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On some apllications of Lie Algebroids in Geometry and Physics

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DOCTORAL THESIS

On some applications of Lie Algebroids in Geometry and Physics

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Chapter 1

Introduction

The properties of the Lie algebroid structure have been investigated in the last years and used in the field of mechanics where they are playing a relevant role as they gather under the same formalism a variety of different kinds of mechanical systems, see e.g. [35]. They have also been proved to be interesting in other problems in control theory and in classical field theory, for which we refer the reader to [42], [40].

The attention received by the Lie algebroid structure in different branches of mathematics is due to the fact that the concept of Lie algebroid is a generalization of two important mathematical structures, Lie algebras on one side and tangent bundle of a manifold on the other, and appear in a natural way in the process of reduction of such structures. Then, in the field of physics, these structures appear as generalizing the velocity phase space or simply when we deliberately forget the tangent bundle structure of the velocity phase space as happens when using the so-called quasi-coordinates. Moreover, the dual of a Lie algebroid is endowed in a natural way with a Poisson structure, and such structures are almost ubiquitous in physics.

The notion of Lie groupoid and Lie algebroid were introduced by Pradines in his work [48], while an important reference for the context of our work is [56], as Weinstein was the pioneer in introducing Hamiltonian dynamical systems on Lie algebroids using the Poisson structure on the dual of a Lie algebroid. Some few years later Lagrangian dynamics was introduced by Martínez [41] by extending to the framework of Lie algebroids the usual symplectic formalism, and later on the variational calculus on Lie algebroids developed in [43]. A particular case of variational calculus on Lie algebroids was studied by Boucetta in [5], where Riemannian metrics on Lie algebroids are introduced. This context of Riemannian Lie algebroids will be the one we will work on.

In classical Riemannian geometry the variational vector field of a variation of geodesics (solutions of dynamical systems), called the Jacobi vector field, satisfies the so called Jacobi equation and has received a special attention. As a reference for it we can mainly consider the books by Michor [46] and Do Carmo [26]. We will study the generalization of the Jacobi field in the context of variations on Lie algebroids and more specifically on Riemannian Lie algebroids. Use will be made of the machinery developed in [13] and [44] where the concept of Ehresmann connection associated to a second order differential equation (hereafter shortened asSODE) is reminded and concepts like those of non-linear connection and Jacobi endomorphism associated to a SODE are introduced.

On the other hand we will analyze in this work a very useful theorem in physics called the virial theorem, which is interesting for both constrained and unconstrained systems whose configuration space is a manifold as it offers useful relations coming from the fact that the time average of the action of the dynamical vector field upon the virial function vanishes. In particular, for systems of mechanical type, a relation between the time averages of the potential energy and of the kinetic energy appears. This theorem was introduced by Clausius in [19], and it has been shown in a recent work [7] that it can be generalized from a configuration space $\mathbb{R}^n$ to an arbitrary differentiable manifold. Seeger in [49] and more recently Papastavridis in [47] considered the problem of extending the virial theorem to nonholonomic mechanics. Here we present a generalization to the framework of Lie algebroids and illustrate the theory with several examples.

In summary, the objective of this work is the study of some applications of the Lie algebroid theory in some concrete mathematical and physical problems related to the matters included in the references mentioned before.

More specifically, our objective for the mathematical part is to generalize some well known properties from Riemannian manifolds to the Lie algebroid framework enlarging the previously considered results by Boucetta.

In the part concerning the physical applications we have centered our attention on the virial theorem, with the main objective of showing that it admits a generalization to both unconstrained and nonholonomic mechanical systems on Lie algebroids and to display some of its possible applications.

In order for our work to be more selfcontained we have included in Chapter 2 some preliminaries containing a short introduction to the structure of Lie algebroid with the most relevant and useful information concerning our work, like the concept of admissible curves, connection, prolongation of a Lie algebroid, and including various illustrative examples. Moreover, the symplectic approach to the classical mechanics and its generalization to the Lie algebroid structure is reminded, as well as the variational formalism, defining the notion of morphism and homotopy for Lie algebroids, and the first variation formulae of the energy of a dynamical system as presented in [43]. Some basic information about nonholonomic systems and on Killing and conformal vector fields associated to a Riemannian metric is also presented.

In the third chapter we generalize the concept of Jacobi field for general second-order differential equations on a manifold and on a Lie algebroid. We look at a variational differential equation on a manifold from a 'new perspective', we later apply it for the case of a SODE on a manifold, and we introduce the corresponding concept of Jacobi field. Such vector fields satisfy the Jacobi equation. The non-linear connection and the Jacobi endomorphism associated to a SODE on a manifold will be used then to express the Jacobi equation for a SODE on a manifold. We generalize theses objects for SODEs on Lie algebroids and then the Jacobi equation for the Jacobi sections associated to a SODE on a Lie algebroid.

Afterwards we present the definition of Riemannian metric on a Lie algebroid and consider the particular case of a SODE, and that of a geodesic spray defined with the help of the Riemannian metric. We will use the first and the second variational formulae of the energy functional on the Lie algebroid and the theory developed in Chapter 3 for the Jacobi sections in order to analyze some minimizing results in the Riemannian geometry on Lie algebroids related to the conjugate points of the geodesic spray.

The virial theorem will be generalized in the last chapter for nonholonomic systems on the tangent bundle and for unconstrained dynamical systems on Lie algebroids in the context of Lie algebroids – i.e. the generalization of the virial theorem to the framework of Lie algebroids is the main purpose.

We will begin the chapter by developing virial-like results for the particular case of mechanical type Lagrangians, and then conformal Killing vector fields will be shown to play a relevant role. For mechanical systems, $L = T_g - V$, finding infinitesimal symmetries of the metric, i.e. Killing fields, is relatively easy. As it is well known, if such a vector field is also a symmetry of the potential we are able to get a constant of the motion, which simplifies the problem, while if the Killing vector field is not a symmetry of the potential the virial theorem also provides relevant information, namely the average value of the derivative of the potential vanishes. With more generality, for a homothetic or a conformal Killing vector field the virial theorem allows us to establish relations between the averages of the kinetic energy and those of certain derivatives of the potential. We consider the case of affine virial functions corresponding to special vector fields on the configuration manifold, as Killing, homothetic, or conformal vector fields and present the expressions of the virial theorems obtained for these types of virial functions. We will make use of quasi-coordinates as presented in [14] to rewrite all these instances of the virial theorem.

Our next objective will be the analysis and generalization of some results from [49] and [47] on the virial theorem for nonholonomic systems by using the appropriate differential geometric tools of geometric mechanics. We will follow the results in [7, 9] in which the virial theorem is understood in terms of the time average of the Poisson bracket of the energy and a virial function and we will apply it here by using the equivalent of the Poisson bracket for the nonholonomic case, the so called nonholonomic bracket. We will do it in two approaches: the Lagrange multipliers and the distributional approach method.

Finally, the extension of some of such results to the framework of mechanics in Lie algebroids is carried out. The generalization of such results to the framework of Lie algebroids is a plus, due to the variety of dynamical systems that can be defined on Lie algebroids. For all these cases the time average of the energy of systems will be thus available and we will explicitly write some of them in a couple of examples.

Before proceeding, we mention that all objects considered in this work, as manifolds, vector fields, tensors, etc are assumed to be of C^∞ class, unless explicitly mentioned.

Chapter 2

Preliminaries

2.1 A symplectic approximation to classical mechanics

The use of geometric methods in classical mechanics of last forty years [1, 2, 12, 23, 24], has been very useful to get a better understanding of different problems offering us new related questions and answers. The existence of constraints forces us to replace affine spaces by differentiable manifolds, coordinates becoming then a local concept. Differential equations are replaced by vector fields, a global concept, in such a way that the integral curves of such vector fields are the solutions of a system of differential equations in a coordinate system. Geometric structures that are compatible with the vector field responsible of the dynamics are playing a relevant role. In particular we will fix next our attention on symplectic structures, which provide a common geometric framework to deal with both Hamiltonian and Lagrangian mechanics (in the regular case).

In this section we will shortly describe the geometrical approach to classical mechanics making use of the theory of symplectic manifolds and Poisson bracket. For more details we refer the reader to [1, 2, 12].

2.1.1 Hamiltonian approach

The configuration space Q of a classical system with n degrees of freedom is a n-dimensional differentiable manifold, and if $\mathrm{pr}^i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, \ldots, n$, denotes the projection on the i-th coordinate, then a local chart (U, ϕ) of M introduces local coordinates $q^i = \mathrm{pr}^i \circ \phi$. Such a chart is usually denoted $(U, q^1, \ldots, q^n)$. These coordinate charts of M have associated $2n$-dimensional local charts of the tangent bundle TQ and the cotangent bundle T^*Q and we can consider the coordinate basis of $\mathfrak{X}(U)$ usually denoted $\{\partial/\partial q^j \mid j = \overline{1,n}\}$ and its dual basis for $\Omega^1(U)$, $\{dq^j \mid j = \overline{1,n}\}$. Then a vector in a point $q \in U$ is $v = v^j\,(\partial/\partial q^j)|_q$ and a covector is $\zeta = p_j\,(dq^j)|_q$, with $v^j = \langle dq^j, v\rangle$ and $p_j = \langle \zeta, \partial/\partial q^j\rangle$ being the usual velocities and momenta.

The Hamiltonian approach to the classical mechanics can be given using a canonical exact symplectic structure of the cotangent bundle T^*Q, determined by the canonical 1-form θ_0 on T^*Q, called the Liouville 1-form. Its exterior differential $\omega_0 = -d\theta_0$, is a canonical symplectic form on T^*Q, i.e. it is a nondegenerate closed 2-form; non degeneracy means that for each point in M the associated linear map $\widehat{\omega}_0 : T_qQ \to T_q^*Q$ defined by $\widehat{\omega}_0(v) = \omega_0(v, \cdot)$, is invertible. This bundle map over the identity extends to the set of sections of both bundles, i.e. there exists a $C^\infty(M)$-linear map $\widehat{\omega}_0 : \mathfrak{X}(T^*Q) \to \bigwedge^1(T^*Q)$ given by $\widehat{\omega}_0(X) = \omega_0(X, \cdot)$ which is invertible and provides an identification of the set of vectors and the set of covectors at a point $q \in Q$, and by extension, between vector fields on T^*Q and 1-forms on T^*Q.

In a coordinate chart adapted to to the structure of the cotangent bundle T^*Q, i.e. induced from a local chart on Q as indicated above, the canonical 1-form has the local expression $\theta_0 = p_i dq^i$, while $\omega_0 = dq^i \wedge dp_i$. Given a function $H \in C^\infty(T^*Q)$ the unique solution of the symplectic equation $i_{X_H}\omega_0 = dH$ is a vector field X_H, called Hamiltonian vector field associated to H, which gives the Hamiltonian dynamics of the system.

In adapted local coordinates the vector field X_H takes the expression:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i},$$

and therefore its integral curves are solutions of the Hamiltonian equations

$$\left\{\begin{array}{rcl} \dot{q}^i & = & \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i & = & -\dfrac{\partial H}{\partial q^i} \end{array}\right. .$$

More generally we can consider a symplectic manifold (M, ω), where ω is a symplectic form, i.e. a non degenerate closed 2-form. Darboux Theorem asserts that M is of even dimension $2n$ and there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that the local expression of ω is like that of ω_0 in the cotangent bundle case, that therefore is but a particular case of symplectic manifold. With full similarity, a Hamiltonian dynamical system is a triplet (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$. In particular, the triplet (T^*Q, ω_0, H) is called a Hamiltonian dynamic system with ω_0 the canonical symplectic form on the differential manifold T^*Q. Thus, the Hamiltonian formalism of the classical mechanics is a particular case of Hamiltonian dynamical system, for $M = T^*\mathbb{R}^n$ together with its canonical symplectic form. We will see next that the Lagrangian formalism (in the regular case) is another particular case, and moreover more general symplectic manifolds appear when doing reduction by constants of motion or symmetries.

The concept of Poisson bracket plays an important role in the Hamiltonian formulation of the classical mechanics. A Poisson bracket on a differentiable manifold M is a skew-symmetric $\mathbb{R}$-bilinear map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that satisfies the Jacobi identity and the Leibniz rule. In particular, when (M, ω) is a symplectic manifold the Poisson bracket of two functions $f, g \in C^\infty(M)$ is a new function $\{f, g\}$ defined by:

$$\{f, g\} := \omega(X_f, X_g) = df(X_g) = X_g f,$$

where X_f and X_g are the Hamiltonian vector fields associated to the functions $f, g \in C^\infty(M)$.

In the case of a Poisson bracket defined by a symplectic form, the Jacobi identity follows as a consequence of the closedness of the symplectic form, and it is nondegenerate in the sense that the unique functions f such that $\{f, g\} = 0, \forall g \in C^\infty(M)$ are constant functions.

Moreover, it can be shown that $[X_f, X_g] = X_{\{g,f\}}$.

2.1.2 Lagrangian approach

Lagrangian mechanics is also a particular case of Hamiltonian dynamical system on the tangent space of the configuration space Q when the Lagrangian is regular, (TQ, ω_L, E_L), but both the symplectic structure ω_L and the energy function L, which plays the role of Hamiltonian, depend on the Lagrangian function L. The dynamical vector field Γ_L giving the dynamics is a second-order differential equation vector field, whose definition is recalled in next paragraph.

More explicitly, recall that the tangent bundle $\tau_Q : TQ \to Q$ is a vector bundle and then there is a dilation vector field, here called Liouville vector field, $\Delta \in \mathfrak{X}(TQ)$, generator of dilations along the fibers, given by

$$\Delta f(q, v) = \frac{d}{dt} f(q, e^t v)|_{t=0},$$

for all $(q, v) \in TQ$ and $f \in C^\infty(TQ)$. This vector field allows us to define an energy function associated to the Lagrangian L by means of $E_L = \Delta L - L$.

The structure of the tangent bundle is characterized by the existence of an additional object, the vertical endomorphism $S : TTQ \to VTQ$, with local expression in tangent bundle coordinates [23, 24]:

$$S = \frac{\partial}{\partial v^i} \otimes dq^i.$$

The Louville 1-form is defined by $\theta_L = dL \circ S$ and we say that the Lagrangian L is regular when the associated exact 2-form, called the Cartan 2-form, $\omega_L = -d(dL \circ S)$, is non-degenerate and therefore is a symplectic form [23, 24]. In this case the Lagrangian dynamics is given by the uniquely defined vector field Γ_L, called Lagrangian vector field, satisfying $i_{\Gamma_L}\omega_L = dE_L$. Actually, one can check that Γ_L is a second order differential equation vector field, which we will abbreviate by SODE, i.e. $S(\Gamma_L) = \Delta$, and that the projections on Q of the integral curves of Γ_L satisfy the well-known second order Euler-Lagrange equations.

In order to get an intrinsic definition of S we start by defining the vertical lift map. Given a vector $v \in T_qQ$ the application $\xi^v : T_qQ \to T_vTQ$ defined by:

$$\xi^v(w)f = \frac{d}{dt} f(v + tw)\Big|_{t=0}, \forall f \in C^\infty(TQ),$$

is called the vertical lift map. To any vector $v \in T_qQ$ the application ξ^v associates to every vector $w \in T_qQ$, the tangent vector to the curve $t \mapsto v + tw$ at $t = 0$. The element $\xi^v(w)$ is called the vertical lift of w to v and will be sometimes denoted by $\overset{\vee}{w}_v$. Their local expressions are related by

$$w = w^i \frac{\partial}{\partial q^i}\Big|_q \Longleftrightarrow \xi^v(w) = w^i \frac{\partial}{\partial v^i}\Big|_{(q,v)}.$$

The vertical endomorphism S is then defined by $S = \xi \circ T\tau_Q$.The vertical lift of a vector field in the base $X = X^i(q)\partial/\partial q^i$ is

$$X^{\vee}(q,v) = X^i(q)\frac{\partial}{\partial v^i}.$$

Recall that given a (hyper-)regular Lagrangian in TQ we can define a related Hamiltonian formulation on T^*Q by means of the Legendre transformation. In fact, if $L \in C^\infty(TQ)$ we denote by $L_q : T_qQ \to \mathbb{R}$ the function $L_q(w) = L(q,w)$. For every $v \in T_qQ$ the application $dL_q(v) \circ \xi^v : T_qQ \to \mathbb{R}$ is linear and therefore defines a 1-form. The vector bundle map $\mathcal{F}L : TQ \to T^*Q$ defined by:

$$\mathcal{F}L(q,v) = (q, dL_q(v) \circ \xi^v).$$

is called Legendre transformation and allows to associate to every vector on a point of Q an unique covector on the same point. The pullback of the canonical symplectic form on T^*Q is such that $\mathcal{F}L^*\omega_0 = \omega_L$.

From now on we assume that L is a regular Lagrangian, which means one of the three equivalent conditions:

i) the fibre derivative (Legendre transformation) $\mathcal{F}L\colon TQ \to T^*Q$ is a local diffeomorphism;

ii) the Lagrange 2-form ω_L is a symplectic form;

iii) its fibre Hessian $\mathcal{F}^2L = G^L\colon TQ \to T^*Q \otimes T^*Q$ is everywhere a nondegenerate bilinear form. Given $u, a, b \in T_qQ$, the fibre Hessian of the Lagrangian can also be expressed as $G^L_u(a,b) = \omega_L(\tilde{a}, b^{\vee}{}_u)$, where $\tilde{a} \in T_uTQ$ is any vector projecting to a, and $b^{\vee}{}_u$ is the vertical lift of b on the point u.

In the hyper regular case of $\mathcal{F}L$ being a global diffeomorphism, one can define the uniquely defined Hamiltonian function $H \in C^\infty(T^*Q)$ by $H \circ \mathcal{F}L = E_L$ and then, $\mathcal{F}L_*\Gamma_L = X_H$.

We already defined the vertical lift, and now we will remember some basic informations about the complete lift.

The *complete lift* of a vector field $X \in \mathfrak{X}(Q)$ with local expression $X = X^i\partial/\partial x^i$, to be denoted X^c, is the vector field in TQ whose flow is $T\phi_t$, where ϕ_t is the flow of the vector field $X \in \mathfrak{X}(Q)$. The local coordinate expression of X^c is

$$X^c(q,v) = X^i(q)\,\frac{\partial}{\partial q^i} + v^j\,\frac{\partial X^i}{\partial q^j}(q,v)\,\frac{\partial}{\partial v^i} = X^i(q)\,\frac{\partial}{\partial q^i} + (DX^i)(q,v)\frac{\partial}{\partial v^i},$$

for any second order differential equation vector field D.

Complete lifts are determined by the action on functions on TQ that are linear in the fibre coordinates and are associated to 1-forms as follows. For a 1-form α on Q, let $\widehat{\alpha}$ denote the associated linear function on TQ given by $\widehat{\alpha}(v) = \langle \alpha_{\tau_Q(v)}, v\rangle$, for $v \in TQ$. In local tangent bundle coordinates, if $\alpha = \alpha_i(q)\,dq^i$, the function $\widehat{\alpha}$ is $\widehat{\alpha}(q,v) = \alpha_i(q)\,v^i$. In particular, for an exact 1-form $\alpha = df$ with $f \in C^\infty(Q)$, the associated linear function is $\widehat{df}(q,v) = v^i(\partial f/\partial q^i)_q$, i.e. $\widehat{df}$ looks like the total derivative of the function f, and we denote $\dot{f} = \widehat{df}$, which can also be obtained by $\dot{f} = \mathcal{L}_D(\tau_Q^* f)$ for an arbitrary second order differential equation vector field D. Given a vector field X on Q its complete lift X^c is the only vector field on TQ which projects on X and satisfies

$$\mathcal{L}_{X^c}\widehat{\alpha} = \widehat{\mathcal{L}_X\alpha} \tag{2.1}$$

for every 1-form α on Q. It is clear that the vertical components of X^c are determined by the above condition by simply considering as 1-form α each one of 1-forms dq^i. Then $\mathcal{L}_X(dq^i) = dX^i = (\partial X^i/\partial q^j)dq^j$, and $\widehat{\alpha} = \widehat{\mathcal{L}_X(dq^i)} = (\partial X^i/\partial q^j)v^j$, while the left hand side of the preceding relation is the corresponding vertical component of X^c.

Let us show that the complete lift satisfies such a relation. If ϕ_t is the flow of X then the flow of X^c is $T\phi_t$, so that for $v \in TQ$, with $q = \tau_Q(v)$, we have

$$\begin{aligned}(\mathcal{L}_{X^c}\widehat{\alpha})(v) &= \frac{d}{dt}\widehat{\alpha}(T\phi_t(v))\Big|_{t=0} = \tfrac{d}{dt}\langle\alpha_{\tau_Q(T\phi_t(v))}, T\phi_t(v)\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle\alpha_{\phi_t(q)}, T\phi_t(v)\rangle\Big|_{t=0} = \tfrac{d}{dt}\langle(\phi_t^*\alpha)_q, v\rangle\Big|_{t=0} \\ &= \langle(\mathcal{L}_X\alpha)_q, v\rangle = \widehat{\mathcal{L}_X\alpha}(v).\end{aligned}$$

In particular, for $\alpha = df$ we have $\mathcal{L}_{X^c}\dot{f} = (\dot{\mathcal{L}_X f})$.

A remarkable property to be used later on is that for a given vector field $X \in \mathfrak{X}(Q)$, $[X^c, D]$ is a vertical vector field in TQ for any second order differential equation vector field D, because $X^cD(q^i) = D(X^i) = DX^c(q^i) = v^k\partial X^i/\partial q^k$. The preceding property (2.1) can also be used to give an intrinsic proof as follows. Indeed, the action on basic functions is

$$\begin{aligned}\mathcal{L}_{[D,X^c]}(\tau_Q^*f) &= \mathcal{L}_D\mathcal{L}_{X^c}(\tau_Q^*f) - \mathcal{L}_{X^c}\mathcal{L}_D(\tau_Q^*f) \\ &= \mathcal{L}_D(\tau_Q^*\mathcal{L}_Xf) - \mathcal{L}_{X^c}\dot{f} = (\dot{\mathcal{L}_Xf}) - \mathcal{L}_{X^c}\,\dot{f} = 0\end{aligned}$$

from where it follows that $[X^c, D]$ is vertical.

This property may be used to show that for any Lagrangian, if X is a vector field on Q and X^c its complete lift, then the function G defined by $G = \langle\theta_L, X^c\rangle$ is such that $\mathcal{L}_{\Gamma_L}G = \mathcal{L}_{X^c}L$, that is,

$$\Gamma_L(G) = X^c(L). \tag{2.2}$$

In fact, as L is assumed to be regular the vector field Γ_L satisfies $\mathcal{L}_{\Gamma_L}\theta_L = dL$ and then $\langle\mathcal{L}_{\Gamma_L}\theta_L - dL, X^c\rangle = 0$. Using a well-known property of the Lie derivative,

$$\langle\mathcal{L}_{\Gamma_L}\theta_L, X^c\rangle = i_{X^c}\mathcal{L}_{\Gamma_L}\theta_L = \mathcal{L}_{\Gamma_L}i_{X^c}\theta_L + i_{([X^c,\Gamma_L])}\theta_L,$$

we have

$$\Gamma_L(\langle\theta_L, X^c\rangle) - \langle\theta_L, [\Gamma_L, X^c]\rangle - \langle dL, X^c\rangle \quad = \quad 0.$$

But the Cartan 1-form θ_L is a semi-basic 1-form and $[X^c, \Gamma_L]$ is a vertical vector field because Γ_L is a second order vector field and then $\langle\theta_L, [\Gamma_L, X^c]\rangle = 0$. Therefore, $\Gamma_L(\langle\theta_L, X^c\rangle) = \Gamma_L(G) = \langle dL, X^c\rangle = X^c(L)$.

2.2 Nonholonomic systems

A constraint on a dynamic system is called holonomic if it only limits the possible positions of the system and correspondingly the velocities by tangency velocities. Otherwise constraints are said to be nonholonomic.

In this section, we will recall some elements from the theory of dynamic systems subjected to nonholonomic constraints on the tangent space, using both Lagrange multipliers approach and a distributional approach.

The geometric approach of the unconstrained systems was extended to nonholonomic mechanic systems, using the symplectic [15, 36], Hamiltonian [52] and Lagrangian [33] approach and also by means of the almost-Poisson bracket [16]. For symmetry and reduction of the dynamics we refer to [3, 17, 18].

2.2.1 Lagrange multipliers approach

We consider an n-dimensional manifold Q, and its tangent bundle $\tau_Q\colon TQ \to Q$. We also consider a set of linear constraints which defines a vector subbundle $\mathcal{D} \subset TQ$ of rank r, and which is called the constraint submanifold. The admissible velocities are the elements of $\mathcal{D}$, and a curve in Q is said to be admissible if its velocity vectors take values in $\mathcal{D}$. From the annihilator $\mathcal{D}^\circ \subset T^*Q$ of $\mathcal{D}$, i.e. the set of linear 1-forms vanishing on the elements of $\mathcal{D}$, we construct the set $\widetilde{\mathcal{D}^\circ} \subset T^*(TQ)$ defined by $\widetilde{\mathcal{D}^\circ} = \{\, \alpha \circ T\tau_Q \in T^*(TQ) \mid \alpha \in \mathcal{D}^\circ \,\}$. It is a vector bundle over TQ, whose fibre at a point $v \in TQ$, such that $\tau_Q(v) = q$, is more explicitly described as

$$\widetilde{\mathcal{D}^\circ_v} = \left\{\, \lambda_v \in T^*_v(TQ) \;\middle|\; \text{there exists } \alpha_q \in \mathcal{D}^\circ_q \text{ such that } \lambda_v = \alpha_q \circ T_v\tau_Q \,\right\}. \tag{2.3}$$

Given a Lagrangian function $L \in C^\infty(TQ)$, we consider the nonholonomic system defined by the Lagrangian L and the linear constraints given by the vector subbundle $\mathcal{D}$. The evolution of the nonholonomic system is determined by the Lagrange–d'Alembert principle, which states that the dynamics of the system is given by the integral curves (with initial condition in $\mathcal{D}$) of the vector field $\Gamma_{\mathrm{nh}} \in \mathfrak{X}(TQ)$tangent to $\mathcal{D}$ satisfying the second-order condition and the Lagrange–d'Alembert equation (see for instance [34])

$$(i_{\Gamma_{\mathrm{nh}}}\omega_L - dE_L)|_{\mathcal{D}} \in \mathrm{Sec}(\widetilde{\mathcal{D}^\circ}) \,. \tag{2.4}$$

In this expression ω_L is the Cartan 2-form associated with L, defined by $\omega_L = -d\theta_L$ as explained in Subsection 2.1.2.This equation above means that at every point of $\mathcal{D}$ the 1-form $i_{\Gamma_{\mathrm{nh}}}\omega_L - dE_L$ takes value in the codistribution $\widetilde{\mathcal{D}^\circ}$. This value is the reaction force exerted by nonholonomic constraints, the constraint forces.

From now on we assume that L is a regular Lagrangian, and, moreover, in order to avoid unnecessary complications we will assume that the constrained system is regular, in the sense that the equation (2.4) has a unique solution Γ_{nh} tangent to $\mathcal{D}$.

For the local description of the problem, we take local coordinates (x^i) on the base manifold Q and induced coordinates (x^i, v^i) on TQ. If we choose a local basis of 1-forms $\{\omega^A = \omega^A_i\, dx^i\}$ of the annihilator of the constraint distribution, the elements of $\mathcal{D}^\circ$ are of the form $\lambda = \lambda_A\, \omega^A$ and hence the local expression of Lagrange-D'Alembert equations is

$$\begin{cases} \dot{x}^i = v^i \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial v^i} - \dfrac{\partial L}{\partial x^i} = \lambda_A\, \omega_i^A \\ \omega_i^A v^i = 0 \end{cases} \tag{2.5}$$

Under our regularity assumption, these equations determine the values of λ_A and hence define a unique second order differential equation.

In terms of the free dynamics, i.e. the Γ_L solution of the unconstrained problem $i_{\Gamma_L}\omega_L = dE_L$, the constrained dynamical vector field Γ_{nh} can be written in the form

$$\Gamma_{\mathrm{nh}} = \Gamma_L + \lambda_A Z_A, \tag{2.6}$$

where Z_A are the vector fields given by $i_{Z_A}\omega_L = -\widetilde{\omega}^A$, with $\widetilde{\omega}^A = \omega^A \circ T\tau_Q$. These vector fields are vertical, since $\widetilde{\omega}^A \in \widetilde{\mathcal{D}^\circ}$ are semibasic and the vertical distribution is Lagrangian for ω_L.

In local coordinates the vector field Γ_{nh} is given by:

$$\Gamma_{\mathrm{nh}} = v^i \frac{\partial}{\partial x^i} + W^{ij}\left(\frac{\partial L}{\partial x^j} - v^k \frac{\partial^2 L}{\partial x^k \partial v^j} + \lambda_A \omega_j^A\right)\frac{\partial}{\partial v_i}, \tag{2.7}$$

where W^{ij} represents the inverse matrix entries of the matrix $[\partial^2 L/\partial v^i \partial v^j]$ and the multipliers λ_A are determined by

$$W^{ij}\omega_j^A \omega_i^B \lambda_A = -(v^i v^j \partial\omega_j^B/\partial x^i + F^i \omega_i^B),$$

with F^i being the forces of the unconstrained system (the coefficients of $\partial/\partial v^i$ in the above expression with $\lambda_A = 0$).

We notice that in practice the constrained dynamics Γ_{nh} and the vector fields Z_A are considered as vector fields defined on an open neighborhood of the constraint submanifold $\mathcal{D}$, but only their values on $\mathcal{D}$ are relevant.

2.2.2 Distributional approach

Regularity

As we said above, the nonholonomic system $(L, \mathcal{D})$ is said to be *regular* if there is a unique solution to Lagrange–d'Alembert equation. In the present context 'uniqueness' must be understood as follows: two SODE solutions are considered equal if they coincide when restricted to $\mathcal{D}$.

There are several equivalent ways to ensure regularity of the constrained system. We define the subbundle $\mathcal{T}^{\mathcal{D}}\mathcal{D} \subset T\mathcal{D} \to \mathcal{D}$ by

$$\mathcal{T}^{\mathcal{D}}\mathcal{D} = \{\, V \in T\mathcal{D} \mid T\tau_Q(V) \in \mathcal{D}\,\}. \tag{2.8}$$

We also consider the restriction $G^{L,\mathcal{D}}$ of the fibre Hessian G^L to the distribution $\mathcal{D}$.

THEOREM 1: *The following properties are equivalent:*

1. *The constrained Lagrangian system $(L, \mathcal{D})$ is regular,*

2. $\operatorname{Ker} G^{L,\mathcal{D}} = \{0\}$.

3. $TTQ|_{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp}$.

where $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp}$ *denotes the orthogonal complement of* $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ *with respect to the symplectic form* ω_L.

For the proof, see for instance [22] and references there in.

REMARK 1: In the case of a constrained mechanical system $L = T_g - V$, the tensor G^L is given by $G^L[a](b,c) = g_{\tau(a)}(b,c)$, so that it is positive definite at every point. Thus the restriction to any subbundle $\mathcal{D}$ is also positive definite and hence regular. Thus, nonholonomic mechanical systems are always regular. $\diamond$

The manifold $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ has a double vector bundle structure over $\mathcal{D}$ with the projections $\tau_{TQ}|_{\mathcal{T}^{\mathcal{D}}\mathcal{D}}$ and $T\tau_Q|_{\mathcal{T}^{\mathcal{D}}\mathcal{D}}$. The rank of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is $\operatorname{rank} \mathcal{T}^{\mathcal{D}}\mathcal{D} = 2 \operatorname{rank} \mathcal{D}$. By a SODE in $\mathcal{D}$ we mean a section Γ of the vector bundle $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ such that $T\tau_Q(\Gamma(v)) = v$ for every $v \in \mathcal{D}$. It follows that a SODE in $\mathcal{D}$ can be extended (in a non unique way) to a SODE on TQ which is tangent to $\mathcal{D}$, and conversely, a SODE vector field on TQ which is tangent to $\mathcal{D}$ restricts to a SODE in $\mathcal{D}$.

Projection to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$

As a consequence of the above Theorem 1 we get that if the constrained system $(L,\mathcal{D})$ is regular we can obtain the constrained dynamics by projection of the free dynamics according to the decomposition given in item 3. Let us denote by $\bar{P}$ and $\bar{Q}$ the complementary projectors defined by the decomposition $T_aTQ = \mathcal{T}^{\mathcal{D}}_a\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}_a\mathcal{D})^{\perp}$ for $a \in \mathcal{D}$, that is,

$$\bar{P}_a \colon T_aTQ \to \mathcal{T}^{\mathcal{D}}_a\mathcal{D} \qquad \text{and} \qquad \bar{Q}_a \colon T_aTQ \to (\mathcal{T}^{\mathcal{D}}_a\mathcal{D})^{\perp}, \quad \text{for all } a \in \mathcal{D}.$$

Then, we have the following result.

THEOREM 2: *Let* $(L,\mathcal{D})$ *be a regular constrained Lagrangian system and let* Γ_L *be the solution of the free dynamics, i.e.,* $i_{\Gamma_L}\omega_L = dE_L$. *Then the solution of the constrained dynamics is the* SODE Γ_{nh} *obtained by:* $\Gamma_{nh} = \bar{P}(\Gamma_L|_{\mathcal{D}})$.

For the proof, see [22] and references in there.

The distributional approach

A second consequence of Theorem 1 is that we can write Lagrange-d'Alembert equations as symplectic equations entirely in terms of objects defined on the manifold $\mathcal{T}^{\mathcal{D}}\mathcal{D}$. Indeed, since $(L,\mathcal{D})$ is regular, from item (3) of Theorem 1, we have that $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is a symplectic subbundle of (TTQ, ω_L). Hence the restriction $\omega^{L,\mathcal{D}}$ of ω_L to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is a symplectic form on the vector bundle $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ (i.e. it is a regular skew-symmetric bilinear form). We denote by $\bar{d}E_L$ the restriction of dE_L to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$. Then, taking the restriction of Lagrange-d'Alembert equations to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, we get the following equation

$$i_{\Gamma_{\mathrm{nh}}}\omega^{L,\mathcal{D}} = \bar{d}E_L, \tag{2.9}$$

which uniquely determines the section Γ_{nh}. Indeed, the unique solution Γ_{nh} of the above equations is the solution of Lagrange-d'Alembert equations: if we denote by λ the constraint force, we have for every $u \in \mathcal{T}^{\mathcal{D}}_a\mathcal{D}$ that

$$\omega_L(\Gamma_{\rm nh}(a), u) - \langle\, dE_L(a)\,, u\,\rangle = \langle\, \lambda(a)\,, T\tau_Q(u)\,\rangle = 0,$$

where we have taken into account that $T\tau_Q(u) \in \mathcal{D}$ and $\lambda(a) \in \mathcal{D}^\circ$.

This approach, the so called distributional approach, was initiated by Bocharov and Vinogradov [53] and further developed by Śniatycki and coworkers [3, 25, 50].

The nonholonomic bracket

The symplectic section $\omega^{L,\mathcal{D}}$ allows us to define an almost-Poisson bracket in $\mathcal{D}$ which is known as the nonholonomic bracket. An almost-Poisson bracket on a manifold P is a bracket $\{\,\cdot, \cdot\}$ of functions on P which is $\mathbb{R}$-bilinear, skew-symmetric, a derivation in each argument with respect to the usual product of functions but it does not necessarily satisfies the Jacobi identity.

For every function $f \in C^\infty(\mathcal{D})$ we consider the restriction $\bar{d}f$ to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ of its differential $\bar{d}f = df|_{\mathcal{T}^{\mathcal{D}}\mathcal{D}} \in \mathrm{Sec}((\mathcal{T}^{\mathcal{D}}\mathcal{D})^*)$. Since $\omega^{L,\mathcal{D}}$ is regular, we have a unique section $\bar{X}_f \in \mathrm{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D})$ such that $i_{\bar{X}_f}\omega^{L,\mathcal{D}} = \bar{d}f$.

Definition 1: *The nonholonomic bracket of two functions $f, g \in C^\infty(\mathcal{D})$ is the function $\{f, g\}_{nh} \in C^\infty(\mathcal{D})$ given by*

$$\{f, g\}_{nh} = \omega^{L,\mathcal{D}}(\bar{X}_f, \bar{X}_g). \tag{2.10}$$

Alternatively the nonholonomic bracket can be defined as follows. We first notice that if $\widetilde{f} \in C^\infty(TQ)$ is an extension to TQ of f then $\bar{X}_f = \bar{P}(X_{\widetilde{f}}\big|_{\mathcal{D}})$. Let f, g be two smooth functions on $\mathcal{D}$ and take arbitrary extensions $\widetilde{f}, \widetilde{g}$ to TQ. Let $X_{\widetilde{f}}$ and $X_{\widetilde{g}}$ the associated Hamiltonian vector fields

$$i_{X_{\widetilde{f}}}\,\omega_L = d\widetilde{f} \qquad \text{and} \qquad i_{X_{\widetilde{g}}}\,\omega_L = d\widetilde{g}.$$

Then the nonholonomic bracket of f and g is

$$\{f, g\}_{\rm nh} = \omega_L(\bar{P}(X_{\widetilde{f}}|_{\mathcal{D}}), \bar{P}(X_{\widetilde{g}}|_{\mathcal{D}})). \tag{2.11}$$

Indeed, the result follows by noticing that if $\widetilde{\widetilde{f}}$ is another extension of f, then $(X_{\widetilde{f}} - X_{\widetilde{\widetilde{f}}})_{|\mathcal{D}}$ is a section of $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp$, and therefore the result does not depend on the choice of extensions.

In what follows, the nonholonomic bracket of two functions on TQ should be understood as the nonholonomic bracket of their restrictions to $\mathcal{D}$

As a consequence of the above we have the following result, which is fundamental for our purposes.

Theorem 3: *If $f, g \in C^\infty(\mathcal{D})$ then*

$$\{f, g\}_{nh} = \bar{X}_g f = -\bar{X}_f g. \tag{2.12}$$

Moreover, for any function $f \in C^\infty(\mathcal{D})$ we have

$$\dot{f} = \{f, E_L\}_{nh}. \tag{2.13}$$

PROOF. If $\widetilde{f}$ and $\widetilde{g}$ are extensions of f and g, then at every point of $\mathcal{D}$ we have

$$\{f,g\}_{\rm nh} = \omega_L(\bar{P}(X_{\widetilde{f}}), \bar{P}(X_{\widetilde{g}})) = \omega_L(X_{\widetilde{f}}, \bar{P}(X_{\widetilde{g}})) = \langle d\widetilde{f}, \bar{P}(X_{\widetilde{g}})\rangle = \bar{X}_g f,$$

where we have used that $\omega_L(\bar{Q}(X_{\widetilde{f}}), \bar{P}(X_{\widetilde{g}})) = 0$ and $\bar{X}_f = \bar{P}(X_{\widetilde{f}}\big|_{\mathcal{D}})$. The second statement follows from the first one by taking into account that $\Gamma_{\rm nh} = \bar{X}_{E_L}$. □

In particular, equation (2.13) implies the conservation of the energy (by the skew-symmetric character of the nonholonomic bracket).

2.3 Riemannian structure

This section recalls some properties of Riemannian manifolds which will be used in following chapters.

Let (Q, g) be a (pseudo-)Riemann manifold, i.e. g is a non-degenerate symmetric two times covariant tensor field on Q. Nondegeneracy means that the map $\widehat{g} : TQ \to T^*Q$ from the tangent bundle $\tau_Q : TQ \to Q$ to the cotangent bundle $\pi_Q : T^*Q \to Q$, defined by $\langle \widehat{g}(v), w\rangle = g(v, w)$, where $v, w \in T_xQ$, is regular. The map $\widehat{g}$ is a fibred map over the identity on Q and induces the corresponding map between the spaces of sections of the tangent and cotangent bundles, to be denoted by the same letter $\widehat{g} : \mathfrak{X}(Q) \to \Omega^1(Q)$, i.e. $\langle \widehat{g}(X), Y\rangle = g(X, Y)$.

In local coordinates for Q, $(q^1, \ldots, q^n)$, the expression for g is

$$g = g_{ij}(q)\, dq^i \otimes dq^j, \quad g_{ij} = g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right). \tag{2.14}$$

Given a symmetric covariant 2-tensor field K in Q we denote by $T_K \in C^\infty(TQ)$ the function

$$T_K(v) = \frac{1}{2}K(v, v), \qquad v \in TQ.$$

This rule identifies symmetric covariant 2-tensor fields with quadratic homogeneous functions on the fibre coordinates. In particular when g is a Riemann structure in Q,

$$T_g(v) = \frac{1}{2}\, g(v, v), \qquad v \in TQ,$$

is the kinetic energy defined by the metric. Later on, on Chapter 4 we will consider the case of Lagrangians of a mechanical type, where in particular we will consider the kinetic energy defined by the metric g.

DEFINITION 2: *A linear connection on a differential manifold Q is a mapping*

$$\nabla : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$$

which satisfies the following properties:

$$\textit{i)}\ \nabla_{f\,X+g\,Y} Z = f\,\nabla_X Z + g\,\nabla_Y Z$$
$$\textit{ii)}\ \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$$
$$\textit{iii)}\ \nabla_X(fY) = X(f)Y + f\nabla_X Y,$$

where $X, Y, Z \in \mathfrak{X}(Q)$, and $f, g \in C^\infty(Q)$.

DEFINITION 3: *Let Q be a differentiable manifold endowed with a linear connection. There exists an unique correspondence which associates to each vector field along the differential curve $c : I \to Q$, V, another vector field $\dfrac{DV}{dt}$, called the covariant derivative of V along the curve c, such that:*

i) $\dfrac{D(V+W)}{dt} = \dfrac{DV}{dt} + \dfrac{DW}{dt}$,

ii) $\dfrac{D}{dt}(fV) = \dfrac{df}{dt}V + f\dfrac{DV}{dt}$,

iii) If V is induced by a vector field $Y \in \mathfrak{X}(Q)$, i.e. $V(t) = Y(c(t))$, then $\dfrac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$.

A linear connection on a Riemann manifold (Q, g) is metric, or compatible with the Riemann structure g, i.e. the parallel transport along any curve is an isometry, if and only if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \qquad \forall X, Y, Z \in \mathfrak{X}(Q). \tag{2.15}$$

The main result is that there exists a unique torsion-free metric connection on a Riemann manifold (Q, g), called Levi-Civita connection, which is given by Koszul formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &- g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned} \tag{2.16}$$

In particular, when a coordinate chart is considered, the Christoffel symbols of the second kind, Γ^i_{jk}, defined by

$$\nabla_{\partial/\partial q^j}\left(\frac{\partial}{\partial q^k}\right) = \Gamma^i_{jk}\frac{\partial}{\partial q^i}$$

are given by

$$\Gamma^i_{jk}(q) = \frac{1}{2} g^{il}(q)\left(\frac{\partial g_{lj}}{\partial q^k}(q) + \frac{\partial g_{lk}}{\partial q^j}(q) - \frac{\partial g_{jk}}{\partial q^l}(q)\right), \tag{2.17}$$

where g^{ij} are the inverse matrix entries of the Riemann structure g.

Then, the linear connection is given by

$$\nabla_X Y = X^i\left(\frac{\partial Y^k}{\partial q^i} + Y^j\, \Gamma^k_{ij}(q)\right)\frac{\partial}{\partial q^k},$$

and correspondingly,

$$\nabla_X \alpha = X^k\left(\frac{\partial \alpha_j}{\partial q^k} - \alpha_i\, \Gamma^i_{jk}\right) dq^j.$$

Another remarkable relation is that if α is the 1-form $\alpha = \widehat{g}(X)$, where $X \in \mathfrak{X}(Q)$, then, using that the relation $\nabla_Z\langle\alpha, Y\rangle = \langle\nabla_Z\alpha, Y\rangle + \langle\alpha, \nabla_Z Y\rangle$, for any two vector fields $Y, Z \in \mathfrak{X}(Q)$, can be rewritten as

$$Z(g(X,Y)) = \langle\nabla_Z\alpha, Y\rangle + g(X, \nabla_Z Y),$$

and having in mind the property of the compatibility of the connection with the metric, we see that

$$\langle\nabla_Z\alpha, Y\rangle = g(\nabla_Z X, Y). \tag{2.18}$$

2.4 Killing and conformal vector fields

A diffeomorphism $F : Q \to Q$ induces a new (pseudo-) Riemann structure F^*g on Q. Such a transformation F is called a conformal symmetry when there exists a function $f \in C^\infty(Q)$ such that $F^*g = f\,g$. In particular when f is a constant (different from one) F is said to be a (proper) homothethy and, finally, when $F^*g = g$, the map F is called isometry.

In the infinitesimal approach we say that a vector field $X \in \mathfrak{X}(Q)$ is either a conformal, a homothetic, or a Killing vector field, when its flow ϕ_t is made of conformal maps, homothethies or isometries, respectively:

$$\begin{array}{lll} \text{conformal vector field}: & \mathcal{L}_X g = f\,g, & f \in C^\infty(Q), \\ \text{homothetic vector field}: & \mathcal{L}_X g = \lambda\,g, & \lambda \in \mathbb{R}, \\ \text{Killing vector field}: & \mathcal{L}_X g = 0. & \end{array}$$

Proper conformal vector fields are those vector fields for which the conformal factor f is non constant and similarly a proper homothetic vector field is when $\lambda \neq 0$. Using the well known property $\mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}$ one sees that the set of conformal vector fields is a Lie algebra and those of homothetic and Killing vector fields are subalgebras. For more details see e.g. [30, 31, 37, 51].

Let us see in local coordinates, the condition of a vector fields to be Killing with respect to the Riemannian metric g. Given a vector field on Q,

$$X = X^i(q)\,\frac{\partial}{\partial q^i} \in \mathfrak{X}(Q), \tag{2.19}$$

the Lie derivative with respect to the vector field X of the metric tensor field g is

$$\mathcal{L}_X g = X^k\,\frac{\partial g_{ij}}{\partial q^k}\,dq^i \otimes dq^j + g_{ij}\left(\frac{\partial X^i}{\partial q^k}\,dq^k \otimes dq^j + \frac{\partial X^j}{\partial q^k} dq^i \otimes dq^k\right),$$

or using the symmetry property of the metric tensor field,

$$\mathcal{L}_X g = \left(X^k\,\frac{\partial g_{ij}}{\partial q^k} + g_{ik}\,\frac{\partial X^k}{\partial q^j} + g_{jk}\,\frac{\partial X^k}{\partial q^i}\right) dq^i \otimes dq^j, \tag{2.20}$$

and then the condition for X to be a Killing vector field, i.e. $\mathcal{L}_X g = 0$, is written in the above mentioned local coordinates as

$$\left(X^k\,\frac{\partial g_{ij}}{\partial q^k} + g_{ik}\,\frac{\partial X^k}{\partial q^j} + g_{jk}\,\frac{\partial X^k}{\partial q^i}\right)\,dq^i \otimes dq^j = 0.$$

Therefore, the set of conditions for the vector field $X \in \mathfrak{X}(Q)$ given by (2.19) to be a Killing symmetry are:

$$X^k\,\frac{\partial g_{ij}}{\partial q^k} + g_{ik}\,\frac{\partial X^k}{\partial q^j} + g_{jk}\,\frac{\partial X^k}{\partial q^i} = 0, \qquad i, j = 1, \ldots, n. \tag{2.21}$$

Next, we want to express the condition for a vector field to be Killing with respect to a Riemannian structure, i.e. (2.21), in terms of the Levi-Cevita connection. This condition $\mathcal{L}_X g = 0$, can be written in an intrinsic way as the condition for the covariant derivative of the vector field X to be a skew-symmetric endomorphism with respect to the metric g, that is (see e.g. Proposition 4.10 of [54]), for every $Y, Z \in \mathfrak{X}(Q)$,

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0. \tag{2.22}$$

Next, we prove the following relation concerning the kinetic energy and the Killing vector fields defined by a Riemannian structure are recalled, that we will use later on for establishing the virial theorem for a Killing vector field:

$$X^c T_g = T_{\mathcal{L}_X g}. \tag{2.23}$$

In fact,

$$\begin{aligned}(X^c T_g)(q, v) &= \frac{1}{2}\left(X^k(q)\frac{\partial g_{ij}}{\partial q^k}(q)\, v^i v^j + g_{ij}(q)\,\frac{\partial X^i}{\partial q^k}(q)\, v^k v^j + g_{ij}(q)\,\frac{\partial X^j}{\partial q^k}(q)\, v^i v^k\right) \\ &= \frac{1}{2}\left(X^k(q)\frac{\partial g_{ij}}{\partial q^k}(q) + g_{kj}(q)\,\frac{\partial X^k}{\partial q^i}(q) + g_{ik}(q)\,\frac{\partial X^k}{\partial q^j}(q)\right) v^i v^j\end{aligned}$$

and therefore, according to (2.20), the relation (2.23) follows. This relation may also be proved intrinsically by using the definitions of Lie derivative and of T_g mentioned earlier in the text: for all $v \in TQ$,

$$\begin{aligned} X^c T_g(v) &= \frac{d}{dt} T_g \circ T\phi_t(v)\big|_{t=0} = \frac{d}{dt}\left(\frac{1}{2} g(T\phi_t(v), T\phi_t(v))\right)_{|t=0} \\ &= \frac{1}{2}\frac{d}{dt}(\phi_t^* g)(v, v)\big|_{t=0} = \frac{1}{2}(\mathcal{L}_X g)(v, v) = T_{\mathcal{L}_X g}(v, v).\end{aligned}$$

Consequently, $X \in \mathfrak{X}(Q)$ is a Killing vector field for the Riemann structure g if and only if $X^c \in \mathfrak{X}(TQ)$ is a symmetry for the corresponding kinetic energy, i.e. the conditions for X^c to be a symmetry of T_g are given by (2.21).

2.5 Lie algebroids

In this section, we introduce the structure of Lie algebroid, and after exemplifying it, we define some other related notions which will be useful in what follows. For details about it and its importance we refer to [39].

Definition 4: *A Lie algebroid A over a smooth manifold M is a vector bundle $\tau : A \to M$ with a real Lie algebra structure on the $C^\infty(M)$-module of sections* $(\mathrm{Sec}(A), [\cdot, \cdot])$ *and with a vector bundle homomorphism, called the anchor map, $\rho : A \to TM$ such that if we also denote by $\rho : \mathrm{Sec}(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by it, the following Leibniz rule holds:*

$$[\sigma_1, f\sigma_2] = f[\sigma_1, \sigma_2] + (\rho(\sigma_1)f)\sigma_2, \tag{2.24}$$

$\forall \sigma_1, \sigma_2 \in \mathrm{Sec}(A),\ \forall f \in C^\infty(M)$.

It can be shown, by using the properties of the Lie bracket $[\cdot, \cdot]$ and the compatibility condition (2.24), that the induced map $\rho : \mathrm{Sec}(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism, that is, that is, it satisfies:

$$\rho[\sigma_1, \sigma_2] = [\rho(\sigma_1), \rho(\sigma_2)]_{TM}.$$

Local coordinates on A: In what follows we will assume that A is finite dimensional, and we will denote the dimension of the base manifold and of the fibres $A_p = \tau^{-1}(p)$, $p \in M$, by: $\dim M = n$, $\dim A_p = m$, $\forall p \in M$. Then take a local coordinate system $(x^i)_{i=\overline{1,n}}$, on the base manifold and a local basis $\{e_\alpha \mid \alpha = 1, \dots, m\}$, of sections of A. This determines a local coordinate system (x^i, y^α) on A.

In this coordinate system, the functions ρ^i_α and $C^\alpha_{\beta\gamma}$ such that:

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \text{and} \qquad [e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\, e_\gamma.$$

determine locally the anchor and the bracket, and are called *structure functions.* They contain the local information of the Lie algebroid.

The structure functions satisfy the following relations:

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \quad \text{and} \quad \sum_{\text{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\alpha\nu} C^\nu_{\beta\gamma} \right] = 0,$$

which are called the *structure equations* of the Lie algebroid.

The local structure of a Lie algebroid is described by the following theorem:

Theorem 4: *(Local splitting theorem). Let $\tau : A \to M$ be a Lie algebroid and $\bar{m} \in M$ a point where $\rho_{\bar{m}}$ has rank q. Then, there exists a neighborhood U of $\bar{m}$, a coordinate system in U, $(x_1, ..., x_q, \bar{x}_1, \dots, \bar{x}_{n-q})$ and a basis of sections $\{e_1, ..., e_m\}$ of the vector bundle A over U such that:*

$$\begin{cases} \rho(e_i) &= \dfrac{\partial}{\partial x_i}, \quad i = \overline{1, q}, \\ \rho(e_i) &= \rho^j_i \dfrac{\partial}{\partial \bar{x}_j}, \quad i = \overline{q+1, m}, \end{cases}$$

where $\rho^j_i \in C^\infty(U)$ are smooth functions depending on the coordinates $\bar{x}_1, \dots, \bar{x}_{n-q}$ and vanishing at $\bar{m}$, i.e. $\rho^j_i = \rho^j_i(\bar{x}^s)$ and $\rho^j_i(\bar{m}) = 0$, with $s = \overline{1, (n-q)}$.

From the Local splitting theorem, it can be deduced that the image of the anchor map, $\operatorname{Im}\rho$, defines a smooth integrable generalized distribution on M. The corresponding foliation is called the *characteristic foliation of the Lie algebroid A.*

In what follows, we will denote by L the leaves of the characteristic foliation.

Examples

1. Lie algebra: A Lie algebra $\mathfrak{g}$ can be seen as a Lie algebroid over a one point space $M = \{p\}$, $\tau : \mathfrak{g} \to M$ with the anchor map ρ being identically zero.

2. Standard Lie algebroid: The tangent bundle of a manifold M, $\tau : TM \to M$ is a Lie algebroid, where the anchor map is the identity map $\rho = \mathrm{Id}_{TM}$.

The linear space of sections for τ is that of the vector fields $\mathfrak{X}(M)$, and the Lie bracket is just the usual Lie bracket defined on $\mathfrak{X}(M)$.

3. Regular distribution: Let $\mathcal{F}$ be a regular foliation of a manifold M and let $\tau_{\mathcal{F}} : T\mathcal{F} \to M$ be the tangent bundle to the foliation. Then this vector bundle can be seen as a Lie algebroid with anchor map the canonical inclusion $\rho_{\mathcal{F}} : T\mathcal{F} \to TM$ and with the Lie bracket on its sections, coming from the restriction of the standard Lie bracket on $\mathfrak{X}(M)$ to $\mathrm{Sec}(T\mathcal{F})$.

That is, regular foliations can be thought as Lie algebroids with injective map.

Remark 2: Notice that, in general, given a Lie algebroid $\tau : A \to M$ whose anchor map is injective, then its image $\operatorname{Im}\rho \subset TM$ is an involutive subbundle, i.e. a constant rank smooth distribution, which is closed for the usual Lie bracket, and so M is a foliation. ◇

4. Action Lie algebroid: Consider an action of a Lie algebra $\mathfrak{g}$ on a differentiable manifold M, i.e. there is a Lie algebra homomorphism $\Psi : \mathfrak{g} \to \mathfrak{X}(M)$. The the trivial vector bundle $A = M \times \mathfrak{g} \to M$ admits a Lie algebroid structure as follows: the anchor map $\rho : A \to TM$ is given by $\rho(x, v) = \Psi(v)|_x \in T_xM$, while the Lie bracket on the space of sections of A is:

$$[v, w](x) = [v(x), w(x)]_{\mathfrak{g}} + \Psi(v(x))w|_x - \Psi(w(x))v|_x, \ \forall x \in M,$$

where we have identified sections of $M \times \mathfrak{g}$ with $\mathfrak{g}$–valued functions.

5. Atiyah algebroid: Let G be a Lie group and $\pi : P \to M$ be a principal $G-$bundle. Let $\Phi : G \times P \to P$ the transitive free action of G on P and $T\Phi : G \times TP \to TP$ the corresponding tangent action of G on TP. The quotient vector bundle $A(P) := TP/G \to M$, can be endowed with a Lie algebroid structure by defining:

- The anchor map is the map induced by $T\pi$, i.e. $\rho([v]) = T\pi(v)$, which is well defined.
- The Lie bracket on $\mathrm{Sec}(A(P))$ is the one induced by the Lie bracket on G-invariant (under the action of Φ) vector fields. Here we use the one-to-one correspondence between sections of $A(P)$ and G-invariant vector fields (under the action of Φ).

With this Lie algebroid structure, $A(P)$ over M is called *the Atiyah algebroid* associated to the principal G-bundle $\pi : P \to M$.

6. Cotangent Lie Algebroid of a Poisson Manifold:
A Poisson structure on M is a Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ such that:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \ \forall f, g, h \in C^\infty(M).$$

Equivalently, a Poisson structure on M is given by a Poisson bivector, $\Pi \in \mathrm{Sec}(\bigwedge^2 TM)$, i.e. which satisfies $[\Pi, \Pi]_{SN} = 0$, where $[\cdot, \cdot]_{SN}$ denotes the Schouten-Nijenhuis bracket.

The relation between Π and $\{\cdot, \cdot\}$ is given by:

$$\{f, g\} = \Pi(df, dg),$$

where $f, g \in C^\infty(M)$.

To any Poisson manifold (M, Π) it can be associated a Lie algebroid structure on $\pi : T^*M \to M$ called the cotangent Lie algebroid of the Poisson manifold (M, Π). The anchor is $\rho = \Pi^\# : T^*M \to TM$ defined by: $\rho(\alpha)(\beta) = \pi(\alpha, \beta), \forall \alpha, \beta \in T^*M$ and the Lie bracket on sections of $A = T^*M$, is defined as the Koszul bracket:

$$[\alpha, \beta] = \mathcal{L}_{\Pi^\#\alpha}\beta - \mathcal{L}_{\Pi^\#\beta}\alpha - d(\Pi(\alpha, \beta)),\ \forall\, \alpha, \beta \in \mathrm{Sec}(T^*M),$$

where is the unique bracket on $\mathrm{Sec}(T^*M)$ such that $[df, dg] = d\{f, g\}$.

7. The Lie algebroid of a Lie groupoid:

Very much like in Lie group theory, where every Lie group has associated a Lie algebra, every Lie groupoid has an associated Lie algebroid.

For a Lie groupoid $\mathcal{G} \rightrightarrows M$, with source map α, target map β and unity section ε, it can be associated a Lie algebroid $\tau\colon \mathcal{A}\mathcal{G} \to M$ as follows. At each point $x \in M$, the fibre $\mathcal{A}_x\mathcal{G}$ is the vector space $\mathrm{Ker}\, T_{\varepsilon(x)}\alpha$ and the anchor map ρ on $\mathcal{A}_{\varepsilon(x)}\mathcal{G}$ is identified with the restriction of $T_{\varepsilon(x)}\beta$ at $\mathrm{Ker}\, T_{\varepsilon(x)}\alpha$. It is easy to prove that there exists a bijection between $\mathrm{Sec}(\mathcal{A}\mathcal{G})$ and the set of left-invariant (resp., right-invariant) vector fields on $\mathcal{G}$. If X is a section of τ, the corresponding left-invariant vector field on $\mathcal{G}$ will be denoted X^L, where

$$X^L(g) = T_{\varepsilon(\beta(g))}L_g(X_{\beta(g)})\,,$$

for $g \in \mathcal{G}$. The Lie bracket of the Lie algebroid structure is defined by

$$[X, Y]^L = [X^L, Y^L]\,,$$

for $X, Y \in \mathrm{Sec}(\mathcal{A}\mathcal{G})$ and the anchor map $\rho(X)(x) = (T_{\varepsilon(x)}\beta)X(x)$.

Exterior differential operator on A

As indicated above a Lie algebroid structure is a generalization of the tangent bundle structure over M. Sections of the vector bundle play the role of vector fields. Correspondingly sections of the dual bundle will correspond to differential 1-forms for M and similarly what corresponds to p-forms on A are the elements of $\bigwedge^p(A^*) = \mathrm{Sec}((A^*)^{\wedge p} \to M)$, which can be called p-forms for the Lie algebroid. Moreover, the remarkable fact is that because of the properties a Lie algebroid one can define an *exterior differential operator* on the space of sections of the bundle $\bigwedge^p(A^*)$, taking values in $\bigwedge^{p+1}(A^*)$. It is defined by:

- if $f : M \to \mathbb{R}$ then $df_m \in A^*_m$ is definď by: $\langle\, df_m\,, a\,\rangle = \rho(a)f, \forall a \in A_m$.
- if $\omega \in \bigwedge^p(A^*)$, then $d\omega \in \bigwedge^{p+1}(A^*)$ is given by

$$\begin{aligned} d\omega(\sigma_0, \sigma_1, \ldots, \sigma_p) &= \sum_{i=0}^{p}(-1)^i\rho(\sigma_i)(\omega(\sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \sigma_p)) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([\sigma_i, \sigma_j], \sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \widehat{\sigma}_j, \ldots, \sigma_p), \end{aligned}$$

  for $\sigma_0, \ldots, \sigma_p \in \mathrm{Sec}(A)$.

In local coordinates defined by a coordinate set in M and a local basis of sections of A, $\{e_\alpha \mid \alpha = \overline{1,m}\}$, the differential d is determined by:

$$dx^i = \rho^i_\alpha e^\alpha \qquad \text{and} \qquad de^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta,$$

where $\{e^\alpha \mid \alpha = \overline{1,m}\}$ is the corresponding dual basis of sections of A^*. Notice that d is a cohomology operator, that is, $d^2 = 0$ and that the structural equations are exactly: $d^2 x^i = 0$ and $d^2 e^\alpha = 0$.

In particular, the differential of a function $f : M \to \mathbb{R}$ has the local expression:

$$df = \frac{\partial f}{\partial x^i} \rho^i_\alpha e^\alpha.$$

On the other side, a section ω of A^*, say $\omega = \omega_\alpha e^\alpha$, has associated a linear function $\widehat{\omega}$ in A given by:

$$\widehat{\omega}(x, y) = \omega_\alpha(x)\, y^\alpha.$$

Remark 3: The existence of a Lie algebroid structure is equivalent to the existence of a exterior differential operator d on $\bigwedge^p(A)$, because both the anchor map and the structure constants can be derived from it. $\diamond$

Definition 5: *The Lie derivative with respect to a section $\sigma \in \mathrm{Sec}(A)$ is the operator $d_\sigma : \bigwedge^k A^* \to \bigwedge^k A^*$ given by $d_\sigma = i_\sigma \circ d + d \circ i_\sigma$.*

Admissible vectors. Admissible curves

In a Lie algebroid, the notion of natural prolongation is replaced by the notion of admissible curve.

Definition 6: *i) A tangent vector X at a point a on a Lie algebroid $\tau : A \to M$ with anchor $\rho : A \to TM$ is called admissible if the tangent vector to M obtained by projecting X under $T\tau : TA \to TM$ is equal to the tangent vector to M, $\rho(a)$.*

ii) A curve $\alpha : [t_0, t_1] \to A$ is said to be admissible if $\dot{\gamma}(t) = \rho(\alpha(t))$, where $\gamma(t) = \tau(\alpha(t))$ is the base curve.

In local coordinates, if $\alpha(t) = (x^i(t), y^\alpha(t))$ is a curve in A, the curve is admissible if its coordinates are related by: $\dot{x}^i = \rho^i_\alpha y^\alpha$.

Definition 7: *Let α be an admissible curve on A. A smooth curve $\beta : [0,1] \to A$ that has the same projection on M as α is called an $\alpha-$ section.*

We will denote by $\mathrm{Sec}_\alpha(A)$ the set of α-sections.

Let us give some examples of admissible curves, considering two particular cases of Lie algebroids:

EXAMPLE 1: Let $\tau : A \to M$ be a Lie algebroid whose anchor map ρ is injective. In this case A can be thought of as an integrable subbundle of the tangent bundle.

Let us determine the admissible curves in this case: Let be $(x_1, \ldots, x_q)$ be a local coordinate system on an open set $U \subset M$, $\{e_\alpha \mid \alpha = 1, \ldots, m\}$ a local basis of sections over U, and denote by y^α the corresponding linear coordinates.

A curve in A, $\alpha(t) = y^\alpha(t)e_\alpha$, is an admissible curve if

$$\frac{d(\tau \circ \alpha(t))}{dt} = \rho(\alpha(t)),$$

or equivalently,

$$\dot{x}^i(t)\frac{\partial}{\partial x^i} = y^\alpha(t)\rho(e_\alpha) = y^\alpha(t)\rho^i_\alpha\frac{\partial}{\partial x^i} = y^\alpha(t)\delta^i_\alpha\frac{\partial}{\partial x^i} = y^i(t)\frac{\partial}{\partial x^i} \Rightarrow$$

$$\dot{x}^i(t) = y^i(t).$$

So the admissible curves in A are the curves that can be identified with the curves in TM tangent to the leaves: $(x^i(t), \dot{x}^i(t))$.

$\triangleleft$

EXAMPLE 2: For the case of a Lie algebra $\tau : \mathfrak{g} \to \{*\}$ seen as a Lie algebroid over a point, where the anchor map is 0, any curve $\alpha : I \to \mathfrak{g}$ is an admissible one, as $\frac{d}{dt}(\tau \circ \alpha)(t) = \rho(\alpha(t)) = 0$. $\triangleleft$

2.5.1 Prolongation of a fibered manifold with respect to a Lie algebroid

Let $\pi : P \to M$ be a fibred manifold with base manifold M and $\tau : A \to M$ a Lie algebroid. In order to describe a dynamic system on P, we would like to work on a space that uses also the information given by the tangent space to P and also the one given by the Lie algebroid A, which can be thought as a substitute of the tangent bundle to M. The TP is not an appropriate space to describe the dynamics on, as the projection of a vector from it is a vector from TM, and we would like instead an element from A,to 'save' the extra information coming from it.

A space which takes into account these requirements, conserving all this information is the A-tangent bundle of P, also called the prolongation of P with respect to A, which we denote by $\mathcal{T}^A P$. It is defined as the vector bundle $\tau^A_P : \mathcal{T}^A P \to P$ whose fiber at a point $p \in P_m$ is the vector space:

$$\mathcal{T}^A_p P = \{(b, v) \in A_m \times T_p P \mid \rho(b) = T_p\pi(v)\},$$

or, in other words, it can be defined as the vector bundle over P whose total space is the pullback of the fiber bundle $T\pi : TP \to TM$ by the anchor map $\rho : A \to TM$.

An element (b, v) can be written as (p, b, v), where p is the point in which v is tangent to P and with this notation the fiber of the total space of $\mathcal{T}^A P$ is written as:

$$\mathcal{T}^A_p P = \{(p, b, v) \in P \times A_m \times T_p P \,|\, \pi(p) = \tau(b);\ \rho(b) = T_p\pi(v),\ v \in T_p P\}.$$

The projection $\tau^A_P : \mathcal{T}^A P \to P$ is given by $\tau^A_P(b, v) = p$ and it will often be denoted in particular cases, by τ_1. There is another projection $\tau_2 : \mathcal{T}^A P \to A$ defined by $\tau_2(b, v) = b$, and the elements from $\mathcal{T}^A P$ whose image through τ_2 is zero, are called *vertical*.

Local coordinates on $\mathcal{T}^A P$

The local coordinate system (x^i, u^A) on P, with $i = \overline{1, n}$ and $A = \overline{1, \dim P}$, determines local coordinates on $\mathcal{T}^A P$ in the following way: consider an element $(p, b, v) \in \mathcal{T}^A P$ and take for p the coordinates $(m_i, \overline{u}^A)$ and for b the coordinates: (m_i, b^α). Then v, as $\rho(b) = T\pi(v)$, will have the form: $\rho^i_\alpha b^\alpha \frac{\partial}{\partial x^i}\Big|_p + v^A \frac{\partial}{\partial u^A}\Big|_p$. So, in the considered coordinates (p, b, v) will be written as $(m^i, \overline{u}^A, b^\alpha, v^A)$. Generically we will denote the local coordinates of $\mathcal{T}^A P$ by $(x^i, u^A, z^\alpha, v^A)$.

For the local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of $\mathrm{Sec}(A)$, it can be defined a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of $\mathrm{Sec}(\mathcal{T}^A P)$ by :

$$\mathcal{X}_\alpha(p) = \Big(p, e_\alpha(\pi(p)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_p\Big) \qquad \text{and} \qquad \mathcal{V}_A(p) = \Big(p, 0, \frac{\partial}{\partial u^A}\Big|_p\Big).$$

In this base, the element (p, b, v) is written as $z^\alpha \mathcal{X}_\alpha(p) + v^A \mathcal{V}_A(p)$, when $b = z^\alpha e_\alpha$ and v has in consequence the form: $v = \rho^i_\alpha z^\alpha \frac{\partial}{\partial x^i} + v^A \frac{\partial}{\partial u^A}$.

A section Z in $\mathcal{T}^A P$ which in coordinates has the expression:

$$Z(x, u) = (x^i, u^A, Z^\alpha(x, u), V^A(x, u)),$$

has the following expression in terms of the base $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$:

$$Z = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha.$$

Lie algebroid structure of $\mathcal{T}^A P$

If A carries a Lie algebroid structure, then so does $\mathcal{T}^A P$. The associated Lie bracket can be easily defined in terms of projectable sections, defining it only for this kind of sections, as the set of projectable sections is a generating set of $\mathrm{Sec}(\mathcal{T}^A P)$, that is, any section of $\mathcal{T}^A P$ can be locally written as a linear combination of projectable sections.

A section Z of $\mathcal{T}^A P$ is said to be projectable if there exists a section σ of $\tau : A \to M$ such that $\tau_2 \circ \eta = \sigma \circ \pi$. We say that Z is a lifting of σ.

Equivalently, a section Z is projectable if and only if it is of the form $Z(p) = (p, \sigma(\tau(p)), X(p))$, for some section $\sigma \in \mathrm{Sec}(A)$ and some vector field $X \in \mathfrak{X}(A)$, which projects to $\rho(\sigma)$ i.e.: $T\tau(X(a)) = \rho(\sigma(\tau(p)))$.

The Lie bracket of two projectable sections Z_1 and Z_2 is then given by

$$[Z_1, Z_2](p) = (p, [\sigma_1, \sigma_2](m), [X_1, X_2](p)), \qquad p \in A, \quad m = \tau(p),$$

where σ_1, σ_2 are the sections of A and X_1, X_2 are the vector fields on P that we said there exist for every projectable section Z_1, Z_2 of $\mathcal{T}^A P$ such that they can be written as $Z_{1,2}(p) = (p, \sigma_{1,2}(\tau(p)), X_{1,2}(p))$. It is easy to see that $[Z_1, Z_2](p)$ is an element of $\mathcal{T}^A P$ for every $p \in A$.

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma \qquad [\mathcal{X}_\alpha, \mathcal{V}_A] = 0 \qquad \text{and} \qquad [\mathcal{V}_A, \mathcal{V}_B] = 0,$$

from where we get the structural functions $C^\gamma_{\alpha\beta}$. The other structural functions of $\mathcal{T}^A A$, ρ^i_α are given by the following formulas:

$$\rho^1(\mathcal{X}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad \rho^1(\mathcal{V}_A) = \frac{\partial}{\partial y^A},$$

as we consider the structure of Lie algebroid of $\mathcal{T}^A A$ with the anchor map the one given by ρ^1.

Exterior differential on $\mathcal{T}^A P$

Next to the Liouville section and the vertical endomorphism that we will present in section 2.7.1, the exterior differential on $\mathcal{T}^A A$ is one of the basis elements to be defined on $\mathcal{T}^A A$ to make the Lagrangian formalism of mechanics on Lie algebroid possible.

Denote by $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ the basis of $(\mathcal{T}^A P)^*$, dual to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

Then the local expression of the differential of a function on A, is

$$dF = \rho^i_\alpha \frac{\partial F}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial F}{\partial u^A} \mathcal{V}^A.$$

In particular we have:

$$dx^i = \rho^i_\alpha \mathcal{X}^\alpha \qquad du^A = \mathcal{V}^A$$

The differential of the sections of the basis $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ of $(\mathcal{T}^A P)^*$ is given by :

$$d\mathcal{X}^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} \mathcal{X}^\beta \wedge \mathcal{X}^\gamma \qquad d\mathcal{V}^A = 0$$

2.5.2 Prolongation of a Lie algebroid

The prolongation of a Lie algebroid is a particular case of the space described above is when $\pi : P \to M$ is the Lie algebroid $\tau : A \to M$ itself and it was used in [41] to develop the geometrical formalism of Lagrangian mechanics on Lie algebroids. It can be simply called the *prolongation of the Lie algebroid* A and it plays the role of $\tau_{TM} : TTM \to TM$ in the ordinary Lagrangian Mechanics, as the total space of the new structure reduces to TTM when $A = TM$. The sections of it and of its dual vector bundle will be a substitute for the vector fields and respectively for the differential forms in the classical case.

There is a certain subset of it which will play the role of diagonal of TTM, that is: $T^2M = \{v \in TTM \,|\, \tau_{TM}(v) = T\tau_M(v)\}$, where $\tau_M : TM \to M$ and $\tau_{TM} : TTM \to TM$.

We say a vector $v \in T_aA$ is admissible if $T_a\tau(v) = \rho(a)$ and then the set of admissible vectors is $\mathrm{Adm}(A) = \{v \in TA \,|\, T_a\tau(v) = \rho(a), v \in T_aA\}$. It can be easily seen that v is admissible if and only if $(a, a, v) \in \mathcal{T}^A A$, so then we can also denote by $\mathrm{Adm}(A)$ the subset of $\mathcal{T}^A A$ given by:

$$\mathrm{Adm}(A) = \{z \in \mathcal{T}^A A \,|\, \tau_1(z) = \tau_2(z)\}.$$

Being the equivalent of T^2M, this space will be later on used, between others, to characterize a second order differential equation.

Lifts of sections of A to sections of $\mathcal{T}^A A$

Some canonical lifting procedures of sections of the vector bundle A over M to sections of $\mathcal{T}^A A$ over A will be presented here:

As we said, an element of $\mathcal{T}^A A$ is vertical if it is in the kernel of the τ_2 projection, and therefore it is of the form $(a, 0, v)$, with v a vertical vector tangent to A at a. The set of vertical vectors forms a vector bundle of $\mathcal{T}^A A$, denoted by $\mathrm{Ver}(\mathcal{T}^A A)$.

We can define the vertical lift on any vector bundle and in particular on a Lie algebroid, in a similar manner to the tangent bundle case described in Subsection 2.1.2. Let F be an arbitrary function defined on A and take a, b two elements in the same fiber of it. We define then the vertical lift $b \mapsto b^{\mathsf{v}}_a \in T_aA$ by:

$$b^{\mathsf{v}}_a F = \frac{d}{dt} F(a + tb)\Big|_{t=0}.$$

By this map the fibers of the vector bundle A can be identified with the vertical tangent space.

The map that associates to a pair of elements from the same fiber of A an element in $\mathcal{T}^A A$, by $\xi^{\mathsf{v}}(a, b) = (a, 0, b^{\mathsf{v}}_a)$ is called the vertical lifting map $\xi^{\mathsf{v}} : A \times_M A \to \mathrm{Ver}(\mathcal{T}^A A)$.

The *vertical lift of a section* of $\tau : A \to M$, say σ, is the section denoted by σ^{v} of $\tau_1 : \mathcal{T}^A A \to A$ defined by $\sigma^{\mathsf{v}}(a) = \xi^{\mathsf{v}}(a, \sigma(\tau(a))) = (a, 0, \sigma(\tau(a))^{\mathsf{v}}_a)$.

The expression of the vertical lift of a section $\sigma = \sigma^\alpha e_\alpha$, is given by $\sigma^{\mathsf{v}} = \sigma^\alpha \mathcal{V}_\alpha$.

Complete lift: Given a section $\sigma \in \mathrm{Sec}(A)$ there exists one and only one section $\sigma^{\mathsf{c}} \in \mathrm{Sec}(\mathcal{T}^A A)$ that projects to σ and satisfies for every section $\omega \in A^*$:

$$\rho^1(\sigma^{\mathsf{c}})(\widehat{\omega}) = \widehat{d_\sigma \omega}.$$

The expression of the complete lift of a section σ is:

$$\sigma^{\mathsf{c}} = \sigma^\alpha \mathcal{X}_\alpha + (\dot\sigma^\alpha + C^\alpha_{\beta\gamma}\sigma^\beta y^\gamma)\mathcal{V}_\alpha.$$

In the particular case of $\mathcal{T}^A A$, the Lie bracket is defined in terms of the brackets of the vertical and complete lifts. For $\sigma, \eta \in \mathrm{Sec}(A)$ we have:

$$[\sigma^{\mathsf{V}}, \eta^{\mathsf{V}}] = 0;\;\; [\sigma^{\mathsf{V}}, \eta^{\mathsf{c}}] = [\sigma, \eta]^{\mathsf{V}};\;\; [\sigma^{\mathsf{c}}, \eta^{\mathsf{c}}] = [\sigma, \eta]^{\mathsf{c}}.$$

The canonical involution map

There exists a canonical map $\chi_A \colon \mathcal{T}^A A \to \mathcal{T}^A A$ such that $\chi_A^2 = \mathrm{Id}$. It is defined by $\chi_A(a, b, v) = (b, a, \bar v)$, for every $(a, b, v) \in \mathcal{T}^A A$, where $\bar v \in T_b A$ is the vector which projects to $\rho(a)$ and satisfies

$$\bar v \hat\theta = v\hat\theta + d\theta(a, b)$$

for every section θ of A^*.

The canonical involution is locally given by

$$\chi_A(x^i, y^\alpha, z^\alpha, v^\alpha) = (x^i, z^\alpha, y^\alpha, v^\alpha + C^\alpha_{\beta\gamma} z^\beta y^\gamma).$$

The canonical involution for a Lie algebroid plays a similar role to that from the classical case, for $A = TM$, where this application relates the second mixed derivatives of $\gamma : \mathbb{R}^2 \to M$ by: $\frac{\partial}{\partial s}\frac{\partial \gamma}{\partial t} = \chi_{TM}(\frac{\partial}{\partial t}\frac{\partial \gamma}{\partial s})$.

The complete lift in terms of the map χ

The complete lift of a section $\eta \in \mathrm{Sec}(A)$ can be expressed in terms of the canonical involution map:

$$\eta^{\mathsf{c}}(a) = \chi_A\big(\eta(m), a, T_m\eta(\rho(a))\big) \in \mathcal{T}^A A,$$

where $m = \tau(a)$,

and if α is an admissible curve whose projection on M is γ, then it is written:

$$\eta^{\mathsf{c}}(\alpha(t)) = \chi_A\big(\eta(\gamma(t)), \alpha(t), \frac{d}{dt}(\eta(\gamma(t)))\big).$$

It follows that for any admissible curve α we have: $\eta^{\mathsf{c}} \circ \alpha = (\eta \circ \gamma)^{\mathsf{c}}_\alpha$

If $\eta = \eta^\alpha e_\alpha$ is a local section of A, then the vector field associated to its complete lift has the local expression

$$\rho^1(\eta^{\mathsf{c}}) = \rho^i_\alpha \eta^\alpha \frac{\partial}{\partial x^i} + \Big(\rho^i_\beta y^\beta \frac{\partial \eta^\alpha}{\partial x^i} + C^\alpha_{\beta\gamma} y^\beta \eta^\gamma\Big)\frac{\partial}{\partial y^\alpha} \in TA.$$

Notice that if η is an admissible curve and its expression in local coordinates is: $\eta(t) = (x^i(t), y^\alpha(t))$, then: $\rho^i_\beta y\beta = \dot x^i = \frac{dx^i}{dt}$, so $\rho^i_\beta y^\beta \frac{\partial \eta^\alpha}{\partial x^i} = \dot\eta^\alpha$, and then:

$$\rho^1(\eta^{\mathsf{c}}(t)) = \rho^i_\alpha(x^i(t))\eta^\alpha(t)\frac{\partial}{\partial x^i} + \Big(\dot\eta^\alpha(t) + C^\alpha_{\beta\gamma}(x^i(t)) y^\beta \eta^\gamma(t)\Big)\frac{\partial}{\partial y^\alpha} \in TA.$$

The map Ξ

Given an admissible curve $\alpha\colon \mathbb{R}\to A$ over $\gamma=\tau\circ\alpha$ we consider the map Ξ_α defined from the sections of A whose projections on the base is γ to the ones of TA whose projection on the base is α, is given by:

$$\Xi_\alpha(\beta)=\rho^1(\chi_A(\beta,\alpha,\dot\beta))$$

that is: $\chi_A(\beta,\alpha,\dot\beta)=(\alpha,\beta,\Xi_\alpha(\beta))$.

The local expression of the map Ξ_α is:

$$\Xi_\alpha(\beta)(t)=\rho^i_a(x(t))\beta^a(t)\frac{\partial}{\partial x^i}\Big|_{\alpha(t)}+\left(\dot\beta^a(t)+C^a_{bc}(x(t))\alpha^b\beta^c(t)\right)\frac{\partial}{\partial y^a}\Big|_{\alpha(t)},$$

where α and β have the local expression $\alpha(t)=(x^i(t),\alpha^\alpha(t))$ and $\beta(t)=(x^i(t),\beta^\alpha(t))$.

REMARK 4: The following property takes place:

$$\Xi_\alpha(f\beta)=f\Xi_\alpha(\beta)+\dot f\beta^{\mathsf{v}}_\alpha,$$

for every function $f\in C^\infty(\mathbb{R})$. $\diamond$

2.6 Connections

Let $\tau:A\to M$ be a Lie algebroid with anchor map ρ, and $E\to M$ a vector bundle.

DEFINITION 8: *An $A-$connection on a vector bundle $E\to M$ is an operator $\nabla:\mathrm{Sec}(A)\times\mathrm{Sec}(E)\to\mathrm{Sec}(E)$ that is $\mathbb{R}$-bilinear and satisfies:*

$$\nabla_{f\alpha}s=f\nabla_\alpha s \text{ and } \nabla_\alpha(fs)=f\nabla_\alpha s+(\rho(a)f)s,$$

any $\alpha\in\mathrm{Sec}(A)$, $s\in\mathrm{Sec}(E)$, $f\in C^\infty(M)$.

The curvature of a linear connection ∇ is defined by:

$$R(\alpha,\beta)s=\nabla_\alpha\nabla_\beta s-\nabla_\beta\nabla_\alpha s-\nabla_{[\alpha,\beta]}s. \tag{2.25}$$

The torsion of a linear connection ∇ is defined by:

$$T(\alpha,\beta)=\nabla_\alpha\beta-\nabla_\beta\alpha-[\alpha,\beta].$$

DEFINITION 9: *Let $\alpha:[t_0,t_1]\to A$ be an admissible curve. There is a unique map $\nabla^\alpha:\mathrm{Sec}_\alpha(E)\to\mathrm{Sec}_\alpha(E)$, where $\mathrm{Sec}_\alpha(E)$ is the space of curves in E with the same projection on M as α, satisfying:*

1.$\nabla^\alpha(c_1s_1+c_2s_2)=c_1\nabla^\alpha s_1+c_2\nabla^\alpha s_2,\ c_1,c_2\in\mathbb{R}$;

2.$\nabla^\alpha fs=f's+f\nabla^\alpha s$, where $f:[t_0,t_1]\to\mathbb{R}$ is a smooth function;

3. if $\overline{s}$ is a local section of A which extends s and $\rho(\alpha(t))\neq 0$, $\forall t\in[t_0,t_1]$, then

$$\nabla^{\alpha} s(t) = \nabla_{\alpha(t)}\overline{s};$$

4. is $\overline{s}$ is a local section of A which extends s and α is vertical, i.e. $\alpha(t) \in \mathrm{Ker}(\rho_{\tau(\alpha(t_0))})$, then:

$$\nabla^{\alpha} s(t) = \nabla_{\alpha(t)}\overline{s} + \frac{d}{dt}s(t).$$

Remark that if α is an admissible curve and $s \in \mathrm{Sec}_{\alpha}(E)$, then we denote $\nabla_t s$ the derivative of $s(t)$ along this admissible curve, instead of $\nabla^{\alpha} s$.

The particular case when the vector bundle is $\tau : A \to M$ itself, then the A-connection over A is called an A-linear connection. If we denote by Γ_{ij}^{k} the Cristophell symbols of an A-linear connection, i.e: $\nabla_{e_i} e_j = \Gamma_{ij}^{k} e_k$, then the expression in coordinates of the covariant derivative of a α-section β is:

$$\nabla_t \beta = (\frac{d\beta^k}{dt} + \Gamma_{ij}^{k}\alpha^i\beta^j)e_k.$$

Definition 10: *Let A be a Lie algebroid with ∇ an $A-$connection. An admissible curve $\alpha : [t_0, t_1] \to A$ is a geodesic for the connection ∇, if $\nabla_t\alpha = 0$.*

In local coordinates $\alpha = \alpha^i e_i$ is a geodesic for the A-connection ∇ if it satisfies the geodesic equation:

$$\frac{d\alpha^k}{dt} + \Gamma_{ij}^{k}\alpha^i\alpha^j = 0.$$

Remark 5: We recall that a spray Γ is a SODE vector field, i.e. $S(\Gamma) = \Delta$, who moreover satisfies $[\Delta, \Gamma] = \Gamma$.

For every symmetric connection there exists a spray whose integral curves project onto its geodesics, called the *geodesic spray* associated to the symmetric connection. ◇

2.7 Lagrangian Mechanics on Lie Algebroids

Given a Lagrangian $L \in C^{\infty}(A)$ it can be defined a dynamical system on the Lie algebroid A. The equations defining such dynamical system are the Euler-Lagrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial y^{\alpha}}\right) + \frac{\partial L}{\partial y^{\gamma}} C_{\alpha\beta}^{\gamma} y^{\beta} = \rho_{\alpha}^{i}\frac{\partial L}{\partial x^{i}} \\ \dot{x}^i = \rho_{\alpha}^{i} y^{\alpha}. \end{cases} \tag{2.26}$$

They can be obtained by pulling back the canonical Poisson structure of the dual A^* by the Legendre transformation when L is regular (see [56]), by variational calculus, taking the energy functional defined on the space of admissible curves from A and finding the extremal points of it or by using in a geometric framework as in [43], a symplectic formalism as presented in [41], when the dynamics can be obtained directly as a solution of a symplectic equation. In this case it is defined a symplectic structure on the bundle $\tau_1 : \mathcal{T}^A A \to A$, meaning a section ω of $(\mathcal{T}^A A)^* \wedge (\mathcal{T}^A A)^* \to A$ non-degenerate as a bilinear form and closed, $d\omega = 0$. Given a regular Lagrangian one can construct a symplectic structure, while if the Lagrangian is singular, the corresponding structure will be pre-symplectic.

We will refer here mostly to the last two ways mentioned for arriving to the Euler-Lagrange equations. However, with reference to the first one, we will describe shortly the natural Poisson structure that the dual of an Lie algebroid carries:

First, $\forall f \in C^\infty(A^*), \forall \theta \in \mathrm{Sec}(A^*)$ it is defined $f_\theta \in \mathrm{Sec}(A), \forall m \in M, \mu_m \in A^*_m$ by:

$$\langle\, \mu_m\,, f_\theta(m)\,\rangle = \frac{d}{dt}\Big|_{t=0} f(\theta(m) + t\mu_m).$$

Then for all $f, g \in C^\infty(A^*)$, we can define $\{f, g\}, \forall \theta \in \mathrm{Sec}(A^*)$ which can be verified to be a Poisson structure, a natural one, given by:

$\{f, g\} \circ \theta = \langle\, \theta\,, [f_\theta, g_\theta]\,\rangle + \rho(f_\theta)(g \circ 0) - \rho(g_\theta)(f \circ 0)$, where $0 : M \to A^*$ zero section.

In local coordinates, if we associate to the dual basis of $A : (e^1, ..., e^m)$, the local system of coordinates $(\xi_1, ..., \xi_m)$ then, the Poisson bracket is given by:

$$\{x_i, x_j\} = 0;\ \{x_i, \xi_s\} = -\rho^i_s;\ \{\xi_s, \xi_t\} = \sum_u C^u_{st}\xi_u.$$

2.7.1 Symplectic formalism

The analogue of the Cartan 1-form and the analogue of the symplectic canonic form, the Cartan 2-section, are introduced, and then the Euler-Lagrange equations are defined in terms of the energy and the symplectic structure.

To present the geometrical formalism for Lagrangian Mechanics on Lie algebroids, gave in [41], similar to Klein's formalism in standard mechanics (on tangent bundle) we need to give some important canonical geometrical objects on $\mathcal{T}^A A$, space that plays the role of TTM in this context:

- The *vertical lifting map* $\xi^{\mathsf{V}} : A \times_M A \to \mathcal{T}^A A$ given by $\xi^{\mathsf{V}}(a, b) = (a, 0, b^{\mathsf{V}}_a)$, where b^{V}_a is the vector tangent to the curve $a + tb$ at $t = 0$, so $b^{\mathsf{V}}_a \in T_a A$.
- The *vertical endomorphism* $S : \mathcal{T}^A A \to \mathcal{T}^A A$ defined as $S = \xi^{\mathsf{V}} \circ \tau_{12}$, that is:

  $$S(a, b, v) = \xi^{\mathsf{V}}(a, b) = (a, 0, b^{\mathsf{V}}_a),$$

  The local expression of S is : $S = \mathcal{V}_\alpha \otimes \mathcal{X}^\alpha$.

  The vertical endomorphism can be thought to map horizontal directions to vertical ones, as:

  $$S(\mathcal{X}_\alpha) = \mathcal{V}_\alpha, \quad S(\mathcal{V}_\alpha) = 0 \tag{2.27}$$

- The *Liouville section* is a vertical section of τ_1, given by:

  $$\Delta(a) = \xi^{\mathsf{V}}(a, a) = (a, 0, a^{\mathsf{V}}_a).$$

In coordinates the expression of Δ is: $\Delta = y^\alpha \mathcal{V}_\alpha$, so its corresponding vector field by ρ^1 is given by: $\rho^1(\Delta) = y^\alpha \frac{\partial}{\partial y^\alpha}$.

Second order differential equation: In the case of $A = TM$ there exists two equivalent definitions of a SODE. As a vectorial field on TM whose integral curves are the natural prolongation of the curves on the base manifold M, and as a vectorial field on TM whose image through the vertical endomorphism is given by the Liouville section. In the case of a general Lie algebroid, in the first definition we only need to replace the notion of the natural prolongation with that of admissible curves.

DEFINITION 11: *A section Γ of $\mathcal{T}^A A$ is said to be a* SODE *section if $S(\Gamma) = \Delta$.*

It can be proven that the definition is equivalent with each one of this conditions:

- Γ takes values in $\mathrm{Adm}(A)$;
- $\tau_2 \circ \Gamma = id_A$.

Thus, in local coordinates, a SODE Γ, of A is a section of the following expression:

$$\Gamma(x, y) = y^\alpha \mathcal{X}_\alpha + f^\alpha(x, y)\mathcal{V}_\alpha,$$

and the integral curves of the SODE section are the integral curves of its associated vector field:

$$\rho^1(\Gamma)(x, y) = \rho^i_\alpha y^\alpha \frac{\partial}{\partial x^i}\Big|_{(x,y)} + f^\alpha(x, y) \frac{\partial}{\partial y^\alpha}\Big|_{(x,y)}$$

that is, they satisfy the differential equations:

$$\dot{x}^i = \rho^i_\alpha y^\alpha \qquad \dot{y}^\alpha = f^\alpha(x, y).$$

The Cartan forms: In a similar manner as presented in the Lagrangian approach from Section 2.1, corresponding Cartan forms can be introduced in this new setting of a Lie algebroid.

DEFINITION 12: *Given a Lagrangian function $L \in C^\infty(A)$ the* Cartan 1-section *$\theta_L \in \mathrm{Sec}((\mathcal{T}^A A)^*)$ is defined by*

$$\theta_L = S \circ dL.$$

Then, its expression in local coordinates will be: $\theta_L = \frac{\partial L}{\partial y^\alpha} \mathcal{X}^\alpha$.

We also remark that θ_L is a semibasic section, meaning that its action upon $Ver(\mathcal{T}^A A)$ is zero. In fact we have:

$$\langle \theta_L , \sigma^{\mathsf{C}} \rangle = d_{\sigma^{\mathsf{V}}} L \qquad \langle \theta_L , \sigma^{\mathsf{V}} \rangle = 0.$$

The *Legendre transformation* $\mathcal{F}_L : A \to A^*$ is given by:

$$\langle \mathcal{F}_L(a)\,,b\,\rangle = \frac{d}{dt}L(a+tb)\Big|_{t=0},\quad a,b\in A,\ \tau(a)=\tau(b).$$

It will sometimes prove useful to identify θ_L, a section of $(\mathcal{T}^A A)^*$, with the *Legendre transformation* $\mathcal{F}_L : A \to A^*$. In local coordinates this means that it will be thought of the 1-Cartan section as: $\theta_L(x,y) = \frac{\partial L}{\partial y^\alpha} e^\alpha \in \mathrm{Sec}(A^*)$.

And it will be convenient to think $\langle \theta_L\,,\tilde{\eta}\,\rangle$ for any section $\tilde{\eta}$ of $\mathcal{T}^A A$ projecting to η, as $\langle \theta_L\,,\eta\,\rangle$, and thus $\langle \theta_L\,,\eta\,\rangle = d_{\eta^{\mathrm{V}}} L$.

Definition 13: *The* Cartan 2-section ω_L *is given by:* $\omega_L = -d\theta_L$.

Then, its local expression will be:

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\rho^i_\alpha + \frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \tag{2.28}$$

Definition 14: *The real function E_L on A defined by $E_L = d_\Delta L - L$ is the* energy functional *of the Lagrangian system.*

Its expression in the local coordinates is:

$$E_L = \frac{\partial L}{\partial y^\alpha}y^\alpha - L. \tag{2.29}$$

By a solution of the Lagrangian system (a solution of the *Euler-Lagrange equations*) we mean a SODE section Γ of $\mathcal{T}^A A$ such that

$$i_\Gamma \omega_L = dE_L. \tag{2.30}$$

The function L is said to be *regular Lagrangian* if ω_L is regular at every point as a bilinear map, that is if and only if the matrix $\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$ is regular at every point.

If L is regular the equation $i_\Gamma\omega_L = dE_L$ has a unique solution the section $\Gamma_L = g^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha$ of $\mathcal{T}^A A$ which satisfies $g^\alpha = y^\alpha$, so Γ is a SODE.

The SODE $\Gamma_L = y^\alpha\mathcal{X}_\alpha + f^\alpha\mathcal{V}_\alpha$ is a solution of the Euler-Lagrange equations if and only if the functions f^α satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha}f^\beta + \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta y^\beta + \frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta}y^\beta - \rho^i_\alpha\frac{\partial L}{\partial x^i} = 0, \text{ for all } \alpha, \tag{2.31}$$

from where:

$$f^\beta = \Big(\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha}\Big)^{-1}\Big(\frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta}y^\beta - \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta y^\beta + \rho^i_\alpha\frac{\partial L}{\partial x^i}\Big). \tag{2.32}$$

So, when a section Γ_L satisfies the Euler-Lagrange equations, the integral curves of the vector field $\rho^1(\Gamma_L)$ satisfy the *Euler-Lagrange differential equations*, which can be written locally in the form:

$$\left\{\begin{array}{l} \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) - \rho^i_\alpha\frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta}y^\beta = 0 \\ \dot{x}^i = \rho^i_\alpha y^\alpha. \end{array}\right. \tag{2.33}$$

Denote by $\mathcal{A}([t_0,t_1],A)$ the space of admissible curves on A. Defining the *Euler-Lagrange operator*: $\delta L:\mathcal{A}([t_0,t_1],A)\to A^*$ by

$$\delta L=\Big(\frac{d}{dt}\Big(\frac{\partial L}{\partial y^\alpha}\Big)+C^\gamma_{\alpha\beta}y^\beta\frac{\partial L}{\partial y^\gamma}-\rho^i_\alpha\frac{\partial L}{\partial x^i}\Big)e^\alpha,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then the *Euler-Lagrange differential equations* can be thought as

$$\delta L=0.$$

2.7.2 Variational formalism

As we mentioned, an alternative to the symplectic formalism for recovering the Euler-Lagrange equations, is the variational formalism. In [43] it was shown that the Euler-Lagrange equations for a Lagrangian system on the Lie algebroid are the equations for critical points of the energy functional defined on the space of admissible curves, imposing some boundary conditions.

In order to give a short review of this variational principle it is needed to present what is the space of curves to be worked on, what it is the considered neighbohood of a curve from this space and what the manifold structure of it, in order to have clear what is the differential of the energy functional and when it is defined.

In the context of Lie algebroids, the finite variations with fixed base endpoints are considered the homotopies on it. We will present here what is the considered the notion of homotopy on the Lie algebroid. First we'll define what a Lie algebroid morphism is and then the notion of A-homotopy-homotopy on a Lie algebroid A-, will be generalized from the classical case.

Morphisms of Lie algebroids

Let $\tau:A\to M$ and $\bar\tau:B\to N$ be two Lie algebroids with the anchor maps ρ and respectively $\bar\rho$.

Definition 15: *A vector bundle map $\phi:A\to B$ over $\varphi:M\to N$ is said to be admissible if it maps admissible curves into admissible curves, that is: $\bar\rho\circ\phi=T\varphi\circ\rho$ and is said to be a morphism of Lie algebroid, if moreover it satisfies: $\phi^*\,d\theta=d\phi^*\theta,\forall\theta\in\mathrm{Sec}(\Lambda^p(B)^*)$.*

Local expression: In local coordinates, a vector bundle map has the expression: $\phi(x,y)=(\varphi^i(x),\phi^\alpha_\beta(x)y^\beta)$ and the condition that ϕ is an admissible map is: $\bar\rho^i_\alpha\frac{\partial\varphi^k}{\partial x^i}=\rho^k_\beta\phi^\beta_\alpha$ and it is a morphism:

$$\bar C^\alpha_{\beta\gamma}\phi^\beta_\mu\phi^\gamma_\nu+\rho^i_\mu\frac{\partial\phi^\alpha_\nu}{\partial x^i}-\rho^i_\nu\frac{\partial\phi^\alpha_\mu}{\partial x^i}=C^\gamma_{\mu\nu}\phi^\alpha_\gamma. \tag{2.34}$$

Particular case: When the morphism is between the Lie algebroids $T\mathbb{R}^2$ over $\mathbb{R}^2$and A over M, ϕ it can be written as: $\alpha(s,t)dt+\beta(s,t)ds$, where $\alpha(s,t)=$

$\phi(\frac{\partial}{\partial t}|_{(s,t)})$ and $\beta(s,t) = \phi(\frac{\partial}{\partial s}|_{(s,t)})$. The condition that ϕ is an admissible map says that the curves: $t \mapsto \alpha(s,t)$ and $s \mapsto \beta(s,t)$ are admissible.

Locally, the condition to be a morphism in this case is translated as:

$$\frac{\partial \alpha^k}{\partial s} - \frac{\partial \beta^k}{\partial t} = C^k_{cd}\alpha^c\beta^d. \tag{2.35}$$

Notice that if $\phi = \alpha(s,t)dt + \beta(s,t)ds$ is a morphism, then the tangent vector to the variation curve $s \mapsto \alpha(s,t)$ is:

$$\frac{\partial \alpha}{\partial s}(s,t) = \Xi_{\alpha_s}\beta_s.$$

Construction of a morphism: Let $\eta \in \mathrm{Sec}(A)$ and take ϕ_s the flow of the vector field $\rho^1(\eta^c) \in \mathfrak{X}(A)$ and φ_s the flow of the vector field $\rho(\eta) \in \mathfrak{X}(M)$. It can be easily observed that ϕ_s projects onto φ_s, so for each fixed s, the map ϕ_s is a vector bundle map which is a morphism of Lie algebroids over φ_s.

Definition 16: *The flow of a section $\eta \in \mathrm{Sec}(A)$ is the pair (ϕ_s, φ_s).*

From a section of A and an admissible curve of it one can construct a morphism from $T\mathbb{R}^2$ to A using the flow of the section, in the following way:

Proposition 1: *Let α_0 be an admissible curve in A, with base path γ_0, and let η be a section of A, with its flow (ϕ_s, φ_s). Then, construct:*

$$\alpha(s,t) = \phi_s(\alpha_0(t)) \qquad \gamma(s,t) = \varphi_s(\gamma_0(t)) \qquad \textit{and} \qquad \beta(s,t) = \eta(\gamma(s,t)).$$

Then $\phi(s,t) = \alpha(s,t)dt + \beta(s,t)ds$ is a morphism from $T\mathbb{R}^2$ to A over γ.

Proof. As ϕ_s projects to φ_s, both $\alpha(s,t)$ and $\beta(s,t)$ project to $\gamma(s,t)$, so ϕ is a vector bundle map over γ.

ϕ_s is a morphism so it maps the admissible curve $\alpha_0(t)$ into admissible curves, such that $t \mapsto \alpha(s,t)$ is admissible. In order to get that ϕ is an admissible map it remains to prove that $s \mapsto \beta(s,t)$ are admissible. Using that φ_s is the flow of $\rho(\eta)$, we have that

$$\frac{\partial \gamma}{\partial s}(s,t) = \frac{\partial}{\partial s}\varphi_s(\gamma_0(t)) = \rho(\eta)(\varphi_s(\gamma_0(t))) = \rho(\eta(\gamma(s,t))) = \rho(\beta(s,t)),$$

Now it is left to prove the morphism condition, that is

$$\frac{\partial \alpha}{\partial s}(s,t) = \Xi_{\alpha_s}\beta_s, \tag{2.36}$$

equivalent to showing that: $\chi_A(\beta, \alpha, \frac{\partial \beta}{\partial t}) = (\alpha, \beta, \frac{\partial \alpha}{\partial s})$.

The first term is equal to:

$$\frac{\partial \alpha}{\partial s}(s,t) = \frac{\partial}{\partial s}(\phi_s(\alpha_0(t))) = \rho^1(\eta^c)(\phi_s(\alpha_0(t))) = \rho^1(\eta^c)(\alpha(s,t)) = \rho^1(\beta^c)(\alpha(s,t))$$

and this is equal to the second term of relation (2.36), as by definition $\Xi_{\alpha_s}\beta_s = \rho^1(\beta, \alpha, \dot{\beta}) = \rho^1(\beta^c)(\alpha(s,t))$.

□

Homotopies

Consider $\gamma(s,t)$ a homotopy on a manifold M. Notice the tangent aplication $T\gamma : T\mathbb{R}^2 \to TM$ has the form $T\gamma = \frac{\partial\gamma}{\partial t}(s,t)dt + \frac{\partial\gamma}{\partial s}(s,t)ds$ and from the definition of homotopy on a manifold it follows: $\frac{\partial\gamma}{\partial s}(s,0) = \frac{\partial\gamma}{\partial s}(s,1) = 0$. It is used this remark in order to generalize the notion of homotopy on a Lie algebroid A, the A-homotopy. Consider the particular case of morphism $\phi : T\mathbb{R}^2 \to A$ written an $\phi(s,t) = a(s,t)dt + b(s,t)ds$ with the additional conditions: $b(s,0) = b(s,1) = 0, \forall s$. This is the idea used to generalize the notion of homotopy to a Lie algebroid. Indeed, remark that in particular, $T\gamma$ is an $TM-$homotopy.

Definition 17: *An A-homotopy between two admissible curves in A denoted α_0, $\alpha_1 : J = [t_0, t_1] \to A$ is a morphism of Lie algebroids $\phi : TI \times TJ \to A$, $\phi = \alpha(s,t)dt + \beta(s,t)ds$, where $s \in I = [0,1]$, such that:*

$$\begin{aligned} \alpha(0,t) &= \alpha_0(t) \qquad & \beta(s,t_0) &= 0 \\ \alpha(1,t) &= \alpha_1(t) \qquad & \beta(s,t_1) &= 0. \end{aligned}$$

We say that ϕ is an $A-$homotopy from the admissible curve α_0 to α_1.

In general, the infinitesimal variation of any homotopy of a curve α has the form given by $\Xi_\alpha(\beta)$, where β is any α-section whose endpoints are zero.

Construction of an A-homotopy: Using the above presented construction of a morphism it can be given the construction of an A-homotopy as a corollary to proposition 1. We just need to use the section η, with compact support, with the supplementary condition that $\eta(m_0) = \eta(m_1) = 0$.

Corollary 1: *Let $\alpha_0 : [t_0, t_1] \to A$ be a curve in A, with base path γ_0 such that $\gamma(t_0) = m_0$ and $\gamma(t_1) = m_1$, where m_0, m_1 are two points in M and let η be a section of A, with compact support such that $\eta(m_0) = \eta(m_1) = 0$ of flow denoted by (ϕ_s, φ_s). Then a map ϕ can be constructed as in Proposition 1, that is an A-homotopy from α_0 to $\alpha_1 = \phi_1 \circ \alpha_0$.*

Proof. The condition of compact support ensures that the flow of η to be globally defined, so that ϕ_1 be defined.

In order to have ϕ a A-homotopy, we just need that $\alpha(0,t) = \alpha_0$ and $\alpha(1,t) = \alpha_1$, which is obviously satisfied, and that $\beta(s,t_0) = 0, \beta(s,t_1) = 0$.

As $\eta(m_0) = 0$ and $\eta(m_1) = 0$, then $\rho(\eta(m_0)) = \rho(\eta(m_1)) = 0$ from where $\varphi_s(m_0) = m_0, \varphi_s(m_1) = m_1$, so:

$$\beta(s,t_0) = \eta(\gamma(s,t_0)) = \eta(\varphi_s(\gamma(t_0))) = \eta(\varphi_s(m_0)) = \eta(m_0) = 0 \text{ and}$$

$$\beta(s,t_1) = \eta(\gamma(s,t_1)) = \eta(\varphi_s(\gamma(t_1))) = \eta(\varphi_s(m_1)) = \eta(m_1) = 0.$$

□

To anticipate, the neighborhood of a curve considered in the variation principle described in [43] is going to be an A-homotopy and we will further describe what topology is going to be considered on the space of admissible curves for the variational principle.

The space of admissible curves

In this section, we will look at the properties of the space of curves that are going to be considered for the variational principle, that is the space of admissible curves from a Lie algebroid. We will present the two manifold structures that can be considered on it and each advantages.

One of this structures is related to the A-homotopy equivalence relation and this one will be used in the variational principle. We will explain why the neighborhood of a curve considered in it will be given by a A-homotopy class and we will give the form of an infinitesimal variation of the variations that are going to be considered.

The set of admissible curves:

$$\mathcal{A}([t_0,t_1],A) = \left\{ a\colon [t_0,t_1] \to A \;\middle|\; \rho \circ a = \frac{d}{dt}(\tau \circ a) \right\},$$

is a subset of the space of all the curves in a vector bundle, who are of C^1-class, and their projection is of C^2-class, space which is a Banach manifold.

The A-homotopy being an equivalence relation defines a partition of the space of admissible curves into disjoint sets. It was proven that any A-homotopy class is a smooth Banach manifold and that such a partition is a foliation.

In fact, the A-homotopy classes are Banach submanifolds as the space of admissible paths is a Banach manifold.

The foliation we mentioned induces a differentiable manifold structure on $\mathcal{A}([t_0,t_1],A)$. This space considered with this induced differentiable manifold structure by the explained foliation is denoted by $\mathcal{P}([t_0,t_1],A)$.

Also, the set of admissible paths admits the natural differentiable structure given by the fact that it is a submanifold of the set of C^1-paths in A, and with this structure will be denoted also with $\mathcal{A}([t_0,t_1],A)$.

The structure of $\mathcal{A}([t_0,t_1],A)$ is used when working in problems which are related to relation between neighbor A-homotopy classes, while the structure of $\mathcal{P}([t_0,t_1],A)$ is used when one is not interested to pass from one A-homotopy class to another.

Let $m_0, m_1 \in M$ be two fixed points. Consider:

$$\mathcal{P}([t_0,t_1],A)_{m_0}^{m_1} = \{\, \alpha \in \mathcal{P}([t_0,t_1],A) \mid \tau(\alpha(t_0)) = m_0 \quad \text{and} \quad \tau(\alpha(t_1)) = m_1 \,\}.$$

This set $\mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$ is a Banach submanifold of $\mathcal{P}([t_0,t_1],A)$, which can be thought as the disjoint union of the A-homotopy classes of curves with base path connecting two fixed points.

Remark also about the set $\mathcal{A}([t_0,t_1],A)_{m_0}^{m_1}$ that one can not be sure if it has a manifold structure.

Let $\alpha \in \mathcal{A}([t_0,t_1],A)$, and denote by

$$\Sigma_\alpha = \left\{ \beta \in \mathrm{Sec}_\alpha(A) \;\middle|\; \beta \text{ is } C^2(A) \text{ with } \beta(t_0) = 0 \text{ and } \beta(t_1) = 0 \right\},$$

and by

$$F_\alpha = \big\{ v \in T_\alpha \mathcal{A}([t_0,t_1],A) \,|\, \text{there exists } \beta \in \mathrm{Sec}_\alpha(A) \text{ such that } \beta(t_0)=0,\ \beta(t_1)=0 \text{ and } v = \rho^1(\chi_A(\beta,\alpha,\dot\beta)) \big\}.$$

Then $F_\alpha = \Xi_\alpha(\Sigma_\alpha), F_\alpha \subset T_\alpha\mathcal{A}([t_0,t_1],A)$ and $F = \cup_{a\in\mathcal{A}([t_0,t_1],A)} F_a \subset T\mathcal{A}([t_0,t_1],A)$ is the integrable subbundle of the tangent bundle to $\mathcal{A}([t_0,t_1],A)$. The leaves defined by it are the $A-$homotopy classes.

That is, if L is the leaf containing α, $L = \mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$, when $\tau(\alpha(t_0)) = m_0$ and $\tau(\alpha(t_1)) = m_1$, then

$$T_\alpha L = \{\Xi_\alpha\beta \,|\, \beta \in \mathrm{Sec}_\alpha A,\ \beta(t_0) = \beta(t_1) = 0\}.$$

That is, the tangent space contains the vectors of the form $\Xi_\alpha\beta = \rho^1(\chi_A(\beta,\alpha,\dot\beta))$ (tangent to the curves $s \mapsto \alpha_s$), for β α-sections vanishing at the end-points. This vectors define an integrable distribution whose leaves are precisely the homotopy classes, and as a consequence, such kind of vectors span the whole tangent space to the given homotopy class.

Notice that as for any $\alpha \in \mathcal{A}([t_0,t_1],A)$ the restriction of Ξ_α to Σ_α is injective, there is a isomorphism between the real vector spaces Σ_α and F_α.

Variational formulation

Having specified the space of curves on which the variational principle is given, its manifold structure, its topology and the variations that are going to be considered in its formulation on a Lie algebroid, we can now present the theorem that shows how crtical points of the energy functional on the Banach space $\mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$ satisfies the Euler-Lagrange equations.

THEOREM 5: *Let $L \in C^\infty(A)$ be a Lagrangian function on the Lie algebroid A and fix two points $m_0, m_1 \in M$. Consider the action functional $\mathcal{E}\colon \mathcal{P}([t_0,t_1],A)_{m_0}^{m_1} \to \mathbb{R}$ given by $\mathcal{E}(\alpha) = \int_{t_0}^{t_1} L(\alpha(t))dt$. The critical points of $\mathcal{E}$ on the connected Banach manifold $\mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$ are precisely those elements in this space which satisfy Lagrange's equations.*

PROOF. The action functional $\mathcal{E}$ is a smooth function on $\mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$. Take a curve $\alpha \in \mathcal{P}([t_0,t_1],A)_{m_0}^{m_1}$. Then, it is known that the tangent vectors to it will have the form $\Xi_\alpha(\beta)$, where $\beta \in \Sigma_\alpha$.

Taking into account that $\Xi_\alpha(f\beta) = f\Xi_\alpha(\beta) + \dot f \beta_\alpha^{\mathsf{v}}$, for every function $f\colon [t_0,t_1] \to$

$\mathbb{R}$, we have

$$
\begin{aligned}
0 = \langle\, d\mathcal{E}(\alpha)\,, \Xi_\alpha(f\beta)\,\rangle &= \int_{t_0}^{t_1} [f(t)\langle\, dL\,, \Xi_\alpha(\beta)\,\rangle + \dot{f}\langle\, dL\,, \beta^{\mathsf{V}}_\alpha\,\rangle]dt \\
&= \int_{t_0}^{t_1} f(t)\Big[\langle\, dL\,, \Xi_\alpha(\beta)\,\rangle + \frac{d}{dt}\langle\, \theta_L \circ \alpha\,, \beta\,\rangle\Big]dt + f\langle\, \theta_L \circ \alpha\,, \beta\,\rangle\Big|_0^1 \\
&= \int_{t_0}^{t_1} f(t)\Big((d_{\beta^{\mathsf{C}}}L)(\alpha(t)) - \frac{d}{dt}\langle\, \theta_L\,, \beta^{\mathsf{C}}\,\rangle(\alpha(t))\Big)dt \\
&= \int_{t_0}^{t_1} f(t)\Big((d_{\beta^{\mathsf{C}}}L)(\alpha(t)) - \frac{d}{dt}(d_{\beta^{\mathsf{V}}}L)(\alpha(t))\Big)dt \\
&= \int_{t_0}^{t_1} f(t)\langle\, \delta L(\dot{\alpha}(t))\,, \beta(t)\,\rangle dt.
\end{aligned}
$$

Since this holds for every function f and every section $\beta \in \Sigma_\alpha$ it follows that the critical points are determined by the equation $\delta L(\dot{\alpha}(t)) = 0$, that is, by the Lagrange's equations. □

An advantage of working on Lie algebroids is that it can appear connections between different dynamical systems and their solutions even when this systems are defined on different Lie algebroids, when there are morphisms between them. In particular, when the morphism is fiberwise surjective, the variational principle can be reduced, and the symplectic equations as well.

When two Lie algebroids and the Lagrangians defined on them are connected through a morphism of Lie algebroids, then it was shown in [43] that there can be found a correspondence between the solution of the Lagrange equations on the two Lie algebroids. Also, there is a connection between the energy functionals the two Lagrangians defined on their corresponding space.

Consider a morphism $\phi\colon A \to B$ of Lie algebroids and the induced map between the spaces of paths $\hat{\phi}\colon \mathcal{P}([t_0, t_1], A) \to \mathcal{P}([t_0, t_1], B)$, defined by $\hat{\phi}(a) = \phi \circ a$, which is a smooth map for this particular case of ϕ being a Lie algebroid morphism. Consider a Lagrangian L on A and a Lagrangian L' on B which are related by ϕ, that is, $L = L' \circ \phi$. Then the associated energy functionals $\mathcal{E}$ on $\mathcal{P}([t_0, t_1], A)$ and $\mathcal{E}'$ on $\mathcal{P}([0, 1], B)$ are related by $\hat{\phi}$, that is $\mathcal{E}' \circ \hat{\phi} = \mathcal{E}$. Indeed,

$$
\mathcal{E}'(\hat{\phi}(\alpha)) = \mathcal{E}'(\phi \circ \alpha) = \int_{t_0}^{t_1} (L' \circ \phi \circ \alpha)(t)\, dt = \int_{t_0}^{t_1} (L \circ \alpha)(t)\, dt = \mathcal{E}(\alpha).
$$

THEOREM 6: *Let $\phi\colon A \to B$ be a morphism of Lie algebroids. Consider a Lagrangian L on A and a Lagrangian L' on B such that $L = L' \circ \phi$. If α is an admissible curve and $\alpha' = \phi \circ \alpha$ is a solution of Lagrange's equations for L' then α itself is a solution of Lagrange's equations for L.*

PROOF. Since $\mathcal{E}' \circ \hat{\phi} = \mathcal{E}$ we have that $\langle\, d\mathcal{E}'(\hat{\phi}(\alpha))\,, T_\alpha\hat{\phi}(v)\,\rangle = \langle\, d\mathcal{E}(\alpha)\,, v\,\rangle$ for every $v \in T_a\mathcal{P}([t_0, t_1], A)^{m_1}_{m_0}$. If $\hat{\phi}(\alpha)$ is a solution of Lagrange's equations for L' then $d\mathcal{E}'(\hat{\phi}(\alpha)) = 0$, from where it follows that $d\mathcal{E}(\alpha) = 0$. □

THEOREM 7: *Let $\phi\colon A \to B$ be a fiberwise surjective morphism of Lie algebroids. Consider a Lagrangian L on A and a Lagrangian L' on B such that $L = L' \circ \phi$. If α is a solution of Lagrange's equations for L then $\alpha' = \phi \circ \alpha$ is a solution of Lagrange's equations for L'.*

PROOF. Since $\mathcal{E}' \circ \hat{\phi} = \mathcal{E}$ we have that $\langle\, d\mathcal{E}'(\hat{\phi}(\alpha))\,, T_\alpha\hat{\phi}(v)\,\rangle = \langle\, d\mathcal{E}(\alpha)\,, v\,\rangle$ for every $v \in T_\alpha\mathcal{P}([t_0, t_1], A)_{m_0}^{m_1}$. If ϕ is fiberwise surjective, then $\hat{\phi}$ is a submersion, from where it follows that $\hat{\phi}$ maps critical points of $\mathcal{E}$ into critical points of $\mathcal{E}'$, i.e. solutions of Lagrange's equations for L into solutions of Lagrange's equations for L'. $\square$

Chapter 3

Applications in Mathematics

A second order differential equation can be seen geometrically as a vector field $\Gamma \in \mathfrak{X}(TQ)$ that satisfies: $S(\Gamma) = \Delta$. If γ is an integral curve on the base of Γ, then $\ddot{\gamma} = \Gamma \circ \dot{\gamma}$ and the local expression of Γ in natural coordinates is

$$\Gamma(q, v) = v^i \partial_{q^i} + f^i(q, v)\partial_{v^i}.$$

A spray is a SODE where the functions f^i are homogeneous polynomial functions of degree 2 in v, that is: $f^i(q, v) = f^i_{jk}(q)v^i v^j$. On a Riemannian manifold Q, the geodesic curves are the integral curves on the base of a spray: $\Gamma(q, v) = v^i \partial_{q^i} + \Gamma^i_{jk}(q)v^i v^j \partial_{v^i}$. Thus the geodesic curves satisfy the geodesic equations: $\ddot{q}^i = \Gamma^i_{jk}(q)\dot{q}^i \dot{q}^j$, when in local coordinates on Q, the curve $\gamma = (q^i)$ and where Γ^i_{jk} the coefficients of the Levi-Civita connection.

One of our objectives for this chapter is to study variations of integral curves of a general SODE on a manifold, their variational vector fields and the differential equation satisfied by such vector fields. We would like to find this equation in a way that resembles as much as possible the original Jacobi equation in Riemannian geometry.

There, a Jacobi field W along a geodesic γ is defined as a solution of the Jacobi equation $D_{\dot{\gamma}} D_{\dot{\gamma}} W + \mathrm{Rie}(W, \dot{\gamma})\dot{\gamma} = 0$, where D is the Levi-Civita connection associated to a given Riemannian metric and Rie is the curvature tensor, associated to the Levi-Civita connection. Jacobi fields are interpreted as infinitesimal variations of the geodesic γ by geodesics, or in other words, as the infinitesimal variation vector field associated to a 1-parameter family of geodesics.

As our initial objective is to find similar Jacobi equation for this more general context of a SODE on a manifold, and as in the Riemannian manifold the Jacobi equation is given with the help of the Levi-Civita connection and the Jacobi endomorphism, we will like to use objects who generalize them. In Section 3.1 we introduce a non-linear connection and the Jacobi endomorphism associated to a SODE on a manifold, given in [44]. We will also present other properties concerning Lie transported vector fields and 1-parameter family of integral curves of a vector field that will allow us to define Jacobi fields for a general SODE. In the end we give the Jacobi generalized equation. In Section 3.2 we generalize this results to the framework of SODE on Lie algebroids.

Afterwards we will generalize this results to the frame of Lie algebroids, that is for general SODE on Lie algebroids and having done that we will consider the case of a geodesic spray on a special kind of Lie algebroid, a Riemannian Lie algebroid, introduced by Boucetta in [5].

Remember that in the Riemannian manifold, a geodesic, an integral curve of a geodesic spray, minimizes the energy functional if it has no conjugate points along it [26]. Conjugate points in that frame are points along a geodesic, for which there exists a Jacobi field that annuls in them.

What we want to do in Section 3.3 is to generalize this result for the case of a Riemannian Lie algebroid. This will situate us in the particular case of the theory for SODE on Lie algebroids,from Section 3.2, where the SODE will be a geodesic spray associated to the Levi-Civita connection or, in other words, a lagrangian SODE for $L(\alpha) = \frac{1}{2}g(\alpha, \alpha)$, where g is the Riemannian metric on the Lie algebroid. We will thus relate the conjugate points corresponding to geodesic spray to second variation of the energy functional.

In Section 3.3 we will introduce a Riemannian metric on a Lie algebroid and its corresponding Levi-Civita connection as done in [5]. For the particular case of a geodesic spray on a Lie algebroid, we will take a look at its integral curves, the geodesics, at the variation of its integral curves with their corresponding Jacobi sections and conjugate points. For the energy functional we will consider, as in the classical case, its first and second variation along variations of integral curves of the sprays in Section 3.4, as they are fundamental tools to study minimizing results.

An advantage of our generalizing results on Lie algebroids from Section 3.2 is that we will obtain an equation which is valid for second order systems with holonomic constraints, systems defined on Lie algebras and systems with symmetry, in addition to the standard case. For applications of the theory of Lie algebroids in Classical Mechanics, Control Theory and Field Theory we refer to the reader to [41, 35, 42, 40].

3.1 The standard case of sodes on tangent bundles

In this section we will review the basic results about the variational equation and we reformulate the Jacobi equation in a way suitable for the generalization to SODE on Lie algebroids that will be given in Section 3.2.

In Subsection 3.1.1 we will study the properties satisfied by the infinitesimal variation vector field of a 1-parameter family of solutions of a general first-order differential equation, which corresponds to the linear variational differential equation. These results will be applied to the case of a second-order differential equation (SODE), in Subsection 3.1.2 clarifying the geometrical meaning of the concept of Jacobi field in terms of the geometry of the tangent bundle. In Subsection 3.1.3 by using the non-linear connection associated to the SODE, we will find the equation satisfied by the variation vector fields, which generalizes the Jacobi equation. The form of this equation, $\nabla\nabla W + \Phi(W) = 0$, is similar to the original Jacobi equation.

3.1.1 The variational differential equation and its geometric interpretation

We consider a vector field $X \in \mathfrak{X}(M)$ on a manifold M and we denote by $\{\varphi_t\}$ its local flow. We fix an integral curve $\zeta_0 \colon I \subset \mathbb{R} \to M$ of X defined on a compact interval $I = [0, T]$. We set $m = \zeta_0(0)$ the initial point, so that $\zeta_0(t) = \varphi_t(m)$.

A vector field along the curve ζ_0 is a map $Z \colon I \to TM$ such that $Z(t) \in T_{\zeta_0(t)}M$ for all $t \in I$. The following definitions are from [**?**].

Definition 18: *A vector field Z along ζ_0 is said to be Lie transported along the flow of X if there exists $\xi \in T_mM$ such that $Z(t) = T\varphi_t(\xi)$ for every $t \in I$.*

Definition 19: *The Lie derivative of a vector field Z along ζ_0 with respect to X is the vector field $\mathcal{L}_X Z$ along ζ_0 defined by*

$$\mathcal{L}_X Z(t) = \frac{d}{ds} T\varphi_{-s} Z(t+s)\Big|_{s=0} = \lim_{h \to 0} \frac{1}{h}[T\varphi_{-h} Z(t+h) - Z(t)].$$

Remark 6: As a consequence of the above definition, if $\tilde{Z} \in \mathfrak{X}(M)$ is an extension of Z then $\mathcal{L}_X Z(t) = [X, \tilde{Z}](\zeta_0(t))$. $\diamond$

We recall that given a vector field W along a curve ζ_0 the complete lift of W is the vector field W^{c} along $\dot{\zeta}_0$ given by $W^{\mathsf{c}}(t) = \chi_{TM}(\dot{W}(t))$. Similarly, if Y is a vector field on M, the complete lift of Y is the vector field on TM defined by $Y^{\mathsf{c}} = \chi_{TM} \circ TY$. Both definitions are consistent in the sense that the complete lift of the restriction of Y to the curve ζ_0 is the restriction of the complete lift to the curve $\dot{\zeta}_0$, that is, $Y^{\mathsf{c}} \circ \dot{\zeta}_0 = (Y \circ \zeta_0)^{\mathsf{c}}$. As it is well known, if $\{\phi_t\}$ is the flow of Y then the flow of Y^{c} is $\{T\phi_t\}$.

Proposition 2: *The following properties are equivalent:*

1. *Z is Lie transported along the flow of X.*
2. *For every $t \in I$ and every s such that $s + t \in I$ we have $T\varphi_s(Z(t)) = Z(t+s)$.*
3. *The Lie derivative of Z vanishes identically, $\mathcal{L}_X Z(t) = 0$ for all $t \in I$.*
4. *The curve $Z \colon I \to TM$ is an integral curve of the complete lift $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ of X.*

In local natural coordinates (x^i, v^i) in TM, the above properties express that the components $x^i = \zeta_0^i(t)$, $v^i = Z^i(t)$ of Z satisfy the linear variational equation

$$\begin{cases} \dot{x}^i = X^i(x) \\ \dot{v}^i = \dfrac{\partial X^i}{\partial x^j}(x) v^j. \end{cases}$$

The proof will be postponed to Section 3.2 where it will be given for the more general case of Lie algebroids.

Our first aim is to prove that a Lie transported vector field along the flow of a vector field X, is obtained via variation of integral curves of the vector field.

Definition 20: *A 1-parameter family of integral curves of X is a map $\zeta \colon (-\epsilon, \epsilon) \times I \subset \mathbb{R}^2 \to M$ such that for every $s \in (-\epsilon, \epsilon)$ the curve $\zeta_s \colon I \to M$, given by $\zeta_s(t) = \zeta(s, t)$ is an integral curve of X. The vector field Z along ζ_0 defined by $Z(t) = \frac{\partial \zeta}{\partial s}(0, t)$ is said to be the infinitesimal variation vector field defined by the 1-parameter family.*

We will also say that ζ is a finite variation of ζ_0 by integral curves of X.

PROPOSITION 3: *A vector field Z along an integral curve ζ_0 of X is the infinitesimal variation vector field defined by a 1-parameter family of integral curves of X if and only if it is Lie transported by the flow of X.*

PROOF. If $\zeta(s,t)$ is a 1-parameter family of integral curves of X then we have that $\zeta(s,t) = \varphi_t(\zeta(s,0))$. Taking the partial derivative with respect to s at $s=0$ we get that the variational vector field $Z(t) = \frac{\partial\zeta}{\partial s}(0,t)$ satisfies

$$Z(t) = \frac{\partial\zeta}{\partial s}(0,t) = T\varphi_t\frac{\partial\zeta}{\partial s}(0,0) = T\varphi_t(Z(0)),$$

so that Z is Lie transported.

Conversely, let $Z(t)$ be Lie transported along the integral curve $\zeta_0(t) = \varphi_t(m)$, that is $Z(t) = T\varphi_t(\xi)$. Consider any curve $\alpha(s)$ in M such that $\alpha(0) = \zeta_0(0) = m$ and $\dot\alpha(0) = \xi$. The 1-parameter family $\zeta(s,t) = \varphi_t(\alpha(s))$ is a family of integral curves of X satisfying the given properties. Indeed, for every fixed s we have that $\zeta_s(t) = \varphi_t(\alpha(s))$ is an integral curve of X; for $s=0$ we have $\zeta(0,t) = \varphi_t(\alpha(0)) = \varphi_t(m) = \zeta_0(t)$; and the variational vector field is

$$\frac{\partial\zeta}{\partial s}(0,t) = T\varphi_t(\dot\alpha(0)) = T\varphi_t(\xi) = Z(t),$$

where in the last equality we have used that Z is Lie transported. □

3.1.2 The case of second-order differential equations

As Γ is a SODE if it satisfies $T\tau_Q(\Gamma(v)) = v$ for every $v \in TQ$, it follows from this property that the integral curves of Γ are natural or tangent lifts $\dot\gamma$ of curves γ in the manifold Q, which are said to be integral curves of Γ in the base.

We consider now the special case of the constructions above with $M \equiv TQ$ and $X \equiv \Gamma$ a SODE on Q. We have that a variation by integral curves of Γ is of the form $\zeta(s,t) = \frac{\partial\gamma}{\partial t}(s,t)$, for $\gamma(s,t)$ a 1-parameter family of curves in Q (each one is an integral curve in the base of Γ). We will denote by $W(t)$ the variational vector field of the base family,

$$W(t) = \frac{\partial\gamma}{\partial s}(0,t).$$

Then, the variational vector field for the family $\zeta(s,t)$ is

$$Z(t) = \frac{\partial\zeta}{\partial s}(0,t) = \frac{\partial}{\partial s}\frac{\partial\gamma}{\partial t}(s,t)\Big|_{s=0} = \chi_{TQ}\frac{\partial}{\partial t}\frac{\partial\gamma}{\partial s}(0,t) = \chi_{TQ}\dot W(t) = W^{\mathrm{c}}(t).$$

DEFINITION 21: *Given a second-order vector field $\Gamma \in \mathfrak{X}(TQ)$, a vector field $W(t)$ along an integral curve in the base γ_0 of Γ is a* Jacobi field *of the* SODE *Γ if $Z = W^{\mathrm{c}}$ is a variation vector field along the integral curve $\dot\gamma_0$ by integral curves of Γ.*

It follows that a vector field $W(t)$ is a Jacobi field if and only if it satisfies the equation $\mathcal{L}_\Gamma W^{\mathrm{c}} = 0$, or equivalently W^{c} is Lie transported along the flow ϕ_t of Γ or equivalently $T\phi_s\big(W^{\mathrm{c}}(t)\big) = W^{\mathrm{c}}(t+s)$.

3.1.3 The Jacobi equation

The equation that W needs to satisfy to be a Jacobi field for the considered SODE $\mathcal{L}_\Gamma W^c = 0$, is a second-order linear differential equation, as it will be presented in local coordinates in Section 3.1.4. In the particular case when Γ is a geodesic spray one can rewrite such equation in terms of the associated covariant derivative. A general SODE does not define such a covariant derivative, but has an associated canonical non-linear connection. We will use such nonlinear connection to find an explicit expression of that equation. The reference [44] offers a detailed construction of the objects in this section. We will present in the following paragraphs first the notion of sections of a fiber bundle along a map, then an Ehresmann connection associated to a SODE and other properties that will be useful in rewriting the Jacobi equation in a convenient way for SODEs on a manifold, that will further allow us a smooth generalization for the Jacobi equation for SODEs on Lie algebroids.

Sections of a fiber bundle along a map. Consider the differential map $\phi : N \to M$. A section of the fiber bundle $\pi : E \to M$ along ϕ is a map $\sigma : N \to E$ that satisfies $\pi \circ \sigma = \phi$. The space of sections along ϕ will be denoted by $\Sigma_\phi(\pi)$ and when E is a vector bundle it has a $C^\infty(N)-$module structure and so does the space of the pullback of E by ϕ. In this case between them there exists a module isomorphism.

Classical examples of vector fields along maps are the restriction of a vector field $X \in \mathfrak{X}(M)$ to a curve $\gamma : I \to M$, which is a vector field along γ, and the tangent vector field to the curve γ, which is a section $\dot\gamma : \mathbb{R} \to TM$ of the tangent fiber bundle to M along γ.

The Ehresmann connection associated to a sode. Any SODE Γ determines on the fiber bundle of a manifold Q an Ehresmann connection. If S is the vertical endomorphism on TQ, it is well known that the tensor $\mathcal{L}_\Gamma S$, satisfies $(\mathcal{L}_\Gamma S)^2 = I$. The subbundle corresponding to the eigenvalue $+1$ coincides with the vertical subbundle and the projector on it is $P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$, $P_V : TTQ \to \mathrm{Ver}(TQ)$. Therefore, at every point, the eigenspace of eigenvalue -1 is a subbundle complementary to the vertical subspace, that is, a connection on $\tau_Q : TQ \to Q$ and hence it defines a splitting $TTQ = \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$. The projector onto the horizontal subbundle is $P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$.

In local coordinates, if $\Gamma = v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i}$ then $\mathcal{L}_\Gamma S$ is identified with the matrix $\begin{pmatrix} 1 & 0 \\ -\frac{\partial f^i}{\partial v^a} & 0 \end{pmatrix}$ and then the expression of P_H in coordinates is:

$$P_H = \frac{\partial}{\partial x^i} \otimes dx^i + \frac{1}{2}\frac{\partial f^j}{\partial v^i}\frac{\partial}{\partial v^j} \otimes dx^i.$$

The coefficients of the connection are given by:

$$\Gamma^i_j = -\frac{1}{2}\frac{\partial f^i}{\partial v^j}. \tag{3.1}$$

A local basis of vector fields adapted to the distribution is given by:

$$\Big\{ H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial v^j},\ V_i = \frac{\partial}{\partial v^i} \Big\},$$

where the brackets of the basis elements are:

$$\begin{cases} [H_i, H_j] = \Big\{ \frac{\partial \Gamma^k_i}{\partial x^j} - \frac{\partial \Gamma^k_j}{\partial x^i} - \frac{\partial \Gamma^k_i}{\partial v^l}\Gamma^l_j + \frac{\partial \Gamma^k_j}{\partial v^l}\Gamma^l_i \Big\} V_k, \\ [H_i, V_j] = [H_j, V_i] = \frac{\partial \Gamma^k_i}{\partial v^j} V_k = \frac{\partial \Gamma^k_j}{\partial v^i} V_k, \\ [V_i, V_j] = 0. \end{cases}$$

and its dual basis of the dual space $\bigwedge^1(TQ)$ is given by:

$$\{\, H^i = dx^i,\; V^i = dv^i + \Gamma^i_j dv^j \}.$$

Remark 7: We know that the curvature of a nonlinear connection is given by $R(U,V) = [U^{\mathsf{H}}, V^{\mathsf{H}}] - [U,V]^{\mathsf{H}}$, where $U, V \in \mathfrak{X}(Q)$ and where U^h denotes the horizontal lift of the vector field U. Thus $R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = [H_i, H_j]$, from where:

$$R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \Big(\frac{\partial \Gamma^k_i}{\partial x^j} - \frac{\partial \Gamma^k_j}{\partial x^i} - \frac{\partial \Gamma^k_i}{\partial v^l}\Gamma^l_j + \frac{\partial \Gamma^k_j}{\partial v^l}\Gamma^l_i \Big) V_k.$$

$\diamond$

The dynamical covariant derivative and the Jacobi endomorphism associated to a sode. Both subbundles $\mathrm{Hor}(TQ)$ and $\mathrm{Ver}(TQ)$ are isomorphic to $\tau_Q^*(TQ) = TQ \times_Q TQ$ via horizontal and vertical lift, respectively. It follows that every vector field $Z \in \mathfrak{X}(TQ)$ can be written in an unique way as $Z = X^{\mathsf{H}} + Y^{\mathsf{V}}$, with X, Y vector fields along the map τ_Q.

Taking into account this decomposition the Lie derivative of a horizontal lift defines two geometrical objects: the generalized Jacobi endomorphism $\Phi : \Sigma_{\tau_Q}(\tau_Q) \to \Sigma_{\tau_Q}(\tau_Q)$ and the covariant derivative $\nabla : \Sigma_{\tau_Q}(\tau_Q) \to \Sigma_{\tau_Q}(\tau_Q)$, by means of

$$\mathcal{L}_\Gamma X^{\mathsf{H}} = (\nabla X)^{\mathsf{H}} + (\Phi(X))^{\mathsf{V}},$$

where by $\Phi(X)$ we understand $\Phi(X)(v) = \Phi_v(X(v))$.

Applying this relation to the vector field fX for a function $f \in C^\infty(TQ)$ we find that ∇ is a derivation along Γ, that is, it satisfies $\nabla(fX) = (\mathcal{L}_\Gamma f)X + f\nabla X$, while Φ is a $(1,1)$-tensor field, $\Phi(fX) = f\Phi(X)$. Indeed, on one hand we have $\mathcal{L}_\Gamma(fX)^{\mathsf{H}} = (\nabla(fX))^{\mathsf{H}} + (\Phi(fX))^{\mathsf{V}}$ and on the other:

$$\begin{aligned} \mathcal{L}_\Gamma(fX)^{\mathsf{H}} &= (\Gamma f)X^{\mathsf{H}} + f\mathcal{L}_\Gamma(X)^{\mathsf{H}} = \\ &= (\Gamma f)X^{\mathsf{H}} + f(\nabla X)^{\mathsf{H}} + f(\Phi(X))^{\mathsf{V}} = \\ &= (\Gamma f X + f\nabla X)^{\mathsf{H}} + (f\Phi(X))^{\mathsf{V}}, \end{aligned}$$

from where the conclusion follows.

The Lie derivative of a vertical lift does not define any new object, but can be expressed in terms of ∇,

$$\mathcal{L}_\Gamma X^{\mathsf{V}} = -X^{\mathsf{H}} + (\nabla X)^{\mathsf{V}}.$$

We can extend the same relations for a vector field W along an integral curve in the base γ of Γ

$$\mathcal{L}_\Gamma W^{\mathsf{H}} = (\nabla W)^{\mathsf{H}} + (\Phi(W))^{\mathsf{V}} \qquad \text{and} \qquad \mathcal{L}_\Gamma W^{\mathsf{V}} = -W^{\mathsf{H}} + (\nabla W)^{\mathsf{V}},$$

where $\Phi(W)$ stands for $\Phi(W)(t) = \Phi_{\dot\gamma(t)}(W(t))$.

In the basis $\{H_i, V_i\}$ the local expression of Γ becomes:

$$\Gamma = v^i H_i + (f^i + \Gamma^i_j v^j)V_i,$$

and the Lie derivatives with respect to Γ of the vector fields of the basis are:

$$\begin{cases} \mathcal{L}_\Gamma H_i = [\Gamma, H_i] = \Gamma^j_i H_j + \Phi^j_i V_j \\ \mathcal{L}_\Gamma V_i = [\Gamma, V_i] = -H_i + \Gamma^j_i V_j \end{cases}$$

where the functions Φ^j_i are the components of the Jacobi endomorphism and are given by:

$$\Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k \Gamma^k_j - \Gamma(\Gamma^i_j). \tag{3.2}$$

In local coordinates (x^i, v^i) on TQ, if the SODE vector field is $\Gamma = v^i\partial_{x^i} + f^i\partial_{v^i}$, the horizontal lift of a coordinate vector field $\partial/\partial x^i$ is H_i. If W is of the form $W(t) = W^i(t)\partial_{x^i}$, then the dynamical covariant derivative of W takes the expression:

$$\nabla W(t) = \Big(\dot W^i(t) + \Gamma^i_j(\dot\gamma(t))W^j(t)\Big)\frac{\partial}{\partial x^i},$$

Taking in consideration Remark 6 a similar formula is valid for a vector field X along τ_Q :

$$\nabla X(q,v) = (\Gamma(X^i(q,v)) + \Gamma^i_j X^j(q,v))\frac{\partial}{\partial x^i}.$$

The expression of the Jacobi endomorphism acting upon W, a vector field along the curve $\gamma(t)$ is:

$$\Phi(W)(t) = \Phi^i_j(\dot\gamma(t))W^j(t)\frac{\partial}{\partial x^i}.$$

Jacobi equation for sode on manifold. Since we are interested in the equation $\mathcal{L}_\Gamma W^{\mathsf{c}} = 0$, we need to decompose the complete lift W^{c} in its horizontal and vertical components. We have that:

$$W^{\mathsf{c}} = W^{\mathsf{H}} + (\nabla W)^{\mathsf{V}}.$$

Indeed, the difference between X^{c} and X^{H} is a vertical vector field $X^{\mathsf{c}} - X^{\mathsf{H}} = Y^{\mathsf{V}}$. Applying $\mathcal{L}_\Gamma S$ to this expression we find:

$$\begin{aligned} Y^{\mathsf{V}} = (\mathcal{L}_\Gamma S)Y^{\mathsf{V}} &= (\mathcal{L}_\Gamma S)X^{\mathsf{c}} - (\mathcal{L}_\Gamma S)X^{\mathsf{H}} = \\ &= \mathcal{L}_\Gamma(SX^{\mathsf{c}}) - S(\mathcal{L}_\Gamma X^{\mathsf{c}}) + X^{\mathsf{H}} = \mathcal{L}_\Gamma X^{\mathsf{V}} + X^{\mathsf{H}} = (\nabla X)^{\mathsf{V}}, \end{aligned}$$

where we have used that $\mathcal{L}_\Gamma X^{\mathsf{c}}$ is vertical (as it can be proved easily in coordinates).

From all this facts it is now easy to prove the following result.

THEOREM 8: *A vector field W along an integral curve in the base γ_0 of Γ is a Jacobi field if and only if it satisfies the second-order differential equation*

$$\nabla\nabla W + \Phi(W) = 0.$$

This equation will be called the (generalized) Jacobi equation.

PROOF. Taking the Lie derivative of $W^{\mathsf{C}} = W^{\mathsf{H}} + (\nabla W)^{\mathsf{V}}$ with respect to Γ we have

$$\begin{aligned}\mathcal{L}_\Gamma W^{\mathsf{C}} &= \mathcal{L}_\Gamma W^{\mathsf{H}} + \mathcal{L}_\Gamma (\nabla W)^{\mathsf{V}} \\ &= \big((\nabla W)^{\mathsf{H}} + (\Phi(W))^{\mathsf{V}}\big) + \big(-(\nabla W)^{\mathsf{H}} + (\nabla\nabla W)^{\mathsf{V}}\big) \\ &= [\nabla\nabla W + \Phi(W)]^{\mathsf{V}}.\end{aligned}$$

From where the result follows immediately. □

Jacobi fields are related to infinitesimal symmetries of the SODE Γ. If $Y \in \mathfrak{X}(TQ)$ is an infinitesimal symmetry of Γ, that is $\mathcal{L}_\Gamma Y = 0$, then it is of the form $Y = X^{\mathsf{H}} + (\nabla X)^{\mathsf{V}}$ for some vector field X along the projection τ_Q. Therefore, if γ is an integral curve in the base of Γ, the vector field $W(t) = X \circ \gamma$ satisfies $Y \circ \dot\gamma = W^{\mathsf{C}}$ and $\mathcal{L}_\Gamma W^{\mathsf{C}} = 0$, and hence W is a Jacobi field. This result already appeared in this form in [13].

3.1.4 The variational sode

The Jacobi equation $\mathcal{L}_\Gamma W^{\mathsf{C}} = 0$ for a given SODE on Q, is itself a second order differential equation as it can be seen from the equivalence of item 3 with item 4 from Proposition 2, and it corresponds to a SODE vector field. More exactly, in the local coordinates (x^i, v^i, w^i, u^i) on TTQ this property is written as:

$$\begin{cases} \dot x^a = v^a \\ \dot v^a = f^a \\ \dot w^a = u^a \\ \dot u^a = w^b \dfrac{\partial f^a}{\partial x^b} + u^b \dfrac{\partial f^a}{\partial v^b}. \end{cases} \tag{3.3}$$

which reduces to

$$\begin{cases} \ddot x^a = f^a \\ \ddot w^a = w^b \dfrac{\partial f^a}{\partial x^b} + u^b \dfrac{\partial f^a}{\partial v^b}. \end{cases} \tag{3.4}$$

from where it is obviously it is a SODE in the coordinates (x^i, w^i).

It is therefore natural to look for a SODE on TQ whose solutions are the Jacobi fields together with the solutions of the original SODE on Q.

PROPOSITION 4: *With the fixed notation, if W is a Jacobi field then $\dot W$ is an integral curve of the vector field* $\Gamma^{\mathsf{VAR}} = T\chi_{TQ} \circ \Gamma^{\mathsf{C}} \circ \chi_{TQ}$.

PROOF. If W is a Jacobi field, then, from Definition 21, W^{C} is the variational vector field along the integral curve $\dot\gamma$ by the integral curves of Γ, from where using Proposition 3 and the equivalent condition 4 from Proposition 2 we get that W^{C} is an integral curve of Γ^{C}. Taking into account that $W^{\mathsf{C}} = \chi_{TQ} \circ \dot W$ we have that the derivative of $\dot W$ satisfies:

$$\frac{d}{dt}\dot W = \frac{d}{dt}(\chi_{TQ} \circ W^{\mathsf{C}}) = T\chi_{TQ} \circ \frac{d}{dt} W^{\mathsf{C}} = T\chi_{TQ} \circ \Gamma^{\mathsf{C}} \circ W^{\mathsf{C}} = T\chi_{TQ} \circ \Gamma^{\mathsf{C}} \circ \chi_{TQ} \circ \dot W,$$

from where the conclusion follows.

□

THEOREM 9: *The map* $\Gamma^{\mathsf{VAR}} = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ}$ *is a vector field on* TTQ *and satisfies the following properties:*

1. Γ^{VAR} *is* χ_{TQ}*-related to* Γ^{C}.
2. Γ^{VAR} *is a* SODE *vector field on* TQ.
3. Γ^{VAR} *is* $T\tau_Q$*-related to* Γ.
4. *If* ϕ_t *is the local flow of* Γ, *then the local flow of* Γ^{VAR} *is* $\phi_t^{\mathsf{VAR}} = \chi_{TQ}\circ T\phi_t\circ\chi_{TQ}$.

If $W\colon \mathbb{R}\to TQ$ *is an integral curve in the base of* $\Gamma^{\mathsf{VAR}}\in\mathfrak{X}(TTQ)$, *then* $\gamma = \tau_Q\circ W$ *is an integral curve on the base of* Γ *and* W *is a Jacobi field along* γ. *Conversely, if* γ *is an integral curve in the base of* Γ *and* W *is a Jacobi field along* γ *then* W *is an integral curve on the base of* Γ^{VAR}.

PROOF. We have that $\Gamma^{\mathsf{VAR}} = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ} = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ}^{-1}$, which proves the first. As a consequence, the flow of Γ^{VAR} is $\chi_{TQ}\circ T\phi_t\circ\chi_{TQ}$ where ϕ_t is the flow of Γ, which proves 4. Taking into account that $\tau_{TQ}\circ\chi_{TQ} = T\tau_Q$, we have

$$T(T\tau_Q)\circ\Gamma^{\mathsf{VAR}} = T(T\tau_Q\circ\chi_{TQ})\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ} = T\tau_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ} = \Gamma\circ\tau_{TQ}\circ\chi_{TQ} = \Gamma\circ T\tau_Q,$$

which proves 3. Finally item 2 follows easily from the coordinate expression of Γ^{C}:

$$\Gamma^{\mathsf{C}} = v^a\frac{\partial}{\partial x^a} + f^a\frac{\partial}{\partial v^a} + u^a\frac{\partial}{\partial w^a} + \Big(w^b\frac{\partial f^a}{\partial x^b} + u^b\frac{\partial f^a}{\partial v^b}\Big)\frac{\partial}{\partial u^a}.$$

[$\Rightarrow$]For the direct implication we will apply in $\dot W$ the expression corresponding to Γ^{VAR} being $T\tau_Q$-related to Γ , i.e.:

$$[T(T\tau_Q)\circ\Gamma^{\mathsf{VAR}}](\dot W) = [\Gamma\circ T\tau_Q](\dot W).$$

Then, on one hand, using the hypothesis that $\Gamma^{\mathsf{VAR}}(\dot W) = \ddot W$, in the first term we have:

$$T(T\tau_Q)\circ\Gamma^{\mathsf{VAR}}(\dot W) = T(T\tau_Q)\circ\ddot W = T(T\tau_Q\circ\dot W) = T(T(\tau_Q\circ W)) = \ddot\gamma$$

and on the other side: $\Gamma\circ T\tau_Q(\dot W) = \Gamma(T(\tau_Q\circ W)) = \Gamma(\dot\gamma)$

Equalizing the two parts we get : $\Gamma(\dot\gamma) = \ddot\gamma$, which proves that γ is an integral curve on the base of Γ.

Now using the definition of W being an integral curve in the base for $\Gamma^{\mathsf{VAR}} = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ}$, i.e.: $T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ}(\dot W) = \ddot W$, so on one side we will have that:

$$\Gamma^{\mathsf{VAR}}(\dot W) = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}\circ\chi_{TQ}(\dot W) = T\chi_{TQ}\circ\Gamma^{\mathsf{C}}(W^{\mathsf{C}})$$

while on the other side:

$$\ddot W = (\chi_{TQ}\dot{\circ} W^{\mathsf{C}}) = T\chi_{TQ}\circ\dot W^{\mathsf{C}}$$

and it results: $\Gamma^{\mathsf{C}}(W^{\mathsf{C}}) = \dot W^{\mathsf{C}}$, so W^{C} is the integral curve of Γ^{C} and so W is a Jacobi field along γ.

[$\Leftarrow$]For the inverse implication, from the Proposition 4 we have that if W Jacobi field then $\dot W$ is the integral curve of Γ^{VAR} and using item 2 of this theorem, it results W is an integral curve in the base of Γ^{VAR}.

$\square$

3.2 The case of sodes on Lie algebroids

In what follows we consider here the generalization of the results from the previous section to the case of SODEs on Lie algebroids. After defining what a Lie transported section of a Lie algebroid is, we will prove the property similar to Proposition 2 from the classical case. In Subsection 3.2.1 we will define the 1-parameter family of integral curves of a section of a Lie algebroid and in Subsection 3.2.2 the Jacobi section, where we will also present the generalization of the Jacobi equation to the context of Lie algebroid.

Let $\tau : A \to M$ be a Lie algebroid and $\sigma \in \mathrm{Sec}(A)$. We will denote the flow of the section σ by (ϕ_t, φ_t). An integral curve of a section $\sigma \in \mathrm{Sec}(A)$ is an integral curve of the vector field $\rho(\sigma) \in \mathfrak{X}(M)$. We set a point $m \in M$ and consider the integral curve $\zeta_0(t) = \varphi_t(m)$ of $\rho(\sigma)$, starting at m and defined on an interval I.

DEFINITION 22: *A section Z of A along ζ_0 is said to be obtained by Lie transport along the flow of σ if there exists $\xi \in A_m$ such that $Z(t) = \phi_t(\xi)$ for all $t \in I$.*

DEFINITION 23: *The Lie derivative of a section Z of A along ζ_0 with respect to σ is the section along ζ_0 given by*

$$\big(d_\sigma Z\big)(t) = \frac{d}{ds}\phi_{-s}Z(t+s)\Big|_{s=0} = \lim_{h\to 0}\frac{1}{h}[\phi_{-h}Z(t+h) - Z(t)].$$

It follows from the definition that if $\eta \in \mathrm{Sec}(A)$ then $[\sigma, \eta] \circ \zeta_0 = d_\sigma(\eta \circ \zeta_0)$.

PROPOSITION 5: *The following properties are equivalent:*

1. *Z is Lie transported along the flow of σ.*
2. *For every $t \in I$ and every s such that $s+t \in I$ we have $\phi_s(Z(t)) = Z(t+s)$.*
3. *The Lie derivative of Z vanishes identically, $d_\sigma Z(t) = 0$ for all $t \in I$.*
4. *The curve $Z\colon I \to A$ is an integral curve of the complete lift vector field $\rho^1(\sigma^{\mathrm{c}}) \in \mathfrak{X}(A)$ of the section σ.*

In local coordinates (x^i, y^α) in A, the above properties express that the components $x^i = \zeta_0^i(t)$, $y^\alpha = Z^\alpha(t)$ of Z satisfy the variational equation

$$\begin{cases} \dot{x}^i = \rho^i_\alpha \sigma^\alpha \\ \dot{y}^\alpha = \dfrac{\partial \sigma^\alpha}{\partial x^i}\rho^i_\beta y^\beta + C^\alpha_{\beta\theta} y^\beta \sigma^\theta. \end{cases}$$

PROOF. We first prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and later on (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1). The final statement follows from (4) and the local expression of $\rho^1(\sigma^{\mathrm{c}}) \in \mathfrak{X}(A)$.
(1) $\Rightarrow$ (2) If $Z(t) = \phi_t(\xi)$ then $\phi_s(Z(t)) = \phi_s(\phi_t(\xi)) = \phi_{s+t}(\xi) = Z(t+s)$.
(2) $\Rightarrow$ (3) Aplying ϕ_{-s} we get $\phi_{-s}(Z(t+s)) = Z(t)$ so that $d_\sigma Z(t) = 0$.
(3) $\Rightarrow$ (1) The derivative of $\phi_{-t}Z(t)$ is

$$\begin{aligned} \frac{d}{dt}\phi_{-t}Z(t) &= \lim_{h\to 0}\frac{1}{h}[\phi_{-t-h}Z(t+h) - \phi_{-t}Z(t)] \\ &= \phi_{-t}\lim_{h\to 0}\frac{1}{h}[\phi_{-h}Z(t+h) - Z(t)] \\ &= \phi_{-t} d_\sigma Z(t) = 0. \end{aligned}$$

Therefore it is constant, $\phi_{-t}Z(t) = \xi$ and hence $Z(t) = \phi_t(\xi)$.
(2) $\Rightarrow$ (4) The flow of the complete lift vector field $\rho^1(\sigma^{\mathrm{c}})$ is $\{\phi_t\}$. Taking the derivative of $\phi_s(Z(t)) = Z(t{+}s)$ with respect to s at $s = 0$ we get $\rho^1(\sigma^{\mathrm{c}})(Z(t)) = \dot{Z}(t)$. Therefore $Z(t)$ is an integral curve of $\rho^1(\sigma^{\mathrm{c}})$.
(4) $\Rightarrow$ (1) If $Z(t)$ is integral curve of $\rho^1(\sigma^{\mathrm{c}})$, then $Z(t) = \phi_t(Z(0))$ because $\{\phi_t\}$ is the flow of $\rho^1(\sigma^{\mathrm{c}})$. Taking $\xi = Z(0)$ we get that $Z(t)$ is Lie transported. □

For the standard Lie algebroid $A = TM$ we have that $\phi_t = T\varphi_t$, and we recover the standard definitions and properties, and in particular a proof of Proposition 2.

3.2.1 One-parameter families of solutions

We now consider a section $\sigma \in \mathrm{Sec}(A)$ and the integral curve ζ_0 of the vector field $\rho(\sigma)$ starting at $m \in M$, that is $\zeta_0(t) = \varphi_t(m)$.

Definition 24: *A 1-parameter family of integral curves of $\sigma \in \mathrm{Sec}(A)$ is a morphism of Lie algebroids $\theta\colon T\mathbb{R}^2 \to A$ over $\zeta\colon (-\epsilon, \epsilon) \times I{\subset}\mathbb{R}^2 \to M$ of the form $\theta = \sigma(\zeta(s,t))dt{+}\beta(s,t)ds$. The section Z along ζ_0 defined by $Z(t) = \beta(0,t)$ is said to be the infinitesimal variation defined by the 1-parameter family.*

It is implicit in this definition that the base map of θ is ζ and that for every fixed s, the curve $\zeta_s(t) = \zeta(s,t)$ is an integral curve of σ, i.e for $\rho(\sigma)$. Indeed,

$$\rho(\sigma)(\zeta_s(t)) = \rho(\sigma(\zeta(s,t))) = \frac{\partial\zeta}{\partial t}(s,t) = \dot{\zeta}_s(t),$$

where we have used that $t \mapsto \sigma(\zeta(s,t))$ is admissible, because θ is a morphism.

Theorem 10: *A section Z along ζ_0, an integral curve of σ is the infinitesimal variation defined by a 1-parameter family of integral curves of σ if and only if it is Lie transported along the flow of σ.*

Proof. [$\Leftarrow$] Let $Z(t) = \phi_t(\xi)$ for some $\xi \in A_m$. Consider an admissible curve $\mu\colon (-\epsilon, \epsilon) \to A$ such that $\mu(0) = \xi$, and denote by ν the base path, $\nu = \tau \circ \mu$. The map

$$\theta(s,t) = \alpha(s,t)dt + \beta(s,t)ds = \sigma(\varphi_t(\nu(s)))dt + \phi_t(\mu(s))ds,$$

is a 1-parameter family of integral curves of σ. The base family is $\zeta(s,t) = \tau(\alpha(s,t)) = \varphi_t(\nu(s))$. To prove it, we recall (see [43]) that θ is a morphism of Lie algebroids if and only if $a_s(t) \equiv \alpha(s,t)$ is admissible, $b_t(s) = \beta(s,t)$ is admissible and $\chi_A\big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\big) = \big(\beta, \alpha, \frac{\partial\beta}{\partial t}\big)$.

Indeed, from $[\sigma, \sigma] = 0$ we get that $\phi_t{\circ}\sigma = \sigma{\circ}\varphi_t$, and therefore we can rewrite $a_s(t) = \phi_t(\sigma(\nu(s)))$, which is admissible because it is an integral curve of $\rho^1(\sigma^{\mathrm{c}})$. On the other hand, $b_t(s) = \phi_t(\mu(s))$ is admissible because ϕ_t is a morphism of Lie algebroids (morphisms transform admissible curves into admissible curves). Finally, to prove $\chi_A\big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\big) = \big(\beta, \alpha, \frac{\partial\beta}{\partial t}\big)$ it is enough to show that $\rho^1\big(\chi_A\big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\big)\big) = \frac{\partial\beta}{\partial t}$. On one hand we have

$$\frac{\partial\beta}{\partial t}(s,t) = \frac{d}{dt}\phi_t(\mu(s)) = \rho^1(\sigma^{\mathrm{c}})(\phi_t(\mu(s))) = (\rho^1(\sigma^{\mathrm{c}}){\circ}\beta)(s,t)$$

and on the other

$$\chi_A\Big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\Big) = \chi_A\Big(\sigma{\circ}\zeta, \beta, T\sigma{\circ}\frac{\partial\zeta}{\partial s}\Big) = \chi_A(\sigma{\circ}\zeta, \beta, T\sigma{\circ}\rho{\circ}\beta) = \sigma^{\mathrm{c}}{\circ}\beta,$$

and applying ρ^1 to it we get $\rho^1\big(\chi_A\big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\big)\big) = \rho^1(\sigma^c \circ \beta) = \rho^1(\sigma^c) \circ \beta = \frac{\partial\beta}{\partial t}$.

Therefore θ is a morphism of Lie algebroids. The infinitesimal variation defined by θ is $\beta(0,t) = \phi_t(\mu(0)) = \phi_t(\xi) = Z(t)$.

[$\Rightarrow$] *Conversely*, given a 1-parameter family $\theta(s,t) = \sigma(\zeta(s,t))dt + \beta(s,t)ds$ of integral curves of σ we will prove that $Z(t) = \beta(0,t)$ is Lie transported by showing that it is an integral curve of $\rho^1(\sigma^c)$. Since θ is a morphism we have $\frac{\partial\beta}{\partial t} = \rho^1\big(\chi_A\big(\alpha, \beta, \frac{\partial\alpha}{\partial s}\big)\big)$, where $\alpha(s,t) = \sigma(\zeta(s,t))$, which at $s = 0$ gives

$$\begin{aligned}\dot{Z}(t) = \frac{\partial\beta}{\partial t}(0,t) &= \rho^1\Big(\chi_A\Big(\sigma(\zeta(0,t)), \beta(0,t), T\sigma(\frac{\partial\zeta}{\partial s}(0,t))\Big)\Big) = \\ &= \rho^1\Big(\chi_A\Big(\sigma(\zeta_0(t)), Z(t), T\sigma(\rho(Z(t)))\Big)\Big) = \rho^1(\sigma^c)(Z(t)),\end{aligned}$$

where we have used that $\sigma^c(a) = \chi_A(\sigma(\tau(a)), a, T\sigma(\rho(a)))$, for $a \in A$. □

From the proof we deduce that every 1-parameter family of integral curves is necessarily of the form $\theta(s,t) = \sigma(\varphi_t(\nu(s)))dt + \phi_t(\mu(s))ds$ and we have $\nu(s) = \zeta(s,0)$, $\mu(s) = \beta(s,0)$ and $Z(t) = \phi_t(\mu(0))$.

3.2.2 Jacobi equations

Jacobi sections. Our results in Subsection 3.1.2 for a SODE on a manifold extend easily to the case of a SODE Γ on a Lie algebroid.

DEFINITION 25: *A 1-parameter family of integral curves of a* SODE $\Gamma \in \mathrm{Sec}(\mathcal{T}^A A)$ *is a morphism of the form* $\Theta(s,t) = \Gamma(\alpha(s,t))dt + B(s,t)ds$.

Let us analyze the expression in coordinates of the morphism Θ:

If we set the expression of $\Gamma(\alpha(s,t)) = \big(\alpha(s,t), \overline{\beta}(s,t), \overline{V}(s,t)\big)$, then as Γ is a SODE whose base is $\alpha(s,t)$, we have that: $\alpha(s,t) = \overline{\beta}(s,t)$. From the condition of the curves $t \mapsto \Gamma(\alpha(s,t))$ to be admissible, we get the expresion of $V(s,t)$:

$$V(s,t) = \rho^1(\Gamma(\alpha(s,t))) = \frac{\partial(\tau \circ \Gamma(\alpha(s,t)))}{\partial t} = \frac{\partial\alpha}{\partial t}(s,t).$$

Thus:

$$\Gamma(\alpha(s,t)) = \Big(\alpha(s,t), \alpha(s,t), \frac{\partial\alpha}{\partial t}(s,t)\Big).$$

Now consider the expression of $B(s,t) = \big(\alpha(s,t), \beta(s,t), V(s,t)\big)$. From the condition of the curves $s \mapsto B(s,t)$ to be admissible, we get that $V(s,t) = \frac{\partial\alpha}{\partial s}(s,t)$, thus:

$$B(s,t) = \big(\alpha(s,t), \beta(s,t), \frac{\partial\alpha}{\partial s}(s,t)\big).$$

Next let us relate the infinitesimal variational section of this 1-parameter family to the infinitesimal variational section of the corresponding 1-parameter family in the base. Recall that the projection onto the second component of elements from the prolongation of a Lie algebroid defined in Section 2.5.1, denoted by τ_2 is also a morphism of Lie algebroids. Thus the composition of the morphism Θ with τ_2 will also be a morphism of Lie algebroids. With the expressions that we obtained above we have:

$$\tau_2 \circ \Theta(s,t) = \theta(s,t) = \alpha(s,t)dt + \beta(s,t)ds,$$

is a morphism of Lie algebroids.

Denote by $W(t) = \beta(0,t)$ the infinitesimal variational section associated to θ and by $a_0 = \alpha(0,t)$. Therefore we have the following relation between W and the infinitesimal section corresponding to Θ at $s = 0$:

$$Z(t) = B(0,t) = \Big(a_0(t), \beta(0,t), \frac{\partial \alpha}{\partial s}(0,t)\Big) = \\ = \chi_A\Big(W(t), a_0(t), \frac{\partial \beta}{\partial t}(0,t)\Big) = \chi_A\Big(W(t), a_0(t), \dot{W}(t)\Big) = W^{\mathsf{c}}_{a_0}(t).$$

DEFINITION 26: *Given an integral curve a_0 of a* SODE Γ *on a Lie algebroid A, a section W of A along $\gamma_0 = \tau \circ a_0$ is said to be a Jacobi field along a_0 if its complete lift $Z = W^{\mathsf{c}}_{a_0}$ is an infinitesimal variational section.*

Equivalently, W is a Jacobi section if it is an integral curve of $\rho^1(\Gamma^{\mathsf{c}})$, or equivalently $W^{\mathsf{c}}_{a_0}$ is Lie transported, or equivalently if $d_\Gamma W^{\mathsf{c}}_{a_0} = 0$.

The Ehresmann connection associated to a sode on a Lie algebroid. In the same way as in the standard case, any SODE on a Lie algebroid determines an Ehresmann connection. The horizontal distribution associated to the SODE $\Gamma = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha$ is also constructed as the eigenspace of eigenvalue -1 of $d_\Gamma S$. Thus, in local coordinates, the connections coefficients are given by:

$$\Gamma^\alpha_\beta = -\frac{1}{2}\left(\frac{\partial f^\alpha}{\partial y^\beta} - C^\alpha_{\beta\gamma} y^\gamma\right). \tag{3.5}$$

Also, related to this coefficients we introduce the following functions to be used later:

$$\gamma^\alpha_\beta = \Gamma^\alpha_\beta - C^\alpha_{\beta\gamma} y^\gamma = -\frac{1}{2}\left(\frac{\partial f^\alpha}{\partial y^\beta} + C^\alpha_{\beta\gamma} y^\gamma\right). \tag{3.6}$$

A local basis corresponding to the splitting associated to this SODE is:

$$\{H_\alpha = \mathcal{X}_\alpha - \Gamma^\beta_\alpha \mathcal{V}_\beta, \;\; V_\alpha = \mathcal{V}_\alpha\},$$

and the brackets of the elements from this basis are given by:

$$\begin{cases} [H_\alpha, H_\beta] = C^\gamma_{\alpha\beta} H_\gamma + \Big(C^\gamma_{\alpha\beta}\Gamma^a_\gamma - \rho^i_\alpha \dfrac{\partial \Gamma^a_\beta}{\partial x^i} + \rho^i_\beta \dfrac{\partial \Gamma^a_\alpha}{\partial x^i} + \Gamma^\gamma_\alpha \dfrac{\partial \Gamma^a_\beta}{\partial y^\gamma} - \Gamma^b_\beta \dfrac{\partial \Gamma^a_\alpha}{\partial y^b}\Big) V_a, \\ [H_\alpha, V_\beta] = \dfrac{\partial \Gamma^\gamma_\alpha}{\partial y^\beta} V_\gamma, \\ [V_\alpha, V_\beta] = 0. \end{cases}$$

REMARK 8: The curvature of this nonlinear connection is given by: $R(e_\alpha, e_\beta) = [H_\alpha, H_\beta] - [e_\alpha, e_\beta]^{\mathsf{H}}$, and similar to Remark 7 we observe that its expression in local coordinates is:

$$R(e_\alpha, e_\beta) = (C^\gamma_{\alpha\beta}\Gamma^a_\gamma - \rho^i_\alpha \frac{\partial \Gamma^a_\beta}{\partial x^i} + \rho^i_\beta \frac{\partial \Gamma^a_\alpha}{\partial x^i} + \Gamma^\gamma_\alpha \frac{\partial \Gamma^a_\beta}{\partial y^\gamma} - \Gamma^b_\beta \frac{\partial \Gamma^a_\alpha}{\partial y^b}) V_a.$$

$\diamond$

The dynamical covariant derivative and the Jacobi endomorphism associated to a sode on a Lie algebroid. Using the same arguments as in the standard case we have that there is a derivation ∇ and a tensor field Φ satisfying:

$$d_\Gamma \eta^{\mathsf{H}} = (\nabla\eta)^{\mathsf{H}} + \Phi(\eta)^{\mathsf{V}} \qquad \text{and} \qquad d_\Gamma \eta^{\mathsf{V}} = -\eta^{\mathsf{H}} + (\nabla\eta)^{\mathsf{V}},$$

where η is a section of $\tau^*A = A \times_M A$. Similar expressions hold for a section W along the base curve of an integral curve of Γ :

$$d_\Gamma W^{\mathsf{H}} = (\nabla W)^{\mathsf{H}} + \Phi(W)^{\mathsf{V}} \qquad \text{and} \qquad d_\Gamma W^{\mathsf{V}} = -\eta^{\mathsf{H}} + (\nabla W)^{\mathsf{V}},$$

If the local expression of a section W along γ_0 is $W(t) = W^\alpha(t) e_\alpha$ then:

$$\nabla W(t) = \big[\dot W^\alpha(t) + \gamma^\alpha_\beta(a_0(t)) W^\beta(t)\big] e_\alpha \qquad \text{and} \qquad \Phi(W) = \Phi^\alpha_\beta(a_0(t)) W^\beta(t) e_\alpha,$$

where the components of the Jacobi endomorphism are:

$$\Phi^\alpha_\beta = -\rho^i_\beta \frac{\partial f^\alpha}{\partial x^i} - d_\Gamma \Gamma^\alpha_\beta - \gamma^\alpha_\theta \Gamma^\theta_\beta - \Gamma^\alpha_\theta C^\theta_{\beta\mu} y^\mu. \tag{3.7}$$

Jacobi equation for a sode on a Lie algebroid. On the other hand, the same arguments as in the standard case show that the complete lift of a section W (with respect to an integral curve a of Γ, which we omit in the notation) has the expression

$$W^{\mathsf{C}} = W^{\mathsf{H}} + (\nabla W)^{\mathsf{V}}.$$

From these facts, it follows that $d_\Gamma W^{\mathsf{C}}$ is vertical and has the expression

$$d_\Gamma W^{\mathsf{C}} = [\nabla\nabla W + \Phi(W)]^{\mathsf{V}},$$

and we have proved the following result.

Theorem 11: *A section W along the base curve of an integral curve a of the* sode Γ *is a Jacobi section along a if and only if it satisfies the second-order differential equation*

$$\nabla\nabla W + \Phi(W) = 0.$$

This equation is the generalized Jacobi equation.

3.3 Riemannian geometry on Lie algebroids

A Riemannian metric can be introduced on the Lie algebroid structure. This was done in [21], and later on in [5], where the Riemannian metric has been defined as a fibre metric on a vector bundle. To its associated Levi-Civita connection it corresponds a geodesic spray, whose integral curves are the geodesics of the mentioned given connection.

Definition 27: *A Riemannian metric on a Lie algebroid $\tau : A \to M$ is a family of scalar products, one for each point $p \in M$ $\langle \cdot , \cdot \rangle_p$ on the fibre A_p such that for any local sections $\alpha, \beta \in \mathrm{Sec}(A)$ the function $p \mapsto \langle \alpha(p), \beta(p) \rangle_p$ is smooth.*

Consider now $\tau : A \to M$ a Riemannian Lie algebroid, where the Riemannian metric is denoted by $\langle \cdot , \cdot \rangle$. For any leaf L of the characteristic foliation described in Section 2.5, $\forall p \in L$ we have $A_p = \mathcal{G}_p \oplus \mathcal{G}_p^{\perp}$, where $\mathcal{G}_p^{\perp}$ is the orthogonal to $\mathcal{G}_p = \operatorname{Ker} \rho_p$ with respect to $\langle \cdot , \cdot \rangle_p$.

The restriction of the anchor ρ to $\mathcal{G}_p^{\perp}$ is an isomorphism on T_pL, and hence it induces a scalar product on T_pL :

$$\langle \rho(a) , \rho(b) \rangle_L = \langle a , b \rangle,$$

where $a, b \in \mathcal{G}_p^{\perp}$. So, $\langle \cdot , \cdot \rangle$ induces a Riemannian metric $\langle \cdot , \cdot \rangle_L$ on L. We call it the *induced Riemannian metric on the leaf* L.

As in the classical case, to a Riemannian metric it can be associated an unique A-connection, denoted by D, compatible with the metric, that is, it satisfies: $\rho(\alpha)\langle \beta , \eta \rangle = \langle D_\alpha \beta , \eta \rangle + \langle D_\alpha \eta , \beta \rangle$ and that is symmetric, i.e., torsion free: $D_\alpha\beta - D_\beta\alpha = [\alpha, \beta]$. This connection is called the *Levi-Civita A-connection* associated to the Riemannian metric $\langle \cdot , \cdot \rangle$. It is uniquely determined by the relation:

$$\begin{aligned} 2\langle D_\alpha \beta , \eta \rangle = \rho(\alpha)\langle \beta , \eta \rangle + \rho(\beta)\langle \alpha , \eta \rangle - \rho(\eta)\langle \alpha , \beta \rangle + \\ + \langle [\eta, \alpha] , \beta \rangle + \langle [\eta, \beta] , \alpha \rangle + \langle [\alpha, \beta] , \eta \rangle. \end{aligned}$$

As in the case of Riemannian geometry on manifolds, from this relation the expression of the Christoffel coefficients of the Levi-Civita A-connection follows:

$$\Gamma^f_{cd} = \frac{1}{2} g^{f\alpha}\Big(\rho^i_c \frac{\partial g_{d\alpha}}{\partial x_i} + \rho^i_d \frac{\partial g_{c\alpha}}{\partial x_i} - \rho^i_\alpha \frac{\partial g_{cd}}{\partial x_i}\Big) +$$

$$+ \frac{1}{2} g^{f\alpha}(C^i_{cd} g_{i\alpha} + C^i_{\alpha c} g_{id} + C^i_{\alpha d} g_{ic}).$$

Denote the coefficients of the associated curvature defined by relation (2.25) by R^s_{ijk}, where $R(e_i, e_j)e_k = R^s_{ijk} e_s$. From the following relations:

$$\nabla_{e_i}(\nabla_{e_j} e_k) = \nabla_{e_i}(\Gamma^a_{jk} e_a) = \rho^b_i \frac{\partial \Gamma^m_{jk}}{\partial x^b} e_m + \Gamma^a_{jk}\Gamma^m_{ia} e_m$$

$$\nabla_{e_j}(\nabla_{e_i} e_k) = \nabla_{e_j}(\Gamma^n_{ik} e_n) = \rho^b_j \frac{\partial \Gamma^m_{ik}}{\partial x^b} e_m + \Gamma^n_{ik}\Gamma^m_{jn} e_m$$

$$\nabla_{[e_i, e_j]} e_k = \nabla_{C^a_{ij} e_a} e_k = C^a_{ij}\Gamma^m_{ak} e_m = (\Gamma^a_{ij} - \Gamma^a_{ji})\Gamma^m_{ak} e_m$$

we get their expression:

$$R^s_{ijk} = \rho^a_i \frac{\partial \Gamma^s_{jk}}{\partial x^a} - \rho^a_j \frac{\partial \Gamma^s_{ik}}{\partial x^a} + \Gamma^l_{jk}\Gamma^s_{il} - \Gamma^l_{ik}\Gamma^s_{jl} - \Gamma^l_{ij}\Gamma^s_{lk} + \Gamma^l_{ji}\Gamma^s_{lk}. \tag{3.8}$$

Remark 9: As from now on we will work on Riemannian Lie algebroids, we will consider the regular curves parametrized by arc length and thus defined on the standard interval $[0, 1]$.

The arc length for all continuous curve α which we denote by: $s(t) = \int_{t_0}^{t} \sqrt{g(\alpha(t'), \alpha(t'))}dt'$ is an increasing function as $s'(t) > 0, \forall t$. In consequence the function $s(t)$ is invertible, and its inverse $\tau(s)$ is also an increasing function. Denote the base curve of α by $\gamma(t)$. If we reparametrize γ by $t = \tau(s)$ we get the curve $\eta(s) = \gamma(\tau(s))$. Consider the curve $\beta(s)$ along η :

$$\beta(s) = \frac{d\tau}{ds}(s)\alpha(\tau(s)) = \frac{1}{\sqrt{g(\alpha(\tau(s), \alpha(\tau(s))}}\alpha(\tau(s)),$$

which is an admissible curve, as:

$$\rho(\beta(s)) = \rho(\alpha(\tau(s)))\frac{d\tau}{ds}(s) = \frac{d\gamma}{dt}(\tau(s))\frac{d\tau}{ds}(s) = \frac{d(\gamma(\tau(s)))}{ds} = \frac{d\eta}{ds}(s).$$

In this case it can be verified that $\beta(s)$ is parametrized by arc length, in the sense that:

$$g(\beta(s), \beta(s)) = \frac{1}{g(\alpha(\tau(s), \alpha(\tau(s))}g(\alpha(\tau(s), \alpha(\tau(s)) = 1.$$

◇

We recall the notation $D_t\alpha$ for the covariant derivative associated to the $A-$connection, as specified in Section 2.6, and we give the following definition:

Definition 28: *Let $\tau : A \to M$ be a Riemannian Lie algebroid and denote by D the associated Levi-Civita connection. An admissible curve $\alpha : [0,1] \to A$ is a geodesic for D if it satisfies $D_t\alpha = 0$.*

Then the geodesic spray for this Levi-Civita $A-$connection (see Remark 5) is written in coordinates as follows:

$$\Gamma_{geod} = y^\alpha \mathcal{X}_\alpha - \Gamma^f_{cd} y^c y^d \mathcal{V}_f. \tag{3.9}$$

We consider, now, the relation between the dynamical covariant derivative and the connection.

Proposition 6: *Let D be the Levi-Civita connection associated to the metric and let ∇ be the dynamical covariant derivative associated to the geodesic spray Γ_{geod}. Then*

$$\nabla = D_t.$$

Proof. Take β a section along the integral curve α of the geodesic spray. Then:

$$D_t\beta = (\dot\beta^k + \Gamma^k_{ij}\alpha^k\beta^j)e_k \text{ and } \nabla\beta = (\dot\beta^a(t) + \gamma^a_m(\alpha(t))\beta^m(t))e_a,$$

where

$$\gamma^a_\beta = -\frac{1}{2}(\frac{\partial f^a}{\partial y^\beta} + C^a_{\beta\gamma}y^\gamma) = -\frac{1}{2}(\frac{\partial(-\Gamma^a_{mn}y^m y^n)}{\partial y^\beta} + C^a_{\beta\gamma}y^\gamma) = \Gamma^a_{\gamma\beta}y^\gamma,$$

and so $\gamma^a_m(\alpha(t)) = \Gamma^a_{\gamma m}\alpha^\gamma$, from where the conclusion follows. □

Moreover, the Jacobi endomorphism can be expressed in terms of the curvature of the connection.

THEOREM 12: *The Jacobi endomorphism associated to the geodesic spray of the Levi-Civita connection and the curvature tensor of such connection are related by:*

$$\Phi_\alpha(\beta) = R(\beta, \alpha)\alpha.$$

PROOF. From the expression of the geodesic spray given in (3.9) we get that $f^f = -\Gamma^f_{cd} y^c y^d$, and substituting it in formula (3.5) we have the coefficients of its associated connection $\Gamma^i_j = \Gamma^i_{jb} y^b$. On the other hand, from (3.7), using (3.6) we get that the coefficients of the Jacobi endomorphism associated to Γ_{geod} are given by:

$$\begin{aligned}
\Phi^a_b &= -\rho^i_b \frac{\partial f^a}{\partial x^i} - d_\Gamma \Gamma^a_b - \gamma^a_\theta \Gamma^\theta_b - \Gamma^a_\theta C^\theta_{b\mu} y^\mu = \\
&= \rho^i_b \frac{\partial \Gamma^a_{mn}}{\partial x^i} y^m y^n - \rho^i_m \frac{\partial \Gamma^a_{bn}}{\partial x^i} y^n y^m + \Gamma^l_{mn} \Gamma^a_{bl} y^m y^n + \\
&\quad + \frac{1}{2}(-\Gamma^a_{lm} - \Gamma^a_{ml} + C^a_{lm}) \Gamma^l_{bn} y^n y^m - \Gamma^a_{ln} C^l_{bm} y^n y^m = \\
&= (\rho^i_b \frac{\partial \Gamma^a_{mn}}{\partial x^i} - \rho^i_m \frac{\partial \Gamma^a_{bn}}{\partial x^i} + \Gamma^l_{mn} \Gamma^a_{bl} - \Gamma^l_{bn} \Gamma^a_{ml} + \Gamma^l_{mb} \Gamma^a_{ln} - \Gamma^l_{bm} \Gamma^a_{ln}) y^m y^n.
\end{aligned}$$

Taking into consideration the formula of the curvature coefficients given by (3.8) we get the wanted conclusion. □

It follows from the above results that a section β is a Jacobi section along a geodesic α if and only if it satisfies

$$D_t D_t \beta + R(\beta, \alpha)\alpha = 0,$$

as in the case of standard Riemannian geometry. Therefore, in this case, we always have the following Jacobi sections:

- $\beta(t) = \alpha(t)$,
- $\beta(t) = t\alpha(t)$,

which are trivial consequences of the relations $D_t \alpha = 0$ and $R(\alpha, \alpha)\alpha = 0$.

REMARK 10: The equations of the geodesics of the Levi-Civita A-connection coincide with the Euler-Lagrange equations for the regular Lagrangian L associated to the Riemannian metric considered on the Lie algebroid, by the expression:

$$L(\alpha(t)) = \frac{1}{2}\langle\, \alpha(t)\,, \alpha(t)\,\rangle = \frac{1}{2} g_{\alpha\beta}(x(t)) y^\alpha y^\beta, \tag{3.10}$$

where $g_{\alpha\beta}$ is the matrix of a Riemannian metric on A. Indeed the system (2.33) of the Euler-Lagrange equations in (x^i, y^α)-a local system of coordinates on A, becomes:

$$\begin{cases} \dot{y}^f(t) = -\Gamma^f_{cd} y^c(t) y^d(t) \\ \dot{x}^i = \rho^i_\alpha y^\alpha. \end{cases} \tag{3.11}$$

For this particular Lagrangian we have that $\frac{\partial L}{\partial y^\alpha} = g_{\alpha\beta}(x(t))y^\beta$ and we then compute: $\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) = \frac{\partial g_{\alpha c}}{\partial x^i}\dot{x}^i(t)y^c(t)+g_{\alpha f}(x(t))\dot{y}^f(t)$ and $\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{cd}}{\partial x^i}(x(t))y^c(t)y^d(t)$.

Substituting in the first equation of the differential equations system (2.33) we get:

$$\frac{\partial g_{\alpha c}}{\partial x^i}\dot{x}^i(t)y^c(t) + g_{\alpha f}(x(t))\dot{y}^f(t) + C^\gamma_{\alpha c}y^c g_{\gamma d}y^d = \frac{1}{2}\rho^i_\alpha \frac{\partial g_{cd}}{\partial x^i}(x(t))y^c(t)y^d(t).$$

The Lagrangian used is regular, that is, the matrix $g_{\alpha f}$ is invertible. We separate $\dot{y}^f$ from the above equation, denoting by $g^{\alpha f}$ the inverse of the metric matrix, and we indeed get the first equation from (3.11):

$$\dot{y}^f = g^{\alpha f}\Big(\frac{1}{2}\rho^i_\alpha\frac{\partial g_{cd}}{\partial x^i}y^c y^d - \frac{\partial g_{\alpha c}}{\partial x^i}\rho^i_d y^d y^c - C^\gamma_{\alpha c}g_{\gamma d}y^c y^d\Big) =$$

$$= -\frac{1}{2}g^{\alpha f}\Big(-\rho^i_\alpha\frac{\partial g_{cd}}{\partial x^i} + 2\frac{\partial g_{\alpha c}}{\partial x^i}\rho^i_d\Big)y^c y^d - g^{\alpha f}C^\gamma_{\alpha c}g_{\gamma d}y^c y^d =$$

$$= -\frac{1}{2}g^{\alpha f}\Big(\rho^i_c\frac{\partial g_{d\alpha}}{\partial x^i}+\rho^i_d\frac{\partial g_{c\alpha}}{\partial x^i}-\rho^i_\alpha\frac{\partial g_{cd}}{\partial x^i}\Big)y^c y^d-\frac{1}{2}g^{\alpha f}\Big(C^i_{cd}g_{i\alpha}+C^i_{\alpha d}g_{id}+C^i_{\alpha d}g_{ic}\Big)y^c y^d =$$

$$= -\Big[\frac{1}{2}g^{\alpha f}\Big(\rho^i_c\frac{\partial g_{d\alpha}}{\partial x^i}+\rho^i_d\frac{\partial g_{c\alpha}}{\partial x^i}-\rho^i_\alpha\frac{\partial g_{cd}}{\partial x^i}\Big)+\frac{1}{2}g^{\alpha f}\Big(C^i_{cd}g_{i\alpha}+C^i_{\alpha d}g_{id}+C^i_{\alpha d}g_{ic}\Big)\Big]y^c y^d =$$

$$= -\Gamma^f_{cd}y^c(t)y^d(t).$$

◇

REMARK 11: From this calculus it is also easy to notice that the Lagrangian SODE for this L is the geodesic spray for the Levi-Civita connection given by (3.9). ◇

3.4 Variational formulae

Let $\tau : A \to M$ be a Riemannian Lie algebroid, with Riemann metric denoted by $\langle\,\cdot\,,\cdot\,\rangle$. Then we can consider the action functional $\mathcal{E} : \mathcal{P}([0,1], A) \to \mathbb{R}$ defined by the metric Lagrangian:

$$\mathcal{E}(\alpha) = \frac{1}{2}\int_0^1 \langle\,\alpha(t)\,,\alpha(t)\,\rangle dt,$$

which in this context will be called the energy functional, as in the case of standard Riemannian geometry.

We already saw in Chapter 2 the first variation formula of the energy functional of a Lagrangian system. For the particular case of the Lagrangian defined by (3.10) we will rewrite this formula that appears in the proof of the Theorem 5, in terms of the Levi-Civita connection and moreover, we will give the formula of the second variation of the energy for this particular Lagrangian.

First we will see the form the morphism condition (2.35) takes in terms of the Levi-Civita connection.

PROPOSITION 7: *Consider a vector bundle ϕ map between $T\mathbb{R}^2$ over $\mathbb{R}^2$ and A over M, written as $\phi = \alpha(s,t)dt + \beta(s,t)ds$, and assume that $t \mapsto \alpha(s,t)$ and $s \mapsto \beta(s,t)$ are admisible curves. Let D be a symmetric A-connection. Then ϕ is a Lie algebroid morphism if and only if*

$$D_t\beta = D_s\alpha.$$

PROOF. Indeed, using the expressions of the covariant derivative:

$$D_t\beta = \Big(\frac{\partial \beta^k}{\partial t} + \Gamma^k_{cd}\alpha^c\beta^d\Big)e_k \text{ and } D_s\alpha = \Big(\frac{\partial \alpha^k}{\partial s} + \Gamma^k_{dc}\alpha^c\beta^d\Big)e_k,$$

we have that:

$$D_t\beta - D_s\alpha = \Big(\frac{\partial \beta^k}{\partial t} - \frac{\partial \alpha^k}{\partial s} + \alpha^c(\Gamma^k_{cd} - \Gamma^k_{dc})\beta^d\Big)e_k.$$

Since the connection D is symmetric, the condition $\Gamma^k_{cd} - \Gamma^k_{dc} = C^k_{cd}$ takes place, from where we get: $D_t\beta - D_s\alpha = (\frac{\partial \beta^k}{\partial t} - \frac{\partial \alpha^k}{\partial s} + C^k_{cd}\alpha^c\beta^d)e_k$,. It follows that $D_t\beta - D_s\alpha = 0$ is equivalent to condition (2.35) characterizing such morphisms. □

First variation formula:

As in chapter 2, we consider variations $\alpha_s(t) = \alpha(s,t)$ of a given admissible curve $\alpha_0(t)$, with fixed base endpoints, and the associated morphism of Lie algebroids $\phi = \alpha(s,t)dt + \beta(s,t)ds$. We will use the Levi-Civita connection D to find a expression of the fist derivative of the energy functional.

PROPOSITION 8: *For any morphism $\phi : [0,1] \times [0,1] \to A$ of Lie algebroids, $\phi = \alpha(s,t)dt + \beta(s,t)ds$ we have the following expression of the first derivative of the energy functional:*

$$\frac{d}{ds}\mathcal{E}(\alpha_s) = \langle\, \beta(s,1)\,, \alpha(s,1)\,\rangle - \langle\, \beta(s,0)\,, \alpha(s,0)\,\rangle - \int_0^1 \langle\, \beta\,, D_t\alpha\,\rangle dt. \tag{3.12}$$

PROOF. Indeed, from the definition of the Levi-Civita connection we have

$$\frac{d}{ds}\mathcal{E}(\alpha_s) = \frac{d}{ds}\int_0^1 \frac{1}{2}\langle\, \alpha\,, \alpha\,\rangle dt = \int_0^1 \langle\, D_s\alpha\,, \alpha\,\rangle dt$$

and using the relation given by Proposition 7 we get:

$$\frac{d}{ds}\mathcal{E}(\alpha_s) = \int_0^1 \langle\, D_s\alpha\,, \alpha\,\rangle dt = \int_0^1 \langle\, D_t\beta\,, \alpha\,\rangle.$$

Finally integrating by parts, we arrive to

$$\frac{d}{ds}\mathcal{E}(\alpha_s) = \int_0^1 \langle\, D_t\beta\,, \alpha\,\rangle = \int_0^1 D_t\langle\, \alpha\,, \beta\,\rangle dt - \int_0^1 \langle\, \beta\,, D_t\alpha\,\rangle dt =$$

$$= \langle\, \beta(s,1)\,, \alpha(s,1)\,\rangle - \langle\, \beta(s,0)\,, \alpha(s,0)\,\rangle - \int_0^1 \langle\, \beta\,, D_t\alpha\,\rangle dt.$$

□

REMARK 12: When ϕ is an E-homotopy, the critical points of the energy functional are the geodesics of the Levi-Civita connection, which as we mentioned in Remark 10, satisfy the Euler-Lagrange equations for the Lagrangian associated to the Riemannian metric by the expression (3.10). This is a particular case of Theorem 5. ◇

Second variation formula:

Let us consider know the second derivative of the energy functional.

PROPOSITION 9: *Let $\phi = \alpha(s,t)dt + \beta(s,t)ds$ be a smooth homotopy such that the curve $\alpha_0(t) = \alpha(0,t)$ is a geodesic. We have the following expressions for the second derivative of the energy functional,*

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha)\Big|_{s=0} = \int_0^1 [\langle D_t\beta_0 , D_t\beta_0 \rangle dt - \langle \beta_0 , R(\beta_0,\alpha_0)\alpha_0 \rangle] dt, \tag{3.13}$$

and

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha)\Big|_{s=0} = -\int_0^1 \langle \beta_0 , D_tD_t\beta_0 + R(\beta_0,\alpha_0)\alpha_0 \rangle dt, \tag{3.14}$$

where R is the curvature tensor of the Levi-Civita connection, and β_0 denotes the curve $\beta_0(t) = \beta(0,t)$ for all $t \in [0,1]$.

PROOF. Following the steps in the proof of proposition 8, we have that the first derivative of the energy functional is

$$\frac{d}{ds}\mathcal{E}(\alpha_s) = \int_0^1 \langle D_t\beta , \alpha \rangle.$$

Taking the derivative with respect to s in this expression we get

$$\begin{aligned}
\frac{d^2}{ds^2}\mathcal{E}(\alpha) &= \int_0^1 [\langle D_sD_t\beta , \alpha \rangle + \langle D_t\beta , D_s\alpha \rangle]\, dt \\
&= \int_0^1 [\langle D_tD_s\beta + R(\beta,\alpha)\beta , \alpha \rangle + \langle D_t\beta , D_t\beta \rangle]\, dt \\
&= \int_0^1 \langle D_tD_s\beta , \alpha \rangle\, dt + \int_0^1 [\langle R(\beta,\alpha)\beta , \alpha \rangle + \langle D_t\beta , D_t\beta \rangle]\, dt,
\end{aligned}$$

where, in the second step, we have used $D_sD_t\beta - D_tD_s\beta = R(\beta,\alpha)\beta$ in the first term, and we have taken into account that $D_t\beta = D_s\alpha$ in the second term. Integrating by parts in the first term of this expression we get

$$\begin{aligned}
\frac{d^2}{ds^2}\mathcal{E}(\alpha) = \langle D_s\beta , \alpha \rangle\Big|_{t=0}^1 - \int_0^1 \langle D_s\beta , D_t\alpha \rangle\, dt + \\
+ \int_0^1 [\langle D_t\beta , D_t\beta \rangle + \langle R(\beta,\alpha)\beta , \alpha \rangle]\, dt
\end{aligned}$$

At $s = 0$ we have

$$\begin{aligned}\frac{d^2}{ds^2}\mathcal{E}(\alpha)\Big|_{s=0} &= \langle\, D_s\beta|_{s=0}\,, \alpha_0\,\rangle\Big|_{t=0}^{1} - \int_0^1 \langle\, D_s\beta|_{s=0}\,, D_t\alpha_0\,\rangle\, dt + \\ &+ \int_0^1 [\langle\, D_t\beta_0\,, D_t\beta_0\,\rangle + \langle\, R(\beta_0,\alpha_0)\beta_0\,, \alpha_0\,\rangle]\, dt \\ &= \int_0^1 [\langle\, D_t\beta_0\,, D_t\beta_0\,\rangle + \langle\, R(\beta_0,\alpha_0)\beta_0\,, \alpha_0\,\rangle]\, dt\end{aligned}$$

due to the following facts: (1) α_0 is a geodesic, so that $D_t\alpha_0 = 0$, and (2) ϕ is a homotopy, so that $\beta(s,0) = 0$ and $\beta(s,1) = 0$ for all $s \in [0,1]$, and taking the derivative at $s = 0$ we get $D_s\beta(0,0) = 0$ and $D_s\beta(0,1) = 0$. Formula (3.13) follows by taking into account that the endomorphism $R(\alpha_0,\beta_0)$ is skew-symmetric with respect to the scalar product, and hence

$$\langle\, R(\beta_0,\alpha_0)\beta_0\,, \alpha_0\,\rangle = -\langle\, \beta_0\,, R(\beta_0,\alpha_0)\alpha_0\,\rangle.$$

The second expression in the statement, equation (3.14), follows by taking again integration by parts in (3.13),

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha)\Big|_{s=0} = \langle\, \beta_0\,, D_t\beta_0\,\rangle\Big|_0^1 - \int_0^1 [\langle\, \beta_0\,, D_tD_t\beta_0\,\rangle dt + \langle\, \beta_0\,, R(\beta_0,\alpha_0)\alpha_0\,\rangle] dt,$$

and taking into account that $\beta_0(0) = \beta(0,0) = 0$ and $\beta_0(1) = \beta(0,1) = 0$. □

Remark 13: Taking into account the relation expressed in Using that the Jacobi endomorphism $\Phi_{\alpha_0}(\beta_0) = R(\beta_0,\alpha_0)\alpha_0$ the second variation formula can be written as

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha_s)\Big|_{s=0} = \int_0^1 \langle\, D_t\beta_0\,, D_t\beta_0\,\rangle dt - \int_0^1 \langle\, \beta_0\,, \Phi_{\alpha_0}(\beta_0)\,\rangle dt.$$

After integrating by parts we get:

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha_s)\Big|_{s=0} = -\int_0^1 \langle\, \beta_0\,, D_tD_t\beta_0 + \Phi_{\alpha_0}(\beta_0)\,\rangle dt.$$

◇

Remark 14: Taking into account Proposition and Theorem 12, that is $\nabla = D_t$ and $\phi_\alpha(\beta) = R(\beta,\alpha)\alpha$, we can recognize in equation (3.14) the expresion in the left hand side of Jacobi equation, so that we can write

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha)\Big|_{s=0} = -\int_0^1 \langle\, \beta_0\,, \nabla\nabla\beta_0 + \Phi_{\alpha_0}(\beta_0)\,\rangle dt.$$

◇

Therefore, as in the case of standard Riemannian geometry, we have connected the concept of Jacobi sections with the problem of minimizing the energy functional. This is considered in the next section.

3.5 Conjugate points and minimizing properties

In this Section we will study the minimizing properties of the geodesics, using the second variation of the energy functional, in terms of the conjugate points. More exactly we will generalize to Lie algebroids the result we have for the tangent fibre bundle, that a geodesic, an integral curve on the base of a geodesic spray has no conjugate points along it if and only if its integral curve is a weak minimum for the energy functional.

Definition 29: *Let $\alpha : [0,1] \to A$ be a geodesic. The points $\alpha(0), \alpha(1)$ are said to be conjugated along α if there exists a α-Jacobi section β such that $\beta(0) = \beta(1) = 0$.*

We shall need a slight generalization of the formulas for the second variation. Up to now, we have considered only smooth sections. In order to ensure differentiability of the energy functional in the topology of $\mathcal{P}(A, [0,1])^{m_1}_{m_0}$ (we recall that it is a foliated Banach submanifold of the the manifold $\mathcal{A}(A, [0,1])^{m_1}_{m_0}$ of admissible C^1 curves with base C^2) we need to generalize our results to piecewise smooth sections.

Let α be a geodesic over the base curve γ, and consider the set of continuous sections of A along γ which are smooth on $[t_i, t_{i+1}]$ for $0 < t_1 < \cdots < t_k = 1$, and which vanish at $t = 0$ and $t = 1$. Then taking care about the computations at the points t_i one can find that the second differential of the energy functional at a geodesic (i.e. a critical point) α is given by

$$\mathcal{E}''(\alpha)(\Xi_\alpha \beta_1, \Xi_\alpha \beta_2) = -\int_0^1 \langle\, \beta_2\,, D_t D_t \beta_1 + R(\beta_1, \alpha)\alpha \,\rangle dt + \\ + \sum_{i=0}^{k} \langle\, D_t \beta_1(t_{i+1}^-) - D_t \beta_1(t_i^+)\,, \beta_2(t_{i+1}) \,\rangle.$$

We will not prove this result, and we will take this expression as the definition of a bilinear form that will denoted $d^2\mathcal{E}(\alpha)(\beta_1, \beta_2)$. In other words, we define

$$d^2\mathcal{E}(\alpha)(\beta_1, \beta_2) := -\int_0^1 \langle\, \beta_2\,, D_t D_t \beta_1 + R(\beta_1, \alpha)\alpha \,\rangle dt + \\ + \sum_{i=0}^{k} \langle\, D_t \beta_1(t_{i+1}^-) - D_t \beta_1(t_i^+)\,, \beta_2(t_{i+1}) \,\rangle.$$

It should be clear that $d^2\mathcal{E}$ is continuous in the given topology, and that for a smooth homotopy we have

$$\frac{d^2}{ds^2}\mathcal{E}(\alpha_s)\Big|_{s=0} = d^2\mathcal{E}(\alpha)(\beta_0, \beta_0).$$

This will be important later on, since we will approximate piecewise smooth sections by smooth ones.

Theorem 13: *Let $\alpha : [0,1] \to A$ be a geodesic of a Riemannian Lie algebroid A. If α minimizes the energy functional, then α does not have conjugated points.*

Proof. By contradiction, suppose that $0, t_1$ are conjugated values along α, $0 < t_1 < 1$. Then, there exists a $\alpha|_{[0,t_1]}$-Jacobi section $\beta \neq 0, \beta(0) = 0, \beta(t_1) = 0$.

Let $0 < t_0 < t_1 < t_2 < 1$ and let ζ be any $\alpha-$section such that:

$$\zeta_{[0,t_0]} = 0, \zeta_{[t_2,a]} = 0, \zeta(t_1) = -(D_t\beta)(t_1).$$

Let $\tilde{\beta}$ a continuous piecewise smooth α-section given by the conditions $\tilde{\beta}|_{[0,t_1]} = \beta|_{[0,t_1]}$, and $\tilde{\beta}|_{[t_1,a]} = 0$.

Then, for any $\eta \in \mathbb{R}$, the curve $V = \tilde{\beta} + \eta\zeta$ is a continuous piecewise smooth α-section, which is not differentiable at t_1 and it vanishes at 0 and 1. From bilinearity we have

$$d^2\mathcal{E}(\alpha)(V,V) = d^2\mathcal{E}(\alpha)(\tilde{\beta},\tilde{\beta}) + 2\eta d^2\mathcal{E}(\alpha)(\tilde{\beta},\zeta) + \eta^2 d^2\mathcal{E}(\alpha)(\zeta,\zeta).$$

We now calculate each term in this expression.

On one hand

$$\begin{aligned}
d^2\mathcal{E}(\alpha)(\tilde{\beta},\zeta) &= \int_0^a \langle D_t\tilde{\beta}\,, D_t\zeta\,\rangle dt - \int_0^a \langle\, R(\tilde{\beta},\alpha)\alpha\,,\zeta\,\rangle dt \\
&= \int_0^{t_1} \langle D_t\tilde{\beta}\,, D_t\zeta\,\rangle dt + \int_{t_1}^a \langle D_t\tilde{\beta}\,, D_t\zeta\,\rangle dt + \\
&\qquad - \int_0^{t_1} \langle\, R(\tilde{\beta},\alpha)\alpha\,,\zeta\,\rangle dt - \int_{t_1}^a \langle\, R(\tilde{\beta},\alpha)\alpha\,,\zeta\,\rangle dt \\
&= \int_0^{t_1} (D_t\langle\, D_t\beta\,,\zeta\,\rangle - \langle\, D_tD_t\beta\,,\zeta\,\rangle)dt + \\
&\qquad + \int_{t_1}^a (D_t\langle\, D_t\tilde{\beta}\,,\zeta\,\rangle - \langle\, D_tD_t\tilde{\beta}\,,\zeta\,\rangle)dt + \\
&\qquad - \int_0^{t_1} \langle\, R(\beta,\alpha)\alpha\,,\zeta\,\rangle dt - \int_{t_1}^a \langle\, R(\tilde{\beta},\alpha)\alpha\,,\zeta\,\rangle dt \\
&= \langle\, (D_t\beta)(t_1^-)\,,\zeta(t_1)\,\rangle - \langle\, (D_t\beta)(0^+)\,,\zeta(0)\,\rangle + \\
&\qquad + \langle\, (D_t\tilde{\beta})(a^-)\,,\zeta(a)\,\rangle - \langle\, (D_t\tilde{\beta}(t_1^+)\,,\zeta(t_1)\,\rangle + \\
&\qquad - \int_{t_1}^a \langle\, D_tD_t0 + R(0,\alpha)\alpha\,,\zeta\,\rangle dt + \\
&\qquad - \int_0^{t_1} \langle\, D_tD_t\beta + R(\beta,\alpha)\alpha\,,\zeta\,\rangle dt \\
&= \langle\, (D_t\beta)(t_1)\,, -D_t\beta(t_1)\,\rangle \\
&= -|D_t\beta(t_1)|^2 < 0
\end{aligned}$$

On the other

$$
\begin{aligned}
d^2\mathcal{E}(\alpha)(\tilde\beta,\tilde\beta) &= \int_0^1 \langle\, D_t\tilde\beta \,, D_t\tilde\beta \,\rangle dt - \int_0^1 \langle\, \tilde\beta \,, R(\tilde\beta,\alpha)\alpha \,\rangle dt \\
&= \int_0^{t_1} \langle\, D_t\beta \,, D_t\beta \,\rangle dt + \int_{t_1}^{a} \langle\, D_t\tilde\beta \,, D_t\tilde\beta \,\rangle dt + \\
&\qquad - \int_0^{t_1} \langle\, \beta \,, R(\beta,\alpha)\alpha \,\rangle dt - \int_{t_1}^{a} \langle\, 0 \,, R(0,\alpha)\alpha \,\rangle dt \\
&= \int_0^{t_1} (D_t\langle\, \beta \,, D_t\beta \,\rangle - \langle\, \beta \,, D_tD_t\beta \,\rangle) dt + \\
&\qquad + \int_{t_1}^{a} (D_t\langle\, \tilde\beta \,, D_t\tilde\beta \,\rangle - < \langle\, \tilde\beta \,, D_tD_t\tilde\beta \,\rangle) dt + \\
&\qquad - \int_0^{t_1} \langle\, \beta \,, R(\beta,\alpha)\beta \,\rangle dt - 0 \\
&= \langle\, \beta \,, D_t\beta \,\rangle \Big|_0^{t_1} - \int_0^{t_1} \langle\, \beta \,, D_tD_t\beta - R(\beta,\alpha)\alpha \,\rangle dt + \langle\, \tilde\beta \,, D_t\tilde\beta \,\rangle \Big|_{t_1}^{a} - 0 \\
&= 0.
\end{aligned}
$$

Therefore

$$
d^2\mathcal{E}(\alpha)(V,V) = \eta^2 d^2\mathcal{E}(\alpha)(\zeta,\zeta) - 2\eta |D_t\beta(t_1)|^2,
$$

from where it follows that $d^2\mathcal{E}(\alpha)(V,V) < 0$ for $\eta > 0$ small enough.

As $d^2\mathcal{E}(\alpha)$ is continuous, we can approximate $\tilde\beta + \eta\zeta$ by a smooth α-section $\tilde V$, vanishing at 0 and 1, and such that $d^2\mathcal{E}(\alpha)(\tilde V,\tilde V) < 0$.

Let $\phi = \alpha(s,t)dt + \beta(s,t)ds$ be a homotopy, defined on some interval $(-\epsilon,\epsilon)\times[0,1]$, such that $\alpha_0(t) \equiv \alpha(t)$ and $\beta(0,t) = \tilde V$. Taking into account that $\alpha \equiv \alpha_0$ is a critical point of $\mathcal{E}$ we have the Taylor expansion

$$
\mathcal{E}(\alpha_s) = \mathcal{E}(\alpha) + \frac{s^2}{2} d^2\mathcal{E}(\alpha)(\tilde V,\tilde V) + s^3 h(s),
$$

where $h(s) = \int_0^1 \frac{(1-u)^2}{2} \frac{d^3}{du^3}\mathcal{E}(\alpha_u)\Big|_{v=us} du.$

It follows that $\mathcal{E}(\alpha_s) < \mathcal{E}(\alpha)$ for all $s \neq 0$ sufficiently small, which contradicts the hypothesis. $\square$

From now on we will consider a local basis of sections of A along the base curve γ of a geodesic α defined as follows. We consider an orthonormal basis of $A_{\gamma(0)}$ and we extend it to an orthonormal basis of $A_{\gamma(t)}$ by parallel transport. Then we get an orthonormal basis of the module of sections of A along γ. We recall that a section σ along γ is parallel (with respect to α) if $D_t\sigma = 0$ for all $t \in [0,1]$. This is a linear differential equation for σ which has a global unique solution once fixed the initial condition for $\sigma(0)$. Therefore the above procedure is well defined, and produces a basis $\{e_\alpha\}$. Moreover, such a basis is orthonormal because

$$
D_t\langle\, e_\alpha \,, e_\beta \,\rangle = \langle\, D_te_\alpha \,, e_\beta \,\rangle + \langle\, e_\alpha \,, D_te_\beta \,\rangle = 0,
$$

so that $\langle\, e_\alpha \,, e_\beta \,\rangle$ is constant, and equal to the initial value $\delta_{\alpha,\beta}$.

In such a basis, if $\sigma = \sigma^\alpha(t)e_\alpha$ then $D_t\sigma = \dot\sigma^\alpha(t)e_\alpha$, so that D_t reduces to the usual derivative. The Jacobi endomorphism will be represented by a matrix $\Phi(t)$ so that $R(\sigma,\alpha)\alpha = \Phi^\alpha_\beta(t)\sigma^\beta(t)e_\alpha$.

From now on we fix one of such basis and we identify a section β with the tuple of its components, which will be denoted by the same symbol. With this notation, a section β is a Jacobi section if and only if $\ddot\beta + \Phi\beta = 0$. The second variation along a smooth section takes the expression

$$\mathcal{I}(\beta_1,\beta_2) \equiv d^2\mathcal{E}(\alpha(t))(\beta_1,\beta_2) = \int_0^1 [\dot\beta_1(t)^T\dot\beta_2(t) - \beta_1(t)^T\Phi(t)\beta_2(t)]dt.$$

With this convention we can now proceed as in the standard case.

Theorem 14: *Let $\alpha : [0,1] \to A$ be a geodesic with $\alpha(0) = p$, $\alpha(1) = q$. Then, p has no conjugate points along α if and only if the quadratic form*

$$\mathcal{I}(\beta,\beta) = \int_0^1 [\dot\beta(t)^T\dot\beta(t) + \beta(t)^T\Phi\beta(t)]dt,$$

is positive definite on the vector space Σ_0 of the C^1 functions $\beta : [0,1] \to \mathbb{R}^m$ such that $\beta(0) = \beta(1) = 0$.

Proof. [$\Leftarrow$] In order to demonstrate this implication we suppose that there exists a $t_0 \in (0,1]$ such that $\alpha_0(t_0)$ is conjugate to p along α and β is a Jacobi α-section such that $\beta(0) = \beta(t_0) = 0$, β not identically zero.

If we define $\overline\beta$ to be the α-section such that

$$\overline\beta = \begin{cases} \beta(t) & t \in [0,t_0] \\ 0 & t \in [t_0,1] \end{cases}$$

then $\bar\beta(t)$ is a α-section, not identically zero, $\bar\beta \in \Sigma_0$.

For this $\bar\beta$, which is derivable on $[0,t_0]$ and on $[t_0,1]$, but whose derivatives will not be the same in t_0, so we'll have to integrate it by parts, we have that:

$$\begin{aligned}\int_0^1 [\dot{\overline\beta}^T\dot{\overline\beta} - \overline\beta^T\Phi\overline\beta]dt &= \\ &= \int_0^{t_0} [\dot\beta^T\dot\beta - \beta^T\Phi_\alpha\beta]dt + \int_{t_0}^1 0dt \\ &= 0 - \int_0^{t_0} [\ddot\beta + \Phi\beta]^T\beta dt + 0 = 0,\end{aligned}$$

as β is a Jacobi $\alpha|_{[0,t_0]}$-section, that vanishes in t_0.

So there exists an α-section, $\bar\beta(t)$, not identically zero, such that $\int_0^1[\dot{\overline\beta}^T\dot{\overline\beta} - \overline\beta^T\Phi\overline\beta]dt = 0$, so the functional is not positive defined, statement that contradicts the hypotesis.

[$\Rightarrow$] For the direct implication, we start by observing that for any differentiable symmetric matrix W and for all $\beta \in \Sigma_0$ it takes place:

$$\begin{aligned} 0 &= \int_0^1 \frac{d}{dt}(\beta^T W \beta) dt \\ &= \int_0^1 (\beta^T \dot{W} \beta + \beta^T W \dot{\beta} + \dot{\beta}^T W \beta) dt \\ &= \int_0^1 (\beta^T \dot{W} \beta) dt + 2 \int_0^1 (\dot{\beta}^T W \beta) dt, \end{aligned}$$

so, adding this term to $\mathcal{I}(\beta, \beta)$, won't change the value of it:

$$\int_0^1 [\dot{\beta}^T \dot{\beta} - \beta^T \Phi \beta + \beta^T \dot{W} \beta + 2\dot{\beta}^T W \beta] dt. \tag{2}$$

Showing that $\mathcal{I}$ is positive definite is equivalent to showing that the above expression (2) is positive definite.

An idea would be writing the integrand of (2) as a perfect square (step a) and show it can never be zero (step b). One detail stays in the fact that the integrand of (2) can be written as a perfect square if the matrix W can be chosen such that

$$-\Phi_{\alpha_0} + \dot{W} = W^2 \tag{3}$$

that is, we need W to be a solution of this Ricatti equation.

We will concentrate on weather this equation has or not an useful solution later (step c).

(a) Now, let us see that indeed, if we use (3) what becomes of the integrand of (2) can be written as a perfect square. Substituting, we get:

$$\beta^T \Phi \beta + \beta^T \dot{W} \beta + \dot{\beta}^T \dot{\beta} + 2\dot{\beta}^T W \beta \tag{4}$$

which can be written as a perfect square:

$$(\dot{\beta} + W\beta)^T (\dot{\beta} + W\beta).$$

(b) Moreover we must show that $\dot{\beta} + W\beta$ can not vanish on $(0, 1]$, unless β is identically zero.

It is obvious that β identically zero is a solution of the equation $\dot{\beta} + W\beta = 0$. As β has to fulfill the boundary condition $\beta(0) = 0$ as $\beta \in \Sigma_0$, then by the uniqueness theorem for first order differential equations, we have that the unique solution of $\dot{\beta} + W\beta = 0$ that satisfy the mentioned boundary condition is β identically zero.

(c) All we need now to show in order to conclude the demonstration is that the Ricatti equation (3) has a solution W defined on $(0, 1]$ such that with this choice of W we have that the functional $\mathcal{I}$ is positive definite.

In order to show this, we'll use the hypothesis that α_0 contains no conjugate points to p along $\alpha_0|_{(0,1]}$.

The equation (3), a first order differential equation quadratic in the unknown function, can be reduced to a linear equation of second order by making the following change of variables $W = -\dot{U}U^{-1}$, where U is the new unknown invertible matrix. Then the equation turns into:

$$-\ddot{U} - \Phi U = 0. \tag{5}$$

(we need at least a solution of this equation with determinant different than zero on (0,1])

The solution of the equation above with the initial condition $U(0) = O_n$, $\dot{U}(0) = I_n$, is a solution whose determinant does not vanish on $(0, 1]$, due to the fact that α contains no conjugate point to $\alpha(0)$ along it.

This solution of the Ricatti equation, W converted the functional into perfect square in the way described, from where we get the wanted conclusion. $\square$

Chapter 4

Applications in Physics: Virial Theorem

4.1 Introduction

The virial theorem was originally introduced by Clausius in 1870 [19] in the field of the classical statistical mechanics, but soon it became more and more important in many other different branches of physics. Nowadays the virial theorem has a wide range of applicability, as it is applicable to dynamical and thermodynamical systems, it can also be formulated to deal with relativistic (in the sense of special relativity) systems, and it is also applicable to systems with velocity dependent forces and for viscous systems. Even if it provides less information than the equations of motion, it is simpler to apply and can provide information concerning systems whose complete analysis may defy description. For instance, in astronomy, the virial theorem finds applications in the theory of dust and gas of interstellar space as well as cosmological considerations of the universe as a whole and in other discussions concerning the stability of clusters of galaxies. For an excellent historical account one can see [20].

In the particular case of the motion of a particle of mass m under the action of a force F we can introduce a (virial) function $G(\mathbf{x}, \mathbf{v}) = m\,\mathbf{x} \cdot \mathbf{v}$, and using the Newton second law, one easily sees that when either the motion is periodic of period T, or when the possible values of the function G remain bounded, the limit of T going to infinity of the time average over the time interval T of G is zero:

$$\langle\!\langle G \rangle\!\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{G}\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T [m\,\mathbf{v} \cdot \mathbf{v} + \mathbf{x} \cdot \mathbf{F}] = 0,$$

and as a consequence he obtained the following relation, where E_c is the kinetic energy, $E_c = \frac{1}{2}\mathbf{v} \cdot \mathbf{v}$,

$$\langle\!\langle 2\, E_c(\mathbf{v}) + \mathbf{x} \cdot \mathbf{F} \rangle\!\rangle = 0.$$

Then, for the particular case of a conservative force, i.e. when there exists a potential function $V(\mathbf{x})$ such that $\mathbf{F} = -\boldsymbol{\nabla} V$:

$$\langle\!\langle 2\, E_c(\mathbf{v}) - \mathbf{x} \cdot \nabla V \rangle\!\rangle = 0.$$

Moreover, when the potential V is homogenous of degree k, Euler's theorem for homogenous functions establishes that $\mathbf{x} \cdot \nabla V = k\, V$, and then the values for the time averages of the kinetic and the potential energies can be written in terms of the total energy, E, as follows:

$$\langle\!\langle E_c(\mathbf{v})\rangle\!\rangle = \frac{k\,E}{k+2} \qquad \text{and} \qquad \langle\!\langle V(\mathbf{x})\rangle\!\rangle = \frac{2E}{k+2}.$$

When non-holonomic constraints are present, the forces are $\mathbf{F} = -\nabla V + \mathbf{F}_c$, where $\mathbf{F}_c$ are the constraint forces due to the nonholonomic constraints. In this case, to be considered in section 4.4, a similar calculation shows that:

$$\langle\!\langle 2\,E_c(\mathbf{v}) - \mathbf{x}\cdot\nabla V + \mathbf{x}\cdot\mathbf{F}_c\rangle\!\rangle = 0.$$

Assuming ideal constraints, so that the energy E is conserved, and in the particular case of a homogeneous potential of degree k we get:

$$\begin{aligned}
\langle\!\langle E_c(\mathbf{v})\rangle\!\rangle &= \frac{k}{k+2}E - \frac{1}{k+2}\langle\!\langle \mathbf{F}_c\cdot\mathbf{x}\rangle\!\rangle \\
\langle\!\langle V(\mathbf{x})\rangle\!\rangle &= \frac{2}{k+2}E + \frac{1}{k+2}\langle\!\langle \mathbf{F}_c\cdot\mathbf{x}\rangle\!\rangle.
\end{aligned}$$

It has recently been shown in [7] that there exist geometric virial-like theorems that are generalizations for systems with a configuration space different from $\mathbb{R}^n$. These generalizations are based on the use of symplectic formalism as an approach both in the Hamiltonian and the regular Lagrangian case. More specifically it was proved that if (M,ω) is a symplectic manifold, and X_F denotes the Hamiltonian vector field defined by the function F on M, i.e. the one such that $i_{X_F}\omega = dF$, then for a function G remaining bounded in its time evolution, using the property that the flow of the Hamiltonian vector field X_H on the symmetric manifold commutes with the action of X_H, one easily proves that:

$$\langle\!\langle X_G H\rangle\!\rangle = 0, \tag{4.1}$$

where, as indicated before, by the time average we mean:

$$\langle\!\langle F\rangle\!\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T F(\gamma(t))\,dt,$$

where γ is the evolution curve. In particular, when the motion of the dynamical system is periodic with period τ the time average reduces to:

$$\langle\!\langle F\rangle\!\rangle = \frac{1}{\tau}\int_0^\tau F(\gamma(t))\,dt.$$

It is also remarkable that if $\langle\!\langle F_1\rangle\!\rangle$ and $\langle\!\langle F_2\rangle\!\rangle$ do exist, then $\langle\!\langle F_1+F_2\rangle\!\rangle$ also exists and $\langle\!\langle F_1+F_2\rangle\!\rangle = \langle\!\langle F_1\rangle\!\rangle + \langle\!\langle F_2\rangle\!\rangle$.

When the configuration space is $\mathbb{R}^3$, the phase space is the cotangent bundle $T^*\mathbb{R}^3$ which is endowed with its canonical symplectic structure ω_0. Then, for the Clausius function in terms of Cartesian coordinates, $G(\mathbf{x},\mathbf{p}) = \mathbf{x}\cdot\mathbf{p}$, the Hamiltonian vector field X_G is given by

$$X_G = x^i\frac{\partial}{\partial x^i} - p_i\frac{\partial}{\partial p_i}$$

and represents the canonical lift to $T^*\mathbb{R}^3$ of the generator of dilatations in $\mathbb{R}^3$, i.e. of the vector field in $\mathbb{R}^3$,

$$D = x^i \frac{\partial}{\partial x^i}.$$

When the Hamiltonian is natural, i.e. of the form:

$$H(\mathbf{x}, \mathbf{p}) = H_0(\mathbf{p}) + V(\mathbf{x}) = \frac{\mathbf{p}\cdot\mathbf{p}}{2} + V(\mathbf{x}),$$

and the function G remains bounded in its time evolution, the above virial relation reduces to the usual one $\langle\!\langle 2H_0 - \mathbf{x}\cdot\nabla V\rangle\!\rangle = 0$, therefore the standard virial theorem appears as a particular case of the more general property stated for Hamiltonian systems in symplectic manifolds.

For the particular case of Hamiltonian systems, a regular Lagrangian system (TQ, ω_L, E_L), the virial relation for an appropriate function $G \in C^\infty(TQ)$ is:

$$\langle\!\langle X_G(E_L)\rangle\!\rangle = 0. \tag{4.2}$$

When the Lagrangian function is of the mechanical type, $L(\mathbf{x}, \mathbf{v}) = \frac{1}{2} m\, \mathbf{v}\cdot\mathbf{v} - V(\mathbf{x})$, the energy and the symplectic form corresponding to L are respectively given by:

$$E_L(\mathbf{x}, \mathbf{v}) = \frac{1}{2} m\, \mathbf{v}\cdot\mathbf{v} + V(\mathbf{x}), \qquad \omega_L = \sum_{i=1}^n m\, dx^i \wedge dv^i,$$

and then if $G(\mathbf{x}, \mathbf{v}) = m\, \mathbf{x}\cdot\mathbf{v}$ the corresponding Hamiltonian vector field is

$$X_G = x^i \frac{\partial}{\partial x^i} - v^i \frac{\partial}{\partial v^i},$$

from which we recover the original virial theorem, because:

$$X_G(E_L)(\mathbf{x}, \mathbf{v}) = \mathbf{x}\cdot\boldsymbol{\nabla} V(\mathbf{x}) - m\, \mathbf{v}\cdot\mathbf{v}.$$

The geometric version of the virial theorem for Hamiltonian systems given in [7] can be expressed in terms of the Poisson bracket. On a Poisson manifold $(M', \{\cdot\,, \cdot\})$, every function $H \in C^\infty(M')$ defines a dynamical system by $\dot{x} = X_H(x) = \{x\,, H\}$, i.e. $X_H = \{\cdot\,, H\}$. In the case of a Hamiltonian system defined in a symplectic manifold, the Poisson bracket of two functions F_1, F_2 on the symplectic manifold is defined as $\{F_1, F_2\} = \omega(X_{F_1}, X_{F_2})$. For a virial function $G \in C^\infty(M)$, as $X_H G = \{G, H\}$ the virial theorem (4.1) states that the time average of the Poisson bracket $\{G, H\}$ vanishes:

$$\langle\!\langle \{G, H\}\rangle\!\rangle = 0.$$

This relation is simply (4.1) in the case of a Hamiltonian dynamical system, and it becomes (4.2) in the Lagrangian case where the Poisson bracket is given by $\{F, G\} = \omega_L(X_F, X_G)$.

In this chapter leaving from the geometric approach for the generalized virial theorem given in [7], we pretend to both particularize some of the results obtained there, for standard Lagrangians (i.e. of a the mechanical type), and to generalize them first to mechanical systems on Lie algebroids, and then to nonholonomic systems both on the tangent bundle and on Lie algebroids. Moreover it will be made use of quasi-coordinates to state in these instances the virial theorem.

In Section 4.2 we will restrict to the Lagrangian formalism, and write intrinsically and in local coordinates the virial theorem for a Lagrangian system of mechanical type on a Riemannian manifold. An important case we study is that of an affine virial function associated to a vector field on the configuration manifold. The special cases of a virial function associated to a Killing, a homothetic and a conformal Killing vector field are considered and the corresponding virial theorems are established for this type of functions.

Then, in Section 4.3 the geometric approach to the virial theorem developed in [7] is written in terms of quasi-velocities in the Lagrangian case (see [14]), and respectively in quasi-momenta in the Hamiltonian case.

In Section 4.4 we approach the virial theorem for nonholonomic mechanical systems, using the Lagrange multipliers method and afterwards the distributional method for the description of this kind of systems. The second one permits us to create a similarity between the holonomic and the nonholonomic case, as it will make possible the writing of the virial theorem using a nonholonomic bracket. In the end it will be given a description of the main results using quasi-velocities.

Afterwards, the extension of similar results to the framework of mechanics in Lie algebroids is made in Section 4.5, and to nonholonomic mechanical systems on Lie algebroids, in Section 4.6.

4.2 Virial Theorem for Mechanical Lagrangians on TQ

In Subsection 4.2.1 the form of mechanical Lagrangians is reminded. Then, the virial theorem for Lagrangian systems of mechanical type is presented, in Subsection 4.2.2 both in intrinsic form and in terms of local coordinates. As a particular case, a special virial function G, and then a spherical geometry problem, are used in order to show some examples.

In Subsection 4.2.3 we consider a particularly important case of an affine on the velocities virial function, as associated to a vector field on the configuration manifold. The cases where the vector field is either a Killing, a homothetic or a conformal Killing vector field, are considered in Subection 4.2.4. Several examples are used to illustrate the theory.

4.2.1 Mechanical Lagrangians

We can define Lagrangians of mechanical type for systems with configuration space Q, $L \in C^\infty(TQ)$, by choosing a (pseudo-)Riemann structure g on Q and a potential function $V \in C^\infty(Q)$ as follows:

$$L_{g,V}(q,v) = \frac{1}{2}\, g_q(v,v) - (\tau_Q^* V)(q,v) = \frac{1}{2}\, g_q(v,v) - \tilde{V}(q), \tag{4.3}$$

i.e. the Lagrangian function is of the form $L_{g,V} = T_g - \tau_Q^* V$, where the function $T_g \in C^\infty(TQ)$ represents the kinetic energy for the Riemannian metric g given above, which can be rewritten as:

$$T_g = \frac{1}{2}\, g(T\tau_Q \circ D, T\tau_Q \circ D),$$

with D being any second order differential equation vector field, i.e. a vector field on TQ such that $\tau_{TQ} \circ D = \mathrm{id}_{TQ}$, while the potential energy $\widetilde{V} = \tau_Q^* V$ is a basic function, i.e. the pull-back of a smooth function V on the base manifold Q.

Given a Riemann structure g on a manifold Q with local expression in a local chart (2.14), the expression for the corresponding free (i.e. $V = 0$) Lagrangian, i.e. the function T_g is:

$$T_g(q, v) = \frac{1}{2}\, g_{ij}(q)\, v^i v^j, \tag{4.4}$$

while the coordinate expression of an arbitrary second order vector field is:

$$D(q, v) = v^i\, \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}\,. \tag{4.5}$$

4.2.2 Virial theorem for Mechanical Lagrangians

The energy of a Lagrangian system is defined by $E_L = \Delta L - L$, where Δ is the Liouville vector field, generator of dilations along the fibres.

If the Lagrangian is of a mechanical type $L_{g,V}$, then, as $\Delta(T_g) = 2\,T_g$ and $\Delta(V) = 0$, the total energy of a Lagrangian system of mechanical type is $E_L = T_g + V$. The coordinate expression of the Cartan 1-form $\theta_L = dL \circ S$ for such Lagrangian (4.3) is given by:

$$\theta_L(q, v) = g_{ij}(q)\, v^j\, dq^i$$

and the symplectic form $\omega_L = -d\theta_L$ by:

$$\omega_L = g_{ij}\, dq^i \wedge dv^j + \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial q^k} v^j - \frac{\partial g_{kj}}{\partial q^i} v^j \right) dq^i \wedge dq^k.$$

The dynamics is then given by the dynamical vector field Γ_L defined for a regular Lagrangian L by:

$$i_{\Gamma_L} \omega_L = dE_L, \tag{4.6}$$

and the solution of such dynamical equation (4.6) turns out to be

$$\Gamma_L(q, v) = v^i \frac{\partial}{\partial q^i} - \left(\Gamma^i_{jk}(q) v^j v^k + g^{ij}(q) \frac{\partial V}{\partial q^j}(q) \right) \frac{\partial}{\partial v^i},$$

where Γ^i_{jk} are the Christoffel symbols of the second kind with respect to the Levi-Civita connection defined by the metric g, as given by (2.17).

The Hamiltonian vector field of a smooth function G on TQ is determined by the equation $i_{X_G}\omega_L = dG$ and in local coordinates is given by

$$\begin{aligned} X_G(q,v) &= g^{ij}(q)\frac{\partial G}{\partial v^j}(q,v)\frac{\partial}{\partial q^i} \\ &+ g^{ik}(q)\left[\left(\frac{\partial g_{ln}}{\partial q^k}(q)\,v^n - \frac{\partial g_{kn}}{\partial q^l}(q)\,v^n\right) g^{lj}(q)\frac{\partial G}{\partial v^j}(q,v) - \frac{\partial G}{\partial q^k}(q,v)\right]\frac{\partial}{\partial v^i}. \end{aligned} \tag{4.7}$$

Since the total energy of the system is $E_L = T + V$, then,

$$X_G(E_L) = -\Gamma_L(G) = \frac{\partial G}{\partial v^l}\left(\Gamma^l_{jk} v^j v^k + g^{il}\frac{\partial V}{\partial q^i}\right) - \frac{\partial G}{\partial q^k} v^k. \tag{4.8}$$

The *virial theorem*, $\langle\langle X_G(E_L)\rangle\rangle = 0$ (see e.g. [7] for a geometric approach), establishes the following relation of time average:

$$\langle\langle \frac{\partial G}{\partial v^l}\left(\Gamma^l_{jk} v^j v^k + g^{il}\frac{\partial V}{\partial q^i}\right) - \frac{\partial G}{\partial q^k} v^k \rangle\rangle = 0.$$

We will see that the preceding expression is much simpler when the vector field X_G is a complete lift.

From the expression $\Gamma_L(G) = X^c(L)$, evaluating on the time evolution and averaging on the interval $[0,T]$, in the limit when $T \to \infty$, we get as we did in [7] in an analogous case, the *virial theorem* stating that if G remains bounded, then:

$$\langle\langle X^c(L)\rangle\rangle = 0 \Longleftrightarrow \langle\langle X^c(T_g) - X(V)\rangle\rangle = 0,$$

whose local coordinate expression is

$$\langle\langle X^k \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k} v^i v^j + \frac{\partial X^k}{\partial q^l} g_{kj} v^l v^j - X^k \frac{\partial V}{\partial q^k}\rangle\rangle = 0. \tag{4.9}$$

A particular case studied in [7], is when there exists a nonzero real number a such that $X^c L = a\,L$, and then we recover the result $\langle\langle L\rangle\rangle = 0$, i.e. $\langle\langle T - V\rangle\rangle = 0$.

Example 3 (Spherical geometry): Consider as an illustrative example the motion of a unity mass point on a sphere of radius $R = 1/\sqrt{\lambda}$ centred at the origin and the usual spherical polar coordinates, i.e. a point P on the sphere is fixed by two coordinates (θ, ϕ) such that

$$\mathbf{x}(\theta,\phi) = (R\,\sin\theta\,\cos\phi, R\,\sin\theta\,\sin\phi, R\,\cos\theta),$$

and then

$$g_{\theta\theta} = R^2, \quad g_{\theta\phi} = 0, \quad g_{\phi\phi} = R^2\,\sin^2\theta,$$

i.e. the arc-length is

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{4.10}$$

Suppose that the motion is under the action described by a potential function $V(\theta)$ that does not depend on ϕ but only on the distance to the North pole. Then, if X is the vector field on the base $X = \tan\theta\,\partial/\partial\theta$, with complete lift

$$X^c = \tan\theta\frac{\partial}{\partial\theta} + \sec^2\theta\, v_\theta\frac{\partial}{\partial v_\theta},$$

as the kinetic energy is $T = \frac{1}{2}R^2(v_\theta^2 + \sin^2\theta\, v_\phi^2)$ and

$$X^c(T) = R^2(\sec^2\theta\, v_\theta^2 + \sin^2\theta\, v_\phi^2), \qquad X(V) = \tan\theta\,\frac{\partial V}{\partial\theta},$$

the Virial Theorem establishes that

$$\langle\!\langle R^2(\sec^2\theta\, v_\theta^2 + \sin^2\theta\, v_\phi^2)\rangle\!\rangle = \langle\!\langle \tan\theta\,\frac{\partial V}{\partial\theta}\rangle\!\rangle.$$

The points of the lower half sphere can be described by the points obtained by central projection onto the tangent plane $x_3 = -R$, i.e. points $(q_1, q_2, -R)$ such that

$$\begin{cases} q_1 &= \dfrac{x_1\,R}{-x_3} = -\dfrac{R^2\,\sin\theta\,\cos\phi}{R\,\cos\theta} = -R\tan\theta\,\cos\phi \\ q_2 &= \dfrac{x_2\,R}{-x_3} = -\dfrac{R^2\,\sin\theta\,\sin\phi}{R\,\cos\theta} = -R\tan\theta\,\sin\phi \end{cases}$$

or eliminating the South pole and using polar coordinates (r, ϕ) centred at $(0, 0, -R)$, i.e. $r = -R\tan\theta$, having in mind that

$$\frac{d\theta}{dr} = -\frac{1}{R}\,\frac{1}{1 + (r/R)^2} = -\frac{1}{R}\,\frac{1}{1 + \lambda\, r^2},$$

the expression of the arc-length becomes

$$ds^2 = \frac{1}{(1 + \lambda r^2)^2}dr^2 + \frac{r^2}{(1 + \lambda r^2)}d\phi^2.$$

In terms of the new coordinates, as $\tan\theta = -r/R$,

$$\sec^2\theta = 1 + \lambda r^2, \qquad \sin^2\theta = \frac{r^2}{R^2}(1 + \lambda r^2)^{-1}, \qquad v_r = R(1 + \lambda r^2)v_\theta$$

and then we can rewrite the preceding equation as

$$\langle\!\langle (1 + \lambda r^2)^{-1}(v_r^2 + r^2 v_\phi^2)\rangle\!\rangle = \langle\!\langle r\,(1 + \lambda r^2)\frac{\partial V}{\partial r}\rangle\!\rangle, \tag{4.11}$$

which coincides with the expression (14) of [55]. However, in [55] such expression was only proved for two special cases and it was proposed as a guess for the general case. ◁

4.2.3 Virial Theorem for Killing vector fields

As mentioned earlier, the Virial Theorem for a given smooth bounded function G is but $\langle\langle X_G(E_L)\rangle\rangle = 0$, which for systems of mechanical type reduces to $\langle\langle X_G(T) + X_G(V)\rangle\rangle = 0$. A particularly simple case would be when X_G is a complete lift and this property constraints the possible form of G.

Note first that looking at expression (4.7) and asking for the vector field X_G to be τ_Q projectable, a necessary and sufficient condition for this to happen is that $\partial G/\partial v^i$ be a basic function, i.e. G is an affine in velocities function, or in more geometric language, there must be a 1-form $\alpha = \alpha_k(q)\, dq^k$ on Q and a function φ on Q such that

$$G = \widehat{\alpha} + \tau_Q^*\varphi,$$

and then the τ_Q-related vector field is $\widehat{g}^{-1}(\alpha)$.

With this form of G, in order for X_G to be a complete lift, the τ_Q-related vector field must be $\widehat{g}^{-1}(\alpha)$, i.e.: $d\tau_Q(X_G) = \widehat{g}^{-1}(\alpha)$ and the n functions α_k and the function φ on the base manifold must satisfy, for any index i:

$$\frac{\partial}{\partial q^k}\left(g^{ij}\alpha_j\right) v^k = g^{ik}\left[\left(\frac{\partial g_{ln}}{\partial q^k}v^n - \frac{\partial g_{kn}}{\partial q^l}v^n\right) g^{lj}\alpha_j - v^j\frac{\partial \alpha_j}{\partial q^k} - \frac{\partial \varphi}{\partial q^k}\right].$$

These conditions can be rewritten for any pair of indices (i,k), as:

$$\alpha_j\,\frac{\partial g^{ij}}{\partial q^k} + g^{ij}\,\frac{\partial \alpha_j}{\partial q^k} = g^{ij}\,\frac{\partial g_{lk}}{\partial q^j}\,g^{lm}\,\alpha_m - g^{im}\,\frac{\partial g_{mk}}{\partial q^l}\,g^{lj}\,\alpha_j - \frac{\partial \alpha_k}{\partial q^j}\,g^{ij}, \qquad \frac{\partial \varphi}{\partial q^k} = 0,$$

and therefore as follows

$$g^{ij}\left(\frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^j}\right) = \alpha_n\left(-\frac{\partial g^{in}}{\partial q^k} + g^{ij}g^{ln}\frac{\partial g_{lk}}{\partial q^j} - g^{im}g^{ln}\frac{\partial g_{mk}}{\partial q^l}\right), \qquad \frac{\partial \varphi}{\partial q^k} = 0.$$

Using now that

$$\frac{\partial g^{ij}}{\partial q^k} = -g^{il}g^{jm}\frac{\partial g_{lm}}{\partial q^k},$$

the preceding equation becomes

$$g^{ij}\left(\frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^j}\right) = \alpha_n\left(g^{ir}g^{ns}\frac{\partial g_{rs}}{\partial q^k} + g^{ij}g^{ln}\frac{\partial g_{lk}}{\partial q^j} - g^{im}g^{ln}\frac{\partial g_{mk}}{\partial q^l}\right),$$

or equivalently

$$g^{ij}\left(\frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^j}\right) = \alpha_n g^{ij}g^{ln}\left(\frac{\partial g_{jl}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right) = 2\alpha_n g^{ln}\Gamma^i_{lk}.$$

which can be rewritten as

$$\frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^j} = 2\,\alpha_i\,\Gamma^i_{jk},$$

or in other words, for any pair of indices i, k,

$$\left(\frac{\partial \alpha_j}{\partial q^k} - \alpha_i\,\Gamma^i_{jk}\right) + \left(\frac{\partial \alpha_k}{\partial q^j} - \alpha_i\,\Gamma^i_{kj}\right) = 0.$$

Multiplying both sides by Z^jY^k and summing on repeated indices we see that this equation is the coordinate expression of the intrinsic one

$$\langle\nabla_Y\alpha, Z\rangle + \langle\nabla_Z\alpha, Y\rangle = 0, \qquad \forall Y, Z \in \mathfrak{X}(Q),$$

so that the 2-covariant tensor field $\nabla\alpha$ is skew-symmetric. But as $\alpha = \widehat{g}(X)$, the relation (2.18) allows us to express this condition as $g(\nabla_Y X, Z) + g(Z, \nabla_Y X) = 0$, which means that X satisfies the Killing condition (2.22). The preceding result can be summarized in the following proposition whose intrinsic proof is also given:

PROPOSITION 10: *The vector field $X \in \mathfrak{X}(Q)$ is a Killing vector w.r.t. the Riemann structure g iff $X_{\widehat{\alpha}} = X^c$, where $\widehat{\alpha}$ is the linear in the fibers function defined by the 1-form $\alpha = \widehat{g}(X)$.*

PROOF. The linear in the fibers function $G = \langle \theta_{T_g}, X^c\rangle$ is nothing but the function $\widehat{\alpha}$, because:

$$\langle \theta_{T_g}, X^c\rangle = \langle dT_g \circ S, X^c\rangle = \langle dT_g, S(X^c)\rangle = X^{\mathrm{v}}(T_g),$$

where the vector field X^{v} is the vertical lift of X [23, 24], and therefore,

$$\langle \theta_{T_g}, X^c\rangle(v) = \frac{d}{ds} T_g(v + sX(\tau_Q(v))\big|_{s=0} = g(X(\tau_Q(v)), v) = \widehat{\alpha}(v),$$

for every $v \in TQ$.

If the Hamiltonian vector field X_G is the complete lift X^c, then the relation (2.2) shows that $X^c(E_L) = -X^c(L)$, because $X^c(L) = \Gamma_L G = -X_G(E_L) = -X^c(E_L)$. Therefore, $X^c(T_g) - X^c(V) = -X^c(T_g) - X^c(V)$, i.e. $X^c(T_g) = 0$, and then X is a Killing vector.

On the other hand, if X is a Killing vector we have that $T_{\mathcal{L}_X g} = 0$. Since $i_{(X_G - X^c)}\omega_{T_g} = \theta_{T_{\mathcal{L}_X g}} = 0$, then $X_G = X^c$. □

Let X be a Killing vector field, and $\alpha = \widehat{g}(X)$ the associated 1-form. As we have seen, $X_{\widehat{\alpha}} = X^c$, from where we have:

$$\{E_L, \widehat{\alpha}\} = X_{\widehat{\alpha}} E_L = X^c E_L = E_{X^c L} = T_{\mathcal{L}_X g} + \tau_Q^*\left(\mathcal{L}_X V\right) = \tau_Q^*\left(\mathcal{L}_X V\right).$$

Taking mean values we get that for every Killing vector field X:

$$\langle\!\langle \mathcal{L}_X V \rangle\!\rangle = 0.$$

Therefore, if X is not a symmetry of the potential energy then the mean value of the derivative $\mathcal{L}_X V$ vanishes along any trajectory of the Lagrangian dynamical system.

EXAMPLE 4 (Spherical geometry revisited): Coming back to the case of the spherical geometry, we can say that the vector field

$$X = X_\theta \frac{\partial}{\partial\theta} + X_\phi \frac{\partial}{\partial\phi}$$

is a Killing vector field if and only if its complete lift

$$X^c = X_\theta \frac{\partial}{\partial\theta} + X_\phi \frac{\partial}{\partial\phi} + \left(\frac{\partial X_\theta}{\partial\theta} v_\theta + \frac{\partial X_\theta}{\partial\phi} v_\phi\right) \frac{\partial}{\partial v_\theta} + \left(\frac{\partial X_\phi}{\partial\theta} v_\theta + \frac{\partial X_\phi}{\partial\phi} v_\phi\right) \frac{\partial}{\partial v_\phi}$$

is a symmetry of the kinetic energy

$$T(\theta, \phi, v_\theta, v_\phi) = \frac{1}{2}(v_\theta^2 + \sin^2\theta\, v_\phi^2).$$

From the condition

$$\left(\frac{\partial X_\theta}{\partial\theta}\,v_\theta+\frac{\partial X_\theta}{\partial\phi}\,v_\phi\right)v_\theta+\sin^2\theta\left(\frac{\partial X_\phi}{\partial\theta}\,v_\theta+\frac{\partial X_\phi}{\partial\phi}\,v_\phi\right)v_\phi+X_\theta\,\sin\theta\,\cos\theta\,{v_\phi}^2=0,$$

we obtain the conditions:

$$\begin{aligned}&\frac{\partial X_\theta}{\partial\theta}=0,\\&\frac{\partial X_\theta}{\partial\phi}+\sin^2\theta\frac{\partial X_\phi}{\partial\theta}=0\\&\sin\theta\left(\cos\theta\,X_\theta+\sin\theta\frac{\partial X_\phi}{\partial\phi}\right)=0\end{aligned}$$

One solution is given by $X_\theta=0$ and $X_\phi=1$, i.e. the vector field $X_3=\partial/\partial\phi$ is a Killing vector field. Another particular solution is $X_\theta=\cos\phi$ and $X_\phi=-\sin\phi\,\mathrm{cotan}\,\theta$, and then another Killing vector field is

$$X_1=\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\,\mathrm{cotan}\,\theta\frac{\partial}{\partial\phi}.$$

The corresponding virial theorem is

$$\langle\!\langle\mathcal{L}_{X_1}V\rangle\!\rangle=0\Longleftrightarrow\langle\!\langle\cos\phi\frac{\partial V}{\partial\theta}\rangle\!\rangle=\langle\!\langle\sin\phi\,\mathrm{cotan}\,\theta\frac{\partial V}{\partial\phi}\rangle\!\rangle.$$

◁

Example 5 (Periodic Toda lattice with n particles): A periodic Toda lattice system with n particles without impurities (each particle as the same mass m), is defined by a mechanical Lagrangian $L=T-V$ on $T\mathbb{R}^n$. The kinetic energy is the quadratic function defined by the Euclidian metric on $\mathbb{R}^n$,

$$T(q,v)=\frac{1}{2}\sum_{i=1}^{n}m\,v_i^2,$$

and the potential is given by

$$V(q)=\sum_{i=1}^{n}e^{q_i-q_{i+1}},$$

where $q_{n+1}=q_1$. Consider the following vector field, for a fixed $k=1,\dots,n$,

$$X_k=\frac{\partial}{\partial q_k}.$$

The vector field is a Killing vector w.r.t. the Euclidean metric.

Then the Virial Theorem implies that $\langle\!\langle\mathcal{L}_{X_k}V\rangle\!\rangle=\langle\!\langle e^{q_k-q_{k+1}}-e^{q_{k-1}-q_k}\rangle\!\rangle=0$. Therefore, $\langle\!\langle e^{q_k-q_{k+1}}\rangle\!\rangle=\langle\!\langle e^{q_{k-1}-q_k}\rangle\!\rangle$ for every k and hence $\langle\!\langle V\rangle\!\rangle=n\langle\!\langle e^{q_1-q_2}\rangle\!\rangle$.

◁

Example 6 (Kepler problem in polar coordinates): Consider a particle P of mass m moving in a plane under the action of a central force $F(r)=-\gamma\,m\,m'/r^2$ on the direction of a fixed point O of mass $m'\gg m$, where γ is a positive constant and r represents the distance between O and the point particle P. Let ϕ be the angle that the line OP makes with a fixed direction on the plane. In polar coordinates the arc-length is given by $ds^2 = dr^2 + r^2 d\phi^2$. The kinetic energy of the particle is given by

$$T(r, \phi, v_r, v_\phi) = \frac{m}{2}\left(v_r^2 + r^2\, v_\phi^2\right)$$

and the potential is the function $V(r) = -\gamma\, m\, m'/r$. The vector field

$$X = \cos\phi\, \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\, \frac{\partial}{\partial \phi}$$

is a Killing vector field of the Euclidean metric in polar coordinates. Then the Virial Theorem tell us that $\langle\!\langle \mathcal{L}_X V \rangle\!\rangle = 0$, that is, $\langle\!\langle -\cos(\phi)\gamma\, m\, m'/r^2 \rangle\!\rangle = 0$.

$\triangleleft$

4.2.4 Virial Theorem for conformal and homothetic vector fields

Conformal Killing vector fields and in particular homothetic vector fields have also been relevant in many problems in physics and more particularly in space-time geometry (see e.g, [37, 6, 29]). We now explore the information that we can extract from them in the problem of virial theorem we are considering. With this aim we first find the difference between the Hamiltonian vector field $X_{\widehat{\alpha}}$ associated to the 1-form $\alpha = \widehat{g}(X)$, where X is a vector field on Q and X^c the complete lift of X.

Definition 30: *A vector field X on Riemannian manifold (Q, g) is a conformal Killing vector field if there exists a function $f \in C^\infty(Q)$ such that $\mathcal{L}_X g = f\, g$.*

Proposition 11: *If X is the vector field on Q associated to the 1-form α, $\alpha = \widehat{g}(X)$, and as before $\widehat{\alpha} \in C^\infty(TQ)$ is the function $\widehat{\alpha}(v) = g(X(\tau_Q(v)), v)$, for $v \in TQ$, then the difference of the complete lift X^c of X and the Hamiltonian vector field $X_{\widehat{\alpha}}$ associated to $\widehat{\alpha}$ with respect to the symplectic form ω_{T_g} is the vertical vector field whose contraction with the symplectic form ω_{T_g} is the semi-basic 1-form $\theta_{T_{\mathcal{L}_X g}}$.*

Proof. Notice first that as both vector fields, X^c and $X_{\widehat{\alpha}}$, are projectable on the vector field $X = \widehat{g}^{-1}(\alpha)$, the difference vector is vertical. Moreover, taking into account the above mentioned relation $\langle \theta_{T_g}, X^c \rangle = \widehat{\alpha}$, we have

$$i_{(X_{\widehat{\alpha}} - X^c)}\omega_{T_g} = i_{X_{\widehat{\alpha}}}\omega_{T_g} - i_{X^c}\omega_{T_g} = d\widehat{\alpha} + i_{X^c} d\theta_{T_g}, \tag{4.12}$$

and then

$$i_{(X_{\widehat{\alpha}} - X^c)}\omega_{T_g} = d(i_{X^c}\theta_{T_g}) + i_{X^c} d\theta_{T_g} = \mathcal{L}_{X^c}\theta_{T_g} = \theta_{X^c T_g} = \theta_{T_{\mathcal{L}_X g}}, \tag{4.13}$$

where the last equality follows from (2.23). $\square$

It is also well known (see e.g. [15]) that contraction with the symplectic forms ω_L defined by a regular Lagrangian L establishes a one-to-one correspondence of vertical vector fields with semi basic 1-forms. More explicitly, in the particular case we are considering of $L = T_g$, the semi basic 1-form corresponding to the Liouville vector field Δ, generating dilation along the fibers of TQ, is $-\theta_{T_g}$ because, as θ_{T_g} is semi-basic,

$$i_\Delta \omega_{T_g} = -i_\Delta d\theta_{T_g} = -\mathcal{L}_\Delta \theta_{T_g},$$

and as θ_{T_g} is homogeneous of degree one in velocities, we find that

$$i_\Delta \omega_{T_g} = -\theta_{T_g}. \tag{4.14}$$

This allows us to write:

$$i_{(X_{\widehat{\alpha}} - X^c)} \omega_{T_g} = -i_\Delta \omega_{T_{\mathcal{L}_X g}}.$$

As a consequence, in the case of a conformal Killing vector field, we have the following result.

THEOREM 15: *A vector field X on the Riemannian manifold (Q, g) is a conformal Killing vector field if and only if $X_{\widehat{\alpha}} = X^c - f\,\Delta$, where α is the 1-form $\alpha = \widehat{g}(X)$.*

PROOF. Indeed, if X is a conformal Killing vector field, there exists a function $f \in C^\infty(Q)$ such that $\mathcal{L}_X g = fg$, and then $\theta_{T_{\mathcal{L}_X g}} = f\,\theta_{T_g}$. The relation (4.13) reduces in this case to $i_{(X_{\widehat{\alpha}} - X^c)} \omega_{T_g} = f\,\theta_{T_g}$, and then using (4.14), to $i_{(X_{\widehat{\alpha}} - X^c)} \omega_{T_g} = -i_{(f\,\Delta)} \omega_{T_g}$. As ω_{T_g} is nondegenerate we find $X_{\widehat{\alpha}} - X^c = -f\,\Delta$.

Conversely, if there exists a function $f \in C^\infty(Q)$ such that $X_{\widehat{\alpha}} - X^c = -f\Delta$, then

$$i_{(X_{\widehat{\alpha}} - X^c)} \omega_{T_g} = -i_{(f\,\Delta)} \omega_{T_g} = f\,\theta_{T_g},$$

and as a consequence of (4.13) we obtain that $\theta_{T_{\mathcal{L}_X g}} = f\,\theta_{T_g}$, which implies $\mathcal{L}_X g = f\,g$ and then X is a conformal Killing vector field. □

This result is in agreement with the meaning of being a conformal Killing vector field: its flow transforms geodesics in re-parameterized geodesics, the responsible for reparametrization is the term $f\,\Delta$. Of course, for $f = 0$ we recover the result of Proposition 1.

Virial Theorem for conformal Killing vector fields: The preceding result allows us to now state:

THEOREM 16: *Let us consider a Lagrangian of mechanical type $L = L_{g,V} = T_g - \tau_Q^* V$, a conformal Killing vector field X for g, and the associated 1-form $\alpha = \widehat{g}(X)$. Then we have that*

$$\langle\!\langle f T_g - \mathcal{L}_X V \rangle\!\rangle = 0.$$

PROOF. If $\alpha = \widehat{g}(X)$ is the associated 1-form, from the relation $X_{\widehat{\alpha}} = X^c - f\Delta$ it follows that

$$\{E_L, \widehat{\alpha}\} = X_{\widehat{\alpha}} E_L = X^c E_L - f\,\Delta E_L = E_{X^c L} - 2fT_g = -fT_g + \tau_Q^*(\mathcal{L}_X V),$$

where we have used that $E_{X^c L} = T_{\mathcal{L}_X g} + \tau_Q^*(\mathcal{L}_X V) = f\,T_g + \tau_Q^*(\mathcal{L}_X V)$. Applying the virial theorem $\langle\!\langle \{E_L, \widehat{\alpha}\} \rangle\!\rangle = 0$ we obtain the result. □

Example 7: Consider now the spherical geometry metric (4.10) for $R = 1$ and look for a conformal vector field of the form $X = X_\theta(\theta)\,\partial/\partial\theta$. From the relationships

$$\mathcal{L}_X(d\theta^2) = 2\dot{X}_\theta\, d\theta^2, \qquad \mathcal{L}_X(\sin^2\theta\, d\phi^2) = 2\,\sin\theta\,\cos\theta\, X_\theta\, d\phi^2$$

we see that in order to be a conformal vector field one must have:

$$2\,\dot{X}_\theta = 2\,\mathrm{cotan}\,\theta\, X_\theta = f(\theta),$$

from where we obtain $X_\theta = \sin\theta$ and $f(\theta) = 2\,\cos\theta$. Therefore the corresponding Virial relation reads

$$\langle\!\langle 2\,\cos\theta\, T_g\rangle\!\rangle = \langle\!\langle\, \cos\theta\,(v_\theta^2 + \sin^2\theta\, v_\phi^2)\rangle\!\rangle = \langle\!\langle \sin\theta \frac{\partial V}{\partial\theta}\rangle\!\rangle.$$

◁

Example 8: Another example with three degrees of freedom is the metric

$$ds^2 = h(r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad h(r) > 0.$$

If we look for a conformal vector field of the form $X = X_r(r)\,\partial/\partial r$ we arrive to the relationship

$$\mathcal{L}_X\, g = (\dot{h}X_r + 2\,h\,\dot{X}_r)\, dr^2 + 2\,r\,X_r(d\theta^2 + \sin^2\theta\, d\phi^2) = f\, g,$$

and we see that in order to be a conformal vector field one must have:

$$\frac{\dot{h}}{h}X_r + 2\,\dot{X}_r = \frac{2X_r}{r} = f,$$

from where we can conclude that X_r is a solution of the differential equation

$$\dot{X}_r + \left(\frac{1}{2}\frac{\dot{h}}{h} - \frac{1}{r}\right) X_r = 0 \Longrightarrow X_r = C\frac{r}{h^{1/2}}.$$

and $f = 2C/h^{1/2}$. In particular for $h(r) = 1$, the Euclidean metric, we have the homothetic dilation vector field $X = r\,\partial/\partial r$, with $f = 2$ while for $h(r) = r^2$ we find the conformal vector field $X = \partial/\partial r$ with a conformal factor $f = 2/r$.Therefore the corresponding Virial relations read

$$\langle\!\langle 2T_g\rangle\!\rangle = \langle\!\langle \mathcal{L}_X V\rangle\!\rangle \Longrightarrow \langle\!\langle v_r^2 + r^2(v_\theta^2 + \sin^2\theta\, v_\phi^2)\rangle\!\rangle = \langle\!\langle r\frac{\partial V}{\partial r}\rangle\!\rangle.$$

and

$$\langle\!\langle \frac{2}{r}T_g\rangle\!\rangle = \langle\!\langle \mathcal{L}_X V\rangle\!\rangle \Longrightarrow \langle\!\langle r\,\left(v_r^2 + v_\theta^2 + \sin^2\theta\, v_\phi^2\right)\rangle\!\rangle = \langle\!\langle \frac{\partial V}{\partial r}\rangle\!\rangle.$$

◁

We can prove a similar result when we have two Riemann metrics g and g' on Q and the vector field $X \in \mathfrak{X}(Q)$ relates them in the following way $\mathcal{L}_X g = f\, g'$.

Theorem 17: *Consider a Lagrangian of mechanical type $L = L_{g,V} = T_g - \tau_Q^* V$. If there exists a function $f \in C^\infty(Q)$ such that $\mathcal{L}_X g = f\, g'$, and $\alpha = \widehat{g}(X)$, then,*

$$\langle\!\langle f T_{g'} - \mathcal{L}_X V\rangle\!\rangle = 0.$$

Proof. Since $\widehat{\alpha} = \langle \theta_L, X^c \rangle$, then $\Gamma_L(\widehat{\alpha}) = X^c(L)$. Hence,

$$\begin{aligned} \{E_L, \widehat{\alpha}\} = X_{\widehat{\alpha}} E_L = -X^c(L) = -X^c(T_g) + \tau_Q^*(\mathcal{L}_X V) = \\ = -T_{\mathcal{L}_X g} + \tau_Q^*(\mathcal{L}_X V) = -f T_{g'} + \tau_Q^*(\mathcal{L}_X V). \end{aligned}$$

The Virial Theorem implies that $\langle\!\langle -f T_{g'} + \mathcal{L}_X V \rangle\!\rangle = 0$, and the result follows. □

Example 9 (Spherical geometry): In example 3 the vector field $X = \tan(\theta)\partial_\theta$ defines the virial function. In polar coordinates, this vector is given by

$$X = r(1 + \lambda r^2)\partial_r.$$

The vector field X is not a conformal vector field of the Euclidian metric g' given by $ds^2 = dr^2 + r^2 d\phi^2$, but $\mathcal{L}_X g = 2(1 + \lambda r^2)^{-1} g'$. In this case, we have the formula: $\langle\!\langle 2(1 + \lambda r^2)^{-1} T_{g'} \rangle\!\rangle = \langle\!\langle \mathcal{L}_X V \rangle\!\rangle$ which is equivalent to (4.11). ◁

Virial theorem for homothetic vector fields: When the vector field X is homothetic, i.e. $f = \mu$ is a real constant, $\mathcal{L}_X g = \mu\, g$, then $\mathcal{L}_{X^c} T_g = \mu\, T_g$, where T_g is the kinetic energy T.

In example 9, when $\lambda \to 0$, the limit vector field is the infinitesimal generator of dilations on $\mathbb{R}^2$ written in polar coordinates, and it is a $2-$homothetic vector field of the Euclidian metric, so in the limit the Virial Theorem implies that $2\langle\!\langle T_g \rangle\!\rangle = \langle\!\langle r\partial_r V \rangle\!\rangle$.

If V is a X-homogeneous function of degree ν, i.e. $\mathcal{L}_X V = \nu V$, then $\langle\!\langle \mu T_g - \nu V \rangle\!\rangle = 0$ because of $\langle\!\langle X^c(T_g) - \mathcal{L}_X V \rangle\!\rangle = 0$. Using that the energy is a constant E along a trajectory we also have $\langle\!\langle T_g + V \rangle\!\rangle = E$, from where

$$\langle\!\langle T_g \rangle\!\rangle = \frac{\nu}{\nu + \mu} E \qquad \text{and} \qquad \langle\!\langle V \rangle\!\rangle = \frac{\mu}{\nu + \mu} E.$$

As a particular case, if both degrees of homogeneity are equal $\nu = \mu \equiv a$ then we have that

$$\langle\!\langle T_g \rangle\!\rangle = \langle\!\langle V \rangle\!\rangle = \frac{1}{2} E.$$

On the other hand, this condition is equivalent to $\mathcal{L}_{X^c} L = aL$, and hence we can apply directly a result in [7] obtaining $\langle\!\langle L \rangle\!\rangle = 0$, from where we also get $E = 2\langle\!\langle T_g \rangle\!\rangle = 2\langle\!\langle V \rangle\!\rangle$.

4.3 Virial Theorem for Mechanical systems on TQ in quasi-coordinates

In this section, using this time quasi-coordinates, we will present the modern geometric approach to the virial theorem developed in [7], where a Hamiltonian formalism is being used, and as a particular case is obtained the virial theorem for a regular Lagrangian system.

In many problems in classical mechanics and control theory it is useful to consider quasi-velocities. For instance, in studying the rotation of a rigid body, it is traditional to use Euler's angles to parametrize the orientation of the body while using body angular velocities to describe the dynamics. Similarly, for a system with nonholonomic constraints (i.e. constraints on the velocities that are not derivable from position constraints) one can define quasi-velocities in such a way that some of them coincide with the constraints, obtaining in this way fewer equations to solve.

An important fact of using quasi-coordinates in determining the virial theorem on TQ, is that, as the quasi-coordinates are not related with the tangent structure of $\tau_Q : TQ \to Q$, using only the vector bundle structure of it and the Lie algebra structure on $\mathfrak{X}(Q)$, we will be able to naturally generalize in Section 4.5 and Section 4.6 the virial theorem obtained in quasi-coordinates on TQ to a Lie algebroid, where, as there is no preferred basis of sections, there is no preferred choice of coordinates.

We consider a configuration manifold Q where a mechanical system is evolving. The traditional concept of velocities and momenta are obtained when considering a local chart $(U, q^1, \ldots, q^n)$, the coordinate basis $\{\partial/\partial q^j\}$ and its dual $\{dq^j\}$. Then, as we mentioned in section 2.1, if v, respectively ζ are written in this basis as: $v = v^j \partial/\partial q^j$ and $\zeta = p_j\, dq^j$, then $v^j = \langle dq^j, v\rangle$ and $p_j = \langle \zeta, \partial/\partial q^j\rangle$ are the usual velocities and momenta.

Alternatively we can chose a local basis of vector fields on Q, $\{X_1, \ldots, X_n\}$, and the dual basis $\{\alpha^1, \ldots, \alpha^n\}$. Any tangent vector $v \in T_qQ$ can be expressed uniquely as $v = w^j X_j(q)$. The real numbers $(w^1, \ldots, w^n)$ are called the *quasi-velocities* of v in the given basis. In terms of the dual basis $w^j = \langle \alpha^j(q), v\rangle$. Similarly a covector $\zeta \in T^*_qQ$ can be expressed as $\zeta = \pi_j \alpha^j(q)$, and then $(\pi_1, \ldots, \pi_n)$ are called the quasi-momenta of ζ in the given basis, which can be obtained as $\pi_j = \langle \zeta, X_j(q)\rangle$. The pair (q^i, w^i) is called the quasi-coordinates of $v \in TQ$ and the pair (q^i, π_k) is called the quasi-coordinates of $\zeta \in T^*Q$ (see [14]).

The relation between standard velocities and quasi-velocities is given by the well-known basis change formulas. If $X_j = \beta^k_j(q)\frac{\partial}{\partial q^k}$ is the coordinate expression of the vector field X_j in the coordinate basis, where $\frac{\partial}{\partial q^k} = \frac{\partial}{\partial q^k}\Big|_v - \frac{\partial \beta^j_a}{\partial q^k}\alpha^a_j v^j \frac{\partial}{\partial v^j}$, then $dq^j = \beta^j_k(q)\alpha^k$ and it follows that $v^i = w^j\beta^i_j(q)$ and $\pi_k = p_i\beta^i_k(q)$. A system of quasi-coordinates has an associated set of local functions on Q called Hamel's symbols given by $\gamma^k_{ml} = \beta^j_m\beta^i_l\big(\frac{\partial \alpha^k_j}{\partial q^i} - \frac{\partial \alpha^k_i}{\partial q^j}\big)$, where α^i_m is the inverse matrix of β^m_j, i.e. $\alpha^i_m\beta^m_j = \delta^i_j$. They can be defined by means of $d\alpha^k = -\frac{1}{2}\gamma^k_{ml}\alpha^m \wedge \alpha^l$, or alternatively by $[X_m, X_l] = \gamma^k_{ml}X_k$.

4.3.1 Lagrangian formalism

Consider now a dynamical system defined by a regular Lagrangian $L \in C^\infty(TQ)$. As mentioned in Subsection 2.1.2, the dynamical vector field $\Gamma_L \in \mathfrak{X}(TQ)$ is determined by the dynamical equation $i_{\Gamma_L}\omega_L = dE_L$, where $\omega_L = -d\theta_L$ is the Cartan 2-form associated to the Lagrangian and E_L is the energy function defined by L. In quasi-coordinates (q^i, w^i) on the tangent bundle TQ, the differential of an arbitrary function $G \in C^\infty(TQ)$ is given by $dG = X_j(G)\alpha^j + \frac{\partial G}{\partial w^j}dw^j$, and the expression in quasi-coordinates of the Cartan 2-form $\omega_L = -d\theta_L$ by:

$$\omega_L = \frac{1}{2}\left[\gamma^k_{ml}\frac{\partial L}{\partial w^k} + X_l\left(\frac{\partial L}{\partial w^m}\right) - X_m\left(\frac{\partial L}{\partial w^l}\right)\right]\alpha^m \wedge \alpha^l + \frac{\partial^2 L}{\partial w^j \partial w^k}\,\alpha^k \wedge dw^j\ .$$

Therefore, the dynamical vector field $\Gamma_L = X_{E_L}$ is given by

$$\Gamma_L = w^j X_j + W^{rl}\left[w^m \gamma^k_{ml}\frac{\partial L}{\partial w^k} - w^m X_m\left(\frac{\partial L}{\partial w^l}\right) + X_l(L)\right]\frac{\partial}{\partial w^r},$$

where $[W^{rl}]$ is the inverse matrix of $[\partial^2 L/\partial w^l \partial w^r]$, and the Hamiltonian vector field of the function G is

$$X_G = W^{jl}\frac{\partial G}{\partial w^l}X_j + W^{jl}\left\{\left[\frac{\partial L}{\partial w^k}\gamma^k_{ml} + X_l\left(\frac{\partial L}{\partial w^m}\right) - X_m\left(\frac{\partial L}{\partial w^l}\right)\right]W^{mr}\frac{\partial G}{\partial w^r} - X_l(G)\right\}\frac{\partial}{\partial w^j}.$$

For a virial function G the *virial theorem on TQ in quasi-coordinates* takes the form

$$\langle\!\langle \frac{\partial G}{\partial w^r}W^{rl}\left[w^m X_m\left(\frac{\partial L}{\partial w^l}\right) - X_l(L) - w^m\gamma^k_{ml}\frac{\partial L}{\partial w^k}\right] - w^j X_j(G)\rangle\!\rangle = 0 \quad (4.15)$$

The above equation provides a geometric interpretation of the Boltzmann's formalism of the virial theorem.

An important case is that of the function $G = \langle \theta_L, X^c\rangle$, where X is a vector field on Q and X^c is its complete lift to TQ. It was proved in [7] that $\{G, E_L\} = \Gamma_L G = X^c L$, from where it follows that from the condition of the virial theorem we have $\langle\!\langle X^c L\rangle\!\rangle = 0$. In quasi-coordinates, if $X = f^i X_i$ then the expression of the complete lift is

$$X^c = f^i X_i + \left[X_k(f^i) + \gamma^i_{kj} f^j\right] w^k \frac{\partial}{\partial w^i},$$

and therefore

$$\langle\!\langle f^i X_i(L) + \left[X_k(f^i) + \gamma^i_{kj} f^j\right] w^k \frac{\partial L}{\partial w^i}\rangle\!\rangle = 0.$$

If moreover the Lagrangian is of mechanical type, $L = T - V$, then the virial theorem has the form $\langle\!\langle X^c(T)\rangle\!\rangle = \langle\!\langle X(V)\rangle\!\rangle$. In coordinates, turns out to be

$$\langle\!\langle f^i X_i(T) + \left[X_k(f^i) + \gamma^i_{kj} f^j\right] w^k \frac{\partial T}{\partial w^i}\rangle\!\rangle = \langle\!\langle f^j X_j(V)\rangle\!\rangle. \quad (4.16)$$

Example 10 (Kepler problem and quasi-velocities): Let us consider a particle P of mass m moving in a plane under the action of a central force $F(r) = -\gamma m m'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where γ is a positive constant and r represents the distance between O and the particle P. The configuration space of the system is $Q = \mathbb{R}^2 - \{O\}$. Let θ be the angle that the line OP makes with a fixed direction on the plane. Consider as quasi-velocities $w^1 = \dot r$ and $w^2 = r^2\,\dot\theta$, corresponding to twice the area swept-out per time unit. Then,

$$L(r,\theta,w^1,w^2) = \frac{m}{2}\left[(w^1)^2 + \frac{1}{r^2}(w^2)^2\right] + \frac{\gamma m m'}{r}.$$

Let $X = r\partial_r$ be the infinitesimal generator of dilations on the space $\mathbb{R}^2$ written in polar coordinates. The complete lift of X is the vector field $X^c = r\partial_r + w^1\partial_{w^1} + 2w^2\partial_{w^2}$ on the tangent bundle $T\mathbb{R}^2$. If the virial function is defined by $G = \langle\theta_L, X^c\rangle$, that is, $G(r,\theta,w^1,w^2) = m r w^1$, then the Hamiltonian vector field of G turns out to be $X_G = r\partial_r - w^1\partial_{w^1}$. Applying formula (4.16), we obtain $\langle\!\langle r\partial_r V\rangle\!\rangle = \langle\!\langle m(w^1)^2 + m\frac{(w^2)^2}{r^2}\rangle\!\rangle$, that is, $\langle\!\langle -V\rangle\!\rangle = \langle\!\langle 2T\rangle\!\rangle$ as expected. $\triangleleft$

4.3.2 Hamiltonian formalism

As specified in Subsection 2.1.1, a function in the phase space, $H \in C^\infty(T^*Q)$, determines an associated Hamiltonian vector field X_H by the dynamical equation $i_{X_H}\omega_0 = dH$, where $\omega_0 = -d\theta_0$ is the canonical symplectic form on T^*Q. The motions of the system are the integral curves of X_H. In quasi-coordinates (q^i, π_i) on the cotangent bundle T^*Q, the differential of an arbitrary function $G \in C^\infty(T^*Q)$ is given by: $dG = X_j(G)\alpha^j + \frac{\partial G}{\partial \pi_j} d\pi^j$. The canonical 1-form θ_0 has the expression $\theta_0 = \pi_k \alpha^k$ and the canonical symplectic form $\omega_0 = -d\theta_0$ is locally given by $\omega_0 = \alpha^i \wedge d\pi_i + \frac{1}{2}\pi_k \gamma_{ij}^k \alpha^i \wedge \alpha^j$. Therefore, the Hamiltonian vector field associated to the function G is given by:

$$X_G = \frac{\partial G}{\partial \pi_i} X_i - \Big(\beta_i^j \frac{\partial G}{\partial q^j} + \pi_k \gamma_{ij}^k \frac{\partial G}{\partial \pi_j}\Big) \frac{\partial}{\partial \pi_i}.$$

Given a virial function G on T^*Q, the *virial theorem in the Hamiltonian formulation written in quasi-coordinates* is:

$$\langle\!\langle \beta_i^j \frac{\partial G}{\partial \pi_i}\frac{\partial H}{\partial q^j} - \beta_i^j \frac{\partial G}{\partial q^j}\frac{\partial H}{\partial \pi_i} - \pi_k \gamma_{ij}^k \frac{\partial G}{\partial \pi_j}\frac{\partial H}{\partial \pi_i} \rangle\!\rangle = 0.$$

This equation provides a geometric interpretation of the virial theorem as presented in [28] by using the Poincaré's formalism.

Particularly important are fibrewise linear virial functions. Every vector field X on the base manifold Q is associated with a linear function $G \in C^\infty(T^*Q)$ defined by $G(\zeta) = \langle \zeta, X(q)\rangle$ for $\zeta \in T_q^*Q$. The associated Hamiltonian vector field is the complete lift X^c of X to T^*Q. In quasi-coordinates (q^i, π_i) on T^*Q, if X has the expression $X = f^i X_i$, then $G(q, \pi) = \pi_k f^k(q)$ and the Hamiltonian vector field has the expression:

$$X_G = X^c = f^i X_i - \Big(\beta_i^j \frac{\partial f^k}{\partial q^j} + \gamma_{ij}^k f^j\Big)\pi_k \frac{\partial}{\partial \pi_i}.$$

For such a function the virial theorem can be expressed in the form:

$$\langle\!\langle \beta_i^j f^i \frac{\partial H}{\partial q^j} - \beta_i^j \frac{\partial f^k}{\partial q^j}\pi_k \frac{\partial H}{\partial \pi_i} - \pi_k \gamma_{ij}^k f^j \frac{\partial H}{\partial \pi_i} \rangle\!\rangle = 0. \qquad (4.17)$$

4.4 Virial Theorem for Nonholonomic Mechanical systems

We will first use the standard description of the nonholonomic systems in terms of Lagrange multipliers, using D'Alambert principle, and later on we will pose the problem in the more modern language of the distributional approach in which the Lagrange multipliers are eliminated by considering the appropriate manifolds, in order to study the virial theorem for nonholonomic mechanical systems. This second approach permits us to get similar results as in the unconstrained case, using this time the nonholonomic bracket, which is a Poisson bracket that does not satisfy the Jacobi identity, i.e. an almost-Poisson bracket. Finally we will make again appeal to the use of quasi-velocities.

4.4.1 Lagrange multipliers approach

For the regular nonholonomic system $(L, \mathcal{D})$, with L a regular Lagrangian, the following elementary result is the basis of the Virial Theorem. By a virial function we mean any function on TQ whose time evolution is bounded. This of course depends on the initial conditions of the solution, and in what follows we will implicitly assume the hypothesis that we have selected a set of initial conditions satisfying this property.

PROPOSITION 12: *If G is a virial function on TQ, then the time average of $\mathcal{L}_{\Gamma_{nh}} G$ vanishes:*

$$\langle\langle \mathcal{L}_{\Gamma_{nh}} G \rangle\rangle = 0. \tag{4.18}$$

PROOF. Let $\gamma(t)$ be a solution of the constrained dynamics, $\Gamma_{\mathrm{nh}} \circ \gamma = \frac{d\gamma}{dt}$. Then, $\gamma^*(\mathcal{L}_{\Gamma_{\mathrm{nh}}} G) = \frac{d}{dt}(\gamma^* G)$. From the definition of the time average we have:

$$\langle\langle \mathcal{L}_{\Gamma_{\mathrm{nh}}} G \rangle\rangle = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \frac{d}{dt}(\gamma^* G)\, dt = \lim_{\tau \to +\infty} \frac{(\gamma^* G)(\tau) - (\gamma^* G)(0)}{\tau}.$$

Since $\gamma^* G$ is bounded we conclude that the limit is zero. □

As an immediate consequence we have the following result.

THEOREM 18: *Under the conditions stated above, if G is a virial function on TQ, we have that*

$$\langle\langle \Gamma_L(G) + \lambda_A\, Z_A(G) \rangle\rangle = 0. \tag{4.19}$$

In local coordinates, taking in account (2.7) this relation is:

$$\langle\langle v^i \frac{\partial G}{\partial x^j} + W^{ij} \left(\frac{\partial L}{\partial x^j} - v^k \frac{\partial^2 L}{\partial x^k \partial v^j} + \lambda_A \omega^A_j \right) \frac{\partial G}{\partial v^i} \rangle\rangle = 0. \tag{4.20}$$

In applications, the virial function G is generally chosen as the Hamiltonian function associated to a vector field which generates a 1-parameter group of transformation of interest (for instance the dilation group in the case $Q = \mathbb{R}^n$) and we pretend to write the consequences of the virial theorem in the nonholonomic case directly in terms of such a vector field.

THEOREM 19: *Let X^c be the complete lift of a vector field X on Q, and $G = \langle \theta_L, X^c \rangle$ be the virial function. Then,*

$$\langle\langle X^c(L) + \lambda_A \omega^A(X) \rangle\rangle = 0. \tag{4.21}$$

PROOF. Since the solution $\Gamma_{\mathrm{nh}} = \Gamma_L + \lambda_A Z_A$ is a SODE vector field we can rewrite Lagrange-D'Alembert equations in the form:

$$\mathcal{L}_{\Gamma_{\mathrm{nh}}} \theta_L = dL + \lambda_A\, \tilde{\omega}^A.$$

Contracting with X^c we obtain:

$$\begin{aligned} \Gamma_{\mathrm{nh}}(G) &= \mathcal{L}_{\Gamma_{\mathrm{nh}}}(\langle \theta_L, X^c \rangle) \\ &= \langle \mathcal{L}_{\Gamma_{\mathrm{nh}}} \theta_L, X^c \rangle - \langle \theta_L, \mathcal{L}_{\Gamma_{\mathrm{nh}}} X^c \rangle \\ &= dL(X^c) + \lambda_A\, \omega^A(X) \end{aligned}$$

where we have taken into account that $\mathcal{L}_{\Gamma_{\mathrm{nh}}} X^c$ is vertical and θ_L is semibasic. It follows from Proposition 12 that $\langle\langle \Gamma_{\mathrm{nh}}(G) \rangle\rangle = 0$, i.e. $\langle\langle X^c(L) + \lambda_A \omega^A(X) \rangle\rangle$ vanishes. □

We remark that, if X is a section of the vector bundle $\mathcal{D}$, the constraint manifold described in detail in the Section 2.2.1, then the above Virial Theorem reduces to the simpler form $\langle\langle X^c(L) \rangle\rangle = 0$.

EXAMPLE 11 (Nonholonomic harmonic oscillator): To illustrate the theory we will consider the nonholonomic dynamical system known as the nonholonomic harmonic oscillator.

Consider an isotropic harmonic oscillator moving in $Q = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3). The Lagrangian function

$$L = T - V = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

We have $\omega_L = dx_1 \wedge d\dot{x}_1 + dx_2 \wedge d\dot{x}_2 + dx_3 \wedge d\dot{x}_3$ and $dE_L = \dot{x}_1 d\dot{x}_1 + \dot{x}_2 d\dot{x}_2 + \dot{x}_3 d\dot{x}_3 + x_1 dx_1 + x_2 dx_2 + x_3 dx_3$, so the unconstrained dynamics is the well-known dynamics described by the vector field

$$\Gamma_L = \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{x}_3 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial \dot{x}_1} - x_2 \frac{\partial}{\partial \dot{x}_2} - x_3 \frac{\partial}{\partial \dot{x}_3}.$$

We constraint the motion of the particle by introducing the nonholonomic constraint

$$\phi = \dot{x}_3 - x_2 \dot{x}_1 = 0.$$

The constraint submanifold is given by

$$\mathcal{D} = \{(x_1, x_2, x_3; \dot{x}_1, \dot{x}_2, \dot{x}_3) \in TQ \mid \dot{x}_3 = x_2 \dot{x}_1\}.$$

Applying Lagrange-D'Alembert's principle we find

$$\Gamma_{\mathrm{nh}} = \Gamma_L + \frac{\dot{x}_1 \dot{x}_2 - x_1 x_2 + x_3}{1 + x_2^2} \left(\frac{\partial}{\partial \dot{x}_3} - x_2 \frac{\partial}{\partial \dot{x}_1} \right).$$

We consider the dilation vector field

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$$

and we apply the virial theorem. The virial function is $G = x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3$ and we get

$$\langle\langle 2T - X(V) + x_1 x_2 \dot{x}_1 \dot{x}_2 \rangle\rangle = 0,$$

or in other words

$$\langle\langle 2T - 2V + \frac{\dot{x}_1 \dot{x}_2 - x_1 x_2 + x_3}{1 + x_2^2} (x_3 - x_1 x_2) \rangle\rangle = 0,$$

and taking into account conservation of the energy we finally get

$$\langle\langle T \rangle\rangle = \frac{E}{2} - \frac{1}{4} \langle\langle \frac{\dot{x}_1 \dot{x}_2 - x_1 x_2 + x_3}{1 + x_2^2} (x_3 - x_1 x_2) \rangle\rangle$$
$$\langle\langle V \rangle\rangle = \frac{E}{2} + \frac{1}{4} \langle\langle \frac{\dot{x}_1 \dot{x}_2 - x_1 x_2 + x_3}{1 + x_2^2} (x_3 - x_1 x_2) \rangle\rangle.$$

◁

4.4.2 Distributional approach

In the above treatment using the Lagrange multipliers, it is not explicit that the different objects have to be defined in the constraint distribution. Moreover, in the unconstrained case, the virial theorem, as well as many other interesting results, are stated in terms of the Poisson bracket associated to the symplectic form. So, in what follows we will study how the theory can be developed only in terms of objects intrinsically defined in $\mathcal{D}$ and in terms of an almost-Poisson bracket, the nonholonomic bracket.

The virial theorem

With the help of the nonholonomic bracket defined by relation (2.10), or alternatively by (2.11), and taking into account Theorem 3, we can rewrite the nonholonomic virial theorem in the following form.

THEOREM 20: *For any virial function G we have that*

$$\langle\langle \{G, E_L\}_{nh} \rangle\rangle = 0. \tag{4.22}$$

In this way the nonholonomic virial theorem can be expressed in a similar manner to the holonomic virial theorem with the only difference that the bracket is the nonholonomic bracket instead of the Poisson bracket associated to the symplectic form.

In particular, if X is a section of $\mathcal{D}$, i.e. a vector field on Q taking values in $\mathcal{D}$, then for $G = \langle\, \theta_L \,, X^c \,\rangle$ a simple calculation shows that $\{G, E_L\}_{\mathrm{nh}} = \mathcal{L}_{\Gamma_{\mathrm{nh}}} G = \mathcal{L}_{X^c} L|_{\mathcal{D}}$, so that $\langle\langle \mathcal{L}_{X^c} L \rangle\rangle = 0$.

4.4.3 Virial theorem for nonholonomic mechanical systems in quasi-velocities

In nonholonomic mechanics the use of quasi-velocities is highly convenient [14]. Consider a local basis $\{\mathcal{X}_a\}$ of vector fields spanning the distribution $\mathcal{D} \subset TQ$ and complete with a family of vector fields $\{\mathcal{X}_A\}$ to a local basis $\{\mathcal{X}_\alpha\} = \{\mathcal{X}_a, \mathcal{X}_A\}$ of $\mathfrak{X}(Q)$. Taking a local coordinate system (x^i) on the manifold Q we have that

$$\mathcal{X}_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i}, \tag{4.23}$$

for some local functions $\rho^i_\alpha \in C^\infty(Q)$. The brackets of the vector fields in such a basis are $[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\mathcal{X}_\gamma$, where the functions $C^\alpha_{\beta\gamma} \in C^\infty(Q)$ are the so called Hammel's transpositional symbols, which are determined by

$$\rho^i_\alpha \frac{\partial \rho^k_\beta}{\partial x^i} - \rho^i_\beta \frac{\partial \rho^k_\alpha}{\partial x^i} = \rho^k_\gamma C^\gamma_{\alpha\beta}. \tag{4.24}$$

Associated to this choice of coordinates in Q and the local basis of vector fields in Q there is a coordinate system (x^i, y^α) in TQ where y^α are the coordinates of a vector in the basis $\{\mathcal{X}_\alpha\}$. For vector fields on TQ we have a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ given by

$$\mathcal{X}_\alpha = \rho^i_\alpha \frac{\partial}{\partial x^i} \qquad\qquad \mathcal{V}_\alpha = \frac{\partial}{\partial y^\alpha}. \tag{4.25}$$

Notice that we have denoted with the same symbol $\mathcal{X}_\alpha$ the local vector fields on Q and on TQ which have the same coordinate expression. The dual basis will be denoted by $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$, and it is related to the differential of the coordinates by means of $dx^i = \rho^i_\alpha \mathcal{X}^\alpha$ and $dy^\alpha = \mathcal{V}^\alpha$. Notice that $\mathcal{X}^A = \omega^A$ are the constraint 1-forms that we used in section 4.4.1.

The local expressions for the Lagrangian energy and the Cartan 2-form are [8, 14]

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L \tag{4.26}$$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta. \tag{4.27}$$

In the coordinates $(x^i, y^\alpha) = (x^i, y^a, y^A)$ on TQ the equations for $\mathcal{D}$, i.e. the constraints, are simply $y^A = 0$, or in other words, (x^i, y^a) are coordinates for $\mathcal{D}$. In what respect to the decomposition $TTQ|_{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp$ we have that $\{\mathcal{X}_a, \mathcal{V}_a\}$ is a local basis of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, and a simple calculation shows that a local basis $\{Y_A, Z_A\}$ of sections of $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp$ is given by

$$Z_A = \mathcal{V}_A - Q^a_A \mathcal{V}_a, \qquad Y_A = \mathcal{X}_A - Q^a_A \mathcal{X}_a + \mathcal{C}^{bc}(M_{Ab} - M_{ab} Q^a_A)\mathcal{V}_c, \tag{4.28}$$

where $Q^a_A = W_{Ab}\mathcal{C}^{ab}$ and $\mathcal{C}^{ab}$ are the components of the inverse of the matrix $\mathcal{C}_{ab} = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, y^A = 0)$, and $M_{\alpha\beta} = \omega_L(\mathcal{X}_\alpha, \mathcal{X}_\beta)$. Therefore the expression of the projector onto $(\mathcal{T}^{\mathcal{D}}\mathcal{D})^\perp$ is $\bar{Q} = Z_A \otimes \mathcal{V}^A + Y_A \otimes \mathcal{X}^A$.

For the constrained dynamics, we look for a section $\Gamma_{\rm nh}$ of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, so that it is of the local form $\Gamma_{\rm nh} = g^a \mathcal{X}_a + f^a \mathcal{V}_a$. Assuming a regular constrained system, from the local expression (4.27) of the Cartan 2-form and the local expression (4.26) of the energy function, we get that $g^a = y^a$ and the functions f^a are the solution of the linear equations

$$\frac{\partial^2 L}{\partial y^b \partial y^a} f^b + \frac{\partial^2 L}{\partial x^i \partial y^a} \rho^i_b y^b + \frac{\partial L}{\partial y^\gamma} C^\gamma_{ab} y^b - \rho^i_a \frac{\partial L}{\partial x^i} = 0, \tag{4.29}$$

where all the partial derivatives of the Lagrangian are to be evaluated on $y^A = 0$.

The differential equations for the integral curves of $\Gamma_{\rm nh}$, i.e. Lagrange-d'Alembert differential equations, are in quasi-velocities

$$\begin{cases} \dot{x}^i = \rho^i_a y^a, \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial y^a}\right) + \dfrac{\partial L}{\partial y^\gamma} C^\gamma_{ab} y^b - \rho^i_a \dfrac{\partial L}{\partial x^i} = 0, \\ y^A = 0. \end{cases} \tag{4.30}$$

Finally, the contraction of $i_{\Gamma_{\rm nh}}\omega_L - dE_L$ with $\mathcal{X}_A$ just gives the value of the Lagrange multipliers $\lambda_A = \langle i_{\Gamma_{\rm nh}}\omega_L - dE_L\, , \mathcal{X}_A \rangle|_{y^A=0}$, i.e. the components of the constraint forces $\lambda = \lambda_A \mathcal{X}^A$.

In what respect to the virial theorem, if X is a section of $\mathcal{D}$ with local expression $X = X^a \mathcal{X}_a$, then

$$\langle\!\langle \rho^i_a X^a \frac{\partial L}{\partial x^i} + \left[\rho^j_b \frac{\partial X^a}{\partial x^j} + C^a_{bd} X^d\right] y^b \frac{\partial L}{\partial y^a} + C^A_{bd} X^d y^b \frac{\partial L}{\partial y^A} \rangle\!\rangle = 0. \tag{4.31}$$

where all partial derivatives must be taken at $y^A = 0$.

Example 12 (The nonholonomic harmonic oscillator): Consider again an isotropic harmonic oscillator in $Q = \mathbb{R}^3$, with Lagrangian function

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),$$

subjected to the nonholonomic constraint $\phi = \dot{x}_3 - x_2 \dot{x}_1 = 0$.

As a basis $\{\mathcal{X}_1, \mathcal{X}_2\}$ of sections of $\mathcal{D}$ we can take,

$$\mathcal{X}_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \qquad \mathcal{X}_2 = \frac{\partial}{\partial x_2},$$

which we can complete with the vector field $\mathcal{X}_3 = \frac{\partial}{\partial x_3}$. The only non-vanishing bracket is $[\mathcal{X}_1, \mathcal{X}_2] = -\mathcal{X}_3$, so that $C^3_{21} = -C^3_{12} = 1$, and for other indices $C^\gamma_{\alpha\beta} = 0$.

The associated quasivelocities are related to the velocities by

$$(y_1, y_2, y_3) = (\dot{x}_1, \dot{x}_2, \dot{x}_3 - x_2 \dot{x}_1),$$
$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = (y_1, y_2, y_3 + x_2 y_1),$$

and substituting in the Lagrangian we get

$$L(x_1, x, x_3, y_1, y_2, y_3) = \frac{1}{2}\big(y_1^2 + y_2^2 + (y_3 + x_2 y_1)^2\big) - \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

Taking the vector field $X = x_1 \mathcal{X}_1 + x_2 \mathcal{X}_2$ so that

$$X^c = x_1 \mathcal{X}_1 + x_2 \mathcal{X}_2 + y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + (y_2 x_1 - x_2 y_1) \frac{\partial}{\partial y_3}$$

and applying the virial theorem we get $\langle\!\langle X^c L \rangle\!\rangle = 0$ which, after taking into account that $y_3 = 0$, reads

$$\langle\!\langle 2T - X(V) + x_1 x_2 y_1 y_2) \rangle\!\rangle = 0,$$

or equivalently

$$\langle\!\langle 2T - 2V + x_1 x_2 y_1 y_2 + x_3(x_3 - x_1 x_2) \rangle\!\rangle = 0.$$

◁

4.5 Virial Theorem for Mechanical systems on Lie algebroids

In this section, a generalization of the virial theorem for mechanical systems on Lie algebroids is given, using the geometric tools of Lagrangian and Hamiltonian mechanics on the prolongation of the Lie algebroid A in the Lagrangian case, respectively of A^* with respect to the Lie algebroid A for the Hamiltonian case. (see [35, 41])

The two Lie algebroids $\mathcal{T}^A A \to A$ and $\mathcal{T}^A A^* \to A$ are both particular cases of symplectic Lie algebroids, i.e a Lie algebroid with a regular bilinear 2-form, closed with respect to the exterior differential operator defined on it. For any symplectic Lie algebroid, $E \to M$, with anchor map denoted by ρ, every function $H \in C^\infty(M)$ defines a dynamical system on the base manifold M as follows. Given the function H, there is a unique section σ_H of E, called Hamiltonian section of H, such that $i_{\sigma_H}\omega = dH$. The vector field $X_H = \rho(\sigma_H)$ is the infinitesimal generator of such a dynamical system. In both particular instances considered here, the basis manifold is A, so any smooth function on A defines a dynamical system on it, and so, unique Hamiltonian sections of $T^A A$, respectively of $T^A A^*$.

The Hamiltonian vector field X_H can also be obtained in terms of a Poisson bracket on the base manifold M. Indeed, given two function $F, G \in C^\infty(M)$, the bracket defined by $\{F, G\} = \omega(\sigma_F, \sigma_G)$ is a Poisson bracket on M. We clearly see the relations $\{F, G\} = i_{\sigma_G} dF = \rho(\sigma_G)F = X_G F = -X_F G$.

We remark that the generalization of the virial theorem to the framework of the Lie algebroids can be done naturally. In the tangent bundle case, we have proved these results using quasi-coordinates. The geometrical interpretation for quasi-coordinates has been given in [14]. As it forgets the tangent structure and uses only the vector bundle structure $\tau_Q : TQ \to Q$ and the Lie algebra structure $[\cdot\,,\cdot]$ on the set of vector fields on Q, it follows naturally to consider these results to the more general framework of Lie algebroids. In this formalism we also do not have a preferred choice of coordinates on the base manifold, as we do not have a canonical choice of basis for sections.

4.5.1 Lagrangian formalism

As we presented in Section 2.7.1, the dynamical section $\Gamma_L \in \mathrm{Sec}(\mathcal{T}^A A)$ determined by the equation $i_{\Gamma_L}\omega_L = dE_L$, is given by: $\Gamma_L = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha$ with $f^\alpha = W^{\alpha\theta}\Big(\rho^i_\theta \frac{\partial L}{\partial x^i} - \rho^i_\beta y^\beta \frac{\partial^2 L}{\partial x^i \partial y^\theta} - C^\gamma_{\theta\beta} y^\beta \frac{\partial L}{\partial y^\gamma}\Big)$, where $[W^{\alpha\beta}]$ is the inverse matrix of $\big[\partial^2 L/\partial y^\alpha \partial y^\beta\big]$. In the above expressions $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ denotes a basis of $\mathcal{T}^A A$ constructed as in Section 2.5.1 and $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha\}$ denotes its dual basis.

The differential of a function $G \in C^\infty(A)$ is $dG = \rho^i_\alpha \frac{\partial G}{\partial x^i}\mathcal{X}^\alpha + \frac{\partial G}{\partial y^\alpha}\mathcal{V}^\alpha$ and therefore the *virial theorem* in this case, which states that $\langle\!\langle \rho^1(\Gamma_L)G \rangle\!\rangle = 0$, which locally amounts to $\langle\!\langle \rho^i_\alpha y^\alpha \frac{\partial G}{\partial x^i} + f^\alpha \frac{\partial G}{\partial y^\alpha} \rangle\!\rangle = 0$, or explicitly

$$\langle\!\langle \rho^i_\alpha y^\alpha \frac{\partial G}{\partial x^i} + W^{\alpha\theta}\Big(\rho^i_\theta \frac{\partial L}{\partial x^i} - \rho^i_\beta y^\beta \frac{\partial^2 L}{\partial x^i \partial y^\theta} - C^\gamma_{\theta\beta} y^\beta \frac{\partial L}{\partial y^\gamma}\Big)\frac{\partial G}{\partial y^\alpha} \rangle\!\rangle = 0.$$

EXAMPLE 13: Consider a Lagrangian function L on a finite-dimensional Lie algebra $\mathfrak{g}$, that we consider as a Lie algebroid over a point. For a constant vector $a \in \mathfrak{g}$ we consider the virial function $G(y) = a^\beta \frac{\partial L}{\partial y^\beta}$. The virial theorem becomes $\langle\!\langle \mathrm{ad}^*_y \frac{\partial L}{\partial y} \rangle\!\rangle = 0$, which in quasi-coordinates reads $\langle\!\langle \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\alpha \rangle\!\rangle = 0$, where we already took into account that a is arbitrary.

In the particular case of a free rigid body, we have $\mathfrak{g} = \mathfrak{so}(3)$ and the Lagrangian is $L(\omega) = \frac{1}{2}\omega \cdot I\omega$, where I are the inertia tensor. It follows that, $\langle\!\langle \omega \times I\omega \rangle\!\rangle = 0$, in concordance with the result in the Hamiltonian formalism. ◁

Let σ be a section of A, σ^c its complete lift to $\mathcal{T}^A A$, and take as virial function $G = \langle \theta_L, \sigma^c \rangle$. Then, as it was proved in [41] that we have $d_\Gamma G = d_{\sigma^c} L$, or in other words $\{G, E_L\} = d_{\sigma^c} L$. Therefore we can prove the following result.

Theorem 21: *Let σ be a section on the Lie algebroid A and let σ^c be its complete lift to $\mathcal{T}^A A$. Assume that $G = \langle \theta_L, \sigma^c \rangle$ is bounded on its time evolution. Then $\langle\!\langle \rho^1(\sigma^c)(L) \rangle\!\rangle = 0$.*

Example 14: A heavy top can be modeled on the Lie algebroid $S^2 \times \mathfrak{so}(3) \to S^2$ with Lagrangian $L = \frac{1}{2}\omega \cdot I\omega - mgl\gamma \cdot e$ (see [41] for the notation and other details). Taking the linear function $G = a \cdot \gamma$ and applying the virial theorem we get $\langle\!\langle a \cdot (\gamma \times \omega) \rangle\!\rangle = 0$, and since a is arbitrary we arrive to $\langle\!\langle \gamma \times \omega \rangle\!\rangle = 0$.

On the other hand, we consider a constant vector a on $\mathbb{R}^3 \equiv \mathfrak{so}(3)$ and the associated constant section of A given by $\sigma(\gamma) = (\gamma, a)$. The complete lift of σ is $\sigma^c = a^i \mathcal{X}_i + (a \times \omega)^i \mathcal{V}_i$. Applying theorem 21 we get that $\langle\!\langle \rho(\sigma^c) L \rangle\!\rangle = 0$, and after an straightforward computation and taking into account that a is arbitrary we arrive at $\langle\!\langle \omega \times I\omega \rangle\!\rangle = mgl \langle\!\langle \gamma \times e \rangle\!\rangle$. ◁

4.5.2 Hamiltonian formalism

Let $\tau : A \to M$ be a Lie algebroid over a manifold M, with anchor ρ and bracket $[\cdot,\cdot]$. In the construction of the prolongation a fibered manifold with respect to a Lie algebroid, from Section 2.5.1, as the fibre bundle P we will consider $\nu : A^* \to M$, the dual bundle of A. Thus we will work on the A-tangent to A^*, who is a Lie algebroid $(\mathcal{T}^A A^*, [\cdot,\cdot], \rho^1)$. Taking local coordinates (x^i) on M and choosing a basis $\{e_\alpha\}$ of sections of A and the dual basis $\{e^\alpha\}$, we have the local coordinates (x^i, μ_α) on the bundle A^*, and we can define the local basis $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$ of sections of $\mathcal{T}^A A^*$ as explained in Section 2.5.1. We will denote by $\{\mathcal{X}^\alpha, \mathcal{P}_\alpha\}$ the dual basis. We then have, $\rho^1(\mathcal{X}_\alpha) = \rho^i_\alpha \partial_{x^i}$ and $\rho^1(\mathcal{P}^\alpha) = \partial_{\mu_\alpha}$, and for a function $f \in C^\infty(A^*)$ its differential is $df = \rho^i_\alpha \frac{\partial f}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial f}{\partial \mu_\alpha} \mathcal{P}_\alpha$.

In the A-tangent of A^* there is a canonical section θ_A of $(\mathcal{T}^A A^*)^*$, called the Liouville section, defined by $\theta_A(a^*)(b, v) = a^*(b)$, for $(b, v) \in \mathcal{T}_{a^*} A^*$, and a canonical symplectic section $\omega_A = -d\theta_A$. In coordinates, they are given by

$$\theta_A = \mu_\alpha \mathcal{X}^\alpha \quad \text{and} \quad \omega_A = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} C^\gamma_{\alpha\beta} \mu_\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

The Hamiltonian section $\Gamma_H \in \mathrm{Sec}(\mathcal{T}^A A^*)$ defined by a function $H \in C^\infty(A^*)$ is written in local coordinates

$$\Gamma_H = \frac{\partial H}{\partial \mu_\alpha} \mathcal{X}_\alpha - \left(C^\gamma_{\alpha\beta} \mu_\gamma \frac{\partial H}{\partial \mu_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i} \right) \mathcal{P}^\alpha \in \mathrm{Sec}(\mathcal{T}^A A^*),$$

and then, the *virial theorem* in the Hamiltonian formalism on a Lie algebroid $\langle\langle \rho^1(\Gamma_H)G \rangle\rangle = 0$ in local coordinates is:

$$\langle\langle \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha}\frac{\partial G}{\partial x^i} - \rho^i_\alpha \frac{\partial H}{\partial x^i}\frac{\partial G}{\partial \mu_\alpha} - C^\gamma_{\alpha\beta}\mu_\gamma \frac{\partial H}{\partial \mu_\beta}\frac{\partial G}{\partial \mu_\alpha} \rangle\rangle = 0, \quad G \in C^\infty(A^*). \tag{4.32}$$

EXAMPLE 15: Consider a finite-dimensional Lie algebra $\mathfrak{g}$ as a Lie algebroid over a singleton $M = \{e\}$. For a Hamiltonian $H \in C^\infty(\mathfrak{g}^*)$ and a linear virial function $G(\mu) = \langle a, \mu \rangle$, with $a \in \mathfrak{g}$ a constant vector, the virial theorem becomes $\langle\langle \mathrm{ad}^*_{\frac{\partial H}{\partial \mu}} \mu \rangle\rangle = 0$. Taking a local basis on $\mathfrak{g}$ and the corresponding linear coordinates on $\mathfrak{g}^*$ we get $\langle\langle \mu_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial \mu_\beta} a^\alpha \rangle\rangle = 0$, where $C^\gamma_{\alpha\beta}$ are the structure constants. Since a is arbitrary we get $\langle\langle \mu_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial \mu_\beta} \rangle\rangle = 0$ for every $\alpha = 1, \ldots, \dim \mathfrak{g}$.

An important particular case is that of a free rigid body. The Lie algebra is $\mathfrak{g} = \mathfrak{so}(3)$ and the Hamiltonian is defined by $H(\mu) = \frac{1}{2}\mu \cdot I^{-1}\mu$, where I is the inertia tensor. The virial theorem tell us that each component of the cross product $I^{-1}\mu \times \mu$ has vanishing time average. $\triangleleft$

4.6 Virial Theorem for Nonholonomic Mechanical systems on Lie algebroids

Finally, in this section, we present another application of the Lie algebroids, the virial theorem for nonholonomic systems on this structure, using again the two approaches considered in the tangent bundle case: the Lagrangian multipliers approach and the distributional approach.

4.6.1 Lagrange multipliers approach

Consider a dynamical system defined by a regular Lagrangian L on the Lie algebroid A. Suppose the system has k linear nonholonomic constraints $\hat{\phi}_a = 0$, each one associated to a 1-form $\phi_a \in \Omega(A)$, defined by:

$$\hat{\phi}_a(x, y) = \langle\, \phi_a(x)\, , y\, \rangle = \phi_{ab} y^b, \forall (x, y) \in A.$$

Consider a vector subbundle $\mathcal{D} \to M$ of A, where $\mathcal{D}$ is the constrained manifold. The dynamics of the nonholonomic system is defined by a section Γ_{nh} on the $\mathcal{D}$-tangent bundle to the bundle $\mathcal{D} \to M$, $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, and obeys the d'Alambert-Chetaev principle (i.e. the work of the reaction forces is null for all virtual displacements):

$$\begin{cases} i_{\Gamma_{\mathrm{nh}}}\omega_L - dE_L \in \mathrm{Sec}(\tilde{\mathcal{D}}^0) \\ \Gamma_{\mathrm{nh}}|_{\mathcal{D}} \in \mathrm{Sec}(\mathcal{T}^{\mathcal{D}}\mathcal{D}) \end{cases}$$

where $\mathcal{D}^0 = \{\phi_a |\, a = \overline{1,k}\}$ is the annihilator of $\mathcal{D}$ and $\tilde{\mathcal{D}}^0 = \{p_2^*(\phi_a) |\, a = \overline{1,k}\}$ is a vector bundle given by the lift of the elements of $\mathcal{D}^0$ to elements of the dual bundle of $\mathcal{T}^A A$, where $\tau_2 : \mathcal{T}^A A \to A$ is the projection defined by $\tau_2(a, b, c) = b$.

The dynamics is given by the equation

$$i_{\Gamma_{\rm nh}}\omega_L - dE_L = -\lambda_a \tau_2^*(\phi^a),$$

where the semi-basic sections $\tau_2^*(\phi^a) = \phi_{a\beta}\mathcal{X}^\beta$ are the reaction forces and the Lagrange multipliers $\lambda_a \in C^\infty(A)$ are determined by the tangency conditions $\rho^1(\Gamma_{\rm nh})\hat{\phi_a} = 0, \forall a = \overline{1,k}$.

In order to determine the multipliers we will assume the compatibility condition that: $\mathcal{C}_{ab} = \rho^1(Z_a)\hat{\phi_b}$ is a regular matrix for all points in the constrained manifold $\mathcal{D}$. The nonholonomic system $(L, \mathcal{D})$ is called regular when the compatibility condition is satisfied, and we will further assume this to be satisfied.

The solution can be written as $\Gamma_{\rm nh} = \Gamma_L + \lambda_a Z_a$, where Γ_L is the solution of the unconstrained system and Z_a is the vertical section of the $\mathcal{T}^A A$ given by $i_{Z_a}\omega_L = -p_2^*(\phi_a)$; in coordinates the vertical section is given by $Z_a = W^{b\gamma}\phi_{a\gamma}\mathcal{V}_b$.

Under the previous conditions of regularity, we can establish a *virial theorem for nonholonomic systems.*

For a virial function G on the Lie algebroid A, that is, a bounded smooth function on the algebroid, we have

$$\langle\langle \rho^1(\Gamma_{\rm nh})G \rangle\rangle = 0.$$

THEOREM 22: *Let $(L, \mathcal{D})$ be a regular nonholonomic system defined on a Lie algebroid. If G is a virial function on the Lie algebroid, then*

$$\langle\langle \rho^1(\Gamma_L)G + \lambda_a \rho^1(Z_a)G \rangle\rangle = 0,$$

where Γ_L is the solution of the unconstraint system, Z_a is the vertical section of $\mathcal{T}^A A$ given by $i_{Z_a}\omega_L = -\tau_2^*(\phi_a)$ and λ_a represents the Lagrange multipliers determined by the tangency conditions.

PROOF. If X_G is τ_2- projectable, that is, $\tau_2 \circ X_G = \sigma \circ \tau$, then we can prove that $\rho^1(Z_a)G = \phi_a(\sigma)$.

In fact, $\rho^1(Z_a)G = i_{Z_a}i_{X_G}\omega_L = -i_{X_G}i_{Z_a}\omega_L = i_{X_G}\tau_2^*(\phi_a) = \phi_a(\sigma)$.

Then, the theorem implies $\langle\langle \rho^1(\Gamma_L)G + \lambda_a\phi_a(\sigma) \rangle\rangle = 0$. □

The section $\Gamma_{\rm nh} = \Gamma_L + \lambda_a Z_a$ is a SODE because $S(\Gamma_{\rm nh}) = \Delta$, and then, $\mathcal{L}_{\Gamma_{\rm nh}}\theta_L = dL + \lambda_a\tau_2^*\phi_a$.

Let σ^c be the complete lift to $\mathcal{T}^A A$ of a section σ of A.

THEOREM 23: *Let $G = \langle\, \theta_L\,, \sigma^c \,\rangle$ be the virial function. Then the virial theorem implies that $\langle\langle \rho^1(\sigma^c)L + \lambda_a\phi_a(\sigma) \rangle\rangle = 0$.*

The proof is an immediate generalization of the proof of Theorem 19.

Example 16 (The Suslov system):

A rigid body about a fixed point moves under the action of the following nonholonomic constraint: the angular velocity vector of the body is orthogonal to a fixed direction of the space, given by the vector V.

The configuration space of the problem is the Lie group $SO(3)$. The Lie algebra in this case, $so(3)$ will be identified with $\mathbb{R}^3$, and under this identification the Lie bracket on $so(3)$ will be the cross product $\times$ on $\mathbb{R}^3$. We will think of this Lie algebra as a Lie algebroid over a one point basis.

Denote an element of the Lie algebroid $so(3)$ by ω which by the identification with $\mathbb{R}^3$ can be seen as: $\omega = (\omega_1, \omega_2, \omega_3)$. Then a basis of the prolongation of this Lie algebroid is given by: $\mathcal{X}_\alpha(a) = (a, e_i, \rho^i_\alpha \frac{\partial}{\partial x^i})$ and $\mathcal{V}_\alpha(a) = (a, 0, \frac{\partial}{\partial \omega^\alpha})$. The Lagrangian of the system is given by $L(\omega) = \frac{1}{2}\langle\, I(\omega)\,, \omega\,\rangle$, while the constraint is : $\langle\, \omega\,, V\,\rangle = 0$.

The dynamic section is: $\Gamma_{\mathrm{nh}} = \omega^a \mathcal{X}_a + [I^{-1}(\omega \times (I\omega))]^\alpha \mathcal{V}_\alpha + \lambda (I^{-1}V)^\alpha \mathcal{V}_\alpha$, where $\lambda = -\frac{\langle\,(I\omega)\times\omega\,, I^{-1}V\,\rangle}{\langle\, V\,, I^{-1}V\,\rangle}$.

Then we have:

$$\rho^1(\Gamma_{\mathrm{nh}}) = [I^{-1}(\omega \times (I\omega)) - \frac{\langle\,(I\omega)\times\omega\,, I^{-1}V\,\rangle}{\langle\, V\,, I^{-1}V\,\rangle}(I^{-1}V)]^\alpha \frac{\partial}{\partial \omega_\alpha}.$$

a) For $\sigma = \omega^i e_i$ as $\theta_L = \frac{\partial L}{\partial \omega^\alpha}\mathcal{X}^\alpha = I_{\alpha\beta}\omega^\beta \mathcal{X}^\alpha$ then $G = \langle\, \theta_L\,, \sigma^c\,\rangle = L$. Then if G is bounded the Virial theorem states that:

$$\langle\!\langle \rho^1(\Gamma_{\mathrm{nh}})L \rangle\!\rangle = 0 \to \langle\!\langle -\frac{\langle\,(I\omega)\times\omega\,, I^{-1}V\,\rangle}{\langle\, V\,, I^{-1}V\,\rangle} V\cdot\omega \rangle\!\rangle = 0.$$

b) For $\sigma = a^i e_i$ the function G is $G = I_{\alpha\beta}\omega^\alpha a^\beta$ and from the virial theorem if G is bounded, it results:

$$\langle\!\langle [I^{-1}(\omega \times (I\omega))]^\alpha I_{\alpha\beta}a^\beta - \frac{\langle\,(I\omega)\times\omega\,, I^{-1}V\,\rangle}{\langle\, V\,, I^{-1}V\,\rangle}(I^{-1}V)^\alpha I_{\alpha\beta}a^\beta \rangle\!\rangle =$$

$$= \langle\!\langle \langle\, \omega \times (I\omega)\,, Ia\,\rangle - \frac{\langle\,(I\omega)\times\omega\,, I^{-1}V\,\rangle}{\langle\, V\,, I^{-1}V\,\rangle} V\cdot a \rangle\!\rangle = 0.$$

◁

4.6.2 Distributional approach

Regularity

In what follows we will also assume the considered Lagrangian is regular in at least a neighborhood of D, and that the constrained system $(L, \mathcal{D})$ is regular, i.e: the Lagrange-d'Alembert equations have a unique solution.

An important geometric object in this case, is the subbundle $F \subset T^A A|_{\mathcal{D}} \to \mathcal{D}$ of $\mathcal{T}^A A \to A$, whose fiber at a point $a \in \mathcal{D}$ is $F_a = \omega_L^{-1}(\tilde{\mathcal{D}}^\circ_{\tau(a)})$.

$$F_a = \{v \in \mathcal{T}^A_a A|\ \text{exists } \zeta \in \mathcal{D}^\circ_{\tau(a)} \text{ such that } \omega_L(v,u) = \langle\, \zeta\,, T\tau_M|_{T^{\mathcal{D}}\mathcal{D}}(u)\,\rangle, \forall u \in \mathcal{T}^A_a A\}.$$

In this case it will also be useful to give a geometrical characterization for the nonholonomic regular systems, equivalent to Theorem 1.

THEOREM 24: *The following properties are equivalent:*
(1) The constrained Lagrangian system $(L, \mathcal{D})$ *is regular;*
(2) Ker $G^{L,\mathcal{D}} = \{0\}$;
(3) $\mathcal{T}^A\mathcal{D} \cap F = \{0\} \Leftrightarrow (3')\ \mathcal{T}^A A|_{\mathcal{D}} = \mathcal{T}^A\mathcal{D} \oplus F$;
(4) $\mathcal{T}^{\mathcal{D}}\mathcal{D} \cap (\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp} = \{0\} \Leftrightarrow (4')\ \mathcal{T}^A A|_{\mathcal{D}} = \mathcal{T}^{\mathcal{D}}\mathcal{D} \oplus (\mathcal{T}^{\mathcal{D}}\mathcal{D})^{\perp}$,
where $(T^{\mathcal{D}}\mathcal{D})^{\perp} \subset T^A A|_{\mathcal{D}}$ *the orthogonal to* $T^{\mathcal{D}}\mathcal{D}$ *with respect to the symplectic form* ω_L *and where as in the* TQ *counterpart* $G^{L,\mathcal{D}}$ *is the restriction of the Hessian fiber* G^L *to* $\mathcal{D}$.

For the complete proof see [22].

Due to this theorem and using the same reasoning as in the Section 2.2.2, it results that any nonholonomic mechanical system on a Lie algebroid is always regular.

Projectors

Using the equivalent conditions for the regularity of a constrained Lagrangian system from Theorem 24 we will be able to arrive to the constrained dynamical section in terms of the free dynamical section, in two manners. For it, correspondingly we will define two projections onto $T^A\mathcal{D}$ and $T^{\mathcal{D}}\mathcal{D}$, corresponding to the decomposition (3'), respectively (4') of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$.

Projection to $\mathcal{T}^A\mathcal{D}$: As a consequence of the assumption that the constrained system $(L, \mathcal{D})$ is regular, we have the direct sum decomposition given by (3'): $T^A_a A = T^A_a\mathcal{D} \oplus F_a$, for every $a \in \mathcal{D}$.

This allows us to define the following complementary projectors, defined by the decomposition from above: $P_a : \mathcal{T}^A_a A \to \mathcal{T}^A_a\mathcal{D}$ and $Q_a : \mathcal{T}^A_a A \to F_a, \forall a \in \mathcal{D}$.

THEOREM 25: *Let* $(L, \mathcal{D})$ *be a regular constrained Lagrangian system and let* Γ_L *be the solution of the free dynamics, i.e.:* $i_{\Gamma_L}\omega_L = dE_L$. *Then the solution of the constrained dynamics is the* SODE, Γ_{nh}, *obtained by projection* $\Gamma_{nh} = P(\Gamma_L|_{\mathcal{D}})$.

Local expression in adapted coordinates: Let (x^i) be local coordinates on an open set $U \subset M$, and $\{e_a\}$ a local basis of sections of $\mathcal{D} \subset A$ and complete it to a basis of local sections of A, defined on U : $\{e_a, e_A\}$. In this case we denote the associated coordinates on A to this local basis by (x^i, y^a, y^A). In this set of coordinates, the constraints imposed by the submanifold $\mathcal{D} \subset A$ are just $y^A = 0$. If $\{e^a, e^A\}$ is the dual basis of $\{e_a, e_A\}$, then a basis for the annihilator $\mathcal{D}^\circ$ of $\mathcal{D}$ is $\{e^A\}$ and a basis for $\widetilde{\mathcal{D}^\circ}$ is $\mathcal{X}^A$.

A basis $\{Z_A\}$ of the local sections of F is given by $Z_A = \mathcal{X}_A - Q^a_A\mathcal{V}_a$, where $Q^a_A = W_{Ab}\mathcal{C}^{ab}$, with $\mathcal{C}^{ab}$ is the inverse matrix of $\mathcal{C}_{ab}(x^i, y^c) = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, 0)$.

The local expression of the projector Q_a over F is then: $Q = Z_A \otimes \mathcal{V}^A$ and the expression of the constrained dynamic section is given by:

$$\Gamma_{\mathrm{nh}} = y^a\mathcal{X}_a + (f^a + f^A Q^a_A)\mathcal{V}_a,$$

where all the functions f^α are evaluated at $y^A = 0$ and the expression of free dynamic section is: $\Gamma_L = y^\alpha\mathcal{X}_\alpha + f^\alpha\mathcal{V}_\alpha$.

Projection to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$: Again according to the regular condition for the constrained system $(L, \mathcal{D})$ it takes place (4'). This decomposition gives birth to the following two projections, $\overline{P}_a : \mathcal{T}^A_a A \to \mathcal{T}^{\mathcal{D}}_a \mathcal{D}$ and $\overline{Q}_a : \mathcal{T}^A_a A \to (\mathcal{T}^{\mathcal{D}}_a \mathcal{D})^{\perp}$, for all $a \in \mathcal{D}$. They are the equivalent to the ones presented in section 2.2.2 and with their help it can be arrived to the same result of the nonholonomic dynamical section, as using the preceding projections. The equivalent result to Theorem 2 takes place for a nonholonomic mechanical system on a Lie algebroid:

THEOREM 26: *Let $(L, \mathcal{D})$ be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e.: $i_{\Gamma_L}\omega_L = dE_L$. Then the solution of the constrained dynamics is the* SODE , Γ_{nh}, *obtained the projection $\Gamma_{nh} = \overline{P}(\Gamma_L|_{\mathcal{D}})$.*

Local expression: In the adapted coordinates presented above, a basis of $(T^{\mathcal{D}}\mathcal{D})^{\perp}$ is $\{Y_A, Z_A\}$ where the sections Z_A are given by 3.9!!, and the sections Y_A are: $Y_A = \mathcal{X}_A - Q^a_A \mathcal{X}_a + \mathcal{C}^{bc}(M_{Ab} - M_{ab}Q^a_A)\mathcal{V}_c$, with $M_{\alpha\beta} = \omega_L(\mathcal{X}_\alpha, \mathcal{X}_\beta)$. Then it can be given the expression of the projector onto $(T^{\mathcal{D}}\mathcal{D})^{\perp}$:

$$\overline{Q} = Z_A \otimes \mathcal{V}^A + Y_A \otimes \mathcal{X}^A.$$

For complete proofs for Theorem 25 and Theorem 26 see [22].

The distributional approach

Just like in the tangent bundle framework, here the Lagrange-d'Alembert equations can be entirely written in terms of objects in the manifold $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, which is not a Lie algebroid, but is a symplectical subbundle of $(\mathcal{T}^A A, \omega_L)$ if $(L, \mathcal{D})$ is regular. On the bundle $(\mathcal{T}^A A, \omega_L)$, the restriction of ω_L to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, denoted with $\omega^{L,\mathcal{D}}$ is a symplectical section. Again, denoting by $\overline{d}E_L$, the restriction of dE_L to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, the restriction of the Lagrange-d'Alembert equations to $\mathcal{T}^{\mathcal{D}}\mathcal{D}$:

$$i_{\Gamma_{\mathrm{nh}}}\omega^{L,\mathcal{D}} = \overline{d}E_L,$$

which uniquely determines the section Γ_{nh}. The proof of this statement is similar to the one found in Section 2.2.2.

Similar equations, within the framework of Lie algebroids, are the base of the theory proposed in [45].

Remark that if the constraint force, ω_L and dE_L, were restricted to $\mathcal{T}^A\mathcal{D}$, a Lie algebroid, ω_L is closed in this case, but as it is in general degenerated, no advantage over the previous approach would be obtained.

The nonholonomic bracket

Denote by X_f the Hamiltonian section on $\mathcal{T}^A A$ corresponding to a smooth function f on $\mathcal{D}$. The nonholonomic bracket, an almost-Poisson bracket, is defined by

$$\{f, g\}_{\mathrm{nh}} = \omega_L(\overline{P}(X_f), \overline{P}(X_g))$$

and the formulae $\dot{f} = \{f, E_L\}_{\mathrm{nh}}$ takes place, that implies the conservation of the energy in this case.

The virial theorem:

$$\langle\!\langle \{G, E_L\}_{\mathrm{nh}} \rangle\!\rangle = 0.$$

4.6.3 Virial theorem for nonholonomic systems on Lie algebroids in adapted coordinates:

Observe that we could use adapted coordinates to write the local expression of the Lagrange-d'Alembert equations and the virial theorem in this case. However as the equations are so similar to the ones in the TQ framework, gave in section 4.4.3, instead of writing them, we will follow the virial theorem for nonholonomic systems on Lie algebroids in adapted coordinates through the following examples.

Example 17 (The Chaplygin sleigh):

The chaplygin sleigh is a rigid body sliding on a horizontal plane. The body is supported at three points, two of which slide freely without friction while the third is a knife edge, a constraint that allows no motion orthogonal to this edge.

The configuration space of this system is the group $SE(2)$ of Euclidian motions of the two-dimensional plane $\mathbb{R}^2$. As local coordinates we can choose the angular orientation of the blade and the position of the contact point on the blade on the plane. Another choice of coordinates is given by considering the origin at the contact point and the first coordinate axis in the direction of the knife edge. Denoting by ω the angular velocity of the body, and by v_1, v_2 the components of the linear velocity of the contact point (relative to the body frame) we get a new coordinate system called the body frame. Then (ω, v_1, v_2) is regarded as an element of the Lie algebra $se(2)$.

The elements of the $se(2)$ are matrices of the form $\begin{pmatrix} 0 & c_3 & c_1 \\ -c_3 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}$, and the standard basis of the Lie algebra $se(2) \simeq \mathbb{R}^3$ is given by:
$\{E_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}$. For this basis, the Lie brackets are given by: $[E_2, E_0] = E_1; [E_1, E_2] = E_0$ and $[E_0, E_1] = 0$.

The position of the center mass is specified by the coordinates (a, b) relative to the body frame, and through m and J is specified the mass and the moment of inertia of the sleigh relative to the contact point. Then, the corresponding symmetric positive definite inertia operator $I : se(2) \to se(2)^*$, which is also the Hessian matrix of the Lagrangian to be considered, is:

$$I = \begin{pmatrix} J + m(a^2 + b^2) & -bm & am \\ -bm & m & 0 \\ am & 0 & m \end{pmatrix} \text{ and } I^{-1} = \frac{1}{J} \begin{pmatrix} 1 & b & -a \\ b & \frac{J}{m} + b^2 & -ab \\ -a & -ab & \frac{J}{m} + a^2 \end{pmatrix}.$$

The reduced nonholonomic Lagrangian system $(L, \mathcal{D})$ on $se(2)$ is defined by the Lagrangian:

$$\begin{aligned} L(\omega, v_1, v_2) &= \frac{1}{2}\langle\, I(\omega, v_1, v_2)\,, (\omega, v_1, v_2)\,\rangle = \\ &= \frac{1}{2}[(J + m(a^2+b^2))\omega^2 + m(v_1^2+v_2^2) - 2bm\omega v_1 + 2am\omega v_2]. \end{aligned}$$

and by the constrained manifold
$\mathcal{D} = \{(\omega, v_1, v_2) \in se(2) \,|\, v_2 = 0\}$.We will denote the given constraint $v_2 = 0$ with $\phi_1 = v_2 = 0$ and with $\phi_{1\alpha} = \frac{\partial \phi_1}{\partial w^\alpha}$.

The Lagrange-D'Alembert equations are:

$$\begin{cases} \dot{\omega} = \frac{am\omega}{J+ma^2}(b\omega - v_1) \\ \dot{v}_1 = \frac{a\omega}{J+ma^2}((J+m(a^2+b^2))\omega - mbv_1) \\ v_2 = 0. \end{cases}$$

Thus we will consider a basis adapted to the decomposition $\mathcal{D} \oplus \mathcal{D}^\perp$, precisely:

$$\{e_0 = E_2; e_1 = E_0; e_2 = -maE_2 - mabE_0 + (J+ma^2)E_1\}.$$

Then indeed, $\mathcal{D} = \mathrm{span}\{e_0, e_1\}$ while $\mathcal{D}^\perp = \mathrm{span}\{e_2\}$. So the coordinates associated to this basis are: (v_1, v_2, ω). The only Lie bracket that is not zero with this choice of basis is:

$$[e_0, e_1]_{\mathcal{D}} = \frac{ma}{J+ma^2}e_0 + \frac{mab}{J+ma^2}e_1.$$

Corresponding we have the basis of $T^{\mathcal{D}}\mathcal{D}$ given by:

$$\{\mathcal{X}_0(b) = (b, e_0(m), 0);\ \mathcal{X}_1(b) = (b, e_1(m), 0);\ \mathcal{V}_0(b) = (b, 0, \frac{\partial}{\partial \omega});\ \mathcal{V}_1(b) = (b, 0, \frac{\partial}{\partial v_1})\}.$$

And then: $X_L = \omega^a \mathcal{X}_a + I^{\alpha a} C^\gamma_{ba} \omega^b I_{\gamma m} \omega^m \mathcal{V}_\alpha$ and using that $\phi_{10} = 0;\ \phi_{11} = 0;\ \phi_{12} = 1$ we get:

$$\begin{aligned} \Gamma_{\mathrm{nh}} &= X_L + \lambda_1 W^{\alpha\gamma}\phi_{1\gamma}\mathcal{V}_\alpha = X_L + \lambda_1 W^{\alpha 2}\cdot 1 \cdot \mathcal{V}_\alpha = \\ &= X_L + \lambda_1 W^{02}\mathcal{V}_0 + \lambda_1 W^{12}\mathcal{V}_1,\ \text{ so:} \end{aligned}$$

$$\begin{aligned} \rho^1(\Gamma_{\mathrm{nh}}) &= [I^{0a}C^\gamma_{ba}\omega^b I_{\gamma m}\omega^m]\frac{\partial}{\partial \omega} + [I^{1a}C^\gamma_{ba}\omega^b I_{\gamma m}\omega^m]\frac{\partial}{\partial v^1} + \lambda\frac{-a}{J}\frac{\partial}{\partial \omega} + \lambda\frac{-ab}{J}\frac{\partial}{\partial v^1} = \\ &= [I^{0a}C^\gamma_{ba}\omega^b I_{\gamma m}\omega^m + \lambda\frac{-a}{J}]\frac{\partial}{\partial \omega} + [I^{1a}C^\gamma_{ba}\omega^b I_{\gamma m}\omega^m + \lambda\frac{-ab}{J}]\frac{\partial}{\partial v^1} = \\ &= [\frac{ma\omega}{J}(v^1 - b\omega) + \lambda\frac{-a}{J}]\frac{\partial}{\partial \omega} + [a\omega\frac{bm(v^1-b\omega) - J\omega}{J} + \lambda\frac{-ab}{J}]\frac{\partial}{\partial v^1} = \\ &= [S_1]\frac{\partial}{\partial \omega} + [S_2]\frac{\partial}{\partial v^1} \end{aligned}$$

where
$$\lambda = -\frac{m^2a^2(v^1 - b\omega)\omega}{J + a^2m}. \tag{4.33}$$

$$\theta_L = [(J + m(a^2 + b^2))\omega - 2bmv^1]\mathcal{X}^0 + [mv_1 - bm\omega]\mathcal{X}^1.$$

a) For $\sigma = a^\alpha e_\alpha \Rightarrow \sigma^c = a^\alpha \mathcal{X}_\alpha + B^\alpha \mathcal{V}_\alpha$, where the expression of B does not influence the virial function G whose expression is:

$$G = \langle\, \theta_L\, , \sigma^c\, \rangle = \omega[(J + m(a^2 + b^2))a^1 - bma^2] + v^1(ma^2 - 2bma^1).$$

The virial theorem $\langle\!\langle \rho^1(X_{\mathrm{nh}})G \rangle\!\rangle = 0$ says that:

$$\langle\!\langle [\frac{ma\omega}{J}(v^1 - b\omega) + \lambda\frac{-a}{J}][(J + m(a^2 + b^2))a^1 - bma^2]+ \\ + [ma\omega(\frac{bv^1}{J} - (\frac{1}{m} + \frac{b^2}{J})) + \lambda\frac{-ab}{J}][-2bma^1 + ma^2]\rangle\!\rangle = 0,$$

where we can afterwards replace λ by the expression (4.33).

Replace λ:

$$\langle\!\langle [\frac{ma\omega}{J}(v^1 - b\omega) - \frac{m^2a^2(v^1 - b\omega)\omega}{J + a^2m}\frac{-a}{J}][(J + m(a^2 + b^2))a^1 - bma^2]+ \\ + [ma\omega(\frac{bv^1}{J} - (\frac{1}{m} + \frac{b^2}{J})) - \frac{m^2a^2(v^1 - b\omega)\omega}{J + a^2m}\frac{-ab}{J}][-2bma^1 + ma^2]\rangle\!\rangle = 0.$$

◁

b) For $\sigma = \omega e_1 + v^1 e_2$, the virial theorem: $\langle\!\langle \rho^1(X_{\mathrm{nh}})G \rangle\!\rangle = 0$ gives that:

$$\langle\!\langle [S_1][(J + m(a^2 + b^2))2\omega + 3v^1bm] + [S_2]2v^1m \rangle\!\rangle = 0.$$

c) For $\sigma = v^1 e_1 + \omega e_2$, the virial theorem takes the following form:

$\langle\!\langle [S_1][(J + m(a^2 + b^2))v^1 - 2bm\omega + mv^1] + [S_2][\omega m - 4bmv^1 + (J + m(a^2 + b^2))\omega] \rangle\!\rangle = 0.$

Example 18 ($T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$):

Consider the dynamical system on the Lie algebroid $A = T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ determined by the Lagrangian $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{K^2}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2)$ and by the constraints: $\phi_1 = \dot{x} - r\omega_y$ and $\phi_2 = \dot{y} + r\omega_x$. The coordinates $(x, y, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ are associated to the following basis of sections of A: $\{e_1 = (\partial_x, 0);\ e_2 = (\partial_y, 0);\ e_3 = (0, X_3);\ e_4 = (0, X_4);\ e_5 = (0, X_5)\}$.

The structural functions of the Lie algebroid for this basis are given by:

-$[e_3, e_4]_A = -e_5$; $[e_3, e_5]_A = e_4$; $[e_4, e_5]_A = -e_3$; so the corresponding structural functions are: $C_{34}^5 = -1, C_{35}^4 = 1, C_{45}^3 = -1$.

-$\rho(e_1) = \partial_x$; $\rho(e_2) = \partial_y$, so $\rho_1^1 = \rho_2^2 = 1$, while the rest of the $\rho_j^i = 0$.

Then the basis of $\mathcal{T}^A A$ is: $\{\mathcal{X}_1(a) = (a, e_1(m), \partial_x|_a)$; $\mathcal{X}_2(a) = (a, e_2(m), \partial_y|_a)$; $\mathcal{X}_j(a) = (a, e_j(m), 0), j = 3, 4, 5$; $\mathcal{V}_1(a) = (a, 0, \partial_{\dot{x}}|_a)$; $\mathcal{V}_2(a) = (a, 0, \partial_{\dot{y}}|_a)$; $\mathcal{V}_3(a) = (a, 0, \partial_{\omega_x}|_a)$; $\mathcal{V}_4(a) = (a, 0, \partial_{\omega_y}|_a)$; $\mathcal{V}_5(a) = (a, 0, \partial_{\omega_z}|_a)\}$,

while the basis of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$.

However, we will work with the following coordinates adapted to the restrictions the movement has:

$$\begin{cases} w^1 = \dot{x} = v^1, \\ w^2 = \dot{y} = v^2, \\ w^3 = \omega_z = v^5, \\ w^4 = \phi_1 = v^1 - rv^4, \\ w^5 = \phi_2 = v^2 + rv^3, \end{cases}$$

where for simplicity we changed the notation of the initial coordinates in the fiber by v_i.

Let us denote by $\{f_i\}, i = \overline{1, 5}$ the basis of sections of A which correspondes to this change of coordinates. Then the corresponding basis of $\mathcal{T}^A A$ is:

$\{\mathcal{X}'_1(a) = (a, f_1(m), \partial_x|_a)$; $\mathcal{X}'_2(a) = (a, f_2(m), \partial_y|_a)$; $\mathcal{X}'_i(a) = (a, f_i(m), 0), i = 3, 4, 5$; $\mathcal{V}'_i(a) = (a, 0, \partial_{w^i}|_a), i = 1, 5\}$.

Note that $\mathcal{V}'_4, \mathcal{V}'_5$ are not zero, but as for the elements in $\mathcal{T}^{\mathcal{D}}\mathcal{D}$, $w^4 = \phi_1 = 0$ and $w^5 = \phi_2 = 0$, so: $\rho^1(\mathcal{V}'_4) = \rho^1(\mathcal{V}'_5) = 0$.

In this adapted coordinates, the basis of $\mathcal{T}^{\mathcal{D}}\mathcal{D}$ is: $\{\mathcal{X}'_1, \mathcal{X}'_2, \mathcal{X}'_3, \mathcal{V}'_1, \mathcal{V}'_2, \mathcal{V}'_3\}$.

We will use the following formulae to find the expression of the dynamical section:

$$\begin{aligned} \Gamma_{\text{nh}} &= X_L + \lambda_a Z_a = \\ &= w^\alpha \mathcal{X}'_\alpha + b^\alpha \mathcal{V}'_\alpha + \lambda_a Z_a^\alpha \mathcal{V}'_\alpha = \\ &= w^\alpha \mathcal{X}'_\alpha + b^\alpha \mathcal{V}'_\alpha + \lambda_a W^{\alpha\gamma} \phi_{a\gamma} \mathcal{V}'_\alpha, \quad \text{for } \alpha = 1, 5, \end{aligned} \tag{4.34}$$

where $\phi_{a\gamma} = \frac{\partial \phi_a}{\partial w^\gamma}$.

Using $v_4 = (w^1 - w^4)/r$ and $v_3 = (w^5 - w^2)/r$, the expression of the Lagrangian in adapted coordinates follows :

$$L = \frac{1}{2}((w^1)^2 + (w^2)^2) + \frac{K^2}{2}((w^3)^2 + (\frac{(w^5 - w^2)}{r})^2 + (\frac{(w^1 - w^4)}{r})^2),$$

while the elements of the inverse matrix of W are:

$$\begin{cases} W^{11} = W^{22} = W^{41} = W^{52} = W^{14} = W^{25} = 1, \\ W^{33} = \frac{1}{K^2}\ , W^{44} = W^{55} = 1 + \frac{r^2}{k^2}. \end{cases}$$

As the change of coordinate matrix is:

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -r & 0 \\ 0 & 1 & r & 0 & 0 \end{pmatrix} v$$

then the corresponding change of basis is given by:

$$f_i = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & 1 & \frac{-1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{-1}{r} & 0 \\ 0 & 0 & \frac{1}{r} & 0 & 0 \end{pmatrix} e_i$$

so $f_1 = e_1 + \frac{1}{r}e_4$; $f_2 = e_2 - \frac{1}{r}e_3$; $f_3 = e_5, f_4 = -\frac{1}{r}e_4$; $f_5 = \frac{1}{r}e_3$ and remark that as a consequence $\rho'^i_\alpha = \rho^i_\alpha$.

Using the C^i_{jk} for e_i, we get the C'^i_{jk} for f_i.

$$\begin{cases} C'^3_{12} = -\frac{1}{r^2};\ C'^5_{13} = -1;\ C'^3_{15} = \frac{1}{r^2};\ C'^4_{23} = 1;\ C'^3_{24} = -\frac{1}{r^2}; \\ C'^5_{34} = -1;\ C'^4_{35} = 1;\ C'^3_{45} = -\frac{1}{r^2}. \end{cases}$$

Having all this information, from the expression of b^α which is left to $b^\alpha = W^{\alpha\beta}w^c C'^\epsilon_{c\beta}\frac{\partial L}{\partial w^\epsilon}$, we get that $b^\alpha = 0$ and in consequence the dynamical section is: $X_L = w^1\mathcal{X}'_1 + w^2\mathcal{X}'_2 + w^3\mathcal{X}'_3$, more explicitly:

$$X_L(a) = (a, w^1 f_1(m) + w^2 f_2(m) + w^3 f_3(m), w^1\partial_x|_a + w^2\partial_y|_a).$$

Indeed, the Lagrange-d'Alembert equations are:

$$\begin{cases} \dot w^1 + \frac{K^2}{r^2}(\dot w^1 - \dot w^4) = w^j C'^k_{j1}\frac{\partial L}{\partial w^k} = 0 \\ \dot w^2 + \frac{K^2}{r^2}(\dot w^2 - \dot w^5) = w^j C'^k_{j2}\frac{\partial L}{\partial w^k} = 0 \\ K^2\dot w^3 = w^j C'^k_{j3}\frac{\partial L}{\partial w^k} = 0 \\ \frac{K^2}{r^2}(\dot w^4 - \dot w^1) = w^j C'^k_{j4}\frac{\partial L}{\partial w^k} = 0 \\ \frac{K^2}{r^2}(\dot w^5 - \dot w^2) = w^j C'^k_{j5}\frac{\partial L}{\partial w^k} = 0 \end{cases}$$

and therefore we obtain: $\dot w^1 = \dot w^2 = \dot w^3 = \dot w^4 = \dot w^5 = 0$

We will calculate Z_1, Z_2 in order to be able to use formula (4.34), taking into consideration that as $\phi_1 = w^4$ and $\phi_2 = w^5$ and that in consequence we have $\phi_{14} = 1$ and $\phi_{25} = 1$, while all the other ϕ_{ij} are zero.

$$Z_1 = W^{\alpha\beta}\phi_{1\beta}\mathcal{V}_\alpha = W^{\alpha 4}\phi_{14}\mathcal{V}'_\alpha = W^{14}\phi_{14}\mathcal{V}'_1 + W^{44}\phi_{14}\mathcal{V}'_4 =$$
$$= \mathcal{V}'_1 + (1 + \frac{r^2}{K^2})\mathcal{V}'_4.$$

$$\begin{aligned} Z_2 = W^{\alpha\beta}\phi_{2\beta}\mathcal{V}_\alpha = W^{\alpha 5}\phi_{25}\mathcal{V}'_\alpha = W^{25}\phi_{25}\mathcal{V}'_2 + W^{55}\phi_{25}\mathcal{V}'_5 = \\ = \mathcal{V}'_2 + (1+\frac{r^2}{K^2})\mathcal{V}'_5. \end{aligned}$$

Now, to calculate λ_1, λ_2 use:

$$\begin{cases} d\phi_1(X_L) + \lambda_a d\phi_1(Z_a) = 0 \quad (1) \\ d\phi_2(X_L) + \lambda_a d\phi_2(Z_a) = 0 \quad (2) \end{cases}$$

Using $dF = \rho'^i_\alpha \frac{\partial F}{\partial x^i}\mathcal{X}'^\alpha + \frac{\partial F}{\partial y^\alpha}\mathcal{V}'^\alpha$ we get:

$$\begin{aligned} d\phi_1 = &\rho'^i_1\frac{\partial\phi_1}{\partial x^i}\mathcal{X}'^1 + \rho'^i_2\frac{\partial\phi_1}{\partial x^i}\mathcal{X}'^2 + \rho'^i_3\frac{\partial\phi_1}{\partial x^i}\mathcal{X}'^3 + \rho'^i_4\frac{\partial\phi_1}{\partial x^i}\mathcal{X}'^4 + \rho'^i_5\frac{\partial\phi_1}{\partial x^i}\mathcal{X}'^5 + \\ &+ \frac{\partial\phi_1}{\partial w^1}\mathcal{V}'^1 + \frac{\partial\phi_1}{\partial w^2}\mathcal{V}'^2 + \frac{\partial\phi_1}{\partial w^3}\mathcal{V}'^3 + \frac{\partial\phi_1}{\partial w^4}\mathcal{V}'^4 + \frac{\partial\phi_1}{\partial w^5}\mathcal{V}'^5 = \\ &= \frac{\partial\phi_1}{\partial x}\mathcal{X}'^1 + \frac{\partial\phi_1}{\partial y}\mathcal{X}'^2 + \mathcal{V}'^4 = \mathcal{V}'^4. \end{aligned}$$

$$d\phi_2 = \mathcal{V}'^5.$$

We get that $\lambda_1 = \lambda_2 = 0$ from equation (1) and (2) as:

$$(1) \Leftrightarrow \mathcal{V}'^4(X_L) + \lambda_1\mathcal{V}'^4(Z_1) + \lambda_2\mathcal{V}'^4(Z_2) = 0 \Leftrightarrow \lambda_1(1+\frac{r^2}{K^2}) = 0, \text{ so } \lambda_1 = 0.$$

$$(2) \Leftrightarrow \mathcal{V}'^5(X_L) + \lambda_1\mathcal{V}'^5(Z_1) + \lambda_2\mathcal{V}'^5(Z_2) = 0 \Leftrightarrow \lambda_2(1+\frac{r^2}{K^2}) = 0, \text{ so } \lambda_2 = 0.$$

In conclusion $\Gamma_{\mathrm{nh}} = X_L$, thus:

$$\rho^1(X_{\mathrm{nh}}) = w^1\partial_x + w^2\partial_y.$$

With the help of:

$$\begin{aligned} \theta_L =&(w^1 + \frac{K^2}{r^2}(w^1 - w^4))\mathcal{X}'^1 + (w^2 + \frac{K^2}{r^2}(w^2 - w^5))\mathcal{X}'^2 + \\ &+ K^2 w^3\mathcal{X}'^3 + \frac{K^2}{r^2}(w^4 - w^1)\mathcal{X}'^4 + \frac{K^2}{r^2}(w^5 - w^2)\mathcal{X}'^5. \end{aligned}$$

we will construct the function $G = \langle\, \theta_L\,, \sigma^c\,\rangle|_{\mathcal{D}}$, where we remember that for $\sigma = \sigma^i e_i$ the expression of its complete lift is: $\sigma^c = \sigma^\alpha\mathcal{X}_\alpha + (\dot\sigma^\alpha + C^\alpha_{\beta\gamma}\sigma^\beta y^\gamma)\mathcal{V}_\alpha$.

Note that only the coefficients of $\mathcal{X}'_i,\ i = 1,2,3$ of σ^c will count in determining the function G, due to the fact that θ_L has only $\mathcal{X}^a$.

a')$\sigma = xf_1 + yf_2 \Rightarrow G = (1+\frac{K^2}{r^2})(xw^1 + yw^2)$.

$$
\begin{aligned}
G =& \langle \theta_L \,, \sigma^c \rangle|_{\mathcal{D}} = \\
&= x(1+\frac{K^2}{r^2})w^1 + y(1+\frac{K^2}{r^2})w^2 = \\
&= (1+\frac{K^2}{r^2})(xw^1+yw^2).
\end{aligned}
$$

$$
\langle\!\langle \rho^1(\Gamma_{\rm nh})G \rangle\!\rangle = \langle\!\langle (w^1\frac{\partial}{\partial x} + w^2\frac{\partial}{\partial y})[(1+\frac{K^2}{r^2})(xw^1+yw^2)] \rangle\!\rangle
$$
$$
\langle\!\langle (1+\frac{K^2}{r^2})[(w^1)^2+(w^2)^2] \rangle\!\rangle = 0.
$$

b') $\sigma = xyf_3 \Rightarrow G = K^2w^3xy$

$$
\langle\!\langle \rho^1(\Gamma_{\rm nh})G \rangle\!\rangle = \langle\!\langle (w^1\frac{\partial}{\partial x} + w^2\frac{\partial}{\partial y})(K^2w^3xy) \rangle\!\rangle
$$
$$
\langle\!\langle K^2w^3(w^1y+w^2x) \rangle\!\rangle = 0.
$$

c') $\sigma = x^2f_1 + y^2f_2 \Rightarrow G = \langle \theta_L \,, \sigma^c \rangle|_{\mathcal{D}} = (1+\frac{K^2}{r^2})[w^1x^2+w^2y^2]$

$$
\langle\!\langle \rho^1(\Gamma_{\rm nh})G \rangle\!\rangle = \langle\!\langle (w^1\frac{\partial}{\partial x} + w^2\frac{\partial}{\partial y})[(1+\frac{K^2}{r^2})(x^2w^1+y^2w^2)] \rangle\!\rangle
$$
$$
\langle\!\langle (1+\frac{K^2}{r^2})(2x(w^1)^2+2y(w^2)^2) \rangle\!\rangle = 0.
$$

d') $\sigma = yf_1 + xf_2 \Rightarrow G = \langle \theta_L \,, \sigma^c \rangle|_{\mathcal{D}} = (1+\frac{K^2}{r^2})[yw^1+xw^2]$

$$
\langle\!\langle \rho^1(\Gamma_{\rm nh})G \rangle\!\rangle = \langle\!\langle (w^1\frac{\partial}{\partial x} + w^2\frac{\partial}{\partial y})[(1+\frac{K^2}{r^2})(yw^1+xw^2)] \rangle\!\rangle
$$
$$
\langle\!\langle (1+\frac{K^2}{r^2})(2w^1w^2) \rangle\!\rangle = 0.
$$

$\triangleleft$

Conclusions and future work

The main purpose of our work is to present applications of the Lie algebroid structure in both mathematical and physical context. In the first chapter we have introduced the notion of Lie algebroid, presenting a number of examples, and we have presented some useful properties that we used later on.

One of our principal results in the mathematical part was to give a generalization of the notion of Jacobi fields corresponding to SODE on manifolds and on Lie algebroids. We have done that considering a new take on a first order variational equation on a manifold. We also generalized the Jacobi equation for this generalized cases of Jacobi fields associated to SODE. For that we had to generalize the non-linear connection and the Jacobi endomorphism to the context of Lie algebroid. We used this theory in the particular instance of a geodesic spray on a Riemannian Lie algebroid. For this case we have shown that an integral curve of it has no conjugate points along it if and only if it minimizes the energy functional of the system whose solution are given by the geodesic spray. To exemplify the theorem we considered the space of skew-symmetric matrices of dimension 3 who has a Lie algebroid structure.

In Chapter 4, for the physical counterpart, we analyzed the virial theorem in the first place for mechanical systems and nonholonomic systems on the tangent bundle, and afterwards, for unconstrained and nonholonomic systems on Lie algebroids. We could prove that a virial like theorem holds for systems on Lie algebroids, fact that will allow us to obtain information about the time average of the action of the dynamical section upon the virial function for more systems than before due to the wide range of systems that can be described with the help of a Lie algebroid structure. Also in this chapter we have presented in detail instances of this theorem through some examples.

We find interesting for further investigation to see if the minimizing theorem presented here takes place for any Lagrangian, not necessarily a Riemannian one and for the other topology. Precisely see in what conditions the result holds when we look for the geodesic to be a strong minimum for the energy functional.

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