators A for which the Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = x, & x \in X, \end{cases}$$

is *ill-posed* in the sense that A *is not* the infinitesimal generator of any strongly C_0-semigroup, so the solution to the equation, even when it exists, is not given by the action of a semigroup on the initial value $x \in X$.

However, in some important cases of the above situation, it is still possible to work with families of operators which are to be found as generalizations, extensions or variants of the ones formed by semigroups. These are the so-called integrated semigroups. Other one-parameter families like cosine families, integrated cosine families or Mittag-Leffler families appear in the setting of ill-posed Cauchy problems of higher orders. We are here mostly concerned about one-parameter integrated semigroups.

Let assume that A is the generator of an exponentially bounded n-times integrated semigroup $(T_n(t))_{t\geq 0}$ (see definitions in Chapter 1 below). Then the function

$$u(t) := \frac{d^n}{dt^n} T_n(t)x, \quad (t > 0)$$

is the unique solution to the equation (ACP). Thus the ergodic type limit

$$\lim_{t\to\infty} t^{-n} T_n(t)x = \lim_{t\to\infty} \frac{1}{t^n} \int_0^t (t-s)^{n-1} u(s) \frac{ds}{(n-1)!},$$

reflects the asymptotic behaviour of the solution u at infinity.

However, it is not clear what one should handle as the most accurate notion of stability of an integrated semigroup. In [Me], a *once* integrated semigroup $(T_1(t))_{t\geq 0}$ generated by A is called stable when there exists $\lim_{t\to\infty} T_1(t)x$ in X for every $x \in$

$\overline{D(A)}$. Such a condition seems to be fairly suitable at first sight, though it entails a notable restriction on the generator of $(T_1(t))_{t\geq 0}$. In fact, if $(T_1(t))_{t\geq 0}$ is stable then A is invertible. Moreover, $\lim_{t\to\infty} T_1(t)x = -A^{-1}x$, $x \in \overline{D(A)}$; see [Me, Proposition 5.1 and Remark 5.3].

In the above setting, the stability theorem of Arendt-Batty-Lyubich-Vũ (Theorem 3) admits the following integrated version; see [Me, Theorem 5.6].

Theorem 8. *Let A be the generator of a uniformly bounded once integrated semigroup $(T_1(t))_{t\geq 0}$ such that*

(i) $\sigma(A) \cap i\mathbb{R}$ *is countable,*

(ii) $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$,

(iii) A is invertible.

Then $(T_1(t))_{t\geq 0}$ is stable; that is, there exists $\lim_{t\to\infty} T_1(t)x (= -A^{-1}x)$, *for every* $x \in \overline{D(A)}$.

We give an extension of this result to n-times integrated semigroups, with invertible generator and satisfying the growth condition $\sup_{t>0} \omega(t)^{-1}\|T_n(t)\| < \infty$ -where ω is a non-quasianalytic weight-, in Chapter 4 below.

The question now is to find out results like Theorem 8 for n-times integrated semigroups with *non-invertible* generator. A formally straightforward generalization of the semigroup stability property for n-times integrated semigroups satisfying $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$, is the requirement that $\lim_{t\to\infty} t^{-n}T_n(t)x = 0$ for all $x \in X$. However, specialists prefer to consider this requirement as an ergodic type property rather than stability in its own. We establish a result of this type in Chapter 6.

Main results

In this section we give statements of the main theorems presented in the memoir. Our first concern here is about the investigation of new results involving the Laplace transform of vector-valued functions and their direct application to the study of asymptotics of orbits. More precisely, we are interested in the decay rate of stable orbits and in an Euler formula for integrated semigroups.

• Decay rate of stable orbits

As we have pointed out before, one of the methods used in the memoir, in order to study the asymptotics of one-parameter families of bounded operators, consists of studying analytic properties of appropriate vector-valued functions. In this setting, it is interesting to find new results about the Laplace transform, and its inverse, of such functions. The results obtained here locate around this circle of ideas, and are motivated by those given by C. J. K. Batty and T. Duckaerts in [BD]. Our main results are as follows.

Put $e_1(t) := e^{-t}$ for $t \in \mathbb{R}^+$. Let $*$ denote the usual convolution in $\mathbb{R}$ and let $\circ$ denote the convolution product defined by

$$g \circ f(t) := \int_t^{\infty} g(s-t) f(s)\, ds, \quad t > 0,$$

for $g \in L^1(\mathbb{R}^+)$ and $f \in L^\infty(\mathbb{R}^+;X)$.

Theorem 2.1.1, p. 63: *Let X be a Banach space and let $f \in L^\infty(\mathbb{R}^+;X)$. Assume that there exists a continuous function $\mu\colon (0,\infty) \to (0,\infty)$ satisfying:*

(i) The Laplace transform $\mathscr{L}f$ has a holomorphic extension to the region $\Sigma_\mu := \{z \in \mathbb{C} : \Re z > -\mu(|\Im z|)^{-1})\}$ and $\|\mathscr{L}f(z)\| \le \mu(|\Im z|)$ throughout $\Sigma_\mu \cap \mathbb{C}^-$.

(ii) μ *is decreasing on* $(0,1]$ *and increasing on* $[1,\infty)$.

Then there exists positive constants C *and* τ *such that*

$$\|(e_1 - e_1 * e_1)\circ f(t)\| \le C\left(m_{\log}^{-1}(t/4) + \frac{1}{M_{\log}^{-1}(t/4)} + \frac{1}{t}\right), \quad t > \tau,$$

where $M_{\log}^{-1}$ *and* $m_{\log}^{-1}$ *denote the inverse functions of* $M_{\log}$ *and* $m_{\log}$*, respectively, defined by*

$$M_{\log}(\xi) := \mu(\xi)\log(1+\mu(\xi))(1+\xi)), \quad \xi \ge 1,$$

$$m_{\log}(\xi) := \mu(\xi)\log\left(\frac{1+\mu(\xi)}{\xi}\right), \quad 0 < \xi \le 1.$$

The preceding theorem is the basis to obtain the following result on decay rate of semigroup stable orbits.

Let $(T(t))_{t\ge 0}$ be a uniformly bounded C_0-semigroup on the Banach space X, with infinitesimal generator A. Let $M\colon [1,\infty) \to \mathbb{R}^+$ and $m\colon (0,1] \to \mathbb{R}^+$ be the continuous functions given respectively by

$$M(\xi) := \sup_{1\le|\lambda|\le\xi} \|(i\lambda - A)^{-1}\|, \quad \xi \ge 1,$$

and

$$m(\xi) := \sup_{\xi\le|\lambda|\le 1} \|(i\lambda - A)^{-1}\|, \quad 0 < \xi \le 1.$$

Define $\mu\colon (0,\infty) \to (0,\infty)$ by

$$\mu(\xi) := 2m(\xi), \text{ if } 0 < \xi \le 1 \text{ and } \mu(\xi) := 2\mathrm{M}(\xi), \text{ if } \xi \ge 1.$$

Now, let $M_{\log}$ and $m_{\log}$ be defined as in the previous theorem.

Theorem 2.2.1, p. 71: *In the above setting, assume that* $\sigma(A)\cap i\mathbb{R}\subseteq\{0\}$. *Then, for every* $k\in\mathbb{N}$ *there exist positive constants* $C_k,\tau_k>0$ *such that for all* $t>\tau_k$,

$$\|T(t)A^k(1-A)^{-2k}\|\leq C_k\left(m_{\log}^{-1}(t/4k)+\frac{1}{M_{\log}^{-1}(t/4k)}+\frac{k}{t}\right)^k.$$

This theorem admits an extension to semigroups whose infinitesimal generators have boundary spectrum formed by a finite number of points, not necessarily the origin. See Theorem 2.2.1 in Chapter 2.

• Post-Widder integrated formula. Euler formula for α-times integrated semigroups

The link between Laplace transforms of vector valued Bochner-measurable functions and orbits of semigroups can also be taking into account to deal with integrated families. In this direction, we first give an inversion formula of Post-Widder type for λ^α-multiplied vector-valued Laplace transforms ($\alpha>0$), which generalizes Theorem 7 above.

Let X be a Banach space, and let $f\in L^1_{loc}(\mathbb{R}^+;X)$ be such that

$$\sup_{t>0}\|t^{-\gamma}e^{-\omega t}f(t)\|=M<\infty$$

for some $\gamma>-1$ and some $\omega\geq 0$. Clearly, the Laplace transform $\mathscr{L}f(\lambda)$ of f exists at least on the open right half-plane $\Re\lambda>\omega$. For such a function f we have the following.

Theorem 3.1.1, p. 80: *For every* $\alpha\in(0,\gamma+1)$ *and for any Lebesgue point* $t>0$ *of* f,

$$f(t)=\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda^\alpha\mathscr{L}f)\Big|_{\lambda=n/s}ds.$$

Let us point out that one can also obtain an analogous result for Laplace-Stieltjes transforms and α a positive integer; see Corollary 3.1.1 in Chapter 3 below.

The above formula may well be seen as an *α-times integrated Post-Widder formula*. Its interest relies upon the fact that it provides us with an inversion formula for those functions $\varphi : (\omega,\infty) \to X$ which are not necessarily a Laplace transform, but such that $\lambda^{-\alpha}\varphi(\lambda)$ is a Laplace transform for some $\alpha > 0$. Important classes of (vector-valued) functions in this situation are to be found among integrated families of operators. In fact, the formula implies inversion theorems for resolvents of generators of integrated semigroups and integrated cosine functions. Moreover, it recovers and extends for integrated semigroups other previously known results in the literature, see [C, VV]:

Let $(T_\alpha(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ be a (strongly continuous) α-times integrated semigroup, with generator A and existing Laplace transform on $\lambda > \omega$ for some $\omega \in \mathbb{R}$. Then

$$R(\lambda,A) := (\lambda - A)^{-1} = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda t} T_\alpha(t) dt\,,\ \lambda > \omega.$$

Applying Theorem 3.1.1, one gets the following Euler's type formula.

Corollary 3.2.1, p. 87: *Assume that $\|T_\alpha(t)\| \leq Ct^{\gamma}e^{\omega t}$, $t \geq 0$, for some $\gamma > \alpha - 1$ and $\omega \geq 0$. Then, for every $t > 0$ and every $x \in X$,*

$$T_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{n+1} R(\frac{n}{s},A)^{n+1}\, x ds.$$

Examples of integrated semigroups satisfying the assumptions of Corollary 3.2.1 can be found in [H].

One can also obtain a similar result for generators and resolvents of integrated cosine families (see Chapter 1 for the definition of such families).

Corollary 3.2.2, p. 88: *Let* $A : D(A) \subseteq X \to X$ *be the generator of an* α*-times integrated cosine function* $(C_\alpha(t))_{t\geq 0}$ *for which there exist constants* $\gamma > \alpha - 1$ *and* $\omega \geq 0$ *satisfying* $\|C_\alpha(t)\| \leq Ct^\gamma e^{\omega t}$ *for* $t \geq 0$*. Then, for every* $x \in X$ *and* $t > 0$*,*

$$C_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!} \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n} \left(\lambda R(\lambda^2, A)\right)\Big|_{\lambda=n/s} x\, ds.$$

Notice that the two latter theorems tell us that the orbits of integrated semigroups and cosine families are obtained as asymptotic limits of orbits of their resolvent functions.

• Stability of n-times integrated semigroups

We now address the question of stability for n-times integrated semigroups with $n \geq 1$. First of all, in the research line initiated by O. El Mennaoui to generalize the Arendt-Batty-Lyubisch-Vũ theorem, we extend [Me, Theorem 5.6] as follows. Let $\rho(A)$ denote the resolvent set of a closed operator A.

Theorem 4.0.1, p. 93: *Let* A *be the generator of a n-times integrated semigroup* $(T_n(t))_{t\geq 0}$ *such that*

- $\sigma(A) \cap i\mathbb{R}$ *is countable,*
- $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$*,*
- $0 \in \rho(A)$*.*

Assume that $\sup_{t\geq 1} \omega(t)^{-1}\|T_n(t)\| < \infty$ *where* ω *is a nonquasianalytic weight on* $[0,\infty)$ *for which* $\tilde{\omega}(t) = O(t^k)$ *as* $t \to \infty$*, for some* $k \geq 0$*.*

We have:

(i) If $\omega(t)^{-1} = o(t^{-n+1})$ *as* $t \to \infty$*, then*

$$\lim_{t\to\infty} \omega(t)^{-1} T_n(t)x = 0, \quad x \in \overline{D(A)}.$$

(ii) If $\omega(t) \sim t^{-n+1}$ *as* $t \to \infty$*, then there exists*

$$\lim_{t\to\infty} t^{-n+1} T_n(t)x = -\frac{1}{(n-1)!} A^{-1}x, \quad x \in \overline{D(A^n)}.$$

Remark. For $n = 1$ Theorem 4.0.1 (ii) is [Me, Theorem 5.6]. So any n-times integrated semigroup $(T_n(t))_{t\geq 0}$ satisfying the latter equality in Theorem 4.0.1 (ii) might well be called *stable*. Similarly, the ergodic type equality $\lim_{t\to\infty} \omega(t)^{-1} T_n(t)x = 0$, $x \in \overline{D(A^n)}$, for $\omega(t) \sim t^n$ at infinity, defines a property on $T_n(t)$ which corresponds to stability of C_0-semigroups when $n = 0$. Then one could say that an integrated semigroup satisfying Theorem 4.0.1 (i) for $\omega(t) \sim t^n$ as $t \to \infty$ is *stable of order n*, and *stable under* ω in general. However, as we have mentioned before, specialists prefer to use the term *ergodicity* (to 0 in the present case) to refer to the existence of limits like $\lim_{t\to\infty} t^{-n} T_n(t)x$.

The proof of Theorem 4.0.1 above relies upon a nontrivial adaptation of arguments of [Me] and [V1]. In this way, we extend in passing some other auxiliary results of [Me] and [V1].

Remark. A fairly nontrivial problem arising from Theorem 4.0.1 is how to remove condition (iii) on the invertibility of the generator A. The arguments considered in the original proofs of the Arendt-Batty-Lyubisch-Vũ theorem do not seem to fit well in the integrated setting. However, there is a proof of this theorem in [ESZ] -relying on the continuous Katznelson-Tzafriri theorem and harmonic analysis properties of suitable Banach algebra homomorphisms- which turned to be useful, and will allow us to obtain some, partial but interesting, result; see Chapter 6. To start with, one establishes a fairly nontrivial extension of the continuous version of the Katznelson-Tzafriri theorem for semigroups in Chapter 5.

• Extension of the Esterle-Strouse-Zouakia-Vũ theorem: Katznelson-Tzafriri theorem for integrated semigroups.

Let $C_c^\infty(\mathbb{R})$ be the space of test functions on $\mathbb{R}$, and let $C_c^\infty(\mathbb{R}^+)$ be the space of test functions $g\chi_{[0,\infty)}$ where g runs over $C_c^\infty(\mathbb{R})$. For $\alpha > 0$ and function $f \in C_c^\infty(\mathbb{R}^+)$ we put $W_+^\alpha f$ to refer to the so-called Weyl derivative of order α on $(0,\infty)$. One defines also the Weyl derivative $W^\alpha g$ on all on $\mathbb{R}$ for every $g \in C_c^\infty(\mathbb{R})$; see definitions in Chapter 1 (Section 1.3) below.

Let $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ denote the Banach spaces obtained as the completions of $C_c^\infty(\mathbb{R}^+)$, $C_c^\infty(\mathbb{R})$ in the norm given by

$$\nu_\alpha(f) := \int_\Omega |W_+^\alpha f(t)|\,|t|^\alpha\,dt, \qquad f \in C_c^\infty(\Omega)$$

where Ω is equal to $[0,\infty)$, $\mathbb{R}$, respectively. These spaces were introduced in [GM], and are in fact Banach algebras for the usual convolution on $\mathbb{R}$. Moreover, $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ is a regular Banach algebra and $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is a closed (non-regular) subalgebra of it.

Let $(T_\alpha(t))_{t\geq 0}$ be an α-times integrated semigroup in $\mathscr{B}(X)$ (see, once again, the definition in Chapter 1, Section 1.2.2) such that $\sup_{t>0} t^{-\alpha}\,\|T_\alpha(t)\| < \infty$. Then the mapping $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ defined by

$$\pi_\alpha(f)x := \int_0^\infty W_+^\alpha f(t)\,T_\alpha(t)x\,dt, \quad x \in X, f \in \mathscr{T}_+^{(\alpha)}(t^\alpha),$$

is a bounded Banach algebra homomorphism. All the above properties can be found in [GM].

We have the following extension of the Esterle-Strouse-Zouakia-Vũ theorem.

Theorem 5.0.1, p. 103: *For $\alpha>0$, let $(T_\alpha(t))_{t\geq 0}$ be an α-integrated semigroup in $\mathscr{B}(X)$ with generator A such that*

$$\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\|<\infty, \qquad \lim_{t\to 0^+}\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x=x, \quad x\in X.$$

Suppose that $f\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is of spectral synthesis in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ with respect to $i\sigma(A)\cap\mathbb{R}$. Then

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)\,\pi_\alpha(f)\|=0.$$

Theorem 5.0.1 is the main result in [GMM1]. Its proof relies on the harmonic analysis of the mapping π_α and some intricated duality techniques involving distribution spaces.

Now, as similarly as it is indicated just after Theorem 5 (p. 13), one could prove that

$$\lim_{t\to\infty} T_\alpha(t)x=0, \quad x\in X,$$

whenever the subspace $Y:=\{\pi_\alpha(f)x : f\in\mathfrak{S}_\alpha, x\in X\}$ were dense in X and always under the conditions

$$\sigma(A)\cap i\mathbb{R} \text{ countable}, \ \sigma_P(A^*)\cap i\mathbb{R}=\emptyset$$

on the generator A. Here, the set $\mathfrak{S}_\alpha$ denotes the space of functions f in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ which are of spectral synthesis in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ with respect to $i\sigma(A)\cap\mathbb{R}$.

In fact, for usual semigroups this is obtained in [ESZ] on the basis of certain auxiliary results about spectral synthesis and related items. It does not seem simple how to establish part of these results in the integrated setting, but, anyway, the corresponding quoted density of Y holds under other quite reasonable conditions, as we show in the next section.

• Spectral synthesis and stability.

The content of this section corresponds to Chapter 6 of the memory and it is splitted up into two parts, the first one devoted to spectral synthesis and the second one dealing with the integrated semigroups.

Here we restrict our results to $\alpha = n$, any nonnegative integer. Let $(T_n(t))_{t\geq 0}$ be a n-times integrated semigroup in $\mathscr{B}(X)$ generated by A. Let $\mathfrak{S}_n$ denote the vector subspace of functions in $\mathscr{T}_+^{(n)}(t^n)$ which are of spectral synthesis in $\mathscr{T}^{(n)}(|t|^n)$ for $S := i\sigma(A) \cap \mathbb{R}$. In the semigroup case, when $n = 0$, one of the key ingredients to show the density of $\pi_n(\mathfrak{S}_n)$ in X is the fact that closed countable subsets of $\mathbb{R}$ are sets of spectral synthesis in $L^1(\mathbb{R})$. Thus one must investigate whether or not this property still holds in the Banach algebra $\mathscr{T}^{(n)}(|t|^n)$ for general n. Indeed, we will obtain that it does not hold in the latter case. Therefore, one must find out what remains valid in this respect.

Weak spectral synthesis in $\mathscr{T}^{(n)}(|t|^n)$

Let S be a closed subset of $\mathbb{R}$. With $\mathscr{F}(f)$ we refer to the Fourier transform of any $f \in \mathscr{T}^{(n)}(|t|^n)$. Such transforms are n-times differentiable in $\mathbb{R} \setminus \{0\}$ and the derivatives satisfy $\lim_{x\to 0} x^j(\mathscr{F}f)^{(j)}(x) = 0$ for $j = 0, 1, \cdots, n$. Set

$$M_k(S) := \{f \in \mathscr{T}^{(n)}(|t|^n) : x^j \mathscr{F}(f)^{(j)}(x) = 0 \ (x \in S; 0 \leq j \leq k)\}$$

for $k = 0, \ldots, n$; $M(S) := M_0(S)$, and

$$J(S) := \{f \in \mathscr{T}^{(n)}(|t|^n) : \mathscr{F}(f) = 0 \text{ on a neighborhood of } S\}.$$

Then by definition the closed subset S of $\mathbb{R}$ is of spectral synthesis if and only if $J(S)$ is dense in $M(S)$.

Theorem 6.2.1, p. 132: *For every $a \in \mathbb{R}$,*

$$M_n(\{a\}) = \overline{J(\{a\})}.$$

At this point it must be noticed that $M_0(0) = M_k(0) = M_n(0)$ for $0 \leq k \leq n$, and therefore $\{0\}$ is a set of spectral synthesis for the Sobolev algebra $\mathscr{T}^{(n)}(|t|^n)$. Moreover, the theorem tells us that $\{0\}$ is the only singleton with such a property.

Now, standard arguments give us the following.

Theorem 6.2.2, p. 134: *For every countable subset S of $\mathbb{R}$,*

$$\overline{J(S)} = M_n(S).$$

As a consequence of these results, we are ready to prove the following ones on integrated semigroups.

Null ergodicity of semigroups

Put $M_{n,+}(S) := M_n(S) \cap \mathscr{T}_+^{(n)}(t^n)$. We say that S is an interpolation set for $\mathscr{T}_+^{(n)}(t^n)$ in $\mathscr{T}^{(n)}(|t|^n)$ if

$$\mathscr{T}_+^{(n)}(t^n)/M_{n,+}(S) = \mathscr{T}^{(n)}(|t|^n)/M_n(S).$$

Theorem 6.3.2, p. 136: *Let $(T_n(t))_{t \geq 0}$ be a n-times integrated semigroup in $\mathscr{B}(X)$ with generator A such that*

$$\sup_{t>0} t^{-n} \|T_n(t)\| < \infty \quad \textit{and} \quad \lim_{t \to 0} n!\, t^{-n} T_n(t)x = x \quad (x \in X).$$

Assume that

(i) $S := i\sigma(A) \cap \mathbb{R}$ *is a countable compact interpolation set for* $\mathscr{T}_+^{(n)}(t^n)$ *in* $\mathscr{T}^{(n)}(|t|^n)$.

(ii) $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.

Then $\pi_n(M_{n,+}(S))X$ *is dense in* X *and, in consequence,*

$$\lim_{t\to\infty} t^{-n} T_n(t)x = 0 \text{ for all } x \in X.$$

Asking for compactness of the set S in the above theorem is not out of all proportion. Indeed, when one is dealing with general statements about (standard) ideals in $L^1(\mathbb{R}^+)$, for example, compactness of the ideal hulls is usually assumed.

QUESTION.- What real subsets are indeed interpolation sets for the subalgebra $\mathscr{T}_+^{(n)}(t^n)$ in $\mathscr{T}^{(n)}(|t|^n)$?

The above question is not simple to settle in all generality, but at least finite subsets provide a positive answer. So we have:

Theorem 6.3.3, p. 140: *Let* $(T_n(t))_{t\geq 0}$ *be a n-times integrated semigroup in* $\mathscr{B}(X)$ *with generator A. Assume that*

(i) $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$ *and* $\lim_{t\to 0} n!\, t^{-n} T_n(t)x = x$ $(x \in X)$,

(ii) $i\sigma(A) \cap \mathbb{R}$ *is finite and* $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.

Then

$$\lim_{t\to\infty} t^{-n} T_n(t)x = 0 \text{ for all } x \in X.$$

The theorem applies to C_0-semigroups $(T(t))_{t\geq 0}$ satisfying (ii) above, which are not uniformly bounded in norm but for which $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$ for some n, where

$$T_n(t) \equiv \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}T(t)\,dt.$$

In this way, we obtain an abstract result on (null) ergodicity of C_0-semigroups.

This memoir is organized as follows. In Chapter 1, we introduce notation and the main concepts we will use along this text. It contains a couple of new results (Propositions 1.3.1 and 1.3.2) of great importance in other chapters. In Chapter 2, we prove the results concerning to estimates of the rate of decay of vector-valued functions (in terms of its Laplace transforms) and their applications on semigroup theory. The contents of this chapter correspond to the ones in [M]. The (vector-valued) Laplace transform is also the key of the results in Chapter 3. The main result here is the Post-Widder type formula for λ^α-multiplied Laplace transforms. As a consequence, we get some inversion formulas for the resolvent of α-times integrated semigroups and cosine families. These results are contained in [GMM]. The last three chapters are mainly devoted to the study of asymptotic properties of α-times integrated semigroups. The main result of Chapter 4 is concerned to *stability* (under some non-quasianalytics weights) of n-times integrated semigroups. The assumptions on the generator of the integrated semigroup are, in particular, invertibility and the countability of its boundary spectrum. In Chapter 5, we give an extension to the setting of α-times integrated semigroups of the Katznelson-Tzafriri theorem for C_0-semigroups. The results in this chapter can be also found in [GMM1]. Finally, in Chapter 6, we carry on a study of primary ideals and the notion of spectral synthesis in the Sobolev algebras $\mathscr{T}^{(n)}(|t|^n)$. This allows us to prove a result for integrated semigroups in the spirit of the Arendt-Batty-Lyubich-Vũ stability theorem.

Chapter 1

Basic concepts and preliminary results

This chapter is devoted to establish notation and introduce the main concepts we will use along the memoir.

From now on, we take X to be a complex Banach space with generic norm $\|\cdot\|$. We denote by $\mathscr{B}(X)$ the Banach algebra of all bounded linear operators on X endowed with the operator norm.

For a closed operator $(A, D(A))$ on X, we denote by $\rho(A)$ the *resolvent set* of A defined by $\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible with inverse in } \mathscr{B}(X)\}$. The *resolvent* of A is the function $R(\cdot, A) : \rho(A) \to \mathscr{B}(X)$ given by $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. The *spectrum* $\sigma(A)$ of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$, and the *point spectrum* $\sigma_p(A)$ by $\sigma_p(A) := \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda - A) \neq \{0\}\}$.

Let A be a densely defined operator on X. Then, its *adjoint operator* A^* on the dual space X^* is the element of $\mathscr{B}(X^*)$ whose domain is given by

$$D(A^*) := \{x^* \in X^* : \exists y^* \in X^* \text{ such that } \langle Ax, x^*\rangle = \langle x, y^*\rangle \text{ for all } x \in D(A)\},$$

and is defined by $A^*x^* := y^*$ for $x^* \in D(A^*)$. See for instance [EN, Appendix B].

Through the text, we shall consider the following function spaces. Here, I is a real interval and X stands for a Banach space.

$C(I,X) := \{f : I \to X : f \text{ is continuous}\}$,
$C_0(I,X) := \{f \in C(I,X) : \lim_{t\to\pm\infty} \|f(t)\| = 0\}$, if $\pm\infty \in I'$,
$C^n(I,X) := \{f \in C(I,X) : f \text{ is } n\text{-times continuously differentiable}\}$, $n \in \mathbb{N}$,
$C^\infty(I,X) := \{f \in C(I,X) : f \text{ is infinitely many times differentiable}\}$,
$C_c^\infty(I,X) := \{f \in C^\infty(I,X) : f \text{ has compact support}\}$,
$\mathscr{S}(\mathbb{R}) :=$ Schwartz space of rapidly decreasing functions,
$\mathscr{S}'(\mathbb{R}) :=$ Space of tempered distributions,
$L^1_{loc}(I;X) := \{f : I \to X : f \text{ is locally Bochner integrable on } I\}$,
$L^p(I;X) := \{f : I \to X : f \text{ is Bochner p-integrable on } I\}$,
$L^\infty(I;X) := \{f : I \to X : f \text{ is measurable and essentially bounded on } I\}$.

Recall that $\|f\|_p := (\int_I \|f(t)\|^p dt)^{\frac{1}{p}}$ whenever $f \in L^p(I;X)$ for $1 \le p < \infty$ and $\|f\|_\infty := ess\,sup_{t\in I}\|f(t)\|$ if $f \in L^\infty(I;X)$. Notice that when $X = \mathbb{C}$, the spaces $L^p(I;\mathbb{C})$ are the usual Lebesgue spaces which we simply denote by $L^p(I)$.

Also, we briefly write $\mathbb{R}^+ := [0,\infty)$, $\mathbb{R}^- := (-\infty,0]$, $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$ and $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z < 0\}$.

1.1 Vector-valued Laplace and Fourier transforms

1.1.1 The Laplace integral

In this paragraph, we introduce the definition and some basic properties of the vector-valued Laplace transform.

For a function $f \in L^1_{loc}(\mathbb{R}^+; X)$, the *Laplace integral* $\mathscr{L}f$ of f is formally defined by

$$\mathscr{L}f(\lambda) = \int_0^\infty f(s)e^{-\lambda s}ds = \lim_{t\to\infty}\int_0^t f(s)e^{-\lambda s}ds, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

where the latter integral is understood in the Bochner sense.

Now, let $abs(f)$ denote the *abscissa of convergence of* $\mathscr{L}f$ given by

$$abs(f) := \inf\{\Re\lambda \,:\, \mathscr{L}f(\lambda)\,\text{exists}\}.$$

It is shown in [ABHN, Theorem 1.4.1] that the Laplace integral $\mathscr{L}f(\lambda)$ converges if $\Re\lambda > abs(f)$ and diverges if $\Re\lambda < abs(f)$, so that the open right half-plane $\{\Re\lambda > abs(f)\}$ is contained in the interior of the domain of convergence of $\mathscr{L}f(\lambda)$. If $\mathscr{L}f$ converges for every $\lambda \in \mathbb{C}$ means that $abs(f) = -\infty$ and if $\mathscr{L}f$ does not converge for any λ then $abs(f) = \infty$.

Definition 1.1.1. *We say that a function $f \in L^1_{loc}(\mathbb{R}^+; X)$ is Laplace transformable if $abs(f) < \infty$ and then the function $\mathscr{L}f : \{\Re\lambda > abs(f)\} \to X$ is called the Laplace transform of f.*

From the definition (1.1) of Laplace integral, it is straightforward to check that any exponentially bounded function $f \in L^1_{loc}(\mathbb{R}^+; X)$ is Laplace transformable. Recall that a function $f : \mathbb{R}^+ \to X$ is exponentially bounded if there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|f(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. The infimum of all exponents ω

for which such a estimation holds is called *exponential growth bound* (or *type*) of the function and it is usually denoted by $\omega(f)$. Hence, $abs(f) \leq \omega(f)$ but the converse is no longer true (see [ABHN, Example 1.4.4.]). As a matter of fact, the abscissa of convergence $abs(f)$ is determined by the exponential growth of the *antiderivative of* f. In [ABHN, Theorem 1.4.3.], it is shown that f is Laplace transformable if and only if its antiderivative $F(t) = \int_0^t f(s)ds$ is exponentially bounded and, moreover,

$$abs(f) = \omega(F - F_\infty)$$

where $F_\infty := \lim_{t\to\infty} F(t)$ if the limit exists and $F_\infty := 0$ otherwise.

Other remarkable fact about the Laplace integral of a given Laplace transformable function $f \in L^1_{loc}(\mathbb{R}^+;X)$ is that $\mathscr{L}f$ is analytic at least for $\Re\lambda > abs(f)$ and for every $n \in \mathbb{N}\cup\{0\}$,

$$(\mathscr{L}f)^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^n f(t)dt\,, \quad \Re\lambda > abs(f).$$

On the other hand, any *Laplace transformable* function $f \in L^1_{loc}(\mathbb{R}^+;X)$ is uniquely determined by its Laplace transform, as the following result shows.

Theorem 1.1.1. [ABHN, Theorem 1.7.3] *Let $f,g \in L^1_{loc}(\mathbb{R}^+;X)$ be Laplace transformable functions. Assume that $\mathscr{L}f(\lambda) = \mathscr{L}g(\lambda)$ for $\Re\lambda$ sufficiently large. Then, $f = g$ a.e.*

Other important result in this setting is the Post-Widder inversion formula (see Theorem 7, p. 17). This formula allows us to retrieve a function $f \in L^1_{loc}(\mathbb{R}^+;X)$ (a.e.) through derivatives of its Laplace transform.

To finish this paragraph, we focus on Laplace transforms of strongly continuous operator-valued functions. In particular, if $T : \mathbb{R}^+ \to \mathscr{B}(X)$ is strongly continuous and exponentially bounded (so Laplace transformable), the Laplace integral of T is defined for every λ such that $\Re\lambda > abs(T)$ by

$$\mathscr{L}T(\lambda) = \int_0^\infty T(s)e^{-\lambda s}ds = \lim_{t\to\infty}\int_0^t T(s)e^{-\lambda s}ds\,, \tag{1.2}$$

where the limit exists in the operator norm. We recall that the evolution families we will deal with along this memoir are under these conditions.

We refer to [ABHN] for a complete overview on this matter. Other specific results about the Laplace transform will be stated when necessary throughout this text.

1.1.2 The Fourier transform

Definition 1.1.2. *For a function $f \in L^1(\mathbb{R};X)$, its Fourier transform $\mathscr{F}(f):\mathbb{R}\to X$ is given by*

$$\mathscr{F}(f)(s) := \int_{-\infty}^{\infty} f(t)e^{-ist}dt\,,\ s\in\mathbb{R}.$$

Next, we collect some well-known properties of the (vector-valued) Fourier transform (see for instance [ABHN], [EN]). For $f \in L^1(\mathbb{R};X)$ and $g \in L^1(\mathbb{R})$ we have:

(i) **Riemann-Lebesgue Lemma**: $\mathscr{F}(f) \in C_0(\mathbb{R};X)$

(ii) $\mathscr{F}(f*g)(s) = \mathscr{F}(f)(s)\mathscr{F}(g)(s)$ for every $s\in\mathbb{R}$.

(iii) $\int_{-\infty}^{\infty} g(t)\mathscr{F}(f)(t)dt = \int_{-\infty}^{\infty}\mathscr{F}(g)(t)f(t)dt$.

(iv) **Inversion Theorem**: If $\mathscr{F}(f) \in L^1(\mathbb{R};X)$ then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{F}(f)(s)e^{ist}ds\,,\ \text{a.e. } t\in\mathbb{R}.$$

(v) **Plancherel's Theorem**: If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\mathscr{F}(g) \in L^2(\mathbb{R})$ and

$$\|\mathscr{F}(g)\|_2 = \sqrt{2\pi}\|g\|_2 .$$

In order to study the Fourier transform of functionals, we first turn our attention to the space of smooth, rapidly decreasing functions, that is, the Schwartz space given by

$$\mathscr{S}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \,:\, \|\varphi\|_{m,\alpha} := \sup_{x\in\mathbb{R}}(1+|x|)^m|\varphi^{(n)}(x)| < \infty, \text{ for all } m,n \in \mathbb{N}_0\},$$

endowed with its usual locally convex topology defined by the family of seminorms $\|\cdot\|_{m,\alpha}$. Recall that the space of test functions $\mathscr{D}(\mathbb{R}) := C_c^\infty(\mathbb{R})$ is densely contained in $\mathscr{S}(\mathbb{R})$. It is a well known fact that the Fourier transform is a continuous, linear and one-to-one mapping of $\mathscr{S}(\mathbb{R})$ onto itself, whose inverse is also continuous. Moreover, if $f, g \in \mathscr{S}(\mathbb{R})$, we have:

(i) $f * g \in \mathscr{S}(\mathbb{R})$ and

(ii) $\mathscr{F}(fg) = \mathscr{F}(f) * \mathscr{F}(g)$.

Let $\mathscr{S}'(\mathbb{R})$ denote the space of all *tempered distributions*, that is, the space of all continuous linear maps from $\mathscr{S}(\mathbb{R})$ to $\mathbb{C}$. The space $\mathscr{S}'(\mathbb{R})$ is naturally embedded in the space of distributions $\mathscr{D}'(\mathbb{R})$. For instance, every distribution with compact support is tempered. On the other hand, every function $f \in L^1(\mathbb{R})$ may be also regarded as a tempered distribution $f \equiv L_f$ by

$$L_f(\varphi) = \langle f, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,dx, \;\; \varphi \in \mathscr{S}(\mathbb{R}).$$

Starting from convolution of two functions, it is natural to define the convolution of a tempered distribution and a function. For functions $\varphi, \phi : \mathbb{R} \to \mathbb{C}$, the convolution product $\varphi * \phi$ is given by

$$(\varphi * \phi)(x) = \int_{\mathbb{R}} \varphi(y)\phi(x-y)\, dy,$$

provided that the integral exists in the Lebesgue sense a.e. $x \in \mathbb{R}$. Now, for a tempered distribution $u \in \mathscr{S}'(\mathbb{R})$ and a function $\varphi \in \mathscr{S}(\mathbb{R})$, their convolution product $u * \varphi$ is the tempered distribution defined as follows:

$$(u * \varphi)(\phi) := u(\tilde{\varphi} * \phi),\ \ \phi \in \mathscr{S}(\mathbb{R}),$$

where $\tilde{\varphi}(x) = \varphi(-x)$. This definition is consistent when functions are identified with distributions. Moreover, for every $u \in \mathscr{S}'(\mathbb{R})$ and $\varphi, \phi \in \mathscr{S}(\mathbb{R})$,

$$(u * \varphi) * \phi = u * (\varphi * \phi). \tag{1.3}$$

Now, for $u \in \mathscr{S}'(\mathbb{R})$ we define its Fourier transform $\mathscr{F}(u)$ as the tempered distribution given by

$$\mathscr{F}(u)(\varphi) := u(\mathscr{F}(\varphi)) \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

This definition is also consistent when a function is identified with a distribution, that is, given a function $f \in L^1(\mathbb{R})$ the Fourier transform of u_f equals to $u_{\mathscr{F}(f)}$.

The formal properties of the Fourier transform on $\mathscr{S}(\mathbb{R})$ are preserved for tempered distributions. In particular, the Fourier transform is a continuous, linear and one-to-one mapping of $\mathscr{S}'(\mathbb{R})$ onto $\mathscr{S}'(\mathbb{R})$. The topology considered here is the weak*-topology that $\mathscr{S}(\mathbb{R})$ induces on $\mathscr{S}'(\mathbb{R})$. Moreover, for every $u \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$, it holds:

(i) $\mathscr{F}(u * \varphi) = \mathscr{F}(\varphi)\mathscr{F}(u)$.

(ii) $\mathscr{F}(u) * \mathscr{F}(\varphi) = \mathscr{F}(\varphi u)$.

Recall that for arbitraries $u \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$, the multiplication φu is also a tempered distribution such that $(\varphi u)(\phi) := u(\varphi\phi)$.

For a fairly complete overview on convolution and the Fourier transform of functionals, we refer to [Ru].

1.2 C_0-semigroups and integrated families

In this section, we present some basic results of the theory of semigroups and other evolution families related with the abstract Cauchy problem.

1.2.1 Strongly continuous semigroups

Definition 1.2.1. *A family $(T(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ is called a C_0-semigroup if it satisfies:*

(i) $T(0) = I$ *(the identity operator on X).*

(ii) $T(t)T(s) = T(t+s)$ *for all* $t,s \geq 0$.

(iii) $\lim_{t\to 0^+} T(t)x = x$ *for every* $x \in X$*, in the norm of* X.

Observe that conditions (ii) and (iii) imply that the semigroup is *strongly continuous*, that is, the orbit map $t \mapsto T(t)x$ is continuous from $\mathbb{R}^+$ into X for every $x \in X$.

An automatic consequence of the strong continuity is that every C_0-semigroup $T \equiv T(t)$ is exponentially bounded, that is, $\omega(T) < \infty$ (see [ABHN] or [EN]). Hence, every C_0-semigroup is Laplace transformable in the sense of Definition 1.1.1.

Definition 1.2.2. *The generator of a C_0-semigroup $(T(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ is defined as* *the operator A on X whose domain is given by*

$$D(A) := \{x \in X : y = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) \; exists\}.$$

In that case, $Ax = y$.

One may define the generator of a given C_0-semigroup $T \equiv (T(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ alternatively as the unique operator A on X whose resolvent is the Laplace transform of the semigroup in the sense that $(\omega(T), \infty) \subset \rho(A)$ and $\hat{T}(\lambda) = R(\lambda, A)$ for $\lambda > \omega(T)$. Hence, the resolvent of A is given by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x\, ds, \; \Re\lambda > \omega(T), x \in X.$$

As a matter of fact, an operator A generates a C_0-semigroup if and only if its resolvent is a Laplace transform, that is, there exists a strongly continuous function $T : \mathbb{R}^+ \to \mathscr{B}(X)$ such that $R(\lambda, A) = \hat{T}(\lambda)$ for $\Re\lambda$ large enough. The functional equation $T(t)T(s) = T(t+s)$ follows from the fact that $R(\cdot, A)$ is a pseudoresolvent, i.e., it verifies the resolvent identity:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A), \quad \lambda, \mu \in \rho(A).$$

This identity together with the injectivity of $R(\cdot, A)$ implies that $T(0) = I$. Moreover, the semigroup T is *non-degenerate*, which means that if $T(t)x = 0$ for all $t \geq 0$ then $x = 0$.

For the generator A of a C_0-semigroup on X, the following properties hold:

(i) A is a closed, densely defined linear operator.

(ii) **Hille-Yosida condition:** There exist constants $M \geq 0$, $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|(\lambda - \omega)^{n+1} R(\lambda, A)^{(n)}/n!\| \leq M \quad (\lambda > \omega, n \in \mathbb{N} \cup 0).$$

(iii) $R(\lambda, A)T(t) = T(t)R(\lambda, A)$ for all $t \geq 0$ and $\lambda \in \rho(A)$.

(iv) If $x \in D(A)$ then $T(t)x \in D(A)$ and $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$ for all $t \geq 0$.

(v) For all $t \geq 0$ and $x \in X$,

$$\int_0^t T(s)x\,ds \in D(A) \;\text{ and }\; T(t)x - x = A\int_0^t T(s)x\,ds$$

(vi) A is bounded if and only if the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous (i.e. $T(\cdot)$ is continuous from $[0,\infty)$ to $\mathscr{B}(X)$ endowed with the norm topology). In that case, $T(t) = e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ for every $t \geq 0$.

The proof of these and other properties can be found in [ABHN, Proposition 3.1.9].

C_0-semigroups and abstract Cauchy problems

As said in the Introduction, given a closed operator A on a Banach space X, the initial value problem

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in X \end{cases} \tag{ACP}$$

is called the *abstract Cauchy problem* associated to the operator A and the initial value x. A *classical solution* of (ACP) is a function $u \in C^1(\mathbb{R}^+, X)$ such that $u(t) \in D(A)$ for every $t \geq 0$ and for which (ACP) holds. The problem is said to be *well-posed* if for every $x \in D(A)$, there exists a unique classical solution u of (ACP).

Notice that the existence of a classical solution yields $x \in D(A)$. Hence, if the initial value x is an arbitrary point in X, a weaker notion of solution should be considered, that of mild solution. A function $u \in C(\mathbb{R}^+, X)$ is called a *mild solution* of (ACP) if for all $t \geq 0$,

$$\int_0^t u(s)ds \in D(A) \quad \text{and} \quad u(t) - x = A\int_0^t u(s)ds\,.$$

This notion arises by integrating the differential equation in (ACP). Analogously, the problem (ACP) is *mildly well-posed* if for every $x \in X$, there exists a unique mild solution of (ACP). Mild and classical solutions only differ by regularity. As a matter of fact, a mild solution u of (ACP) is classical if and only if $u \in C^1(\mathbb{R}^+, X)$.

The following result shows that the existence of unique solution of (ACP) is the same as saying that the operator A generates a C_0-semigroup:

Theorem 1.2.3. ([ABHN, Theorem 3.1.12], [EN, Theorem 6.7]) *Let A be a closed operator on X. Then, the following assertions are equivalent:*

(i) (ACP) is mildly well-posed.

(ii) The operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

(iii) $\rho(A) \neq \emptyset$ and (ACP) is well-posed.

If these assertions hold, the mild solution of (ACP) is given by $u(t) := T(t)x$ for each $x \in X$.

1.2.2 α-Times integrated semigroups

Definition 1.2.4. *Let $\alpha > 0$. An α-times integrated semigroup is a strongly continuous family $(T_\alpha(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ such that $T_\alpha(0) = 0$ and satisfies*

$$\Gamma(\alpha)T_\alpha(t)T_\alpha(s) = \int_t^{t+s}(t+s-r)^{\alpha-1}T_\alpha(r)dr - \int_0^s (t+s-r)^{\alpha-1}T_\alpha(r)dr \quad (1.4)$$

for every $s,t \geq 0$.

Moreover, $(T_\alpha(t))_{t\geq 0}$ is called *non-degenerate* if $T_\alpha(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

As in the case of C_0-semigroups, integrated semigroups are related to their generators by Laplace transform. Let $T_\alpha : [0,\infty) \to \mathscr{B}(X)$ be an α-times integrated semigroup and assume that it is Laplace transformable, that is, $abs(T_\alpha) < \omega$ for some $\omega \geq 0$. In this case, there exists a unique operator A on X satisfying $(\omega,\infty) \subseteq \rho(A)$ and such that

$$R(\lambda,A) := (\lambda - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} T_\alpha(t)dt\,,\ \lambda > \omega.$$

Such an operator A is called the *generator* of $(T_\alpha(t))_{t\geq 0}$. A 0-times integrated semigroup corresponds to the notion of C_0-semigroup.

Hence, whereas the Laplace tranforms of C_0-semigroups are resolvents $R(\lambda,A)$ of operators A, α-times integrated semigroups are those operator-valued functions whose Laplace transforms are of the form $\lambda^{-\alpha}R(\lambda,A)$. This property corresponds to the functional equation (1.4) for T_α.

Moreover, we have for $\alpha \geq 0$:

(i) $R(\lambda,A)T_\alpha(t) = T_\alpha(t)R(\lambda,A)$ for all $t \geq 0$ and $\lambda \in \rho(A)$.

(ii) If $x \in D(A)$ then $T_\alpha(t)x \in D(A)$ and $T_\alpha(t)Ax = AT_\alpha(t)x$ for all $t \geq 0$.

(iii) For every $t \geq 0$ and $x \in X$, $\int_0^t T_\alpha(s)ds \in D(A)$ and

$$A\int_0^t T_\alpha(s)x\,ds = T_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\,.$$

In particular, $T_\alpha(0) = 0\,.$

(iv) For all $t \geq 0$ and $x \in D(A)$,

$$\int_0^t T_\alpha(s)Ax\,ds = T_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x\,.$$

In particular, $T_\alpha(\cdot)$ is differentiable and $\frac{d}{dt}T_\alpha(t)x = T_\alpha(t)Ax + \frac{t^{\alpha-1}}{\Gamma(\alpha)}x\,.$

The following result shows that A generates an integrated semigroup if and only if the associated Cauchy problem (ACP) admits classical solutions for every initial value x belonging to the domain of some power of A.

Theorem 1.2.5. [ABHN] *Let A be a closed operator on X and let $n \in \mathbb{N}$. Then, the following are equivalent:*

(i) A generates an exponentially bounded n-times integrated semigroup.

(ii) $\rho(A) \neq \emptyset$ and for every $x \in D(A^{n+1})$ there exists a unique classical solution of (ACP) which is exponentially bounded.

If these assertions hold, let $(T_n(t))_{t\geq 0}$ denote the n-times integrated semigroup generated by A. Then, for every $x \in D(A^n)$ the (mild) solution of (ACP) is given by $u(t) = \frac{d^n}{dt^n} T_n(t)x,\ t \geq 0$.

See [ABHN], [AK] and [H] for the general theory of integrated semigroups.

1.2.3 Integrated cosine functions

In Chapter 3, we shall deal with α-times integrated cosine functions ($\alpha > 0$). In particular, we obtain an inversion formula of Post-Widder type for the Laplace transform of these functions, up to the factor λ^α.

Definition 1.2.6. *A strongly continuous family $(C_\alpha(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ is an α-times integrated cosine function if $C_\alpha(0) = I$ and*

$$2C_\alpha(t)C_\alpha(s) = \int_t^{t+s} (t+s-r)^{\alpha-1} C_\alpha(r) \frac{dr}{\Gamma(\alpha)} - \int_0^s (t+s-r)^{\alpha-1} C_\alpha(r) \frac{dr}{\Gamma(\alpha)} \quad (1.5)$$
$$+ \int_{t-s}^t (r-t+s)^{\alpha-1} C_\alpha(r) \frac{dr}{\Gamma(\alpha)} + \int_0^s (r+t-s)^{\alpha-1} C_\alpha(r) \frac{dr}{\Gamma(\alpha)}$$

for every $0 < s < t$.

The family $(C_\alpha(t))_{t\geq 0}$ is called *non-degenerate* if $C_\alpha(t)x = 0$ for every $t \geq 0$ implies $x = 0$. Moreover, if the Laplace transform of $C_\alpha(\cdot) : [0,\infty) \to \mathscr{B}(X)$ converges in (ω,∞) for some $\omega \geq 0$, then there exists a unique operator A on X, called the *generator* of $(C_\alpha(t))_{t\geq 0}$, such that

$$\lambda R(\lambda^2,A) := \lambda(\lambda^2 - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} C_\alpha(t)dt\,,\ \lambda > \omega.$$

Hence, generators of α-times integrated cosine functions are those operators for which $\lambda^{1-\alpha}R(\lambda^2,A)$ is a Laplace transform.

A 0-times integrated cosine function is a usual cosine function. Recall that a *cosine function* is a strongly continuous function $Cos : \mathbb{R}^+ \to \mathscr{B}(X)$ such that $Cos(0) = I$ and

$$2Cos(t)Cos(s) = Cos(t+s) + Cos(t-s) \quad (t \geq s \geq 0).$$

Cosine functions are related with second order Cauchy problems, namely problems defined by closed operators A on a Banach space X in the following way:

$$\begin{cases} u''(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in X, \\ u'(0) = y & y \in X. \end{cases}$$

As in the case of integrated semigroups and (ACP), integrated cosine functions appear when weaker notions of well-posedness of the second order Cauchy problem are considered. See [ABHN, Section 3.14.] and references therein for details.

1.3 Convolution Banach algebras of Sobolev type

As said before, there are certain convolution Banach algebras defined by fractional derivation which are well-suited for the study of integrated semigroups. In this section, we give in detail definitions and basic properties of such algebras, as well as the natural relationship with integrated semigroups via a functional calculus of Hille-Phillips type.

For $\alpha > 0$ and $f \in C_c^\infty(\mathbb{R}^+)$, the *Weyl fractional integral of f of order α on $\mathbb{R}^+$* is defined by

$$W_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)\, ds \quad (t \geq 0), \tag{1.6}$$

and then the *Weyl fractional derivative of f on $\mathbb{R}^+$* is defined as

$$W_+^{\alpha} f(t) := (-1)^n \frac{d^n}{dt^n} W_+^{-(n-\alpha)} f(t) \quad (t \geq 0),$$

where $n := [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

The operator $W_+^{-\alpha} : C_c^\infty(\mathbb{R}^+) \to C_c^\infty(\mathbb{R}^+)$ defined by (1.6) is one-to-one (and continuous for the usual topology of $C_c^\infty(\mathbb{R}^+)$) and then the fractional derivative W_+^α can be seen as the inverse operator of $W_+^{-\alpha}$. These operators satisfy the group law $W_+^{\alpha+\beta} = W_+^\alpha W_+^\beta$ for any $\alpha, \beta \in \mathbb{R}$ where W_+^0 is defined to be the identity operator.

Definition 1.3.1. *We define $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ as the completion of $C_c^\infty(\mathbb{R}^+)$ in the norm ν_α given by*

$$\nu_\alpha(f) := \int_0^\infty |W_+^\alpha f(t)| t^\alpha\, dt, \qquad f \in C_c^\infty(\mathbb{R}^+).$$

For $0 < \beta < \alpha$, we have $\mathcal{T}_+^{(\alpha)}(t^\alpha) \hookrightarrow \mathcal{T}_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_+^{(0)}(t^0) \equiv L^1(\mathbb{R}^+)$, where $\hookrightarrow$ means continuous inclusion.

We have in fact that $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a commutative Banach algebra with respect to the convolution product on $\mathbb{R}^+$ given by

$$f * g(x) := \int_0^x f(x-y)\, g(y)\, dy \quad (\text{a. e. } x > 0;\, f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)).$$

These facts can be found in [GM, Prop. 1.4] and [GMR, Prop. 2.3]. Also, the character space of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ coincides with $\{z \in \mathbb{C} : \Re z \geq 0\}$ and the Gelfand transform of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is the Laplace transform [GMR1]. The fractional algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ extend the corresponding Banach algebras introduced in [AK] for integer α.

We further consider a copy of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ on the negative real half-line. Analogously, the Weyl fractional integral and derivative of a function f in $C_c^\infty(\mathbb{R}^-)$ are given by

$$W_-^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (s-t)^{\alpha-1} f(s)\, ds, \quad (t \leq 0),$$

and

$$W_-^{\alpha} f(t) := \frac{d^n}{dt^n} W_-^{-(n-\alpha)} f(t), \quad (t \leq 0),$$

respectively. Let $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ denote the convolution Banach algebra obtained as the completion of $C_c^\infty(\mathbb{R}^-)$ with respect to the norm

$$\nu_\alpha(f) := \int_{-\infty}^{0} |W_-^{\alpha} f(t)|\, (-t)^\alpha\, dt, \qquad f \in C_c^\infty(-\infty, 0].$$

Of course, $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ enjoys properties similar to those of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Now, we introduce the natural extension of these algebras for functions defined in the whole real line:

Definition 1.3.2. *Let $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm*

$$\nu_\alpha(f) := \int_{-\infty}^{\infty} |W^{\alpha} f(t)|\, |t|^\alpha\, dt, \qquad f \in C_c^\infty(\mathbb{R}),$$

where $W^\alpha f := W_-^\alpha f_- + W_+^\alpha f_+$, $f_- = f\chi_{(-\infty,0]}$ and $f_+ = f\chi_{[0,\infty)}$.

As a matter of fact, we have that $\mathcal{T}^{(\alpha)}(|t|^\alpha) = \mathcal{T}_-^{(\alpha)}((-t)^\alpha) \oplus \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and it is $\mathcal{T}^{(n)}(|t|^n)$ is also a convolution Banach algebra, this time for the convolution on all of $\mathbb{R}$, see [GM, Th. 1.8]. Other properties of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ are that its character space is isomorphic to $\mathbb{R}$ and its Gelfand transform is equal to the Fourier transform. Also, $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ is *regular* on $\mathbb{R}$ since it contains the test functions.

Some additional structure of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ as Banach algebras has been recently studied in [GMR] and [GMR1].

Riesz kernels

Next we pay attention to a couple of distinguished families of functions related to the algebras $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathscr{T}^{(\alpha)}(|t|^\alpha)$. These families are canonical examples of integrated semigroups.

For $\beta > 0$ and $t > 0$, the *Riesz kernel* $R_t^{\beta-1}$ on $\mathbb{R}^+$ is defined by

$$R_t^{\beta-1}(s) := \begin{cases} \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} & \text{if} \quad 0 \le s < t, \\ 0 & \text{if} \quad s \ge t. \end{cases}$$

Then $R_t^{\beta-1} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ whenever $\beta > \alpha$ and, though $R_t^{\alpha-1} \notin \mathscr{T}_+^{(\alpha)}(t^\alpha)$, $R_t^{\alpha-1}$ defines a multiplier of the algebra $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ by convolution. This means that for all $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $t > 0$, then $R_t^{\alpha-1} * f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and the inequality

$$\nu_\alpha(R_t^{\alpha-1} * f) \le C_\alpha t^\alpha \nu_\alpha(f) \tag{1.7}$$

holds. This estimate follows from the formula

$$W_+^\alpha(R_t^{\alpha-1} * f)(s) = \frac{\chi_{(t,\infty)}(s)}{\Gamma(\alpha)} \int_{s-t}^{s} (r+t-s)^{\alpha-1} W_+^\alpha f(r)\, dr \tag{1.8}$$

$$- \frac{\chi_{(0,t)}(s)}{\Gamma(\alpha)} \int_{s}^{\infty} (r+t-s)^{\alpha-1} W_+^\alpha f(r)\, dr,$$

valid for all $s > 0$, which is also the base to prove that the mapping

$$\mathbb{R}^+ \to \mathscr{T}_+^{(\alpha)}(t^\alpha),\ t \mapsto R_t^{\alpha-1} * f$$

is norm continuous; see [GM, pp. 17, 34].

Moreover, the convolution product in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ can be expressed in terms of Riesz kernels [GMR, Lemma 4.2]:

$$f * g = \int_0^\infty W_+^\alpha f(t) R_t^{\alpha-1} * g\, dt \qquad f, g \in \mathscr{T}_+^{(\alpha)}(t^\alpha). \tag{1.9}$$

As a consequence of the above integral representation one gets that for a given closed subspace I in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, I *is an ideal of* $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ if and only if $R_t^{\alpha-1} * f \in I$ for all $f \in I$ and $t > 0$ [GMR, Prop. 4.3].

Therefore closed ideals of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ are characterized as those closed subspaces of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ which are invariant under the action by convolution of kernels $R_t^{\alpha-1}$. Thus the family of Riesz kernels plays a similar role to the one that the translation semigroup $(\delta_t)_{t>0}$, formed by the Dirac masses on $\mathbb{R}^+$, has with respect to $L^1(\mathbb{R}^+)$. This fact seems to be of interest because the algebras $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ are not invariant under translations [GMR, p. 5].

In the next result, we show that the family $\Gamma(\alpha+1) t^{-\alpha} R_t^{\alpha-1}$, $t > 0$, is a summability kernel for $\mathscr{T}_+^{(\alpha)}(t^\alpha)$:

Proposition 1.3.1. [GMM1, Proposition 1.1] *Let* $\alpha > 0$. *For all* $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$,

$$\lim_{t \to 0^+} \Gamma(\alpha+1) t^{-\alpha} R_t^{\alpha-1} * f = f.$$

Proof. Let $f \in C_c^{(\infty)}[0,\infty)$ and $t > 0$. By using the formula (1.8) and the equality $\alpha t^{-\alpha} \int_{s-t}^{s} (r+t-s)^{\alpha-1} dr = 1$, we obtain that

$$\begin{aligned}
&\nu_\alpha \left(\Gamma(\alpha+1) t^{-\alpha} (R_t^{\alpha-1} * f) - f\right) \\
&\quad \le \alpha t^{-\alpha} \int_0^t \int_s^\infty (r+t-s)^{\alpha-1} |W_+^\alpha f(r)| s^\alpha dr ds + \int_0^t |W_+^\alpha f(s)| s^\alpha ds \\
&\qquad\qquad + \alpha t^{-\alpha} \int_t^\infty \int_{s-t}^{s} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha dr ds.
\end{aligned}$$

By Fubini's theorem and the dominated convergence theorem, it readily follows that the first integral on the right-hand member of the above inequality converges to zero as $t \to 0^+$. The second integral also tends to 0 as $t \to 0^+$ by a straightforward argument. For the third integral we apply again Fubini's theorem to get

$$\begin{aligned}\alpha t^{-\alpha} \int_t^\infty \int_{s-t}^s (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, drds \\ = \alpha t^{-\alpha} \int_0^t \int_t^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, dsdr \\ + \alpha t^{-\alpha} \int_t^\infty \int_r^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, dsdr.\end{aligned}$$

Now, taking into account that $W_+^\alpha f \in C_c^{(\infty)}[0,\infty)$ and the fact that $\int_t^{r+t}(r+t-s)^{\alpha-1} s^\alpha \, ds \le C_\alpha t^{2\alpha}$ for some $C_\alpha > 0$, we get that

$$t^{-\alpha} \int_0^t \int_t^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha dsdr \le C_\alpha \|W_+^\alpha f\|_\infty t^{\alpha+1}$$

and therefore the integral tends to zero as $t \to 0^+$.

As regards the double integral $\int_t^\infty \int_r^{r+t}$, setting $C := \sup[\mathrm{supp}(W_+^\alpha f)]$, we have

$$\begin{aligned}\alpha t^{-\alpha} \int_t^\infty \int_r^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, dsdr \\ \le \alpha t^{-\alpha} \int_t^C \sup_{s\in(r,r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| \int_r^{r+t} (r+t-s)^{\alpha-1} ds \, (r+t)^\alpha dr \\ = \int_t^C (r+t)^\alpha \sup_{s\in(r,r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| \, dr,\end{aligned}$$

for $0 < t < C$. Since $W_+^\alpha f$ is uniformly continuous, given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < t < \delta$ then

$$\int_t^C (r+t)^\alpha \sup_{s\in(r,r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| dr \le \varepsilon \int_t^C (r+t)^\alpha dr \le \frac{(2C)^{\alpha+1}}{\alpha+1} \varepsilon.$$

Thus putting all the above bounds together, we have shown that

$$\lim_{t\to 0^+}\Gamma(\alpha+1)\,t^{-\alpha}R_t^{\alpha-1}*f=f$$

in the norm ν_α, for every $f\in C_c^\infty[0,\infty)$. To conclude the proof, we only have to apply the density of $C_c^\infty[0,\infty)$ in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and the estimate (1.7). □

As noticed before, we must deal with convolution on the whole real line. Thus a version of Riesz kernels on all of $\mathbb{R}$ is needed, which extends the family of Riesz kernels $R_t^{\alpha-1}$ defined previously for $t>0$. We maintain the same notation for such an extension.

For $\alpha>0$ and $t\in\mathbb{R}$, put

$$R_t^{\alpha-1}(s):=\begin{cases}\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if}\quad 0\le s<t;\\[2mm] \dfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if}\quad t<s\le 0;\\[2mm] 0, & \text{otherwise}.\end{cases}$$

The above family satisfies similar properties to those summarized prior to Proposition 1.3.1; see [GM] and [GMR]. In particular we have:

(i) $R_t^{\alpha-1}*f\in\mathscr{T}^{(\alpha)}(|t|^\alpha)$ for every $t\in\mathbb{R}$ and $f\in\mathscr{T}^{(\alpha)}(|t|^\alpha)$, with

$$\nu_\alpha(R_t^{\alpha-1}*f)\le C_\alpha\,|t|^\alpha\,\nu_\alpha(f).$$

(ii) For any $f,g\in\mathscr{T}^{(\alpha)}(|t|^\alpha)$,

$$f*g=\int_{-\infty}^{\infty}W^\alpha f(t)\,(R_t^{\alpha-1}*g)\,dt,$$

where the integral converges in the norm topology of $\mathscr{T}^{(\alpha)}(|t|^\alpha)$.

Banach algebra homomorphisms and α-times integrated semigroups

The link existing between the above algebras and integrated semigroups relies upon the following fact:

Let us assume that $(T_\alpha(t))_{t\geq 0}$ is an α-times integrated semigroup of homogeneous growth t^α, that is, such that $\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty$. Then the mapping $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ defined by

$$\pi_\alpha(f)x := \int_0^\infty W_+^\alpha f(t)\, T_\alpha(t)x\, dt, \quad x \in X, f \in \mathscr{T}_+^{(\alpha)}(t^\alpha), \tag{1.10}$$

is a bounded Banach algebra homomorphism [Mi].

Under some additional assumption and via the family of Riesz kernels, we can establish the following one-to-one correspondence between α-times integrated semigroups and bounded Banach algebra homomorphisms from $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ to $\mathscr{B}(X)$:

Proposition 1.3.2. [GMM1, Proposition 5.1] *Let X be a Banach space and let $\alpha > 0$. If $T_\alpha(t)$ is an α-times integrated semigroup on a Banach space X satisfying*

$$\|T_\alpha(t)\| \leq Ct^\alpha \quad (t > 0), \tag{1.11}$$

and

$$\lim_{t\to 0} \Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x = x \quad (x \in X), \tag{1.12}$$

then the bounded homomorphism $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ given by

$$\pi_\alpha(f)x = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)x\, dt \quad (x \in X) \tag{1.13}$$

is such that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X.

Conversely, for every bounded homomorphism $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ such that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X, the family defined by

$$T_\alpha(t)x := \pi_\alpha(R_t^{\alpha-1} * g)y,\ (x = \pi_\alpha(g)y \in \mathscr{T}_+^{(\alpha)}(t^\alpha);\ t \geq 0),$$

is an α-times integrated semigroup on $\mathscr{B}(X)$ satisfying (1.11) and (1.12) whose associated homomorphism defined by the integral expression (1.13) is π_α.

Proof. Let $(T_\alpha(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ be an α-times semigroup on X under the assumptions in (1.11) and (1.12). Since the mapping $f \mapsto W_+^\alpha(f)t^\alpha$, $\mathscr{T}_+^{(\alpha)}(t^\alpha) \to L^1(\mathbb{R}^+)$ is an isometric isomorphism of Banach spaces, we can take a sequence $(e_n)_{n\in\mathbb{N}} \subseteq \mathscr{T}_+^{(\alpha)}(t^\alpha)$ such that $\Gamma(\alpha+1)^{-1}W_+^\alpha(e_n)(t)t^\alpha = n\chi_{[0,\frac{1}{n}]}(t)$, $t>0$, for each $n \in \mathbb{N}$. Then, for any $x \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} x-\pi_\alpha(e_n)x &= x-n\int_0^{\frac{1}{n}} \Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)xdt \\ &= n\int_0^{\frac{1}{n}} \big(x-\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x\big)\,dt\,. \end{aligned}$$

From this and the fact that $\lim_{t\to 0}\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x = x$ for every $x\in X$, we conclude that

$$\|x-\pi_\alpha(e_n)x\| \leq \sup_{t\in(0,\frac{1}{n})} \|x-\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x\| \to 0, \text{ as } n\to\infty.$$

This shows that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X.

Conversely, let $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ be an arbitrary bounded homomorphism such that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X. Then X is a Banach (bi-)module on $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ through the action π_α such that $\mathscr{T}_+^{(\alpha)}(t^\alpha)X = X$, by the Cohen's factorization theorem, since $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ has a bounded approximate identity (see [GM]). It means that for every $x\in X$ there exist $g\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $y\in X$ such that $x=\pi_\alpha(g)y$. For a given multiplier μ of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ define

$$\pi_\alpha(\mu)x = \pi_\alpha(\mu * g)y.$$

The expression above does not depend on the decomposition $x=\pi_\alpha(g)y$, it gives rise to a bounded linear operator on X, and makes π_α a bounded algebra homomorphism from the Banach algebra of multipliers of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathfrak{B}(X)$ [E, Proposition 5.2]. In particular one can define $T_\alpha(t) := \pi_\alpha(R_t^{\alpha-1})$, $t \geq 0$, since $R_t^{\alpha-1}$ is a multiplier of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$.

The fact that π_α is a bounded homomorphism and that $R_t^{\alpha-1}$ is moreover an α-times integrated semigroup implies that $(T_\alpha(t))$ is an α-times integrated semigroup, as well. Also, $(T_\alpha(t))$ verifies the growth condition (1.11) because $\|R_t^{\alpha-1}\| \leq Ct^\alpha$ $(t>0)$. Now, for $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $x = \pi_\alpha(g)y \in X$,

$$\begin{aligned}\pi_\alpha(f)\pi_\alpha(g)y = \pi_\alpha(f*g)y &= \pi_\alpha\left(\int_0^\infty W_+^\alpha f(t) R_t^{\alpha-1} * g\, dt\right) y \\ &= \int_0^\infty W_+^\alpha f(t)\pi_\alpha(R_t^{\alpha-1} * g)y\, dt = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)\pi_\alpha(g)y\, dt,\end{aligned}$$

where we have used the representation (1.9) in the second equality and the continuity of π_α in the third one. Hence

$$\pi_\alpha(f)x = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)x\, dt \quad \forall f \in \mathscr{T}_+^{(\alpha)}(t^\alpha), x \in X.$$

Finally notice that

$$\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)\pi_\alpha(f) = \pi_\alpha\left(\Gamma(\alpha+1)t^{-\alpha}R_t^{\alpha-1} * f\right)$$

for any $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $t>0$. Then, by the factorization $X = \pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$, Proposition 1.3.1 and the continuity of π_α we obtain that

$$\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x = x \quad (x \in X).$$

Thus we have shown that $(T_\alpha(t))_{t\geq 0}$ is a C_α-semigroup, and the proof is over. □

The corresponding result for $\alpha = 0$ is well known and can be seen in [ESZ], [Ki] and [CT1] for instance.

Remark 1.3.1. Given any α-times integrated semigroup $(T_\alpha(t))$ on X of homogeneus growth t^α, there always exists a closed subspace X_0 of X such that the family $(T_\alpha(t)\,|_{X_0})$ is an α-times integrated semigroup on $\mathfrak{B}(X_0)$ and verifies condition (1.12) for every $x \in X_0$. In fact, X_0 can be taken as the closure of $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ in X, where $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$ is as in (1.13).

Remark 1.3.2. For α-times integrated cosine functions there is a similar result, with the only difference that the homomorphism π_α has to be replaced by a homomorphism $\gamma_\alpha\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$ with respect to the cosine convolution product $*_c$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ defined for $f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ by

$$f *_c g(t) := \frac{1}{2}\left(f * g(t) + \int_t^\infty f(s-t)g(s)ds + \int_t^\infty g(s-t)f(s)ds\right),\ t > 0.$$

Actually, the family $(R_t^{\alpha-1})_{t\geq 0}$ is an α-times integrated semigroup and an α-times integrated cosine family since it verifies the corresponding functional equations (namely, (1.4) and (1.5) in Chapter 1). Thus it seems reasonable to consider the Riesz kernels as canonical integrated families, in the present setting. For all the above facts we refer the reader to [GM], [GMM1] and [Mi1].

Duality in the Sobolev algebras

We shall also need to consider duality in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. In order to do that, we take into account the following fact:

The operator $W_+^{-\alpha}$ extends as a surjective isometry $W_+^{-\alpha}\colon L^1(t^\alpha) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)$ where $L^1(t^\alpha)$, endowed with the usual norm, is the Banach space of integrable functions on $\mathbb{R}^+$ with respect to the weight t^α; see [GMR, Section 2]. Thus the mapping $W_+^\alpha\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to L^1(t^\alpha)$ is defined as the inverse of $W_+^{-\alpha}\colon L^1(t^\alpha) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)$, so that a function f in $L^1(\mathbb{R}^+)$ belongs to $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if there exists

$F \in L^1(t^\alpha)$ such that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} F(y)\, dy \quad \text{for a. e. } x > 0.$$

On account of the isomorphism between $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $L^1(t^\alpha)$, the dual Banach space $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is automatically identified with the set of almost everywhere defined functions $\phi : \mathbb{R}^+ \to \mathbb{C}$ which satisfy $t^{-\alpha}\phi \in L^\infty(\mathbb{R}^+)$. The duality is implemented by the formula

$$L_\phi(f) \equiv \langle L_\phi\,, f \rangle := \int_0^\infty W_+^\alpha f(t)\phi(t)\, dt, \quad f \in \mathscr{T}_+^{(\alpha)}(t^\alpha),$$

for every $\phi \equiv L_\phi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$.

Moreover, $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ becomes a (dual) Banach $\mathscr{T}_+^{(\alpha)}(t^\alpha)$-module through the action of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ on itself as a Banach algebra. We denote the dual Banach module product by $\bullet$, so that

$$(L_\varphi, f) \mapsto L_\varphi \bullet f,\ \mathscr{T}_+^{(\alpha)}(t^\alpha)^* \times \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$$

is defined by $(L_\varphi \bullet f)(g) := L_\varphi(f * g)$ for all $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$.

Analogous facts hold about duality in $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ and functions ϕ, now supported on $(-\infty, 0)$. Then we shall represent the dual $\mathscr{T}^{(n)}(|t|^n)^*$ as the (topological) direct sum $\mathscr{T}^{(n)}(|t|^n)^* = \mathscr{T}_-^{(\alpha)}((-t)^\alpha)^* \oplus \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, so that the continuous linear functionals of $\mathscr{T}^{(n)}(|t|^n)$ correspond to functions φ, defined almost everywhere on $\mathbb{R}$, such that $|t|^{-\alpha}\varphi \in L^\infty(\mathbb{R})$. An important observation here is that every functional L_φ of $\mathscr{T}^{(n)}(|t|^n)^*$ can be considered as a tempered distribution on $\mathbb{R}$ since the Schwartz class $\mathscr{S}(\mathbb{R})$ is continuously and densely contained in $\mathscr{T}^{(n)}(|t|^n)$ [GMR, Prop. 2.3].

1.4 Spectral synthesis on regular Banach algebras

A commutative semisimple Banach algebra $\mathfrak{A}$ with character space $\mathscr{M}(\mathfrak{A})$ is called *regular* if for every closed subset $E \subset \mathscr{M}(\mathfrak{A})$ and every point $p \notin E$ there exists an element $f \in \mathfrak{A}$ such that $\widehat{f} = 0$ on E and $\widehat{f}(p) \neq 0$, where $\widehat{f}$ is the Gelfand transform of f.

An element f of $\mathfrak{A}$ is said to be *of spectral synthesis* with respect to a closed subset $S \subseteq \mathscr{M}(\mathfrak{A})$ if there exists a sequence (f_n) in $\mathfrak{A}$ such that the Gelfand transform $\widehat{f_n}$ vanishes in a neighbourhood $\mathscr{U}_n$ of S for each n, and

$$\lim_{n\to\infty} \|f - f_n\|_{\mathfrak{A}} = 0.$$

Then a closed subset S of $\mathscr{M}(\mathfrak{A})$ is called a set of spectral synthesis for $\mathfrak{A}$ if every $f \in \mathfrak{A}$ such that $\widehat{f}$ vanishes on S is of spectral synthesis with respect to S; see [K] and [L].

It is said that $\mathfrak{A}$ satisfies the *Ditkin-Wiener* property at $p \in \mathscr{M}(A)$ if $\{p\}$ is of spectral synthesis for $\mathfrak{A}$. In a Banach algebra $\mathfrak{A}$ satisfying the Ditkin-Wiener condition at every point of $\mathscr{M}(A)$ any closed subset of $\mathscr{M}(\mathfrak{A})$ with scattered boundary is a set of spectral synthesis, so is any closed countable subset; see [RS, p. 37].

An example of Banach algebra which satisfies the Ditkin-Wiener property at every point of its character space $\mathbb{R}$ is $L^1(\mathbb{R})$. In particular, every closed set $E \subseteq \mathbb{R}$ with countable boundary is of spectral synthesis (see [N, Theorem 5.4.3]).

Chapter 2

Decay estimates of functions through singular extensions of vector-valued Laplace transforms

In the main result of the present chapter (Theorem 2.1.1), we obtain estimates for the rate of decay of certain slight modification of a given function $f \in L^\infty(\mathbb{R}^+;X)$ in terms of the growth of $\mathscr{L}f$ on the imaginary axis. In more detail we shall assume that $\mathscr{L}f$ has an analytic extension to some region containing $i\mathbb{R}\setminus\{0\}$, where it satisfies *suitable* bounds. Hence, we allow $\mathscr{L}f$ to have a pole at the origin. Then, we estimate the decay of $(e_1 * e_1 - e_1)\circ f$ where $e_1(t) := e^{-t}$ for $t \in \mathbb{R}^+$ and $\circ$ denotes the convolution product defined by

$$g\circ f(t) := \int_t^\infty g(s-t)f(s)\,ds\,,\; t>0\,,\, g\in L^1(\mathbb{R}^+)\,,\, f\in L^\infty(\mathbb{R}^+;X).$$

This result is in the spirit of [BD, Theorem 4.1] and [AB, Proposition 1.1].

The first section is devoted to prove the above-mentioned result (Theorem 2.1.1).

In the second section, Theorem 2.1.1 is applied to C_0-semigroups under the assumption that the boundary spectrum of their infinitesimal generators is empty or consists of the origin. Such semigroups appear in applications to wave equations (see [Bu], [DFMP], [DFMP1] and references therein). We estimate the decay of certain orbits in terms of the norm of the resolvent operator along the imaginary axis (Proposition 2.2.1). Notice that if $(T(t))_{t\geq 0}$ is such a C_0-semigroup and we consider $f(t) = T(t)x$ for any $x \in X$ then

$$(e_1 * e_1 - e_1) \circ f(t) = T(t)A(1-A)^{-2}x, \quad t \geq 0,$$

where A denotes the generator of $(T(t))$. This result is in the spirit of Theorem 6 (see p. 14) mentioned in the Introduction.

Finally, in Theorem 2.2.1 we show that similar estimates can be given for the rate of decay of certain orbits of bounded semigroups having arbitrary finite boundary spectrum. This result extends Proposition 2.2.1 and completes [BD, Proposition 4.3]. The added interest of this result is that such estimates are given in an explicit way in terms of the resolvent operator along the imaginary axis.

The results in this chapter correspond to those of the paper [M].

2.1 The decay rate of functions

Put $e_z(t) := e^{-zt}$ for every $z \in \mathbb{C}$ and $t \in \mathbb{R}^+$. Then $e_z \in L^1(\mathbb{R}^+)$ whenever z belongs to the open right half plane $\mathbb{C}^+$, and $\|e_z\|_1 = 1/\Re z$. The family $(e_z)_{z\in\mathbb{C}}$ verifies the resolvent identity

$$(z-\omega)(e_z * e_\omega) = e_\omega - e_z, \quad z, \omega \in \mathbb{C}.$$

Given $f \in L^\infty(\mathbb{R}^+;X)$ and $g \in L^1(\mathbb{R}^+)$, let $g \circ f \in L^\infty(\mathbb{R}^+;X)$ denote the convolution product given by

$$g \circ f(t) := \int_t^\infty g(s-t) f(s)\, ds\,, \qquad t > 0\,,$$

where the integral is understood in the sense of Bochner. This product is the *adjoint convolution* to the usual one in $L^1(\mathbb{R}^+)$ in the sense that for any $g,h \in L^1(\mathbb{R}^+)$ and $f \in L^\infty(\mathbb{R}^+;X)$ we have

$$\int_0^\infty (g*h)(t) f(t)\, dt = \int_0^\infty h(t)(g \circ f)(t)\, dt\,,.$$

However, this product $\circ$ is neither commutative nor associative. Some of the properties of this convolution that will be used in the sequel are the following (see for instance [Mi2] and [CT1]):

(i) $g \circ (h \circ f) = (g*h) \circ f = h \circ (g \circ f)$ for f,g,h as above.

(ii) $e_z \circ e_\omega = \dfrac{1}{z+\omega} e_\omega$ for every $z, \omega \in \mathbb{C}^+$.

Lemma 2.1.1. *Let* $f \in L^\infty(\mathbb{R}^+;X)$. *Then,*

$$(e_1 - z e_z * e_1) \circ f = e_z \circ (f - e_1 \circ f)\,, \quad z \in \mathbb{C}^+\,. \tag{2.1}$$

Proof. Taking into account that $(z-1)(e_z * e_1) = e_1 - e_z$ for all $z \in \mathbb{C}^+$, it is readily seen that

$$(e_1 - z e_z * e_1) \circ f = e_1 \circ f - z(e_z * e_1) \circ f = e_z \circ f - (e_z * e_1) \circ f\,.$$

Now, the claim follows trivially from the basic properties of $\circ$ mentioned above. □

Lemma 2.1.2. *Let $f \in L^\infty(\mathbb{R}^+;X)$ be such that its Laplace transform $\mathscr{L}f$ extends to an analytic function in some region Ω containing $\mathbb{C}^+$. Then, the function*

$$z \mapsto (e_1 - ze_z * e_1) \circ f(t), \ \mathbb{C}^+ \to X$$

also extends analytically to Ω for every $t > 0$. Moreover, if $z \in \Omega \setminus \{1\}$ and $t > 0$ then

$$(e_1 - ze_z * e_1) \circ f(t) = \frac{1}{1-z}\left(zg_z(t) + e_1 \circ f(t)\right) + \frac{z}{1-z} e^{zt} \mathscr{L}f(z) \tag{2.2}$$

where $g_{(\cdot)}(t)$ denotes the entire function given by

$$g_z(t) := e^{tz} \int_0^t e^{-sz} f(s) ds, \quad z \in \mathbb{C}.$$

Proof. Let $t > 0$ be fixed and $z \in \mathbb{C}^+$. Notice that

$$e_z \circ f(t) = \int_t^\infty e^{-z(s-t)} f(s) ds = e^{zt} \mathscr{L}f(z) - g_z(t).$$

By (2.1) and the expression of $e_z \circ f$ above, we get that

$$(e_1 - ze_z * e_1) \circ f(t) = e^{zt} \mathscr{L}f(z) - g_z(t) - e_z \circ (e_1 \circ f)(t).$$

It is also straightforward from the definition of $\circ$ that

$$e_z \circ (e_1 \circ f)(t) = e^{zt} \int_0^\infty e^{-sz} (e_1 \circ f)(s) ds - e^{zt} \int_0^t e^{-sz} (e_1 \circ f)(s) ds.$$

Considering the integral expression of $e_1 \circ f$ and applying Fubini Theorem in both integrals, we obtain that if $z \neq 1$ then

$$e_z \circ (e_1 \circ f)(t) = \frac{e^{zt}}{1-z} \mathscr{L}f(z) - \frac{1}{1-z} g_z(t) - \frac{1}{1-z} (e_1 \circ f)(t).$$

From this, we get that, for any $z \in \mathbb{C}^+ \setminus \{1\}$,

$$(e_1 - ze_z * e_1) \circ f(t) = \frac{1}{1-z}\,(zg_z(t) + e_1 \circ f(t)) + \frac{z}{1-z} e^{zt} \mathscr{L} f(z).$$

Now, observe that the right-hand side of this equality has an analytic extension to the region $\Omega \setminus \{1\}$. Thus, by the identity theorem for analytic functions, we obtain that the function

$$z \mapsto (e_1 - ze_z * e_1) \circ f(t)\,,\ \mathbb{C}^+ \to X$$

extends analytically to the region Ω and, also, the last equality holds in $\Omega \setminus \{1\}$, which proves (2.2) and concludes the proof. □

In order to state the main result, let us introduce some notation. Given a continuous function $\mu : (0,\infty) \to (0,\infty)$, we will denote

$$\Sigma_\mu := \{z \in \mathbb{C} : \Re z > -\frac{1}{\mu(|\Im z|)}\}\,.$$

In Theorem 2.1.1, we assume that the Laplace transform of a given function admits an analytic extension to a region Σ_μ. This assumption on the Laplace transform is certainly natural and it has been considered in other settings, see for example [CT1, Section 4] and references therein. The outline of the proof of Theorem 2.1.1 is inspired by that of [BD, Theorem 1.5.] mentioned above, which is based on the contour integral method introduced by Newman and Korevaar ([Ne] and [Ko]). We shall apply a suitable adaptation of this technique, similarly to [AFR] and [AB] where some singularities are considered.

Theorem 2.1.1. [M, Theorem 2.3.] *Let X be a Banach space and let $f \in L^\infty(\mathbb{R}^+;X)$. Assume that there exists a continuous function $\mu : (0,\infty) \to (0,\infty)$ verifying the following conditions:*

(i) The Laplace transform $\mathscr{L}f$ *has a holomorphic extension to the region* Σ_μ *and* $\|\mathscr{L}f(z)\| \leq \mu(|\Im z|)$ *throughout* $\Sigma_\mu \cap \mathbb{C}^-$*.*

(ii) μ *is decreasing on* $(0,1]$ *and increasing on* $[1,+\infty)$*.*

Then, there exist positive constants C and T such that

$$\|(e_1 - e_1 * e_1)\circ f(t)\| \leq C\left(m_{log}^{-1}(t/4) + \frac{1}{M_{log}^{-1}(t/4)} + \frac{1}{t}\right) \quad t > T,$$

where m_{log}^{-1} *and* M_{log}^{-1} *denote the inverse functions of* m_{log} *and* M_{log}*, respectively, defined by*

$$M_{log}(\xi) \;:=\; \mu(\xi)\log((1+\mu(\xi))(1+\xi)),\ \xi \geq 1 \tag{2.3}$$

$$m_{log}(\xi) \;:=\; \mu(\xi)\log\left(\frac{1+\mu(\xi)}{\xi}\right),\ 0 < \xi \leq 1\,. \tag{2.4}$$

Proof. Under the assumptions of the statement, notice that the functions $M_{\log}$ and $m_{\log}$ are strictly increasing and decreasing, respectively. We are then allowed to consider the inverse functions $M_{\log}^{-1} : [M_{\log}(1),\infty) \to [1,\infty)$ and $m_{\log}^{-1} : [m_{\log}(1),\infty) \to (0,1]$, which satisfy:

$$\lim_{t\to\infty} m_{\log}^{-1}(t) = 0 = \lim_{t\to\infty} \frac{1}{M_{\log}^{-1}(t)}\,.$$

For $d > 0$, let γ_d^+ and γ_d^- denote the right and left-hand half of the circle $|z| = d$, respectively. Let $t > 0$ and let any $R > 1$ and $0 < r < \frac{1}{2}$. Set $\gamma := \gamma_R^+ \cup \gamma_r^+ \cup \gamma'$ where γ' is a path in $\Sigma_\mu \cap \mathbb{C}^-$, which is to be chosen later, so that γ is closed, rectifiable and homotopic to zero. Hence, Cauchy's Theorem yields

$$(e_1 - e_1 * e_1)\circ f(t) = \frac{N_{R,r}}{2\pi i}\int_\gamma (1+\frac{z^2}{R^2})(1+\frac{r^2}{z^2})\,(e_1 - ze_z * e_1)\circ f(t)\frac{dz}{z-1}$$

where $N_{R,r} := (1+1/R^2)^{-1}(1+r^2)^{-1} \leq 1$. The factors $1+z^2/R^2$ and $1+r^2/z^2$ play an important role in order to get suitable upper estimates of the integral above. We will write C to denote any positive constant, which may change from line to line, depending only on $\|f\|_\infty$ and the function μ.

First, we prove that the norm of $(e_1 - e_1 * e_1) \circ f(t)$ is bounded by

$$r + \frac{1}{R-1} + \| \int_{\gamma'} (1 + \frac{z^2}{R^2})(1 + \frac{r^2}{z^2}) \frac{z}{1-z} e^{zt} \hat{f}(z) \frac{dz}{z-1} \| \tag{2.5}$$

up to some constant $C > 0$.

Note that if $|z| = R$, $|1 + z^2/R^2| = 2|\Re z|/R$ and $|1 + r^2/z^2| \leq 2$. Taking into account the equality in (2.1), we observe that

$$\|(e_1 - ze_z * e_1) \circ f(t)\| \leq 2\|f\|_\infty \|e_z\|_1 \leq C/\Re z$$

whenever $z \in \mathbb{C}^+$. Therefore,

$$\| \int_{\gamma_R^+} (1 + \frac{z^2}{R^2})(1 + \frac{r^2}{z^2}) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \| \leq \frac{C}{R-1}. \tag{2.6}$$

On the other hand, if $|z| = r$ we have $|1 + z^2/R^2| \leq 2$ and $|1 + r^2/z^2| = |1 + z^2/r^2| = 2|\Re z|/r$. In addition, if $z \in \gamma_r^+$,

$$\|(e_1 - ze_z * e_1) \circ f(t)\| \leq \|f\|_\infty \|e_1 - ze_z * e_1\|_1 \leq C \frac{r}{\Re z}$$

and, as a consequence,

$$\| \int_{\gamma_r^+} (1 + \frac{z^2}{R^2})(1 + \frac{r^2}{z^2}) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \| \leq Cr. \tag{2.7}$$

Up to now, we then have that the norm of $(e_1 - e_1 * e_1) \circ f(t)$ is bounded by

$$r + \frac{1}{R-1} + \| \int_{\gamma'} (1 + \frac{z^2}{R^2})(1 + \frac{r^2}{z^2}) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \|,$$

up to some constant $C > 0$.

Now, we consider the expression of $(e_1 - ze_z * e_1) \circ f(t)$ given by (2.2). Thus, the norm of the last integral is bounded by

$$\| \int_{\gamma'} (1+\frac{z^2}{R^2})(1+\frac{r^2}{z^2})\,\frac{1}{1-z}\,(zg_z(t)+e_1\circ f(t))\,\frac{dz}{z-1}\|$$

$$+\|\int_{\gamma'}(1+\frac{z^2}{R^2})(1+\frac{r^2}{z^2})\frac{z}{1-z}e^{zt}\mathscr{L}f(z)\frac{dz}{z-1}\|.$$

Denote for simplicity

$$G_t(z) := (1+\frac{z^2}{R^2})(1+\frac{r^2}{z^2})\,\frac{1}{1-z}\,(zg_z(t)+e_1\circ f(t))\,\frac{1}{z-1}\,,\ z\in\mathbb{C}^-.$$

By Cauchy's Theorem,

$$\int_{\gamma'\cup\gamma_r^-} G_t(z)\,dz = \int_{\gamma_R^-} G_t(z)\,dz$$

since both paths $\gamma'\cup\gamma_r^-$ and γ_R^- go from iR to $-iR$. Next, we see that the integrals of G_t along γ_R^- and γ_r^- are bounded similarly to (2.6) and (2.7), respectively. First, it is easy to check that

$$\frac{|\Re z|}{|z|}\|zg_z(t)+e_1\circ f(t)\|\le C,\quad z\in\mathbb{C}^-.$$

Thus, acting as in (2.6), we get that

$$\|\int_{\gamma_R^-} G_t(z)\,dz\| \le \frac{C}{R-1}$$

and, analogously to (2.7),

$$\|\int_{\gamma_r^-} G_t(z)\,dz\| \le C\,r.$$

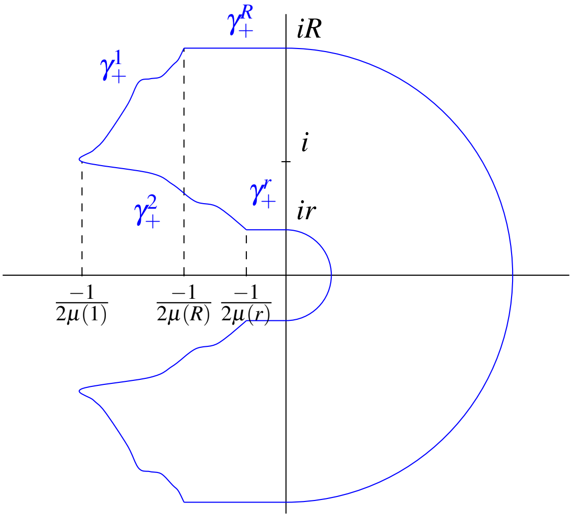
Therefore,

$$\|\int_{\gamma'} G_t(z)\,dz\| \le C\left(r+\frac{1}{R-1}\right).$$

Finally, putting together all the estimates above, we get the estimate in (2.5).

Next, we consider a suitable choice of γ'. In particular, let γ_+ and γ_- to be the two paths in $\Sigma_\mu \cap \mathbb{C}^-$ such that $\gamma' = \gamma_+ \cup \gamma_-$. We take $\gamma_\pm := \gamma_\pm^R \cup \gamma_\pm^1 \cup \gamma_\pm^2 \cup \gamma_\pm^r$ where

$$
\begin{aligned}
\gamma_\pm^R(s) &= s \pm iR, & \frac{-1}{2\mu(R)} \le s \le 0,\\
\gamma_\pm^1(\tau) &= \frac{-1}{2\mu(\tau)} \pm i\tau, & 1 \le \tau \le R,\\
\gamma_\pm^2(\tau) &= \frac{-1}{2\mu(\tau)} \pm i\tau, & r \le \tau \le 1,\\
\gamma_\pm^r(s) &= s \pm ir, & \frac{-1}{2\mu(r)} \le s \le 0.
\end{aligned}
$$



Moreover, we know that

$$
\begin{aligned}
\|\mathscr{L}f(\gamma_\pm^R(s))\| &\le \mu(R), & \frac{-1}{2\mu(R)} \le s < 0,\\
\|\mathscr{L}f(\gamma_\pm^1(\tau))\| &\le \mu(\tau), & 1 \le \tau \le R,\\
\|\mathscr{L}f(\gamma_\pm^2(\tau))\| &\le \mu(\tau), & r \le \tau \le 1,\\
\|\mathscr{L}f(\gamma_\pm^r(s))\| &\le \mu(r), & \frac{-1}{2\mu(r)} \le s < 0.
\end{aligned}
$$

We can consider γ' to be piecewise smooth (if it is not, it can be approximated by smooth paths). In particular, γ' remains rectifiable.

We shall now estimate the norm of the integral in (2.5). The factor $z/(z-1)$ is uniformly bounded along γ'. Straightforward calculations show that both factors $1+r^2/z^2$ and $1+z^2/R^2$ are also bounded on γ' by some constant depending only on the value $\mu(1)$. We do not need sharper bounds on these factors in order to get suitable estimates, except for $\gamma^r_\pm$.

First, the integral in (2.5) over $\gamma^R_\pm$ is bounded by

$$C\int_{-(2\mu(R))^{-1}}^{0} \mu(R)\,e^{st}\,\frac{ds}{|\gamma^R_\pm(s)-1|} \le \frac{C}{2R}\,.$$

On $\gamma^r_\pm$, we have $|1+r^2/z^2| \le 2|\Re z|/|z|$ and $|z-1|\ge 1$, so that

$$\|(1+\frac{r^2}{z^2})\frac{z}{z-1}\mathscr{L}f(z)\| \le 2\frac{|\Re z|}{|z-1|}\|\mathscr{L}f(z)\| \le 2\frac{1}{2\mu(r)}\,\mu(r) = 1.$$

Hence, the integral in (2.5) over $\gamma^r_\pm$ is bounded by

$$C\int_{-(2\mu(r))^{-1}}^{0} e^{st}\,ds \le \frac{C}{t}\,(1-e^{-t/2\mu(r)}) \le \frac{C}{t}\,.$$

On the other hand, the integral over $\gamma^1_\pm$ can be bounded by

$$C\int_1^R \mu(\tau)\,e^{-t/2\mu(\tau)}\,d\tau \le C\,\mu(R)\,(R-1)\,e^{-t/2\mu(R)}$$

$$\le \frac{C}{R-1}\left((1+\mu(R))^2(1+R)^2\,e^{-t/2\mu(R)}\right).$$

Now, we see that if we consider $t > 4M_{\log}(1)$ and we set $R = M_{\log}^{-1}(t/4)$, exactly as in [BD], then $R>1$ and, moreover,

$$(1+\mu(R))^2(1+R)^2 = e^{2M_{\log}(R)/\mu(R)} = e^{t/2\mu(R)}\,.$$

Thus, the norm of the integral over $\gamma_{\pm}^{1}$ is bounded by $C/(R-1)$. Finally, we may bound the integral over $\gamma_{\pm}^{2}$ by

$$C\int_{r}^{1} \mu(\tau)\, e^{-t/2\mu(\tau)}\, \frac{d\tau}{|\gamma_{\pm}^{2}(\tau)-1|} \leq C\,\mu(r)\, e^{-t/2\mu(r)}$$

$$\leq C\, r \left(\frac{(1+\mu(r))^2}{r^2}\, e^{-t/2\mu(r)} \right).$$

Analogously to the preceding case, we notice that if we set $r = m_{\log}^{-1}(t/4)$ then $r < 1/2$ whenever $t > 4m_{\log}(1/2)$. Furthermore,

$$\frac{(1+\mu(r))^2}{r^2} = e^{2m_{\log}(r)/\mu(r)} = e^{t/2\mu(r)},$$

so that the integral over $\gamma_{\pm}^{2}$ is bounded by Cr.

In conclusion, choosing $R = M_{\log}^{-1}(t/4)$ and $r = m_{\log}^{-1}(t/4)$, the claim of Theorem 2.1.1 holds for some $T \geq 4\max\{\, M_{\log}(1)\,,\, m_{\log}(1/2)\,\}$. □

Remark 2.1.1. By looking at the estimates in detail, one realizes that the final constant $C > 0$ appearing in Theorem 2.1.1 is of the form $C = C_{\mu}\, \|f\|_{\infty}$ where $C_{\mu} > 0$ is some constant only depending on μ.

Remark 2.1.2. Assumption (ii) in Theorem 2.1.1 is not strictly necessary in order to get similar decay estimates. In particular, given $f \in L^{\infty}(\mathbb{R}^{+};X)$ and a continuous function $\mu : (0,\infty) \to (0,\infty)$ such that (i) is satisfied, then the function $\nu : (0,\infty) \to (0,\infty)$ defined as

$$\nu(t) := \begin{cases} \sup\limits_{t \leq s \leq 1} \mu(s) & \text{if} \quad 0 < t \leq 1, \\ \sup\limits_{1 \leq s \leq t} \mu(s) & \text{if} \quad t \geq 1 \end{cases}$$

is continuous and verifies both conditions (i) and (ii). The claim of Theorem 2.1.1 then holds for $M_{\log}$ and $m_{\log}$ defined in terms of ν.

To illustrate Theorem 2.1.1, we next consider some particular cases of growth of the Laplace transform near its singularities on the imaginary axis. Some examples for the growth at infinity are considered in [BD] and remain valid in this setting. Indeed, set $M \equiv \mu|_{[1,\infty)}$ and let $\alpha, \beta > 0$:

- If $M(\xi) = \beta e^{\alpha\xi}$ then $M_{\log}^{-1}(t) \sim \frac{1}{\alpha}\log(t), \ \ t \to +\infty$.
- If $M(\xi) = \beta(1+\xi)^{\alpha}$ then $M_{\log}^{-1}(t) \sim C_{\alpha,\beta}\left(\dfrac{t}{\log(t)}\right)^{\frac{1}{\alpha}}, \ \ t \to +\infty$.
- If M is bounded then $M_{\log}^{-1}(t) \sim e^{Ct}$ $(C > 0)$, $\ \ t \to +\infty$.

As examples of growth for $m \equiv \mu|_{(0,1]}$, we can consider the following:

- If $m(\xi) = \beta e^{\alpha\xi^{-1}}$ then $m_{\log}^{-1}(t) \sim \dfrac{\alpha}{\log(t)}, \ \ t \to +\infty$.
- If $m(\xi) = \beta\xi^{-\alpha}$ then $m_{\log}^{-1}(t) \sim C_{\alpha,\beta}\left(\dfrac{\log(t)}{t}\right)^{\frac{1}{\alpha}}, \ \ t \to +\infty$.
- If m is bounded then $m_{\log}^{-1}(t) \sim e^{-Ct}$ $(C > 0)$, $\ \ t \to +\infty$.

2.2 Applications to semigroup theory

2.2.1 One point in the boundary spectrum

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X and let A denote its infinitesimal generator. See definitions and properties in Chapter 1. Assume that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. Then, the function $\tau \mapsto \|(i\tau - A)^{-1}\|$ is continuous on $\mathbb{R} \setminus \{0\}$.

Let $M : [1,\infty) \to \mathbb{R}^+$ be the continuous increasing function defined by

$$M(\xi) := \sup_{1\leq|\tau|\leq\xi} \|(i\tau - A)^{-1}\|, \ \xi \geq 1\,.$$

Consider also the continuous decreasing function given by

$$m(\xi) := \sup_{\xi \le |\tau| \le 1} \|(i\tau - A)^{-1}\|,\ 0 < \xi \le 1\,.$$

Now, we define $\mu : (0,\infty) \to (0,\infty)$ as

$$\mu(\xi) := \begin{cases} 2m(\xi) & \text{if} \quad 0 < \xi \le 1; \\ 2M(\xi) & \text{if} \quad \xi \ge 1\,. \end{cases} \tag{2.8}$$

Theorem 2.2.1. [M, Proposition 3.1] *Let $(T(t))_{t\ge 0}$ be a bounded C_0-semigroup on a Banach space X and let A be its generator. Assume that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. Let $\mu : (0,\infty) \to (0,\infty)$ be the continuous function defined as in (2.8) above and let M_{log} and m_{log} be defined in terms of μ as in (2.3) and (2.4). Then, for any $k \in \mathbb{N}$ there exist positive constants C_k and T_k such that for all $t > T_k$,*

$$\left\|T(t)A^k(1-A)^{-2k}\right\| \le C_k \left(m_{log}^{-1}(t/4k) + \frac{1}{M_{log}^{-1}(t/4k)} + \frac{k}{t} \right)^k \tag{2.9}$$

where M_{log}^{-1} and m_{log}^{-1} denote the inverse functions of M_{log} and m_{log}, respectively.

Proof. Since $\left\|T(t)A^k(1-A)^{-2k}\right\| \le \left\|T(t/k)A(1-A)^{-2}\right\|^k, t \ge 0$, it suffices to prove (2.9) for $k = 1$. Let $x \in X$ such that $\|x\| = 1$ and set $f(t) := T(t)x$ for $t \ge 0$. Since $(1-A)^{-1} - I = A(1-A)^{-1}$, we have that

$$T(t)A(1-A)^{-2}x = T(t)\left((1-A)^{-2}x - (1-A)^{-1}x\right) =$$

$$= T(t)\int_0^\infty (e_1 * e_1 - e_1)(s)T(s)x\,ds = \int_0^\infty (e_1 * e_1 - e_1)(s)T(t+s)x\,ds$$
$$= \int_t^\infty (e_1 * e_1 - e_1)(s-t)f(s)\,ds = -(e_1 - e_1 * e_1) \circ f(t).$$

Now, the claim follows from Theorem 2.1.1 and Remark 2.1.1 just by checking that μ satisfies the assumptions in this theorem. It is clear from the definition that μ

is decreasing on $(0,1]$ and increasing on $[1,+\infty)$, as well as that μ is continuous. On the other hand, we know that $i\mathbb{R}\setminus\{0\}\subseteq\rho(A)$ so that we may extend the resolvent operator into the left-half plane by means of standard Neumann series. In particular, if $z\in\Sigma_\mu\cap\mathbb{C}^-$ then $i\Im z\in\rho(A)$ and

$$|i\Im z-z|=|\Re z|\leq\frac{1}{\mu(|\Im z|)}\leq\frac{1}{2\|(i\Im z-A)^{-1}\|}$$

so that $z\in\rho(A)$ and

$$(z-A)^{-1}=\sum_{n=0}^{\infty}(i\Im z-A)^{-(n+1)}(i\Im z-z)^n.$$

Now, it follows easily that $\|(z-A)^{-1}x\|\leq 2\|(i\Im z-A)^{-1}x\|\leq\mu(|\Im z|)$. □

Remark 2.2.1. Under the assumptions of Theorem 2.2.1, the operator $A(1-A)^{-2}$ is a particular case of the functional calculus given by

$$\pi(h)x:=\int_0^\infty h(t)\,T(t)x\,\mathrm{d}t\,,\qquad x\in X,\,h\in L^1(\mathbb{R}^+). \tag{2.10}$$

Indeed, $A(1-A)^{-2}=\pi(e_1*e_1-e_1)$. Moreover, observe that the function $e_1*e_1-e_1\in L^1(\mathbb{R}^+)$ is of spectral synthesis with respect to $(i\sigma(A))\cap\mathbb{R}$ since its Fourier transform vanishes at $\{0\}$. Therefore, the Katznelson-Tzafriri type theorem for C_0-semigroups (see [ESZ] and [V]) yields

$$\lim_{t\to\infty}\|T(t)A^k(1-A)^{-2k}\|=0\,,\ \text{ for all } k\in\mathbb{N}.$$

Definitions and details about sets and functions of spectral synthesis can be seen in Chapter 1 and references therein.

Corollary 2.2.1. *Let $(T(t))_{t\geq0}$ be a bounded C_0-semigroup on a Banach space X generated by an operator B such that $\sigma(B)\cap i\mathbb{R}\subseteq\{ip\}$ for some $p\in\mathbb{R}$. Let μ :*

$(0,\infty) \to (0,\infty)$ *be defined in terms of* $A := B - ip$ *as in (2.8). Then, there are constants* $C_k, T_k > 0$ *such that for all* $t \geq T_k$,

$$\left\|T(t)(ip-B)^k(1+ip-B)^{-2k}\right\| \leq C_k \left(m_{log}^{-1}(t/4k) + \frac{1}{M_{log}^{-1}(t/4k)} + \frac{k}{t} \right)^k$$

where M_{log} *and* m_{log} *are given by (2.3) and (2.4), respectively.*

Proof. The claim follows immediately from Theorem 2.2.1 by noticing that the semigroup $(e^{-ipt}T(t))_{t\geq 0}$ is a bounded C_0-semigroup generated by $B - ip$ such that $\sigma(B-ip) \cap i\mathbb{R} \subseteq \{0\}$. □

2.2.2 Arbitrary finite boundary spectrum

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X. Assume that the boundary spectrum of its generator A is at most finite, that is, $\sigma(A) \cap i\mathbb{R} \subseteq \bigcup_{j=1}^n \{ip_j\}$ for some $n \in \mathbb{N}$ and some $\bigcup_{j=1}^n \{p_j\} \subseteq \mathbb{R}$.

Under these hypothesis, the bounded operator

$$(1-A)^{-1} \prod_{j=1}^n \left((ip_j - A)(1-A)^{-1}\right) \tag{2.11}$$

is a particular case of the functional calculus given by (2.10). Indeed, let

$$g(x) := \frac{1}{1+ix} \prod_{j=1}^n \frac{ip_j + ix}{1+ix}, \quad x \in \mathbb{R}.$$

Observe that $g \in C_0(\mathbb{R})$ and vanishes on $i\sigma(A) \cap \mathbb{R}$. Note also that

$$\frac{ip_j + ix}{1+ix} = 1 - \frac{1}{1+ix} + \frac{ip_j}{1+ix} = (\delta - (1-ip_j)u)\hat{}(x), \quad \forall x \in \mathbb{R},$$

where δ denotes the Dirac measure at the origin and ˆ denotes the Fourier transform. Then, the function

$$f := u * \prod_{j=1}^n (\delta - (1-ip_j)u) \in L^1(\mathbb{R}^+)$$

is such that $\hat{f} = g$. From this and the fact that finite sets of $\mathbb{R}$ are of spectral synthesis, it follows that f is of spectral synthesis with respect to $i\sigma(A) \cap \mathbb{R}$. Furthermore, it is easy to check that the operator in (2.11) is equal to $\pi(f)$. Hence, it follows from the continuous version of the Katznelson-Tzafriri Theorem above mentioned that

$$\lim_{t\to+\infty} \|T(t)(1-A)^{-1}\prod_{j=1}^{n}\left((ip_j - A)(1-A)^{-1}\right)\| = 0.$$

In the following Theorem 2.2.1, we estimate this decay in terms of certain functions M and m, similar to those considered in the last case, defined by means of the resolvent operator along the imaginary axis when avoiding the possible singularities ip_j.

In order to define these functions, assume without loss of generality that $n \geq 2$ and $p_l < p_j$ whenever $1 \leq l < j \leq n$. Let

$$d := \min_{1\leq j<n}\{\frac{p_{j+1}-p_j}{2}\} \text{ and } D \geq \max\{|p_n + d|, |p_1 - d|\}.$$

Let $K := [-D, D] \setminus \bigcup_{j=1}^{n}(p_j - d, p_j + d)$ and denote

$$m_K := \sup_{\tau\in K}\|(i\tau - A)^{-1}\| < \infty.$$

Now, let M be the continuous positive increasing function given by

$$M(\xi) := \sup_{D\leq|\tau|\leq\xi}\{\|(i\tau - A)^{-1}\|, m_K\}, \;\; \xi \geq D.$$

Also, for each $j = 1, \dots, n$, we define

$$m_j(\xi) := \sup_{\xi\leq|\tau-p_j|\leq d}\{\|(i\tau - A)^{-1}\|, m_K\}, \;\; 0 < \xi \leq d,$$

and set

$$m(\xi) := \sup_{1\leq j\leq n} m_j(\xi), \;\; 0 < \xi \leq d,$$

which is continuous, positive and decreasing. Moreover, we define

$$M_{\log}(\xi) := M(\xi)\log\left((1+M(\xi))(1+\xi)\right), \ \xi \geq D, \tag{2.12}$$

and

$$m_{\log}(\xi) := m(\xi)\log\left(\frac{1+m(\xi)}{\xi}\right), \ 0 < \xi \leq d. \tag{2.13}$$

Theorem 2.2.1. *Let $(T(t))_{t\geq 0}$ be a bounded (C_0)-semigroup on a Banach space X and let A be its generator. Assume that $\sigma(A)\cap i\mathbb{R} \subseteq \bigcup_{j=1}^{n}\{ip_j\}$ for some $n \in \mathbb{N}$ and $\{p_j : 1 \leq j \leq n\} \subseteq \mathbb{R}$. Then, there exist positive constants C and T such that for all $t \geq T$,*

$$\|T(t)(1-A)^{-1}\prod_{j=1}^{n}(ip_j - A)(1-A)^{-1}\| \leq C\left(m_{log}^{-1}(t/4) + \frac{1}{M_{log}^{-1}(t/4)} + \frac{1}{t}\right)$$

where M_{log} and m_{log} are defined as in (2.12) and (2.13), respectively.

Proof. We shall follow an argument analogous to that of Theorem 2.1.1 so we just present a sketch of the proof. We will write C to denote a positive constant, which may change from line to line, depending only on n, $\tilde{C} := \sup_{t\geq 0}\|T(t)\|$, the functions M and m and the absolute values of $\{p_j\}$.

Let $t > 0$ and let R and r be any positive constants such that

$$R > \max\{1, D\} \ \text{ and } \ r < \min\{\frac{1}{2}, d\}.$$

For $p \in \mathbb{R}$ and $d > 0$, let $\gamma_{p,d}^{+}$ and $\gamma_{p,d}^{-}$ denote the right and left-hand half of the circle $|z - ip| = d$, respectively. Let γ' be any path in $\rho(A)\cap\mathbb{C}^{-}$ such that $\gamma := \gamma_{0,R}^{+} \cup (\bigcup_{j=1}^{n}\gamma_{p_j,r}^{+}) \cup \gamma'$ is closed, rectifiable and homotopic to zero. Thus, by Cauchy's Theorem,

$$T(t)(1-A)^{-1}\Pi_A = \frac{1}{2\pi i}\int_{\gamma}\frac{F(z)}{F(1)}T(t)(z-A)^{-1}\Pi_A\frac{dz}{z-1}$$

where F is the function given for every $z \in \mathbb{C} \setminus \{0\} \cup (\sigma(A) \cap i\mathbb{R})$ by

$$F(z) := (1 + \frac{z^2}{R^2}) \prod_{j=1}^{n} (1 + \frac{r^2}{(z - ip_j)^2})$$

and $\Pi_A := \prod_{j=1}^{n} (ip_j - A)(1-A)^{-1}$.

As a consequence of the choice of R and r and after straightforward calculations, we observe that

$$|F(z)| \leq \frac{\tilde{C}|\Re z|}{R} \text{ if } |z| = R \text{ and } |F(z)| \leq \frac{\tilde{C}|\Re z|}{r} \text{ if } |z - ip_j| = r.$$

Thus, acting as in the first part of the proof of Theorem 2.1.1, it is not difficult to see that $\|T(t)(1-A)^{-1}\Pi_A\|$ is bounded by

$$\frac{1}{R-1} + r + \| \int_{\gamma'} F(z) e^{zt} (z-A)^{-1} \Pi_A \frac{dz}{z-1} \| \tag{2.14}$$

up to a positive constant C.

To estimate the latest integral, we denote $\gamma' = \eta \cup (\bigcup_{j=1}^{n} \eta_j) \cup (\bigcup_{j=0}^{n} \alpha_j)$ where $\eta := \eta_-^R \cup \eta_- \cup \eta_+ \cup \eta_+^R$ and $\eta_j := \eta_{j,-} \cup \eta_{j,-}^r \cup \eta_{j,+}^r \cup \eta_{j,+}$ may be taken by means of Neumann series as follows:

$$\begin{aligned} \eta_\pm^R(s) &= s \pm iR, & \frac{-1}{2M(R)} \leq s \leq 0, \\ \eta_\pm(\tau) &= \frac{-1}{2M(\tau)} \pm i\tau, & D \leq \tau \leq R. \end{aligned}$$

For each $j = 1, \ldots, n$,

$$\begin{aligned} \eta_{j,\pm}^r(s) &= s + i(p_j \pm r), & \frac{-1}{2m(r)} \leq s \leq 0, \\ \eta_{j,+}(\tau) &= \frac{-1}{2m(\tau - p_j)} + i\tau, & p_j + r \leq \tau \leq p_j + d, \\ \eta_{j,-}(\tau) &= \frac{-1}{2m(|\tau - p_j|)} + i\tau, & p_j - d \leq \tau \leq p_j - r, \end{aligned}$$

and

$$\begin{aligned}\alpha_j &= [-1/2m(d)+i(p_j+d)\,,\,-1/2m(d)+i(p_{j+1}-d)]\,,\; j=1,\ldots,n-1\,,\\ \alpha_n &= [-1/2m(d)+i(p_n+d)\,,\,-1/2M(D)+iD]\,,\\ \alpha_0 &= [-1/2M(D)-iD\,,\,-1/2m(d)+i(p_1-d)]\,,\end{aligned}$$

where $[z,w]$ denotes the closed line segment connecting z and w. Moreover,

$$\begin{aligned}\|(z-A)^{-1}\| &\le 2M(R)\,, &&\text{if} \quad z\in\eta\,,\\ \|(z-A)^{-1}\| &\le 2m(r)\,, &&\text{if} \quad z\in\eta_j\,,\\ \|(z-A)^{-1}\| &\le 2m_K\,, &&\text{if} \quad z\in\alpha_j\,.\end{aligned}$$

One can easily check that the function F is bounded by a constant C on the whole path γ'. This bound is sharp enough to our purposes except for the paths $\eta^r_{j,\pm}$, $j=1,\ldots,n$.

It is straightforward to see that the integral in (2.14) over $\eta^R_\pm$ is bounded by C/R.

Moreover, proceeding as in (2.1), the integral over $\eta_\pm$ may be estimated by

$$\frac{C}{R-1}\left((1+M(R))^2(1+R)^2e^{-t/2M(R)}\right).$$

By the equality $(ip_j-A)(z-A)^{-1}=I-(z-ip_j)(z-A)^{-1}$ and noticing that $|F(z)|\le C|\Re z|/|z-ip_j|$ whenever $z\in\eta^r_{j,\pm}$, we obtain that

$$|F(z)|\|(ip_j-A)(z-A)^{-1}\|\le C \quad \text{on} \;\; \eta^r_{j,\pm}.$$

so that the integral in (2.14) over $\eta^r_{j,\pm}$ is bounded by

$$C\int_{-1/2m(r)}^{0} e^{st}\,ds \le \frac{C}{t}\,.$$

Acting as in (2.1), the integral over $\eta_{j,\pm}$ can be estimated by

$$C\,r\left(\frac{1+m(r)}{r^2}\,e^{-t/2m(r)}\right).$$

On the other hand, for $j=1,\dots,n-1$, we can bound the integral in (2.14) over α_j by

$$C\,e^{-t/2m(d)}\le C\,e^{-t/2m(r)}.$$

For $j\in\{0,n\}$, observe that $\Re z\le\max\{-1/2m(d)\,,\,-1/2M(D)\}$ whenever $z\in\alpha_j$. If $M(D)\le m(d)$ then the integral over α_j may be bounded similarly as above by $C\,e^{-t/2m(r)}$. Conversely, if $m(d)\le M(D)$, the integral over α_j may be estimated by $C\,e^{-t/2M(R)}$.

Now, we realize that if

$$t>\max\{\,4M_{\log}(\max\{1,D\})\,,\,2m_{\log}(\min\{1/2,d\})\,\}$$

and setting $R=M_{\log}^{-1}(t/4)$ and $r=m_{\log}^{-1}(t/2)$, then

$$(1+M(R))^2(1+R)^2\,e^{-t/2M(R)}\,=\,1\,=\,\left(\frac{1+m(r)}{r^2}\right)\,e^{-t/2m(r)}.$$

In particular, $e^{-t/2m(r)}\le r$ and $e^{-t/2M(R)}\le 1/R$.

Finally, putting together all estimations above, we obtain the claim. $\square$

Chapter 3

Integrated version of the Post-Widder inversion formula for Laplace transforms

Let X be a Banach space. It is a well known fact that every *Laplace transformable* function $f \in L^1_{loc}(\mathbb{R}^+;X)$ is determined by its Laplace transform. As a matter of fact, if $\mathscr{L}f(\lambda)$ converges for some $\lambda \in \mathbb{C}$ then

$$f(t) = \lim_{n\to\infty} (-1)^n \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} (\mathscr{L}f)^{(n)} \left(\frac{n}{t}\right) \tag{3.1}$$

for every Lebesgue point $t > 0$ of f. See [ABHN, Theorem 1.7.7]. Recall that $t > 0$ is a Lebesgue point of a function $f \in L^1_{loc}([0,\infty);X)$ if

$$\lim_{h\to 0} \int_t^{t+h} \|f(s) - f(t)\| ds = 0.$$

Every point of continuity is a Lebesgue point of f and almost all points are Lebesgue points of f (see [ABHN, p. 16]).

The main result of this chapter, see Theorem 3.1.1 below, is an integrated Post-Widder formula for λ^{α}-multiplied Laplace transforms (and Laplace-Stieltjes transforms) of vector-valued functions. This theorem allows us to obtain inversion formulae for resolvents of generators of (α-times) integrated semigroups and integrated cosine families of operators. Such formulae in particular recover and extend for α-times integrated semigroups other previously known results in the literature, see [C, VV]. The chapter ends with a discussion about the canonical example of integrated family formed by the so-called Riesz kernels. Let us recall that the results in this chapter can also be found in [GMM].

3.1 The main result

Let $f:(0,\infty)\to X$ be a measurable function such that

$$\sup_{t>0}\|t^{-\gamma}e^{-\omega t}f(t)\| = M < \infty$$

for some $\gamma > -1$ and $\omega \geq 0$. Clearly, the Laplace transform $\mathscr{L}f$ exists at least on the open right half-plane $\Re\lambda > \omega$. The following is the main result of the chapter.

Theorem 3.1.1. *Let γ, ω and f be as above. Then, for every $\alpha \in (0,\gamma+1)$ and every Lebesgue point $t>0$ of f,*

$$f(t) = \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda^{\alpha}\mathscr{L}f)\Big|_{\lambda=n/s}\,ds.$$

In the next lemma it is shown that the conditions on f and on α ensure that the Post-Widder approximant

$$L_{n,s}[\lambda^{\alpha}\mathscr{L}f(\lambda)] := \frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda^{\alpha}\mathscr{L}f)\Big|_{\lambda=n/s}\quad (s>0)$$

is Bochner integrable near the origin for n sufficiently large, so that the integral in Theorem 3.1.1 is actually convergent.

Lemma 3.1.1. *Let $f\colon (0,\infty) \to X$, γ, ω and α be as in the assumptions of Theorem 3.1.1. Then the function $L_{n,(\cdot)}[\lambda^{\alpha}\mathscr{L}f(\lambda)]$ is Bochner integrable in $(0,t)$ for every $t > 0$ and for every $n > \omega t$.*

Proof. First of all, notice that, due to the growth conditions on f, the integral $\int_0^\infty f(u)u^k e^{-\lambda u}du$ is Bochner convergent for every $\lambda > \omega$ and $k \geq 0$. Now, take $t > 0$ and $n > \omega t$. Hence if $s \in (0,t)$ then $n > \omega s$, so we get

$$L_{n,s}[\lambda^{\alpha}\mathscr{L}f(\lambda)] = \frac{(-1)^n}{n!}\sum_{k=0}^{n} C_{k,n}^{\alpha}\left(\frac{n}{s}\right)^{\alpha+1+k}\int_0^\infty f(u)u^k e^{-(n/s)u}du,$$

where $C_{k,n}^{\alpha} := (-1)^k\binom{n}{k}(n-k)!\binom{\alpha}{n-k}$ for $k = 0,\ldots,n$. Then note that, for some $M > 0$,

$$\begin{aligned}\left(\frac{n}{s}\right)^{\alpha+1+k}&\int_0^\infty |f(u)|u^k e^{-(n/s)u}du\\ &\leq M\left(\frac{n}{s}\right)^{\alpha+1+k}\int_0^\infty u^{\gamma+k}e^{-((n/s)-\omega)u}du\\ &= M\frac{(n/s)^{\alpha+1+k}}{((n/s)-\omega)^{\gamma+1+k}}\Gamma(\gamma+k+1)\\ &\leq M\Gamma(\gamma+k+1)n^{\alpha+k+1}s^{\gamma-\alpha} \quad (k=0,\ldots,n),\end{aligned}$$

provided that $\gamma > -1$. Therefore, the function $L_{n,s}[\lambda^{\alpha}\mathscr{L}f(\lambda)]$ is integrable in $(0,t)$ whenever $\alpha \in (0,\gamma+1)$. □

Remark 3.1.1. In order to ensure the Bochner integrability of $L_{n,(\cdot)}[\lambda^{\alpha}\mathscr{L}f(\lambda)]$ near the origin, it is enough to assume that the given function f is in $L^1_{loc}(\mathbb{R}^+;X)$, it is Laplace transformable, and its Laplace transform $\mathscr{L}f$ satisfies

$$\int_R^\infty \lambda^{\alpha+k+1}(\mathscr{L}f)^{(k)}(\lambda)d\lambda < \infty \text{ for every } k \in \mathbb{N} \text{ and } R > 0.$$

Under these weaker assumptions, the inversion formula in Theorem 3.1.1 also holds.

Proof of Theorem 3.1.1. Let $t>0$ be a Lebesgue point of f. Denote

$$\mathscr{I}_n(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_{n,s}\left[\lambda^\alpha \mathscr{L} f(\lambda)\right] ds.$$

The convergence of this integral for $n>\omega t$ follows from Lemma 3.1.1. As before, write

$$L_{n,s}\left[\lambda^\alpha \mathscr{L} f(\lambda)\right] = \frac{(-1)^n}{n!} \sum_{k=0}^{n} C_{k,n}^{\alpha} \left(\frac{n}{s}\right)^{\alpha+1+k} \int_0^\infty f(u) u^k e^{-(n/s)u} du$$

for $s\in(0,t)$ and $n>\omega t$. Using Fubini's Theorem we get

$$\mathscr{I}_n(t) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} C_{k,n}^{\alpha} \int_0^\infty u^k f(u)\, \mathscr{K}_n(u)\, du$$

where

$$\mathscr{K}_n(u) := \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{\alpha+k+1} e^{-(n/s)u} ds \quad (u>0).$$

Making the change of variable $z=(n/s)u-(n/t)u$, we obtain

$$\begin{aligned} \mathscr{K}_n(u) &= \frac{ne^{-(n/t)u}}{u^{\alpha+k}} t^{\alpha-k-1} \int_0^\infty z^{\alpha-1}(tz+nu)^k e^{-z} dz \\ &= \frac{ne^{-(n/t)u}}{u^{\alpha+k}} t^{\alpha-1} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{nu}{t}\right)^{k-i} \Gamma(\alpha+j). \end{aligned}$$

Then

$$\mathscr{I}_n(t) = \frac{(-1)^n}{n!} n t^{\alpha-1} \int_0^\infty u^{-\alpha} f(u) e^{-(n/t)u} \Phi_{n,t,\alpha}(u)\, du$$

where, for $u>0$,

$$\Phi_{n,t,\alpha}(u) := \sum_{k=0}^{n} C_{k,n}^{\alpha} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{nu}{t}\right)^{k-j} \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} = (-1)^n \left(\frac{nu}{t}\right)^n;$$

see [VV, Lemma 3.1] for the general formula. Hence we get

$$\mathscr{I}_n(t) = \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty u^{n-\alpha} e^{-(n/t)u} f(u) du. \tag{3.2}$$

Notice that for every non-negative integer $n > \alpha + 1$,

$$\frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty u^{n-\alpha} e^{-(n/t)u} du = \frac{n^\alpha}{n!} \Gamma(n+1-\alpha),$$

which tends to 1 ($n \to \infty$), since $\Gamma(u+1) \sim u^{u+1/2} e^{-u} \sqrt{2\pi}$ as $u \to \infty$ (see [T]). Thus, to obtain the assertion of the theorem, it is enough to check that

$$\mathscr{J}_n(t) := \mathscr{I}_n(t) - \frac{n^\alpha}{n!} \Gamma(n+1-\alpha) f(t) \to 0, \text{ as } n \to \infty.$$

So, to proceed with it, set

$$G(s) := \int_t^s (f(u) - f(t))\, du = F(s) - F(t) - f(t)(s-t),$$

where $F(s) := \int_0^s f(u)du$, $s \geq 0$. Then $\|F(s)\| \leq M s^{\gamma+1} e^{\omega s}$ for some $M > 0$ and every $s \geq 0$. This readily implies that the function G is exponentially bounded, so there exist some constants $\mu \geq 0$ and $C > 0$ such that $\|G(s)\| \leq C e^{\mu s}$ for every $s \geq 0$. We may assume that $\mu \geq \omega$. On the other hand, the fact that t is a Lebesgue point of f implies that $\|G(s)\| = o(|s-t|)$, as $s \to t$.

By integration by parts, we have for $n > \max\{\mu t, \alpha\}$ that

$$\begin{aligned}
\mathscr{J}_n(t) &= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty u^{n-\alpha} e^{-(n/t)u} (f(u) - f(t))\, du \\
&= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty \left(\frac{n u^{n-\alpha}}{t} - (n-\alpha) u^{n-\alpha-1} \right) e^{-(n/t)u} G(u) du \\
&= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty \left(\frac{nu}{t} - (n-\alpha) \right) u^{n-\alpha-1} e^{-(n/t)u} G(u) du \\
&= \frac{n^{n+2}}{n!} \frac{1}{t} \int_0^\infty \left(y - \frac{n-\alpha}{n} \right) y^{n-\alpha-1} e^{-ny} G(ty) dy.
\end{aligned}$$

Let now $\varepsilon > 0$ and choose $0 < \delta < 1$ such that if $|y-1| < \delta$ then

$$\frac{1}{t} \|G(ty)\| < \varepsilon |y-1|. \tag{3.3}$$

Divide $\mathscr{I}_n(t)$ into three integrals $\mathscr{I}_{1,n}(t)$, $\mathscr{I}_{2,n}(t)$ and $\mathscr{I}_{3,n}(t)$ whose intervals of integration are $(0,1-\delta)$, $(1-\delta,1+\delta)$ and $(1+\delta,\infty)$, respectively.

First, we are going to estimate $\mathscr{I}_{1,n}(t)$. Take $n>(\alpha+1)/\delta$. In this case, the function $y\mapsto y^{n-\alpha-1}e^{-ny}$ is increasing on $(0,1-\delta)$, and therefore

$$\begin{aligned}\|\mathscr{I}_{1,n}(t)\| &\le \frac{n^{n+2}}{n!}\frac{1}{t}\int_0^{1-\delta}|y-\frac{n-\alpha}{n}|\,y^{n-\alpha-1}e^{-ny}\|G(ty)\|dy\\ &\le \frac{n^{n+2}}{n!}\frac{1}{t}(1-\delta)^{n-\alpha-1}e^{-n(1-\delta)}\int_0^{1-\delta}\|G(ty)\|dy =: a_n,\end{aligned}$$

where we have used that $\delta/(\alpha+1)\le(n-\alpha)/n-y<1$ for all $y\in(0,1-\delta)$. Then, by Stirling's formula,

$$a_n = O\left(n^{3/2}\left((1-\delta)e^{\delta}\right)^n\right) \text{ as } n\to\infty,$$

and therefore $a_n\to 0$ as $n\to\infty$, since $(1-\delta)e^{\delta}<1$. Therefore, $\|\mathscr{I}_{1,n}(t)\|<\varepsilon$ for n large enough.

Now, applying to $\mathscr{I}_{2,n}(t)$ the estimate (3.3), we get

$$\begin{aligned}\|\mathscr{I}_{2,n}(t)\| &\le \varepsilon\frac{n^{n+2}}{n!}\int_{1-\delta}^{1+\delta}|y-\frac{n-\alpha}{n}|\,|y-1|\,y^{n-\alpha-1}e^{-ny}dy\\ &=\varepsilon\frac{n^{n+2}}{n!}\int_{1-\delta}^{1+\delta}|y-1+1-\frac{n-\alpha}{n}|\,|y-1|\,y^{n-\alpha-1}e^{-ny}dy\\ &\le\varepsilon\frac{n^{n+2}}{n!}\int_{1-\delta}^{1+\delta}\left((y-1)^2+(1-\frac{n-\alpha}{n})(y+1)\right)y^{n-\alpha-1}e^{-ny}dy\\ &=\varepsilon\frac{n^{n+2}}{n!}\int_0^{\infty}\left(y^2-(1+\frac{n-\alpha}{n})y+(2-\frac{n-\alpha}{n})\right)y^{n-\alpha-1}e^{-ny}dy\\ &=\varepsilon\frac{n^{\alpha}}{n!}\left(\Gamma(n-\alpha+2)-(1+\frac{n-\alpha}{n})n\Gamma(n-\alpha+1)+(2-\frac{n-\alpha}{n})n^2\Gamma(n-\alpha)\right)\\ &=\varepsilon\frac{n^{\alpha}}{n!}\left(\Gamma(n-\alpha+1)+2\alpha n\Gamma(n-\alpha)\right).\end{aligned}$$

Thus, the fact that $\lim_{n\to\infty}\frac{n^\beta}{n!}\Gamma(n-\beta+1)=1$ for all $\beta\geq 0$ implies that

$$\lim_{n\to\infty}\frac{n^\alpha}{n!}\left(\Gamma(n-\alpha+1)+2\alpha n\Gamma(n-\alpha)\right)=1+2\alpha.$$

Hence, $\|\mathscr{I}_{2,n}(t)\|<2(1+\alpha)\varepsilon$ for all sufficiently large n.

To estimate $\mathscr{I}_{3,n}(t)$, take $n_0\in\mathbb{N}$ such that $n_0>\mu t$ and let $n>n_0$. Thus, the function $y\mapsto y^{n-n_0-\alpha}e^{-(n-n_0)y}$ is decreasing on $(1+\delta,\infty)$. Then we have

$$\begin{aligned}
\|\mathscr{I}_{3,n}(t)\| &\leq \frac{n^{n+2}}{n!}\frac{1}{t}\int_{1+\delta}^{\infty}(y-\frac{n-\alpha}{n})y^{n-\alpha-1}e^{-ny}\|G(ty)\|dy\\
&\leq \frac{n^{n+2}}{n!}\frac{C}{t}\int_{1+\delta}^{\infty}y^{n-\alpha}e^{-ny}e^{\mu ty}dy\\
&= \frac{n^{n+2}}{n!}\frac{C}{t}\int_{1+\delta}^{\infty}y^{n-n_0-\alpha}e^{-(n-n_0)y}y^{n_0}e^{-(n_0-\mu t)y}dy\\
&\leq \frac{n^{n+2}}{n!}\frac{C}{t}\frac{(1+\delta)^{n-n_0-\alpha}}{e^{(n-n_0)(1+\delta)}}\int_{1+\delta}^{\infty}y^{n_0}e^{-(n_0-\mu t)y}dy=:b_n.
\end{aligned}$$

Similarly as before, the Stirling's formula applies to show that $b_n\to 0$ as $n\to\infty$, and we have that $\|\mathscr{I}_{3,n}(t)\|<\varepsilon$ for large enough n. The proof is completed. □

Remark 3.1.2. There are some particular cases in which the inversion formula in Theorem 3.1.1 can be obtained as a consequence of the classical (vector-valued) Post-Widder inversion formula 3.1. For example, it occurs when the function is the integral of order $\alpha>0$ of some suitable function:

For $\alpha>0$, set $j_\alpha(t):=t^{\alpha-1}\Gamma(\alpha)^{-1}$, $t>0$. Let $g\in L^1_{loc}(\mathbb{R}^+;X)$ be an exponentially bounded function. Thus, the function $f:=j_\alpha * g$ satisfies the assumptions of Theorem 3.1.1, where $*$ is the usual convolution on $\mathbb{R}^+$. Notice that $\lambda^\alpha\mathscr{L}f(\lambda)=\mathscr{L}g(\lambda)$ for suitable complex values of λ. Therefore, applying the formula in 3.1

and the dominated convergence theorem, we have that for every $t > 0$,

$$\begin{aligned} f(t) &= j_\alpha * \left(\lim_{n\to\infty} (-1)^n \frac{1}{n!} \left(\frac{n}{(\cdot)}\right)^{n+1} \hat{g}^{(n)} \left(\frac{n}{(\cdot)}\right)\right)(t) \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!} \left(\frac{n}{s}\right)^{n+1} \hat{g}^{(n)} \left(\frac{n}{s}\right) ds \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!} \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n} (\lambda^\alpha \mathscr{L} f)\Big|_{\lambda=n/s} ds. \end{aligned}$$

Thus the interest of Theorem 3.1.1 relies upon the fact that it provides an inversion formula for those functions $\varphi : (\omega, \infty) \to X$ which are not necessarily a Laplace transform, but such that $\lambda^{-\alpha}\varphi(\lambda)$ is a Laplace transform for some $\alpha > 0$; see [ABHN, Example 2.2.4]. Important classes of functions in this situation involve general α-times integrated semigroups or integrated cosine functions (see next section).

To end this section, we point out that there exists a well known version of the Post-Widder inversion formula 3.1 in which the *Laplace-Stieltjes transform* $\mathscr{L}_S$ of vector-valued Lipschitz continuous functions is considered. If $F : \mathbb{R}^+ \to X$ is a Lipschitz continuous function, that is,

$$\sup_{t,s\geq 0} \frac{\|F(t)-F(s)\|}{|t-s|} < \infty,$$

then the Laplace-Stieltjes transform of F is given by

$$\mathscr{L}_S(F)(\lambda) := -F(0) + \lambda \int_0^\infty e^{-\lambda t} F(t) dt$$

for those λ greater than the exponential growth bound of F. It follows from 3.1 that if $F : \mathbb{R}^+ \to X$ is a Lipschitz continuous function such that $F(0) = 0$ then

$$F(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \frac{d^n}{d\lambda^n} \left(\frac{\mathscr{L}_S(F)(\lambda)}{\lambda}\right)\Big|_{\lambda=n/s}, \ t > 0.$$

See [ABHN, Theorem 2.3.1].

As a consequence of Theorem 3.1.1, we further obtain the following inversion formula for Laplace-Stieltjes transforms:

Corollary 3.1.1. *Let $F : \mathbb{R}^+ \to X$ be a Lipschitz continuous function such that $F(0) = 0$. Let $t > 0$. Then,*

$$F(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \int_0^t \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n} \left(\mathscr{L}_S(F)(\lambda)\right)\Big|_{\lambda=n/s} ds.$$

Proof. For F under these assumptions, we have that $\mathscr{L}F(\lambda) = \lambda^{-1}\mathscr{L}_S(F)(\lambda)$ for λ large enough. Moreover, $\|F(t)\| \leq Ct$ for every $t \geq 0$ and some $C > 0$. Then, it suffices to apply Theorem 3.1.1 for $\alpha = 1$. □

3.2 Applications to α-times integrated families

We show here that Theorem 3.1.1 applies to α-times integrated semigroups and α-times integrated cosine families, obtaining in such a way appropriate inversion formulae of Euler's type for these families.

3.2.1 Euler's exponential type formula for α-times integrated semigroups

Let X be a Banach space and let $\alpha > 0$.

Corollary 3.2.1. *Let $A : D(A) \subseteq X \to X$ be the generator of an α-times integrated semigroup $(T_\alpha(t))_{t\geq 0}$ such that $\|T_\alpha(t)\| \leq Ct^\gamma e^{\omega t}$, $t \geq 0$, for some $\gamma > \alpha - 1$ and $\omega \geq 0$. Then, for every $t > 0$ and every $x \in X$,*

$$T_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{n+1} R(\frac{n}{s}, A)^{n+1} x ds.$$

Proof. Let $x \in X$. Set $f(t) := T_\alpha(t)x$ for $t \geq 0$. Notice that f is continuous on $[0,\infty)$ since $(T_\alpha(t))_{t\geq 0}$ is strongly continuous. By definition, $R(\lambda,A)x = \lambda^\alpha \mathscr{L} f(\lambda)$ for λ large enough. Moreover, the resolvent equation gives us

$$((-1)^n/n!)(d^n/d\lambda^n)R(\lambda,A)x = R(\lambda,A)^{n+1}x.$$

Now, the claim follows automatically from Theorem 3.1.1 since $\alpha \in (0,\gamma+1)$. □

The above corollary extends previous results in this setting (see [C, Theorem 3.1] for n-times integrated semigroups, $n \in \mathbb{N}$, and [VV, Theorem 3.1] for exponentially bounded α-times integrated semigroups and $0 < \alpha < 1$), and provides a unified proof for them.

A large number of examples of α-times integrated semigroups satisfying the assumptions of Corollary 3.2.1 can be found in [H].

3.2.2 α-Times integrated cosine functions

Another consequence of Theorem 3.1.1 is the following result, which seems to be new.

Corollary 3.2.2. *Let $A : D(A) \subseteq X \to X$ be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\geq 0}$ for which there exist constants $\gamma > \alpha - 1$ and $\omega \geq 0$ satisfying $\|C_\alpha(t)\| \leq Ct^\gamma e^{\omega t}$ for $t \geq 0$. Then, for every $x \in X$ and $t > 0$,*

$$C_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!} \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n} \left(\lambda R(\lambda^2,A)\right)\Big|_{\lambda=\frac{n}{s}} ds.$$

Proof. Similar to the proof of Corollary 3.2.1. □

Particular examples of generators of α-times integrated cosine functions are provided by generators of α-times integrated semigroups. In fact, if an operator

B on a Banach space is such that B and $-B$ are both generators of corresponding α-times integrated semigroups then $A = B^2$ is the generator of an α-times integrated cosine function; see [AK, MeK]. In this case the explicit calculation of $(d^n/d\lambda^n)\left(\lambda R(\lambda^2, A)\right)$ is simple:

$$\frac{d^n}{d\lambda^n}\left(\lambda R(\lambda^2, A)\right) = \frac{1}{2}[R(\lambda, -iB)^{n+1} + R(\lambda, iB)^{n+1}].$$

3.2.3 On Riesz kernels

Let us recall that, for $\alpha, t > 0$, the *Riesz kernel* $R_t^{\alpha-1}$ is the function defined by

$$R_t^{\alpha-1}(s) := \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{(0,t)}(s), \quad s > 0.$$

As we have shown in Chapter 1, these kernels play a central role in the study of Banach algebras of Sobolev type $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, which are in close relationship with α-times integrated semigroups and integrated cosine functions. In particular, the function $R_t^{\alpha-1}$ is a multiplier of the Banach algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ with respect to either the usual convolution product $*$, or even the cosine convolution product $*_c$. In both cases we have that, as a multiplier, $\|R_t^{\alpha-1}\| \leq Ct^\alpha e^{\omega t}$ $(t > 0)$.

In view of Theorem 3.1.1 we have the following:

Corollary 3.2.3. *Let $\alpha > 0$ and $\omega \geq 0$. Then for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $t > 0$ we have*

$$R_t^{\alpha-1} \bullet g = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{n+1} e_{n/s}^{*(n+1)} \bullet g\, ds$$

in the norm of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, where

$$e_\lambda^{*(n+1)}(r) = \frac{r^n}{n!} e_\lambda(r), \quad (r \geq 0)$$

and $\bullet$ is either the usual convolution $*$ or the cosine convolution $*_c$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Proof. Note that for every $\lambda > \omega$ and $n \in \mathbb{N}$ one has

$$e_\lambda := e^{-\lambda(\cdot)} = \lambda^\alpha \int_0^\infty R_t^{\alpha-1} e^{-\lambda t}\, dt.$$

and

$$\frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} e_\lambda = e_\lambda^{*(n+1)}.$$

Hence it is enough to take $f(t) = R_t^\alpha \bullet g$ in the formula of Theorem 3.1.1 to obtain the result. □

Remark 3.2.1. The formula in the preceding corollary serves to illustrate Theorem 3.1.1 in a canonical situation, as regards α-times integrated semigroups. For simplicity, assume $\alpha > 1$. The equality

$$R_t^{\alpha-1}(r) = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{n+1} \frac{r^n}{n!} e^{-(n/s)r}\, ds, \quad t > 0,$$

holds as a particular case of the fact that formula 3.1 remains true when one replaces functions like f with Dirac masses:

For $r > 0$,

$$\delta_r = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{\cdot}\right)^{n+1} \hat{\delta}_r^{(n)} \left(\frac{n}{\cdot}\right) = \lim_{n\to\infty} \left(\frac{n}{\cdot}\right)^{n+1} \frac{r^n}{n!} e^{-(n/\cdot)r}$$

in the sense of weak convergence of measures. In fact, for each continuous function F on $[0,\infty)$ with $\lim_{t\to\infty} F(t) = 0$ we have

$$F_n(r) := \frac{1}{n!} \int_0^\infty \left(\frac{n}{s}\right)^{n+1} r^n e^{-(n/s)r} F(s)\, ds = \frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt} F\left(\frac{r}{t}\right) dt,$$

with

$$\frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt}\, dt = 1$$

Therefore

$$F_n(r) - F(r) = \frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt} \left[F(r/t) - F(r)\right] dt,$$

and so by standard methods involving the partition of the integration domain $(0,\infty)$ into two parts $\{|t-1|\le\varepsilon\}$ and $\{|t-1|>\varepsilon\}$, for suitable small $\varepsilon>0$, one gets $\lim_{n\to\infty}F_n(r)=F(r)$. In this connection, for the sake of completeness, let us point out that integration by parts gives us

$$\frac{n^{n+1}}{n!}\int_{1+\varepsilon}^{\infty}t^{n-1}e^{-nt}\,dt=\frac{1}{(n-1)!}\int_{n(1+\varepsilon)}^{\infty}y^{n-1}e^{-y}\,dy$$

$$=\sum_{k=1}^{n-1}\frac{(1+\varepsilon)^{n-k}n^{n-k}}{(n-k)!}e^{-(1+\varepsilon)n}+\int_{(1+\varepsilon)n}^{\infty}e^{-y}\,dy,$$

and this expression tends to 0 as $n\to\infty$ by the Stirling formula and the fact that $y\mapsto e^{-y}$ is integrable.)

Corollary 3.2.3 tells us that the above numerical limit holds indeed for convolution and in the norm of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$.

Notice that starting from Corollary 3.2.3, with a direct proof independent of Theorem 3.1.1, one can prove Corollary 3.2.1 and Corollary 3.2.2 (for $\gamma=\alpha$ and $\omega=0$) by just considering the image of $R_t^{\alpha-1}\bullet g$ and of its integral expression through the homomorphisms π_α and γ_α, respectively, introduced in Chapter 1 (see Proposition 1.3.2 and Remark 1.3.2).

Chapter 4

On stability of integrated semigroups with nonquasianalytic growth

The main result of this chapter extends [Me, Theorem 5.6] to n-times integrated semigroups for every natural n and a fairly wide boundedness condition involving nonquasianalytic weights (see definition in Chapter 1, p.11). The result is the following.

Theorem 4.0.1. *Let A be the generator of a n-integrated semigroup $T_n(t)$ such that $\sigma(A) \cap i\mathbb{R}$ is countable, $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$ and $0 \in \rho(A)$. Assume that*

$$\sup_{t \geq 1} \omega(t)^{-1} \|T_n(t)\| < +\infty,$$

for some nonquasianalytic weight ω on $[0,\infty)$ for which $\widetilde{\omega}(t) = O(t^k)$, as $t \to \infty$, for some $k \geq 0$.

We have:

(i) *If* $\omega(t)^{-1} = o(t^{-n+1})$, *as* $t \to \infty$, *then*

$$\lim_{t\to\infty} \omega(t)^{-1} T_n(t)x = 0, \quad x \in \overline{D(A^n)}.$$

(ii) *If* $\omega(t) \sim t^{n-1}$, *as* $t \to \infty$, *then*

$$\lim_{t\to\infty} t^{-n+1} T_n(t)x = -\frac{1}{(n-1)!} A^{-1}x, \quad x \in \overline{D(A^n)}.$$

The proof of Theorem 4.0.1 is a combination of arguments and ideas of [Me] and [V1]. In passing, we give extensions of results of [Me] and [V1]. Lemma 4.0.1 below provides a slight improvement of [V1, Theorem 7], which is in turn an extension of the Arendt-Batty-Lyubich-Vũ theorem.

Lemma 4.0.1. *Let* $U(t))_{t\geq 0} \subset \mathscr{B}(Y)$ *be a strongly continuous* C_0*-semigroup of positive exponential type with generator* L*. Let* β *be a nonquasianalytic weight on* $[0,\infty)$ *such that* $\widetilde{\beta}(t) = O(t^k)$ *as* $t \to \infty$*, for some* $k \geq 0$*. Assume that there exists* $R \in \mathscr{B}(Y)$ *such that* $U(t)R = RU(t)$ *for all* $t \geq 0$ *and* $\|U(t)R\| \leq \beta(t)$ $(t \geq 0)$*.*

If $\sigma(L) \cap i\mathbb{R}$ *is countable and* $\sigma_P(L^*) \cap i\mathbb{R} = \emptyset$ *then*

$$\lim_{t\to\infty} \frac{1}{\beta(t)} U(t)Ry = 0 \quad (y \in Y).$$

The overall argument goes along similar lines as in [V1, Theorem 7], lemmata included. Here we outline that argument for convenience of prospective readers.

Proof. Put

$$q(y) := \limsup_{t\to\infty} \beta(t)^{-1} \|U(t)Ry\|_Y \quad (y \in Y).$$

Then q is a seminorm on Y such that $q(y) \leq \|y\|_Y$ for all $y \in Y$. Moreover, $q(U(s)y) \leq \widetilde{\beta}(s)q(y)$ for every $s \geq 0$, $y \in Y$, and so $N := \{y \in Y : q(y) = 0\}$ is a $U(t)$-invariant closed subspace of Y. Hence one can define a norm $\widehat{q}$ on Y/N given by

$$\widehat{q}(\pi(y)) := q(y), \quad y \in Y,$$

and an operator $\widehat{U}(t)$ on Y/N given by

$$\widehat{U}(t)(\pi(y)) := \pi(U(t)y), \quad y \in Y, t \geq 0,$$

where π is the projection $Y \to Y/N$.

It is straightforward to show that $\widehat{U}(t)$ is a strongly continuous semigroup in the norm $\widehat{q}$ on Y/N. Let Z be the $\widehat{q}$-completion of Y/N, and let $V(t)$ be the continuous extension on Z of $\widehat{U}(t)$. Then:

(a) $\|\pi(y)\|_Z = \limsup_{t\to\infty} \frac{1}{\beta(t)} \|U(t)Ry\|_Y$, $y \in Y$. This is obvious.

(b) $\|V(t)\| \leq \widetilde{\beta}(t)$, $t \geq 0$, and from this one readily obtains that $V(t)$ is a strongly continuous C_0-semigroup in $\mathscr{B}(Z)$. The above bound follows by continuity and density by the estimate

$$\begin{aligned}\widehat{q}(\widehat{U}(t)\pi(y)) &= \widehat{q}(\pi(U(t)y)) = q(U(t)y) \\ &\leq \widetilde{\beta}(t)q(y) \leq \widetilde{\beta}(t)\widehat{q}(\pi(y), \quad y \in Y, t \geq 0.\end{aligned}$$

(c) $\|V(t)z\|_Z \geq \|z\|_Z$ for all $z \in Z$: For $y \in Y$ and $t \geq 0$,

$$\widehat{q}(\widehat{U}(t)\pi(y)) = \limsup_{t\to\infty} \frac{\beta(t+s)}{\beta(t)} \frac{\|U(t+s)Ry\|_Y}{\beta(t+s)} \geq \widehat{q}(\pi(y)).$$

Then apply continuity and density.

(d) $V(t) \circ \pi = \pi \circ U(t)$ $(t \geq 0)$ and then one easily obtains that $\pi(D(L) \subseteq D(H)$ and $H \circ \pi = \pi \circ L$ on $D(L)$, where H is the infinitesimal generator of $V(t)$.

(e) $\sigma(H) \subseteq \sigma(L)$: By hypothesis, $U(t)$ is of exponential type $\delta > 0$, whence as is well known, for $y \in Y$ and $\lambda \in \mathbb{C}$, $\Re\lambda > \delta$,

$$R(\lambda, L)y := -(\lambda - L)^{-1}y = -\int_0^\infty e^{-\lambda t} U(t)y \, dt.$$

Similarly, since $\|V(t)\| \leq \widetilde{\beta}(t)$, $t \geq 0$, the semigroup $V(t)$ is of exponential type 0, and therefore we have for $z \in Z$ and $\lambda \in \mathbb{C}$, $\Re\lambda > 0$,

$$R(\lambda, H)z := -(\lambda - H)^{-1}z = -\int_0^\infty e^{-\lambda t} V(t)z \, dt$$

On the other hand, R commutes with $(U(t)$ by assumption and so R commutes with $R(\lambda, L)$, $\Re\lambda > \delta$. Then $q(R(\lambda, L)y) \leq \|R(\lambda, L)\| q(y)$ for all $y \in Y$, which implies that N is $R(\lambda, L)$-invariant. Hence one can define the bounded operator $\widehat{R}(\lambda, L)$ on Z given by $\widehat{R}(\lambda, L)(\pi(y)) := \pi(R(\lambda, L)y)$, $y \in Y$. Thus,

$$\begin{aligned} \widehat{R}(\lambda, L)\pi(y) = \pi(R(\lambda, L)y) &= -\int_0^\infty e^{-\lambda t} \pi(U(t)y) \, dt \\ &= -\int_0^\infty e^{-\lambda t} V(t)\pi(y) dt = R(\lambda, H)\pi(y); \end{aligned}$$

where (d) has been applied in the last but one equality. Hence $\widehat{R}(\lambda, L) = R(\lambda, H)$, $\Re\lambda > \delta$.

Now, for $\Re\lambda > \delta$ and any $\mu \in \rho(L)$, by using the resolvent identity

$$R(\lambda, L) - R(\mu, L) = (\lambda - \mu)R(\lambda, L)R(\mu, L)$$

and its corresponding identity for $\widehat{R}(\lambda, L)$, $\widehat{R}(\mu, L)$ one readily finds that there exists $R(\lambda, H)$ and that

$$R(\mu, H) = \widehat{R}(\mu, L);$$

see [V1, p. 234]. Thus $\mu \in \rho(H)$. Hence $\rho(L) \subseteq \rho(H)$ as it was claimed.

(f) $\sigma_P(H^*) \subseteq \sigma_P(L^*)$. This is straightforward to see, using restricitions of functionals; see [V1, p. 234]

Suppose, if possible, that $Z \neq \{0\}$. By (e) above we have that $\sigma(H) \cap i\mathbb{R}$ is countable and then $i\mathbb{R} \setminus \sigma(H) \neq \emptyset$. So, by (c) above and [V1, Lemma 2] the semigroup $V(t)$ can be extended to a C_0-group $\widetilde{V}(t)$ on $\mathbb{R}$ such that $\|\widetilde{V}(-t)\| \leq 1$ $(t > 0)$ and $\|\widetilde{V}(t)\| = O(t^k)$, as $t \to +\infty$. Also, $\sigma(H)$ is nonempty by (b) above and [V1, Lemma 5].

Then reasoning as in [V1, Theorem 7] one gets $\sigma_P(H^*) \cap i\mathbb{R} \neq \emptyset$ whence $\sigma_P(L^*) \cap i\mathbb{R} \neq \emptyset$ by (f) above. This is a contradiction and so we have proved that $Z = \{0\}$. By (a) above we get the statement. □

Lemma 4.0.2. *Let ω be a nonquasianalytic weight such that $\widetilde{\omega}$ is of polynomial growth at infinity. Let X be a Banach space and $(T_n(t)_{t\geq 0})$ be a n-times integrated semigroup in $\mathscr{B}(X)$ with generator A such that $\|T_n(t)\| \leq \omega(t)$, $t \geq 0$. Let assume that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.*

For every $\mu > 0$ we have:

(i) *If $\omega(t)^{-1} = o(t^{-(n+1)})$, as $t \to \infty$, then*

$$\lim_{t\to\infty} \omega(t)^{-1} T_n(t) A^n (\mu - A)^{-n} x = 0, \quad x \in X.$$

(ii) *If $\omega(t) \sim t^{n-1}$, as $t \to \infty$, then*

$$\lim_{t\to\infty} \frac{1}{t^{n-1}} T_n(t) A^n (\mu - A)^{-n} x = -\frac{A^{n-1}(\mu - A)^{-2n} x}{(n-1)!}, \quad x \in X.$$

Proof. Here we follow arguments of [Me, Theorem 5.6] suitably adapted to our setting. Take $\mu > \delta > 0$. For $x \in X$ define

$$\|x\|_Y := \sup_{t\geq 0} \|e^{-\delta t}(T_n(t) A^n (\mu - A)^{-n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-n} x)\|_X.$$

Note that $A(\mu-A)^{-1} = -I+\mu(\mu-A)^{-1}$ is a bounded operator on X and $T_n(0)=0$, so $\|\cdot\|_Y$ is a norm on X and there exists a constant $M_\delta > 0$ such that

$$\|(\mu-A)^{-n}x\|_X \le \|x\|_Y \le M_\delta \|x\|_X, \quad x\in X. \tag{4.1}$$

Let Y be the Banach space obtained as the completion of X in the norm $\|\cdot\|_Y$. By the extrapolation theorem [ANS, Theorem 0.2], there exists a closed operator B on Y which generates a strongly continuous C_0-semigroup $(S(t))_{t\ge 0} \subset \mathscr{B}(Y)$ of positive exponential type such that $D(B^n) \hookrightarrow X \hookrightarrow Y$, $A = B_X$ where the operator B_X is given by the conditions $D(B_X) := \{x\in D(B)\cap X : Bx \in X\}$, $B_X(x) := B(x)$ $(x\in X)$. Moreover, $\sigma_P(B^*) \subseteq \sigma_P(A^*)$, and also $\rho(A) = \rho(B)$ with

$$(\lambda-A)^{-1}x = (\lambda-B)^{-1}x, \quad \lambda\in\rho(A)=\rho(B), x\in X; \tag{4.2}$$

see [ANS, Remark 3.1].

Let $S_n(t)$ be the n-times integrated semigroup defined by B on Y, given by

$$S_n(t)y := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} S(s)\, ds, \quad y\in Y.$$

Then $S_n(t)x = T_n(t)x$ for all $x\in X$. To see this, note that $T_n(t)$ and $S_n(t)$ are of exponential type so one can rewrite (4.2) above in terms of the Laplace transforms of $T_n(t)$ and $S_n(t)$ respectively, for $\Re\lambda$ large enough. Then it suffices to apply the uniqueness of the Laplace transform.

From the above identification between $T_n(t)$ and $S_n(t)$, it readily follows that

$$\|S_n(u)x\|_Y \le \|T_n(u)\|\|x\|_Y \le \omega(u)\|x\|_Y, \quad u\ge 0, x\in X, \tag{4.3}$$

which is to say, by density, that $\|S_n(u)\| \le \omega(u)$, for all $u\ge 0$.

Now, by reiteration of the well known relation between a semigroup and its generator $S(t)y - y = \int_0^t BS(s)y\,ds$, for $t \geq 0$ and $y \in D(B)$, we have

$$S(t)y = S_n(t)B^n y + \sum_{j=0}^{n-1} \frac{t^j}{j!} B^j y, \quad \forall y \in D(B^n).$$

Hence, for every $y \in Y$,

$$S(t)(\mu - B)^{-n} y = S_n(t) \left(\frac{B}{\mu - B} \right)^n y + \sum_{j=0}^{n-1} \frac{t^j}{j!} \left(\frac{B}{\mu - B} \right)^j (\mu - B)^{-(n-j)} y \qquad (4.4)$$

and therefore there exists a constant $C_\mu > 0$ such that

$$\|S(t)(\mu - B)^{-n}\|_{Y \to Y} \leq C_\mu \omega(t), \quad t \geq 0. \qquad (4.5)$$

Then, by applying Lemma 4.0.1 with $U(t) = S(t)$, $B = L$ and $R = (\mu - A)^{-n}$, we obtain

$$\lim_{t \to \infty} \frac{1}{\omega(t)} \|S(t)(\mu - B)^{-n} y\|_Y = 0, \quad \forall y \in Y,$$

whence, by (4.1), (4.2) and (4.4),

$$\begin{aligned} 0 &= \lim_{t \to \infty} \frac{1}{\omega(t)} \|T_n(t)A^n(\mu - A)^{-n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-n} x\|_Y \\ &\leq \limsup_{t \to \infty} \frac{1}{\omega(t)} \|T_n(t)A^n(\mu - A)^{-2n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-2n} x\|_X, \end{aligned}$$

for every $x \in X$.

Thus we get

$$\lim_{t \to \infty} \frac{1}{\omega(t)} T_n(t)A^n(\mu - A)^{-2n} x = -\lim_{t \to \infty} \frac{1}{\omega(t)} \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-2n} x$$

in X, and the statement follows readily. $\square$

Proof of Theorem 4.0.1. In the setting of the Lemma 4.0.2, let assume in addition that $0 \in \rho(A)$. Since the resolvent function of A is holomorphic, so continuous, in the open subset $\rho(A) \subseteq \mathbb{C}$, we have that

$$\lim_{\mu\to 0^+} A^n(\mu - A)^{-n} = \lim_{\mu\to 0^+} (-I + \mu(\mu - A)^{-1}) = (-1)^n I.$$

Now, to prove (i) and (ii) of the theorem it suffices to notice that

$$\sup_{t>0} \omega(t)^{-1}\|T_n(t)\| < \infty$$

in both cases. □

Remark 4.0.1. It looks desirable to find out the behavior of a n-integrated semigroup at infinity when its generator A is not assumed to be invertible. According to the discussion prior to Theorem 4.0.1 the existence of $\lim_{t\to\infty} T_n(x)$ entails invertibility of A. Thus the type of convergence at infinity of $T_n(t)$, if there is some, that one can expect if A is not invertible must be weaker than the existence of limit. In Chapter 5, under the assumptions

$$\sup_{t>0} t^{-n}\|T_n(t)\| < \infty \text{ and } \lim_{t\to 0^+} n!t^{-n}T_n(t)x = x \quad (x \in X),$$

it is proved that

$$\lim_{t\to\infty} t^{-n}T_n(t)\pi_n(f) = 0, \quad f \in \mathfrak{S}_n,$$

in the operator norm, where $\mathfrak{S}_n$ is the subspace of functions of $\mathcal{T}_+^{(n)}(t^n)$ which are of spectral synthesis in $\mathcal{T}^{(n)}(|t|^n)$ with respect to the subset $i\sigma(A) \cap \mathbb{R}$, and $\pi_n(f) = (-1)^n \int_0^\infty f^{(n)}(t)T_n(t)\,dt$. Here, $\mathcal{T}^{(n)}(|t|^n)$ and $\mathcal{T}_+^{(n)}(t^n)$ are the convolution Banach algebras defined in Chapter 1, Section 1.3. This result is an extension of the Esterle-Strouse-Vũ-Zouakia theorem, which corresponds to the case $n = 0$ (see [ESZ] and [V]).

We wonder if in the case when A is not invertible, $\pi_n(\mathfrak{S}_n)X$ is dense in X (under the conditions $\sigma(A)\cap i\mathbb{R}$ countable and $\sigma_P(A^*)\cap i\mathbb{R}=\emptyset$). This would give us the ergodic type property

$$\lim_{t\to\infty} t^{-n}T_n(t)x=0, \quad x\in X. \tag{4.6}$$

In Chapter 6, we obtain such result in the case that $\sigma(A)\cap i\mathbb{R}$ finite.

Notice that (4.6) is a consequence of Theorem 4.0.1 (i) when A is invertible; on the other hand, the ergodicity of a n-times integrated semigroup $T_n(t)$ such that $\sup_{t\geq 1} t^{-n}\|T_n(t)\|<\infty$ is characterized in [Me] in terms of Abel-ergodicity or/and ergodic decompositions of the Banach space X.

Chapter 5

Katznelson-Tzafriri type theorem for integrated semigroups

We call C_α-*integrated semigroup* on X any α-times integrated semigroup $(T_\alpha(t))_{t\geq 0}$ which, in addition, satisfies

$$\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x = x \quad (x\in X). \tag{5.1}$$

Notice that a C_0-semigroup $(T(t))_{t\geq 0}$ in $\mathscr{B}(X)$ satisfies in particular

$$\lim_{t\to 0^+} T(t)x = x \quad (x\in X).$$

The main result of the present chapter is the following:

Theorem 5.0.1. *For $\alpha > 0$, let $(T_\alpha(t))_{t\geq 0} \subseteq \mathscr{B}(X)$ be a C_α-integrated semigroup with generator A such that*

$$\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty.$$

Suppose that $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ *is of spectral synthesis in* $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ *with respect to* $i\sigma(A) \cap \mathbb{R}$*. Then*

$$\lim_{t\to\infty} t^{-\alpha}\,\|T_\alpha(t)\,\pi_\alpha(f)\| = 0.$$

Recall that $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ is a regular Banach algebra whose Gelfand transform is the Fourier transform. Hence, a function $f \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$ is of spectral synthesis respect to a closed set $E \subseteq \mathbb{R}$ if there exists a sequence (f_n) in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ such that

(i) $f = \lim_{n\to\infty} f_n$ in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$.

(ii) The Fourier transform $\mathscr{F} f_n$ vanishes on a neighbourhood U_n of E.

To prove the above theorem, we carry through a detailed analysis of the homomorphism π_α along the same lines as it is done for the homomorphism π_0 in the case of C_0-semigroups considered in [ESZ]. The procedure followed in the present chapter has to deal with a number of new and fairly non-trivial technicalities with respect to the proof given in [ESZ]. The stuff needed is organized in sections as follows.

In Section 5.1, we collect results about the relationships between duality and convolution in the algebras of Sobolev type $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathscr{T}^{(\alpha)}(|t|^\alpha)$. Section 5.2 contains a couple of results involving extensions of Laplace transforms of functionals (of the algebras $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathscr{T}^{(n)}(|t|^n)$) through open subsets of the imaginary axis. The analysis of bounded homomorphisms of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is carried out in Section 5.3. First, we establish some formulae for Laplace transforms, of functionals associated to such bounded homomorphisms, which are natural extensions of those given in [ESZ, Section 2]. Then we give the main result of the section, Theorem 5.3.1, which relates homomorphisms of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ to functions of spectral synthesis. Finally, the results of previous sections are translated in terms of integrated semigroups to prove the main theorem of the paper, Theorem 5.0.1, in Section 5.4. All these results can be also found in [GMM1].

The family of Riesz kernels $R_t^{\alpha-1}$ plays a key role in this chapter. Recall that the convolution product in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ can be expressed in terms of Riesz kernels as follows ([GMR, Lemma 4.2]):

$$f*g=\int_0^\infty W_+^\alpha f(t)\,R_t^{\alpha-1}*g\,dt\,,\quad f,g\in\mathscr{T}_+^{(\alpha)}(t^\alpha)\,. \tag{5.2}$$

As a consequence of the above integral representation one gets that closed ideals of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ are characterized as those closed subspaces of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ which are invariant under the action by convolution of kernels $R_t^{\alpha-1}$. Thus the family of Riesz kernels plays a similar role to the one that the translation semigroup $(\delta_t)_{t>0}$, formed by the Dirac masses on $\mathbb{R}^+$, has with respect to $L^1(\mathbb{R}^+)$. Recall also from Section 1.3 (Proposition 1.3.1) that the family $\Gamma(\alpha+1)t^{-\alpha}R_t^{\alpha-1}$, $t>0$, is a summability kernel for $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, that is,

$$\lim_{t\to0^+}\Gamma(\alpha+1)t^{-\alpha}R_t^{\alpha-1}*f=f\,,\quad f\in\mathscr{T}_+^{(\alpha)}(t^\alpha). \tag{5.3}$$

5.1 Duality and convolutions in the Sobolev algebras

We shall also need to consider duality in $\mathscr{T}^{(n)}(|t|^n)$ and $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, as well as its relationship with convolution.

Recall that the dual Banach space $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is identified with the set of almost everywhere defined functions $\phi:\mathbb{R}^+\to\mathbb{C}$ which satisfy $t^{-\alpha}\phi\in L^\infty(\mathbb{R}^+)$. The duality is implemented by the formula

$$L_\phi(f)\equiv\langle L_\phi\,,f\rangle:=\int_0^\infty W_+^\alpha f(t)\phi(t)\,dt,\quad f\in\mathscr{T}_+^{(\alpha)}(t^\alpha),$$

for every $\phi \equiv L_\phi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$. Analogous facts hold about duality in $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ and $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ and functions ϕ, now supported on $\mathbb{R}^-$ and $\mathbb{R}$, respectively. See Section 1.3 above for details and references.

The following result translates to our setting the well-known property that the function $\phi * g$ lies in $C_0[0,\infty)$ provided $\phi \in C_0[0,\infty)$ and $g \in L^1(\mathbb{R}^+)$. It will be used in the proof of Theorem 5.3.1 below.

Proposition 5.1.1. *If $L_\phi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ is such that $t^{-\alpha}\phi \in C_0[0,\infty)$, then*

$$\lim_{t\to\infty} t^{-\alpha}\left\langle L_\phi\,, R_t^{\alpha-1} * g\right\rangle = 0$$

for every $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$.

Proof. Let ϕ be as above and let $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$. Using the formula (1.8) and then applying Fubini's theorem, we obtain that

$$\begin{aligned} t^{-\alpha}\left\langle L_\phi, R_t^{\alpha-1} * g\right\rangle &= \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_0^t\int_t^{r+t}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\,dsdr \\ &\quad+\frac{t^{-\alpha}}{\Gamma(\alpha)}\int_t^\infty\int_r^{r+t}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\,dsdr \\ &\quad-\frac{t^{-\alpha}}{\Gamma(\alpha)}\int_0^t\int_0^r(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\,dsdr \\ &\quad-\frac{t^{-\alpha}}{\Gamma(\alpha)}\int_t^\infty\int_0^t(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\,dsdr. \end{aligned}$$

First, let us see that the first of the above four integrals tends to 0 as $t\to\infty$. Since $s^{-\alpha}\phi \in L^\infty(\mathbb{R}^+)$, we have

$$t^{-\alpha}\int_t^{r+t}(r+t-s)^{\alpha-1}|\phi(s)|\,ds \le \alpha^{-1}2^\alpha\|s^{-\alpha}\varphi\|_\infty r^\alpha \quad (0<r<t).$$

On the other hand, the change of variables $s=(r+t)x$ gives us, for $0<r<t$

and some $C(r) > 0$,

$$t^{-\alpha}\int_t^{r+t}(r+t-s)^{\alpha-1}|\phi(s)|\,ds \le C(r)\sup_{\frac{t}{r+t}\le x\le 1}((r+t)x)^{-\alpha}|\phi((r+t)x)|$$
$$\le C(r)\sup_{\frac{1}{2}\le x\le 1}((r+t)x)^{-\alpha}|\phi((r+t)x)|,$$

Moreover,

$$\lim_{t\to\infty}\sup_{\frac{1}{2}\le x\le 1}((r+t)x)^{-\alpha}|\phi((r+t)x)| = 0$$

since $s^{-\alpha}\phi \in C_0(\mathbb{R}^+)$. The claim then follows from the dominated convergence theorem (taking sequential limits $t_n \to \infty$), since $|W_+^\alpha g(r)|r^\alpha$ is integrable on $\mathbb{R}^+$.

For the second integral, it is straightforward to check that, as in the preceding case, there exists some $C > 0$ for which

$$t^{-\alpha}\int_r^{r+t}(r+t-s)^{\alpha-1}|\phi(s)|\,ds \le C\,r^\alpha \quad (r > t > 0),$$

and from here it is clear that the second integral converges to 0 as $t \to \infty$.

Now note that the third and fourth integrals are bounded respectively by

$$\frac{C}{\Gamma(\alpha+1)}\int_0^\infty \chi_{(0,t)}(r)\left[1-\left(1+\frac{r}{t}\right)^\alpha\right]|W_+^\alpha g(r)|r^\alpha dr$$

and

$$\frac{C}{\Gamma(\alpha+1)}\int_t^\infty [(r+t)^\alpha - r^\alpha]\,|W_+^\alpha g(r)|dr,$$

which in turn go clearly to 0 as $t \to \infty$. The proof is over. □

Now, we introduce several convolutions relating functions and functionals of the algebras $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $\mathscr{T}^{(\alpha)}(|t|^\alpha)$. Recall that $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ is becomes a Banach $\mathscr{T}_+^{(\alpha)}(t^\alpha)$-module through the action of $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ on itself, so that

$$(L_\varphi, f) \mapsto L_\varphi \bullet f,\ \mathscr{T}_+^{(\alpha)}(t^\alpha)^* \times \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{T}_+^{(\alpha)}(t^\alpha)^*,$$

where the module product $\bullet$ is defined by $(L_\varphi \bullet f)(g) := L_\varphi(f*g)$ for all $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$.

This module product can be expressed in terms of Riesz kernels as follows:

Proposition 5.1.2. *For any $L_\varphi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ and $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$, we have*

$$L_\varphi \bullet f = L_\psi \text{ where } \psi(t) := L_\varphi(R_t^{\alpha-1} * f),\ t>0\,.$$

Proof. Let ψ be as in the statement. By properties of the Riesz kernels pointed out formerly, we obtain that $t^{-\alpha}\psi \in L^\infty(\mathbb{R}^+)$ and therefore $L_\psi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$. Moreover, for every $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$,

$$L_\psi(g) = L_\varphi\left(\int_0^\infty W_+^\alpha g(t)\, R_t^{\alpha-1} * f\, dt\right) = L_\varphi(f*g) = L_\varphi \bullet f(g),$$

where (5.2) has been used in the next-to-last equality. □

For a complex function F defined a. e. on $\mathbb{R}$ we put $\tilde{F}(x) = F(-x)$. Now, from the distribution theory we borrow the convolution product of tempered distributions and functions and define, for $L_\varphi \in \mathscr{T}^{(\alpha)}(|t|^\alpha)^*$ and $f \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$, the functional $L_\varphi * f$ by

$$(L_\varphi * f)(g) := L_\varphi(\tilde{f} * g), \qquad g \in \mathscr{T}^{(\alpha)}(|t|^\alpha).$$

Clearly, $L_\varphi * f \in \mathscr{T}^{(\alpha)}(|t|^\alpha)^*$ and the mapping $f \mapsto L_\varphi * f$, $\mathscr{T}^{(\alpha)}(|t|^\alpha) \to \mathscr{T}^{(\alpha)}(|t|^\alpha)^*$ is linear and bounded. Also, it is readily seen that $L_\varphi * f = (L_{\tilde{\varphi}} * \tilde{f})\tilde{}$, that is,

$$(L_\varphi * f)(g) = L_{\tilde{\varphi}}(f * \tilde{g}), \qquad g \in \mathscr{T}^{(\alpha)}(|t|^\alpha).$$

From here and (5.2) we get that

$$L_\varphi * f(g) = \int_{-\infty}^\infty W^\alpha g(s)\, L_{\tilde{\varphi}}(R_{-s}^{\alpha-1} * f)\, ds, \tag{5.4}$$

for every $f \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$, $L_\varphi \in \mathscr{T}^{(\alpha)}(|t|^\alpha)^*$ and $g \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$.

Definition 5.1.1. *For $f \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$ and $L_\varphi \in \mathscr{T}^{(\alpha)}(|t|^\alpha)^*$, we set*

$$L_\varphi \circ f := L_{\psi_-} \qquad \textit{and} \qquad L_\varphi \diamond f := L_{\psi_+},$$

where $\psi(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$, $\psi_- := \psi\chi_{(-\infty,0]}$, $\psi_+ := \psi\chi_{[0,\infty)}$.

By the growth rate of ψ it follows that $L_\varphi \circ f$ and $L_\varphi \diamond f$ are elements of $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, respectively. Moreover,

$$L_\varphi * f = L_\varphi \circ f + L_\varphi \diamond f \quad \text{in } \mathscr{T}^{(\alpha)}(|t|^\alpha)^*,$$

so that $L_\varphi \circ f = (L_\varphi * f)\,|_{\mathscr{T}_-^{(\alpha)}((-t)^\alpha)}$ and $L_\varphi \diamond f = (L_\varphi * f)\,|_{\mathscr{T}_+^{(\alpha)}(t^\alpha)}$.

Thus we have that if $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $L_{\widetilde{\varphi}} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ then $L_\varphi \circ f$ and $L_{\tilde{\varphi}} \bullet f$ are elements of $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, respectively, which are related by

$$L_\varphi \circ f = L_\psi\,, \quad L_{\widetilde{\varphi}} \bullet f = L_{\widetilde{\psi}}$$

where $\psi(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$ for $s < 0$.

Proposition 5.1.3. *For any sequences $(f_n)_{n=1}^\infty \subseteq \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $(L_{\widetilde{\varphi}_n})_{n=1}^\infty \subseteq \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, we have that*

$$\lim_{n\to\infty} L_{\widetilde{\varphi}_n} \bullet f_n = 0 \;\; \textit{if and only if} \;\; \lim_{n\to\infty} L_{\varphi_n} \circ f_n = 0.$$

Proof. For $n \in \mathbb{N}$, $L_{\varphi_n} \circ f_n = L_{\psi_n}$ and $L_{\widetilde{\varphi}_n} \bullet f_n = L_{\widetilde{\psi}_n}$ where $\psi_n(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f_n)$ for $s < 0$, as in the remark prior to this proposition. Then, for any $g \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)$,

$$\begin{aligned} L_{\psi_n}(g) &= \int_{-\infty}^0 W_-^\alpha g(t)\psi_n(t)\,dt \\ &= \int_0^\infty W_-^\alpha g(-s)\widetilde{\psi}_n(s)\,ds = \int_0^\infty W_+^\alpha \tilde{g}(s)\widetilde{\psi}_n(s)\,ds = L_{\widetilde{\psi}_n}(\tilde{g}). \end{aligned}$$

We conclude the argument by noticing that $g \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ if and only if $\tilde{g} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$. □

5.2 Laplace transform of functionals

Recall that every element in $\mathscr{T}^{(\alpha)}(|t|^\alpha)^*$ is a tempered distribution. In particular, if we set $e_z := e^{-z\cdot}$, then the Laplace transform of a functional $L_\varphi \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ is given by

$$\mathscr{L}(L_\varphi)(z) := L_\varphi(e_z) = \int_0^\infty W_+^\alpha(e_z)(t)\,\varphi(t)\,dt = z^\alpha \mathscr{L}(\varphi)(z)\,,\; z\in\mathbb{C}^+,$$

since $e_z \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ whenever $z\in\mathbb{C}^+$ and $W_+^\alpha(e_z) = z^\alpha e_z$ on $\mathbb{R}^+$. As a matter of fact, $\mathscr{L}(L_\varphi)$ is analytic on $\mathbb{C}^+$. On the other hand, the Laplace transform of an element $L_\varphi \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ is well defined and analytic on $\mathbb{C}^- := \{z\in\mathbb{C} : \Re z < 0\}$. Now, we have that $W_-^\alpha(e_z) = (-z)^\alpha e_z$ for $z\in\mathbb{C}^-$, so that

$$\mathscr{L}(L_\varphi)(z) = L_\varphi(e_z) = (-z)^\alpha \mathscr{L}(\varphi)(z)\,,\; z\in\mathbb{C}^-.$$

Proposition 5.2.1. *Let $L_\rho \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ such that $\mathscr{L}(L_\rho)$ extends continuously to $\mathbb{C}^+\cup U$, where U is some open subset of $i\mathbb{R}$. For any $\Psi\in\mathscr{T}^{(\alpha)}(|t|^\alpha)$ such that $\mathscr{F}(\Psi)$ has compact support contained in $-iU$, we have that $L_\rho * \Psi \in C^\infty(\mathbb{R})\cap L^\infty$ with, in fact,*

$$L_\rho * \Psi(t) = \frac{1}{2\pi}\int_{supp\mathscr{F}(\Psi)} \mathscr{F}(\Psi)(y)\,\mathscr{L}(L_\rho)(iy)\,e^{iyt}\,dy \quad (t>0).$$

Proof. For every $x>0$, let e_xL_ρ denote the functional given by

$$\left\langle e_xL_\rho\,,\,g\right\rangle := \left\langle L_\rho\,,\,e_xg\right\rangle\,,\; g\in\mathscr{T}^{(\alpha)}(|t|^\alpha).$$

Let Ψ satisfy the assumptions of the proposition. Then $\tilde\Psi * h \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$ for any $h\in\mathscr{S}(\mathbb{R})$, and $\mathscr{F}(\tilde\Psi * h) = \mathscr{F}(\tilde\Psi)\mathscr{F}(h) \in L^2(\mathbb{R})\cap L^1(\mathbb{R})$ because $\mathscr{F}(\tilde\Psi)$ is of compact support. We can then apply the Fourier inversion formula to get that

$$(\tilde\Psi * h)(t) = \frac{1}{2\pi}\int_{\mathrm{supp}\mathscr{F}(\tilde\Psi)} \mathscr{F}(\tilde\Psi)(y)\mathscr{F}(h)(y)e^{iyt}dy \quad (t>0),$$

so that

$$e^{-xt}\,(\tilde{\Psi}*h)(t)=\frac{1}{2\pi}\int_{\operatorname{supp}\mathscr{F}(\Psi)}\mathscr{F}(\Psi)(y)\,\mathscr{F}(h)(-y)\,e^{-(x+iy)t}\,dy\,.$$

The last integral converges in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ since $e_{x+iy}\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$. Therefore,

$$\begin{aligned}\left\langle e_xL_\rho,\tilde{\Psi}*h\right\rangle &= \frac{1}{2\pi}\int_{\operatorname{supp}\mathscr{F}(\Psi)}\mathscr{F}(\Psi)(y)\mathscr{F}(h)(-y)L_\rho(e_{x+iy})\,dy\\ &= \frac{1}{2\pi}\int_{\operatorname{supp}\mathscr{F}(\Psi)}\mathscr{F}(\Psi)(y)\mathscr{F}(h)(-y)\mathscr{L}(L_\rho)(x+iy)\,dy.\end{aligned}$$

Since $\mathscr{L}(L_\rho)$ is continuous on the subset $i\operatorname{supp}\mathscr{F}(\Psi)$ of $i\mathbb{R}$, the dominated convergence theorem applies and we get

$$\lim_{x\to 0}\left\langle e_xL_\rho,\tilde{\Psi}*h\right\rangle=\frac{1}{2\pi}\int_{\operatorname{supp}\mathscr{F}(\Psi)}\mathscr{F}(\Psi)(y)\mathscr{F}(h)(-y)\mathscr{L}(L_\rho)(iy)\,dy.$$

On the other hand, $\lim_{x\to 0}e_x\,g=g$ in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ for every $g\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$ (see [GMR, Lemma 3.6 (ii)]). Therefore,

$$\lim_{x\to 0}\left\langle e_xL_\rho,\tilde{\Psi}*h\right\rangle=\left\langle L_\rho,\tilde{\Psi}*h\right\rangle.$$

Applying Fubini's theorem, we finally obtain that

$$\left\langle L_\rho*\Psi,h\right\rangle=\int_{\mathbb{R}}\left(\frac{1}{2\pi}\int_{\operatorname{supp}\mathscr{F}(\Psi)}\mathscr{F}(\Psi)(y)\mathscr{L}(L_\rho)(iy)e^{iyt}dy\right)h(t)\,dt$$

for all $h\in\mathscr{S}(\mathbb{R})$. Then the statement follows automatically. □

Remark 5.2.1. An analogous statement to the one of the above proposition holds for elements $L_\rho\in\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ whose Laplace transform extends continuously to $\mathbb{C}^-\cup U$.

Proposition 5.2.2. *Let U be an open subset of $i\mathbb{R}$. Let $L_{\rho_1} \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $L_{\rho_2} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ be such that $\mathscr{L}(L_{\rho_1})$ and $\mathscr{L}(L_{\rho_2})$ have continuous extensions to $\mathbb{C}^- \cup U$ and $\mathbb{C}^+ \cup U$, respectively, and verify $\mathscr{L}(L_{\rho_1}) \mid_U = -\mathscr{L}(L_{\rho_2}) \mid_U$. Then,*

$$(L_{\rho_1} + L_{\rho_2}) * \Psi = 0,$$

for every $\Psi \in \mathscr{T}^{(\alpha)}(|t|^\alpha)$ such that $\operatorname{supp}\mathscr{F}(\Psi) \subseteq -iU$.

Proof. Take $\varphi \in \mathscr{S}(\mathbb{R})$ such that its Fourier transform has compact support in $\mathbb{R}$ and $\int_{-\infty}^{\infty} \varphi = 1$. Then the family $\varphi_n := n\varphi(n\cdot)$, $n \in \mathbb{N}$, is a bounded approximate identity in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ (the proof of this fact is similar to the one given in [GM, Theorem 1.11(ii)]). Hence $\Psi = \lim_{n\to\infty} \Psi * \varphi_n$ in $\mathscr{T}^{(\alpha)}(|t|^\alpha)$ in particular.

Now, $\mathscr{F}(\Psi * \varphi_n)$ is a compact subset of $\mathscr{F}(\Psi)$, so it is a compact subset of $-iU$. By Proposition 5.2.1, $(L_{\rho_1} + L_{\rho_2}) * (\Psi * \varphi_n) = 0$ for all n. Then, as $n \to \infty$, the weak* continuity in $(\mathscr{T}^{(\alpha)}(|t|^\alpha))^*$ implies that $(L_{\rho_1} + L_{\rho_2}) * \Psi = 0$. □

5.3 Bounded homomorphisms from $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and functions of spectral synthesis

For any element $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$, let $\sigma(f)$ denote its spectrum in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$. Let u be the function e_1 in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$. The spectrum of u is $\sigma(u) = \{(1+z)^{-1} : z \in \overline{\mathbb{C}^+}\} \cup \{0\}$, and its n-th convolution product is $u^{*n}(t) = \frac{1}{(n-1)!} t^{n-1} e^{-t}$ $(t > 0)$. It is shown in [GMR1, Prop. 1.1] that $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ is polynomially generated by u; that is, $\mathscr{T}_+^{(\alpha)}(t^\alpha) = \overline{\operatorname{span}}\{u^{*n} : n \in \mathbb{N}\}$.

Let $\mathfrak{B}$ be a Banach algebra with a unit e and let π be a bounded algebra homomorphism $\pi\colon \mathscr{T}_+^{(\alpha)}(t^\alpha) \oplus \mathbb{C}\delta_0 \to \mathfrak{B}$ such that $\pi(\delta_0) = e$, where δ_0 is the Dirac mass at the origin. Let $\mathscr{A}$ be the closure of the image $\pi(\mathscr{T}_+^{(\alpha)}(t^\alpha))$ in $\mathfrak{B}$. We denote by

π^* the adjoint mapping of $\pi\colon \mathscr{T}_+^{(\alpha)}(t^\alpha)\to\mathfrak{A}$, so that $\pi^*:\mathscr{A}^*\to\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, and by $\sigma(\pi(u))\equiv\sigma_{\mathfrak{B}}(\pi(u))$ the spectrum of $\pi(u)$ in $\mathfrak{B}$.

Set $Z_1(\pi):=\left\{z\in\mathbb{C}:(z+1)^{-1}\in\sigma(\pi(u))\right\}$ and denote by $Z(\pi)$ the complement of the connected component of $\mathbb{C}\backslash Z_1(\pi)$ which contains $\mathbb{C}^-$. Notice that $Z(\pi)$ is well-defined since $Z_1(\pi)\subseteq\overline{\mathbb{C}^+}$.

Proposition 5.3.1. *Under the above assumptions and notations, the following holds:*

(i) $(e-(z+1)\pi(u))^{-1}\in\mathscr{A}\oplus\mathbb{C}e$ *for all* $z\in\mathbb{C}\backslash Z(\pi)$.

(ii) Let $L_{\widetilde{\varphi}}:=\pi^*(T)$ *where* $T\in\mathscr{A}^*$. *Then the mapping*

$$z\mapsto\left\langle T,\pi(u)(e-(z+1)\pi(u))^{-1}\right\rangle,\ \mathbb{C}\setminus Z(\pi)\to\mathbb{C} \tag{5.5}$$

is an analytic extension of $\mathscr{L}(L_\varphi)$ *to* $\mathbb{C}\setminus Z(\pi)$, *which we continue denoting by* $\mathscr{L}(L_\varphi)$.

Proof. (i) Put $\lambda\equiv\lambda(z):=(z+1)^{-1}$. Clearly, if $z\in\mathbb{C}\setminus Z(\pi)$ then $\lambda\in\mathbb{C}\setminus\sigma(\pi(u))$. Moreover, $\{-1\}\subseteq\mathbb{C}^-\subseteq\mathbb{C}\setminus Z(\pi)$ and therefore $\lambda(\mathbb{C}\setminus Z(\pi))$ is contained in the unbounded connected component of $\mathbb{C}\setminus\sigma(\pi(u))$. Since the boundary of $\sigma_{\mathfrak{A}\oplus\mathbb{C}e}(\pi(u))$ is included in the boundary of $\sigma(\pi(u))$ we obtain that $\lambda(\mathbb{C}\setminus Z(\pi))\subseteq\mathbb{C}\setminus\sigma_{\mathfrak{A}\oplus\mathbb{C}e}(\pi(u))$, as claimed.

(ii) Let $z\in\mathbb{C}^-:=\{w\in\mathbb{C}:\Re w<0\}$. Since $(z+1)^{-1}\notin\sigma(u)$, we have that $\delta_0-(z+1)u$ is invertible in $\mathscr{T}_+^{(\alpha)}(t^\alpha)\oplus\mathbb{C}\delta_0$. Indeed, if $\nu_\alpha((z+1)u)<1$ then

$$(\delta_0-(z+1)u)^{-1}=\delta_0+\sum_{n=1}^{\infty}(z+1)^n u^{*n},$$

so that

$$u*(\delta_0-(z+1)u)^{-1}(x)=e^{-x}+\sum_{n=1}^{\infty}(z+1)^n\frac{x^n e^{-x}}{n!}=e^{zx}.$$

Now, let $T \in \mathscr{A}^*$ and set $L_{\tilde{\varphi}} = \pi^*(T)$. Since $L_\varphi \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$, the Laplace transform of L_φ is well-defined and it is analytic on $\mathbb{C}^-$. Moreover,

$$\mathscr{L}(L_\varphi)(z) \;=\; \int_{-\infty}^0 W_-^\alpha(e_z)(t)\,\varphi(t)\,dt \;=\; \int_0^\infty W_+^\alpha(e_{-z})(s)\,\tilde{\varphi}(s)\,ds \;=\; L_{\tilde{\varphi}}(e_{-z})$$

for any $z \in \mathbb{C}^-$. In particular, if $|z+1| < \nu_\alpha(u)^{-1}$ we have

$$\begin{aligned}\mathscr{L}(L_\varphi)(z) &= \left\langle \pi^*(T), u*(\delta_0-(z+1)u)^{-1}\right\rangle \\ &= \left\langle T, \pi(u*(\delta_0-(z+1)u)^{-1})\right\rangle = \left\langle T, \pi(u)(e-(z+1)\pi(u))^{-1}\right\rangle.\end{aligned}$$

By the identity principle for analytic functions, the above equality holds on the whole left half-plane $\mathbb{C}^-$. Now, observe that the mapping

$$z \mapsto \left\langle T, \pi(u)(e-(z+1)\pi(u))^{-1}\right\rangle,\; \mathbb{C}\setminus Z(\pi) \to \mathbb{C}$$

is analytic, so it is an analytic extension of $\mathscr{L}(L_\varphi)$ to $\mathbb{C}\setminus Z(\pi)$. □

A first consequence of the proposition is an analytic extension result for the Laplace transform linked to π and the product $\circ$.

Corollary 5.3.1. *If $L_{\tilde{\varphi}} = \pi^*(T) \in \pi^*(\mathscr{A}^*)$ then $\mathscr{L}(L_\varphi \circ f)$ extends to an analytic function on $\mathbb{C}\setminus Z(\pi)$ for every $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$. In particular, for $z \in \mathbb{C}^+\setminus Z(\pi)$ we have*

$$\mathscr{L}(L_\varphi \circ f)(z) = \left\langle T, \pi(f)\pi(u)(e-(z+1)\pi(u))^{-1}\right\rangle.$$

Proof. Let $L_{\tilde{\varphi}} = \pi^*(T)$ for some $T \in \mathscr{A}^*$. Recall that $L_\varphi \circ f = L_{\psi_-}$ with $\psi(s) := L_{\tilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$ (see Definition 5.1.1). Next, we show that $L_{\widetilde{\psi_-}} \in \pi^*(\mathscr{A}^*)$:

For every $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$,

$$\begin{aligned}L_{\widetilde{\psi_-}}(g) = \int_0^\infty W_+^\alpha g(t) L_{\tilde{\varphi}}(R_t^{\alpha-1}*f)dt = L_{\tilde{\varphi}}\left(\int_0^\infty W_+^\alpha g(t) R_t^{\alpha-1}*f\,dt\right) = L_{\tilde{\varphi}}(g*f) \\ = \langle \pi^*(T), g*f\rangle = \langle T, \pi(g)\pi(f)\rangle.\end{aligned}$$

Let $T\pi(f)$ denote the element of $\mathscr{A}^*$ defined by $\langle T\pi(f), a\rangle = \langle T, a\pi(f)\rangle$ for all $a \in \mathscr{A}$. Then one has that

$$L_{\widetilde{\psi_-}}(g) = \langle T, \pi(g)\pi(f)\rangle = \langle T\pi(f), \pi(g)\rangle = \langle \pi^*(T\pi(f)), g\rangle$$

and therefore $L_{\widetilde{\psi^-}} = \pi^*(T\pi(f)) \in \pi^*(\mathscr{A}^*)$ as claimed.

The statement follows now from Proposition 5.3.1. In addition, we have that

$$\mathscr{L}(L_\varphi \circ f)(z) = \left\langle T, \pi(f)\pi(u)(e-(z+1)\pi(u))^{-1}\right\rangle$$

if z is taken in $\mathbb{C}^+ \setminus Z(\pi)$. □

Now we relate the products $\circ$ and $\diamond$ for functionals associated with the homomorphism π. The result extends [ESZ, Proposition 2.5].

Corollary 5.3.2. *For every $L_{\widetilde{\varphi}} \in \pi^*(\mathscr{A}^*)$ and $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$,*

$$\mathscr{L}(L_\varphi \circ f) + \mathscr{L}(L_\varphi \diamond f) = \mathscr{L}(L_\varphi)\,\mathscr{L}(f) \quad \text{in } \mathbb{C}^+ \setminus Z(\pi). \tag{5.6}$$

Proof. Let $L_{\widetilde{\varphi}} \in \pi^*(\mathscr{A}^*)$ and $z \in \mathbb{C}^+ \setminus Z(\pi)$ be fixed. By Proposition 5.3.1 and Corollary 5.3.1, the mapping $\Phi : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathbb{C}$ given for each $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ by

$$\Phi(f) := \mathscr{L}(L_\varphi \circ f)(z) + \mathscr{L}(L_\varphi \diamond f)(z) - \mathscr{L}(L_\varphi)(z)\mathscr{L}(f)(z)$$

is well-defined and linear. Furthermore, it is continuous. To prove that we note firstly that $f \to \mathscr{L}(f)(z)$ is clearly continuous. Secondly, recall that $L_\varphi \diamond f = L_{\psi_+} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ with $\psi(s) = L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$. Hence,

$$\begin{aligned} |\mathscr{L}(L_\varphi \diamond f)(z)| = |L_{\psi_+}(e_z)| \le |z|^\alpha \int_0^\infty e^{-\Re z t}\, |L_{\widetilde{\varphi}}(R_{-t}^{\alpha-1} * f)|\, dt \\ \le C\, \nu_\alpha(f)\, |z|^\alpha \int_0^\infty t^\alpha e^{-t\Re z}\, dt = C(z)\nu_\alpha(f). \end{aligned}$$

Finally, by Corollary 5.3.1 we have that

$$\mathscr{L}(L_\varphi \circ f)(z) = \left\langle T, \pi(f)\pi(u)(e-(z+1)\pi(u))^{-1} \right\rangle$$

for all $z \in \mathbb{C}^+ \setminus Z(\pi)$. Thus $|\mathscr{L}(L_\varphi \circ f)(z)| \le C'(z)\nu_\alpha(f)$. In conclusion, Φ is bounded on $\mathscr{T}_+^{(\alpha)}(t^\alpha)$.

Now we want to prove that $\Phi = 0$. Clearly, it will be enough to show that Φ vanishes on $C_c^\infty(\mathbb{R}^+)$. So take $f \in C_c^\infty(\mathbb{R}^+)$. Then we have that $L_\varphi * f \in C^{(\infty)}(\mathbb{R})$ and $\operatorname{supp}(L_\varphi * f) \subseteq \operatorname{supp}(L_\varphi) + \operatorname{supp}(f) \subseteq (-\infty, a]$ where $a := \sup(\operatorname{supp}(f)) \ge 0$ [Ru, Th. 6.30 b) and Th. 6.37 b)]. Hence, $\mathscr{L}(L_\varphi * f)$ exists and it is analytic at least on $\mathbb{C}^-$. Moreover, $\mathscr{L}(L_\varphi * f) = \mathscr{L}(L_\varphi)\mathscr{L}(f)$ on $\mathbb{C}^-$, where all factors have sense simultaneously. On the other hand, we have by definition that $L_\varphi * f = L_\varphi \circ f + L_\varphi \diamond f$, where $L_\varphi \circ f$ and $L_\varphi \diamond f$ are supported on $(-\infty, 0]$ and $[0, \infty)$, respectively. Since $\operatorname{supp}(L_\varphi * f) \subseteq (-\infty, a]$, it follows that $\operatorname{supp}(L_\varphi \diamond f) \subseteq [0, a]$, so that its Laplace transform is an entire function. Therefore, $\mathscr{L}(L_\varphi * f) = \mathscr{L}(L_\varphi \circ f) + \mathscr{L}(L_\varphi \diamond f)$ on $\mathbb{C}^-$. Then, $\mathscr{L}(L_\varphi \circ f) + \mathscr{L}(L_\varphi \diamond f) = \mathscr{L}(L_\varphi)\mathscr{L}(f)$ on $\mathbb{C}^-$. Indeed, this equality holds on $\mathbb{C} \setminus Z(\pi)$ by the identity theorem for analytic functions. This concludes the proof. □

Next, we give the key result on functions in $\mathscr{T}_+^{(\alpha)}(t^\alpha) \subseteq \mathscr{T}^{(\alpha)}(|t|^\alpha)$ which are of spectral synthesis with respect to $(-iZ(\pi)) \cap \mathbb{R}$. Such a result is in the spirit of [ESZ, Théorème 2.7].

Theorem 5.3.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathscr{A}^*$ and denote $L_{\widetilde{\varphi}_n} := \pi^*(T_n)$, $n \in \mathbb{N}$. If $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ is a function of spectral synthesis with respect to $(-iZ(\pi)) \cap \mathbb{R}$ then*

$$\lim_{n\to\infty} t_n^{-\alpha} L_{\varphi_n} \circ (R_{t_n}^{\alpha-1} * g) \circ f = 0,$$

*in the weak** *topology of* $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$, *for every* $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ *and* $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ *such that* $\lim_{n\to\infty} t_n = \infty$.

Proof. For $g \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ and (t_n) as above, set $g_n := t_n^{-\alpha} R_{t_n}^{\alpha-1} * g$ ($n \in \mathbb{N}$). Note that $\nu_\alpha(g_n) \leq C\nu_\alpha(g)$ for all n. We want to prove that $\lim_{n\to\infty} L_{\varphi_n} \circ g_n \circ f = 0$ weakly* in $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$.

The fact that $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded implies that the sequences $(L_{\varphi_n} \circ g_n)$ and $(L_{\varphi_n} \diamond g_n)$ are uniformly bounded in $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$, respectively. Indeed, for all n,

$$\|L_{\varphi_n} \circ g_n\|,\ \|L_{\varphi_n} \diamond g_n\| \leq C \sup_n \|\pi^*(T_n)\| \nu_\alpha(g).$$

Also, since $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ and $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ are separable Banach spaces their bounded subsets are weak* metrizable and then by the Banach-Alaouglu theorem there exist (taking subsequences if necessary) the two weak* limits

$$\lim_{n\to\infty} L_{\varphi_n} \circ g_n =: L_{\rho_1} \in \mathscr{T}_-^{(\alpha)}((-t)^\alpha)^* \tag{5.7}$$

and

$$\lim_{n\to\infty} L_{\varphi_n} \diamond g_n =: L_{\rho_2} \in \mathscr{T}_+^{(\alpha)}(t^\alpha)^* \tag{5.8}$$

On the other hand, the family $(\mathscr{L}(L_{\varphi_n} \circ g_n))$ is a normal family of analytic functions in $\mathbb{C}\backslash Z(\pi)$: By Proposition 5.3.1, $\mathscr{L}(L_{\varphi_n} \circ g_n)(z)$ extends holomorphically to $z \in \mathbb{C} \setminus Z(\pi)$ as

$$\mathscr{L}(L_{\varphi_n} \circ g_n)(z) := \left\langle T_n, \pi(g_n)\pi(u)(e-(z+1)\pi(u))^{-1} \right\rangle.$$

Since the sequences (T_n) and $(\nu_\alpha(g_n))$ are uniformly bounded and the mapping $z \in \mathbb{C}^+\backslash Z(\pi) \mapsto \pi(u)(e-(z+1)\pi(u))^{-1} \in \mathscr{A}$ is analytic in $\mathbb{C}\backslash Z(\pi)$, we have that

$(\mathscr{L}(L_{\varphi_n} \circ g_n))$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash Z(\pi)$ as well. In other words, $(\mathscr{L}(L_{\varphi_n} \circ g_n))$ is a normal family on $\mathbb{C}\backslash Z(\pi)$.

Therefore, by the Montel theorem we can assume (by passing to a subsequence if necessary) that there exists an analytic function H in $\mathbb{C}\backslash Z(\pi)$ such that

$$\lim_{n\to\infty} \mathscr{L}(L_{\varphi_n} \circ g_n) = H \tag{5.9}$$

uniformly on compact subsets of sets of $\mathbb{C}\backslash Z(\pi)$. Also, it is readily seen that $(\mathscr{L}(L_{\varphi_n} \diamond g_n))_{n\in\mathbb{N}}$ converges to an analytic function on $\mathbb{C}^+$ uniformly on compact subsets.

Now, we are going to prove that $\lim_{n\to\infty} \mathscr{L}(g_n)(z) = 0$ for any $z \in \mathbb{C}^+$. Note that

$$\mathscr{L}(g_n)(z) = \int_0^\infty g_n(t)e^{-zt}\,dt = \int_0^\infty W_+^\alpha g_n(t) D^{-\alpha}(e_z)(t)\,dt$$

for every $z \in \mathbb{C}^+$, where

$$D^{-\alpha}(e_z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_z(s)\,ds\,,\ t > 0.$$

Then $t^{-\alpha}D^{-\alpha}(e_z) \in C_0([0,\infty)$ and so $\lim_{n\to\infty} \mathscr{L}(g_n)(z) = 0$ $(z \in \mathbb{C}^+)$, by Proposition 5.1.1.

Applying formula (5.6) to φ_n and g_n, and then using (5.9) and (5.8), we obtain

$$H = \lim_{n\to\infty} \mathscr{L}(L_{\varphi_n} \circ g_n) = -\lim_{n\to\infty} \mathscr{L}(L_{\varphi_n} \diamond g_n) = -\mathscr{L}(L_{\rho_2})$$

in $\mathbb{C}^+\backslash Z(\pi)$. Hence, the function given by

$$F(z) := \begin{cases} H(z)\,, & \text{if} \quad z \in \mathbb{C}\backslash Z(\pi); \\ -\mathscr{L}(L_{\rho_2})(z)\,, & \text{if} \quad z \in \mathbb{C}^+ \end{cases}$$

is well-defined and analytic on $\mathbb{C}\backslash(Z(\pi) \cap i\mathbb{R})$. In addition, $F\,|_{\mathbb{C}^-} = \mathscr{L}(L_{\rho_1})$ by (5.7) and (5.9), and also $F\,|_{\mathbb{C}^+} = -\mathscr{L}(L_{\rho_2})$.

Denote $U := i\mathbb{R}\backslash(Z(\pi)\cap i\mathbb{R})$. Recall that f is assumed to be a function of spectral synthesis with respect to $(-iZ(\pi))\cap\mathbb{R}$. Thus there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathscr{T}^{(\alpha)}(|t|^\alpha)$ such that $\operatorname{supp}\mathscr{F} f_n \subseteq -iU$ and $\lim_{n\to\infty}\nu_\alpha(f-f_n)=0$. Clearly,

$$\lim_{n\to\infty}(L_{\rho_1}+L_{\rho_2})*f_n=(L_{\rho_1}+L_{\rho_2})*f$$

in the weak* topology of $\mathscr{T}^{(\alpha)}(|t|^\alpha)^*$. On the other hand, $\mathscr{L}(L_{\rho_2})$ can be continuously extended to U as H, and so we get that $\mathscr{L}(L_{\rho_2})\mid_U := -\mathscr{L}(L_{\rho_1})\mid_U$. Then, by Proposition 5.2.2 we have that $(L_{\rho_1}+L_{\rho_2})*f_n=0$ for every $n\in\mathbb{N}$. Therefore, $(L_{\rho_1}+L_{\rho_2})*f=0$ in $\mathscr{T}^{(\alpha)}(|t|^\alpha)^*$. In particular, if $h\in\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ then

$$(L_{\rho_1}*f)(h)=-(L_{\rho_2}*f)(h)=-L_{\rho_2}(\tilde{f}*h)=0$$

since $\operatorname{supp}(L_{\rho_2})\subseteq[0,\infty)$ and $\operatorname{supp}(\tilde{f}*h)\subseteq(-\infty,0]$. In other words, $L_{\rho_1}\circ f=0$.

In conclusion, we have proved that any weak* cluster point of the sequence $(L_{\varphi_n}\circ g_n\circ f)_{n\in\mathbb{N}}$ is 0. This implies (recall again the Banach-Alaouglu theorem and the metrizability of bounded weak* subsets of $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$) that $\lim_{n\to\infty}t_n^{-\alpha}L_{\varphi_n}\circ(R_{t_n}^{\alpha-1}*g)\circ f=0$, as we wanted to show. □

5.4 Katznelson-Tzafriri theorem for integrated semigroups

On the base of what has been established in previous sections, we go on to prove the main result of the paper (Theorem 5.0.1).

Recall that we have defined a C_α-semigroup as an α-times integrated semigroup on X such that

$$\lim_{t\to0}\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x=x \quad (x\in X).$$

In Proposition 1.3.2, we have proved that C_α-semigroups on X of growth t^α are in a one-to-one correspondence with bounded Banach algebra homomorphisms

$\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ for which $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X. In particular, if $(T_\alpha(t))_{t\geq 0}$ is a C_α-semigroup on X such that $\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty$ then the bounded homomorphism $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ given by the formula in (1.10) is such that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X. Conversely, for every bounded homomorphism $\pi_\alpha : \mathscr{T}_+^{(\alpha)}(t^\alpha) \to \mathscr{B}(X)$ such that $\pi_\alpha(\mathscr{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X, the family defined by

$$T_\alpha(t)x := \pi_\alpha(R_t^{\alpha-1} * g)y, \quad (x = \pi_\alpha(g)y \in \mathscr{T}_+^{(\alpha)}(t^\alpha); t \geq 0),$$

is a C_α-semigroup on X verifying $\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty$ whose associated homomorphism defined by formula (1.10) is π_α.

Next, we give the proof of Theorem 5.0.1. Let us point out first the following lemma:

Lemma 5.4.1. *Let A be the generator of a sub-homogeneous α-times integrated semigroup $T_\alpha(t)$. Let π_α be defined as in (1.10). Then, $-iZ(\pi_\alpha) \cap \mathbb{R} = i\sigma(A) \cap \mathbb{R}$.*

Proof. Set $B := -A$ and $D(B)$ for the domain of B. Note that

$$(B+I)^{-1} = \int_0^\infty e^{-t} T_\alpha(t)\, dt = \pi_\alpha(u) \in \mathscr{B}(X).$$

In particular, $X = (B+I)D(B)$. These facts imply that

$$[I-(z+1)\pi_\alpha(u)]X = (B-zI)D(B)$$

for every $z \in \mathbb{C}$. As a result, $Z_1(\pi_\alpha) = \sigma(B)$. Therefore, $Z(\pi_\alpha) \cap i\mathbb{R} = Z_1(\pi_\alpha) \cap i\mathbb{R} = -\sigma(A) \cap i\mathbb{R}$. □

Proof of Theorem 5.0.1. Let $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ be a function of spectral synthesis with respect to the subset $i\sigma(A) \cap \mathbb{R}$. Since $T_\alpha(t) = \pi_\alpha(R_t^{\alpha-1})$, we have to prove that

$\lim_{t\to\infty}\pi_\alpha(t^{-\alpha}R_t^{\alpha-1}*f)=0$ in norm. Take $g\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$ and $(t_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}^+$ be such that $\lim_{n\to\infty}t_n=\infty$. Put $g_n:=t_n^{-\alpha}R_{t_n}^{\alpha-1}*g$, $n\in\mathbb{N}$. For $h\in\mathscr{T}_+^{(\alpha)}(t^\alpha)$, by the Hahn-Banach theorem there exists a sequence $(T_n)_{n\in\mathbb{N}}\subseteq\mathscr{A}^*$ such that $\|T_n\|=1$, and

$$\|\pi_\alpha(h*f*g_n)\|=\langle T_n,\pi_\alpha(h*f*g_n)\rangle\,,\ \text{for all } n.$$

Hence, if $L_{\widetilde{\varphi}_n}:=\pi_\alpha^*(T_n)$, we have that

$$\|\pi_\alpha(h*f*g_n)\|=\left\langle L_{\widetilde{\varphi}_n}\bullet g_n\bullet f,h\right\rangle,\ \text{for all } n.$$

By Proposition 5.1.3 one has that $\lim_{n\to\infty}L_{\widetilde{\varphi}_n}\bullet\vartheta_n\bullet f=0$ weakly* in $\mathscr{T}_+^{(\alpha)}(t^\alpha)^*$ if and only if $\lim_{n\to\infty}L_{\varphi_n}\circ g_n\circ f=0$ weakly* in $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)^*$. The latter limit holds as a consequence of Theorem 5.3.1, since f is of spectral synthesis with respect to $i\sigma(A)\cap\mathbb{R}=-iZ(\pi_\alpha)\cap\mathbb{R}$. Therefore

$$\lim_{n\to\infty}t_n^{-\alpha}\|\pi_\alpha(R_{t_n}^{\alpha-1}*f*g*h)\|=\lim_{n\to\infty}\|\pi_\alpha(h*f*g_n)\|=0,$$

that is,

$$\lim_{n\to\infty}t_n^{-\alpha}\|T_\alpha(t_n)\pi_\alpha(f*F)\|=0\ \text{for every } F\in\mathscr{T}_+^{(\alpha)}(t^\alpha),$$

since $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ factorizes and $T_\alpha(t_n)=\pi_\alpha(R_{t_n}^{\alpha-1})$. Finally, letting F running over a a bounded approximate identity family in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, we get the desired result:

$$\lim_{n\to\infty}t_n^{-\alpha}\|T_\alpha(t_n)\pi_\alpha(f)\|=0\,.$$

□

Remark 5.4.1. There are proofs of the Esterle-Strouse-Vũ-Zouakia theorem on the basis of the Ingham's tauberian theorem or using the notion of complete trajectories, see [CT, pp. 84, 85]. We wonder if such arguments admit analogues, in the setting of integrated semigroups, which could be fruitfully employed to give alternative proofs of Theorem 5.0.1.

The Esterle-Strouse-Zouakia and Vũ's theorem can be regarded as a result on stability of orbits $T(\cdot)x$, for $x \in \mathrm{Im}\, \pi_0(f)$ and appropriate f in $L^1(\mathbb{R}^+)$. Further, it is proven in [ESZ] that the aforementioned theorem implies (in a certainly non-trivial way) the Arendt-Batty-Lyubich-Vũ stability theorem.

Here, we do not deal with stability of integrated semigroups. Instead, we consider another type of asymptotic behaviour (which is of clear ergodic nature when the integrated semigroup $T_\alpha(t)$ comes from a C_0-semigroup $T(t)$ as given by $T_\alpha(t) = \Gamma(\alpha)^{-1} \int_0^t (t-s)^{\alpha-1} T(s)\, ds$).

Definition 5.4.1. *Given an α-times integrated semigroup $(T_\alpha(t))_{t\geq 0}$ on X, we call an orbit $T_\alpha(\cdot)x$ $(x \in X)$ $o(t^\alpha)$ -ergodic if*

$$\lim_{t\to\infty} t^{-\alpha} \|T_\alpha(t)x\| = 0\,.$$

We say that $(T_\alpha(t))_{t\geq 0}$ is $o(t^\alpha)$ - ergodic if $T_\alpha(\cdot)x$ is $o(t^\alpha)$ - ergodic for every $x \in X$.

Proposition 5.4.1. *Let $T_\alpha(t)$, A and π_α be under the assumptions of Theorem 5.0.1. Let $\rho(A)$ be the resolvent set of A. If $\sigma(A) \cap i\mathbb{R} = \emptyset$ then*

$$\lim_{t\to\infty} t^{-\alpha} \left\|T_\alpha(t)(\lambda - A)^{-1}\right\| = 0 \quad \forall \lambda \in \rho(A).$$

In consequence, $(T_\alpha(t))_{t\geq 0}$ is $o(t^\alpha)$ - ergodic; that is,

$$\lim_{t\to\infty} t^{-\alpha} \|T_\alpha(t)x\| = 0 \quad \forall x \in X.$$

Proof. Take λ with $\Re\lambda > 0$, so that $\lambda \in \rho(A)$ in particular. Recall that the function $e_\lambda(t) = e^{-\lambda t}$ $(t > 0)$ is in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ with $W_+^\alpha e_\lambda = \lambda^\alpha e_\lambda$ (see Section 4, prior to

Proposition 5.2.1). Hence, for $x \in X$,

$$(\lambda - A)^{-1}x = \int_0^\infty (W_+^\alpha e_\lambda)(t)\, T_\alpha(t) x\, dt = \pi_\alpha(e_\lambda)x.$$

Notice now that every function in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, so e_λ, is of spectral synthesis for the empty set. Hence, by Theorem 5.0.1, one gets

$$\lim_{t\to\infty} t^{-\alpha} \|T_\alpha(t)(\lambda - A)^{-1}\| = \lim_{t\to\infty} t^{-\alpha} \|T_\alpha(t)\pi_\alpha(e_\lambda)\| = 0.$$

For arbitrary μ in $\rho(A)$ it is enough to apply the resolvent identity for μ and λ with $\Re\lambda > 0$ to obtain $\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)(\mu - A)^{-1}\| = 0$.

Finally, the $o(t^\alpha)$-ergodicity of $T_\alpha(t)$ follows by factorising

$$x = (\mu - A)^{-1}(\mu - A)x$$

if x belongs to the domain $D(A)$ of A, and then for arbitrary $x \in X$ by the density of $D(A)$ in X and the uniform boundedness of $t^{-\alpha}T_\alpha(t)$. □

Remark 5.4.2. (1) The assumption $\sigma(A) \cap i\mathbb{R} = \emptyset$ implies that $0 \in \rho(A)$ and therefore we obtain from the above proposition that

$$\lim_{t\to\infty} t^{-\alpha} \left\|T_\alpha(t)A^{-1}\right\| = 0.$$

This extends [V, Corollary 3.3].

(2) For a bounded C_0-semigroup $T(t)$ the necessary property given in the first part of Proposition 5.4.1 is in fact an equivalence:

$$\lim_{t\to\infty} t^{-\alpha} \left\|T_\alpha(t)(\lambda - A)^{-1}\right\| = 0 \quad \forall\lambda \in \rho(A) \Leftrightarrow \sigma(A) \cap i\mathbb{R} = \emptyset;$$

see [N, Cor. 5.2.6]. Thus we wonder if this equivalence also holds for integrated semigroups. One of the ingredients to prove the above result is that if $f \in L^1(\mathbb{R}^+)$

is such that $\lim_{t\to\infty} \|T(t)\pi_0(f)\| = 0$ then its Fourier transform $\mathscr{F}(f)$ vanishes on $\sigma(A)\cap i\mathbb{R}$. Unfortunately, the argument used in [N, Th. 5.2.6] to show this property of $\mathscr{F}(f)$ does not work for integrated semigroups. The reason is that in the latter case one cannot appeal to translations $\delta_s * f$ in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$. The natural substitute in $\mathscr{T}_+^{(\alpha)}(t^\alpha)$ for δ_s is the family of Riesz kernels $R_s^{\alpha-1}$, but then by taking convolutions $R_s^{\alpha-1} * f$ one gets that the lower bound obtained for semigroups in [N, Th. 5.2.6] fails for integrated semigroups.

(3) An alternative way to show the $o(t^\alpha)$-ergodicity of $T_\alpha(t)$ in Proposition 5.4.1 relies on the idea employed in [ESZ] to prove the Arendt-Batty-Lyubich-Vũ stability theorem from [ESZ, Théorème 3.4].

Let $\mathfrak{S}_\alpha$ be the set formed by all functions $f \in \mathscr{T}_+^{(\alpha)}(t^\alpha)$ which are of spectral synthesis with respect to $i\sigma(A)\cap\mathbb{R}$. By Theorem 5.0.1 it follows that the subset Y of X given by

$$Y := \{\pi_\alpha(f)x : f \in \mathfrak{S}, x \in X\}, \tag{5.10}$$

defines a family of $o(t^\alpha)$ - ergodic orbits of a C_α-semigroup $(T_\alpha(t))_{t\geq 0}$. Trivially, if the set Y is dense in X then the integrated semigroup is $o(t^\alpha)$ - ergodic. From this observation, and since the empty set is of spectral synthesis, using Proposition 1.3.2 and the Cohen's factorization theorem we obtain that $\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)x\| = 0$ for every $x \in X$, whenever $\sigma(A)\cap i\mathbb{R} = \emptyset$.

(Note that if one assumes additionally that the integrated semigroup $T_\alpha(t)$ is Lipschitz continuous then that conclusion follows automatically by [Me, Th. 2.4].)

Chapter 6

Weak spectral synthesis and ergodicity of semigroups

One of the purposes of the present chapter is to show that, at least in some cases, the argument considered in [ESZ] to deduce the Arendt-Batty-Lyubich-Vũ theorem works (partially) for integrated semigroups. More precisely, we prove that

$$\pi_n(\mathfrak{S}_n)X \text{ is dense in } X, \tag{6.1}$$

which, by applying Theorem 5.0.1 (p. 103), gives $\lim_{t\to\infty} t^{-n}T_n(t)x = 0$ for every $x \in X$. The proof in [ESZ] appeals to the Hahn-Banach theorem and methods of harmonic analysis, and relies on the fact that countable sets in $\mathbb{R}$ are sets of spectral synthesis for $L^1(\mathbb{R})$. Thus, we first need to analyse the type of spectral synthesis properties that the algebra $\mathscr{T}^{(n)}(|t|^n)$ enjoys, and related items.

The organization of the chapter is as follows. In Section 6.1 we collect some specific lemmata on convolution and derivations of Fourier transforms which will be needed in subsequents sections. In Section 6.2, we study primary ideals and (weak) spectral synthesis properties of the algebra $\mathcal{T}^{(n)}(|t|^n)$. In contrast with the L^1-case, not even points in $\mathbb{R}$ are of spectral synthesis in $\mathcal{T}^{(n)}(|t|^n)$ for $n \geq 1$, with the only exception of $t = 0$. This is proved in Theorem 6.2.1 together with a characterization of the primary closed ideals of $\mathcal{T}^{(n)}(|t|^n)$. In particular, we prove that a function $f \in \mathcal{T}^{(n)}(|t|^n)$ is of spectral synthesis in $\mathcal{T}^{(n)}(|t|^n)$ with respect to a countable subset S of $\mathbb{R}$ if $x^j \mathcal{F} f^{(j)}(x) = 0$ for every $x \in S$ and $0 \leq j \leq n$. This implies a sort of weak spectral synthesis in the algebra $\mathcal{T}^{(n)}(|t|^n)$ for closed countable subsets, see Theorem 6.2.2. In Section 6.3 we apply these results to the asymptotic behavior of (integrated) semigroups, and prove the density result (6.1) and its consequences, see Theorem 6.3.2 and Theorem 6.3.3.

6.1 Derivatives, convolution and Fourier transform

Lemma 6.1.1. *For $g \in \mathcal{S}(\mathbb{R})$, $n \in \mathbb{N}$ and $k = 0, \ldots, n$,*

$$(ix)^k (\mathcal{F} g)^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!} \mathcal{F}(t^j g^{(j)})(x), \quad x \in \mathbb{R}. \tag{6.2}$$

Proof. Let $k = 1, \ldots, n$. Using integration by parts k times and the Leibniz's derivation rule we get, for $x \in \mathbb{R}$,

$$\begin{aligned}(ix)^k (\mathcal{F} g)^{(k)}(x) &= (ix)^k \int_{-\infty}^{\infty} (t^k g)^{(k)}(t) \frac{e^{-ixt}}{(ix)^k} \\ &= \sum_{j=0}^{k} \binom{k}{j} \int_{-\infty}^{\infty} (t^k)^{(k-j)} g^{(j)}(t) e^{-ixt}\, dt \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!} \mathcal{F}(t^j g^{(j)})(x),\end{aligned}$$

as we wanted to show. □

Let $\mathscr{D}(0,\infty)$ denote the space of test functions with compact support in $(0,\infty)$. The following lemma is part of folklore. We include a proof for the sake of completeness, since it will be significantly used in Section 6.2 (Proposition 6.2.2) and Section 6.3.

Lemma 6.1.2. *Let n and N be nonnegative integers. For $a_m \in \mathbb{R}$, $m = 0,1,\dots,N$, and $c_{jk} \in \mathbb{C}$, $j = 0,1\dots,N$, there exist functions $u_j \in \mathscr{D}(0,\infty)$, $j = 0,1,\dots,N$, such that*

$$(\mathscr{F}u_j)^{(k)}(a_m) = \delta_{j,m}c_{jk} \quad (j,m = 0,1,\dots,n; k = 0,1,\dots,n).$$

Proof. For $0 \le j \le N$ and $0 \le k \le n$ define the distribution $\Phi_{jk} \in \mathscr{D}'(0,\infty)$ by

$$\Phi_{jk}(u) := u^{(k)}(a_m) = (-i)^k \int u(y) y^k e^{-ia_m y} dy, \quad u \in \mathscr{D}(0,\infty).$$

The above family Φ_{jk} is linearly independt: If for some $d_{jk} \in \mathbb{R}$ one has $0 = \sum_{j,k} d_{jk}\Phi_{jk}(u)$ for all $u \in \mathscr{D}(0,\infty)$ then

$$0 = \sum_{j,k} (-i)^k d_{jk} y^k e^{-ia_m y} = \sum_{k=0}^{n} \left(\sum_{j=0}^{N} d_{jk} e^{-ia_m y} \right) (-i)^k y^k,$$

for all $m = 0,1,\dots,N$ and every $y > 0$. Dividing by y^n in the above equality one obtains that

$$\lim_{y\to+\infty} \sum_{j=0}^{N} d_{jn} e^{-ia_m y} = 0.$$

Since $\sum_{j=0}^{N} d_{jn} e^{-ia_m y}$ is almost periodic it follows that $d_{jn} = 0$ for all $j = 0,1,\dots,N$. Now, by recurrence on k from $k = n$ to $k = 0$ one ends finding that $c_{jk} = 0$ for all $j = 0,1,\dots,N$ and $k = 0,1,\dots,n$.

Hence $F_{jk} := \operatorname{span}\{\Phi_{m,l} : (m,l) \neq (j,k)\}$ is a finite-dimensional subspace of $\mathscr{D}'(0,\infty)$ and Φ_{jk} does not belong to F_{jk}. By Hahn-Banach theorem there is $v_{jk} \in$

$\mathscr{D}'(0,\infty)' = \mathscr{D}(0,\infty)$, since $\mathscr{D}(0,\infty)$ is reflexive, such that $\Phi_{ml}(v_{jk}) = \langle v_{jk}, \Phi_{ml}\rangle = 0$, if $j \neq m$ or $k \neq l$, and $\Phi_{jk}(v_{jk}) = \langle v_{jk}, \Phi_{jk}\rangle = 1$.

For every $j = 0,1,\ldots,N$, put $u_j := \sum_{l=0}^{n} c_{jl} v_{jl}$. Then,

$$u_j(\Phi_{jk}) = \sum_{l=0}^{n} c_{jl} v_{jl}(\Phi_{jk}) = c_{jk} v_{jk}(\Phi_{jk}) = c_{jk},$$

whereas

$$u_j(\Phi_{mk}) = \sum_{l=0}^{n} c_{jl} v_{jl}(\Phi_{mk}) = 0 \text{ if } m \neq j, \text{ for every } k, 0 \leq k \leq n.$$

It follows that $(\mathscr{F}u_j)^{(k)}(a_m) = \delta_{j,m} c_{jk}$ as required. □

Lemma 6.1.3. *Given $a_0, a_1, \ldots, s_N \in \mathbb{R}$ and $c_{jk} \in \mathbb{C}$ for $j = 0,1,\ldots,N$ and $k = 0,1,\ldots,n$ there exists $u \in \mathscr{D}'(o,\infty)$ such that*

$$(\mathscr{F}u)^{(k)}(a_j) = c_{jk}, \quad (0 \leq j \leq N; 0 \leq k \leq n).$$

Proof. It suffices to take $u := u_0 + u_1 + \cdots + u_N$ with $u_0, u_1, \ldots, u_N$ the functions in the statement of Lemma 6.1.2. □

Lemma 6.1.4. *Let f be a complex function on $\mathbb{R}$ such that $t^k f \in L^1(\mathbb{R})$ for $k = 0,1,\ldots,n$. Then, for every $g \in \mathscr{S}$,*

$$t^n (f * g)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (t^{n-j} f * t^j g^{(n)}). \tag{6.3}$$

Proof. This is straightforward. For f and g as in the statement, and $t \in \mathbb{R}$,

$$\begin{aligned} t^n (f * g)^{(n)}(t) = t^n (f * g^{(n)})(t) &= \int_{-\infty}^{\infty} (t - s + s)^n f(t-s) g^{(n)}(s) ds \\ &= \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{\infty} (t-s)^{n-j} f(t-s)\, s^j g^{(n)}(s) ds, \end{aligned}$$

as we wanted to show. □

Let $h \in L^1(\mathbb{R})$ and put $h_\rho(x) := \rho h(\rho x)$, a.e. $x \in \mathbb{R}$, $\rho > 0$. It is well known that, for every $g \in L^1(\mathbb{R})$,

$$\|h_\rho * g - \mathscr{F}h(0)g\|_{L^1} \to 0, \text{ as } \rho \to \infty. \tag{6.4}$$

See [RS, p. 8] for details. Notice that $(h_\rho)_{\rho>0}$ is a summability kernel in $L^1(\mathbb{R})$ if $\mathscr{F}h(0) = 1$. In order to study primary ideals in $\mathscr{T}^{(n)}(|t|^n)$ we need a version of the above convergence for derivatives of functions.

Lemma 6.1.5. *Let h be a locally integrable function on $\mathbb{R}$ with compact support and let $g \in \mathscr{S}(\mathbb{R})$. Then*

$$\|h_\rho * g - \sum_{j=0}^{m} \frac{(-i)^j}{j!} \rho^{-j} \, (\mathscr{F}h)^{(j)}(0) g^{(j)}\|_{L^1} = \mathrm{O}(\rho^{-(m+1)}), \text{ as } \rho \to \infty, \tag{6.5}$$

for every $m \in \mathbb{N} \cup \{0\}$.

Proof. For $\rho > 0$ and $0 \le j \le m$, $(\mathscr{F}h)^{(j)}(0) = (-i)^j \rho^j \int_{-\infty}^{\infty} y^j h_\rho(y) dy$. Then, for $x \in \mathbb{R}$,

$$\begin{aligned} R(x,\rho) &:= (h_\rho * g)(x) - \sum_{j=0}^{m} \rho^{-j} \frac{(-i)^j}{j!} (\mathscr{F}h)^{(j)}(0)\, g^{(j)}(x) \\ &= \int_{-\infty}^{\infty} h_\rho(y) \left(g(x-y) - \sum_{j=0}^{m} \frac{g^{(j)}(x)}{j!} (-y)^j \right) dy \\ &= \int_{\mathrm{supp}(h)} h(t) \left(g(x-t/\rho) - \sum_{j=0}^{m} \frac{g^{(j)}(x)}{j!} (-t/\rho)^j \right) dt \\ &= \frac{1}{(m+1)!} \int_{\mathrm{supp}(h)} h(t)(-t/\rho)^{m+1} g^{(m+1)}(\xi_{x,t})\, dt \end{aligned}$$

where we have applied the Taylor formula with Lagrange remainder, so that $\xi_{x,t}$ inside the last integral is a point between $x - t/\rho$ and x.

Since h has compact support, one can choose $\rho > 0$ large enough so that $\xi_{x,t} \in (x-1, x+1)$ for all $x \in \mathbb{R}$ and $t \in \mathrm{supp}(h)$. Moreover, as $g \in \mathscr{S}(\mathbb{R})$ there is a constant C such that $|g^{(m+1)}(\xi_{x,t})| \le C(1+\xi_{x,t}^2)^{-1}$.

Hence, for all $\rho > 0$,

$$\rho^{m+1}\|R(\cdot,\rho)\|_{L^1} \leq \frac{C}{(m+1)!}\left(\int_{\mathrm{supp}(h)} |h(t)|\,|t|^{m+1}\,dt\right) \times\left(2\int_1^\infty \frac{dx}{1+|x-1|^2}+\int_{-1}^1 dx\right) = \frac{C(\pi+2)}{(m+1)!}\|h\|_{L^1(|t|^{m+1})}.$$

This proves (6.5). □

6.2 Weak spectral synthesis in Sobolev algebras

Recall from Chapter 1 that the character space of $\mathscr{T}^{(n)}(|t|^n)$ is isomorphic to $\mathbb{R}$ and its Gelfand transform is equal to the Fourier transform $\mathscr{F}$. Moreover, $\mathscr{T}^{(n)}(|t|^n)$ is regular on $\mathbb{R}$ since it contains the test functions. See Chapter 1

Next, we study the range space of $\mathscr{F}$ on $\mathscr{T}^{(n)}(|t|^n)$. Let $C_0^{(n)}(x^n)$ denote the space formed by the continuous functions $\varphi\colon \mathbb{R}\to\mathbb{C}$ such that, for every $k = 0,1\ldots n$:

(i) There exists the k-th derivative $\varphi^{(k)}$ on $\mathbb{R}\setminus\{0\}$.

(ii) The function $\Phi_k\colon x\mapsto x^k\varphi^{(k)}(x)$ belongs to $C_0(\mathbb{R})$.

(iii) $\Phi_k(0) = 0$ for $k = 1,\ldots,n$.

It is readily seen that $C_0^{(n)}(x^n)$ is a Banach algebra endowed with pointwise multiplication and norm $\|\cdot\|_{\infty,(n)}$ given by

$$\|\varphi\|_{\infty,(n)} := \sum_{k=0}^{n}\|x^k\varphi^{(k)}\|_\infty, \quad \varphi\in C_0^{(n)}(\mathbb{R}).$$

Proposition 6.2.1. *For every $f\in\mathscr{T}^{(n)}(|t|^n)$, we have $\mathscr{F}(f)\in C_0^{(n)}(x^n)$ and then the Fourier transform $\mathscr{F}:\mathscr{T}^{(n)}(|t|^n)\to C_0^{(n)}(x^n)$ is a bounded Banach algebra homomorphism.*

Proof. Obviously, $\mathscr{F}\varphi \in C_0^{(n)}(x^n)$ for every $\varphi \in \mathscr{S}(\mathbb{R})$. Moreover, for $\varphi \in \mathscr{S}(\mathbb{R})$, and $0 \le j \le n$ there is a constant C_j

$$\|\mathscr{F}(t^j\varphi)^j\|_\infty \le \|t^j\varphi^{(j)}\|_1 = \|\varphi\|_{1,(j)} \le C_j\|\varphi\|_{1,(n)},$$

where the last inequality reflects the continuous inclusion $\mathscr{T}^{(n)}(|t|^n) \hookrightarrow \mathscr{T}^{(j)}(|t|^j)$. Thus, for some constant C, we get $\|\mathscr{F}\varphi\|_{\infty,(n)} \le C\|\varphi\|_{1,(n)}$ by (6.2). Then the result follows by density of $\mathscr{S}(\mathbb{R})$ in $\mathscr{T}^{(n)}(|t|^n)$. □

Remark 6.2.1. Since the norm in $C_0^{(n)}(x^n)$ implies pointwise convergence of all the weighted derivatives and $\mathscr{S}(\mathbb{R})$ is dense in $\mathscr{T}^{(n)}(|t|^n)$, the equality (6.2) in Lemma 6.1.1 holds for functions f in $\mathscr{T}^{(n)}(|t|^n)$ as well. Also, it is readily seen that the coefficient matrix implicit in the system of $n+1$ equations defined by (6.2) is self-invertible, so that we get the reverse equality

$$\mathscr{F}(t^k f^{(k)})(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!} x^j (\mathscr{F}f)^{(j)}(x), \quad x \in \mathbb{R},$$

for every $f \in \mathscr{T}^{(n)}(|t|^n)$ and $k = 0, 1, \ldots, n$.

Now, let S be a closed subset of $\mathbb{R}$. Notice that a function $f \in \mathscr{T}^{(n)}(|t|^n)$ is of spectral synthesis for S if there is a sequence $(\tau_k)_{k=1}^\infty$ in $\mathscr{T}^{(n)}(|t|^n)$, with $\mathscr{F}\tau_k \equiv 1$ on U_k, such that $\lim_{k\to\infty} f * \tau_k = 0$ in $\mathscr{T}^{(n)}(|t|^n)$. Also, the spectral synthesis property of S can be rewritten in term of ideals. Set

$$M_k(S) := \{f \in \mathscr{T}^{(n)}(|t|^n) : \Phi_j(x) := x^j(\mathscr{F}f)^{(j)}(x) = 0\ (x \in S; 0 \le j \le k)\}$$

for $k = 0, \ldots, n$; $M(S) := M_0(S)$, and

$$J(S) := \{f \in \mathscr{T}^{(n)}(|t|^n) : \mathscr{F}f = 0 \text{ on a neighborhood of } S\}.$$

Then the closed subset S of $\mathbb{R}$ is of spectral synthesis if and only if $J(S)$ is dense in $M(S)$. For singletons $S=\{a\}$, $a\in\mathbb{R}$, we put $M_k(a)=M_k(\{a\})$, if $0\le k\le n$, and $J(a)=J(\{a\})$. Note that $M_0(0)=M_k(0)=M_n(0)$ for $0\le k\le n$ since $\mathscr{F}f\in C_0^{(n)}(x^n)$. We will show that $\overline{J(a)}=M_n(a)$ for every $a\in\mathbb{R}$. In the case when $a=0$, one gets that $\{0\}$ is a set of spectral synthesis for the Sobolev algebra $\mathscr{T}^{(n)}(|t|^n)$. To see this, we first give a density result of functions of compact support.

Proposition 6.2.2. *In the above setting,*

$$M_n(a)\cap\mathscr{D} \text{ is dense in } M_n(a) \text{ for all } a\in\mathbb{R}.$$

Proof. Assume $a\neq 0$. Let $f\in M_n(a)$ and $\varepsilon>0$. Take $h\in\mathscr{D}$ such that $\|f-h\|_{1,(n)}<\varepsilon$. By Lemma 6.1.2, there are functions $u_0,\dots u_n\in\mathscr{D}$ such that $(\mathscr{F}u_j)^{(k)}(a)=\delta_{j,k}$, $j,k=0,1\dots,n$. Take $g:=h-\sum_{j=0}^n(\mathscr{F}h)^{(j)}(a)u_j$ in $\mathscr{D}$. Clearly, $(\mathscr{F}g)^{(k)}(a)=0$ ($k=0\dots n$), and so $g\in M_n(a)\cap\mathscr{D}$. On the other hand,

$$\begin{aligned}|a^j(\mathscr{F}h)^{(j)}(a)| &= |a^j\mathscr{F}(f-h)^{(j)}(a)|\le\|x^j\mathscr{F}(f-h)^{(j)}\|_\infty\\ &\le\|\mathscr{F}(f-h)\|_{\infty,(n)}\le\|f-h\|_{1,(n)}\end{aligned}$$

where the last inequality is Proposition 6.2.1. Hence,

$$\begin{aligned}\|f-g\|_{1,(n)} &\le \|f-h\|_{1,(n)}+\sum_{j=0}^n|(\mathscr{F}h)^{(j)}(a)|\|u_j\|_j\|_{1,(n)}\\ &\le(1+\sum_{j=0}^n a^{-j}\|u_j\|_{1,(n)})\varepsilon.\end{aligned}$$

This proves the proposition for $a\neq 0$. The case $a=0$ is similar and easier. □

We now proceed to describe the closed primary ideals of $\mathscr{T}^{(n)}(|t|^n)$.

Theorem 6.2.1. *For every $a\in\mathbb{R}$,*

$$M_n(a)=\overline{J(a)}.$$

Proof. Let $a \in \mathbb{R}$, $a \neq 0$, and let $f \in \mathscr{D} \cap M_n(a)$. Take $\tau \in \mathscr{S}(\mathbb{R})$ such that $\mathscr{F}\tau \equiv 1$ on a fixed neighborhood of 0. Let $e_a, E_{a,\rho}$ denote the functions given by $e_a(x) := e^{-iax}$, $E_{a,\rho}(x) := e_{-a}(x)\tau_{1/\rho}(x)$, $x \in \mathbb{R}$, respectively, where $\rho > 0$. Clearly, $\mathscr{F}(E_{a,\rho}) = \mathscr{F}\tau(\cdot - a)$, so $\mathscr{F}(E_{a,\rho}) \equiv 1$ in a neighborhood $\mathscr{U}_\rho$ of a.

Claim:

$$\|f * E_{a,\rho}\|_{1,(n)} \to 0, \text{ as } \rho \to \infty. \tag{6.6}$$

If the above assertion holds then $f - f * E_{a,\rho}$ tends to f in $\mathscr{T}^{(n)}(|t|^n)$ and its Fourier transform vanishes on $\mathscr{U}_\rho$, so (ii) is true for functions in $\mathscr{D} \cap M_n(a)$. Since $\mathscr{D} \cap M_n(a)$ is dense in $M_n(a)$ by Proposition 6.2.2, we have done. Thus let us prove the claim.

The following elementary equality will be used later on: For every nonnegative integers p, m,

$$\mathscr{F}(e_a t^p f)^{(m)}(0) = i^p (\mathscr{F} f)^{(p+m)}(a). \tag{6.7}$$

We have $\|f * E_{a,\rho}\|_{1,(n)} = \|t^n (f * E_{a,\rho})^{(n)}\|_{L^1}$, and by formula (6.3) in Lemma 6.1.4

$$t^n (f * E_{a,\rho})^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (t^{n-j} f * t^j E_{a,\rho}^{(n)}),$$

so we must show that

$$\lim_{\rho \to \infty} \|t^{n-j} f * t^j E_{a,\rho}^{(n)}\|_{L^1} = 0 \quad (0 \leq j \leq n). \tag{6.8}$$

Thus let j be fixed such that $j \in \{0, 1, \ldots, n\}$. Then

$$t^{n-j} f * t^j E_{a,\rho}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (ia)^{n-k} \rho^{j-k} (e_a t^{n-j} f)_\rho * (t^j \tau^{(k)}). \tag{6.9}$$

For $0 \leq k \leq n$ such that $k > j$, we use the estimate

$$\rho^{j-k} \|(e_a t^{n-j} f)_\rho * (t^j \tau^{(k)})\|_{L^1} \leq \rho^{j-k} \|t^{n-j} f\|_{L^1} \|t^j \tau^{(k)}\|_{L^1},$$

whence

$$\lim_{\rho\to\infty} \rho^{j-k}\|(e_a t^{n-j} f)_\rho * (t^j \tau^{(k)})\|_{L^1} = 0, \text{ if } k > j. \tag{6.10}$$

For $0 \le k \le n$ such that $k \le j$, we apply (6.7) so that

$$\mathscr{F}(e_a t^{n-j} f)^{(m)}(0) = i^{n-j}(\mathscr{F} f)^{(n-j+m)}(a) = 0$$

for every $m = 0, 1 \ldots, j-k$, since $0 \le n-j+m \le n-k \le n$ and $f \in M_n(a)$. Then, by (6.5) in Lemma 6.1.5, for some constant C

$$\rho^{j-k}\|(e_a t^{n-j} f)_\rho * (t^j \tau^{(k)})\|_{L^1} \le C\rho^{j-k}\rho^{-j+k+1} = C\rho^{-1}$$

and therefore

$$\lim_{\rho\to\infty} \rho^{j-k}\|(e_a t^{n-j} f)_\rho * (t^j \tau^{(k)})\|_{L^1} = 0, \text{ if } k \le j. \tag{6.11}$$

Now, (6.9) (6.10) and (6.11) implies (6.8). Thus the claim (6.6) follows and we have completed the proof for $a \neq 0$. The case $a = 0$ is simpler and it is left to the reader. □

Theorem 6.2.2. *For every countable subset S of* $\mathbb{R}$*,*

$$\overline{J(S)} = M_n(S).$$

Proof. The argument is an adaptation of [RS, Th. 2.5.9(iii)] to our setting. Let f be in $M_n(S)$. Since the Sobolev algebra $\mathscr{T}^{(n)}(|t|^n)$ possesses (bounded) approximate identities formed by functions whose Fourier transforms are of compact support, one can assume without loss of generality that $\operatorname{supp}\mathscr{F} f$ is compact.

Let enumerate the elements of S, say $S = \{a_m : m = 1, 2, \ldots\}$. For every $m \in \mathbb{N}$ let μ_m be the measure given by

$$\mu_m := \delta_0 - E_{a_m,\rho_m},$$

where ρ_m is to be chosen later, in the notation of the proof of Theorem 6.2.1. For all $m \in \mathbb{N}$, one has $f * \mu_m \in \mathscr{S}(\mathbb{R})$ and $\mathscr{F}\mu_m = 0$ on the open interval $\mathscr{U}_m := (a_m - \rho_m^{-1}, a_m - \rho_m^{-1})$.

Take now $\varepsilon > 0$. By (6.6),

$$\|f - f * \mu_1\|_{1,(n)} \leq \varepsilon/2$$

for ρ_1 large enough. Since $f * \mu_1 \in M_n(a_2)$, by (6.6) again there exists $\rho_2 > \rho_1$ big enough such that

$$\|f * \mu_1 - f * \mu_1 * \mu_2\|_{1,(n)} \leq \varepsilon/2^2.$$

In fact, using induction, we find a sequence $(\rho_k)_{k=1}^{\infty}$ such that for every $k = 1, 2, \ldots,$

$$\|f * \mu_1 * \ldots * \mu_k - f * \mu_1 * \ldots * \mu_k * \mu_{k+1}\|_{1,(n)} \leq \varepsilon/2^{k+1}.$$

Therefore, for every $m \in \mathbb{N}$,

$$\begin{aligned}
&\|f - f * \mu_1 * \ldots * \mu_m\|_{1,(n)} \\
&\qquad \leq \|f - f * \mu_1\|_{1,(n)} \\
&\qquad + \|f * \mu_1 * \ldots * \mu_k - f * \mu_1 * \ldots * \mu_k * \mu_{k+1}\|_{1,(n)} \\
&\qquad \leq \varepsilon \sum_{k=0}^{m-1} \frac{1}{2^{k+1}} = \varepsilon(1 - \frac{1}{2^m}).
\end{aligned}$$

Set $K := S \cap \operatorname{supp}\mathscr{F}f$. As K is compact there exists $m \in \mathbb{N}$ such that $K \subseteq \mathscr{U} := \cup_{k=1}^{m} \mathscr{U}_k$. Take then $g := f * \mu_1 * \cdots * \mu_m \in \mathscr{S}(\mathbb{R})$. Clearly, $\mathscr{F}g$ vanishes on the open subset $\mathscr{U} \cup (\mathbb{R} \setminus \operatorname{supp}\mathscr{F}f)$ of $\mathbb{R}$. Moreover, $S = K \cup (S \cap (\mathbb{R} \setminus \operatorname{supp}\mathscr{F}f)) \subseteq \mathscr{U} \cup (\mathbb{R} \setminus \operatorname{supp}\mathscr{F}f)$.

In conclusion, $g \in J(S)$ with $\|f - g\|_{1,(n)} < \varepsilon(1 - \frac{1}{2^m})$ and the proof is over. $\square$

6.3 Null ergodicity of semigroups

For S a closed set of real numbers, put $M_{n,+}(S) := M_n(S) \cap \mathscr{T}_+^{(n)}(t^n)$, where $M(S)$ is as in Section 6.2.

Definition 6.3.1. *We say that S is an interpolation set for $\mathscr{T}_+^{(n)}(t^n)$ in $\mathscr{T}^{(n)}(|t|^n)$ if*

$$\mathscr{T}_+^{(n)}(t^n)/M_{n,+}(S) = \mathscr{T}^{(n)}(|t|^n)/M_n(S),$$

that is, for every $f \in \mathscr{T}^{(n)}(|t|^n)$ there exists $g \in \mathscr{T}_+^{(n)}(t^n)$ such that $(\mathscr{F}g)^{(k)}(a) = (\mathscr{F}f)^{(k)}(a)$ for every $a \in S \setminus \{0\}$, $k = 0, 1, \ldots, n$, and $\mathscr{F}g(0) = \mathscr{F}f(0)$ if $0 \in S$.

Set $Y := \overline{\pi_n(M_{n,+}(S))X}$ in X. By the functional law of integrated semigroups, $T_n(t)$ commutes with π_n and therefore Y is $T_n(t)$-invariant. Thus the prescription

$$\widetilde{T}_n(t)[x] := T_n(t)x + Y \text{ where } [x] = x + Y \in X/Y,$$

is well defined. It is readily seen that $\widetilde{T}_n(t)$ is a n-times integrated semigroup on X/Y which satisfies

$$\sup_{t>0} t^{-n} \|\widetilde{T}_n(t)\| < \infty \text{ and } \lim_{t\to 0^+} n! t^{-n} \widetilde{T}_n(t)[x] = [x] \quad ([x] \in X/Y), \tag{6.12}$$

and its generator is the well defined closed operator $\widetilde{A}[x] := Ax + Y$, for $x \in D(A)$, with dense domain $D(\widetilde{A}) = \{[x] : x \in D(A)\}$ in X/Y. Moreover, $\sigma(\widetilde{A}) \subseteq \sigma(A)$ and $\sigma_P(\widetilde{A}^*) \subseteq \sigma_P(A^*)$.

We are ready to give the main result of this section:

Theorem 6.3.2. *Let $T_n(t)$ be a n-times integrated semigroup in $\mathscr{B}(X)$ with generator A such that*

$$\sup_{t>0} t^{-n} \|T_n(t)\| < \infty \text{ and } \lim_{t\to 0} n!\, t^{-n} T_n(t)x = x \quad (x \in X). \tag{6.13}$$

Assume that

(i) $S := i\sigma(A) \cap \mathbb{R}$ *is a countable compact interpolation set for* $\mathscr{T}_+^{(n)}(t^n)$ *in* $\mathscr{T}^{(n)}(|t|^n)$.

(ii) $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.

Then $\pi_n(M_{n,+}(S))X$ *is dense in* X *and, in consequence,*

$$\lim_{t\to\infty} t^{-n} T_n(t)x = 0 \text{ for all } x \in X.$$

Proof. Let X' be the topological dual of X. As before, set $Y := \overline{\pi_n(M_{n,+}(S))X}$. In order to prove that $Y = X$ we want to show that $Y^\perp = 0$ where $Y^\perp = \{\ell \in X' : \ell(Y) = 0\}$ from which the equality will follow by the Hahn-Banach theorem. We identify $Y^\perp$ with the dual $(X/Y)'$.

Take $\ell \in Y^\perp$ and $x \in X$ and define $\ell \otimes x \in \mathscr{T}_+^{(n)}(t^n)'$ by

$$(\ell \otimes x)(g) := \langle \pi_n(g)x, \ell \rangle, \quad g \in \mathscr{T}_+^{(n)}(t^n).$$

Since $\ell \in Y^\perp$ it is clear that $(\ell \otimes x)(M_{n,+}(S)) = 0$. Besides this, $\mathscr{T}_+^{(n)}(t^n)/M_{n,+}(S) = \mathscr{T}^{(n)}(|t|^n)/M_n(S)$ because S is a set of interpolation, so $\ell \otimes x$ defines a continuous functional in $(\mathscr{T}^{(n)}(|t|^n)/M_n(S))'$ given by $(\ell \otimes x)([f]) := (\ell \otimes x)([g]) \equiv (\ell \otimes x)(g)$ where $g \in \mathscr{T}_+^{(n)}(t^n)$ is such that $g + M_n(S) = f + M_n(S) = [f]$. By composition with the projection $\mathscr{T}^{(n)}(|t|^n) \to \mathscr{T}^{(n)}(|t|^n)/M_n(S)$ the functional $\ell \otimes x$ extends to $\mathscr{T}^{(n)}(|t|^n)'$. On the other hand, $\ell \otimes y = 0$ on $\mathscr{T}_+^{(n)}(t^n)$ for all $y \in Y$, whence one readily sees that, for some constant K, $\|\ell \otimes x\| \le K\|\ell\| \, \|[x]\|$ where $[x] = x + Y \in X/Y$.

Then there exists an almost everywhere defined mapping $\varphi \colon \mathbb{R} \to \mathbb{C}$ such that the function $t \mapsto t^{-n}\varphi(t; x, \ell) \equiv t^{-n}\varphi(t; [x], \ell)$ is in $L^\infty(\mathbb{R})$ and

$$(\ell \otimes x)(f) = (-1)^n \int_{-\infty}^{\infty} f^{(n)}(t)\varphi(t; x, \ell)\, dt, \quad f \in \mathscr{T}^{(n)}(|t|^n). \tag{6.14}$$

Notice that for $g \in \mathscr{T}_+^{(n)}(t^n)$,

$$(\ell \otimes x)(g) = \langle \pi_n(g)x, \ell \rangle = (-1)^n \int_0^\infty g^{(n)}(t) \langle T_n(t)x, \ell \rangle \, dt,$$

from which one deduces that $\varphi(t;x,\ell) = \langle T_n(t)x, \ell \rangle$ for all $t \geq 0$ a. e., and that

$$\sup_{t \neq 0} |t|^{-n} |\varphi(t;x,\ell)| \leq K \|\ell\| \|[x]\|. \tag{6.15}$$

Abbreviate $\varphi(t;x,\ell) = \varphi(t)$ for a moment. The integral in (6.14) means that $\ell \otimes x = \varphi^{(n)}$ in the distributional sense (recall that $\mathscr{T}^{(n)}(|t|^n)$ contains $\mathscr{S}(\mathbb{R})$ densely). As $(\ell \otimes x)(M_n(S)) = 0$ the Fourier transform $(\cdot)^n \mathscr{F}\varphi$ of $\ell \otimes x$ is concentrated on S by Theorem 6.2.2, and therefore $\mathrm{supp}(\mathscr{F}\varphi) \subseteq S \cup \{0\}$. By the Paley-Wiener theorem ([Ru, Th.]), φ is an entire function such that

$$|\varphi(z;x,\ell)| \leq C(1+|z|^N) e^{r|\Im z|}, \quad z \in \mathbb{C}, \tag{6.16}$$

for some $C, r > 0$ and $N \in \mathbb{N}$, $N \geq n$.

Define $F(z) := (z+i)^{-N} e^{irz} \varphi(z;x,\ell)$ for $\Im z \geq 0$. Then F is analytic and bounded on $\{\Im z > 0\}$ with $|F(t)| \leq K\|\ell\| \|[x]\|$, for all $t \in \mathbb{R}$, by (6.15). Then a version of the Phragmen-Lindelöff theorem applied to the function F implies that $|F(z)| \leq K\|\ell\| \|[x]\|$, for all $z \in \mathbb{C}$. Similarly, taking $G(z) := (z-i)^{-N} e^{-irz} \varphi(z;x,\ell)$ for $\Im z \leq 0$ and using the same argument as above for G on $\{\Im z < 0\}$, one eventually deduces altogether that, for all $z \in \mathbb{C}$,

$$|\varphi(z;x,\ell)| \leq K\|\ell\| \|[x]\| \left((\Re z)^2 + (1+|\Im z|)^2 \right)^{N/2} e^{r|\Im z|}. \tag{6.17}$$

Let now write φ as a power series

$$\varphi(z;x,\ell) = \sum_{k=0}^{\infty} \frac{\beta_k(x,l)}{k!} z^k, \quad z \in \mathbb{C}, \tag{6.18}$$

with $\beta_k(x,l) \in \mathbb{C}$ independent of z and depending on $[x]$ but not on y in $[x]$.

Since $T_n(t)$ (and $\widetilde{T}_n(t)$) satisfies

$$T_n(t)x = \int_0^t T_n(s)Ax\,ds + \frac{t^n}{n!} \quad (t \geq 0) \tag{6.19}$$

one has $\beta_k = 0$ for $k = 0, 1, \ldots, n-1$ and $\beta_n(x,\ell) = \langle x,\ell \rangle \equiv \langle [x],\ell \rangle$ for every $x \in X$ and $\ell \in Y^{\perp}$ (it suffices to apply the L'Hopital rule, for instance). Also, using the Cauchy integral formula for derivatives one obtains from (6.17) that

$$|\beta_k(x,\ell)| \leq \frac{(k-1)!}{\delta^k} K_\delta \|\ell\| \|[x]\|, \quad \text{for } k \geq n+1, \text{ and } \delta > 0, \tag{6.20}$$

where K_δ is a constant depending on δ.

Now, by derivation in (6.19) one obtains

$$(d/dt)\varphi(t;x,\ell) = \varphi(t;Ax,\ell) + (t^{n-1})/(n-1)!.$$

Identifying the coefficients of the corresponding power series one then finds for $k = n, n+1, \ldots$ and $x \in D(A)$,

$$\beta_{k+1}(x,\ell) = \beta_k(Ax,\ell) = \beta_k(\widetilde{A}[x],\ell).$$

This in particular means that for all $[x] \in D(\widetilde{A})$,

$$\langle \widetilde{A}[x],\ell \rangle = \langle Ax,\ell \rangle = \beta_{n+1}(x,\ell),$$

and it follows by (6.20) that $\widetilde{A}$ is a bounded operator on X/Y since $D(\widetilde{A})$ is dense in X/Y. By induction one gets also that $\beta_k([x],\ell) = \langle \widetilde{A}^{k-n}[x],\ell \rangle$ for all $[x] \in X$ and $k \geq n$. Hence,

$$\widetilde{T}_n(t) = \sum_{k=n}^{\infty} \widetilde{A}^{k-n} \frac{t^k}{k!}, \quad t \geq 0,$$

in $\mathscr{B}(X/Y)$. In other words,

$$e^{t\widetilde{A}} = \widetilde{A}^n \widetilde{T}_n(t) + I_{X/Y} + \widetilde{A}t + \cdots + \widetilde{A}^{n-1}\frac{t^{n-1}}{(n-1)!} \quad (t \geq 0).$$

Thus by the estimate assumed in (6.13) we have that the (holomorphic) C_0-semigroup $e^{t\widetilde{A}}$ on X/Y is of polynomial growth along the nonnegative half-line. One needs to control the growth of $e^{t\widetilde{A}}$ on $(-\infty, 0)$ too. For this, we apply [V1, Lemma 3 & proof of Theorem 7]: There exists a Banach space E, a bounded homomorphism $\Theta\colon X/Y \to E$ with dense range, and a group $(V(t))_{t\in\mathbb{R}}$ in $\mathscr{B}(E)$ such that $V(t)\Theta(\xi) = e^{t\widetilde{A}}\xi$ for every $\xi \in X/Y$ and

$$\|V(t)\| = O(t^n), \text{ as } t \to +\infty, \; \|V(t)\| = O(1), \text{ as } t \to -\infty. \tag{6.21}$$

Moreover, if H is the infinitesimal generator of $V(t)$ then $\sigma(H) \subseteq \sigma(\widetilde{A})$ and $\sigma_P(H^*) \subseteq \sigma_P(\widetilde{A}^*)$.

We now finish the proof adapting the argument of [V1, p. 236]: Suppose if possible that $\ell \neq 0$. Then $\widetilde{T}_n(t) \neq 0$ so the semigroup $e^{t\widetilde{A}}$, and so the group $V(t)$, are nontrivial. By (6.21) one gets that $\sigma(H)$ is a subset of $i\mathbb{R}$, and then it is also countable by on eof the assumptions of the statement. Moreover, it is nonempty by [V1, Lemma 5]. Hence there is an isolated point $i\omega$ in $\sigma(H)$. Using the projection on E associated with $i\omega$ by the holomorphic functional calculus one arrives at the conclusion that $H^*\phi = i\omega\,\phi$ for some nonzero $\phi \in E^*$. Therefore $\emptyset \neq \sigma_P(H^*) \cap i\mathbb{R} \subseteq \sigma_P(\widetilde{A}^*) \cap i\mathbb{R} \subseteq \sigma_P(A^*) \cap i\mathbb{R}$, which contradicts one of the assumptions. It follows that $\ell = 0$ as we wanted to show. $\square$

A clear consequence of Lemma 6.1.3 is that finite sets in $\mathbb{R}$ are interpolation sets for $\mathscr{T}_+^{(n)}(t^n)$ in $\mathscr{T}^{(n)}(|t|^n)$. This implies immediately the result which follows.

Theorem 6.3.3. *Let $T_n(t)$ be a n-times integrated semigroup in $\mathscr{B}(X)$ with generator A. Assume that*

(i) $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$ *and* $\lim_{t\to 0} n!\, t^{-n}T_n(t)x = x$ $(x \in X)$,

(ii) $i\sigma(A) \cap \mathbb{R}$ *is finite and* $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.

Then

$$\lim_{t\to\infty} t^{-n}T_n(t)x = 0 \text{ for all } x \in X.$$

Remark 6.3.1. (i) Both preceding theorems have interest only when $\{0\} \subseteq i\sigma(A) \cap \mathbb{R}$, in view of Theorem 4.0.1.

(ii) The compactness of the boundary spectrum $\sigma(A) \cap i\mathbb{R}$ in the statement of Theorem 6.3.2 is not, morally, a very strong condition in our setting. For instance, compactness of hulls is a kind of usual assumption when one wants to establish general statements on the (standard) ideals structure in L^1 spaces. On the other hand, the finiteness of boundary spectra of generators is not generally an easy-to-handle condition about semigroups; see [BD] and [M].

One would like to remove the compactness assumption, anyway, but it does not seem to be simple.

We finish the chapter with a result on ergodic behaviour of (certain) non necessarily uniformly bounded C_0-semigroups.

Corollary 6.3.1. *Let $T(t) = e^{tA}$ be a C_0-semigroup in $\mathscr{B}(X)$, which is not necessarily uniformly bounded in $t > 0$, such that $i\sigma(A) \cap \mathbb{R}$ is finite and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$. Assume moreover that there exists $n \in \mathbb{N}$ for which* $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$ *where*

$$T_n(t)x := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}T(s)x\,ds, \quad x \in X:$$

Then

$$\lim_{t\infty} t^{-n} T_n(t)x = 0, \qquad \text{for all } x \in X.$$

Proof. The generator A of $T(t)$ is also the generator of $T_n(t)$. Moreover, since $T(t)$ is a C_0-semigroup one gets that $T_n(t)$ satisfies the limit property in condition (6.13). Thus it is enough to apply Theorem 6.3.3 to obtain the corollary. □

Examples of non-bounded C_0-semigroups $T(t)$ but which satisfy the estimate $\sup_{t>0} t^{-n}\|T_n(t)\| < \infty$ for some n are given in [CSS].

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Notation

	Subsets of $\mathbb{R}$ or $\mathbb{C}$	
$\mathbb{R}^+$	set of non-negative real numbers	p. 34
$\mathbb{R}^-$	set of non-positive real numbers	p. 34
$\mathbb{C}^+$	open right half-plane	p. 34
$\mathbb{C}^-$	open left half-plane	p. 34
$\mathbb{T}$	unit circle	p. 12

	Function, Distribution and Operator Spaces	
$\mathscr{B}(X)$	space of all bounded linear operators on X	p. 33
$C(I,X)$	space of continuous functions	p. 34
$C_0(I,X)$	space of continuous functions vanishing at infinity	p. 34
$C^n(I,X)$	space of n times continuously differentiable functions	p. 34
$C^\infty(I,X)$	space of infinitely differentiable functions	p. 34
$C_c^\infty(I,X)$, $C_c^\infty(I)$	space of infinitely differentiable functions having compact support	p. 34
$L^1_{loc}(I;X)$	space of locally Bochner integrable functions	p. 34

$L^p(I;X)$	space of Bochner p-integrable functions	p. 34
$L^p(I)$	space of Lebesgue p-integrable functions	p. 34
$L^\infty(I;X)\,,L^\infty(I)$	space of measurable, essentially bounded functions	p. 34
$\mathscr{T}_+^{(\alpha)}(t^\alpha)$	Banach algebra of Sobolev type on $\mathbb{R}^+$	p. 47
$\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$	Banach algebra of Sobolev type on $\mathbb{R}^-$	p. 48
$\mathscr{T}^{(\alpha)}(\lvert t\rvert^\alpha)$	Banach algebra of Sobolev type on $\mathbb{R}$	p. 48
$\mathscr{D}(\mathbb{R})$	space of test functions	p. 38
$\mathscr{S}(\mathbb{R})$	Schwartz space of rapidly decreasing functions	p. 38
$\mathscr{S}'(\mathbb{R})$	space of tempered distributions	p. 34
$\mathscr{D}'(\mathbb{R})$	space of distributions	p. 38

Convolutions, Norms and Dualities

$\lVert\cdot\rVert_p$	Lebesgue-Bochner norm	p. 34
ν_α	Lebesgue norm for the algebras $\mathscr{T}_+^{(\alpha)}(t^\alpha)$, $\mathscr{T}_-^{(\alpha)}((-t)^\alpha)$ or $\mathscr{T}^{(\alpha)}(\lvert t\rvert^\alpha)$	p. 47
$ess\,sup$	essential supremum	p. 34
$f*g$	convolution product of two functions (distributions)	p. 39 (39)
$f\circ g$	adjoint convolution product of two functions	p. 61
$f*_c g$	cosine convolution product of two functions	p. 56
X^*	dual space of X	p. 34
$\langle\cdot,\cdot\rangle$	duality between a space X and its dual X^*	p. 38

Functions and Transformations

$\mathscr{L}(f)$	Laplace transform of f	p. 35
$abs(f)$	abscissa of convergence of $\mathscr{L}f$	p. 35

$\omega(f)$	exponential growth bound of f	p. 36
$\mathscr{F}(f)\,,\,\hat{f}$	Fourier transform	p. 37
$\mathscr{L}_S$	Laplace-Stieltjes transform	p. 86
$W_+^{-\alpha}$	Weyl fractional integral of order $\alpha > 0$ on $\mathbb{R}^+$	p. 47
W_+^{α}	Weyl fractional derivative of order $\alpha > 0$ on $\mathbb{R}^+$	p. 47
$W_-^{-\alpha}$	Weyl fractional integral of order $\alpha > 0$ on $\mathbb{R}^-$	p. 48
W_-^{α}	Weyl fractional derivative of order $\alpha > 0$ on $\mathbb{R}^-$	p. 48
W^{α}	Weyl fractional derivative of order $\alpha > 0$ on $\mathbb{R}$	p. 48
$R_t^{\beta-1}$	Riesz function	p. 49
e_z	exponential function $t \mapsto e^{-zt}$	p. 60

Operators

$\rho(A)$	resolvent set of an operator A	p. 33
$R(\lambda,A)$	resolvent of an operator A in λ	p. 33
$\sigma(A)$	spectrum of an operator A	p. 33
$\sigma_p(A)$	point spectrum of an operator A	p. 33
A^*	adjoint operator of A	p. 33
(ACP)	abstract Cauchy problem	p. 42