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SOLVMANIFOLDS WITH HOLOMORPHICALLY TRIVIAL CANONICAL BUNDLE

(Solvariedades con fibrado canónico holomórficamente trivial)

Memoria presentada por

ANTONIO OTAL GERMÁN

para optar al Grado de

Doctor por la Universidad de Zaragoza

Dirigida por el doctor

Luis Ugarte Vilumbrales, Universidad de Zaragoza (España)

Primum principium indemonstrabile est, quod "non est simul affirmare et negare", quod fundatur supra rationem entis et non entis, et super hoc principio omnia alia fundatur ut dicit Philosophus in IV Metaph., cap.3.

S. Thomas. Summa Theologica, *I*[a] *II*[æ] qæstio 94, art.2.

Agradecimientos

Cuando se lee el nombre del autor de una tesis doctoral se corre el peligro de pensar que ese trabajo se ha debido exclusivamente al propio empeño personal del mismo. Esto, siendo parte de la verdad, no es toda la verdad, ya que tratándose de un trabajo que se extiende a lo largo de cinco años, necesariamente ha involucrado a muchas personas e instituciones hasta su conclusión. Aún a costa de no disponer de espacio para poder citarlas a todas ellas, sirvan estas líneas como un humilde reconocimiento a su ayuda.

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Antonio Otal Germán,
Zaragoza, 10 de julio del año 2014.

Contents

Resumen

Las variedades de Calabi-Yau constituyen una de las más importantes clases en geometría. Estas variedades, que pueden ser consideradas como la generalización a varias dimensiones de las superficies K3, son variedades complejas compactas (M, J) de dimensión compleja n dotadas de una $SU(n)$ estructura (F, Ψ), de manera que la 2-forma fundamental F es cerrada y la forma Ψ de tipo $(n, 0)$ es holomorfa. Por tanto, la holonomía de la métrica $g(\cdot, \cdot) = F(\cdot, J\cdot)$ se reduce a un subgrupo de $SU(n)$, lo que a su vez implica que g es Ricci llana, y el fibrado canónico de (M, J) es holomórficamente trivial.

Las anteriores condiciones que caracterizan las variedades de Calabi-Yau han sido debilitadas en varias direcciones de tal forma que las geometrías resultantes siguen jugando un papel importante en diferentes aspectos de la geometría compleja. En esta memoria nos centramos en la geometría de variedades complejas compactas (M, J) con fibrado canónico holomórficamente trivial dotadas de métricas Hermitianas que son menos restrictivas que las métricas Kähler.

En el caso de las superficies complejas compactas, las posibilidades de que admitan fibrado canónico holomórficamente trivial se reducen, salvo isomorfismo, a una superficie $K3$, un toro complejo o una superficie de Kodaira, donde las dos primeras son Kähler y la última es un ejemplo de nilvariedad $M = G/\Gamma$, es decir, un cociente compacto de un grupo de Lie conexo y simplemente conexo G por un lattice Γ de rango máximo en G. Sin embargo, no hay ninguna clasificación en dimensión compleja 3 o superior, por lo que es natural comenzar a estudiar esta geometría compleja en algunas clases particulares de variedades compactas de dimensión real 6. La clase de nilvariedades con estructura compleja invariante es una buena candidata para este estudio, ya que Salamon [82] probó que cualquier variedad compleja de este tipo tiene fibrado canónico holomórficamente trivial. Salamon proporciona a su vez en [82] una clasificación de nilvariedades de dimensión real 6 admitiendo este tipo de estructuras complejas, siendo la (nil)variedad de Iwasawa un ejemplo bien conocido que juega un papel relevante en geometría compleja (véase por ejemplo [6] y las referencias allí contenidas).

Aunque la geometría compleja de nilvariedades constituye una fuente importante de ejemplos en geometría diferencial, estos espacios nunca satisfacen el $\partial\bar{\partial}$-lema porque no son formales a excepción de los toros complejos [26, 44]. Sin embargo, la investigación de determinadas propiedades en geometría compleja requiere variedades complejas compactas cumpliendo el $\partial\bar{\partial}$-lema, por lo que es preciso considerar una clase más amplia

de espacios homogéneos $M = G/\Gamma$. La primera extensión natural de las nilvariedades viene dada por los cocientes compactos de grupos de Lie G que son resolubles en vez de nilpotentes. Un ejemplo destacable de este tipo es la variedad de Nakamura [67], la cual presenta una geometría muy rica. Esta clase de variedades, denominadas solvariedades, es el objeto central de estudio en esta tesis. Más concretamente, describimos las solvariedades de dimensión 6 que poseen una estructura compleja invariante con fibrado canónico holomórficamente trivial, así como una clasificación de tales estructuras.

Otro de los objetivos de esta memoria es el estudio de métricas Hermitianas especiales que son menos restrictivas que las métricas Kähler. Es bien sabido que la existencia de una métrica Kähler sobre una variedad compacta impone fuertes restricciones topológicas. Por contra, en la clase conforme de cualquier métrica Hermitiana sobre una variedad compleja compacta (M, J) de dimensión compleja n existe siempre una métrica standard (también llamada Gauduchon), es decir, una métrica Hermitiana cumpliendo $\partial\bar{\partial}F^{n-1} = 0$ [40]. Entre la clase Kähler y la clase Gauduchon, otras clases interesantes de métricas Hermitianas especiales han sido consideradas en relación con diversos problemas en geometría diferencial. Por ejemplo, una métrica Hermitiana se dice equilibrada si la forma fundamental F cumple que F^{n-1} es cerrada, y se dice que es fuertemente Gauduchon si la $(n, n-1)$-forma ∂F^{n-1} es $\bar{\partial}$-exacta. Las métricas fuertemente Gauduchon han sido recientemente introducidas por Popovici en [74] mientras que las equilibradas fueron consideradas previamente por Michelsohn [62].

Por otra parte, Fu, Wang y Wu han introducido en [39] una generalización de las métricas Gauduchon sobre variedades complejas. Para cada entero $1 \leq k \leq n-1$, se dice que una métrica Hermitiana es k-Gauduchon si $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$. Se observa que por definición las métricas $(n-1)$-Gauduchon coinciden con las métricas Gauduchon, y para $k = 1$ se tiene que la clase de métricas 1-Gauduchon contiene las métricas Kähler con torsión, también conocidas como métricas pluricerradas. Streets y Tian introdujeron en [88] un flujo de Ricci Hermitiano bajo el cual la condición Kähler con torsión es preservada. La geometría Kähler con torsión también ha sido estudiada por diversos autores (véase por ejemplo [33, 34, 90]). Las nilvariedades de dimensión 6 que admiten métricas invariantes Kähler con torsión, 1-Gauduchon o equilibradas han sido determinadas en [33, 35, 95]. En esta tesis estudiamos la existencia de estas métricas así como la existencia de métricas fuertemente Gauduchon en la clase más amplia de las solvariedades de dimensión 6 provistas con una estructura compleja invariante cuyo fibrado canónico es holomórficamente trivial.

Asociada a cualquier variedad compleja compacta M existen diversos invariantes complejos que miden varios aspectos específicos de M. Entre ellos distinguimos las cohomologías de Dolbeault, Bott-Chern [14], Aeppli [1] y la sucesión espectral de Frölicher $\{E_r(M)\}_{r\geq 1}$ que relaciona la cohomología de Dolbeault con la cohomología de de Rham de la variedad [38]. Si M es una variedad Kähler compacta entonces todos estos invariantes complejos coinciden porque M satisface el $\partial\bar{\partial}$-lema, sin embargo la sucesión espectral de Frölicher puede no degenerar en el primer paso para variedades complejas compactas arbitrarias. Un problema interesante en geometría compleja es el estudio del comportamiento de estos invariantes. En esta memoria damos una descripción completa de la sucesión $\{E_r(M)\}_{r\geq 1}$ en el caso de las nilvariedades de dimensión 6, mientras que para las solvariedades de dimensión 6 dotadas de una estructura compleja invariante de tipo splitting [51] con fibrado canónico holomórficamente trivial usamos los resultados de Kasuya y Angella [51, 7] y de Angella y Tomassini [10] para determinar cuándo se cumple el $\partial\bar{\partial}$-lema. A su vez, motivados por el trabajo [76], exploramos en esta tesis las relaciones entre la degeneración de la sucesión espectral de Frölicher, el $\partial\bar{\partial}$-lema y la existencia de métricas equilibradas o fuertemente Gauduchon, así como su comportamiento por deformaciones holomorfas de la estructura compleja.

A continuación describimos con más detalle los contenidos de cada capítulo de esta memoria de tesis.

El Capítulo 1 tiene como finalidad ubicar el objeto de estudio de la tesis dentro del marco más general de las variedades complejas. En la Sección 1.1 se repasan nociones y aspectos básicos sobre geometría compleja en general. Las variedades complejas se presentan bajo dos puntos de vista, por un lado como variedades diferenciables que admiten un atlas holomorfo compatible con la estructura diferenciable de la variedad y por otro como variedades diferenciables junto con un campo de tensores diferenciable $J \in \mathrm{End}(TM)$ tal que $J^2 = -\mathrm{Id}_{TM}$, cumpliendo a su vez la condición de integrabilidad obtenida por Newlander y Nirenberg [69]. Aunque ambos puntos de vista son equivalentes, sin embargo este último enfoque es el que se sigue principalmente en la memoria. La presencia de una estructura compleja da lugar además a la existencia de fibrados vectoriales holomorfos, entre los que destacan el fibrado tangente holomorfo $\mathcal{T}_M$, su dual holomorfo $\Omega^1_M(M)$ y los fibrados de p-formas holomorfas $\Omega^p_M(M) := \wedge^p\Omega^1_M(M)$ con $1 \leq p \leq n$, siendo $n = \dim_{\mathbb{C}} M$. A este último tipo pertenece el llamado fibrado canónico holomorfo $K_M := \Omega^n_M(M)$.

Es posible asociar diversos complejos diferenciales a las variedades complejas que dan lugar a distintas cohomologías. Este aspecto se trata en la Sección 1.2. La presencia de una estructura compleja induce una bigraduación en la complexificación del complejo de formas diferenciales $(\wedge^\bullet M_{\mathbb{C}}, d)$ que da lugar a un álgebra bidiferencial bigraduada $(\wedge^{\bullet,\bullet} M, \partial, \bar{\partial})$, siendo $d = \partial + \bar{\partial}$. A estos complejos diferenciales se asocian los grupos de cohomología de de Rham $H^\bullet_{\mathrm{dR}}(M;\mathbb{C}) = \ker d/\mathrm{im} d$ y de Dolbeault $H^{\bullet,\bullet}_{\bar{\partial}}(M) = \ker\bar{\partial}/\mathrm{im}\bar{\partial}$ de la variedad compleja, cuyas dimensiones $b_\bullet(M) := \dim H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$ y $h^{\bullet,\bullet}_{\bar{\partial}}(M) := \dim H^{\bullet,\bullet}_{\bar{\partial}}(M)$ son finitas cuando la variedad es compacta. A partir del anterior complejo bigraduado se presentan otras cohomologías de interés tales como la cohomología de Aeppli $H^{\bullet,\bullet}_{\mathrm{A}}(M) = \ker\partial\bar{\partial}/(\mathrm{im}\partial + \mathrm{im}\bar{\partial})$ y la cohomología de Bott-Chern $H^{\bullet,\bullet}_{\mathrm{BC}}(M) = (\ker\partial \cap \ker\bar{\partial}))/\mathrm{im}\partial\bar{\partial}$ [1, 14]. La conjugación induce un isomorfismo para ambas cohomologías entre los grupos de bigrados (p,q) y (q,p). Además, es posible desarrollar una teoría de Hodge asociada a estas dos cohomologías [85] por medio de la cual se obtienen diversas propiedades interesantes, como la finitud de las dimensiones $h^{\bullet,\bullet}_{\mathrm{A}}(M)$, $h^{\bullet,\bullet}_{\mathrm{BC}}(M)$ de estos grupos en el caso compacto y que ambas cohomologías son duales en el sentido $h^{p,q}_{\mathrm{A}}(M) = h^{n-q,n-p}_{\mathrm{BC}}(M)$ para todo $p,q \in \mathbb{N}$. De las definiciones de estas cohomologías es posible establecer de manera natural aplicaciones $H^{\bullet,\bullet}_{\mathrm{BC}}(M) \to H^{\bullet,\bullet}_{\bar{\partial}}(M) \to H^{\bullet,\bullet}_{\mathrm{A}}(M)$ bien definidas, que en general no son ni inyectivas ni suprayectivas. La validez del isomorfismo $H^{\bullet,\bullet}_{\mathrm{BC}}(M) \cong H^{\bullet,\bullet}_{\bar{\partial}}(M)$ es equivalente a que la variedad compleja cumpla el $\partial\bar{\partial}$-lema [26]. Más recientemente Angella y Tomassini [10] caracterizan el $\partial\bar{\partial}$-lema en términos de la anulación de ciertos invariantes complejos que involucran los números de Betti y las dimensiones de los grupos de cohomología de Aeppli y de Bott-Chern. Finalmente, se presenta la sucesión espectral de Frölicher $\{E_r^{\bullet,\bullet}(M)\}_{r\geq 1}$ que parte de la cohomología de Dolbeault de una variedad compleja, identificada con el primer término $E_1^{\bullet,\bullet}(M)$ de la sucesión, y converge a su cohomología de de Rham, el término $E_\infty^{\bullet,\bullet}(M)$, siendo éste alcanzado en un número finito de pasos. La desigualdad obtenida por Frölicher [38] acota superiormente los números de Betti de la variedad por una suma de números de Hodge de M.

Como se ha mencionado al principio, algunas de las variedades complejas más interesantes se distinguen por la presencia de algún tipo especial de métrica Hermitiana, las cuales se recuerdan en la Sección 1.3. Las métricas Hermitianas se describen por medio de una 2-forma positiva $F \in \wedge^{1,1}M$, llamada forma fundamental (o forma de Kähler). Es bien sabido que sobre una variedad compleja compacta M de dimensión compleja n siempre existen métricas compatibles con la estructura compleja, sin embargo Gauduchon [40] prueba que además siempre existe una métrica standard (también conocida como métrica Gauduchon), dada por $\partial\bar{\partial}F^{n-1} = 0$, en la clase conforme de cada métrica Hermitiana. La presencia de una métrica Kähler [50, 83], definida por $dF = 0$, impone fuertes restricciones topológicas a la variedad, algunas de las cuales vienen expresadas en términos de invariantes cohomológicos. Por ejemplo, Deligne, Griffiths, Morgan y Sullivan [26] demuestran que la existencia de una tal métrica sobre una variedad compleja compacta implica que ésta cumple el $\partial\bar{\partial}$-lema y por tanto la variedad subyacente es formal.

La condición Kähler puede debilitarse en dos direcciones. Por un lado, cuando la torsión de la conexión de Bismut [12] es cerrada, la métrica Hermitiana se denomina Kähler con torsión [33], siendo estas métricas caracterizadas también por la condición $F \in \ker \partial\bar{\partial}$. La geometría con torsión, además de su importancia en Física Matemática en el contexto de ciertos modelos supersimétricos [41] y en algunos tipos de teorías de cuerdas [89], juega un papel central en el flujo de Ricci Hermitiano introducido por Streets y Tian [88]. Recientemente se han introducido clases de estructuras especiales denominadas Gauduchon generalizadas [39], definidas por la condición $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$ para algún $k \in \mathbb{N}$ tal que $1 \leq k \leq n-1$. Estas estructuras contienen a las métricas Kähler con torsión para $k = 1$ y coinciden con las métricas Gauduchon cuando $k = n-1$. La otra dirección en la que se pueden debilitar las métricas Kähler viene dada por la condición $F^{n-1} \in \ker d$. Este tipo de métricas Hermitianas se denominan equilibradas [62] y pertenecen a la clase $\mathcal{W}_3$ en la clasificación de Gray-Hervella [42]. A su vez desempeñan un papel importante en compactificaciones en teorías heteróticas de cuerdas [89]. Popovici [75] ha introducido una clase intermedia entre las métricas equilibradas y las Gauduchon, denominadas fuertemente Gauduchon, que vienen definidas como $\partial F^{n-1} \in \operatorname{im}\bar{\partial}$. Además, prueba en [72] que para una variedad compleja compacta que cumple el $\partial\bar{\partial}$-lema, la condición Gauduchon y la condición fuertemente Gauduchon coinciden.

Una misma variedad diferenciable puede poseer distintas estructuras complejas de modo que las variedades complejas resultantes son no biholomorfas. Todas las estructuras complejas sobre una variedad dada forman un espacio denominado espacio de moduli de estructuras complejas de la variedad. Conocer dicho espacio de moduli es un problema de enorme dificultad, aunque la teoría de deformaciones holomorfas de estructuras complejas desarrollada por Kodaira, Spencer, Nirenberg [53, 54] y Kuranishi [55] proporciona un medio para afrontar esta cuestión al menos parcialmente. Esta teoría se presenta en la Sección 1.4 que se subdivide en dos partes. La primera parte presenta la noción de deformación holomorfa $\{(M, J_{\mathbf{t}})\}_{\mathbf{t}\in\mathcal{B}}$ de una variedad compleja (M, J) dada. Por una deformación holomorfa se entiende una familia de estructuras complejas $\{J_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ sobre una misma variedad real subyacente M indexadas por un parámetro $\mathbf{t}$ que pertenece a un espacio complejo $\mathcal{B}$. Cuando el parámetro alcanza un valor distinguido $\mathbf{t_0}$ se recupera la estructura compleja original $J_{\mathbf{t_0}} := J$. El teorema de Kuranishi da una descripción local del espacio de deformaciones holomorfas de una variedad compleja, conocido desde entonces como espacio de Kuranishi de M y denotado por $\mathrm{Kur}(M)$, que permite calcular de una manera relativamente sencilla nuevas variedades complejas compactas. El objeto de la segunda parte es presentar uno de los aspectos más interesantes de la teoría de deformaciones consistente en el comportamiento abierto o cerrado de diversas propiedades de la variedad compleja bajo deformaciones. Por un lado, se dice que una propiedad $\mathcal{P}$ es abierta, o estable, bajo deformaciones holomorfas si cuando se cumple $\mathcal{P}$ para (M, J) entonces también se cumple $\mathcal{P}$ para cualquier deformación $(M, J_{\mathbf{t}})$ suficientemente pequeña. Por otro lado, se dice que una propiedad $\mathcal{P}$ es cerrada bajo deformaciones holomorfas si cuando se cumple $\mathcal{P}$ en $(M, J_{\mathbf{t}})$ para todos los valores del parámetro excepto en un valor distinguido $\mathbf{t}_0$, entonces también se cumple $\mathcal{P}$ cuando $\mathbf{t} = \mathbf{t}_0$. Se presentan los principales resultados sobre propiedades abiertas y cerradas, con especial atención al problema de si la propiedad equilibrada y la propiedad fuertemente Gauduchon son cerradas por deformaciones holomorfas [76, Conjectures 1.21 and 1.23]. En el Capítulo 5 se construye un contraejemplo a ambas conjeturas.

El Capítulo 2 tiene como objetivo la geometría compleja invariante en el ámbito de las solvariedades. La Sección 2.1 se dedica a precisar los términos del objeto de estudio de este capítulo, que son por un lado las solvariedades y por otro la geometría compleja invariante. En primer lugar, se considera la clase de variedades compactas a la que pertenecen las solvariedades, que vienen dadas por el cociente de un grupo de Lie G por un subgrupo discreto Γ de manera que la variedad cociente $M = G/\Gamma$ es compacta. A la variedad cociente se le denomina solvariedad cuando el grupo de Lie G es resoluble o nilvariedad si G es nilpotente. El estudio de nilvariedades fue iniciado por Malcev [61] mientras que las solvariedades fueron estudiadas originariamente por Mostow [64]. Un resultado fundamental para este tipo de variedades compactas es el conocido Teorema de Nomizu [68] para nilvariedades, así como sus extensiones para solvariedades debidas a Hattori [46] y Mostow [64], que describen la cohomología de de Rham de M por medio de la cohomología de Eilenberg-Chevalley del álgebra de Lie subyacente.

La submersión $\pi\colon G \to M$ permite definir campos de tensores sobre M que proceden de campos de tensores invariantes por la izquierda definidos sobre G, o equivalentemente, sobre el álgebra de Lie $\mathfrak{g}$ del grupo. A este tipo de tensores, denominados invariantes, pertenecen las estructuras complejas que se consideran en la memoria. Hasegawa [45] clasifica las solvariedades de dimensión 4 que admiten estructura compleja invariante y a su vez prueba que cualquier estructura compleja sobre una solvariedad de esta dimensión es necesariamente invariante. Sin embargo, Hasegawa señala que esto último deja de ser cierto en dimensiones superiores, mostrando un ejemplo de una solvariedad de dimensión real seis con una estructura compleja no invariante obtenida mediante una deformación de la (sol)variedad de Nakamura, cuyo espacio de Kuranishi había sido previamente calculado en [67]. Ya en dimensión 6, existen resultados parciales de clasificación tales como el de las álgebras de Lie resolubles que admiten estructura compleja abeliana obtenido por Andrada, Barberis y Dotti [4] o la clasificación de Salamon [82] de álgebras de Lie nilpotentes $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$ admitiendo estructura compleja. Salamon demuestra a su vez que la geometría compleja invariante de una nilvariedad (M, J) viene caracterizada por la existencia de una sección invariante $\Psi \in \wedge^{n,0} M$ cerrada, siendo $n = \dim_{\mathbb{C}} M$, por medio de la cual se trivializa el fibrado canónico holomorfo K_M. En general la existencia de una sección cerrada $\Psi \in \wedge^{n,0}\mathfrak{g}^*$ supone una condición suficiente para la integrabilidad de J. Estas últimas consideraciones justifican clasificar las álgebras de Lie resolubles $\mathfrak{g}$ de dimensión real 6 que admiten una estructura compleja con una forma cerrada $\Psi \in \wedge^{3,0}\mathfrak{g}^*$ tales que den lugar a solvariedades a partir de ellas. En esta situación, la sección invariante que define Ψ trivializa el fibrado canónico holomorfo de la solvariedad. Además, mediante el proceso de simetrización [11] se prueba en la Proposición 2.1.31 que si una solvariedad M admite una estructura compleja invariante J con fibrado canónico holomórficamente trivial entonces admite una sección invariante $\Psi \in \wedge^{n,0} M$ cerrada. Como una primera consecuencia de este resultado, se prueba en el Teorema 2.1.32 que la propiedad de tener fibrado canónico holomorfo trivial no es estable por deformaciones holomorfas. La demostración se basa en una deformación invariante $\{(M, J_t)\}_{t\in\Delta}$ de una solvariedad compleja (M, J), siendo J invariante con fibrado canónico holomórficamente trivial.

La existencia de un lattice Γ sobre el grupo de Lie G impone que éste sea unimodular [62], lo cual equivale a que su álgebra de Lie $\mathfrak{g}$ sea también unimodular. La Proposición 2.1.31 reduce el problema de clasificar las álgebras de Lie que dan lugar a solvariedades con este tipo de geometría compleja invariante a los dos problemas siguientes:

(i) Clasificar las álgebras de Lie resolubles y unimodulares $\mathfrak{g}$ de dimensión 6 admitiendo una estructura casi-compleja J con una forma $0 \neq \Psi \in \wedge^{3,0}\mathfrak{g}^*$ cerrada.

(ii) Encontrar lattices en los grupos de Lie correspondientes a las álgebras de Lie encontradas en el problema anterior.

La Sección 2.2 se dedica a resolver el problema (i) utilizando para ello el formalismo de formas estables introducido por Hitchin [48]. Dado un espacio vectorial real de dimensión 6 orientado (V, ν), esta técnica permite asociar a una 3-forma $\rho \in \wedge^3 V^*$ un endomorfismo $K_\rho \colon V \to V$ tal que $K_\rho^2 = \lambda(\rho)\mathrm{Id}_V$, siendo el signo de $\lambda(\rho)$ independiente de la orientación $\nu \in \wedge^6 V^*$ escogida. Si $\mathfrak{g}$ es un álgebra de Lie resoluble de dimensión 6, el problema (i) se reduce mediante este formalismo a encontrar las 3-formas cerradas

$\rho \in \wedge^3\mathfrak{g}$ tales que $J_\rho := K_\rho$ es casi-compleja y $d(J_\rho^*\rho) = 0$. En tal caso, la forma $\Psi := \rho + iJ_\rho^*\rho \in \wedge^{3,0}\mathfrak{g}^*$ es cerrada y por tanto J_ρ es una estructura compleja del tipo buscado sobre $\mathfrak{g}$. Además, la existencia de un par (J, Ψ) cumpliendo estas condiciones sobre un álgebra de Lie $\mathfrak{g}$ unimodular impone la condición $b_3(\mathfrak{g}) \geq 2$. Esta última propiedad permite excluir algunas álgebras de Lie resolubles de dimensión 6 de las listas extraídas de Turkowski [91], Shabanskaya [86], Schulte-Hengesbach [84] y Freibert y Schulte-Hengesbach [36]. Sin embargo, ante el elevado número de álgebras de Lie a analizar, se ha dividido el estudio según el álgebra de Lie sea descomponible o no. Las álgebras de Lie resolubles y no nilpotentes $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha\geq 0}, \mathfrak{g}_3, \ldots, \mathfrak{g}_9$ que resuelven el problema (i) están recogidas en el Teorema 2.2.14. Hay que destacar que se trata de una familia infinita de álgebras de Lie, ya que las $\mathfrak{g}_2^{\alpha\geq 0}$ son no isomorfas dos a dos, siendo esto una diferencia con respecto al resultado obtenido por Salamon [82] para las álgebras de Lie nilpotentes. En las tablas del Apéndice B se encuentran las álgebras de Lie consideradas para obtener este resultado de clasificación.

La Sección 2.3 aborda el problema (ii) relativo a la existencia de lattices en los grupos de Lie conexos y simplemente conexos con álgebras de Lie subyacentes $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$, $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. Esta cuestión está resuelta para los grupos de Lie nilpotentes en virtud del teorema de Malcev [61] que caracteriza la existencia de lattices en términos de la existencia de una estructura racional sobre el álgebra de Lie subyacente. Todas las álgebras de Lie $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$ admiten una estructura racional por lo que es posible construir nilvariedades a partir de ellas. Esta cuestión se complica cuando el grupo de Lie no es nilpotente, y es preciso recurrir a resultados que permitan construir explícitamente estos lattices. Siguiendo [13] encontramos lattices para aquellos grupos de Lie que son casi-nilpotentes. En la Proposición 2.3.5 se prueba la existencia de lattices para los grupos de Lie conexos y simplemente conexos correspondientes a las álgebras de Lie $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. Para la familia $\mathfrak{g}_2^{\alpha\geq 0}$ se ha encontrado una familia numerable de álgebras de Lie $\{\mathfrak{g}_2^{\alpha_n}\}_{n\in\mathbb{N}}$ tales que sus correspondientes grupos de Lie admiten lattice, en conformidad con un resultado de Witte [100, Proposition 8.7]. El capítulo concluye con el Teorema 2.3.7 que resume las solvariedades de dimensión 6 con estructura compleja invariante con fibrado canónico holomórficamente trivial en términos del álgebra de Lie real subyacente.

El Capítulo 3 tiene como objetivo proporcionar una clasificación de estructuras complejas invariantes con fibrado canónico holomórficamente trivial sobre solvariedades de dimensión seis. Esta clasificación se realiza salvo equivalencia de estructuras complejas, es decir, dos estructuras $J, J' \colon \mathfrak{g} \to \mathfrak{g}$ son equivalentes si existe un automorfismo $F \colon \mathfrak{g} \to \mathfrak{g}$ del álgebra de Lie tal que $J' \circ F = F \circ J$. Dividimos este capítulo en dos secciones, la primera dedicada a las nilvariedades y la segunda a las solvariedades. Para una mejor comprensión de las estructuras complejas sobre un álgebra de Lie nilpotente, Cordero, Fernández, Gray y Ugarte [24] definen un tipo específico de estructuras denominadas nilpotentes. Las conocidas estructuras compleja-paralelizables, definidas por $[JX, Y] = J[X, Y]$, y las abelianas, dadas por $[JX, JY] = [X, Y]$, para todo $X, Y \in \mathfrak{g}$, son ejemplos de estructuras complejas nilpotentes cuando $\mathfrak{g}$ es nilpotente. A la clase de las compleja-paralelizables pertenecen la célebre (nil)variedad de Iwasawa, cuya álgebra de Lie subyacente es $\mathfrak{h}_5$, y la (sol)variedad de Nakamura [67], con álgebra de Lie subyacente $\mathfrak{g}_8$. Ugarte [95] prueba que sobre una misma nilvariedad de dimensión 6 no pueden coexistir una estructura compleja nilpotente y otra que no lo es. Más concretamente, prueba que todas las estructuras complejas sobre las álgebras de Lie $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$ son nilpotentes mientras que las álgebras de Lie $\mathfrak{h}_{19}^-$ y $\mathfrak{h}_{26}^+$ sólo admiten estructuras no nilpotentes. Las estructuras complejas sobre $\mathfrak{h}_{19}^-$ y $\mathfrak{h}_{26}^+$ son clasificadas por Ugarte y Villacampa en [96] mientras que, como ya se ha mencionado, las estructuras abelianas sobre las álgebras de Lie resolubles de dimensión 6 son clasificadas en [4]. Por tanto, en la Sección 3.1 se plantea como objetivo clasificar las estructuras complejas no abelianas sobre las álgebras de Lie $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$. Para una exposición más clara hemos dividido este estudio según el paso de nilpotencia de las álgebras de Lie, y en la Tabla 3.1 se recoge la clasificación final.

La Sección 3.2 tiene como objeto clasificar las estructuras complejas con una forma de tipo $(3,0)$ cerrada sobre las álgebras de Lie resolubles $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. Para ello se utiliza la técnica de formas estables considerada en la Sección 2.2 para hallar el espacio de este tipo de estructuras sobre cada una de las álgebras de Lie de la lista anterior. Por medio de un proceso de reducción encontramos un representante en cada clase de equivalencia de estructuras complejas. Es destacable (véase Proposición 3.2.7) que para el álgebra de Lie $\mathfrak{g}_8$ correspondiente a la variedad de Nakamura existe una familia infinita de estructuras $\{J^A\}_{A\in\mathbb{C},\mathfrak{Im}\,A\neq 0} \cup \{J'\} \cup \{J''\}$, siendo $J_0 := J^{-i}$ la estructura compleja-paralelizable y $J_1 := J^i$ la abeliana. Estas estructuras complejas junto con la clasificación de las correspondientes a las álgebras de Lie $\mathfrak{g}_1, \ldots, \mathfrak{g}_7$ y $\mathfrak{g}_9$ se encuentran recogidas en la Tabla 3.2.

Los Capítulos 4 y 5 hacen uso de los resultados de clasificación obtenidos en los Capítulos 2 y 3 para el estudio de ciertos invariantes cohomológicos, la existencia de métricas Hermitianas especiales y el comportamiento de diversas propiedades por deformaciones holomorfas. El Capítulo 4 estudia en concreto la sucesión espectral de Frölicher $\{E_r^{\bullet,\bullet}(M)\}_{r\geq 1}$ y el $\partial\bar{\partial}$-lema. Rollenske [80] demuestra que para una nilvariedad $M = G/\Gamma$ de dimensión 6 con una estructura compleja invariante J, la inclusión natural $(\wedge^{\bullet,\bullet}\mathfrak{g}^*, \bar{\partial}) \to (\wedge^{\bullet,\bullet}M, \bar{\partial})$ induce un isomorfismo en la cohomología de Dolbeault, siempre que el álgebra de Lie subyacente no sea isomorfa a $\mathfrak{h}_7$. Cordero, Fernández, Gray y Ugarte [23] demuestran que en tal caso también se cumple que $E_r^{\bullet,\bullet}(M) \cong E_r^{\bullet,\bullet}(\mathfrak{g})$ para todo $r \geq 2$. La Sección 4.1 contiene el cálculo de la sucesión de Frölicher para toda estructura compleja sobre las álgebras de Lie nilpotentes $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. De este estudio y de los resultados previamente mencionados llegamos al Teorema 4.1.4 que contiene una descripción general de la sucesión espectral de Frölicher para las nilvariedades de dimensión 6 con estructura compleja invariante (excepto para el caso $\mathfrak{h}_7$). En cuanto a la validez del $\partial\bar{\partial}$-lema, las nilvariedades no pueden cumplirlo porque no son formales en el sentido de Sullivan [44]. Por otro lado, es conocido que toda variedad compleja compacta M que satisface el $\partial\bar{\partial}$-lema cumple que $E_1(M) \cong E_\infty(M)$ y también la simetría de los números de Hodge $h_{\bar{\partial}}^{p,q}(M) = h_{\bar{\partial}}^{q,p}(M)$ para todo $p, q \in \mathbb{N}$. Estas consideraciones dan lugar a la siguiente cuestión formulada por Angella y Tomassini en [10]: encontrar una variedad compleja compacta que satisfaga las dos últimas condiciones y sin embargo no cumpla el $\partial\bar{\partial}$-lema. En la Proposición 4.1.5 mostramos que la nilvariedad compleja $(M = G/\Gamma, J)$ con álgebra de Lie subyacente $\mathfrak{h}_6$ da una respuesta afirmativa a la anterior cuestión.

En la Sección 4.2 se presentan resultados relativos al cálculo de la sucesión espectral de Frölicher para las solvariedades con estructura compleja invariante cuyo fibrado canónico es holomórficamente trivial. Esta sección se divide a su vez en dos subsecciones. En la primera se realiza el cálculo de $\{E_r^{\bullet,\bullet}(\mathfrak{g})\}_{r\geq 1}$ a nivel del álgebra de Lie $\mathfrak{g}$. Por medio del proceso de simetrización se prueba en la Proposición 4.2.1 que si en el álgebra de Lie se tiene $E_1(\mathfrak{g}) \ncong E_\infty(\mathfrak{g})$ entonces en la solvariedad M también se cumple $E_1(M) \ncong E_\infty(M)$. Esto permite concluir en el Corolario 4.2.8 que sobre una solvariedad (M, J) con álgebra de Lie isomorfa a $\mathfrak{g}_8$ y J equivalente a J_0, J_1, J' ó J'' no se cumple el $\partial\bar{\partial}$-lema. En la segunda subsección se presentan en primer lugar los resultados recientes debidos a Angella y Kasuya [7, 51] que permiten calcular la cohomología de Dolbeault y de Bott-Chern de una solvariedad $(M = G/\Gamma, J)$ con una estructura compleja invariante J de un tipo específico, denominado tipo splitting [51, Assumption 1.1]. La presencia de este tipo de estructuras complejas sobre un grupo de Lie resoluble permite concebirlo como un producto semidirecto de la forma $G = \mathbb{C}^r \ltimes N$, siendo N un grupo de Lie nilpotente con una estructura compleja invariante tal que $H_{\bar{\partial}}^{\bullet,\bullet}(N/\Gamma_N) \cong H_{\bar{\partial}}^{\bullet,\bullet}(\mathfrak{n})$. Además el subgrupo $\Gamma = \Gamma_{\mathbb{C}^r} \ltimes \Gamma_N \subset G$ escogido para construir la solvariedad debe presentar cierta compatibilidad con este producto semidirecto. El cálculo de las cohomologías de Dolbeault y de Bott-Chern de estas solvariedades se realiza por medio de dos complejos de formas diferenciales de dimensión finita $(B_\Gamma^{\bullet,\bullet}, \bar{\partial})$ y $(C_\Gamma^{\bullet,\bullet}, \partial, \bar{\partial})$, de manera que $H_{\bar{\partial}}^{\bullet,\bullet}(M) \cong H_{\bar{\partial}}(B_\Gamma^{\bullet,\bullet})$ y $H_{\mathrm{BC}}^{\bullet,\bullet}(M) \cong H_{\mathrm{BC}}(C_\Gamma^{\bullet,\bullet})$, respectivamente.

De las estructuras complejas sobre las álgebras de Lie resolubles $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ obtenidas en el Capítulo 3 sólo son de tipo splitting las correspondientes a $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$ con $\alpha \geq 0$ y $\mathfrak{g}_8$, exceptuando J' y J''. Además, los ejemplos analizados por Kasuya en [51] se corresponden con la estructura compleja de $\mathfrak{g}_1$ y la compleja-paralelizable de $\mathfrak{g}_8$. Tras construir explícitamente lattices compatibles con las estructuras complejas de tipo splitting anteriores, en la Proposición 4.2.21 se presentan solvariedades complejas con álgebra de Lie $\mathfrak{g}_2^0$ que cumplen el $\partial\bar{\partial}$-lema. Más aún, para una elección concreta de lattice, se obtiene una solvariedad cuya cohomología de de Rham coincide con la cohomología de Chevalley-Eilenberg de $\mathfrak{g}_2^0$, aunque $\mathfrak{g}_2^0$ no es completamente resoluble y por tanto no cumple la hipótesis necesaria para aplicar el resultado de Hattori [46]. Por otro lado, para una familia infinita $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{>0}$, se construyen solvariedades con álgebra de Lie subyacente $\mathfrak{g}_2^{\alpha_n}$ cuyos lattices son compatibles con las dos estructuras complejas no equivalentes $J^\pm$. Para algunos de estos lattices se tiene $E_1(M) \ncong E_\infty(M)$ mientras que para otros se cumple el $\partial\bar{\partial}$-lema (véase Proposición 4.2.22). En relación a $\mathfrak{g}_8$, en la Proposición 4.2.25 se proporcionan lattices compatibles con cada una de las estructuras $\{J^A\}_{A\in\mathbb{C},\mathfrak{Im}\,A\neq 0}$ de manera que las solvariedades resultantes cumplen el $\partial\bar{\partial}$-lema si y sólo si $A = \frac{i}{k}$ con $k \neq 0$ entero.

La Sección 4.3 estudia el comportamiento de invariantes cohomológicos por deformaciones holomorfas. Kodaira y Spencer [53] prueban la semicontinuidad de los números de Hodge, de donde se sigue que la propiedad "$E_1(M) \cong E_\infty(M)$" es abierta por deformaciones holomorfas. Sin embargo, Eastwood y Singer [27] prueban que esta propiedad no es cerrada usando geometría compleja sobre espacios de twistor. Por otro lado, Maclaughlin, Pedersen, Poon y Salamon [60] proporcionan la descripción del espacio de Kuranishi de las nilvariedades nilpotentes en paso 2 con estructura compleja abeliana. Haciendo uso de éste y nuestros resultados anteriores, en el Corolario 4.3.3 se construye otro ejemplo, basado en una deformación invariante de una nilvariedad compleja (M, J) con álgebra de Lie isomorfa a $\mathfrak{h}_4$ y J la estructura abeliana sobre ella, que muestra que "$E_1(M) \cong E_\infty(M)$" no es una propiedad cerrada. Además, se construye una familia de estructuras complejas $\{J_t\}_{t\in\mathbb{R}}$ sobre al álgebra de Lie $\mathfrak{h}_{15}$ que muestra que las dimensiones de los términos $E_2^{\bullet,\bullet}$ no son monótonamente crecientes ni decrecientes. Esta familia nos permite concluir en el Corolario 4.3.6 que la propiedad "$E_2(M) \cong E_\infty(M)$" no es abierta por deformaciones.

En cuanto al $\partial\bar{\partial}$-lema, es bien conocido que se trata de una propiedad abierta por deformaciones holomorfas [98, 101, 10]. Angella y Kasuya proporcionan en [8] una técnica para calcular tanto la cohomología de Dolbeault como la cohomología de Bott-Chern a lo largo de deformaciones holomorfas $(M, J_t)_{t\in\Delta}$ de una solvariedad compleja (M, J) de tipo splitting. Mediante esa técnica demuestran [8] que el $\partial\bar{\partial}$-lema no es una propiedad cerrada por deformaciones. La prueba se basa en una deformación invariante de la variedad de Nakamura. Usando esta técnica y el resultado obtenido en la Proposición 4.2.25, se construye en la Proposición 4.3.9 una familia de deformaciones holomorfas $\{(M_k, J_{k,t})\}_{t\in\Delta}$ para cada una de las variedades complejas compactas de la familia infinita $\{(M_k, J_k := J^{A_k})\}_{k\in\mathbb{Z}}$ siendo $A_k = \frac{i}{2k+1}$. El álgebra de Lie subyacente a todas las M_k es isomorfa a $\mathfrak{g}_8$. Además, estas deformaciones no cumplen el $\partial\bar{\partial}$-lema en los límites centrales $J_k := J_{k,0}$ para todos los $k \in \mathbb{Z}$, pero sí para $J_{k,t}$ con $t \in \Delta^*$, lo que extiende la prueba ideada por Angella y Kasuya a una clase infinita de ejemplos.

En el Capítulo 5 se estudia en su primera Sección 5.1 la existencia de métricas equilibradas, fuertemente Gauduchon, Kähler con torsión y, al estar en dimensión compleja 3, 1-Gauduchon generalizadas sobre las solvariedades complejas obtenidas anteriormente. Por el proceso de simetrización, se tiene que la existencia tanto de métricas Kähler y equilibradas [31] como de métricas Kähler con torsión [95] sobre una variedad compacta $(M = G/\Gamma, J)$ con J invariante se reduce al estudio a nivel del álgebra de Lie. En el ámbito de las nilvariedades con estructura compleja invariante se conocen resultados sobre la existencia de métricas Kähler con torsión [33], 1-Gauduchon generalizadas [35] y equilibradas [95]. Por ello, el estudio se ha centrado para esas métricas en las estructuras complejas sobre las álgebras de Lie resolubles $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ obteniéndose los resultados de clasificación en las Proposiciones 5.1.1, 5.1.6 y 5.1.8. Como consecuencia, en el Teorema 5.1.3 se prueba que una solvariedad dotada de estructura compleja invariante con fibrado canónico holomórficamente trivial admite una métrica Calabi-Yau si y sólo si su álgebra subyacente es isomorfa a $\mathbb{R}^6$ ó $\mathfrak{g}_2^0$. En relación a las métricas fuertemente Gauduchon, se prueba en la Proposición 5.1.9 que su existencia sobre solvariedades con estructura compleja invariante es también reducible a su existencia a nivel del álgebra de Lie. No se conoce ningún estudio previo sobre la existencia de estas métricas en dimensión 6, por lo que lo dividimos en la Proposición 5.1.11 para nilvariedades y en la Proposición 5.1.13 (véase también la Tabla 5.2) para solvariedades. Como ya se ha mencionado antes, la condición equilibrada implica la fuertemente Gauduchon, y el cumplir el $\partial\bar{\partial}$-lema en una variedad compleja compacta implica la existencia de métricas fuertemente Gauduchon. Por ello, siguiendo [76, Theorem 1.10], se estudia en la Proposición 5.1.15 qué nilvariedades complejas admiten métricas fuertemente Gauduchon sin admitir ninguna métrica equilibrada. Los resultados de este estudio se recogen en la Tabla 5.1.

Finalmente, la Sección 5.2 contiene dos resultados relativos al comportamiento de métricas Hermitianas bajo deformaciones holomorfas. El Teorema 5.2.1 encierra el resultado más importante de esta sección al demostrar que tanto la propiedad equilibrada como la propiedad fuertemente Gauduchon no son cerradas por deformaciones holomorfas. En la demostración utilizamos la deformación basada en $(\mathfrak{h}_4, J)$ usada en la prueba del Corolario 4.3.3 para mostrar que $E_1(M) \cong E_\infty(M)$ no es cerrada por deformaciones. Por otro lado, Popovici [72] muestra que si una deformación $\{(M, J_t)\}_{t\in\Delta}$ satisface el $\partial\bar{\partial}$-lema para todo $t \in \Delta^*$ entonces el límite central (M, J_0) admite una métrica fuertemente Gauduchon. Se plantea la cuestión de si se puede asegurar la existencia en el límite central de métricas más restrictivas que las fuertemente Gauduchon. En relación a esto, el Teorema 5.2.4 muestra una deformación invariante de una solvariedad compleja $\{(M, J_t)\}_{t\in\Delta}$ que satisface el $\partial\bar{\partial}$-lema y admite métricas equilibradas para todos los $t \in \Delta^*$, pero que en el límite central no satisface el $\partial\bar{\partial}$-lema ni tampoco admite métricas equilibradas. Por tanto, a diferencia de lo que sucede con las métricas fuertemente Gauduchon, la propiedad del $\partial\bar{\partial}$-lema no hace que existan métricas equilibradas en el límite central de una deformación.

Parte de los resultados de los Capítulos 2, 3, 4 y 5 de esta memoria han sido obtenidos en colaboración con Manuel Ceballos, Anna Fino, Luis Ugarte y Raquel Villacampa y están recogidos en los artículos [18] y [32], mientras que algunos otros resultados más recientes han sido obtenidos con Daniele Angella y Luis Ugarte.

Introduction

As it is well-known Calabi-Yau manifolds constitute one of the most important classes in geometry. These manifolds, which can be thought as higher-dimensional analogues of K3 surfaces, are compact complex manifolds (M, J) of complex dimension n endowed with an $\mathrm{SU}(n)$ structure (F, Ψ) such that the fundamental 2-form F is closed and the $(n, 0)$-form Ψ is holomorphic. Thus, the holonomy of the metric $g(\cdot, \cdot) = F(\cdot, J\cdot)$ reduces to a subgroup of $\mathrm{SU}(n)$, so that g is a Ricci-flat Kähler metric, and the canonical bundle of (M, J) is holomorphically trivial.

The above conditions defining a Calabi-Yau manifold have been weakened in different directions so that the resulting geometries still play an important role in several aspects of complex geometry. In this thesis we focus our attention in the geometry of compact complex manifolds (M, J) with holomorphically trivial canonical bundle endowed with special Hermitian metrics which are less restrictive than the Kähler ones.

Concerning compact complex manifolds with holomorphically trivial canonical bundle, we recall that in complex dimension 2 the possibilities, up to isomorphism, are a K3 surface, a torus or a Kodaira surface, where the first two are Kähler and the latter is an example of a nilmanifold $M = G/\Gamma$, i.e. a compact quotient of a simply connected nilpotent Lie group G by a lattice Γ of maximal rank in G. However, there are no classifications in complex dimension 3 or higher, so it is natural to begin by studying such complex geometry on some particular classes of compact manifolds of real dimension 6. A good candidate is the class consisting of nilmanifolds endowed with an invariant complex structure, as Salamon proved in [82] that any such complex nilmanifold has holomorphically trivial canonical bundle. In (real) dimension 6 a classification of nilmanifolds admitting this kind of complex structures is also provided in [82], where the Iwasawa (nil)manifold is a classical example which plays a relevant role in complex geometry (see for instance [6] and the references therein).

Although the complex geometry of nilmanifolds provides an important source of examples in differential geometry, these spaces never satisfy the $\partial\bar{\partial}$-lemma because they are not formal except for tori [26, 44]. However, the investigation of some properties in complex geometry requires compact complex manifolds satisfying the $\partial\bar{\partial}$-lemma, so one needs to consider a broader class of homogeneous spaces $M = G/\Gamma$. The first natural generalization of nilmanifolds is given by compact quotients of Lie groups G which are solvable instead of nilpotent. For instance, the Nakamura manifold, whose complex geometry is very rich [67], is an example of this type. This class of manifolds, known as solvmanifolds, is the central object of study in this thesis. More concretely, we describe the 6-dimensional solvmanifolds admitting an invariant complex structure with holomorphically trivial canonical bundle, as well as we obtain a classification of such invariant structures.

As we mentioned above, another goal in this thesis is the study of special Hermitian metrics which are less restrictive than the Kähler ones. It is well-known that the existence of a Kähler metric on a compact manifold imposes strong topological obstructions. In contrast, by [40] on a compact complex manifold (M, J) of complex dimension n there always exists a Gauduchon metric, i.e. a metric satisfying $\partial\bar{\partial}F^{n-1} = 0$, in the conformal class of any given Hermitian metric. Between the Kähler class and the Gauduchon class other interesting classes of special Hermitian metrics have been considered in relation to different problems in differential geometry. For instance, a Hermitian metric is called balanced if the fundamental form satisfies that F^{n-1} is closed, and it is said to be strongly Gauduchon if the $(n, n-1)$-form ∂F^{n-1} is $\bar{\partial}$-exact. Strongly Gauduchon metrics have been introduced recently by Popovici in [74], whereas balanced metrics were previously considered in [62].

On the other hand, Fu, Wang and Wu have introduced in [39] a generalization of the Gauduchon metrics on complex manifolds. For each integer $1 \leq k \leq n-1$, a Hermitian metric is called k-Gauduchon if $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$. Then, by definition the notion of $(n-1)$-Gauduchon metric coincides with the one of the usual Gauduchon metric, and for $k = 1$ one has that the class of 1-Gauduchon metrics contains in particular the strong Kähler with torsion (SKT for short) metrics, also known as pluriclosed metrics. Streets and Tian introduced in [88] a Hermitian Ricci flow under which the SKT condition is preserved, and the SKT geometry has been studied by many authors (see for instance [33, 34, 90]). The 6-dimensional nilmanifolds admitting invariant SKT, 1-Gauduchon or balanced metrics have been determined in [33, 35, 95], and in this thesis we study the existence of such metrics, as well as the existence of strongly Gauduchon metrics, on the bigger class of 6-dimensional solvmanifolds endowed with an invariant complex structure with holomorphically trivial canonical bundle.

Associated to any compact complex manifold M there exist several complex invariants which measure some specific aspects of M. Among them, we distinguish the Dolbeault, the Bott-Chern and the Aeppli cohomologies [14, 1], and the Frölicher spectral sequence $\{E_r(M)\}$ relating the Dolbeault to the de Rham cohomology of the manifold [38]. If M is a compact Kähler manifold then all these complex invariants coincide because M satisfies the $\partial\bar{\partial}$-lemma, however the Frölicher sequence may not degenerate at the first step for arbitrary compact complex manifolds. A problem of interest in complex geometry is to study the behaviour of these invariants. In the case of 6-dimensional nilmanifolds a complete picture of the behaviour of the sequence $\{E_r(M)\}_{r\geq 1}$ is given in this thesis, and for solvmanifolds of dimension 6 endowed with an invariant complex structure of splitting type (in the sense of [51]) with holomorphically trivial canonical bundle we use the results by Kasuya and Angella [51, 7] and by Angella and Tomassini [10] to find when the $\partial\bar{\partial}$-lemma is satisfied. Motivated by the paper [76], in this thesis we also explore the relations among the degeneration of the Frölicher spectral sequence, the

$\partial\bar{\partial}$-lemma and the existence of balanced or strongly Gauduchon metrics, as well as their behaviour under small holomorphic deformations of the complex structure.

Next we describe in more detail the contents of each chapter in this thesis.

The goal of Chapter 1 is to place our research in the more general context of complex manifolds. We recall the basic notions and results about complex geometry in Section 1.1. Complex manifolds are introduced from two points of view, on one hand as smooth manifolds admitting a holomorphic atlas compatible with the differentiable structure, and on the other hand as smooth manifolds endowed with a $\mathcal{C}^\infty$ tensor field $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}_{TM}$ and satisfying the integrability condition stated by the Newlander-Nirenberg Theorem [69]. Although both viewpoints are equivalent, in this work we adopt the latter one. The presence of a complex structure gives rise to the existence of holomorphic vector bundles such as the holomorphic tangent bundle $\mathcal{T}_M$, its holomorphic dual bundle $\Omega^1_M(M)$ and, more in general, the bundles of holomorphic p-forms $\Omega^p_M(M) := \wedge^p\Omega^1_M(M)$ with $1 \leq p \leq n$, where $n = \dim_{\mathbb{C}} M$. In particular, the holomorphic canonical bundle $K_M := \Omega^n_M(M)$ is an example of this type.

Several differential complexes are associated to complex manifolds yielding several cohomologies. We deal with these cohomologies in Section 1.2. The presence of a complex structure induces a bigraduation in the complexified complex of differential forms $(\wedge^\bullet M_{\mathbb{C}}, d)$ giving rise to a bidifferential bigraded algebra $(\wedge^{\bullet,\bullet} M, \partial, \bar{\partial})$, where $d = \partial + \bar{\partial}$. The de Rham cohomology groups $H^\bullet_{\mathrm{dR}}(M;\mathbb{C}) = \ker d/\mathrm{im} d$ and the Dolbeault cohomology groups $H^{\bullet,\bullet}_{\bar{\partial}}(M) = \ker \bar{\partial}/\mathrm{im}\bar{\partial}$ of the complex manifold are associated to these complexes and their dimensions, denoted by $b_\bullet(M) := \dim H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$ and $h^{\bullet,\bullet}_{\bar{\partial}}(M) := \dim H^{\bullet,\bullet}_{\bar{\partial}}(M)$, respectively, are finite when M is compact. However, other cohomologies of interest can be defined from the bidifferential bigraded algebra such as the Aeppli cohomology $H^{\bullet,\bullet}_{\mathrm{A}}(M) = \ker \partial\bar{\partial}/(\mathrm{im}\partial + \mathrm{im}\bar{\partial})$ and the Bott-Chern cohomology $H^{\bullet,\bullet}_{\mathrm{BC}}(M) = (\ker \partial \cap \ker \bar{\partial})/\mathrm{im}\partial\bar{\partial}$ [1, 14]. The conjugation map induces an isomorphism between the cohomology groups of bidegree (p, q) and (q, p). In addition, when M is compact, a Hodge theory associated to these cohomologies [85] allows to deduce several interesting properties, such as the finiteness of the dimensions of these groups, denoted by $h^{\bullet,\bullet}_{\mathrm{BC}}(M)$ and $h^{\bullet,\bullet}_{\mathrm{A}}(M)$, and the duality between them, in the sense that $h^{p,q}_{\mathrm{A}}(M) = h^{n-q,n-p}_{\mathrm{BC}}(M)$ for any $p, q \in \mathbb{N}$. From the definitions above, it is possible to set well-defined natural maps $H^{\bullet,\bullet}_{\mathrm{BC}}(M) \to H^{\bullet,\bullet}_{\bar{\partial}}(M) \to H^{\bullet,\bullet}_{\mathrm{A}}(M)$. However, these maps are in general neither injective nor surjective, and it is proved in [26] that the isomorphism $H^{\bullet,\bullet}_{\mathrm{BC}}(M) \cong H^{\bullet,\bullet}_{\bar{\partial}}(M)$ holds if and only if the complex manifold satisfies the $\partial\bar{\partial}$-lemma. Recently, Angella and Tomassini [10] characterize the $\partial\bar{\partial}$-lemma in terms of the vanishing of some complex invariants involving the Betti numbers and the dimensions of the Aeppli and the Bott-Chern cohomology groups. Finally, the Frölicher spectral sequence $\{E^{\bullet,\bullet}_r(M)\}_{r\geq 1}$ of a complex manifold is presented. It links the Dolbeault cohomology, identified with the first term $E^{\bullet,\bullet}_1(M)$ of the sequence, with the de Rham cohomology, the term $E^{\bullet,\bullet}_\infty(M)$, which is reached in a finite number of steps. We recall the Frölicher inequality [38] involving the Betti numbers and the Hodge numbers of the complex manifold.

Some of the most interesting complex manifolds are distinguished by the presence of a special Hermitian metric. We recall some definitions and results concerning these metrics in Section 1.3. Hermitian metrics can be described by means of a positive 2-form $F \in \wedge^{1,1}M$, called fundamental form (also Kähler form). It is well-known that on a compact complex manifold M of complex dimension n one can always find metrics compatible with the complex structure, moreover Gauduchon [40] proves that there exists a standard metric (also called Gauduchon metric), defined by $\partial\bar{\partial}F^{n-1} = 0$, in the conformal class of any Hermitian metric. The existence of Kähler metrics [50, 83], specified by $dF = 0$, imposes strong topological obstructions on the manifold, some of them expressed in terms of cohomological invariants. For instance, Deligne, Griffiths, Morgan and Sullivan proved in [26] that the existence of such metrics on a compact complex manifold implies the $\partial\bar{\partial}$-lemma and hence the underlying manifold has to be formal.

The Kähler condition can be weaken in two directions. On the one hand, when the torsion of the Bismut connection [12] is closed then the Hermitian metric is called strong Kähler with torsion [33]. These metrics are also characterized by the condition $F \in \ker \partial\bar{\partial}$. The geometry with torsion plays a central role in the Hermitian-Ricci flow introduced by Streets and Tian [39] other than its importance in the context of some supersymmetric models [41] and in some types of string theories [89]. Recently, a new type of special Hermitian metrics have been introduced in [39], which are called generalized Gauduchon and are defined by $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$ for some $k \in \mathbb{N}$ such that $1 \leq k \leq n-1$. These structures contain the strong Kähler with torsion metrics for $k = 1$ and coincide with the Gauduchon metrics for $k = n-1$. The Kähler condition can be also weakened by requiring that $F^{n-1} \in \ker d$. This type of metrics are called balanced [62] and belong to the class $\mathcal{W}_3$ in the Gray-Hervella classification [42]. They have importance in compactifications of heterotic string theories [89]. Popovici has introduced an intermediate class between the balanced and the Gauduchon classes [75], called strongly Gauduchon metrics, defined by $\partial F^{n-1} \in \mathrm{im}\bar{\partial}$. In addition, Popovici proves in [72] that, for a compact complex manifold satisfying the $\partial\bar{\partial}$-lemma, the Gauduchon and the strongly Gauduchon conditions coincide.

A smooth manifold can admit several complex structures in such a way that the corresponding complex manifolds are not biholomorphic. All the complex structures on a manifold constitute a space called the moduli space of complex structures of the manifold. It is a very hard problem to describe this space although the holomorphic deformation theory of complex structures developed by Kodaira, Spencer, Nirenberg [53, 54] and Kuranishi [55] provides a partial answer. The basics of this theory are presented in Section 1.4 which is divided in two parts. The first part is devoted to the notion of holomorphic deformation $\{(M, J_{\mathbf{t}})\}_{\mathbf{t}\in\mathcal{B}}$ of a complex manifold (M, J). A holomorphic deformation is conceived as a family of complex structures $\{J_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ on the same underlying smooth manifold M given by a parameter $\mathbf{t} \in \mathcal{B}$ taking values in a complex space $\mathcal{B}$. The original complex structure J is recovered when the parameter rises some distinguished point $\mathbf{t_0}$. The Kuranishi's Theorem provides a local description of the space of holomorphic deformations of a complex manifold, denoted by $\mathrm{Kur}(M)$, allowing to construct, in a relatively simple way, new compact complex manifolds. In the second part several open and closed properties under holomorphic deformations are presented. A property $\mathcal{P}$ is said to be open, or stable, under holomorphic deformations if whenever $\mathcal{P}$ holds for (M, J) then $\mathcal{P}$ also holds for $(M, J_{\mathbf{t}})$ for any $t \in \Delta^*$. Here $\Delta^* = \Delta \backslash \{t_0\}$, where Δ is an open disc in $\mathbb{C}$ around t_0 and $J_{\mathbf{t_0}} = J$. On the other hand, a property $\mathcal{P}$ is said to be closed if whenever the property holds for $(M, J_{\mathbf{t}})$ for any $t \in \Delta^*$ then it holds in the central limit $(M, J_{\mathbf{t_0}} = J)$. The main results concerning open and closed properties are presented with special attention to the problem on closedness of the balanced and the strongly Gauduchon properties under holomorphic deformations [76, Conjectures 1.21 and 1.23]. A counterexample to both conjectures will be constructed in Chapter 5.

Chapter 2 is devoted to invariant complex geometry in the class of solvmanifolds. In Section 2.1 we consider the class of compact complex manifolds constructed by taking a quotient of a Lie group G by a subgroup $\Gamma \subset G$, so that the quotient manifold $M = G/\Gamma$ is compact. These manifolds are called solvmanifolds when G is solvable or nilmanifolds when G is nilpotent. The study of nilmanifolds was started by Malcev [61] whereas solvmanifolds were studied firstly by Mostow [64]. We recall the main result due to Nomizu [68] for nilmanifolds and its extensions due to Hattori [46] and Mostow [64] for solvmanifolds concerning the computation of the de Rham cohomology of G/Γ by means of the Eilenberg-Chevalley cohomology of the underlying Lie algebra $\mathfrak{g}$.

The submersion $\pi\colon G \to M$ allows to define tensor fields on M coming from left-invariant tensor fields on G, or equivalently, encoded in the Lie algebra $\mathfrak{g}$ of the Lie group. The complex structures considered in this work belong to this class of invariant tensor fields. Hasegawa [45] classifies the solvmanifolds of dimension 4 admitting an invariant complex structure and proves that any complex structure on a solvmanifold of this dimension is necessarily invariant. However, Hasegawa shows that the latter is not true for higher dimensions by showing a solvmanifold of dimension six with a non-invariant complex structure obtained by a holomorphic deformation of the Nakamura (solv)manifold, whose Kuranishi space had been previously obtained in [67]. There are partial classification results in dimensions six such as the solvable Lie algebras admitting an abelian complex structure obtained by Andrada, Barberis and Dotti [4] or Salamon's classification [82] of nilpotent Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$ admitting complex structures. Salamon also shows that the invariant complex geometry on a complex nilmanifold (M, J) is characterized by the existence of a closed invariant section $\Psi \in \wedge^{n,0} M$, where $n = \dim_{\mathbb{C}} M$. Hence, the holomorphic canonical bundle K_M of a complex nilmanifold is trivial. In general, the existence of a closed section $\Psi \in \wedge^{n,0}\mathfrak{g}^*$ yields a sufficient condition to the integrability of the complex structure. The latter considerations constitute a reason to classify the solvable Lie algebras $\mathfrak{g}$ of dimension six admitting a complex structure with a closed complex volume form $\Psi \in \wedge^{3,0}\mathfrak{g}^*$ so that the corresponding Lie groups give rise to solvmanifolds. In this situation the invariant section defined by Ψ trivializes the holomorphic canonical bundle of the solvmanifold. In addition, by using the symmetrization process [11] we prove in Proposition 2.1.31 that if a solvmanifold M admits an invariant complex structure J with holomorphically trivial canonical bundle then it admits an invariant closed section $\Psi \in \wedge^{n,0} M$. As a first consequence of this result, we prove in Theorem 2.1.32 that the property of having holomorphically trivial canonical bundle is not open under holomorphic deformations. The proof is based on an invariant deformation $\{(M, J_t)\}_{t\in\Delta}$ of a complex solvmanifold (M, J) where J is invariant with holomorphically trivial canonical bundle.

The existence of a lattice on a Lie group G implies [62] the unimodularity of G, and in particular, of the Lie algebra $\mathfrak{g}$ of the Lie group. Proposition 2.1.31 reduces the problem of classifying the Lie algebras giving rise to solvmanifolds with this type of complex geometry to the following two problems:

(i) Classify the solvable and unimodular Lie algebras $\mathfrak{g}$ of dimension six admitting an almost-complex structure J with a closed form $0 \neq \Psi \in \wedge^{3,0}\mathfrak{g}^*$.

(ii) Find lattices in the Lie groups corresponding to the Lie algebras obtained in the previous problem.

Section 2.2 deals with problem (i) using the formalism of stable forms introduced by Hitchin [48]. Given an oriented six dimensional vector space (V, ν), this technique allows to associate to any 3-form $\rho \in \wedge^3 V^*$ an endomorphism $K_\rho\colon V \to V$ such that $K_\rho^2 = \lambda(\rho)\mathrm{Id}_V$, where the sign of $\lambda(\rho)$ remains independent of the choice of $\nu \in \wedge^6 V^*$. If $\mathfrak{g}$ is a solvable Lie algebra of dimension six, problem (i) reduces with this formalism to find the closed 3-forms $\rho \in \wedge^3\mathfrak{g}$ such that $J_\rho := K_\rho$ is almost-complex and $d(J_\rho^*\rho) = 0$. In this case, the complex form $\Psi := \rho + iJ_\rho^*\rho \in \wedge^{3,0}\mathfrak{g}^*$ is closed and therefore J_ρ is a complex structure of the required type on $\mathfrak{g}$. In addition, the existence of a pair (J, Ψ) satisfying these conditions on an unimodular Lie algebra $\mathfrak{g}$ imposes the condition $b_3(\mathfrak{g}) \geq 2$. The latter allows to exclude some solvable Lie algebras of dimension 6 extracted from the lists of Turkowski [91], Shabanskaya [86], Schulte-Hengesbach [84] and Freibert and Schulte-Hengesbach [36]. However, given the great number of Lie algebras to be considered, we have divided the study according to the Lie algebra is decomposable or not. The (non-nilpotent) solvable Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha\geq 0}, \mathfrak{g}_3, \ldots, \mathfrak{g}_9$ solving problem (i) are obtained in Theorem 2.2.14. It is remarkable that there is an infinite number of Lie algebras because $\mathfrak{g}_2^{\alpha\geq 0}$ are not isomorphic for distinct α's. The complete lists of Lie algebras are included in the tables of Appendix B.

Section 2.3 deals with problem (ii) concerning the existence of lattices in the connected and simply-connected Lie groups with underlying Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$, $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. This problem is solved for the nilpotent Lie groups by Malcev's Theorem [61] which characterizes the existence of lattice by means of the existence of a rational structure on the underlying Lie algebra. In particular, all the Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$ admit a rational structure. When the Lie group is not nilpotent the question becomes more difficult and we must use partial results allowing us to build the lattice explicitly. Following [13] we find lattices for those solvable Lie groups which are almost-nilpotent. We prove in Proposition 2.3.5 the existence of a lattice for the simply-connected Lie groups associated to the Lie algebras in the list, although for $\mathfrak{g}_2^\alpha$ we are able to find a lattice only for a countable number of different values of α. The latter is consistent with a result of Witte [100, Prop. 8.7]. The chapter concludes with Theorem 2.3.7 summing up the solvmanifolds of dimension 6 admitting an invariant complex structure with holomorphically trivial canonical bundle in terms of the underlying real Lie algebra.

The goal of Chapter 3 is to classify the complex structures with holomorphically trivial canonical bundle on solvmanifolds of dimension six. The classification is up to equivalence of complex structures on Lie algebras, that is, two structures $J, J' \colon \mathfrak{g} \to \mathfrak{g}$ are said to be equivalent if there exists an automorphism $F \colon \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra such that $J' \circ F = F \circ J$. This chapter consists of two parts, the first devoted to nilmanifolds and the second to solvmanifolds. For a better understanding of the complex structures on nilpotent Lie algebras, Cordero, Fernández, Gray and Ugarte [24] introduce the notion of nilpotent complex structures. The well-known complex-parallelizable structures, defined by $[JX, Y] = J[X, Y]$, and the abelian complex structures, defined by $[JX, JY] = [X, Y]$, for all $X, Y \in \mathfrak{g}$, belong to the bigger class of nilpotent complex structures when $\mathfrak{g}$ is nilpotent. It is remarkable that to the class of complex-parallelizable manifolds belong the Iwasawa manifold, with underlying Lie algebra $\mathfrak{h}_5$, and the Nakamura manifold [67], with underlying Lie algebra $\mathfrak{g}_8$. Ugarte [95] proves that on a six-dimensional nilmanifold cannot coexist nilpotent and non-nilpotent complex structures. More concretely, all the complex structures on the Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$ are nilpotent whereas the Lie algebras $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ only admit non-nilpotent complex structures. The complex structures on $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ are classified by Ugarte and Villacampa [96] and, as we mentioned previously, the abelian complex structures on the solvable Lie algebras of dimension 6 are classified in [4]. Therefore, the goal of Section 3.1 is to classify the non-abelian complex structures on the Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$. For a clearer exposition we divide the study according to the step of nilpotency of the Lie algebra. We include in Table 3.1 the final classification of all the complex structures on nilpotent Lie algebras of dimension 6.

The aim of Section 3.2 is to classify the complex structures with a closed form of type $(3, 0)$ on the solvable Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. We use the technique of stable forms introduced in Section 2.2 to describe the whole space of these complex structures on the Lie algebras of the previous list. A reduction process leads us to find a representative in each equivalence class of complex structures. It is remarkable (see Proposition 3.2.7) that for the Lie algebra $\mathfrak{g}_8$ corresponding to the Nakamura manifold there is an infinite family $\{J^A\}_{A \in \mathbb{C}, \mathfrak{Im}\, A \neq 0} \cup \{J'\} \cup \{J''\}$, where $J_0 := J^{-i}$ is the unique complex-parallelizable structure and $J_1 := J^i$ is the unique abelian structure. The classification of complex structures on $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ is given in Table 3.2.

Chapters 4 and 5 make use of the results obtained in Chapter 2 and 3 in order to study several cohomological invariants, the existence of special Hermitian metrics and the behaviour of properties under holomorphic deformations. Chapter 4 studies concretely the Frölicher spectral sequence $\{E_r^{\bullet,\bullet}(M)\}_{r \geq 1}$ and the $\partial\bar{\partial}$-lemma. Rollenske [80] proves that for a nilmanifold $M = G/\Gamma$ of dimension 6 endowed with an invariant complex structure the natural inclusion $(\wedge^{\bullet,\bullet}\mathfrak{g}^*, \bar{\partial}) \to (\wedge^{\bullet,\bullet}M, \bar{\partial})$ induces an isomorphism in Dolbeault cohomology, whenever the underlying Lie algebra is not isomorphic to $\mathfrak{h}_7$. Cordero, Fernández, Gray and Ugarte [23] prove that in such case then $E_r^{\bullet,\bullet}(M) \cong E_r^{\bullet,\bullet}(\mathfrak{g})$ for any $r \geq 2$. Section 4.1 contains the computation of the Frölicher sequence for all the complex structures on the nilpotent Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. From this computation and the previous results we state in Theorem 4.1.4 a general description of the behaviour of the Frölicher spectral sequence for six-dimensional nilmanifolds, except for $\mathfrak{g}$ isomorphic to $\mathfrak{h}_7$. As regards the $\partial\bar{\partial}$-lemma, nilmanifolds cannot satisfy it because they are not formal in the sense of Sullivan [44]. On the other hand, it is known that any compact complex manifold M satisfying the $\partial\bar{\partial}$-lemma also satisfies that $E_1(M) \cong E_\infty(M)$ together with the symmetry of the Hodge numbers $h^{p,q}_{\bar{\partial}}(M) = h^{q,p}_{\bar{\partial}}(M)$ for any $p, q \in \mathbb{N}$. These considerations give rise to the following question posed by Angella and Tomassini [10]: find a compact complex manifold satisfying the latter two conditions but not the $\partial\bar{\partial}$-lemma. We provide an example answering this question in Proposition 4.1.5 based on a complex nilmanifold $(M = G/\Gamma, J)$ with underlying Lie algebra $\mathfrak{h}_6$.

Section 4.2 deals with the computation of the Frölicher spectral sequence for six-dimensional solvmanifolds with an invariant complex structure with holomorphically trivial canonical bundle. This section is divided in two parts. The first part is devoted to the computation of $\{E_r^{\bullet,\bullet}(\mathfrak{g})\}_{r\geq 1}$ at the level of the Lie algebra $\mathfrak{g}$. By means of the symmetrization process, we prove in Proposition 4.2.1 that if $E_1(\mathfrak{g}) \ncong E_\infty(\mathfrak{g})$ at the level of the Lie algebra then also holds $E_1(M) \ncong E_\infty(M)$ on the solvmanifold M. As a consequence, we prove in Corollary 4.2.8 that a complex solvmanifold (M, J) with underlying Lie algebra isomorphic to $\mathfrak{g}_8$ and J equivalent to J_0, J_1, J' or J'' does not satisfy the $\partial\bar{\partial}$-lemma. We consider in the second part of this section the results due to Angella and Kasuya [7, 51] concerning the computation of the Dolbeault and Bott-Chern cohomologies of solvmanifolds $(M = G/\Gamma, J)$ endowed with an invariant complex structure of splitting type [51, Assumption 1.1]. The presence of this kind of complex structure on a solvable Lie group allows to conceive it as a semidirect product $G = \mathbb{C}^r \ltimes N$, where N is a nilpotent Lie group endowed with an invariant complex structure such that $H^{\bullet,\bullet}_{\bar{\partial}}(N/\Gamma_N) \cong H^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{n})$. In addition, the lattice $\Gamma = \Gamma_{\mathbb{C}^r} \ltimes \Gamma_N \subset G$ must satisfy certain compatibility with the semidirect product. The computation of the Dolbeault and Bott-Chern cohomologies of these complex solvmanifolds is performed by means of two finite-dimensional differential complexes $(B^{\bullet,\bullet}_\Gamma, \bar{\partial})$ and $(C^{\bullet,\bullet}_\Gamma, \partial, \bar{\partial})$, such that $H^{\bullet,\bullet}_{\bar{\partial}}(M) \cong H_{\bar{\partial}}(B^{\bullet,\bullet}_\Gamma)$ and $H^{\bullet,\bullet}_{\mathrm{BC}}(M) \cong H_{\mathrm{BC}}(C^{\bullet,\bullet}_\Gamma)$, respectively.

It turns out that of the complex structures on the Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ found in Chapter 3 only those corresponding to $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$ for $\alpha \geq 0$ and $\mathfrak{g}_8$ (except J' and J'') are of splitting type. In addition, the examples analyzed by Kasuya in [51] correspond to the complex structure on $\mathfrak{g}_1$ and the complex-parallelizable structure on $\mathfrak{g}_8$. After building explicitly lattices compatible with the complex structures of splitting type, we present in Proposition 4.2.21 complex solvmanifolds with underlying Lie algebra $\mathfrak{g}_2^0$ satisfying the $\partial\bar{\partial}$-lemma. Furthermore, for a concrete choice of the lattice, we get a solvmanifold such that its de Rham cohomology coincides with the Chevalley-Eilenberg cohomology of $\mathfrak{g}_2^0$, although $\mathfrak{g}_2^0$ is not completely solvable and hence it does not satisfy the hypothesis in Hattori's Theorem [46]. For a countable family $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{>0}$, we build solvmanifolds with underlying Lie algebra $\mathfrak{g}_2^{\alpha_n}$ whose lattices are compatible with the non-equivalent complex structures $J^\pm$. Moreover, we get for some lattices that $E_1(M) \ncong E_\infty(M)$ whereas for others we get solvmanifolds satisfying the $\partial\bar{\partial}$-lemma (see Proposition 4.2.22). As regards the Lie algebra $\mathfrak{g}_8$, we provide lattices in Proposition 4.2.25 compatible with each complex structure $\{J^A\}_{A\in\mathbb{C},\mathfrak{Im}\,A\neq 0}$ so that the resulting solvmanifolds satisfy the $\partial\bar\partial$-lemma if and only if $A=\frac{i}{k}$ with $k\neq 0$ integer.

In Section 4.3 we focus on the behaviour of cohomological invariants under holomorphic deformations. Kodaira and Spencer [53] prove the upper semicontinuity of the Hodge numbers and, as a consequence, the openness of the "$E_1(M)\cong E_\infty(M)$" property. However, Eastwood and Singer [27] prove that this property is not closed under holomorphic deformations by using complex geometry on twistor spaces. On the other hand, Maclaughlin, Pedersen, Poon and Salamon [60] provide a description of the Kuranishi space of six-dimensional 2-step nilmanifolds endowed with an abelian complex structure. Making use of this and our previous results, we provide in Corollary 4.3.3 another example based on an invariant holomorphic deformation of a complex nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ and J the abelian structure on $\mathfrak{h}_4$, showing that the property $E_1(M)\cong E_\infty(M)$ is not closed. Moreover, we construct a family of complex structures $\{J_t\}_{t\in\mathbb{R}}$ on the Lie algebra $\mathfrak{h}_{15}$ showing that the dimensions of the terms $E_2^{\bullet,\bullet}$ are neither upper semicontinuous nor lower semicontinuous. This family allows us to conclude in Corollary 4.3.6 that the property "$E_2(M)\cong E_\infty(M)$" is not open under holomorphic deformations.

As regards the $\partial\bar\partial$-lemma, it turns out that it is open under holomorphic deformations [98, 101, 10]. Angella and Kasuya [8] provide a technique to compute the Dolbeault and the Bott-Chern cohomologies of a holomorphic deformation $(M,J_t)_{t\in\Delta}$ of a complex solvmanifold of splitting type. By using these results, they prove [8] that the $\partial\bar\partial$-lemma is not closed under holomorphic deformations. The proof consists of a holomorphic deformation of the Nakamura manifold. Taking into account this technique and the results obtained in Proposition 4.2.25, we provide in Proposition 4.3.9 a family of holomorphic deformations $\{(M_k,J_{k,t})\}_{t\in\Delta}$ for each compact complex manifold of the countable family $\{(M_k,J_k:=J^{A_k})\}_{k\in\mathbb{Z}}$ where $A_k=\frac{i}{2k+1}$. The Lie algebra underlying the complex solvmanifolds M_k is $\mathfrak{g}_8$. In addition, these holomorphic deformations do not satisfy the $\partial\bar\partial$-lemma in the central limits $J_k:=J_{k,0}$ but the $\partial\bar\partial$-lemma holds for the complex structures $J_{k,t}$ with $t\in\Delta^*$. The construction by Angella and Kasuya corresponds to the complex solvmanifold (M_{-i},J_{-i}).

Chapter 5 deals with special Hermitian metrics. In Section 5.1 we study the existence of balanced, strongly Gauduchon, strong Kähler with torsion and 1-st generalized Gauduchon metrics on the complex solvmanifolds obtained above. By the symmetrization process, the existence of Kähler, balanced [31] and strong Kähler with torsion [95] metrics on a compact manifold of the form $(M=G/\Gamma,J)$ with J invariant reduces to the study at the level of the underlying Lie algebra $\mathfrak{g}$. In the context of nilmanifolds there are results of existence of strong Kähler with torsion [33], 1-st generalized Gauduchon [35] and balanced [95] metrics. Hence, our study of these metrics focus on their existence on the solvable Lie algebras $\mathfrak{g}_1,\ldots,\mathfrak{g}_9$ obtaining the classification results in Propositions 5.1.1, 5.1.6 and 5.1.8. As a consequence, we state in Theorem 5.1.3 that a six-dimensional solvmanifold with an invariant complex structure admitting a Calabi-Yau metric has Lie algebra isomorphic to $\mathbb{R}^6$ or $\mathfrak{g}_2^0$. Concerning the strongly Gauduchon metrics, we prove in Proposition 5.1.9 that their existence on a solvmanifold is also reduced to the existence on the Lie algebra. The existence of these metrics on six-dimensional complex solvmanifolds with holomorphically trivial canonical bundle is divided in Proposition 5.1.11 for nilmanifolds and in Proposition 5.1.13 (see also Table 5.2) for solvmanifolds. As we mentioned above, the balanced condition implies the strongly Gauduchon condition, and Popovici proves that on compact complex manifolds the $\partial\bar{\partial}$-lemma implies the existence of strongly Gauduchon metrics. Therefore, following [76, Theorem 1.10] we study in Proposition 5.1.15 which nilmanifolds admit strongly Gauduchon metrics but none balanced metric. The conclusions of this study are included in Table 5.1.

Finally, Section 5.2 contains two results concerning the behaviour of Hermitian metrics under holomorphic deformations. We prove in Theorem 5.2.1 that neither the balanced nor the strongly Gauduchon properties are closed. The proof is based on the holomorphic deformation used in Corollary 4.3.3 to show the non-closedness of the $E_1(M) \cong E_\infty(M)$ property. On the other hand, Popovici [72] shows that if a holomorphic deformation $\{(M, J_t)\}_{t\in\Delta}$ satisfies the $\partial\bar{\partial}$-lemma for all $t \in \Delta^*$, then the central limit (M, J_0) admits a strongly Gauduchon metric. We wonder if the $\partial\bar{\partial}$-lemma assures the existence of Hermitian metrics stronger than the strongly Gauduchon ones. Concerning this, we show in Theorem 5.2.4 an invariant holomorphic deformation of a complex solvmanifold $\{(M, J_t)\}_{t\in\Delta}$ satisfying the $\partial\bar{\partial}$-lemma and admitting balanced metrics for any $t \in \Delta^*$, but the central limit neither satisfies the $\partial\bar{\partial}$-lemma nor admits balanced metrics. Therefore, the $\partial\bar{\partial}$-lemma does not assure the existence of a balanced metric in the central limit.

Some of the results in Chapters 2, 3, 4 and 5 were obtained in collaboration with Manuel Ceballos, Anna Fino, Luis Ugarte and Raquel Villacampa and can be found in the papers [18] and [32], whereas some other more recent results have been obtained with Daniele Angella and Luis Ugarte.

Chapter 1

Complex manifolds

In this introductory chapter we recall basic notions and some classical results about complex geometry and Hermitian geometry on manifolds. In Section 1.1 we start by setting some definitions and notations concerning complex and almost-complex structures on manifolds. In addition, we remind the Newlander-Nirenberg Theorem [69] which characterizes the almost-complex structures giving rise to a holomorphic atlas. Holomorphic vector bundles on complex manifolds are presented, with special attention to the holomorphic canonical bundle. The goal of Section 1.2 is to present basic results about several cohomologies on complex manifolds. The well-known de Rham $H^{\bullet}_{\mathrm{dR}}(M)$ and Dolbeault $H^{\bullet,\bullet}_{\bar{\partial}}(M)$ cohomology groups are recalled as well as other important cohomologies such as the Bott-Chern $H^{\bullet,\bullet}_{\mathrm{BC}}(M)$ and the Aeppli $H^{\bullet,\bullet}_{\mathrm{A}}(M)$ cohomologies [1, 14] and the main results obtained by Schweitzer [85] concerning the cohomology groups $H^{\bullet,\bullet}_{\mathrm{BC}}(M)$ and $H^{\bullet,\bullet}_{\mathrm{A}}(M)$ when M is a compact complex manifold. In addition, we remind some results connecting these cohomology groups with the $\partial\bar{\partial}$-lemma property [10, 26]. We conclude the section recalling the Frölicher spectral sequence $\{E^{\bullet,\bullet}_r(M)\}_{r\geq 1}$ of a complex manifold, which links the Dolbeault cohomology to the de Rham cohomology [38]. Section 1.3 deals with special Hermitian metrics on complex manifolds. The starting point are Kähler metrics [50, 83] defined by the closedness of the fundamental form $F \in \wedge^{1,1}M$ associated to the metric, whose existence imposes several strong topological obstructions on a compact complex manifold. Other special Hermitian metrics of interest are presented such as balanced [62], strongly Gauduchon [75], strong Kähler with torsion [33] and k-th generalized Gauduchon [39], as well as the main results and the relations among them. Finally, we consider in Section 1.4 the theory of small holomorphic deformations of compact complex manifolds developed by Kodaira, Spencer, Nirenberg [53, 54] and Kuranishi [55]. In the first part of the section, we present the main definitions about the theory of holomorphic deformations leading us to the Theorem of Kuranishi [55], which describes the local geometry of the moduli space of complex structures on a compact complex manifold. In the second part, we remind some general results about the openness or closedness of some important properties, such as the Kähler property [47, 53] or the $\partial\bar{\partial}$-lemma [8, 10, 98, 101], among others. We follow for the last part a paper by Popovici [76], paying especial attention to the problems of closedness of the balanced and the strongly Gauduchon properties.

1.1 Almost-complex structures and integrability

A complex manifold is a differentiable manifold of even dimension equipped with a so-called *complex structure*. There are several approaches to this object, but we start by presenting it in terms of the existence of a holomorphic atlas.

Definition 1.1.1. *Let M be a smooth manifold of real dimension $2n$.*

- *A* complex chart *on M is a pair (U, ψ) where $U \subseteq M$ is open and $\psi : U \to \mathbb{C}^n$ is a diffeomorphism between U and an open set of $\mathbb{C}^n$.*
- *A* holomorphic atlas *is a set of complex charts $\{(U_i, \psi_i)\}_{i\in I}$ such that $M = \bigcup_{i\in I} U_i$ and if $U_i \cap U_j \neq \emptyset$, then the transition map $\psi_{ij} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ given by $\psi_{ij} := \psi_j \circ \psi_i^{-1}$ is biholomorphic.*

M is said to be a complex manifold *if it is equipped with a holomorphic atlas. The complex dimension of M is n. If (U, ψ) is a complex chart and $p \in U$, then its complex coordinates in the chart U are $(z_1, \dots, z_n) \in \mathbb{C}^n$, where $z_j : U \to \mathbb{C}$ is given by $z_j = \pi_j \circ \psi$, $\pi_j : \mathbb{C}^n \to \mathbb{C}$ being the projection onto the j-th component.*

Therefore, complex manifolds are a type of differentiable manifolds endowed with an atlas whose transition functions are biholomorphic. Therefore, only even-dimensional smooth manifolds could admit a complex structure. In addition, the existence of a holomorphic atlas requires the orientability of M.

In this work we focus our attention in complex manifolds with *holomorphically trivial canonical bundle*. We briefly recall the definition of holomorphic vector bundle as well as the natural holomorphic vector bundles associated to any complex manifold: the tangent and the cotangent bundles, the bundle of holomorphic p-forms and the holomorphic canonical bundle.

Definition 1.1.2. *Let M be a complex manifold of complex dimension n. A* holomorphic vector bundle of rank r *on M is a complex manifold E, called the* total space*, together with a holomorphic map $\pi : E \to M$ and the structure of an r-dimensional complex vector space on any* fibre *$E_p := \pi^{-1}(p)$ satisfying the following condition: there exists an open covering $M = \bigcup_{i\in I} U_i$ and biholomorphic maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ commuting with the projections to U_i such that the induced map $\pi^{-1}(p) \cong \mathbb{C}^r$ is $\mathbb{C}$-linear.*

Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, the *holomorphic tangent bundle* is the holomorphic vector bundle $\mathcal{T}_M$ on M of rank n which is given by the transition functions

$$\varphi_{ij} : \psi_i(U_i \cap U_j) \times \mathbb{C}^n \to \psi_j(U_i \cap U_j) \times \mathbb{C}^n,$$

where $\varphi_{ij}(\psi_i(p), v) := (\psi_{ij}(\psi_i(p)), (\psi_{ij})_{*\psi_i(p)}v)$, $\{\psi_{ij}\}_{i,j\in I}$ denote the holomorphic transition maps on M, and $(\psi_{ij})_{*\psi_i(p)} \in \mathrm{GL}(n, \mathbb{C})$ denotes the Jacobian of the map ψ_{ij} at point $\psi_i(p) \in \mathbb{C}^n$. The *holomorphic cotangent bundle* Ω^1_M is the dual vector bundle of $\mathcal{T}_M$. The *bundle of holomorphic k-forms* is $\Omega^k_M := \wedge^k \Omega^1_M$ for $0 \leq k \leq n$ and, when

$k = n$, we get the *holomorphic canonical bundle* of M denoted by $K_M := \Omega^n_M$. Hence, by a complex manifold with holomorphically trivial canonical bundle we mean a complex manifold M with a global non-vanishing holomorphic volume form $0 \neq \Psi \in K_M$ which *trivializes* the canonical bundle of M.

Remark 1.1.3. Every holomorphic vector bundle on M is in particular a differentiable (or $\mathcal{C}^\infty$) vector bundle. However, it is possible to define complex vector bundles on M which are not holomorphic. They satisfy that the fibres $\pi^{-1}(p)$ are complex vector spaces for any $p \in M$ and also that the transition functions φ_{ij} are $\mathbb{C}$-linear, but the total space E is not a complex manifold. As a matter of notation, if $\pi : E \to M$ is a $\mathcal{C}^\infty$-vector bundle we denote by $\mathcal{C}^\infty(M; E)$ the set of smooth sections of the bundle. Particularly, we denote the set of $\mathcal{C}^\infty$-vector fields on M by $\mathfrak{X}(M) := \mathcal{C}^\infty(M; TM)$ and the $\mathcal{C}^\infty$-tensor fields of rank (k, l) on M by $\mathcal{T}^k_l(M) := \mathcal{C}^\infty(M; TM \otimes \overset{l}{\ldots} \otimes TM \otimes T^*M \otimes \overset{k}{\ldots} \otimes T^*M)$. More concretely, $\mathcal{T}^k(M) := \mathcal{T}^k_0(M)$ and $\mathrm{End}(TM) := \mathcal{T}^1_1(M)$. The space of sections of the bundle of $\mathcal{C}^\infty$-alternating tensors of rank k are denoted by $\wedge^k M$ and its elements are called *k-forms*. Otherwise stated, every time we deal with any tensor object T on a complex manifold M we will assume that it is $\mathcal{C}^\infty$. When T is a section of a holomorphic vector bundle we will refer to it explicitly by "the holomorphic tensor field T".

Let M be a complex manifold of complex dimension n. The existence of a holomorphic atlas on M yields to the existence of a smooth tensor field $J \in \mathrm{End}(TM)$ defined on the manifold in the following way. Let $(U, (z_j = x_j + iy_j)_{j=1}^n)$ be a complex chart and consider the real chart $(U, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ and the corresponding local basis of coordinate vector fields

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\} \subset TU.$$

We can locally define a section $J \in \mathrm{End}(TU)$ by:

$$J\left(\frac{\partial}{\partial x_i}\right) := \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) := -\frac{\partial}{\partial x_i},$$

for $i = 1, \ldots, n$. It is easy to check that this local definition does not depend on the complex chart (due to the Cauchy-Riemann equations). Hence, the tensor J can be extended to the whole manifold obtaining a global smooth tensor field $J \in \mathrm{End}(TM)$ satisfying $J^2 = -\mathrm{Id}_{TM}$. This tensor field J is called a *complex structure* on the complex manifold M. We usually use the notation (M, J) to refer to a complex manifold, or equivalently, a differentiable manifold M endowed with a complex structure J.

In general, it is interesting to consider a differentiable (not necessarily complex) manifold M endowed with a smooth section $J \in \mathrm{End}(TM)$ satisfying the condition $J^2 = -Id_{TM}$. This kind of tensors are called *almost-complex structures* on M. Similarly, the pair (M, J) is called an *almost-complex manifold*.

Every complex manifold (M, J) admits an almost-complex structure, but there are almost-complex manifolds which do not give rise to a holomorphic atlas on M. The almost-complex structures that yield to a complex structure on M are called *integrable*. The Newlander-Nirenberg theorem [69] characterizes the integrability of an almost-complex structure J in terms of the vanishing of a tensor Nij_J associated to J called the *Nijenhuis tensor of* J.

Theorem 1.1.4 (Newlander and Nirenberg [69, Theorem 1.1]). *Let J be an almost-complex structure on M, then J is integrable if and only if $Nij_J = 0$, where $Nij_J \in \wedge^2 M$ is defined by:*

$$Nij_J(X,Y) := [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY], \quad X,Y \in \mathfrak{X}(M). \tag{1.1}$$

It is worth noticing that on a differentiable manifold there could exist several almost-complex structures satisfying the integrability condition (1.1). More in general, it is required to define a notion of isomorphic complex structures in order to decide whether two complex manifolds are equivalent under the point of view of its complex geometry.

Definition 1.1.5. *Two almost-complex manifolds (M,J), (M',J') are* isomorphic *if there exists a diffeomorphism $F\colon M \to M'$ such that $F_* \circ J = J' \circ F_*$. When (M,J) and (M',J') are both complex manifolds, they are said to be* biholomorphic *as complex manifolds if they are isomorphic as almost-complex manifolds.*

Let (M,J) be a complex manifold and consider the complexified of the tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$. The $\mathbb{C}$-linear extension of J is a diagonalizable endomorphism of $\mathrm{End}(TM_{\mathbb{C}})$ with eigenvalues i and $-i$. Hence, J induces the splitting of the bundle $TM_{\mathbb{C}}$ such that at every point $p \in M$:

$$T_pM_{\mathbb{C}} = T_p^{(1,0)}M \oplus T_p^{(0,1)}M, \tag{1.2}$$

where $T_p^{(1,0)}M := \{X - iJX \mid X \in T_pM\}$ and $T_p^{(0,1)}M := \{X + iJX \mid X \in T_pM\}$ are the J-eigenspaces corresponding to the eigenvalues i and $-i$, respectively. Furthermore, it turns out that the eigenspaces $T_p^{(1,0)}M \cong \mathbb{C}^n \cong T_p^{(0,1)}M$ are irreducible representations of the Lie group $\mathrm{GL}(n,\mathbb{C})$ and that $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$ are complex conjugated (recall that if V is a real vector space and $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is its complexified, then the conjugation map $V_{\mathbb{C}} \to V_{\mathbb{C}}$ is defined by $\overline{v \otimes z} := v \otimes \bar{z}$ for any $v \in V$ and $z \in \mathbb{C}$).

Remark 1.1.6. If (M,J) is an almost-complex manifold, the complexified tangent bundle also admits a splitting like (1.2). It is proved that the almost-complex structure J is uniquely determined by a splitting of the bundle $TM_{\mathbb{C}} = T^{(1,0)}M \oplus T^{(0,1)}M$ satisfying $\overline{T^{(1,0)}M} := T^{(0,1)}M$. In particular, the integrability of J is equivalent to the property of $[T^{(0,1)}M, T^{(0,1)}M] \subseteq T^{(0,1)}M$.

As a matter of notation, we denote by $\mathfrak{X}^{1,0}(M)$ and $\mathfrak{X}^{0,1}(M)$ the sections of the bundles $T^{(1,0)}M$ and $T^{(0,1)}M$, respectively.

Recall that the endomorphism $J^* \in \mathrm{End}(T^*M_{\mathbb{C}})$ dual to J is defined by $(J^*\alpha)X := \alpha(JX)$. Furthermore, the complex structure of the manifold also induces, via the endomorphism J^*, a splitting in the space of the complexified bundle $T^*M_{\mathbb{C}}$:

$$T_p^*M_{\mathbb{C}} = (T_p^*M)^{1,0} \oplus (T_p^*M)^{0,1},$$

where $(T_p^*M)^{1,0} := \{\alpha - iJ^*\alpha \mid \alpha \in T_p^*M\}$ and $(T_p^*M)^{0,1} := \{\alpha + iJ^*\alpha \mid \alpha \in T_p^*M\}$ are the J^*-eigenspaces corresponding to the eigenvalues i and $-i$, respectively. More in general, the extension of the endomorphism J to other complexified bundles of tensors induced splittings on them. In complex geometry, one of the most important ones is the splitting of the space of complexified k-forms $\wedge^k M_{\mathbb{C}}$ given by:

$$\wedge^k M_{\mathbb{C}} = \bigoplus_{p+q=k} \wedge^{p,q} M, \tag{1.3}$$

where $\wedge^{p,q}M := \mathcal{C}^\infty(M; \wedge^p(T^*M)^{1,0} \otimes \wedge^q(T^*M)^{0,1})$. The elements of $\wedge^{p,q}M$ are called *(p,q)-forms* or *complex forms of pure type* (p,q). If $p+q=k$, we define the natural projections $\pi_{p,q}\colon \wedge^k M_{\mathbb{C}} \to \wedge^{p,q}M$.

1.2 Cohomologies of complex manifolds

1.2.1 De Rham and Dolbeault cohomology groups

Let M be a differentiable manifold with $\dim_{\mathbb{R}} M = m$. The *de Rham complex* of M is the differential graded algebra $(\wedge^\bullet M, d)$, where $\wedge^\bullet M := \oplus_{k=0}^m \wedge^k M$ is the algebra, with respect to the wedge product, of differential forms on M called the *exterior algebra of* M and $d\colon \wedge^\bullet M \to \wedge^{\bullet+1}M$ is the exterior derivative. Recall that a k-form α is *closed* if $d\alpha = 0$, and it is *exact* if $\alpha = d\beta$ for some $(k-1)$-form β. As $d^2 = 0$, every exact form is closed. Hence, a cohomology associated to this complex can be defined. The cohomology groups are called the *de Rham cohomology groups* of the manifold:

$$H^\bullet_{\mathrm{dR}}(M) := \frac{\ker\left(d\colon \wedge^\bullet M \to \wedge^{\bullet+1}M\right)}{\operatorname{im}\left(d\colon \wedge^{\bullet-1} M \to \wedge^\bullet M\right)}.$$

The dimensions of these groups are denoted by $b_\bullet(M) := \dim H^\bullet_{\mathrm{dR}}(M)$ when they are finite dimensional. The most important fact concerning the dimensions of these cohomology groups is stated by the well-known de Rham Theorem:

Theorem 1.2.1 (de Rham Theorem)**.** *Let M be a smooth manifold. The de Rham cohomology of M is isomorphic to the singular cohomology.*

Therefore, the dimensions $b_\bullet(M)$ are topological invariants of the manifold coinciding with the Betti numbers of the underlying topological space.

The Hodge theory of smooth, oriented and compact Riemannian manifolds yields to several important results concerning the Betti numbers of compact manifolds. Let (M,g) be an orientable compact Riemannian manifold, and consider a global volume form $0 \neq \nu \in \wedge^m M$. The Riemannian metric defines a smooth inner product on the exterior algebra $\wedge M$ declaring that any two distinct spaces $\wedge^k M$ and $\wedge^{k'}M$ are orthogonal for distinct k, k' and

$$\langle \alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k \rangle_p := \det(g_p^*(\alpha_p^i, \beta_p^j)).$$

Hence, the orientation and the metric on M give rise to the *Hodge star operator* $*\colon \wedge^\bullet M \to \wedge^{m-\bullet}M$ defined point-wise by $(\alpha \wedge *\beta)_p := \langle \alpha, \beta \rangle_p \nu_p$ where $\alpha, \beta \in \wedge^\bullet M$ and $p \in M$.

The theory of integration of forms on manifolds allows to define a global inner product in the space of k-forms of the manifold defined by $\langle\langle \alpha, \beta \rangle\rangle := \int_M \alpha \wedge *\beta$. Hence, it is possible to define the operator $d^*\colon \wedge^\bullet M \to \wedge^{\bullet-1}M$ by the adjoint of the exterior derivative d with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. The *Laplace-Beltrami operator* $\Delta\colon \wedge^\bullet M \to \wedge^\bullet M$ is defined by $\Delta := dd^* + d^*d$. A k-form α is called *harmonic* if $\Delta\alpha = 0$. The space of harmonic forms is denoted by $\mathcal{H}^\bullet(M) := \ker(\Delta\colon \wedge^\bullet M \to \wedge^\bullet M)$. Now, we state the following important results:

Theorem 1.2.2 (Hodge Orthogonal Decomposition Theorem)**.** *Let (M, g) be a compact oriented Riemannian manifold, then* $\dim \mathcal{H}^\bullet(M) < \infty$ *and there is an $\langle\langle \cdot, \cdot \rangle\rangle$-orthogonal decomposition:*

$$\wedge^\bullet M = \mathcal{H}^\bullet(M) \overset{\perp}{\oplus} d\left(\wedge^{\bullet-1}M\right) \overset{\perp}{\oplus} d^*\left(\wedge^{\bullet+1}M\right). \tag{1.4}$$

Furthermore, there is an isomorphism only depending on the metric g such that $H^\bullet_{dR}(M) \cong \mathcal{H}^\bullet(M)$. In particular, $\dim H^\bullet_{dR}(M) < \infty$.

As every smooth compact manifold admits a Riemannian metric (using partitions of unity) there is a proof based on Hodge theory for the following well-known Poincaré Duality Theorem:

Theorem 1.2.3 (Poincaré Duality)**.** *Let M be a compact smooth manifold with* $\dim M = m$*. The pairing*

$$H^{m-\bullet}_{dR}(M) \times H^\bullet_{dR}(M) \to \mathbb{R}, \quad ([\alpha], [\beta]) \to \int_M \alpha \wedge \beta$$

is non-degenerate, i.e. it induces an isomorphism (the Poincaré duality isomorphism) such that $H^{m-\bullet}_{dR}(M) \cong H^\bullet_{dR}(M)$. In particular, $b_{m-\bullet}(M) = b_\bullet(M)$.

Now, let (M, J) be a complex manifold of complex dimension n and consider the complexified de Rham complex $(\wedge M_{\mathbb{C}}, d)$. Attending to the splitting of the forms given by (1.3), it is easy to check that $d(\wedge^{1,0}M) \subseteq \wedge^{2,0}M \oplus \wedge^{1,1}M$. Therefore, the exterior derivative d decomposes as the sum of two differential operators $d = \partial + \bar{\partial}$, where:

$$\partial := \pi_{\bullet+1,\bullet} \circ d\colon \wedge^{\bullet,\bullet} M \to \wedge^{\bullet+1,\bullet}M, \qquad \bar{\partial} := \pi_{\bullet,\bullet+1} \circ d\colon \wedge^{\bullet,\bullet} M \to \wedge^{\bullet,\bullet+1}M. \tag{1.5}$$

As a direct consequence of the property $d^2 = 0$, these differential operators satisfy $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Hence, we can define a bi-differential bi-graded algebra $(\wedge^{\bullet,\bullet}M, \partial, \bar{\partial})$ associated to the complex manifold (M, J) and, in the same manner as for the de Rham complex, we can speak about ∂ or $\bar{\partial}$-closed (p, q)-forms and ∂ or $\bar{\partial}$-exact (p, q)-forms.

Remark 1.2.4. If (M, J) is an almost-complex manifold, then the exterior derivative operator splits into four differential operators $d = A + \partial + \bar{\partial} + \bar{A}$, where

$$A := \pi_{\bullet+2,\bullet-1} \circ d\colon\ \wedge^{\bullet,\bullet} M \to \wedge^{\bullet+2,\bullet-1} M, \quad \bar{A} := \pi_{\bullet-1,\bullet+2} \circ d\colon\ \wedge^{\bullet,\bullet} M \to \wedge^{\bullet-1,\bullet+2} M$$

and ∂ and $\bar{\partial}$ are defined by (1.5). It is proved that the integrability of J is equivalent to the vanishing of the operators $A, \bar{A}$.

Now, for every $p \in \{0, \dots, n\}$ consider the differential graded algebra $(\wedge^{p,\bullet} M, \bar{\partial})$. If $q \in \{0, \dots, n\}$ the (p, q)-*Dolbeault cohomology group* of M, denoted by $H^{p,q}_{\bar{\partial}}(M)$, is defined as the cohomology group associated to the complex $(\wedge^{p,\bullet} M, \bar{\partial})$, namely:

$$H^{p,q}_{\bar{\partial}}(M) := \frac{\ker\left(\bar{\partial}\colon\ \wedge^{p,q} M \to \wedge^{p,q+1} M\right)}{\operatorname{im}\left(\bar{\partial}\colon\ \wedge^{p,q-1} M \to \wedge^{p,q} M\right)}.$$

When they are finite dimensional, we denote their dimensions by $h^{\bullet,\bullet}_{\bar{\partial}}(M) := \dim H^{\bullet,\bullet}_{\bar{\partial}}(M)$. They are invariants of the complex structure, and they are called the *Hodge numbers* of the complex manifold (M, J).

When (M, J) is a compact complex manifold, the Hodge theory of compact Hermitian manifolds provides analytic versions for the operators ∂ and $\bar{\partial}$ of the Hodge Orthogonal Decomposition Theorem and of the Poincaré Duality Theorem. It suffices to replace the de Rham cohomology by the Dolbeault cohomology and the Δ-harmonic forms by $\square$ or $\overline{\square}$-harmonic forms, where $\square := \partial\partial^* + \partial^*\partial$ and $\overline{\square} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ are the Laplace-Beltrami operators for ∂ and $\bar{\partial}$, respectively. In the spirit of Theorems 1.2.2 and 1.2.3, we sum up the results concerning the Hodge numbers of the complex manifold in the following theorems:

Theorem 1.2.5 (Hodge Orthogonal Decomposition Theorem)**.** *Let (M, g) be a compact Hermitian manifold, then* $\dim \mathcal{H}^{\bullet,\bullet}(M) < \infty$ *and there is an* $\langle\langle \cdot, \cdot \rangle\rangle$*-orthogonal decomposition:*

$$\wedge^{\bullet,\bullet} M = \mathcal{H}^{\bullet,\bullet}(M) \overset{\perp}{\oplus} \bar{\partial}\left(\wedge^{\bullet,\bullet-1} M\right) \overset{\perp}{\oplus} \bar{\partial}^*\left(\wedge^{\bullet,\bullet+1} M\right), \tag{1.6}$$

where $\mathcal{H}^{\bullet,\bullet}(M) := \ker\left(\overline{\square}\colon\ \wedge^{\bullet,\bullet} M \to \wedge^{\bullet,\bullet} M\right)$*. Furthermore, there is an isomorphism only depending on the metric g such that* $H^{\bullet,\bullet}_{\bar{\partial}}(M) \cong \mathcal{H}^{\bullet,\bullet}(M)$*.*

Theorem 1.2.6 (Kodaira-Serre duality)**.** *Let M be a compact complex manifold of complex dimension n, then* $H^{n-p,n-q}_{\bar{\partial}}(M) \cong H^{p,q}_{\bar{\partial}}(M)$ *for every bi-degree (p, q). In particular,* $h^{n-p,n-q}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$*.*

1.2.2 Bott-Chern and Aeppli cohomologies and the $\partial\bar{\partial}$-lemma

If (M, J) is a complex manifold, then other interesting cohomologies can be defined. More precisely, the *Bott-Chern cohomology group* [14] is defined by

$$H^{\bullet,\bullet}_{\mathrm{BC}}(M) := \frac{\ker\left(\partial + \bar{\partial}\colon\ \wedge^{\bullet,\bullet} M \to \wedge^{\bullet+1,\bullet} M \oplus \wedge^{\bullet,\bullet+1} M\right)}{\operatorname{im}\left(\partial\bar{\partial}\colon\ \wedge^{\bullet-1,\bullet-1} M \to \wedge^{\bullet,\bullet} M\right)}.$$

On the other hand, the *Aeppli cohomology group* [1] is defined by

$$H_{\mathrm{A}}^{\bullet,\bullet}(M) := \frac{\ker\left(\partial\bar{\partial}\colon \wedge^{\bullet,\bullet} M \to \wedge^{\bullet+1,\bullet+1} M\right)}{\operatorname{im}\left(\partial\colon \wedge^{\bullet-1,\bullet} M \to \wedge^{\bullet,\bullet} M\right) + \operatorname{im}\left(\bar{\partial}\colon \wedge^{\bullet,\bullet-1} M \to \wedge^{\bullet,\bullet} M\right)}.$$

Unlike the case of the Dolbeault cohomology groups, the conjugation map induces the isomorphisms $\overline{H_{BC}^{q,p}(M)} \cong H_{BC}^{p,q}(M)$, $\overline{H_{A}^{q,p}(M)} \cong H_{A}^{p,q}(M)$. Similarly to the de Rham and the Dolbeault cohomologies, a Hodge theory can be defined for the Bott-Chern and the Aeppli cohomologies. Fixed a Hermitian metric g compatible with J, the following differential operators can be considered:

$$\tilde{\Delta}_{\mathrm{BC}} := \left(\partial\bar{\partial}\right)\left(\partial\bar{\partial}\right)^* + \left(\partial\bar{\partial}\right)^*\left(\partial\bar{\partial}\right) + \left(\bar{\partial}^*\partial\right)\left(\bar{\partial}^*\partial\right)^* + \left(\bar{\partial}^*\partial\right)^*\left(\bar{\partial}^*\partial\right) + \bar{\partial}^*\bar{\partial} + \partial^*\partial,$$
$$\tilde{\Delta}_{\mathrm{A}} := \partial\partial^* + \overline{\partial\partial}^* + \left(\partial\bar{\partial}\right)^*\left(\partial\bar{\partial}\right) + \left(\partial\bar{\partial}\right)\left(\partial\bar{\partial}\right)^* + \left(\bar{\partial}\partial^*\right)^*\left(\bar{\partial}\partial^*\right) + \left(\bar{\partial}\partial^*\right)\left(\bar{\partial}\partial^*\right)^*.$$

Schweitzer [85, Sect. 2.c] proves that the operators $\tilde{\Delta}_{\mathrm{BC}}$ and $\tilde{\Delta}_{\mathrm{A}}$ are elliptic. As a consequence of the general theory of partial elliptic operators, it turns out [85, Corollaire 2.3] that the dimensions of the cohomology groups $H_{\mathrm{BC}}^{\bullet,\bullet}(M)$ and $H_{\mathrm{A}}^{\bullet,\bullet}(M)$ are finite for a compact complex manifold M. In this case we denote these dimensions by $h_{\mathrm{BC}}^{\bullet,\bullet}(M)$ and $h_{\mathrm{A}}^{\bullet,\bullet}(M)$, respectively. In addition, the Hodge star operator associated to a Hermitian metric induces an isomorphism $H_{\mathrm{A}}^{n-q,n-p}(M) \cong H_{\mathrm{BC}}^{p,q}(M)$ for any $p, q \in \mathbb{N}$.

The properties $\ker\partial\bar{\partial} \subseteq \ker d$, $\mathrm{im}\partial\bar{\partial} \subseteq \mathrm{im}d$ and $\ker\partial\bar{\partial} \subseteq \ker\bar{\partial}$, $\mathrm{im}\partial\bar{\partial} \subseteq \mathrm{im}\bar{\partial}$ yield to the following natural maps of bi-graded $\mathbb{C}$-vector spaces:

$$\begin{tikzcd} & H_{BC}^{\bullet,\bullet}(M) \arrow[dl] \arrow[d] \arrow[dr] & \\ H_{\bar{\partial}}^{\bullet,\bullet}(M) \arrow[dr] & H_{\mathrm{dR}}^{\bullet}(M;\mathbb{C}) \arrow[d] & H_{\partial}^{\bullet,\bullet}(M) \arrow[dl] \\ & H_{A}^{\bullet,\bullet}(M) & \end{tikzcd}$$

where $H_{\partial}^{\bullet,\bullet}(M)$ denotes the conjugate of the Dolbeault cohomology group $H_{\bar{\partial}}^{\bullet,\bullet}(M)$. However, these maps are in general neither injective nor surjective. Deligne, Griffiths, Morgan and Sullivan [26, Remark 5.16] state that if one of these maps is an isomorphism then the rest are isomorphisms too (for an explicit proof see Angella [6, Theorem 2.1]). In particular, when the identity map $H_{\mathrm{BC}}^{\bullet,\bullet}(M) \to H_{\mathrm{dR}}^{\bullet}(M)$ is injective the compact complex manifold is said to satisfy the *$\partial\bar{\partial}$-lemma*.

Definition 1.2.7 ([26, Lemma 5.15]). *A complex manifold M satisfies the $\partial\bar{\partial}$-lemma if every ∂-closed, $\bar{\partial}$-closed, d-exact form is also $\partial\bar{\partial}$-exact.*

It is an interesting question to know whether a given complex manifold satisfies the $\partial\bar{\partial}$-lemma. Angella and Tomassini [10] provide, whenever M is compact, a Frölicher-type inequality characterizing the validity of the $\partial\bar{\partial}$-lemma in terms of the dimensions of the Bott-Chern and Aeppli cohomology groups.

Theorem 1.2.8 (Angella and Tomassini [10, Theorem pp. 2]). *Let M be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} (h^{p,q}_{BC}(M) + h^{p,q}_{A}(M)) - 2 \dim_{\mathbb{C}} H^k_{dR}(M;\mathbb{C}) \geq 0. \tag{1.7}$$

Moreover, the equality (1.7) *holds for every $k \in \mathbb{N}$ if and only if M satisfies the $\partial\bar{\partial}$-lemma.*

Remark 1.2.9. When M satisfies the $\partial\bar{\partial}$-lemma it turns out that $H^{\bullet,\bullet}_{BC}(M) \cong H^{\bullet,\bullet}_{\bar{\partial}}(M)$. Therefore the $\partial\bar{\partial}$-lemma implies the symmetry of the Hodge numbers $h^{q,p}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$ for every bidegree (p,q).

1.2.3 The Frölicher spectral sequence

As every complex valued d-closed form of pure type (p,q) is also $\bar{\partial}$-closed, it is clear that every class of de Rham cohomology represented by a (p,q)-form defines a class of Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(M)$. However, the converse is not true in general. Considering the bi-differential bi-graded complex $(\wedge^{\bullet,\bullet}M, \partial, \bar{\partial})$, it is possible to define a filtration on the complexified de Rham complex

$$F^p \wedge^k M_{\mathbb{C}} := \{\alpha = \sum_{r \geq p} \alpha_{r,q} \mid \alpha_{r,q} \in \wedge^{r,q} M_{\mathbb{C}}, \text{ such that } r+q=k\}, \tag{1.8}$$

such that $F^0 \wedge^k M_{\mathbb{C}} = \wedge^k M_{\mathbb{C}}$ and $F^{n+r} \wedge^k M_{\mathbb{C}} = \{\mathbf{0}\}$ if $r \geq 1$. This filtration induces the *Frölicher spectral sequence* $\{(E_r^{\bullet,\bullet}(M), d_r)\}_{r\geq 1}$ of the complex manifold (M,J). More precisely, for each $r \geq 1$ there is a sequence of homomorphisms d_r

$$\cdots \xrightarrow{d_r} E_r^{p-r,q+r-1}(M) \xrightarrow{d_r} E_r^{p,q}(M) \xrightarrow{d_r} E_r^{p+r,q-r+1}(M) \xrightarrow{d_r} \cdots \tag{1.9}$$

such that $d_r \circ d_r = 0$ and $E^{p,q}_{r+1}(M) = \ker d_r / \operatorname{im} d_r$. Hence, $\dim E^{p,q}_{r+1}(M) \leq \dim E^{p,q}_r(M)$, and $E^{p,q}_{r+1}(M) \cong E^{p,q}_r(M)$ if and only if $d_r = 0$. The homomorphisms d_r are induced from the ∂ and the $\bar{\partial}$ operators. For $r = 1$, the Frölicher spectral sequence coincides with the Dolbeault cohomology, namely $E_1^{p,q}(M) \cong H^{p,q}_{\bar{\partial}}(M)$. Moreover, the explicit description of each homomorphism d_1 is

$$\begin{array}{ccc} H^{p,q}_{\bar{\partial}}(M) & \xrightarrow{d_1} & H^{p+1,q}_{\bar{\partial}}(M) \\ [\alpha] & \longrightarrow & [\partial\alpha]. \end{array}$$

Hence, for $r = 2$ we have the following explicit expression given by Frölicher [38]:

$$E_2^{p,q}(M) = \frac{\{\alpha_{p,q} \in \wedge^{p,q}(M) \mid \bar{\partial}\alpha_{p,q} = 0,\ \partial\alpha_{p,q} = -\bar{\partial}\alpha_{p+1,q-1}\}}{\{\bar{\partial}\beta_{p,q-1} + \partial\gamma_{p-1,q} \mid \bar{\partial}\gamma_{p-1,q} = 0\}}.$$

Now, the homomorphisms d_2 are given by

$$\begin{array}{ccc} E_2^{p,q}(M) & \xrightarrow{d_2} & E_2^{p+2,q-1}(M) \\ [\alpha_{p,q}] & \longrightarrow & [\partial\alpha_{p+1,q-1}] \end{array}$$

Cordero, Fernández, Gray and Ugarte [22] provide a general description of the terms in the spectral sequence $\{E_r^{p,q}(M)\}$ and of the operators $\{d_r\}$ as follows:

Theorem 1.2.10 ([22, Theorem 1]). *Let M be a complex manifold. Then $E_r^{p,q}(M) \cong \frac{\mathcal{X}_r^{p,q}}{\mathcal{Y}_r^{p,q}}$, where*

$$\mathcal{X}_1^{p,q}(M) := \{\alpha \in \wedge^{p,q}M \mid \bar{\partial}\alpha = 0\}, \qquad \mathcal{Y}_1^{p,q}(M) := \bar{\partial}\left(\wedge^{p,q-1}M,\right)$$

and

$$\mathcal{X}_r^{p,q}(M) := \{\alpha_{p,q} \in \wedge^{p,q}M \mid \bar{\partial}\alpha_{p,q} = 0 \text{ and there exists } \alpha_{p+i,q-i} \in \wedge^{p+i,q-i}M \text{ such that } \partial\alpha_{p+i-1,q-i+1} + \bar{\partial}\alpha_{p+i,q-i} = 0, \text{ for } 1 \leq i \leq r-1\},$$

$$\mathcal{Y}_r^{p,q}(M) := \{\partial\beta_{p-1,q} + \bar{\partial}\beta_{p,q-1} \in \wedge^{p,q}M \mid \text{there exists } \beta_{p-i,q+i-1} \in \wedge^{p-i,q+i-1}M, \; 2 \leq i \leq r-1, \text{ satisfying } \partial\beta_{p-i,q+i-1} + \bar{\partial}\beta_{p-i+1,q+i-2} = 0, \, \bar{\partial}\beta_{p-r+1,q+r-2} = 0\}.$$

Theorem 1.2.11 ([22, Theorem 3]). *For $r \geq 2$ the map $d_r \colon E_r^{p,q}(M) \to E_r^{p+r,q-r+1}(M)$ is given by*

$$d_r\left([\alpha_{p,q}]\right) = [\partial\alpha_{p+r-1,q-r+1}],$$

for $[\alpha_{p,q}] \in E_r^{p,q}(M)$. Furthermore,

$$E_{r+1}^{p,q}(M) = \frac{\mathcal{X}_{r+1}^{p,q}(M)}{\mathcal{Y}_{r+1}^{p,q}(M)} = \frac{\ker\left(d_r \colon E_r^{p,q}(M) \to E_r^{p+r,q-r+1}(M)\right)}{d_r\left(E_r^{p-r,q+r-1}(M)\right)}.$$

Now, the E_∞-terms in the Frölicher spectral sequence are defined as follows. Let $Z_\infty^{p,q}(M) := F^p \wedge^{p+q} M_{\mathbb{C}} \cap \ker d$, $B_\infty^{p,q}(M) := F^p \wedge^{p+q} M_{\mathbb{C}} \cap \operatorname{im} d$ and $H_\infty^{p,q}(M) := \frac{Z_\infty^{p,q}(M)}{B_\infty^{p,q}(M)}$. Making use of the basic isomorphism for modules $\frac{A}{A\cap B} \cong \frac{A+B}{B}$ we have that the E_∞-terms are defined by:

$$E_\infty^{p,q}(M) := \frac{Z_\infty^{p,q}(M)}{Z_\infty^{p+1,q-1}(M) + B_\infty^{p,q}(M)} \cong \frac{H_\infty^{p,q}(M)}{H_\infty^{p+1,q-1}(M)}. \tag{1.10}$$

The complexified k-th de Rham cohomology group also admits a filtration induced by the filtration (1.8) of the de Rham complex:

$$H^k(M;\mathbb{C}) = H_\infty^{0,k}(M) \supset H_\infty^{1,k-1}(M) \supset \cdots \supset H_\infty^{n+1,k-n-1}(M) = 0,$$

where n is the complex dimension of M. Therefore, using (1.10) we get

$$H^k(M;\mathbb{C}) = \bigoplus_{p=0}^{n} \frac{H_\infty^{p,k-p}(M)}{H_\infty^{p+1,k-p-1}(M)} = \bigoplus_{p=0,p+q=k}^{n} \frac{H_\infty^{p,q}(M)}{H_\infty^{p+1,q-1}(M)} \cong \bigoplus_{p+q=k} E_\infty^{p,q}(M). \tag{1.11}$$

Therefore, the Frölicher sequence of a complex manifold M is a spectral sequence $\{(E_r^{\bullet,\bullet}(M), d_r)\}_{r\geq 1}$ whose first term $E_1(M)$ is precisely the Dolbeault cohomology of M and such that after a finite number of steps, the sequence converges to the de Rham cohomology of M. Hence, the Frölicher spectral sequence measures the difference between the Dolbeault cohomology groups and the de Rham cohomology groups.

Remark 1.2.12. For the remainder of this work we introduce the notation $E_r^{|\bullet|}(M) := \bigoplus_{p+q=\bullet} E_r^{p,q}(M)$, $E_r(M) := \bigoplus_{k=0}^n E_r^{|k|}(M)$, $E_\infty^{|\bullet|}(M) := \bigoplus_{p+q=\bullet} E_\infty^{p,q}(M)$ and $E_\infty(M) := \bigoplus_{k=0}^n E_\infty^{|k|}(M)$.

By (1.11), we have $E_\infty^{|\bullet|}(M) \cong H^\bullet_{\mathrm{dR}}(M,\mathbb{C})$ and, whenever M is compact, it is clear that $\dim E_r^{|\bullet|}(M) \geq b_\bullet(M) = \dim H^\bullet_{\mathrm{dR}}(M)$. The equality holds if and only if $E_r(M) \cong E_\infty(M)$, that is, if and only if all the homomorphisms d_{r+k} for $k \geq 0$ are identically zero.

We conclude this section stating the fundamental inequality due to Frölicher:

Theorem 1.2.13 (Frölicher [38, Theorem 2])**.** *Let (M,J) be a compact complex manifold. Then, for every $k \in \mathbb{N}$,*

$$\dim_{\mathbb{C}} H^k_{dR}(M;\mathbb{C}) \leq \sum_{p+q=k} h^{p,q}_{\bar\partial}(M). \tag{1.12}$$

Remark 1.2.14. When the $\partial\bar\partial$-lemma holds on M then every class of the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(M)$ can be represented by a d-closed (p,q)-form. Therefore, for any bidegree (p,q), one has the inclusion $H^{p,q}_{\bar\partial}(M) \hookrightarrow H^{p+q}_{\mathrm{dR}}(M)$. By using the Frölicher inequality (1.12) it turns out that every compact complex manifold satisfying the $\partial\bar\partial$-lemma verifies that the Frölicher sequence degenerates at the first step.

1.3 Special Hermitian geometry

In this section we give the basic definitions about special Hermitian metrics. Hermitian metrics on a complex manifold are a type of Riemannian metrics which are in some sense compatible with the complex structure. Although there are several approaches to these structures, we place them in the bigger class of *almost-Hermitian* structures:

Definition 1.3.1. *Let (M,J) be an almost-complex manifold. An* almost-Hermitian metric *on (M,J) is a Riemannian metric g satisfying the compatibility condition*

$$g(JX,JY) = g(X,Y),$$

for all $X, Y \in \mathfrak{X}(M)$. *The triple* (M, J, g) *is called an* almost-Hermitian manifold *and the pair* (J, g) *is said to be an* almost-Hermitian structure *on* M.

It is very useful to refer to almost-Hermitian structures (J, g) on a manifold by means of a 2-form $F \in \wedge^2 M$ called the *fundamental form*, or *Kähler form*, defined by $F(\cdot, \cdot) := g(J\cdot, \cdot)$. It is worth noticing that the fundamental form is non-degenerate, i.e. $0 \neq F^n \in \wedge^{2n} M$, and if we consider it as a complex valued 2-form then F has bidegree of pure type $(1, 1)$. We can also refer to an almost-Hermitian structure on M with the pair (J, F).

When (M, J) is a complex manifold, it is said that g is a *Hermitian* metric g if it is an almost-Hermitian metric compatible with J. Notice that, even when J is almost-complex, given a Riemannian metric g on M (which exists whenever partitions of the unit exist on the manifold) one can define another Riemannian metric:

$$h(X, Y) := g(X, Y) + g(JX, JY), \quad X, Y \in \mathfrak{X}(M),$$

which is almost-Hermitian. Thus, when J is integrable we have the following proposition:

Proposition 1.3.2. *Every compact complex manifold admits a Hermitian metric compatible with its complex structure.*

Now we present several types of Hermitian metrics distinguished by conditions on the fundamental form F involving the exterior derivative d or the differential operators associated to the complex structure ∂ and $\bar{\partial}$.

The first kind and most important of these Hermitian metrics are the so-called *Kähler metrics* [50, 83] which lay in the intersection of three main branches of differential geometry: Riemannian, complex and symplectic geometry.

Definition 1.3.3. *A Hermitian metric on* (M, J) *is* Kähler *if the fundamental form* $F \in \wedge^{1,1} M$ *is closed. When the complex manifold has a holomorphically trivial canonical bundle, then the Kähler metrics are called* Calabi-Yau *metrics.*

From now on, a complex manifold is said to be Kähler when it admits a compatible Kähler metric. We will adopt the same terminology with any other Hermitian metric.

One of the most important results concerning Kähler geometry is the well-known Hodge Decomposition theorem, which splits the de Rham cohomology into the Dolbeault cohomology groups. It also provides an obstruction to the existence of Kähler structures on a compact complex manifold.

Theorem 1.3.4 (Hodge Decomposition Theorem)**.** *Let* M *be a compact complex manifold endowed with a Kähler metric. Then the following decomposition holds:*

$$H^{\bullet}_{dR}(M; \mathbb{C}) = \bigoplus_{p+q=\bullet} H^{p,q}_{\bar{\partial}}(M; \mathbb{C}),$$

where $H^{q,p}_{\bar{\partial}}(M; \mathbb{C}) = \overline{H^{p,q}_{\bar{\partial}}(M; \mathbb{C})}$ *for any* $p, q \in \mathbb{N}$. *Hence,* $h^{q,p}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$ *for every bi-degree* (p, q).

Deligne, Griffiths, Morgan and Sullivan [26] prove in the following theorem that the validity of the $\partial\bar{\partial}$*-lemma* provides another obstruction to the existence of Kähler metrics on a compact complex manifold.

Theorem 1.3.5 ([26, Lemma 5.11])**.** *Kähler manifolds satisfy the $\partial\bar{\partial}$-lemma property.*

Remark 1.3.6. It follows from the Hodge Decomposition Theorem that if M is Kähler, then the Frölicher spectral sequence degenerates at the first step, namely, $E_1(M) \cong E_\infty(M)$, and the Hodge numbers satisfy $h^{q,p}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$ for any $p, q \in \mathbb{N}$. Recently, Angella and Tomassini [10] posed the question of constructing a compact complex manifold (M, J) with both $E_\infty(M) \cong E_1(M)$ and $h^{q,p}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$ for any $p, q \in \mathbb{N}$, but such that the $\partial\bar{\partial}$-lemma does not hold. We provide an example with these properties in Chapter 4.

In addition, it turns out that the odd Betti numbers of a compact Kähler manifold are even, and in particular, the Hodge number $h^{0,1}_{\bar{\partial}}(M)$ is a topological invariant of the manifold. Moreover, the even Betti numbers of a Kähler manifold do not vanish (if F is the fundamental form, then $0 \neq [F^k] \in H^{2k}_{\mathrm{dR}}(M)$ for $k = 0, \ldots, n$). Another topological obstruction to the existence of a Kähler metric, and more in general, of admitting complex structures satisfying the $\partial\bar{\partial}$-lemma, is that the differential graded algebra $(\wedge^\bullet M, d)$ is *formal* [26, Corollary 1].

We see that the existence of Kähler metrics imposes strong conditions to both the complex structures and the topology of the manifold. It is hence natural to weaken the Kähler condition and study other Hermitian metrics close to the Kähler ones. One of these metrics are called *semi-Kähler Hermitian* or *balanced Hermitian*. We refer to them by *balanced metrics*. Balanced metrics were introduced by Michelsohn [62], although they were characterized in terms of currents. She proved that the existence of a balanced metric in a compact complex manifold M, with $\dim_{\mathbb{C}} M = n$, is equivalent to the non-existence of a non-zero positive current of bi-dimension $(n-1, n-1)$, which is the $(n-1, n-1)$-component of a boundary (this result was later obtained by Harvey and Lawson [43] also in the Kähler case). Here, we use an equivalent definition in terms of the fundamental form F.

Definition 1.3.7. *A Hermitian structure is called* balanced *if the $(n-1, n-1)$-form F^{n-1} is d-closed.*

Balanced metrics are also of interest in Physics because they yield to a geometric interpretation of the solutions of the Strominger system [89].

Given a complex manifold (M, J), recall that the integrability of J produces a decomposition of the exterior differential d of M as $d = \partial + \bar{\partial}$. Several types of Hermitian metrics arise depending on the behaviour of the fundamental form F with the complex differential operators ∂ and $\bar{\partial}$ or some compositions of them. A weaker condition to the balanced condition are the *strongly Gauduchon* metrics introduced by Popovici [75].

Definition 1.3.8. *A Hermitian structure (J, g) is called* strongly Gauduchon *(sG for short) if $\partial F^{n-1} = \bar{\partial} u$ for some $u \in \wedge^{n,n-2} M$.*

It follows from the previous definitions that if M is a complex surface, that is $n = 2$, then the Kähler and the balanced conditions coincide. However, even for $n = 2$, strongly Gauduchon metrics do not need to be Kähler, but any compact complex surface carrying an sG metric also carries a Kähler metric [72, Section 3]. Therefore, a complex surface is Kähler if and only if it is balanced and this also holds if and only if it is strongly Gauduchon. For $n \geq 3$ the situation changes and there are "pure" strongly Gauduchon manifolds (with no compatible balanced metrics) and "pure" balanced manifolds (with no compatible Kähler metrics).

Strongly Gauduchon metrics lie between the balanced metrics and the so-called *standard metrics* introduced by Gauduchon in [40]. A standard metric is a Hermitian metric (J, F) such that the Lee form $\theta := Jd^*F$ is closed, where d^* is the co-differential induced by the Hodge star operator. In this work, we refer to standard metrics by *Gauduchon metrics* and we study them by means of the following equivalent definition:

Definition 1.3.9. *A Hermitian structure* (J, g) *is called* Gauduchon *if* $\partial\bar{\partial}F^{n-1} = 0$.

As we mentioned before, every compact complex manifold admits a compatible Hermitian metric. The most important fact about Gauduchon metrics is the following theorem due to Gauduchon. It asserts that it is possible to associate to every Hermitian metric defined on a compact complex manifold a Gauduchon metric compatible with the underlying complex structure.

Theorem 1.3.10 (Gauduchon [40])**.** *Let* (M, J, F) *be a compact Hermitian manifold, then there is a* J*-Hermitian Gauduchon metric* $\tilde{F}$ *in the conformal class of* F*, namely, there exists* $f \in \mathcal{C}^\infty(M)$ *such that* $\tilde{F} = e^f F$.

Another kind of Hermitian metrics we are interested in are those whose fundamental form F has a specific behaviour with respect to the operator $\partial\bar{\partial}$. In particular, we center our attention in *strong Kähler with torsion* metrics and *generalized k-th Gauduchon* metrics.

Definition 1.3.11. *A Hermitian structure* (J, g) *is called* strong Kähler with torsion *(*SKT *for short) if* ∂F *is a* $\bar{\partial}$*-closed form.*

SKT metrics are also called *pluriclosed* metrics [41] and characterized by the d-closedness of the torsion 3-form $c = JdF$ of the *Bismut connection* ∇^B. Recall that any Hermitian manifold (M, J, F) admits a unique Hermitian connection ∇^B, called Bismut connection, defined by:

$$g(\nabla^B_X Y, Z) := g(\nabla^{\mathrm{LC}}_X Y, Z) - \frac{1}{2} JdF(X, Y, Z).$$

where ∇^{LC} is the Levi-Civita connection of g. SKT metrics have been studied in type II string theory and in 2-dimensional supersymmetric σ-models (see [41, 89]), and they also have relations with generalized Kähler geometry.

Remark 1.3.12. It follows from the definition that any Kähler metric is SKT, but the converse is not true in general. For example, the *Kodaira-Thurston manifold* or *Kodaira primary surface* $M = G/\Gamma$, given by:

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & -\bar{z}_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}, \quad \Gamma = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & -\bar{z}_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2 \in \mathbb{Z}[i] \right\}, \tag{1.13}$$

admits SKT metrics but it cannot admit any Kähler metric, since the first Betti number of the underlying differentiable manifold is odd.

Alexandrov and Ivanov [5] show that the SKT condition and the balanced condition are complementary on compact complex manifolds:

Proposition 1.3.13 (Alexandrov and Ivanov [5])**.** *Let (M, J) be a compact complex manifold. If g is a J-Hermitian metric which is SKT and balanced, then g is Kähler.*

In addition, Streets and Tian introduced a Hermitian Ricci flow under which the SKT condition is preserved [88]. It is clear from the definition that for complex surfaces every SKT metric is Gauduchon and thus, by Theorem 1.3.10, it is possible to find an SKT metric in the conformal class of any Hermitian metric. This fact is not true in higher dimensions.

A weaker condition than the SKT condition is the notion of *1-st generalized Gauduchon* metric. More generally, Fu, Wang, and Wu [39] introduced the notion of *k-th generalized Gauduchon* metrics on M, which are a generalization of Gauduchon metrics for complex manifolds.

Definition 1.3.14. *A Hermitian metric (J, g) is called* generalized k-th Gauduchon metric *if $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$ for k an integer such that $1 \leq k \leq n-1$.*

It follows from the definition that Gauduchon metrics are $(n-1)$-generalized Gauduchon metrics. In addition, k-th and k'-th-generalized Gauduchon metrics are unrelated for distinct k and k'. As in this work we are concerned with complex geometry on six dimensions, it turns out that between the SKT geometry and the Gauduchon condition only lie the 1-*st Gauduchon metrics*, that is, Hermitian metrics represented by a fundamental form F satisfying that:

$$\partial\bar{\partial}F \wedge F^{n-2} = 0.$$

Extending the result obtained by Gauduchon [40] for standard metrics (see Theorem 1.3.10), Fu, Wang and Wu obtain the following result:

Theorem 1.3.15 (Fu, Wang and Wu [39])**.** *For any compact Hermitian manifold (M, J, F) and for any integer $1 \leq k \leq n-1$, there is a unique constant $\gamma_k(F)$ and a (unique up to a constant) function $v \in \mathcal{C}^\infty(M)$ such that $\frac{i}{2}\partial\bar{\partial}(e^v F^k) \wedge F^{n-k-1} = \gamma_k(F) e^v F^n$.*

For $k = n-1$ one gets the classical Gauduchon metric. In addition, if M is Kähler then $\gamma_k(F) = 0$ and v is a constant function for any $1 \le k \le n-1$. Furthermore, the constant $\gamma_k(F)$ is invariant under biholomorphisms depending smoothly on F, and by [39, Proposition 11], its sign is an invariant of the conformal class of F. Therefore, to compute its sign we can use the following:

Proposition 1.3.16 ([39])**.** *Given a Hermitian manifold (M, J, F), the number $\gamma_k(F)$ is > 0 ($= 0$ or < 0) if and only if there exists a J-Hermitian metric $\tilde{F}$ in the conformal class of F such that*

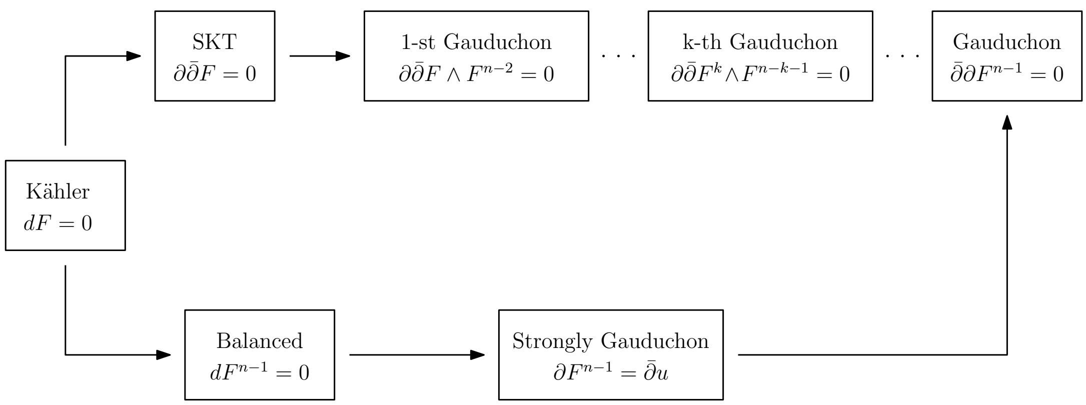
$$\frac{i}{2}\partial\bar{\partial}\tilde{F}^k \wedge \tilde{F}^{n-k-1} > 0\,(= 0,\ or < 0).$$

Recently, Fino and Ugarte [35] extend Proposition 1.3.13 proving that the 1-st generalized Gauduchon and the balanced conditions are complementary for compact complex manifolds of complex dimension greater or equal than 3.

Proposition 1.3.17 (Fino and Ugarte [35, Proposition 2.4])**.** *Let (M, J) be a compact complex manifold of complex dimension $n \geq 3$. If g is a J-Hermitian metric which is 1-st Gauduchon and balanced, then g is Kähler.*

We show in Figure 1.1 the relations between the different special Hermitian metrics presented in this section.

Figure 1.1: Special Hermitian metrics.



1.4 Holomorphic deformations

In this section we present the basics about deformation theory of complex manifolds. In the present work we are mainly concerned with the deformation theory of compact complex manifolds developed by K. Kodaira, D. C. Spencer, L. Nirenberg [53], [54] and

M. Kuranishi [55]. This theory provides a tool to obtain new examples of compact complex manifolds. Furthermore, given a compact complex manifold M, the study of *all* small deformations provides a tool to understand the local geometry of the moduli space of complex structures on M. For the first part of the section, we follow the texts of Angella [6] and Huybrechts [49] and for the second part the survey by Popovici [72], concerning open and closed properties.

1.4.1 The theory of small deformations

Let M be a compact smooth manifold endowed with a complex structure J. Basically, a deformation of J can be understood as a family of complex structures $\{J_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ on M parametrized by a connected complex analytic space $\mathcal{B}$ with a base point $\mathbf{t_0} \in \mathcal{B}$ such that $J = J_{\mathbf{t_0}}$. We adopt the following terminology:

Definition 1.4.1. *Let $\mathcal{B}$ be a connected complex analytic space. A family $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ of compact complex manifolds is said to be an* holomorphic family of compact complex manifolds *if there exists a complex manifold $\mathcal{M}$ and a surjective holomorphic map $\pi\colon \mathcal{M} \to \mathcal{B}$ such that $\pi^{-1}(\mathbf{t}) = M_{\mathbf{t}}$ for every $\mathbf{t} \in \mathcal{B}$ and π is a proper holomorphic submersion.*

It is even possible to speak about *differentiable* families of compact complex manifolds $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$. Thus, it is natural to ask whether two manifolds $M_{\mathbf{t}}$ and $M_{\mathbf{t}'}$ with $\mathbf{t} \neq \mathbf{t}' \in \mathcal{B}$ are diffeomorphic or not. The following theorem due to Ehresmann [28] provides a positive answer to this question only requiring the differentiability of the family.

Theorem 1.4.2 (Ehresmann [28])**.** *Let $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ be a differentiable family of compact complex manifolds. For any $\mathbf{s}, \mathbf{t} \in \mathcal{B}$ the manifolds $M_{\mathbf{s}}$ and $M_{\mathbf{t}}$ are diffeomorphic.*

As a consequence, if $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$ is a holomorphic family of compact complex manifolds, then $M_{\mathbf{t}}$ is diffeomorphic to $M_{\mathbf{t}'}$. Therefore, only the complex structure $J_{\mathbf{t}}$ varies with $\mathbf{t} \in \mathcal{B}$, so the fibres $M_{\mathbf{t}}$ can be identified for any $\mathbf{t} \in \mathcal{B}$ with a fixed smooth manifold M (the one underlying $M_{\mathbf{t_0}}$) endowed with a holomorphic family of complex structures $\{J_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{B}}$.

Consider now another base space $\mathcal{B}'$ with a distinguished point $\mathbf{s_0} \in \mathcal{B}'$. If $f : \mathcal{B}' \to \mathcal{B}$ is a holomorphic map with $f(\mathbf{s_0}) = \mathbf{t_0}$, then the pull-back $\mathcal{M}' = f^*\mathcal{M} := \mathcal{M} \times_{\mathcal{B}} \mathcal{B}'$ gives a complex analytic family of deformations on M. A deformation $\pi\colon \mathcal{M} \to \mathcal{B}$ of the compact complex manifold M is called *complete* if any other deformation $\pi'\colon \mathcal{M}' \to \mathcal{B}'$ of M is obtained by a pull-back under some $f\colon \mathcal{B}' \to \mathcal{B}$. If f is unique, then $\pi\colon \mathcal{M} \to \mathcal{B}$ is called *universal.* If only the differential $df : T_{\mathbf{s_0}}\mathcal{B}' \to T_{\mathbf{t_0}}\mathcal{B}$ is unique, then the deformation is called *semi universal* or *versal.* Taking these definitions into account, Kuranishi proves one of the most important results of this theory:

Theorem 1.4.3 (Kuranishi [55, Theorem 2])**.** *Let M be a compact complex manifold. Then M admits a versal holomorphic family of deformations.*

The previous theorem guarantees the existence of a locally complete space of deformations which is called the *Kuranishi space* of M, and denoted by $\mathrm{Kur}(M)$. It parametrizes all sufficiently small deformations, but not uniquely. The Kuranishi space is written in terms of an open set of the cohomology group $H^1(M;\Theta_M)$, where Θ_M is the sheaf of holomorphic vector fields on M.

Remark 1.4.4. From now on, in order to simplify the presentation, we will consider that the base space is a polydisc $\Delta := \{\mathbf{t} \in \mathbb{C}^N \mid ||\mathbf{t}|| < \epsilon\} \subset \mathbb{C}^N$, for some $N \in \mathbb{N}$ with $\epsilon > 0$ sufficiently small, and the base point is $\mathbf{t_0} = \mathbf{0} \in \mathbb{C}^N$.

We sketch the basic tools to compute the Kuranishi space of a compact complex manifold. Recall that J, considered only as an almost-complex structure, is uniquely determined by the $-i$ eigenspace $T_J^{(0,1)}M \subset TM_{\mathbb{C}}$. Therefore, a family $\{J_\mathbf{t}\}_{\mathbf{t}\in\Delta}$ of complex structures on M such that $J_\mathbf{0} = J$ can also be viewed as a family of $-i$ eigenspaces $\{T_{J_\mathbf{t}}^{(0,1)}M \subset TM_{\mathbb{C}}\}_{\mathbf{t}\in\Delta}$. For a small $\mathbf{t} \in \Delta$ it turns out that the deformation is encoded by a map

$$\Psi(\mathbf{t})\colon \mathfrak{X}_J^{0,1}(M) \to \mathfrak{X}_J^{1,0}(M),$$

satisfying $\Psi(\mathbf{0}) = 0$ and $X + \Psi(\mathbf{t})X \in \mathfrak{X}_{J_\mathbf{t}}^{1,0}(M)$ for all $X \in \mathfrak{X}_J^{0,1}(M)$. Hence, the deformation $J_\mathbf{t}$ is associated to a section $\Psi(\mathbf{t}) \in \mathfrak{X}_J^{1,0}(M) \otimes \wedge_J^{0,1}M$.

Conversely, given an almost-complex structure J and a section $\Psi \in \mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,1}M$, we construct an almost-complex structure $\tilde{J}$ on M with $-i$ eigenspace $T_{\tilde{J}}^{(0,1)}M := (\mathrm{Id}_{T_{\mathbb{C}}M} + \Psi)T_J^{(0,1)}M$. The following proposition states the integrability of a deformation $J_\mathbf{t}$ in terms of a partial differential equation involving the operator $\Psi(\mathbf{t})$:

Proposition 1.4.5. *Let $\{J_\mathbf{t}\}_{\mathbf{t}\in\Delta}$ be a family of almost-complex structures on M. The integrability of $J_\mathbf{t}$ is equivalent to the* Maurer-Cartan equation*:*

$$\bar\partial\Psi(\mathbf{t}) + [\Psi(\mathbf{t}), \Psi(\mathbf{t})] = 0, \tag{1.14}$$

where, $[\cdot,\cdot]\colon \left(\mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,p}M\right)\times\left(\mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,q}M\right) \to \mathfrak{X}_J^{1,0}\otimes\wedge_J^{0,p+q}M$ *is the* Nijenhuis-Schouten bracket *defined by*

$$[X\otimes\alpha, Y\otimes\beta] := X\otimes(\beta\wedge\mathcal{L}_Y\alpha) + Y\otimes(\alpha\wedge\mathcal{L}_X\beta) + [X,Y]\otimes(\alpha\wedge\beta), \tag{1.15}$$

where, $\mathcal{L}_Y\alpha$ is the Lie derivative of α along Y and $\bar\partial\colon \mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,p}M \to \mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,p+1}M$ is the differential operator defined inductively by:

$$\bar\partial(X\otimes\alpha) := \bar\partial(X)\wedge\alpha + (-1)^k X\otimes\bar\partial\alpha, \tag{1.16}$$

whith $\bar\partial(X) := \pi_{1,0}\circ ad_X \mid_{\mathfrak{X}_J^{0,1}(M)}$.

Let Θ_M be the sheaf of holomorphic vector fields on M. As the operator $\bar\partial$ defined by (1.16) satisfies $\bar\partial^2 = 0$, the *sheaf cohomology groups* $H^\bullet(M;\Theta_M)$ of M are the cohomology groups associated to the differential complex:

$$0 \to \mathfrak{X}_J^{1,0}(M) \xrightarrow{\bar\partial} \mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,1}M \xrightarrow{\bar\partial} \mathfrak{X}_J^{1,0}(M)\otimes\wedge_J^{0,2}M \xrightarrow{\bar\partial} \dots$$

Now, let $\{\Psi(\mathbf{t})\}_{\mathbf{t}\in\Delta} \subset \mathfrak{X}^{1,0}_J(M) \otimes \wedge^{0,1}_J M$ be a family of sections with $\Psi(\mathbf{0}) = 0$. Consider the formal power series expansion in $\mathbf{t}$ of $\Psi(\mathbf{t})$:

$$\Psi(\mathbf{t}) = \sum_{k\geq 1} \Psi_k(\mathbf{t}), \tag{1.17}$$

where $\{\Psi_k(\mathbf{t})\}_{k\geq 1} \subset \mathfrak{X}^{1,0}_J(M) \otimes \wedge^{0,1}_J M$ and $\Psi_k(\mathbf{t})$ is homogeneous of degree k in $\mathbf{t}$. Thus, the Maurer-Cartan equation (1.14) yields to the inductive system of equations:

$$\begin{cases} \bar\partial \Psi_1(\mathbf{t}) = 0, \\ \bar\partial \Psi_2(\mathbf{t}) = -[\Psi_1(\mathbf{t}), \Psi_1(\mathbf{t})], \\ \vdots \\ \bar\partial \Psi_{k+1}(\mathbf{t}) = -\displaystyle\sum_{1\leq j\leq k} [\Psi_j(\mathbf{t}), \Psi_{k+1-j}(\mathbf{t})]. \end{cases} \tag{1.18}$$

The first equation of (1.18) reveals that the first-order deformation Ψ_1 defines a cohomology class of the group $H^{0,1}(M;\Theta_M)$ called the *Kodaira-Spencer class* of the deformation. Recalling the Hodge decomposition of the bundles of (p,q)-forms for compact Hermitian manifolds (1.6) we have the following splitting:

$$\mathfrak{X}^{1,0}_J(M) \otimes \wedge^{0,1}_J M = (\mathfrak{X}^{1,0}_J(M) \otimes \ker \overline{\Box}) \oplus (\mathfrak{X}^{1,0}_J(M) \otimes \bar\partial \wedge^{0,0}_J M) \oplus (\mathfrak{X}^{1,0}_J(M) \otimes \bar\partial^* \wedge^{0,2}_J M)$$

together with the projections $H_{\bar\partial}\colon \mathfrak{X}^{1,0}_J(M) \otimes \wedge^{0,1}_J M \to \mathfrak{X}^{1,0}_J(M) \otimes \ker \overline{\Box}$ and $P_{\bar\partial}\colon \mathfrak{X}^{1,0}_J(M) \otimes \wedge^{0,1}_J M \to \mathfrak{X}^{1,0}_J(M) \otimes \bar\partial \wedge^{0,0}_J M$. In order that $\Psi(\mathbf{t})$ satisfies (1.14) we must have:

$$\bar\partial \Psi_{k+1}(\mathbf{t}) = -P_{\bar\partial}\left(\sum_{1\leq j\leq k} [\Psi_j(\mathbf{t}), \Psi_{k+1-j}(\mathbf{t})]\right).$$

Thus, one gets $\bar\partial\Psi(\mathbf{t}) + [\Psi(\mathbf{t}), \Psi(\mathbf{t})] = H_{\bar\partial}([\Psi(\mathbf{t}), \Psi(\mathbf{t})])$.

Now, we have the tools to construct the Kuranishi space of M. We can define the *obstruction map* as follows. Consider $\{X_j \otimes \bar\omega^k\}$ a basis of $H^{0,1}(M;\Theta_M)$. Given $\mu = \sum_{j,k} t_{jk} X_j \otimes \bar\omega^k$. Define the formal power series $\Psi(\mathbf{t}) := \sum_{k\geq 1} \Psi_k(\mathbf{t})$ inductively:

$$\begin{cases} \Psi_1(\mathbf{t}) := \mu, \\ \bar\partial \Psi_{k+1}(\mathbf{t}) = -P_{\bar\partial}\left(\displaystyle\sum_{1\leq j\leq k} [\Psi_j(\mathbf{t}), \Psi_{k+1-j}(\mathbf{t})]\right). \end{cases} \tag{1.19}$$

Define the *obstruction map* $\mathrm{obs}(\mu)\colon H^1(M;\Theta_M) \to H^2(M;\Theta_M)$ by:

$$\mathrm{obs}(\mu) := H_{\bar\partial}([\Psi(\mathbf{t}), \Psi(\mathbf{t})]).$$

Therefore, given $\mu \in H^1(M;\Theta_M)$, an element $\Psi(\mathbf{t})$ contructed with (1.19) defines a small deformation if it satisfies the Maurer-Cartan equation (1.14), that is, if $\mathrm{obs}(\mu) = 0$. Hence, the versal family of Kuranishis's Theorem 1.4.3 is constituted by the *Kuranishi space* defined by:

$$\mathrm{Kur}(M) := \left\{\mu \in H^1(M;\Theta_M) \mid ||\mu|| \ll 1,\ \mathrm{obs}(\mu) = 0\right\},$$

which is an open set of the sheaf cohomology group $H^1(M;\Theta_M)$. It is worth noticing that the Kuranishi space describes the local moduli of complex structures of a given complex manifold and that it can be arbitrarily singular. One of the simplest cases is when M is a compact Calabi-Yau manifold. In that case, Tian [93, Theorem 1] and Todorov [94, Theorem 1] found independently that the local moduli space of deformations of the complex structure of M is again a complex manifold of dimension $h^{n-1,1}_{\bar{\partial}}(M)$, where $n = \dim_{\mathbb{C}} M$.

1.4.2 Open and closed properties

Concerning the study of deformations of complex structures it is a natural question whether a property $\mathcal{P}$ related to the complex structure of the manifold is preserved under any family of holomorphic deformations. This yields to the concept of *stability*, or *openness*, of properties under holomorphic deformations of the complex structure.

Definition 1.4.6. *A given property $\mathcal{P}$ of a compact complex manifold is* open *under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ and for every $\mathbf{t_0} \in \Delta$ the following implication holds:*

if $M_{\mathbf{t_0}}$ has property $\mathcal{P}$, then $M_{\mathbf{t}}$ has property $\mathcal{P}$ for all $\mathbf{t} \in \Delta$ sufficiently close to $\mathbf{t_0}$.

On the other hand, the behaviour of a property related to the complex structure of a complex manifold can be studied under other point of view. It may also be of interest to consider families of holomorphic deformations satisfying certain properties and to ask if the central limit necessarily satisfies the same property. This gives rise to the notion of *closedness* of complex properties under holomorphic deformations:

Definition 1.4.7. *A given property $\mathcal{P}$ of a compact complex manifold is* closed *under holomorphic deformations if for every holomorphic family of compact complex manifolds $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ and for every $\mathbf{t_0} \in \Delta$ the following implication holds:*

if $M_{\mathbf{t}}$ has property $\mathcal{P}$ for all $\mathbf{t} \in \Delta\backslash\{\mathbf{t_0}\}$, then $M_{\mathbf{t_0}}$ has property $\mathcal{P}$.

It is worth noticing that openness and closedness are not opposite, and hence, one property $\mathcal{P}$ can be at the same time open and closed. Furthermore, if a property $\mathcal{P}$ is both open and closed, then given a holomorphic family $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$, all the fibres $M_{\mathbf{t}}$ satisfy $\mathcal{P}$ whenever one of them satisfies $\mathcal{P}$.

Now, we shall summarize the openness and closedness of some of the most important properties. The interest in deformation theory started with the following result concerning the stability of the Kähler property:

Theorem 1.4.8 (Kodaira and Spencer [53, Theorem 15]). *The* Kähler *property of compact complex manifolds is* open *under holomorphic deformations.*

We mentioned before that Kähler metrics are called Calabi-Yau when the complex structure has holomorphically trivial canonical bundle. Tian and Todorov independently found the following result concerning the stability of the Calabi-Yau property:

Theorem 1.4.9 (Tian [93, Theorem 1], Todorov [94, Theorem 1]). *The* Calabi-Yau *property of compact complex manifolds is* open *under holomorphic deformations.*

A complex surface is Kähler if and only if its first Betti number is even [52, 66, 87]. Therefore, the Kähler property is a topological property for complex surfaces. Recalling that the fibres of any holomorphic deformation are diffeomorphic to the central limit, it turns out that the Kähler property is closed for complex surfaces. Therefore, for any holomorphic family of compact complex surfaces, if some fibre is Kähler, then all the fibres are Kähler. However, Hironaka shows that the situation changes for higher dimensions.

Theorem 1.4.10 (Hironaka [47]). *The* Kähler *property of compact complex manifolds of complex dimension* ≥ 3 *is* not closed *under holomorphic deformations.*

A compact complex manifold is said to be *projective* if it admits a closed holomorphic embedding in some projective space $\mathbb{CP}^N$. Projective compact manifolds are a special class of compact Kähler manifolds, and they are characterized by the well-known Kodaira's Embedding Theorem in terms of the *integrality* of the de Rham class $[F] \in H^2_{\mathrm{dR}}(M)$ defined by the fundamental form F.

Recall that a proper bimeromorphic map $F\colon M \to M'$ between two complex manifolds M and M' is called a *modification.* It is said that a compact manifold is *class $\mathcal{C}$ of Fujiki* (respectively, *Moishezon*) if there exists a modification to a compact Kähler (respectively projective) manifold. It turns out that both class $\mathcal{C}$ of Fujiki and Moishezon manifolds satisfy the $\partial\bar{\partial}$-lemma [26, Lemma 5.11,Corollary 5.23]. Hironaka [47] obtains an example of a Moishezon manifold with a holomorphic family of deformations such that its fibres are projective. More recently, Popovici proves the following theorem :

Theorem 1.4.11 (Popovici [72, Theorem 1.1]). *Let $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ be a holomorphic family of compact complex manifolds. If the fibre $M_{\mathbf{t}}$ is projective for every $\mathbf{t} \in \Delta^*$, then $M_{\mathbf{0}}$ is Moishezon.*

The stability of the *class $\mathcal{C}$ of Fujiki* property is stated in the following theorem:

Theorem 1.4.12 (Campana [17], LeBrun and Poon [58]). *The* class $\mathcal{C}$ of Fujiki *property of compact complex manifolds is* not open *under holomorphic deformations.*

However, although the closedness of this property is unknown, Popovici states the following conjecture:

Conjecture 1.4.13 (Standard Conjecture [76, Conjecture 1.19]). *The* class $\mathcal{C}$ *property of compact complex manifolds is* closed *under holomorphic deformations.*

Alessandrini and Bassanelli [3] assert that every class $\mathcal{C}$ of Fujiki manifold admits a balanced metric. Therefore, a way to deal with the former Standard Conjecture is to consider the behaviour of the balanced property under holomorphic deformations. Recall that the *Iwasawa manifold* is the complex parallelizable compact manifold $M = H_3(\mathbb{C})/\Gamma$ given by:
(1.20)

$$H_3(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}, \quad \Gamma = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{Z}[i] \right\},$$

where $H_3(\mathbb{C})$ is the *complex Heisenberg group.* The non-stability of the balanced property is proved by Alessandrini and Bassanelli taking a suitable direction in the Kuranishi space of the Iwasawa manifold computed by Nakamura [67].

Theorem 1.4.14 (Alessandrini and Bassanelli [2])**.** *The* balanced *property of compact complex manifolds is* not open *under holomorphic deformations.*

Therefore, one could approach the Standard Conjecture proving firstly the following:

Conjecture 1.4.15 ([76, Conjecture 1.23])**.** *The* balanced *property of compact complex manifolds is* closed *under holomorphic deformations.*

We show in Chapter 5 a counterexample to this Conjecture. The strongly Gauduchon property presents a different stability behaviour with respect to the balanced property, as the following result due to Popovici shows.

Theorem 1.4.16 (Popovici [73])**.** *The* strongly Gauduchon *property of compact complex manifolds is* open *under holomorphic deformations.*

Popovici also shows [75, Theorem 1.3] that the strongly Gauduchon property is preserved under modification. Hence, it could be an open and closed property under holomorphic deformations and stable under modification. This gives rise to the following conjecture proposed by Demailly:

Conjecture 1.4.17 ([76, Conjecture 1.21])**.** *The* strongly Gauduchon *property of compact complex manifolds is* closed *under holomorphic deformations.*

We show in Section 5.2 a counterexample to this Conjecture. Finally, the following theorem asserts the non-openness of the SKT property:

Theorem 1.4.18 (Fino and Tomassini [34])**.** *The* SKT *property of compact complex manifolds is* not open *under holomorphic deformations.*

Now, we consider some properties related with the topology of the manifold such as the degeneration of the Frölicher sequence at the first step and the $\partial\bar{\partial}$-lemma. The stability of the degeneration of the Frölicher sequence at the first step was proven by

Kodaira and Spencer in [53]. Recalling that the fibres $M_{\mathbf{t}} := (M, J_{\mathbf{t}})$ are diffeomorphic to the central limit $M_{\mathbf{0}} := (M, J_{\mathbf{0}})$, the stability follows from the Frölicher inequality (1.12) and from the upper semicontinuity of the Hodge numbers $h^{\bullet,\bullet}_{\bar\partial}(M_{\mathbf{t}})$, that is, $h^{\bullet,\bullet}_{\bar\partial}(M_{\mathbf{0}}) \geq h^{\bullet,\bullet}_{\bar\partial}(M_{\mathbf{t}})$.

$$b_\bullet(M) = \sum_{p+q=\bullet} h^{p,q}_{\bar\partial}(M_{\mathbf{0}}) \geq \sum_{p+q=\bullet} h^{p,q}_{\bar\partial}(M_{\mathbf{t}}) \geq b_\bullet(M).$$

Thus, $\sum_{p+q=\bullet} h^{p,q}_{\bar\partial}(M_{\mathbf{t}}) = b_\bullet(M)$ and $E_1(M) \cong E_\infty(M)$. On the other hand, Eastwood and Singer prove the non-closedness of the property using the theory of twistor spaces.

Theorem 1.4.19 (Eastwood and Singer [27])**.** *For compact complex manifolds the property of the* Frölicher spectral sequence degenerating at E_1 *is* not closed *under holomorphic deformations.*

Finally, the behaviour of the $\partial\bar\partial$-lemma property under holomorphic deformations comes described by the following theorems. The stability is proven by Voisin:

Theorem 1.4.20 (Voisin [98, Proposition 9.21], Wu [101], Angella and Tomassini [10, Corollary 3.7])**.** *The* $\partial\bar\partial$-lemma *property of compact complex manifolds is* open *under holomorphic deformations.*

Indeed, if $\{M_{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ is a holomorphic family of compact complex manifolds such that the central limit $M_{\mathbf{0}}$ satisfies the $\partial\bar\partial$-lemma, then the stability of this property follows from Theorem 1.2.8 and recalling that the dimensions of the Bott-Chern and Aeppli cohomologies are upper semi continuous [85, Lemme 3.2]:

$$2b_\bullet(M) = \sum_{p+q=\bullet} (h^{p,q}_{BC}(M_{\mathbf{0}}) + h^{p,q}_{A}(M_{\mathbf{0}})) \geq \sum_{p+q=\bullet} (h^{p,q}_{BC}(M_{\mathbf{t}}) + h^{p,q}_{A}(M_{\mathbf{t}})) \geq 2b_\bullet(M).$$

Hence, $2b_\bullet(M) = \sum_{p+q=\bullet}(h^{p,q}_{BC}(M_{\mathbf{t}}) + h^{p,q}_{A}(M_{\mathbf{t}}))$. Concerning the closedness of this property, Angella and Kasuya recently prove the non-closedness of this property.

Theorem 1.4.21 (Angella and Kasuya [8, Corollary 6.1])**.** *The* $\partial\bar\partial$-lemma *property of compact complex manifolds is* not closed *under holomorphic deformations.*

They found that the $\partial\bar\partial$-lemma property is not closed by means of a holomorphic deformation of the so-called *Nakamura manifold*, which is the compact complex manifold $M = G/\Gamma$, where $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^2$ and

$$\varphi\colon \mathbb{C} \to \mathrm{Aut}(\mathbb{C}^2), \quad \varphi(z_1) := \begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix}. \tag{1.21}$$

The subgroup Γ is again the semi-direct product of two subgroups Γ', Γ'' of $\mathbb{C}$ and $\mathbb{C}^2$, respectively. The Kuranishi space of the Nakamura manifold was studied by Nakamura [67] and Angella and Kasuya took a suitable deformation to prove Theorem 1.4.21.

The examples found by Alessandrini and Bassanelli to prove the non-openness of the balanced property, and by Angella and Kasuya to prove the non-closedness of the $\partial\bar{\partial}$-lemma reveal an interesting and rich source of compact manifolds constructed by taking the quotient of a Lie group G by a discrete subgroup $\Gamma \leq G$ called *uniform* so that $M = G/\Gamma$ is compact. In addition, they are endowed with an *invariant* complex structure J, that is, its description can be basically referred to the Lie algebra $\mathfrak{g}$ of the Lie group.

The underlying real Lie group to the Iwasawa manifold is an example of a six-dimensional *nilpotent* Lie group, whereas the underlying real Lie group to the Nakamura manifold is a six-dimensional *solvable* Lie group. In addition, both are endowed with an invariant complex structure such that the holomorphic canonical bundle is trivial, that is, there is a non-vanishing global holomorphic volume form defined on the manifold. These considerations make interesting to study the class of six-dimensional *solvable* Lie algebras admitting an integrable complex structure that gives rise to solvmanifolds with holomorphically trivial canonical bundle. Therefore, our aim in the following chapter will be to achieve a classification of this complex geometry on solvmanifolds, in order to study the behaviour of some cohomological invariants and the existence of special Hermitian metrics. As we mentioned before, this study will lead us to find, among other results, several counterexamples to Conjectures 1.4.15 and 1.4.17, constructing a suitable deformation of an invariant complex structure on a solvmanifold.

Chapter 2
Solvmanifolds and invariant complex geometry

The goal of this chapter is to classify the six-dimensional solvmanifolds admitting an invariant complex structure with holomorphically trivial canonical bundle. Section 2.1 deals with the notions and results concerning solvmanifolds $M = G/\Gamma$ endowed with an invariant complex structure J. The results due to Nomizu [68], Hattori [46] and Mostow [64], for computing the de Rham cohomology of M by means of the Chevalley-Eilenberg cohomology of the Lie algebra underlying M are recalled. As regards the complex geometry of nilmanifolds, Salamon [82] shows that nilmanifolds endowed with an invariant complex structure have holomorphically trivial canonical bundle. We remind partial classifications of solvable Lie algebras admitting complex structures up to dimension six [4, 44, 82, 71]. We complete this section reminding, when M is a nilmanifold, several cases for which the natural map $(\wedge^{\bullet,\bullet}\mathfrak{g}^*, \bar\partial) \to (\wedge^{\bullet,\bullet}M, \bar\partial)$ induces an isomorphism in Dolbeault cohomology [20, 24, 81]. The symmetrization process introduced by Belgun [11] allows us to prove that an invariant complex structure on a solvmanifold M has holomorphically trivial canonical bundle if and only if it admits an invariant non-zero holomorphic volume form. As a consequence, our initial classification problem reduces to classifying the six-dimensional solvable Lie algebras $\mathfrak{g}$ admitting a pair (J, Ψ), where J is an almost-complex structure and Ψ is a non-zero closed form of pure type (3,0) with respect to J, and such that the corresponding connected and simply-connected Lie groups admit a lattice. We conclude the section providing a compact example, based on an invariant holomorphic deformation, of the non-stability of the holomorphically trivial canonical bundle property.

The problem of classifying six-dimensional solvable Lie algebras admitting complex structures with a closed $(3,0)$-form is faced in Section 2.2. For this end, we make use of the formalism of stable forms developed by Hitchin [48], together with ideas in [19, 36, 37, 84]. We adapt this formalism to our problem allowing us to compute the space of almost-complex structures on a six-dimensional Lie algebra $\mathfrak{g}$ by means of a subset of the space $Z^3(\mathfrak{g})$ of closed 3-forms on $\mathfrak{g}$. Hence, we look for the 3-forms $\rho \in Z^3(\mathfrak{g})$ such that the corresponding almost-complex structure J^*_ρ satisfies that $J^*_\rho\rho$ is also closed. In that case, the pair (J, Ψ) with $J := J^*_\rho$ and $\Psi := \rho + iJ^*_\rho\rho$ yields an invariant complex structure with holomorphically trivial canonical bundle on the solvmanifold, whenever the corresponding Lie group admits a lattice. As a consequence, if a solvmanifold admits a complex structure arising from an invariant non-vanishing holomorphic (3,0)-form, then its underlying Lie algebra $\mathfrak{g}$ must be isomorphic to one in the list of Theorem 2.2.14, i.e. $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha}$ $(\alpha \geq 0), \mathfrak{g}_3, \ldots, \mathfrak{g}_8$ or $\mathfrak{g}_9$. The Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha}$ $(\alpha \geq 0)$ and $\mathfrak{g}_3$ are decomposable, whereas $\mathfrak{g}_4, \ldots, \mathfrak{g}_8$ and $\mathfrak{g}_9$ are indecomposable. The Lie algebra $\mathfrak{g}_8$ is precisely the real Lie algebra underlying the Nakamura manifold [67].

Section 2.3 deals with the existence of lattices on solvable Lie groups. We recall the well-known theorem of Malcev [61] for the existence of lattices in nilpotent Lie groups and some previous results following Bock [13] that provide the tools to obtain explicitly lattices for almost-nilpotent Lie groups. Finally, we prove the existence of a lattice for the simply-connected Lie groups associated to the Lie algebras in the list above, although for $\mathfrak{g}_2^{\alpha>0}$ we are able to find a lattice only for a countable number of different values of α, which is consistent with a result by Witte [100].

2.1 Invariant complex structures

In this section we present an important class of compact manifolds $M = G/\Gamma$ arising from taking a quotient of a Lie group G by a discrete and cocompact subgroup $\Gamma \leq G$, as well as the invariant complex structures defined on them. This construction provides a source of interesting examples of compact manifolds in the study of, not only complex, but different geometric structures such as Riemannian or symplectic structures, among others.

2.1.1 Compact homogeneous spaces

Let G be a connected and simply-connected Lie group with $\dim_{\mathbb{R}} G = n$, a subgroup $\Gamma \leq G$ is called *discrete* if the topology of Γ induced by the Lie group G is discrete. If the subgroup $\Gamma \leq G$ is discrete then the space $M = G/\Gamma = \{g\Gamma \mid g \in G\}$ admits a $\mathcal{C}^{\infty}$-atlas such that the surjective map $\pi\colon G \to M = G/\Gamma$ defined by $\pi(g) := [g] = g\Gamma$ is smooth. Hence G is the universal covering of M. In addition, the manifolds of the form $M = G/\Gamma$ are *homogeneous spaces*, that is, differentiable manifolds endowed with a transitive action of a Lie group. Notice that the dimension of M is the dimension of the Lie group G considered as a differentiable manifold. The subgroup $\Gamma \leq G$ is called *cocompact* if the resulting quotient manifold $M = G/\Gamma$ is compact (that is, there is a compact set $K \subset G$, also called a *fundamental domain* of Γ, such that $K\Gamma = G$).

Definition 2.1.1. *A subgroup $\Gamma \leq G$ is said to be* uniform *if it is discrete and cocompact.*

Remark 2.1.2. Uniform subgroups of a given Lie group G are closely related to *lattices*. A discrete subgroup Γ of a locally compact Lie group G is a lattice subgroup, or simply a lattice, if $\mu(G/\Gamma) < \infty$, where μ is the left-invariant Haar measure on G. If G is *unimodular* (for a definition of unimodular Lie group see Section 2.1.4) and $\Gamma \subset G$ is uniform then Γ is a lattice. However, the converse is not true in general. A classic example is the Lie group $G = \mathrm{SL}(2, \mathbb{R})$ and the subgroup $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ because Γ has a non compact fundamental domain, so G/Γ is not compact, although the volume $\mu(G/\Gamma)$ is finite. Hence Γ is a lattice but not a uniform subgroup. This fact does not occur in connected and solvable Lie groups, see [77, Theorem 3.1]. From now on, when we deal with uniform subgroups of solvable Lie groups we refer to them as lattices.

Any $g \in G$ defines a map $L_g : G \to G$ called *left translation* by $L_g(h) := gh$. Furthermore, L_g is smooth for any $g \in G$ and it is a diffeomorphism. We say that a vector field $X \in \mathfrak{X}(G)$ is *left-invariant* if it is invariant under all left translations, namely, $(L_g)_* X_h = X_{gh}$ for all $g, h \in G$. It is direct to check that if $X, Y \in \mathfrak{X}(G)$ are left-invariant vector fields then the vector field $[X, Y] \in \mathfrak{X}(G)$ is left-invariant. Hence, the *Lie algebra associated to* G is defined to be the Lie algebra of left-invariant vector fields of G. We denote it by $\mathfrak{g} := \{X \in \mathfrak{X}(G) \mid (L_g)_* X = X, \forall g \in G\}$. It is proved that $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of $\mathfrak{X}(G)$ and $\dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{R}} G$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathfrak{g}$, then the Lie brackets are expressed by:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k, \quad 1 \le i, j \le n,$$

where $\{c_{ij}^k\} \subset \mathbb{R}$ are the *structure constants* of G with respect to the basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$. We denote by $\mathfrak{g}^*$ the vector space of left-invariant 1-forms on M. The *Eilenberg-Chevalley* complex $(\wedge^\bullet \mathfrak{g}^*, d)$ is a differential graded algebra, where $\wedge^\bullet \mathfrak{g}^* = \oplus_{k=0}^n \wedge^k \mathfrak{g}^*$ is the finite dimensional exterior algebra of $\mathfrak{g}^*$ with respect to the wedge product together with the differential operator $d \colon \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1} \mathfrak{g}^*$ defined by $d\alpha(X, Y) := -\alpha([X, Y])$ for $X, Y \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. Hence, if $\{e^1, \ldots, e^n\}$ is the basis of $\mathfrak{g}^*$ dual to the basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ we get

$$de^k = -\sum_{k=1}^{n} c_{ij}^k e^i \wedge e^j. \tag{2.1}$$

The previous expression is usually known as the *structure equations* of the Lie algebra $\mathfrak{g}$.

Remark 2.1.3. In the practice, we usually refer to the Lie algebras providing the structure equations (2.1) of $\mathfrak{g}$. For instance, the notation $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$ means that there is a basis $\{e^j\}_{j=1}^6$ of $\mathfrak{h}_2^*$ satisfying the structure equations $de^1 = de^2 = de^3 = de^4 = 0$, $de^5 = e^1 \wedge e^2$, $de^6 = e^3 \wedge e^4$; equivalently, the Lie bracket is given in terms of its dual basis $\{e_j\}_{j=1}^6$ by $[e_1, e_2] = -e_5$, $[e_3, e_4] = -e_6$. From now on we also shorten $e^{i_1 \ldots i_k} := e^{i_1} \wedge \ldots \wedge e^{i_k}$ for any set of indices $1 \le i_1 < \ldots < i_k \le n$.

The map $\pi \colon G \to M = G/\Gamma$ enables to define *invariant* vector fields on M. If $X \in \mathfrak{g}$ is a left-invariant vector field on the Lie group G, then the vector field $\pi_* X$ given by $(\pi_* X)_{[g]} := \pi_*(X_g)$ is well defined and hence $\pi_* X \in \mathfrak{X}(M)$. We say that $\tilde{X} \in \mathfrak{X}(M)$ is an *invariant* vector field on M if $\tilde{X} = \pi_* X$ for some $X \in \mathfrak{g}$. The map π is also a local diffeomorphism and as a consequence if we have a left-invariant parallelization $\{X_1, \ldots, X_n\} \subset \mathfrak{g}$ of G then $\{\pi_* X_1, \ldots, \pi_* X_n\} \subset \mathfrak{X}(M)$ is a parallelization of M. Indeed it is immediate to check that the structure equations expressed in the basis

$\{\pi_* X_1, \ldots, \pi_* X_n\}$ on M have the same form that the structure equations expressed in the basis $\{X_1, \ldots, X_n\}$ on G.

The definition of invariant vector fields on M allows us to define invariant k-forms. It is said that $\alpha \in \mathcal{T}^k(M)$ is an invariant k-form on M if $\alpha(X_1, \ldots, X_k) \in \mathbb{R}$ for any $X_1, \ldots, X_k \in \mathfrak{g}$. As a consequence, we can speak in general of invariant tensor fields on the manifold M. Moreover, if $\{X_1, \ldots, X_n\}$ and $\{\alpha^1, \ldots, \alpha^n\}$ are global parallelizations of invariant vector fields and 1-forms on M, then any invariant tensor field $T \in \mathcal{T}_l^k(M)$ admits the following expression:

$$T = \sum_{j_1,\ldots,j_k=1}^{n} \sum_{i_1,\ldots,i_l=1}^{n} T_{j_1\ldots j_k}^{i_1\ldots i_l} X_{i_1} \otimes \ldots \otimes X_{i_l} \otimes \alpha^{j_1} \otimes \ldots \otimes \alpha^{j_k}, \tag{2.2}$$

where $T_{j_1\ldots j_k}^{i_1\ldots i_l} \in \mathbb{R}$. As we saw, some important geometric structures on manifolds are defined by one or several tensor fields such as complex structures, Riemannian metrics or symplectic structures. Thus, in the case of manifolds of type $M = G/\Gamma$, it is possible to speak about invariant complex structures, Riemannian metrics or symplectic structures on M, among others.

Remark 2.1.4. As a matter of notation, when $M = G/\Gamma$, we denote by $\mathfrak{g}$ the space of invariant vector fields, $T_l^k(\mathfrak{g})$ the space of invariant (k,l)-tensors, $\mathrm{End}(\mathfrak{g})$ the space of invariant endomorphisms and $\wedge^k \mathfrak{g}^*$ the space of invariant k-forms on M.

Let $M = G/\Gamma$ be a compact manifold and $\{e^1, \ldots, e^n\}$ be a basis of invariant 1-forms on M. The *Chevalley-Eilenberg* cohomology of M is defined as the cohomology of the Chevalley-Eilenberg complex $(\wedge^\bullet \mathfrak{g}^*, d)$. We denote by $b_\bullet(\mathfrak{g})$ the dimensions of the finite dimensional *Chevalley-Eilenberg cohomology groups*:

$$H^\bullet(\mathfrak{g}) := \frac{\ker\left(d\colon\ \wedge^\bullet \mathfrak{g}^* \to \wedge^{\bullet+1}\mathfrak{g}^*\right)}{\mathrm{im}\left(d\colon\ \wedge^{\bullet-1} \mathfrak{g}^* \to \wedge^\bullet \mathfrak{g}^*\right)}.$$

In some special cases it is possible to compute the de Rham cohomology of M by means of the Chevalley-Eilenberg cohomology of $\mathfrak{g}$. The well-known Nomizu's Theorem [68] (see Theorem 2.1.8) for nilmanifolds is the more representative result in this sense.

2.1.2 Solvmanifolds and nilmanifolds

Solvmanifolds and nilmanifolds are a kind of compact and homogeneous manifolds of the form $M = G/\Gamma$, where G is a connected and simply-connected *solvable* (respectively *nilpotent*) Lie group.

Recall that a Lie group G is nilpotent if its corresponding Lie algebra $\mathfrak{g}$ is nilpotent. The nilpotency of a Lie algebra $\mathfrak{g}$ is defined by means of some descending chain of ideals of $\mathfrak{g}$.

Definition 2.1.5. *A Lie algebra* $\mathfrak{g}$ *is* nilpotent *(*NLA *for short) if the* descending central series $\{\mathfrak{g}^l\}_{l\geq 0}$ *of* $\mathfrak{g}$ *defined by*

$$\mathfrak{g}^0 := \mathfrak{g}, \qquad \mathfrak{g}^{l+1} := [\mathfrak{g}^l, \mathfrak{g}],$$

satisfies that $\mathfrak{g}^l = 0$ *for some* $l \in \mathbb{N}$. *Equivalently, if the* ascending central series $\{\mathfrak{g}_l\}_{l\geq 0}$ *of* $\mathfrak{g}$ *defined by*

$$\mathfrak{g}_0 := 0, \quad \mathfrak{g}_l := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}\},$$

satisfies that $\mathfrak{g}_l = 0$ *for some* $l \in \mathbb{N}$. *If* s *is the first positive integer with this property, then the NLA* $\mathfrak{g}$ *is said to be* s-step *nilpotent.*

The study of nilmanifolds was introduced by Malcev [61]. The simplest examples of nilmanifolds are the *n-dimensional tori.* They arise from the fact that every lattice Γ of the abelian Lie group $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^n$. Hence, the corresponding nilmanifold $\mathbb{R}^n/\mathbb{Z}^n$ is known as the n-dimensional torus. Other important examples of nilmanifolds are the Kodaira-Thurston manifold and the Iwasawa manifold.

Example 2.1.6. The Kodaira-Thurston manifold is the real four-dimensional nilmanifold underlying the compact complex surface given by (1.13). It is defined by $M = G/\Gamma$ where $G = H_3(\mathbb{R})\times\mathbb{R}$ and $\Gamma = \Gamma'\times\mathbb{Z}$. The nilpotent Lie group $H_3(\mathbb{R})$ is the 3-dimensional *Heisenberg group* and Γ' is a lattice of $H_3(\mathbb{R})$ given by:

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}, \quad \Gamma' = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}. \tag{2.3}$$

The Lie group $G = H_3(\mathbb{R}) \times \mathbb{R}$ admits a global chart assigning to each group element $g \in G$ real coordinates $(x, y, z, t) \in \mathbb{R}^4$ and it is straightforward to check that the following 1-forms $e^1 = dx$, $e^2 = dy$, $e^3 = dt$, $e^4 = dz + xdy$ constitute an invariant basis of $\wedge^1 M$. In addition, the structure equations in this basis are

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}. \tag{2.4}$$

Example 2.1.7. Another important example is the real six-dimensional nilmanifold underlying the Iwasawa manifold defined by (1.20). The corresponding Lie group admits a basis of left-invariant 1-forms $\{e^1, \ldots, e^6\}$ satisfying the structure equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0, \\ de^5 = e^{13} + e^{42}, \\ de^6 = e^{14} + e^{23}. \end{cases} \tag{2.5}$$

The Lie algebra satisfying the structure equations (2.5) is denoted by $\mathfrak{h}_5$ (see Theorem 2.1.25).

As we mentioned in the previous section, one of the most appreciated results about nilmanifolds states that the de Rham cohomology of a nilmanifold can be computed at the level of the Lie algebra of the group, or equivalently, by using only invariant forms on the nilmanifold.

Theorem 2.1.8 (Nomizu [68, Theorem 1]). *Let $M = G/\Gamma$ be a nilmanifold, then the inclusion $(\wedge^\bullet \mathfrak{g}^*, d) \hookrightarrow (\wedge^\bullet M, d)$ induces an isomorphism on cohomology.*

The previous theorem states that concerning the calculus of the cohomology of a nilmanifold the choice of the lattice is irrelevant. This situation does not hold in general in the bigger class of solvmanifolds which we pass to present immediately.

The general study of solvmanifolds was started by Mostow in [64]. The solvability of a Lie group G is determined by the solvability of its Lie algebra and it is defined by means of some descending chain of ideals of $\mathfrak{g}$.

Definition 2.1.9. *A Lie algebra $\mathfrak{g}$ is* solvable *if the* descending derived series *defined by*

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \qquad \mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}],$$

satisfies that $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$.

It follows from the previous definition that any nilpotent Lie algebra is solvable. Hence, any nilmanifold is a solvmanifold but the converse is not true in general. We recall several fundamental results on solvmanifolds. If $M = G/\Gamma$ is a solvmanifold of dimension n, then it is a fibre bundle over a torus with fibre a nilmanifold, which is called the *Mostow fibration* of M [64]. In addition, the fundamental group Γ of a solvmanifold can be represented as an extension of a torsion-free nilpotent group Λ of rank $n-k$ by a free abelian group of rank k:

$$0 \to \Lambda \to \Gamma \to \mathbb{Z}^k \to 0, \tag{2.6}$$

where $1 \leq k \leq n$ and $k = n$ if and only if Γ is abelian. Conversely, any abstract group Γ satisfying (2.6) can be the fundamental group of some solvmanifold [99]. Such a group is called *Wang group*. In addition, Mostow [64] proves that two solvmanifolds having isomorphic fundamental groups are diffeomorphic. The same result had been proven previously by Malcev [61] in the realm of nilmanifolds. Moreover nilmanifolds and solvmanifolds G/Γ are *aspherical*, that is, their homotopy groups $\pi_j(G/\Gamma) = \{0\}$ for $j \geq 2$.

Concerning the relation between the de Rham cohomology of the solvmanifold with the Chevalley-Eilenberg cohomology of the Lie algebra it turns out that Theorem 2.1.8 is not true in general for solvmanifolds. Nevertheless, Hattori [46] shows that the natural inclusion $(\wedge^\bullet \mathfrak{g}^*, d) \hookrightarrow (\wedge^\bullet M, d)$ induces an injective map in cohomology

$$H^\bullet(\mathfrak{g}) \to H^\bullet_{\mathrm{dR}}(M).$$

As a consequence of the injectivity of the previous map, and recalling that $H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ and $b_1(\mathfrak{g}) \geq 1$ if $\mathfrak{g}$ is solvable and $b_1(\mathfrak{g}) \geq 2$ if $\mathfrak{g}$ is nilpotent, there is the following lower-bound for the first Betti number of $M = G/\Gamma$ when G is solvable or nilpotent:

Corollary 2.1.10. *Any solvmanifold satisfies $b_1(M) \geq 1$. Any nilmanifold satisfies $b_1(M) \geq 2$.*

Hattori [46] extends Nomizu's Theorem, by using the Mostow Structure Theorem [64, Theorem 2], for $M = G/\Gamma$ when G is *completely solvable*. Recall that a solvable Lie group G is called *completely solvable* if the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$ defined by $\mathrm{ad}_X(Y) := [X, Y]$ has only real eigenvalues for all $X \in \mathfrak{g}$.

Theorem 2.1.11 (Hattori [46]). *Let $M = G/\Gamma$ be a solvmanifold, then the inclusion $(\wedge^\bullet \mathfrak{g}^*, d) \to (\wedge^\bullet M, d)$ induces an injection on cohomology. If G is completely solvable then the inclusion induces an isomorphism on cohomology.*

Finally, the previous result also holds for solvmanifolds satisfying the so-called *Mostow condition* [64, Theorem 8.2, Corollary 8.1].

2.1.3 Invariant complex structures on solvmanifolds

Let $M = G/\Gamma$ be a solvmanifold. In this section, we are concerned concretely with invariant complex structures defined on M.

Definition 2.1.12. *An invariant almost-complex structure on $M = G/\Gamma$ is an endomorphism $J \in End(\mathfrak{g})$ such that $J^2 = -Id_{TM}$.*

As we mentioned before, all kind of invariant geometric structures on $M = G/\Gamma$ defined by an invariant tensor field can be referred to a tensor defined at the level of the Lie algebra $\mathfrak{g}$. In particular, invariant almost-complex structures on $M = G/\Gamma$ are encoded in almost-complex structures on the Lie algebra $\mathfrak{g}$.

Remark 2.1.13. An almost-complex structure J on a Lie algebra $\mathfrak{g}$ induces the splitting in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$, where $\mathfrak{g}^{(1,0)} := \{X - iJX \mid X \in \mathfrak{g}_{\mathbb{C}}\}$ (respectively $\mathfrak{g}^{(0,1)} := \{X + iJX \mid X \in \mathfrak{g}_{\mathbb{C}}\}$) denotes the J-eigenspace with eigenvalue i (respectively $-i$). Actually, every decomposition $\mathfrak{g}_{\mathbb{C}} = V \oplus \overline{V}$ gives rise to a unique almost-complex structure J such that $\mathfrak{g}^{(1,0)} := V$.

The dual almost-complex structure $J^*\colon \mathfrak{g}^* \to \mathfrak{g}^*$ induces a splitting in the dual of the Lie algebra $\mathfrak{g}^*_{\mathbb{C}} = (\mathfrak{g}^*)^{1,0} \oplus (\mathfrak{g}^*)^{0,1}$ and more in general in the complexified tensor products of $\mathfrak{g}$ and $\mathfrak{g}^*$. It is particularly interesting the splitting of the complexified space of alternating tensors:

$$\wedge^\bullet \mathfrak{g}^*_{\mathbb{C}} = \bigoplus_{p+q=\bullet} \wedge^{p,q} \mathfrak{g}^*,$$

where $\wedge^{p,q}\mathfrak{g}^* := \wedge^p (\mathfrak{g}^*)^{1,0} \otimes \wedge^q (\mathfrak{g}^*)^{0,1}$ and $\overline{\wedge^{q,p}\mathfrak{g}^*} = \wedge^{p,q}\mathfrak{g}^*$. When $M = G/\Gamma$ is a solvmanifold endowed with an invariant almost-complex structure J, we denote by $\mathfrak{g}^{(1,0)}$ (respectively $\mathfrak{g}^{(0,1)}$) the space of complexified invariant vector fields with J-eigenvalue i (respectively $-i$) and $\wedge^{p,q}\mathfrak{g}^*$ the space of invariant (p, q)-forms on M. We usually describe an invariant complex structure on a solvmanifold by giving a basis of invariant 1-forms and giving the images by the endomorphism J or also by giving a $(1, 0)$-basis and the complex structure equations expressed in this basis.

By the Newlander-Nirenberg Theorem (see Theorem 1.1.4), an invariant complex structure on $M = G/\Gamma$ is an invariant almost-complex structure J satisfying the integrability condition $\mathrm{Nij}_J = 0$. The following proposition states several equivalent conditions to the integrability of J.

Proposition 2.1.14. *Let J be an invariant almost-complex structure on $M = G/\Gamma$, then the following conditions are equivalent:*

1. *J is integrable.*
2. *$Nij_J(X, Y) = 0$ for any $X, Y \in \mathfrak{g}$.*
3. *$[\mathfrak{g}^{(0,1)}, \mathfrak{g}^{(0,1)}] \subseteq \mathfrak{g}^{(0,1)}$.*
4. *$d\left(\wedge^{1,0}\mathfrak{g}^*\right) \subseteq \wedge^{2,0}\mathfrak{g}^* \oplus \wedge^{1,1}\mathfrak{g}^*$.*

Nilmanifolds endowed with an invariant complex structure play an important role in the study of non-Kähler geometry since Thurston [92] presented a nilmanifold admitting both a complex structure and a symplectic structure but no Kähler metric.

Example 2.1.15. For instance, let $M = G/\Gamma$ be the complex surface given by (1.13). The invariant $(1, 0)$-forms $\varphi := dz_1$ and $\eta := dz_2 + z_1 d\bar{z}_1$ satisfy the complex structure equations

$$d\varphi = 0, \quad d\eta = \varphi \wedge \bar{\varphi}. \tag{2.7}$$

If $\{e^1, e^2, e^3, e^4\}$ is an invariant basis of real 1-forms on M such that $\varphi = e^1 + ie^2$ and $\eta = e^3 - 2ie^4$ then we get the structure equations (2.4) and hence the underlying real manifold is the Kodaira-Thurston manifold. Equivalently, the complex structure given by (2.7) can be seen as an invariant complex structure J on the Kodaira-Thurston manifold defined by

$$Je^1 = -e^2, \quad Je^3 = 2e^4.$$

In addition, $F = e^{13} + e^{24}$ is a symplectic form. However, by using Nomizu's Theorem (see Theorem 2.1.8) it turns out that $b_1(M) = b_1(\mathfrak{g}) = 3$, thus the Kodaira-Thurston manifold admits both complex and symplectic structures but no Kähler metrics.

Furthermore, concerning the problem of characterizing the nilmanifolds admitting Kähler metrics, Hasegawa [44, Theorem 1] shows that the unique formal nilmanifolds are the tori. Therefore, recalling that formality is a necessary condition in order that a compact manifold satisfies the $\partial\bar{\partial}$-lemma [26, Main Theorem] and that any compact Kähler manifold satisfies the $\partial\bar{\partial}$-lemma [26, Corollary 5.23], it holds that if a nilmanifold admits a Kähler metric then it is diffeomorphic to a torus. Therefore, nilmanifolds provide a source of examples of non-Kähler geometry.

The previous considerations make interesting to consider the invariant complex geometry on nilmanifolds. Salamon [82] states the following equivalent condition for the integrability of an almost-complex structure J on a real $2n$-dimensional nilpotent Lie algebra:

Theorem 2.1.16 (Salamon [82, Theorem 1.3])**.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra,* $\dim_{\mathbb{R}} \mathfrak{g} = 2n$*, endowed with an almost-complex structure J. Then J is integrable if and only if $(\mathfrak{g}^*)^{1,0}$ has a basis $\{\omega^j\}_{j=1}^n$ such that $d\omega^1 = 0$ and*

$$d\omega^j \in \mathcal{I}(\omega^1, \dots, \omega^{j-1}), \qquad \text{for } j = 2, \dots, n,$$

where $\mathcal{I}(\omega^1, \dots, \omega^{j-1})$ is the ideal in $\wedge \mathfrak{g}^*_{\mathbb{C}}$ generated by $\{\omega^1, \dots, \omega^{j-1}\}$.

As a consequence of Theorem 2.1.16 the integrability of an almost-complex structure on an NLA is characterized in the following way:

Corollary 2.1.17. *Let $J\colon \mathfrak{g} \to \mathfrak{g}$ be an almost-complex structure on a $2n$-dimensional nilpotent Lie algebra, then J is integrable if and only if exists $0 \neq \Psi \in \wedge^{n,0}\mathfrak{g}^*$ such that $d\Psi = 0$.*

Proof. We must prove that the integrability of J implies the existence of a closed complex volume form. By Theorem 2.1.16 there is a (1,0)-basis $\{\omega^1, \dots, \omega^n\}$ such that $d\omega^1 = 0$ and $d\omega^j \in \mathcal{I}\langle\omega^1, \dots, \omega^{j-1}\rangle$ for $j = 2, \dots, n$. It is direct to check that the $(n, 0)$-form $\Psi = \omega^1 \wedge \ldots \wedge \omega^n$ is a non-zero closed $(n, 0)$-form. □

Remark 2.1.18. It follows from Corollary 2.1.17 that when $M = G/\Gamma$ is a nilmanifold endowed with an invariant complex structure J then the complex manifold (M, J) has holomorphically trivial canonical bundle because the non-zero invariant section $\Psi \in \wedge^{n,0}\mathfrak{g}^*$ is a non-zero holomorphic volume form defined globally on M.

Inspired by Theorem 2.1.16, Cordero, Fernández, Gray and Ugarte [24] suggest a division on the space of complex structures on a nilpotent Lie algebra $\mathfrak{g}$ into *nilpotent* and *non-nilpotent* complex structures.

Definition 2.1.19. *A complex structure J on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called* nilpotent *if there is a basis $\{\omega^j\}_{j=1}^n$ for $(\mathfrak{g}^*)^{1,0}$ satisfying $d\omega^1 = 0$ and*

$$d\omega^j \in \bigwedge^2 \langle \omega^1, \dots, \omega^{j-1}, \omega^{\overline{1}}, \dots, \omega^{\overline{j-1}} \rangle, \qquad \text{for } j = 2, \dots, n. \tag{2.8}$$

Equivalently [24], the ascending series $\{\mathfrak{g}_l^J\}_{l \geq 0}$ for $\mathfrak{g}$ adapted to J*, which is defined inductively by*

$$\mathfrak{g}_0^J := 0, \quad \mathfrak{g}_l^J := \{X \in \mathfrak{g} \mid [J^k(X), \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J \,, k = 1, 2\}, \quad \text{for } l \geq 1, \tag{2.9}$$

satisfies that $\mathfrak{g}_l^J = \mathfrak{g}$ for some positive integer l.

Inside nilpotent complex structures on nilpotent Lie algebras, there are two well-known special classes known as *abelian* and *parallelizable* complex structures, although both abelian and parallelizable can be defined in general on any Lie algebra:

- J is *abelian* if $[JX, JY] = [X, Y]$, for all $X, Y \in \mathfrak{g}$, or equivalently $d(\mathfrak{g}^*)^{1,0} \subset \wedge^{1,1}\mathfrak{g}^*$. They are also characterized by the fact that the subalgebra $\mathfrak{g}^{(1,0)}$ is abelian.

- J is *complex-parallelizable* if $[JX, Y] = J[X, Y]$, for all $X, Y \in \mathfrak{g}$, or equivalently $d(\mathfrak{g}^*)^{1,0} \subset \wedge^{2,0}\mathfrak{g}^*$ or equivalently $[\mathfrak{g}^{(1,0)}, \mathfrak{g}^{(0,1)}] = \{0\}$. These structures are the natural complex structures of the *complex Lie algebras* and give rise to *complex Lie groups*. In addition, the corresponding compact complex manifold $M = G/\Gamma$ has holomorphically trivial tangent bundle.

Remark 2.1.20. It follows from the definition of the ascending series (2.9) that when J is a parallelizable or an abelian structure on a nilpotent Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_l^J = \mathfrak{g}_l$ for $l \geq 0$ hence it is nilpotent. In addition, it is clear that the nilpotency condition for a complex structure (2.8) is preserved under equivalence of complex structures, that is, if J' is equivalent to J then J is nilpotent if and only if J' is.

Some of the most known complex solvmanifolds are endowed with a parallelizable invariant complex structure. On the one hand, the Iwasawa manifold described in (1.20) admits the following invariant $(1,0)$-basis $\omega^1 := dz_1$, $\omega^2 := dz_2$, $\omega^3 := dz_3 + z_1 dz_2$ satisfying the complex structure equations

$$d\omega^1 = 0, \quad d\omega^2 = 0, \quad d\omega^3 = \omega^{12}. \tag{2.10}$$

On the other hand, the Nakamura manifold defined by (1.21) admits the following invariant $(1,0)$-basis $\omega^1 := dz_1$, $\omega^2 := e^{z_1}dz_2$ and $\omega^3 := e^{-z_1}dz_3$. The complex structure equations in this basis are:

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{12}, \quad d\omega^3 = -\omega^{13}. \tag{2.11}$$

It is direct to check that the $(3,0)$-form $\Psi = \omega^{123} = dz_1 \wedge dz_2 \wedge dz_3$ defines a nowhere vanishing invariant holomorphic volume form on M. On the other hand, the same occurs on the Iwasawa manifold by Corollary 2.1.17. Both complex manifolds have holomorphically trivial canonical bundle.

Indeed, the following proposition shows that, given an almost-complex structure on a Lie algebra $\mathfrak{g}$, the existence of a non-zero closed $(n,0)$-form yields the integrability of J.

Proposition 2.1.21. *Let J be an almost-complex structure on a $2n$-dimensional Lie algebra $\mathfrak{g}$ and $0 \neq \Psi \in \wedge^{n,0}\mathfrak{g}^*$. If $d\Psi = 0$ then J is integrable.*

Proof. Recall that J is integrable if and only if $\pi_{0,2}(d\omega) = 0$ for every $\omega \in \wedge^{1,0}\mathfrak{g}^*$, where $\pi_{0,2} : \wedge^2\mathfrak{g}^*_{\mathbb{C}} \to \wedge^{0,2}\mathfrak{g}^*$. Suppose that there is a complex and closed volume form $\Psi \in \Lambda^{n,0}\mathfrak{g}^*$, then there is a (1,0)-basis $\{\omega^1, \ldots, \omega^n\}$ of $\mathfrak{g}^*_{\mathbb{C}}$ such that $\Psi = \omega^1 \wedge \ldots \wedge \omega^n$. When we evaluate the exterior derivative of Ψ we get $0 = d\Psi = \Omega_{n+1,0} + \Omega_{n,1} + \Omega_{n-1,2}$, where $\Omega_{p,q}$ is a complex (p,q)-form which must vanish. If we compute explicitly $\Omega_{n-1,2}$ we get:

$$\Omega_{n-1,2} = \pi_{0,2}(d\omega^1) \wedge \omega^2 \wedge \ldots \wedge \omega^n - \ldots + (-1)^{n+1}\pi_{0,2}(d\omega^n) \wedge \omega^1 \wedge \ldots \wedge \omega^{n-1} = 0,$$

and necessarily $\pi_{0,2}(d\omega^j) = 0$ for every $j \in \{1, \ldots, n\}$ and therefore J is integrable. $\square$

For any $p \in \mathbb{N}$ we can associate a differential graded algebra $(\wedge^{p,\bullet}\mathfrak{g}^*, \bar{\partial})$ of complexified invariant forms and the finite dimensional *invariant* Dolbeault cohomology groups are defined by:

$$H^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{g}) := \frac{\ker\left(\bar{\partial}\colon\ \wedge^{\bullet,\bullet}\ \mathfrak{g}^* \to \wedge^{\bullet,\bullet+1}\mathfrak{g}^*\right)}{\operatorname{im}\left(\bar{\partial}\colon\ \wedge^{\bullet,\bullet-1}\ \mathfrak{g}^* \to \wedge^{\bullet,\bullet}\mathfrak{g}^*\right)}.$$

The dimensions of these groups are denoted by $h^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{g}) := \dim H^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{g})$. The identity map $\wedge^{\bullet,\bullet}\mathfrak{g}^* \hookrightarrow \wedge^{\bullet,\bullet}M$ induces a natural map

$$H^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\bar{\partial}}(M), \tag{2.12}$$

and when M is a nilmanifold Console and Fino [20, Lemma 7] show that this map is injective. It is natural to wonder if there is a kind of Nomizu's Theorem for the Dolbeault cohomology of a nilmanifold endowed with an invariant complex structure. Concerning the map (2.12), Sakane [81] proves that for nilmanifolds endowed with an invariant complex parallelizable structure it is an isomorphism and the same is proven by Console and Fino [20] when the complex structure is abelian. In addition, Cordero, Fernández, Gray and Ugarte [24] show that if the complex nilmanifold (M, J) with J invariant is an iterated principal holomorphic torus bundle then its Dolbeault cohomology can be computed by means of the invariant Dolbeault cohomology. Finally, Console and Fino prove the following:

Theorem 2.1.22 (Console and Fino [20, Theorem A])**.** *The map* (2.12) *is an isomorphism on an open set of any connected component of the moduli space of invariant complex structures on a nilmanifold M.*

It is remarkable that this open set can be empty for a given nilmanifold. In the realm of solvmanifolds, Kasuya [51, Corollary 1.3] has developed recently a technique to compute the Dolbeault cohomology of a solvmanifold endowed with an invariant complex structure of *splitting type* [51, Assumption 1.1] by means of computing the cohomology of a finite dimensional differential bi-graded algebra. This technique has been extended by Angella and Kasuya [8, Theorem 1.1] to compute the Bott-Chern cohomology of solvmanifolds endowed with invariant complex structures of this type. These techniques will be presented in detail in Chapter 4.

Now we are concerned to show the main results related with invariant complex geometry on solvmanifolds up to dimension six. The unique two-dimensional solvable Lie algebra admitting a complex structure is the abelian $\mathbb{R}^2$ and it turns out that any almost-complex structure on it is integrable. On the other hand, Ovando [71] classifies the four-dimensional solvable Lie algebras with complex structure. In addition, Hasegawa shows that the four-dimensional real solvmanifolds only admit invariant complex structures. We present his result adding the underlying real Lie algebra of every solvmanifold:

Theorem 2.1.23 (Hasegawa [45, Theorem 1])**.** *A complex surface is diffeomorphic to a four-dimensional solvmanifold if and only if it is one of the following surfaces: Complex torus ($\mathfrak{s}_1$), Hyperelliptic surface ($\mathfrak{s}_3$), Inoue Surface of type S^0 ($\mathfrak{s}_5$), Primary Kodaira* *surface($\mathfrak{s}_2$), Secondary Kodaira surface ($\mathfrak{s}_4$), Inoue Surface of type $S^{\pm}$ ($\mathfrak{s}_6$). The underlying real Lie algebras are:*

$$
\begin{aligned}
\mathfrak{s}_1 &= (0,0,0,0), & \mathfrak{s}_4 &= (e^{24}, -e^{14}, e^{12}, 0),\\
\mathfrak{s}_2 &= (0,0,e^{12},0), & \mathfrak{s}_5 &= (ae^{14}+be^{24}, -be^{14}+ae^{24}, -2ae^{34}, 0),\ a \neq 0,\ b \in \mathbb{R},\\
\mathfrak{s}_3 &= (e^{24}, -e^{14}, 0, 0), & \mathfrak{s}_6 &= (e^{23}, e^{24}, -e^{34}, 0).
\end{aligned}
$$

Moreover every complex structure on each of these complex surfaces (considered as solvmanifolds) is invariant.

Remark 2.1.24. It is worth noticing that the underlying real solvmanifold of the primary Kodaira surfaces ($\mathfrak{s}_2$) is the Kodaira-Thurston manifold given by (2.3). In addition, Hasegawa [45] shows an example of a six-dimensional solvmanifold endowed with a non-invariant complex structure based on a holomorphic deformation of the Nakamura manifold.

The classification of six-dimensional nilpotent Lie algebras admitting complex structure is obtained by Salamon [82]. In addition, Ugarte [95] extends this result presenting the Lie algebras in terms of the different types of complex structures that they admit.

Theorem 2.1.25 ([82, 95])**.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension 6. Then, $\mathfrak{g}$ has a complex structure if and only if it is isomorphic to one of the following Lie algebras:*

$$
\begin{aligned}
\mathfrak{h}_1 &= (0,0,0,0,0,0), & \mathfrak{h}_{10} &= (0,0,0,e^{12},e^{13},e^{14}),\\
\mathfrak{h}_2 &= (0,0,0,0,e^{12},e^{34}), & \mathfrak{h}_{11} &= (0,0,0,e^{12},e^{13},e^{14}+e^{23}),\\
\mathfrak{h}_3 &= (0,0,0,0,0,e^{12}+e^{34}), & \mathfrak{h}_{12} &= (0,0,0,e^{12},e^{13},e^{24}),\\
\mathfrak{h}_4 &= (0,0,0,0,e^{12},e^{14}+e^{23}), & \mathfrak{h}_{13} &= (0,0,0,e^{12},e^{13}+e^{14},e^{24}),\\
\mathfrak{h}_5 &= (0,0,0,0,e^{13}+e^{42},e^{14}+e^{23}), & \mathfrak{h}_{14} &= (0,0,0,e^{12},e^{14},e^{13}+e^{42}),\\
\mathfrak{h}_6 &= (0,0,0,0,e^{12},e^{13}), & \mathfrak{h}_{15} &= (0,0,0,e^{12},e^{13}+e^{42},e^{14}+e^{23}),\\
\mathfrak{h}_7 &= (0,0,0,e^{12},e^{13},e^{23}), & \mathfrak{h}_{16} &= (0,0,0,e^{12},e^{14},e^{24}),\\
\mathfrak{h}_8 &= (0,0,0,0,0,e^{12}), & \mathfrak{h}_{19}^- &= (0,0,0,e^{12},e^{23},e^{14}-e^{35}),\\
\mathfrak{h}_9 &= (0,0,0,0,e^{12},e^{14}+e^{25}), & \mathfrak{h}_{26}^+ &= (0,0,e^{12},e^{13},e^{23},e^{14}+e^{25}).
\end{aligned}
$$

Moreover:

(a) *Any complex structure on $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ is non-nilpotent.*

(b) *For $1 \leq k \leq 16$, any complex structure on $\mathfrak{h}_k$ is nilpotent.*

(c) *Any complex structure on $\mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_8$ and $\mathfrak{h}_9$ is abelian.*

(d) *There exist both abelian and non-abelian nilpotent complex structures on $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$ and $\mathfrak{h}_{15}$.*

(e) *Any complex structure on $\mathfrak{h}_6$, $\mathfrak{h}_7$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$, $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ and $\mathfrak{h}_{16}$ is not abelian.*

Remark 2.1.26. It follows from Theorem 2.1.25 that, in real dimension six, if $\mathfrak{g}$ admits complex structures then all of them are either nilpotent or non-nilpotent. Cordero, Fernández, Gray and Ugarte [24] show that this is not true in general in higher dimensions.

As we mentioned in the previous section, the well-known Iwasawa manifold and Nakamura manifold admit an invariant closed complex volume form of pure type $(3,0)$ and consequently have holomorphically trivial canonical bundle. This consideration makes interesting to classify the six-dimensional solvmanifolds endowed with an invariant complex structure with holomorphically trivial canonical bundle. We show in the next section that the existence of an invariant closed volume form of pure type $(3,0)$ is not only a sufficient condition but a necessary condition to trivialize the holomorphic canonical bundle. For this purpose we make use of the *symmetrization process* proposed originally by Belgun [11].

2.1.4 Trivialization of the holomorphic canonical bundle

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G, $\Gamma \leq G$ a lattice and $d\mu$ a bi-invariant volume form on the compact manifold $M = G/\Gamma$. Belgun [11] provides a method which reduces in some cases the study of the properties of some geometric structures on $M = G/\Gamma$ to the study of such properties on the spaces of left-invariant geometric structures, equivalently, on the Lie algebra $\mathfrak{g}$ of G. We present the process and several useful properties (for shortening we denote $\mathcal{T}^k(M) := \mathcal{T}^k_0(M)$ and $\mathcal{T}^k(\mathfrak{g}) := \mathcal{T}^k_0(\mathfrak{g})$).

Lemma 2.1.27 (Belgun [11, Theorem 7], Fino and Grantcharov [31, Theorem 2.1])**.** *Let $M = G/\Gamma$ be a compact quotient of a simply-connected Lie group by a lattice Γ, $\mathfrak{g}$ the Lie algebra of G and $d\mu$ a bi-invariant volume form such that $\int_M d\mu = 1$. The symmetrization map $(\cdot)_\nu \colon \mathcal{T}^k(M) \to \mathcal{T}^k(\mathfrak{g})$ defined by*

$$T_\nu(X_1,\ldots,X_k) := \int_{p\in M} T_p(X_{1_p},\ldots,X_{k_p})d\mu, \qquad X_1,\ldots,X_k \in \mathfrak{g}, \tag{2.13}$$

satisfies the following properties:

1. $(\cdot)_\nu \mid_{\mathcal{T}^k(\mathfrak{g})} = Id \mid_{\mathcal{T}^k(\mathfrak{g})}$.
2. *If* $T \in \wedge^k M$ *then* $(dT)_\nu = dT_\nu$.
3. *If* $\alpha \in \wedge^k M$ *and* $\beta \in \wedge^q M$ *then* $(\alpha_\nu \wedge \beta)_\nu = \alpha_\nu \wedge \beta_\nu$.

Remark 2.1.28. If $M = G/\Gamma$ is a compact quotient of a simply-connected nilpotent (resp. completely solvable) Lie group, it follows from Nomizu's Theorem (resp. Hattori's Theorem, see Theorems 2.1.8 and 2.1.11) and from properties *i), ii)* of Lemma 2.1.27 that the restriction of the symmetrization map $(\cdot)_\nu \mid_{\wedge^\bullet M} \colon (\wedge^\bullet M, d) \to (\wedge^\bullet \mathfrak{g}^*, d)$ given by (2.13) to the exterior algebra of M induces an isomorphism in cohomology.

Now, let $(M = G/\Gamma, J)$ be a complex manifold endowed with an invariant complex structure. Firstly, the symmetrization process (see Lemma 2.1.27) is compatible in some sense with the complex structure.

Lemma 2.1.29. *If $\alpha \in \wedge^{\bullet,\bullet} M$ then $\alpha_\nu \in \wedge^{\bullet,\bullet}\mathfrak{g}^*$. Similarly, $(\partial\alpha)_\nu = \partial\alpha_\nu$ and $(\bar\partial\alpha)_\nu = \bar\partial\alpha_\nu$.*

Remark 2.1.30. If $M = G/\Gamma$, it is easy to prove the injectivity of the inclusion $H^\bullet(\mathfrak{g}) \to H^\bullet_{\mathrm{dR}}(M)$ induced by the identity map $\wedge^\bullet\mathfrak{g}^* \hookrightarrow \wedge^\bullet M$. Take a closed invariant form $\alpha \in \wedge^\bullet\mathfrak{g}^*$ and suppose that $\alpha = d\beta$ with $\beta \in \wedge^{\bullet-1}M$, then by using the symmetrization process we find that $\alpha_\nu = (d\beta)_\nu = d\beta_\nu$ and hence $[\alpha] = 0 \in H^\bullet(\mathfrak{g})$. Similarly when M is endowed with an invariant complex structure, the injectivity of the inclusion $H^{\bullet,\bullet}_{\bar\partial}(\mathfrak{g}) \to H^{\bullet,\bullet}_{\bar\partial}(M)$ holds by using the symmetrization process and Lemma 2.1.29.

Now, it is natural to ask whether the existence of a holomorphic form of bidegree $(n, 0)$ with respect to an invariant complex structure on a $2n$-dimensional solvmanifold implies the existence of an invariant non-zero closed $(n, 0)$-form. We show that the answer to this question is positive.

Proposition 2.1.31. *Let $M = \Gamma\backslash G$ be a $2n$-dimensional solvmanifold endowed with an invariant complex structure J. If Ψ is a nowhere vanishing holomorphic $(n, 0)$-form on (M, J), then Ψ is necessarily invariant.*

Proof. Since J is an invariant complex structure on M, we consider a global basis $\{\omega^1, \dots, \omega^n\}$ of invariant $(1, 0)$-forms on (M, J). Then, there is a nowhere vanishing complex-valued function $f \colon M \to \mathbb{C}$ such that $\Psi = f\,\omega^1 \wedge \cdots \wedge \omega^n$. Since Ψ is holomorphic, we have $\bar\partial\Psi = \bar\partial f \wedge \omega^1 \wedge \cdots \wedge \omega^n + f\,\bar\partial(\omega^1 \wedge \cdots \wedge \omega^n) = 0$, that is, $\bar\partial(\omega^1 \wedge \cdots \wedge \omega^n) = -\bar\partial(\log f) \wedge \omega^1 \wedge \cdots \wedge \omega^n$. The latter form is an invariant $(n, 1)$-form on (M, J), so there is an invariant form α of bidegree $(0, 1)$ on (M, J) such that

$$\bar\partial(\log f) = \alpha. \tag{2.14}$$

By Lemma 2.2.1 the Lie group G is unimodular, hence there is a volume element on M induced by a bi-invariant one on the Lie group G (its existence is guaranteed by [63]). Now, we can apply the symmetrization process 2.13 on both sides of equation (2.14) and making use of the properties of Lemma 2.1.29 we get

$$(\bar\partial \log f)_\nu = \bar\partial(\log f)_\nu = \alpha_\nu = \alpha,$$

because α is invariant. But $(\log f)_\nu$ is the symmetrization of a function, so it is a constant and then $\bar\partial(\log f)_\nu = 0$. Therefore, $\alpha = 0$ and by (2.14) we get $\bar\partial(\log f) = 0$. This means that $\log f$ is a holomorphic function on a compact complex manifold, which implies that $\log f = c$, where c is a constant. In conclusion, $f = \exp(c)$ is a constant function, and Ψ is necessarily invariant. □

It is known that the property of having holomorphically trivial canonical bundle is not stable under holomorphic deformations, but as a consequence of the previous result, we provide in the following theorem an example of the non-stability of this property based on an invariant holomorphic deformation.

Theorem 2.1.32. *The property of* having holomorphically trivial canonical bundle *is not stable under holomorphic deformations.*

Proof. We see in Chapter 3 that the real solvmanifold underlying the Nakamura manifold admits an abelian complex structure, denoted by J_1, described in terms of a left-invariant $(1,0)$-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations

$$d\omega^1 = \omega^{1\bar{3}}, \quad d\omega^2 = -\omega^{2\bar{3}}, \quad d\omega^3 = 0.$$

Notice that the $(0,1)$-form $\omega^{\bar{1}}$ defines an invariant Dolbeault cohomology class. Hence, if X_3 is the invariant $(1,0)$-vector field on M dual to ω^3, we can consider the invariant holomorphic deformation of (M, J_1) given by the direction $\Psi(t) = tX_3 \otimes \omega^{\bar{1}}$ where $t \in \Delta = \{t \in \mathbb{C} \mid |t| < \epsilon\}$ for $\epsilon > 0$ enoughly small. The complex structure J_t is described by the following $(1,0)$-basis:

$$\eta_t^1 := \omega^1, \quad \eta_t^2 := \omega^2, \quad \eta_t^3 := \omega^3 - t\omega^{\bar{1}}.$$

It is straightforward to check that the complex structure equations of the invariant complex structure J_t are:

$$d\eta_t^1 = \eta_t^{1\bar{3}}, \quad d\eta_t^2 = \bar{t}\eta_t^{12} - \eta_t^{2\bar{3}}, \quad d\eta_t^3 = t\eta_t^{3\bar{1}},$$

and $d\eta_t^{123} = t\eta_t^{123\bar{1}} \neq 0$ for any $t \in \Delta^*$. By using Proposition 2.1.31, the solvmanifolds $\{(M, J_t)\}_{t \in \Delta^*}$ do not have holomorphically trivial canonical bundle and this concludes the proof. $\square$

2.2 Six-dimensional solvable Lie algebras with complex structures

As a consequence of Proposition 2.1.31, we find that the problem of classifying the Lie algebras underlying the six-dimensional solvmanifolds $M = G/\Gamma$ admitting an invariant complex structure with holomorphically trivial canonical bundle is equivalent to classificate the six-dimensional solvable Lie algebras $\mathfrak{g}$ admitting a pair (J, Ψ) where J is an almost-complex structure and Ψ is a closed form of J-pure type $(n,0)$ and such that the corresponding connected and simply-connected Lie group admits a lattice.

$\exists J \in \mathrm{End}(\mathfrak{g})$, $J^2 = -\mathrm{Id}_{TM}$ with holomorphically trivial canonical bundle	$\longleftrightarrow$	$\exists J \in \mathrm{End}(\mathfrak{g})$, $J^2 = -\mathrm{Id}_{TM}$ such that $\exists \Psi \in \wedge^{n,0}\mathfrak{g}^*$ with $d\Psi = 0$

As we are concerned with studying these complex structures on solvmanifolds $M = G/\Gamma$, we point out that in general not every Lie group G admits a lattice. Milnor [63] states the following necessary condition for the existence of a lattice:

Lemma 2.2.1 (Milnor [63, Lemma 6.2])**.** *If G admits a lattice then G is unimodular.*

Recall, that a Lie group G is called *unimodular* if its left-invariant Haar measure is also right-invariant. The notion of unimodularity is also extended to Lie algebras. A Lie algebra $\mathfrak{g}$ is said *unimodular* if the trace of the adjoint representation ad_X vanishes for all $X \in \mathfrak{g}$. The following lemma shows equivalent conditions for the unimodularity of $\mathfrak{g}$:

Lemma 2.2.2. *Let G be an unimodular Lie group with $\dim_{\mathbb{R}} G = n$ and $\mathfrak{g}$ its Lie algebra, then the following conditions are equivalent:*

1. *The linear transformation $Ad(g)$ has determinant ± 1 for every $g \in G$ [63, Lemma 6.1].*
2. *$\mathfrak{g}$ is unimodular [63, Lemma 6.3].*
3. *$d(\wedge^{n-1}\mathfrak{g}^*) = \{\mathbf{0}\}$.*
4. *$b_n(\mathfrak{g}) = 1$.*

By using the previous characterization of the unimodularity condition, the next lemma shows a simple but useful obstruction to the existence of complex structures with non-zero closed $(n,0)$-volume forms in the unimodular case involving the Betti number $b_n(\mathfrak{g})$.

Lemma 2.2.3. *Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra. If $\mathfrak{g}$ is unimodular and admits a complex structure with a non-zero closed $(n,0)$-form Ψ, then $b_n(\mathfrak{g}) \geq 2$.*

Proof. Let $\Psi_+, \Psi_- \in \wedge^n\mathfrak{g}^*$ be the real and imaginary parts of Ψ, that is, $\Psi = \Psi_+ + i\,\Psi_-$. Since Ψ is closed we have that $d(\Psi_+) = d(\Psi_-) = 0$ and therefore $[\Psi_+], [\Psi_-] \in H^n(\mathfrak{g})$. It is sufficient to see that both classes are non-zero and, moreover, that they are not cohomologous.

Suppose that there exist $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$ such that $a\Psi_+ + b\Psi_- = d\alpha$ for some $\alpha \in \wedge^{n-1}\mathfrak{g}^*$. Since $0 \neq \frac{i}{2}\Psi \wedge \bar{\Psi} = \Psi_+ \wedge \Psi_- \in \wedge^{2n}\mathfrak{g}^*$, we get

$$d(\alpha \wedge (-b\Psi_+ + a\Psi_-)) = (a\Psi_+ + b\Psi_-) \wedge (-b\Psi_+ + a\Psi_-) = (a^2 + b^2)\Psi_+ \wedge \Psi_- \neq 0.$$

But by Lemma 2.2.2 this is in contradiction to the unimodularity of $\mathfrak{g}$. $\square$

The previous Lemmas 2.2.2 and 2.2.3 reduces the problem of classifying the six-dimensional solvmanifolds $M = G/\Gamma$ admitting an invariant complex structure with holomorphically trivial canonical bundle to classificate the six-dimensional unimodular solvable Lie algebras $\mathfrak{g}$ with $b_3(\mathfrak{g}) \geq 2$ admitting a pair (J, Ψ) where J is an almost-complex structure and Ψ is a closed form of J-pure type $(3,0)$ and such that the corresponding connected and simply connected Lie group admits a lattice.

2.2.1 The formalism of stable forms in six dimensions

A complex volume form $\Psi \in \wedge^{3,0}\mathfrak{g}^*$ can be decomposed into its real and imaginary part $\Psi = \Psi_+ + i\Psi_-$, where $\Psi_+, \Psi_- \in \wedge^3\mathfrak{g}^*$ are real 3-forms satisfying $J\Psi_+ = \Psi_-$. In this section, we use a technique to construct an almost-complex structure $J_\rho \colon \mathfrak{g} \to \mathfrak{g}$ on a given six-dimensional Lie algebra $\mathfrak{g}$ by means of a real 3-form $\rho \in \wedge^3\mathfrak{g}^*$. Hence, we obtain the desired pair (J, Ψ) by defining $J := J_\rho$ and $\Psi := \rho + iJ_\rho^*\rho$.

This technique is based in the algebraic formalism of stable forms developed by Hitchin in [48]. Actually, it states a surjective mapping between the space of 3-forms of a six-dimensional vector space V and the set of endomorphisms of the vector space $f\colon V \to V$ satisfying that $f \circ f = \lambda \mathrm{id}_V$. This construction will be very useful in the later classification of 6-dimensional solvable real Lie algebras admitting complex structures.

Let V be a real six-dimensional vector space and fix an orientation $\nu \in \wedge^6 V^*$. A 3-form $\rho \in \wedge^3 V^*$ is *stable* if the orbit $\{g \cdot \rho \mid g \in \mathrm{GL}(V)\}$ is open. Now we want to express this property in an algebraic way. Let $\kappa\colon \wedge^5 V^* \longrightarrow V$ be the isomorphism defined by:

$$\kappa(\eta) := X \text{ where } X \in V \text{ satisfies } \iota_X\nu = \eta.$$

If $\rho \in \wedge^3 V^*$ then for any $X \in V$ we have $\iota_X\rho \wedge \rho \in \wedge^5 V^*$. Hence we can define an endomorphism $K_\rho : V \longrightarrow V$ by:

$$K_\rho(X) := \kappa(\iota_X\rho \wedge \rho). \tag{2.15}$$

The following proposition states the stability of $\rho \in \wedge^3 V^*$ in terms of a scalar associated to the endomorphism K_ρ:

Proposition 2.2.4 (Hitchin [48, Proposition 2])**.** *A 3-form $\rho \in \wedge^3 V^*$ is stable if and only if $\lambda(\rho) := \frac{1}{6} tr(K_\rho^2) \neq 0$. Moreover:*

- *$\lambda(\rho) > 0$ if and only if $\rho = \alpha + \beta$ where $\alpha, \beta \in \wedge^3 V^*$ are decomposable and $\alpha \wedge \beta \neq 0$.*
- *$\lambda(\rho) < 0$ if and only if $\rho = \Psi + \bar{\Psi} = 2\mathfrak{Re}\,\Psi$ where $\Psi \in \wedge^3 V_{\mathbb{C}}^*$ is decomposable and $\Psi \wedge \bar{\Psi} \neq 0$.*

Remark 2.2.5. The vector space $\wedge^3 V^*$ is divided by the hypersurface $\{\rho \in \wedge^3 V^* \mid \lambda(\rho) = 0\}$ into two open subsets corresponding to $\lambda(\rho) > 0$ and $\lambda(\rho) < 0$. Let $\{e^1, \ldots, e^6\}$ be a basis of the space V^*.

- The open set $\Omega_+(V) := \{\rho \in \wedge^3 V^* \mid \lambda(\rho) > 0\}$ is the $\mathrm{GL}(V)$-orbit of the 3-form

  $$\rho = e^{123} + e^{456}. \tag{2.16}$$

- The open set $\Omega_-(V) := \{\rho \in \wedge^3 V^* \mid \lambda(\rho) < 0\}$ is the $\mathrm{GL}(V)$-orbit of the 3-form

  $$\rho = \Psi + \bar{\Psi} = 2\mathfrak{Re}\,\Psi. \tag{2.17}$$

  where $\Psi = (e^1 - ie^2) \wedge (e^3 - ie^4) \wedge (e^5 - ie^6) \in \wedge^3 V_{\mathbb{C}}^*$.

A basis $\{e^1, \ldots, e^6\}$ of V^* in which the 3-form $\rho \in \wedge^3 V^*$ is expressed by (2.16) or (2.17) is called an *adapted basis* to the endomorphism K_ρ. It is easy to check in an adapted basis that $K_\rho^2 = \lambda(\rho)\mathrm{Id}_V$.

From now on we are concerned with the open set $\Omega_-(V) = \{\rho \in \wedge^3 V^* \mid \lambda(\rho) < 0\}$. If $\rho \in \Omega_-$ then the endomorphism $J_\rho \colon V \to V$ defined by

$$J_\rho(X) := \frac{1}{\sqrt{-\lambda(\rho)}} K_\rho(X) \tag{2.18}$$

gives rise to an almost-complex structure on V. Actually, we can see that every almost-complex structure on V may be written as $J = J_\rho$ for some $\rho \in \Omega_-(V)$.

Lemma 2.2.6. *Let $J : V \to V$ be an almost-complex structure on V, then there exists a $\rho \in \Omega_-(V)$ and a volume form $\nu \in \wedge^6 V^*$ such that $J = J_\rho$.*

Proof. Let $J : V \to V$ be an almost-complex structure on V and $\{e_1, \ldots, e_6\}$ an adapted basis of V to J, namely:

$$Je_1 = -e_2,\ Je_3 = -e_4,\ Je_5 = -e_6.$$

Let $\{e^1, \ldots, e^6\}$ be the dual basis and consider the complex 3-form $\Psi = (e^1 - ie^2) \wedge (e^3 - ie^4) \wedge (e^5 - ie^6)$. If we take the real part:

$$\rho = \mathfrak{Re}\,\Psi = e^{135} - e^{146} - e^{236} - e^{245}$$

and the volume form $\nu = 2e^{123456}$, then by (2.15) it is straight to check that $K_\rho = J$. $\square$

As a consequence of the previous lemma, there is a natural and surjective mapping $\Omega_-(V) \to \{J \colon V \to V \mid J^2 = -\mathrm{Id}_V\}$ assigning to each $\rho \in \Omega_-(V)$ the endomorphism $J := J_\rho$ through relation (2.18). From now on, we work better with almost-complex structures $J^* \colon V^* \to V^*$ defined on the dual of the vector space. As a matter of notation, given a volume form $\nu \in \wedge^6 V^*$, we denote by $\tilde{J}^*_\rho \colon V^* \to V^*$ the endomorphism dual to K_ρ and J^*_ρ the corresponding almost-complex structure on V^*.

Remark 2.2.7. Let $\nu \in \wedge^6 V^*$ be a fixed volume form, then the scalar $\lambda(\rho)$ enables to construct a specific volume form $\phi(\rho) := \sqrt{|\lambda(\rho)|}\nu \in \wedge^6 V^*$ such that the action of the dual endomorphism J^*_ρ on 1-forms is given by the formula

$$\big((J^*_\rho \alpha)(X)\big)\, \phi(\rho) = \alpha \wedge \iota_X \rho \wedge \rho, \tag{2.19}$$

for any $\alpha \in V^*$ and $X \in V$.

2.2.2 Complex structures with closed $(3,0)$-form

From now on, let $\mathfrak{g}$ be a real solvable Lie algebra of dimension six. We recall that we are concerned with the complex structures on $\mathfrak{g}$ admitting a non-zero closed (3,0)-form. For this goal we consider ideas in [19, 36, 37, 84]. In the context of symplectic half-flat structures some specific results were obtained in [30] that allowed to classify 6-dimensional solvable Lie algebras admitting such structures.

Let $Z^3(\mathfrak{g}) = \{\rho \in \wedge^3\mathfrak{g}^* \mid d\rho = 0\}$. The map $\Omega_-(\mathfrak{g}) \to \{J : \mathfrak{g} \to \mathfrak{g} \mid J^2 = -\mathrm{Id}_{\mathfrak{g}}\}$ restricts to the surjective mapping

$$\{\rho \in Z^3(\mathfrak{g}) \mid \lambda(\rho) < 0,\ d(J^*_\rho \rho) = 0\} \to \{J : \mathfrak{g} \to \mathfrak{g} \mid J^2 = -\mathrm{Id}_V,\ \exists \Psi \in \wedge^{3,0}\mathfrak{g}^* \text{ closed}\}.$$

The closed (3,0)-form is given by $\Psi = \rho + iJ^*_\rho\rho$. The next result provides an equivalent condition to determine the existence of such complex structures on $\mathfrak{g}$.

Lemma 2.2.8. *Let $\mathfrak{g}$ be a Lie algebra and ν a volume form on $\mathfrak{g}$. Then, $\mathfrak{g}$ admits an almost-complex structure with a non-zero closed $(3,0)$-form if and only if there exists $\rho \in Z^3(\mathfrak{g})$ such that the endomorphism $\tilde{J}^*_\rho : \mathfrak{g}^* \to \mathfrak{g}^*$ defined by*

$$\Big((\tilde{J}^*_\rho \alpha)(X)\Big)\, \nu = \alpha \wedge \iota_X \rho \wedge \rho, \tag{2.20}$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, satisfies that $\tilde{J}^*_\rho \rho$ is closed and $\mathrm{tr}(\tilde{J}^{*2}_\rho) < 0$.

Proof. Let $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an almost-complex structure admitting a non-zero (3,0)-form $\Psi = \Psi_+ + i\Psi_-$ which is closed. Let $\rho = \Psi_+$. Then, $\lambda(\rho) < 0$, $J = J_\rho$ is determined by (2.19) and the form $J^*_\rho \rho = \Psi_-$ is closed. Since the associated form $\phi(\rho)$ is a volume form on $\mathfrak{g}$, we have that $\nu = s\,\phi(\rho)$ for some $s \neq 0$. Now, for the endomorphism $\tilde{J}^*_\rho : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ given by (2.20) we get

$$s\Big((\tilde{J}^*_\rho \alpha)(X)\Big)\,\phi(\rho) = \Big((\tilde{J}^*_\rho \alpha)(X)\Big)\,\nu = \alpha \wedge \iota_X \rho \wedge \rho = \big((J^*_\rho \alpha)(X)\big)\,\phi(\rho),$$

for any $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. This implies that $J^*_\rho = s\tilde{J}^*_\rho$. Therefore, $\mathrm{tr}(\tilde{J}^{*2}_\rho) < 0$ if and only if $\mathrm{tr}(J^{*2}_\rho) < 0$, and moreover, $d(\tilde{J}^*_\rho \rho) = 0$ if and only if $d(J^*_\rho \rho) = 0$. □

As a consequence of Lemma 2.2.3, Theorem 2.1.25 and Corollary 2.1.17 we concentrate on unimodular (non-nilpotent) solvable Lie algebras with $b_3(\mathfrak{g}) \geq 2$. The complete lists of the Lie algebras used to obtain the main result of this section can be found in the Appendix B.

Examples

For the computation of the endomorphism $\tilde{J}^*_\rho$ we use the simplest volume form $\nu = e^{123456}$, where $\{e^1, \ldots, e^6\}$ is the basis of $\mathfrak{g}^*$ in which the Lie algebra is expressed. The next three concrete examples show how we proceed in general in the proofs of Propositions 2.2.12 and 2.2.13 below in order to exclude candidates.

Example 2.2.9. The classification of nilpotent Lie algebras admitting integrable complex structures obtained by Salamon [82] is recovered using this method. For instance the Lie algebra $\mathfrak{h}_8 = (0,0,0,0,0,12)$ admits up to isomorphism a unique complex structure given by $d\omega^1 = d\omega^2 = 0$, $d\omega^3 = \omega^{1\bar{1}}$ (see Table 3.1). The almost-complex structure defined by the $(1,0)$-basis $\{\omega^1 = e^1 - ie^2, \omega^2 = e^3 - ie^4, \omega^3 = e^5 + 2ie^6\}$ satisfies that the complex $(3,0)$-form $\Psi = \omega^{123}$ is closed.

Example 2.2.10. Let us consider the indecomposable solvable Lie algebra $\mathfrak{g} = A_{6,25}^{0,-1} = (e^{23}, e^{26}, -e^{36}, 0, e^{46}, 0)$. Any $\rho \in Z^3(\mathfrak{g})$ is given by

$$\begin{aligned}\rho &= a_1e^{123} + a_2e^{126} + a_3e^{136} + a_4e^{234} + a_5(e^{235} - e^{146}) + a_6e^{236} + a_7e^{246} \\ &\quad + a_8e^{256} + a_9e^{346} + a_{10}e^{356} + a_{11}e^{456},\end{aligned}$$

for $a_1, \ldots, a_{11} \in \mathbb{R}$. Let $\tilde{J}_\rho^*$ be the endomorphism given by (2.20). A direct calculation shows that

$$\mathrm{tr}(\tilde{J}_\rho^{*2}) = 6(a_5^2 - a_1a_{11})^2 \geq 0.$$

In this case it is not worth evaluating the closedness of $\tilde{J}_\rho^*\rho$ because by Lemma 2.2.8 there is no almost-complex structure J_ρ^* coming from a closed 3-form $\rho \in Z^3(\mathfrak{g})$ and in particular $\mathfrak{g}$ does not admit a closed complex volume form.

Example 2.2.11. Let us consider the $5 \oplus 1$ decomposable solvable Lie algebra $\mathfrak{g} = A_{5,15}^{-1} \oplus \mathbb{R} = (e^{15} + e^{25}, e^{25}, -e^{35} + e^{45}, -e^{45}, 0, 0)$. Any $\rho \in Z^3(\mathfrak{g})$ is given by

$$\begin{aligned}\rho &= a_1e^{125} + a_2e^{135} + a_3e^{145} + a_4e^{156} + a_5e^{235} + a_6(e^{236} - e^{146}) + a_7e^{245} \\ &\quad + a_8e^{246} + a_9e^{256} + a_{10}e^{345} + a_{11}e^{356} + a_{12}e^{456},\end{aligned}$$

for $a_1, \ldots, a_{12} \in \mathbb{R}$. Let $\tilde{J}_\rho^*$ be the endomorphism given by (2.20). Then, we have

$$\begin{aligned} d(\tilde{J}_\rho^*\rho) &= 2a_6^2\left(2a_1e^{1256} + a_2(e^{1456} + e^{2356}) + (a_3 + a_5)e^{2456} - 2a_{10}e^{3456}\right), \\ \tfrac{1}{6}\mathrm{tr}(\tilde{J}_\rho^{*2}) &= (a_3 + a_5)^2a_6^2 + 4(a_1a_{10} - a_2a_7)a_6^2 - 2(a_3 - a_5)a_2a_6a_8 + a_2^2a_8^2. \end{aligned}$$

Since the form $\tilde{J}_\rho^*\rho$ must be closed, we distinguish two cases depending on the vanishing of the coefficient a_6. If $a_6 = 0$ then $\mathrm{tr}(\tilde{J}_\rho^{*2}) = 6(a_2a_8)^2 \geq 0$, and if $a_6 \neq 0$ then $a_1 = a_2 = a_3 + a_5 = a_{10} = 0$ and so $\mathrm{tr}(\tilde{J}_\rho^{*2}) = 0$. Consequently, Lemma 2.2.8 assures that there is no almost-complex structure J_ρ^* admitting a non-zero closed (3,0)-form.

2.2.3 The classification

We start the classification problem of finding the solvable and unimodular six-dimensional Lie algebras $\mathfrak{g}$ with $b_3(\mathfrak{g}) \geq 2$ admitting an almost-complex structure with closed complex $(3,0)$-form. Actually, we answer the equivalent question underlying Lemma 2.2.8. Firstly we consider the study in decomposable Lie algebras and then in the indecomposable ones.

The decomposable case

We tackle the classification problem first in the decomposable case. Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$ be a decomposable unimodular solvable six-dimensional real Lie algebra. The unimodularity and solvability of $\mathfrak{g}$ and Lemma 2.2.3 imply restrictions on the factors. In fact, $\mathfrak{g}$ is unimodular, resp. solvable, if and only if $\mathfrak{b}$ and $\mathfrak{c}$ are unimodular, resp. solvable. Moreover, by Lemma 2.2.3 and the well-known formula relating the cohomology of $\mathfrak{g}$ with the cohomologies of the factors, we have

$$b_3(\mathfrak{b})b_0(\mathfrak{c}) + b_2(\mathfrak{b})b_1(\mathfrak{c}) + b_1(\mathfrak{b})b_2(\mathfrak{c}) + b_0(\mathfrak{b})b_3(\mathfrak{c}) = b_3(\mathfrak{g}) \geq 2. \tag{2.21}$$

Proposition 2.2.12. *Let $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}$ be a six-dimensional decomposable unimodular (non-nilpotent) solvable Lie algebra admitting a complex structure with a non-zero closed $(3,0)$-form. Then, $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$, $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ or $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$.*

Proof. Since $\mathfrak{g}$ is decomposable, we divide the proof in the three cases $3 \oplus 3$, $4 \oplus 2$ and $5 \oplus 1$. In the $3 \oplus 3$ case the inequality (2.21) is always satisfied. The $3 \oplus 3$ decomposable unimodular (non nilpotent) solvable Lie algebras are $\mathfrak{e}(2) \oplus \mathfrak{e}(2)$, $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(2) \oplus \mathfrak{h}_3$, $\mathfrak{e}(2) \oplus \mathbb{R}^3$, $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$, $\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$ and $\mathfrak{e}(1,1) \oplus \mathbb{R}^3$ (see Table B.1 in the Appendix B for a description of the Lie algebras). An explicit computation shows that there is no $\rho \in Z^3$ satisfying the conditions $\lambda(\rho) < 0$ and $d(J_\rho^* \rho) = 0$, except for $\mathfrak{g} = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$. We give an example of a closed complex volume form for $\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$ in Appendix B, Table B.1.

Since $\mathbb{R}^2$ is the only 2-dimensional unimodular Lie algebra, the $4 \oplus 2$ case is reduced to the study of $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}^2$ for any 4-dimensional unimodular (non nilpotent) solvable Lie algebra $\mathfrak{b}$ satisfying (2.21), i.e. $b_3(\mathfrak{b}) + 2b_2(\mathfrak{b}) + b_1(\mathfrak{b}) \geq 2$. The resulting Lie algebras are: $A_{4,2}^{-2} \oplus \mathbb{R}^2$, $A_{4,5}^{\alpha,-1-\alpha} \oplus \mathbb{R}^2$ with $-1 < \alpha \leq -\frac{1}{2}$, $A_{4,6}^{\alpha,-\frac{\alpha}{2}} \oplus \mathbb{R}^2$, $A_{4,8} \oplus \mathbb{R}^2$ and $A_{4,10} \oplus \mathbb{R}^2$ (see Table B.1). However, all of them satisfy $\lambda(\rho) \geq 0$ for any $\rho \in Z^3(\mathfrak{g})$.

Finally, the $5 \oplus 1$ case consists of Lie algebras of the form $\mathfrak{g} = \mathfrak{b} \oplus \mathbb{R}$ for any 5-dimensional unimodular (non-nilpotent) solvable Lie algebra $\mathfrak{b}$ such that $(b_2(\mathfrak{b}), b_3(\mathfrak{b})) \neq (0,0), (1,0), (0,1)$. Therefore, the Lie algebras are: $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$, $A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ with $0 < \beta < 1$, $A_{5,8}^{-1} \oplus \mathbb{R}$, $A_{5,9}^{-1,-1} \oplus \mathbb{R}$, $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ with $\gamma > 0$, $A_{5,14}^{0} \oplus \mathbb{R}$, $A_{5,15}^{-1} \oplus \mathbb{R}$, $A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$ with $0 < \gamma < 1$, $A_{5,17}^{\alpha,-\alpha,1}$ with $\alpha \geq 0$, $A_{5,18}^{0} \oplus \mathbb{R}$, $A_{5,19}^{-1,2} \oplus \mathbb{R}$, $A_{5,19}^{1,-2} \oplus \mathbb{R}$, $A_{5,20}^{0} \oplus \mathbb{R}$, $A_{5,26}^{0,\pm 1} \oplus \mathbb{R}$, $A_{5,33}^{-1,-1} \oplus \mathbb{R}$ and $A_{5,35}^{0,-2} \oplus \mathbb{R}$. The explicit computation of each case allows us to distinguish the following three situations:

- If $\mathfrak{g} = A_{5,9}^{-1,-1} \oplus \mathbb{R}$ or $A_{5,26}^{0,\pm 1} \oplus \mathbb{R}$, then $\lambda(\rho) \geq 0$ for all $\rho \in Z^3(\mathfrak{g})$.
- The Lie algebras $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$ admit closed complex volume forms (see Table B.1 for a concrete example).
- For the rest of Lie algebras there is no $\rho \in Z^3(\mathfrak{g})$ satisfying $d(J_\rho^* \rho) = 0$ and $\lambda(\rho) < 0$ simultaneously.

In conclusion, in the $5 \oplus 1$ case the only possibilities are $A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and the family $A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$. □

The indecomposable case

Next we obtain the classification when $\mathfrak{g}$ is indecomposable.

Proposition 2.2.13. *Let $\mathfrak{g}$ be a six-dimensional indecomposable unimodular (non nilpotent) solvable Lie algebra admitting a complex structure with a non-zero closed $(3,0)$-form. Then, $\mathfrak{g}$ is isomorphic to $N_{6,18}^{0,-1,-1}$, $A_{6,37}^{0,0,1}$, $A_{6,82}^{0,1,1}$, $A_{6,88}^{0,0,1}$, $B_{6,4}^{1}$ or $B_{6,6}^{1}$.*

Proof. The Lie algebras $\mathfrak{g}$ such that $b_3(\mathfrak{g}) \geq 2$ are listed in Table B.2 of the Appendix B. The indecomposable case is long to analyse because of the amount of Lie algebras, but after performing the computations we distinguish the following three situations:

- Let $\mathfrak{g}$ be one of the following Lie algebras: $A_{6,13}^{a,-2a,2a-1}$ ($a \in \mathbb{R} - \{-1, 0, \frac{1}{3}, \frac{1}{2}\}$), $A_{6,13}^{a,-a,-1}$ ($a > 0$, $a \neq 1$), $A_{6,14}^{\frac{1}{3},-\frac{2}{3}}$, $A_{6,18}^{a,b}$ with $(a,b) \in \{(-\frac{1}{2}, -2), (-2, 1)\}$, $A_{6,25}^{a,b}$ with $(a,b) \in \{(0,-1), (-\frac{1}{2}, -\frac{1}{2})\}$, $A_{6,32}^{0,b,-b}$ ($b > 0$), $A_{6,34}^{0,0,\epsilon}$ ($\epsilon = 0, 1$), $A_{6,35}^{a,b,c}$ with $a > 0$ and $(b,c) \in \{(-2a, a), (-a, 0)\}$ and $A_{6,37}^{0,0,c}$ ($c > 0$, $c \neq 1$). Then, $\lambda(\rho) \geq 0$ for any $\rho \in Z^3(\mathfrak{g})$.
- The Lie algebras $N_{6,18}^{0,-1,-1}, A_{6,37}^{0,0,1}, A_{6,82}^{0,1,1}, A_{6,88}^{0,0,1}, B_{6,4}^{1}$ and $B_{6,6}^{1}$ admit non-zero closed (3,0)-forms (see Table B.2 for a concrete example).
- For the rest of Lie algebras there is no $\rho \in Z^3(\mathfrak{g})$ such that $d(J_\rho^* \rho) = 0$ and $\lambda(\rho) < 0$. □

Finally, Propositions 2.2.12 and 2.2.13 provide the final classification theorem:

Theorem 2.2.14. *Let $\mathfrak{g}$ be an unimodular (non-nilpotent) solvable Lie algebra of dimension 6. Then, $\mathfrak{g}$ admits a complex structure with a non-zero closed $(3,0)$-form if and only if it is isomorphic to one in the following list:*

$$\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0),$$

$$\mathfrak{g}_2^{\alpha} = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0),\ \alpha \geq 0,$$

$$\mathfrak{g}_3 = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45}),$$

$$\mathfrak{g}_4 = A_{6,37}^{0,0,1} = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$

$$\mathfrak{g}_5 = A_{6,82}^{0,1,1} = (e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0),$$

$$\mathfrak{g}_6 = A_{6,88}^{0,0,1} = (e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$

$$\mathfrak{g}_7 = B_{6,6}^{1} = (e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0),$$

$$\mathfrak{g}_8 = N_{6,18}^{0,-1,-1} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0),$$

$$\mathfrak{g}_9 = B_{6,4}^{1} = (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0).$$

Remark 2.2.15. Only the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_5$ are completely solvable.

2.3 Existence of Lattices

This section deals with the existence of lattices on solvable Lie groups. Good references for this subject are Corwin and Greenleaf [25] and Raghunathan [77].

It is not trivial in general to know whether a Lie group G admits a lattice or not. In the context of nilpotent Lie groups the answer to this question is related with some specific form of the structure equations as it is stated in following well-known theorem obtained by Malcev [61]. A Lie algebra $\mathfrak{g}$ is said to admit a *rational structure* if there is a rational Lie subalgebra $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. Equivalently, $\mathfrak{g}$ has a rational structure if and only if there is an $\mathbb{R}$-basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ having rational structure constants, and then $\mathfrak{g}_{\mathbb{Q}} := \mathbb{Q}\langle X_1, \ldots, X_n\rangle$ provides a rational structure such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. Recall that given a real n-dimensional vector space V, a lattice $\mathcal{L}$ of maximal rank of V is a free abelian group $\mathcal{L} = \mathbb{Z}\langle X_1, \ldots, X_n\rangle$ where $\{X_1, \ldots, X_n\}$ is a $\mathbb{R}$-basis of V.

Theorem 2.3.1 (Malcev [61, Theorem 7],[77, Theorem 2.12])**.** *Let N be a simply connected nilpotent Lie group and let $\mathfrak{n}$ be its Lie algebra. Then N admits a lattice if and only if $\mathfrak{n}$ admits a rational structure. Moreover:*

1. *if $\mathcal{L}$ is a lattice of maximal rank in $\mathfrak{n}$ contained in $\mathfrak{n}_{\mathbb{Q}}$ then the group generated by $\exp^N \mathcal{L}$ is a lattice in N.*
2. *if Γ is a lattice in N then $\mathcal{L} := \mathbb{Z}\langle \log^N \Gamma\rangle$ is a lattice of maximal rank in the underlying vector space of $\mathfrak{n}$ such that the structure constants with respect to any basis contained in $\mathcal{L}$ are rational.*

Remark 2.3.2. Malcev's Theorem states a simple criterion to decide if a given connected and simply connected nilpotent Lie group admits a lattice or not. In particular, it it immediate to check that all the six-dimensional nilpotent Lie algebras obtained in Theorem 2.1.25 give rise to nilmanifolds.

The situation changes when we consider a solvable and non-nilpotent Lie group G. In general it is not easy to decide whether the Lie group admits a lattice or not, but it is possible to find such lattices in some cases. However, most of low-dimensional solvable Lie groups are *almost-nilpotent* and sometimes it is possible to construct one lattice for them. Roughly speaking, *almost-nilpotent* Lie groups are those having nilradical of codimension 1 (recall that the *nilradical* of a Lie group G is the maximal nilpotent Lie subgroup of G).

Definition 2.3.3. *A Lie group G is* almost-nilpotent *if it can be written as $G = \mathbb{R} \ltimes_{\mu} N$ where N is the nilradical of G and $\mu\colon \mathbb{R} \to Aut(N)$ is a one-parameter subgroup of $Aut(N)$. The underlying Lie algebra is $\mathfrak{g} = \mathbb{R} \ltimes_{(\mu(t)_*)_e} \mathfrak{n}$. When the nilradical is abelian, namely $N = \mathbb{R}^n$, then G is called* almost-abelian.

Recall that given a Lie algebra $\mathfrak{g}$, the space of derivations of $\mathfrak{g}$ is the subspace of linear maps of $\mathfrak{g}$ satisfying the Leibnitz rule $\mathfrak{d}(\mathfrak{g}) := \{f\colon \mathfrak{g} \to \mathfrak{g} \mid f([X,Y]) = [f(X),Y] + [X, f(Y)]\} \subset \mathfrak{gl}(\mathfrak{g})$. It is shown in [13] that for every $t \in \mathbb{R}$ is possible to recover the automorphisms $\mu(t)$ of N by means of the derivations of the Lie algebra $\mathfrak{d}(\mathfrak{n})$:

$$\mu(t) = \exp^N \circ \exp^{\mathrm{Aut}(|\mathfrak{n}|)}(t\varphi) \circ \log^N, \quad \forall t \in \mathbb{R}, \quad \varphi \in \mathfrak{d}(\mathfrak{n})$$

and $(\mu(t)_*)_e = \exp^{\mathrm{Aut}(|\mathfrak{n}|)}(t\varphi)$, where e is the identity element of N and $\mathrm{Aut}(|\mathfrak{n}|)$ is the Lie group of automorphisms of the vector space underlying the Lie algebra $\mathfrak{n}$. Recall that if N is abelian then the exponential $\exp^N\colon \mathfrak{n} \to N$ is the identity map. The following result allows to construct a lattice in the case that G is almost-nilpotent.

Lemma 2.3.4 (Bock [13, Chapter 2])**.** *Let $G = \mathbb{R} \ltimes_\mu N$ be an $(n+1)$-dimensional almost-nilpotent Lie group with nilradical N and $\mathfrak{n}$ the Lie algebra of N. If there exists $t_1 \neq 0$ and a rational basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{n}$ such that the coordinate matrix of $d_e(\mu(t_1))$ in such a basis is integer, then $\Gamma = t_1\mathbb{Z} \ltimes_\mu exp^N(\mathbb{Z}\langle X_1, \ldots, X_n\rangle)$ is a lattice of G.*

In this section we show that the simply-connected solvable Lie groups G_k corresponding to the Lie algebras $\mathfrak{g}_k$ in Theorem 2.2.14 admit lattices Γ_k of maximal rank. Therefore, we get compact complex solvmanifolds G_k/Γ_k with holomorphically trivial canonical bundle. Recall that the Lie algebra $\mathfrak{g}_8$ is the underlying real Lie algebra of the Nakamura manifold. Although $\mathfrak{g}_8$ has four-dimensional nilradical and hence is not almost-nilpotent, Yamada [102] shows that it admits a lattice. The rest of Lie algebras are either almost-nilpotent or direct sum of an almost-nilpotent Lie algebra with another one, therefore we can apply the techniques explained previously.

Proposition 2.3.5. *For any $k \neq 2$, the connected and simply-connected Lie group G_k with underlying Lie algebra $\mathfrak{g}_k$ admits a lattice.*

For $k = 2$, there exists a countable number of distinct α's, including $\alpha = 0$, for which the connected and simply-connected Lie group with underlying Lie algebra $\mathfrak{g}_2^\alpha$ admits a lattice.

Proof. The Lie algebra $\mathfrak{g}_8$ is not almost-nilpotent, but its corresponding connected and simply-connected Lie group G_8 admits a lattice by [102]. It is not hard to see that for $k \neq 8$ the Lie algebra $\mathfrak{g}_k$ of Theorem 2.2.14 is either almost-nilpotent or a product of almost-nilpotent Lie algebras. In fact, we find the following correspondence with some of the Lie algebras studied in [13] (we use the notation in that paper in order to compare directly with the Lie algebras therein): $\mathfrak{g}_1 \cong \mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$, $\mathfrak{g}_2^0 \cong \mathfrak{g}_{5,17}^{0,0,1} \oplus \mathbb{R}$, $\mathfrak{g}_3 \cong \mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,4}^{-1}$, $\mathfrak{g}_4 \cong \mathfrak{g}_{6,37}^{0,0,-1}$, $\mathfrak{g}_5 \cong \mathfrak{g}_{6,88}^{0,-1,0}$, $\mathfrak{g}_6 \cong \mathfrak{g}_{6,92}^{0,-1,-1}$ and $\mathfrak{g}_7 \cong \mathfrak{g}_{6,92}^{*}$. For these cases, the existence of lattices in the corresponding Lie groups is already proved in [13]. So, it remains to study $\mathfrak{g}_2^\alpha$ with $\alpha > 0$, and $\mathfrak{g}_9$.

We show first that there exists a countable subfamily of $\mathfrak{g}_2^\alpha$ with $\alpha > 0$ whose corresponding Lie group G_2^α admits lattice. The 5-dimensional factor $A_{5,17}^{\alpha,-\alpha,1}$ in the decomposable Lie algebra $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ is given by

$$[e_1, e_5] = -\alpha e_1 + e_2,\ [e_2, e_5] = -e_1 - \alpha e_2,\ [e_3, e_5] = \alpha e_3 + e_4,\ [e_4, e_5] = -e_3 + \alpha e_4,$$

which is almost abelian since $A_{5,17}^{\alpha,-\alpha,1} = \mathbb{R} \ltimes_{\mathrm{ad}_{e_5}} \mathbb{R}^4$. If we denote by B_α the coordinate matrix of the derivation $\mathrm{ad}_{e_5}\colon \mathbb{R}^4 \to \mathbb{R}^4$ in the basis $\{e_1, e_2, e_3, e_4\}$, then the coordinate matrix of $d_e(\mu(t))$ is the exponential

$$e^{tB_\alpha} = \begin{pmatrix} e^{\alpha t}\cos(t) & e^{\alpha t}\sin(t) & 0 & 0 \\ -e^{\alpha t}\sin(t) & e^{\alpha t}\cos(t) & 0 & 0 \\ 0 & 0 & e^{-\alpha t}\cos(t) & e^{-\alpha t}\sin(t) \\ 0 & 0 & -e^{-\alpha t}\sin(t) & e^{-\alpha t}\cos(t) \end{pmatrix}.$$

If $t_l = l\pi$ with $l \in \mathbb{Z}$ and $l > 0$, then the characteristic polynomial of the matrix $e^{t_l B_\alpha}$ is $p(\lambda) = (1 - (-1)^l(e^{\alpha t_l} + e^{-\alpha t_l})\lambda + \lambda^2)^2$, which is integer if $\alpha_{l,m} = \frac{1}{l\pi}\log(\frac{m+\sqrt{m^2-4}}{2})$ with $m \in \mathbb{Z}$ and $m > 2$. Moreover, $e^{t_l B_\alpha} = P^{-1}C_{l,m}P$, where

$$P^{-1} = \begin{pmatrix} 0 & 0 & \epsilon & \beta^+ \\ \epsilon & \beta^+ & 0 & 0 \\ 0 & 0 & -\epsilon & \beta^- \\ -\epsilon & \beta^- & 0 & 0 \end{pmatrix}, \qquad C_{l,m} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & m(-1)^l & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & m(-1)^l \end{pmatrix},$$

with $\epsilon = \frac{1}{\sqrt{m^2-4}}$ and $\beta^\pm = \frac{m^2-4\pm(-1)^l\sqrt{m^2-4}}{2(m^2-4)}$. Taking the basis

$$X_1 = \epsilon(e_2 - e_4), \quad X_2 = \beta^+e_2 + \beta^-e_4, \quad X_3 = \epsilon(e_1 + e_3), \quad X_4 = \beta^+e_1 + \beta^-e_3,$$

of $\mathbb{R}^4$ and using Lemma 2.3.4 we have that $\Gamma' = l\pi\mathbb{Z} \ltimes_\mu \mathbb{Z}\langle X_1, \ldots, X_4\rangle$ is a lattice of the simply-connected Lie group associated to $A_{5,17}^{\alpha,-\alpha,1}$ with $\alpha = \alpha_{l,m}$. Hence, $\Gamma = \Gamma' \times \mathbb{Z}$ is a lattice in $G_2^{\alpha_{l,m}}$.

The Lie algebra $\mathfrak{g}_9$ can be seen as an almost-nilpotent Lie algebra $\mathfrak{g} = \mathbb{R} \ltimes_{\mathrm{ad}_{e_6}} \mathfrak{h}$, where $\mathfrak{h} = \langle e_1, \ldots, e_5 \,|\, [e_1, e_4] = -e_3,\ [e_1, e_5] = -e_2,\ [e_4, e_5] = -e_1\rangle$ is a 5-dimensional nilpotent Lie algebra. Proceeding in a similar manner as for $\mathfrak{g}_2^\alpha$ and denoting by B the coordinate matrix of the derivation $\mathrm{ad}_{e_6}\colon \mathfrak{h} \to \mathfrak{h}$ in the basis $\{e_1, \ldots, e_5\}$ then the coordinate matrix of $d_e(\mu(t))$ is the exponential e^{tB}:

$$e^{tB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & \sin(t) & \frac{1}{2}(-t\cos(t) + \sin(t)) & \frac{t}{2}\sin(t) \\ 0 & -\sin(t) & \cos(t) & \frac{t}{2}\sin(t) & \frac{1}{2}(t\cos(t) + \sin(t)) \\ 0 & 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}.$$

Hence, we compute the characteristic polynomial of $d_e(\mu(t))$ getting that $p(\lambda) = (\lambda^2 - 2\lambda\cos(t) + 1)^2$. If $t_1 = \pi$ then $p(\lambda) \in \mathbb{Z}[\lambda]$ and the coordinate matrix of $d_e(\mu(t_1))$ in the basis $\{X_1 = \frac{\pi}{2}e_1, X_2 = \sqrt{\frac{\pi}{2}}e_4, X_3 = \sqrt{\frac{\pi}{2}}e_5, X_4 = (\frac{\pi}{2})^{3/2}e_2 + \sqrt{\frac{\pi}{2}}e_4, X_5 = -(\frac{\pi}{2})^{3/2}e_3 + \sqrt{\frac{\pi}{2}}e_5\}$ of $\mathfrak{h}$ is

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Moreover, $\{X_1, \ldots, X_5\}$ is a rational basis of $\mathfrak{h}$ because $[X_1, X_2] = [X_1, X_4] = -X_3 + X_5$, $[X_1, X_3] = [X_1, X_5] = X_2 - X_4$, $[X_2, X_3] = [X_2, X_5] = -[X_3, X_4] = -X_1$. Hence, if we denote by H the simply-connected Lie group corresponding to $\mathfrak{h}$, then using Lemma 2.3.4 we have that $\Gamma = \pi\mathbb{Z} \ltimes_\mu \exp^H(\mathbb{Z}\langle X_1, \ldots, X_5\rangle)$ is a lattice in the Lie group G_9. □

Remark 2.3.6. Bock found a lattice for the Lie group associated to $A_2^{\alpha,-\alpha,1}$ with $\alpha = \alpha_{1,3} = \frac{1}{\pi}\log\frac{3+\sqrt{5}}{2}$, that is, for $l = 1$ and $m = 3$. Notice that our result for $k = 2$ is consistent with the result obtained by Witte in [100, Prop. 8.7], where it is shown that only countably many non-isomorphic simply-connected Lie groups admit a lattice, so that one cannot expect a lattice to exist for any real $\alpha > 0$.

The Lie algebra $\mathfrak{g}_9$ does not appear in [13]. Its nilradical is the 5-dimensional Lie algebra $\mathfrak{h}$, which is isomorphic to $\mathfrak{g}_{5,3}$ (in the notation of [13]), but there are only two solvable and unimodular Lie algebras with nilradical $\mathfrak{g}_{5,3}$ considered in that paper (namely $\mathfrak{g}_{6,76}^{-1}$ and $\mathfrak{g}_{6,78}$) which are both completely solvable, but $\mathfrak{g}_9$ is not.

We summarize the result of Salamon (Theorem 2.1.25) together with the results of Theorem 2.2.14, Proposition 2.3.5 and Proposition 2.1.31 in the following theorem:

Theorem 2.3.7. *Let $M = G/\Gamma$ be a six-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle, then the underlying Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}, \mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$ if $\mathfrak{g}$ is nilpotent or $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha\geq 0}, \mathfrak{g}_3 \ldots, \mathfrak{g}_8$ or $\mathfrak{g}_9$ if $\mathfrak{g}$ is not nilpotent.*

Chapter 3

Invariant complex structures on six-dimensional solvmanifolds

In this chapter we deal with the problem of obtaining a classification of invariant complex structures with holomorphically trivial canonical bundle on six-dimensional solvmanifolds. As we proved in the previous chapter, this problem is equivalent to classify the complex structures with a non-zero closed $(3,0)$-form on the underlying solvable Lie algebras. The chapter is divided in two sections, the first one devoted to nilpotent Lie algebras and the second to solvable Lie algebras. Since the invariant complex structures on nilmanifolds have always holomorphically trivial canonical bundle, the goal of Section 3.1 is to obtain a complete description of such structures on six-dimensional nilmanifolds with underlying real Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$, according to [82]. There have been several partial approaches to this problem. For instance, Andrada, Barberis and Dotti [4] obtain a classification of abelian complex structures on the class of six-dimensional Lie algebras. On the other hand, the non-nilpotent complex structures on six-dimensional nilpotent Lie algebras are classified by Ugarte and Villacampa [96]. Therefore, we study the non-abelian nilpotent complex structures for the nilpotent Lie algebras and the final classification of invariant complex structures on nilmanifolds is summarized in Table 3.1. On the other hand, Section 3.2 deals with the case of solvmanifolds endowed with an invariant complex structure with holomorphically trivial canonical bundle. We have seen in the previous chapter that the underlying real Lie algebras of such solvmanifolds are $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. By using the techniques developed by Hitchin considered in Section 2.2, we describe the whole space of complex structures with a closed $(3,0)$-form on these Lie algebras. The reduced expressions of the complex structures are finally summarized in Table 3.2. In order to lighten the exposition of this part, we have included some long computations in the Appendix A.

3.1 Complex structures on nilmanifolds

We have seen that any invariant complex structure on a compact manifold of the form $M = G/\Gamma$ can be referred to a complex structure on the Lie algebra $\mathfrak{g}$ of the group. Hence, it is natural to define a notion of isomorphic complex structures defined on the same Lie algebra.

Definition 3.1.1. *Let $\mathfrak{g}$ be a Lie algebra endowed with two complex structures J and J'. J and J' are said to be* equivalent *(or* isomorphic*) if there is an automorphism $F\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ of the Lie algebra, that is $F[\cdot,\cdot] = [F\cdot, F\cdot]$, such that $F \circ J = J' \circ F$.*

Remark 3.1.2. It is proved that if $J, J'\colon \mathfrak{g} \to \mathfrak{g}$ are two almost-complex structures on $\mathfrak{g}$ they are equivalent if and only if there exists a linear isomorphism $G\colon (\mathfrak{g}^*)^{1,0}_J \longrightarrow (\mathfrak{g}^*)^{1,0}_{J'}$ commuting with the Chevalley-Eilenberg differential, namely $d \circ G = G \circ d$, where $(\mathfrak{g}^*)^{1,0}_J$ and $(\mathfrak{g}^*)^{1,0}_{J'}$ denote the $(1,0)$-subspaces of $\mathfrak{g}^*_{\mathbb{C}}$ associated to J and J' respectively.

We recall that the abelian [4] and the non-nilpotent [96] complex structures on nilpotent Lie algebras of dimension 6 have already been classified. Hence, we start studying the remaining case, that is, the non-abelian nilpotent complex structures.

3.1.1 The non-abelian nilpotent case

Now we start with the study of the class of non-abelian nilpotent complex structures on six-dimensional nilpotent Lie algebras. In order to provide such classification, our starting point is the following result:

Proposition 3.1.3 (Ugarte [95, Proposition 2])**.** *Let J be a nilpotent complex structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension 6. There is a basis $\{\omega^j\}_{j=1}^3$ for $(\mathfrak{g}^*)^{1,0}$ satisfying*

$$\left\{\begin{aligned} d\omega^1 &= 0, \\ d\omega^2 &= \epsilon\,\omega^{1\bar 1}, \\ d\omega^3 &= \rho\,\omega^{12} + (1-\epsilon)A\,\omega^{1\bar 1} + B\,\omega^{1\bar 2} + C\,\omega^{2\bar 1} + (1-\epsilon)D\,\omega^{2\bar 2}, \end{aligned}\right. \tag{3.1}$$

where $A, B, C, D \in \mathbb{C}$, and $\epsilon, \rho \in \{0, 1\}$.

We recall that by ω^{jk} (resp. $\omega^{j\bar k}$) we mean the wedge product $\omega^j \wedge \omega^k$ (resp. $\omega^j \wedge \omega^{\bar k}$), where $\omega^{\bar k}$ indicates the complex conjugation of ω^k. From now on, we shall use a similar abbreviated notation for "basic" forms of arbitrary bidegree.

Remark 3.1.4. It is worth noticing that equations (3.1) above include the abelian complex structures as those for which $\rho = 0$. On the other hand, the complex parallelizable structures correspond to $\epsilon = 0$ and $A = B = C = D = 0$, and the possible Lie algebras are $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$), that is, a complex torus and the Iwasawa manifold respectively.

Now we start the classification up to equivalence of non-abelian nilpotent complex structures. We divide the study, according to the step of nilpotency of the Lie algebra, in 2-step and 3-step cases.

2-step nilpotent Lie algebras

Let us start with non-abelian nilpotent complex structures on 2-step NLAs $\mathfrak{g}$ of dimension 6. Such a Lie algebra has first Betti number at least 3, and if it is equal to 3 then necessarily the coefficient ϵ in (3.1) is non-zero. We consider firstly the case $\epsilon = 0$, that is, the Lie algebra has first Betti number ≥ 4. We finish this section considering $\epsilon = 1$.

The following proposition provides a further reduction of the equations (3.1) when $\epsilon = 0$.

Proposition 3.1.5. *Let J be a complex structure on a 2-step nilpotent Lie algebra $\mathfrak{g}$ of dimension 6 with first Betti number ≥ 4. If J is not complex-parallelizable, then there is a basis $\{\omega^j\}_{j=1}^3$ of $(\mathfrak{g}^*)^{1,0}$ such that*

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \tag{3.2}$$

where $D \in \mathbb{C}$ with $\mathfrak{Im}\,D \geq 0$ and $\lambda \in \mathbb{R}^{\geq 0}$. Moreover, if we denote $x = \mathfrak{Re}\,D$ and $y = \mathfrak{Im}\,D$, then:

(i) *If $\lambda = \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to*

- (i.1) $\mathfrak{h}_2$, *for $y \neq 0$;*
- (i.2) $\mathfrak{h}_3$, *for $\rho = y = 0$ and $x \neq 0$;*
- (i.3) $\mathfrak{h}_4$, *for $\rho = 1$, $y = 0$ and $x \neq 0$;*
- (i.4) $\mathfrak{h}_6$, *for $\rho = 1$ and $x = y = 0$;*
- (i.5) $\mathfrak{h}_8$, *for $\rho = x = y = 0$.*

(ii) *If $\lambda \neq \rho$, then the Lie algebra $\mathfrak{g}$ is isomorphic to*

- (ii.1) $\mathfrak{h}_2$, *for $4y^2 > (\rho - \lambda^2)(4x + \rho - \lambda^2)$;*
- (ii.2) $\mathfrak{h}_4$, *for $4y^2 = (\rho - \lambda^2)(4x + \rho - \lambda^2)$;*
- (ii.3) $\mathfrak{h}_5$, *for $4y^2 < (\rho - \lambda^2)(4x + \rho - \lambda^2)$.*

Proof. In [95, Lemma 11] it is proved that under these conditions there is a basis $\{\sigma^j\}_{j=1}^3$ for $(\mathfrak{g}^*)^{1,0}$ such that

$$d\sigma^1 = d\sigma^2 = 0, \; d\sigma^3 = \rho\,\sigma^{12} + \sigma^{1\bar{1}} + B\,\sigma^{1\bar{2}} + D\,\sigma^{2\bar{2}}, \tag{3.3}$$

where $B, D \in \mathbb{C}$ and $\rho \in \{0, 1\}$.

If $B \neq 0$ then we can take any non-zero solution z of the equation $\bar{z}\frac{B}{|B|} = z$, and the complex equations (3.3) reduce to (3.2) with $\lambda = |B|$ with respect to the new basis $\{\omega^1 = z\,\sigma^1, \omega^2 = \bar{z}\,\sigma^2, \omega^3 = |z|^2\,\sigma^3\}$.

Consider now $B = \lambda$ with $\lambda \in \mathbb{R}^{\geq 0}$ in (3.3). If $D \neq 0$, then with respect to the new basis $\{\omega^1 = -\bar{D}\,\sigma^2, \omega^2 = \sigma^1 + \lambda\,\sigma^2, \omega^3 = \bar{D}\,\sigma^3\}$ we arrive at (3.2) with $\bar{D}$ instead of D.

Finally, the second part of the proposition follows directly from [95, Proposition 13].

□

Remark 3.1.6. From Proposition 3.1.5 we have that on the Lie algebras $\mathfrak{h}_6$ or $\mathfrak{h}_8$ any two complex structures are equivalent (see Figures 3.1 and 3.2). On the other hand, the complex equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}} \pm \omega^{2\bar{2}}$$

define two non-equivalent complex structures on $\mathfrak{h}_3$, and any complex structure on $\mathfrak{h}_3$ is equivalent to one of them [95, Corollary 16]. More generally, for $\rho = 0$ the complex structures are abelian and the classification problem has been solved in [4] (see Corollary 3.1.23 for details in the 2-step case).

Figure 3.1: Complex structures satisfying (3.2) with $\rho = \lambda = 0$.

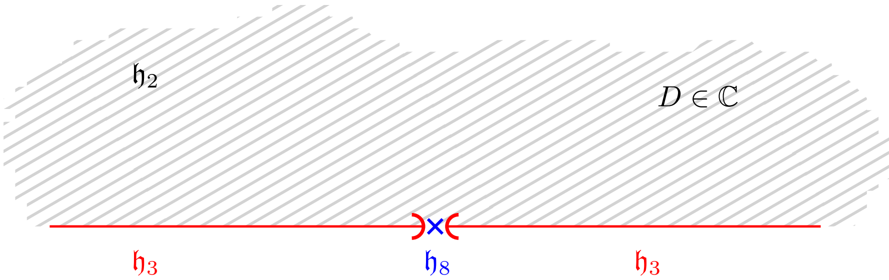
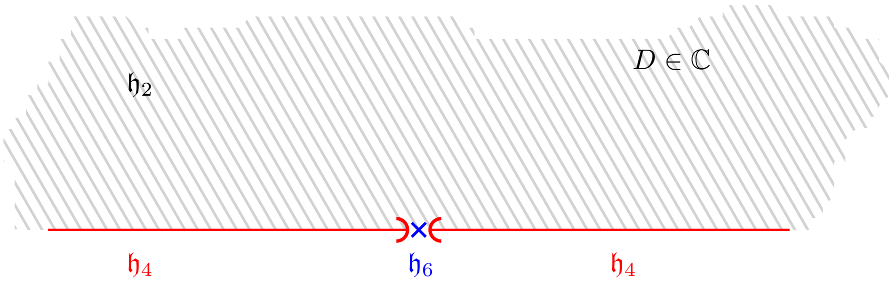


Figure 3.2: Complex structures satisfying (3.2) with $\rho = \lambda = 1$.



As a consequence, it remains to classify in the 2-step case the non-abelian complex structures on the Lie algebras $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$. From now on, we consider in this section that $\rho = 1$ and we use the notation $(1, \lambda, D)$ to refer to a Lie algebra with a complex structure admitting a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex equations (3.2) with parameters $\rho = 1$, $\lambda \geq 0$ and $D \in \mathbb{C}$ with $\mathfrak{Im}\, D > 0$.

We will say that two triples $(1, \lambda, D)$ and $(1, \lambda', D')$ are equivalent, denoted by $(1, \lambda, D) \sim (1, \lambda', D')$, if the corresponding structures J and J' are equivalent. So, the problem reduces to classify triples $(1, \lambda, D)$ up to equivalence.

Lemma 3.1.7. *Let us consider two triples $(1, \lambda, D)$ and $(1, t, E)$ as above.*

(i) *If $D = 0$ then, $(1, t, E) \sim (1, \lambda, 0)$ if and only if $t = \lambda$ and $E = 0$.*

(ii) *If $D \neq 0$ then, $(1,t,E) \sim (1,\lambda,D)$ if and only if there exist non-zero complex numbers e, f such that $E = De/\bar{e}$ and*

$$\left(\frac{|f|^2}{\bar{e}} - 1\right)(\bar{D}\bar{e} - De)^2 = (\lambda\bar{f} - tf)(\lambda\bar{D}\bar{e}f - tDe\bar{f}). \tag{3.4}$$

Proof. The structure equations corresponding to the triples $(1,\lambda,D)$ and $(1,t,E)$ are

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}},$$
$$d\sigma^1 = d\sigma^2 = 0, \quad d\sigma^3 = \sigma^{12} + \sigma^{1\bar{1}} + t\sigma^{1\bar{2}} + E\sigma^{2\bar{2}},$$

where $\lambda, t \geq 0$ and $\mathfrak{Im}\, D, \mathfrak{Im}\, E \geq 0$. Then $(1,t,E) \sim (1,\lambda,D)$ if and only if there exists an automorphism of the Lie algebra preserving the complex equations, i.e. there is $(m_{ij}) \in \mathrm{GL}(3,\mathbb{C})$ such that $\sigma^i = \sum_{j=1}^3 m_{ij}\,\omega^j$ and

$$d\sigma^i = \sum_{j=1}^{3} m_{ij}\, d\omega^j, \quad i = 1,2,3.$$

These conditions are equivalent to

$$\sigma^1 = a\,\omega^1 + b\,\omega^2, \quad \sigma^2 = c\,\omega^1 + f\,\omega^2, \quad \sigma^3 = m_{31}\,\omega^1 + m_{32}\,\omega^2 + e\,\omega^3,$$

and

$$\begin{cases} \text{(I)} & e = af - bc, \\ \text{(II)} & e = |a|^2 + t\,a\bar{c} + E|c|^2, \\ \text{(III)} & \lambda e = a\bar{b} + t\,a\bar{f} + Ec\bar{f}, \\ \text{(IV)} & 0 = \bar{a}b + t\,b\bar{c} + E\bar{c}f, \\ \text{(V)} & De = |b|^2 + t\,b\bar{f} + E|f|^2. \end{cases} \tag{3.5}$$

Notice that $m_{13} = m_{23} = 0$, $e \neq 0$ and the coefficients m_{31} and m_{32} are not relevant.

It is straightforward to see that coefficient f must be non-zero (otherwise $\lambda = t$ and $D = E$) and so we can express a as

$$a = \frac{e + bc}{f}.$$

First of all, let us suppose that $D = 0$. Replacing a in (IV) and using (V) we obtain that $b = 0$ and therefore $E = 0$ by equation (V). Combining (I) and (III) we get that $\lambda f = t\bar{f}$. Since λ and t are real non-negative numbers, we conclude that $\lambda = t$, i.e. $(1,\lambda,0)$ defines an equivalence class for every $\lambda \geq 0$. This completes the proof of (i).

We suppose next that $D \neq 0$. In order to solve (3.5) we transform it into an equivalent system by doing the following substitutions. Replacing a in equation (IV) and using (V) we can express

$$\bar{c} = -\frac{b\bar{e}}{De}.$$

Next, in (II) we can substitute a and c and use again (V) to obtain that

$$De = E\bar{e},$$

which implies in particular $|D| = |E|$. Notice that since $D \neq 0$ we can assume $E \neq \bar{D}$ by Proposition 3.1.5. Now, $\bar{c} = -b/E$. Proceeding in a similar way in equation (III) we get

$$\bar{b} = \frac{\lambda f - t\bar{f}}{1 - D/\bar{E}}.$$

Finally, using the expressions of a, b, c above, equation (V) is equivalent to (3.4). Therefore, given $e, f \in \mathbb{C} - \{0\}$ satisfying $De = E\bar{e}$ and (3.4), it is always possible to find $a, b, c \in \mathbb{C}$ such that system (3.5) is satisfied. □

Remark 3.1.8. As a consequence of Lemma 3.1.7 (ii), when $D \neq 0$ a necessary condition for $(1, t, E)$ to be equivalent to $(1, \lambda, D)$ is that $|D| = |E|$. Moreover, to find an equivalent complex structure $(1, t, E)$ it suffices to find $t \geq 0$ and $e, f \in \mathbb{C} - \{0\}$ satisfying (3.4), because E is necessarily given by $E = De/\bar{e}$.

Corollary 3.1.9. *Let $E \neq \bar{D}$. If $(1, t, E) \sim (1, \lambda, D)$ then, $t = \lambda$ if and only if $E = D$.*

Proof. By hypothesis D cannot be zero, so we are in case (ii) of Lemma 3.1.7. Suppose first that $\lambda = t$ in (3.4), i.e.

$$(\bar{D}\bar{e} - De)^2 \left(\frac{|f|^2}{\bar{e}} - 1 \right) = \lambda^2 (\bar{f} - f)(\bar{D}\bar{e}f - De\bar{f}).$$

The right hand side of the previous equality is a real number. If it is zero then $e = |f|^2$ (otherwise $De = \bar{D}\bar{e}$ would imply $E = \bar{D}$); thus, e is a real number and since $E = De/\bar{e}$ we conclude that $D = E$. On the other hand, if it is a non-zero real number, then $\frac{|f|^2}{\bar{e}} - 1$ must be a real number and then $e \in \mathbb{R}$ and again $D = E$.

Conversely, let us suppose that $E = D \neq 0$. In this case $e \in \mathbb{R}$ and by (3.4) we can express it as

$$e = |f|^2 - \frac{(\lambda\bar{f} - tf)(\lambda\bar{D}f - tD\bar{f})}{(\bar{D} - D)^2}.$$

Notice that by hypothesis $D \neq \bar{E} = \bar{D}$. To ensure that $e \in \mathbb{R}$ it must happen that $(\lambda\bar{f} - tf)(\lambda\bar{D}f - tD\bar{f}) \in \mathbb{R}$ or equivalently,

$$|f|^2(\lambda^2 - t^2)(\bar{D} - D) = 0.$$

As $f(\bar{D} - D) \neq 0$ the only possibility to solve the previous equation is $\lambda = t$. □

From the previous results it follows that it remains to consider the case when $D \neq 0$ and $\lambda \neq t$. The next lemma provides a simplification of equation (3.4).

Lemma 3.1.10. *Let us suppose that $\lambda \neq t$, $D = x + iy \neq 0$ and $e \in \mathbb{C} - \{0\}$. Then, $(1, \lambda, D) \sim (1, t, De/\bar{e})$ if and only if*

$$4y^2 - (t^2 - \lambda^2)(4x + t^2 - \lambda^2) \geq 0. \tag{3.6}$$

Proof. By Lemma 3.1.7 (ii), we know that $(1, \lambda, D) \sim (1, t, De/\bar{e})$ if and only if (3.4) is satisfied. This condition reads, with respect to $H = De$, as

$$(\bar{H} - H)^2 \left(\bar{D}|f|^2 - \bar{H}\right) = \bar{H}(\lambda \bar{f} - tf)(\lambda f \bar{H} - t\bar{f}H).$$

Taking real and imaginary parts in the expression above we obtain

$$\text{(3.7)} \qquad \begin{cases} 4H_2^2(H_1 - x|f|^2) = & |f|^2(t^2 - \lambda^2)H_2^2 + |f|^2(t^2 + \lambda^2)H_1^2 \\ & -2\lambda t(f_1^2 - f_2^2)H_1^2 - 4\lambda t H_1 H_2 f_1 f_2, \\ 4H_2^2(y|f|^2 - H_2) = & 2\lambda H_2 \left[tH_1(f_1^2 - f_2^2) + 2tH_2 f_1 f_2 - \lambda |f|^2 H_1\right], \end{cases}$$

where $H = H_1 + iH_2$ and $f = f_1 + if_2$. Observe that $H_2 \neq 0$, otherwise we get a contradiction using the first equation of (3.7).

Substituting the second equation of (3.7) in the first one and replacing H by De, we can express the system (3.7) as

$$\text{(3.8)} \qquad \begin{cases} e_1^2(t^2 - \lambda^2) + 4ye_1e_2 + e_2^2(t^2 - \lambda^2 + 4x) = 0, \\ 2H_2(y|f|^2 - H_2) = \lambda \left[tH_1(f_1^2 - f_2^2) + 2tH_2 f_1 f_2 - \lambda |f|^2 H_1\right], \end{cases}$$

where $e = e_1 + ie_2$.

To solve the first equation in (3.8) as a second degree equation in e_1 we need the discriminant to be greater than or equal to 0, i.e. $4y^2 - (t^2 - \lambda^2)(4x + t^2 - \lambda^2) \geq 0$, which is precisely condition (3.6).

Now, suppose that (3.6) holds. Then we obtain that

$$e_1 = \frac{e_2\beta}{\lambda^2 - t^2}, \qquad e = e_2\left(\frac{\beta}{\lambda^2 - t^2} + i\right),$$

where $\beta = 2y + \sqrt{4y^2 - (t^2 - \lambda^2)(4x + t^2 - \lambda^2)}$ and e_2 is determined by the second equation in (3.8). □

Summing up the previous results we obtain the following:

Corollary 3.1.11. *Let us suppose that $\lambda \neq t$ and $D = x + iy \neq 0$. If* (3.6) *holds then*

$$(1, \lambda, D) \sim \left(1, t, D\left(\frac{\beta^2 - (\lambda^2 - t^2)^2}{\beta^2 + (\lambda^2 - t^2)^2} + \frac{2\beta(\lambda^2 - t^2)}{\beta^2 + (\lambda^2 - t^2)^2}\, i\right)\right),$$

where $\beta = 2y + \sqrt{4y^2 - (t^2 - \lambda^2)(4x + t^2 - \lambda^2)}$.

Comparing the inequalities (ii.1) and (ii.2) in Proposition 3.1.5 with the condition (3.6), we observe that for $\mathfrak{h}_2$ and $\mathfrak{h}_4$ it is possible to take $t = 1$ in the previous corollary in order to get equivalences with the complex structures (i.1) and (i.3), respectively. Therefore, using Corollary 3.1.9, we conclude:

Proposition 3.1.12. *Let us consider the family of complex structures*

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}, \quad \mathfrak{Im}\,D \geq 0. \tag{3.9}$$

Then:

- *(i) any non-abelian complex structure on $\mathfrak{h}_2$ is equivalent to one and only one structure in* (3.9) *with $\mathfrak{Im}\,D > 0$;*
- *(ii) any non-abelian complex structure on $\mathfrak{h}_4$ is equivalent to one and only one structure in* (3.9) *with $D \in \mathbb{R} - \{0\}$.*

The classification of complex structures on $\mathfrak{h}_5$ requires a more subtle study.

Lemma 3.1.13. *Any non-abelian complex structure on $\mathfrak{h}_5$ which is not complex-parallelizable belongs to one of the following families:*

- *(I) $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + iy\,\omega^{2\bar{2}}$, where $0 \leq 2y < |1 - \lambda^2|$;*
- *(II) $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + (x + iy)\,\omega^{2\bar{2}}$, where $4y^2 < 1 + 4x$.*

Moreover,

- *(i) the structures in family* (I) *are non-equivalent;*
- *(ii) the structures in family* (II) *are non-equivalent;*
- *(iii) a structure $(1, \lambda, iy)$ in family* (I) *is equivalent to a structure in family* (II) *if and only if $2\lambda^2 \in [0, 1)$ and $2y \in [\lambda^2, 1 - \lambda^2)$.*

Proof. Let us consider a complex structure given by $(1, \lambda, D = x + i\,y)$ on $\mathfrak{h}_5$, i.e.

$$4y^2 < (1 - \lambda^2)(4x + 1 - \lambda^2),$$

according to Proposition 3.1.5 (ii.3). If $\lambda^2 \geq 2x$, then $(1, \lambda, D) \sim (1, \sqrt{\lambda^2 - 2x}, i|D|)$ because (3.6) expresses simply as $4|D|^2 \geq 0$ and it trivially holds. On the other hand, if $\lambda^2 < 2x$, then $(1, \lambda, D) \sim (1, 0, E)$, where E is given in Corollary 3.1.11, because in this case $4y^2 + \lambda^2(4x - \lambda^2) \geq 0$, that is, condition (3.6) is satisfied.

To study further equivalences, it is clear that structures in family (I) are non-equivalent and the same holds for structures in family (II). Now let us consider the triples $(1, \lambda, iy)$ and $(1, 0, E)$. Then, (3.6) expresses simply as

$$4y^2 \geq \lambda^4. \tag{3.10}$$

Condition for family (I) implies that $4y^2 < (1 - \lambda^2)^2$, which is equivalent to $4y^2 - \lambda^4 < 1 - 2\lambda^2$, so if $2\lambda^2 \geq 1$ then (3.10) does not hold. Now, if $0 \leq \lambda^2 < \frac{1}{2}$ then the condition for family (I) is equivalent to $y < \frac{1}{2} - \frac{\lambda^2}{2}$, and therefore when $2y \in [\lambda^2, 1 - \lambda^2)$ the triple $(1, \lambda, iy)$ in family (I) is equivalent to the triple $(1, 0, E = -\frac{1}{2}(\lambda^2 - \sqrt{4y^2 - \lambda^4}\,i))$ in family (II). □

Proposition 3.1.14. *Any non-abelian complex structure on $\mathfrak{h}_5$ which is not complex-parallelizable is equivalent to one and only one structure in the following families:*

(I) $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}},$

$$\text{where } \mathfrak{Re}\, D = 0 \text{ and } \begin{cases} 0 \leq 2\,\mathfrak{Im}\, D < \lambda^2, & 0 < \lambda^2 < \frac{1}{2};\ \text{or} \\ 0 \leq 2\,\mathfrak{Im}\, D < |1-\lambda^2|, & \frac{1}{2} \leq \lambda^2. \end{cases}$$

(II) $d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{12} + \omega^{1\bar{1}} + D\,\omega^{2\bar{2}}, \quad$ *where* $4(\mathfrak{Im}\, D)^2 < 1 + 4\,\mathfrak{Re}\, D.$

To finish this section, it remains to study the case of 2-step NLAs $\mathfrak{g}$ with first Betti number equal to 3, which corresponds to $\epsilon = 1$ in (3.1).

Proposition 3.1.15. *Let J be a nilpotent complex structure on a nilpotent Lie algebra $\mathfrak{g}$ given by* (3.1) *with $\epsilon = 1$, i.e.*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar{2}} + C\,\omega^{2\bar{1}},$$

with $\rho \in \{0, 1\}$ and $B, C \in \mathbb{C}$ such that $(\rho, B, C) \neq (0,0,0)$. Then $\mathfrak{g}$ is 2-step nilpotent if and only if $B = \rho = 1$ and $C = 0$. In such case $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_7$ and all the complex structures are equivalent.

Proof. Let Z_1, Z_2, Z_3 be the dual basis of $\omega^1, \omega^2, \omega^3$. It is clear that $[\mathfrak{g}, \mathfrak{g}]$ has dimension at least 2 and is contained in $\langle i(Z_2 - \bar{Z}_2), \mathfrak{Re}\, Z_3, \mathfrak{Im}\, Z_3 \rangle$. Since $\mathfrak{Re}\, Z_3, \mathfrak{Im}\, Z_3$ are central elements and

$$[i(Z_2 - \bar{Z}_2), Z_1] = (\rho - B)i\, Z_3 + \bar{C}i\, \bar{Z}_3,$$

we conclude that $\mathfrak{g}$ is 2-step nilpotent if and only if $B = \rho$ and C vanishes.

Let $(\rho, B, C) = (1,1,0)$ and let us consider a basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by $\omega^1 = \frac{1}{\sqrt{2}}(e^2 + ie^1)$, $\omega^2 = \frac{1}{\sqrt{2}}e^3 + ie^4$ and $\omega^3 = e^6 + ie^5$. Now, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_7$. □

3-step nilpotent Lie algebras

In this section we classify, up to equivalence, nilpotent complex structures on 3-step nilpotent Lie algebras $\mathfrak{g}$ of dimension 6. In this case the coefficient $\epsilon = 1$ in the equations (3.1) given in Proposition 3.1.3. The equivalence of complex structures in terms of the triple (ρ, B, C) is given in the following lemma.

Lemma 3.1.16. *Let $\mathfrak{g}$ be a six-dimensional nilpotent Lie algebra endowed with a nilpotent complex structure* (3.1) *with $\epsilon = 1$ and $(\rho, B, C) \neq (0,0,0)$. Then:*

(i) *if the structure is abelian, then there is a basis $\{\omega^j\}_{j=1}^3$ for $(\mathfrak{g}^*)^{1,0}$ satisfying either*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{2\bar{1}}, \tag{3.11}$$

or

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \tag{3.12}$$

where $c \in \mathbb{R}$, $c \geq 0$.

(ii) *in the non-abelian case there is a basis* $\{\omega^j\}_{j=1}^3$ *for* $(\mathfrak{g}^*)^{1,0}$ *satisfying*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{12} + B\,\omega^{1\bar{2}} + c\,\omega^{2\bar{1}}, \tag{3.13}$$

where $B \in \mathbb{C}$ *and* $c \in \mathbb{R}$ *such that* $c \geq 0$.

Moreover, for any possible choice of parameters B *and* c*, each structure in* (3.11)*,* (3.12) *and* (3.13) *defines a different equivalence class of complex structures.*

Proof. If the complex structure is abelian then the pair $(B, C) \neq (0, 0)$ since $\rho = 0$. If $B = 0$ then it is clear that one arrives at equation (3.11). If $B \neq 0$ then with respect to the basis $\{z\,\omega^1, |z|^2\,\omega^2, \frac{z|z|^2}{B}\,\omega^3\}$, where z is any non-zero solution of $\frac{|C|}{|B|}\,\bar{z} = \frac{C}{B}\,z$, the equations (3.1) reduce to the form (3.12).

For the proof of (ii), we observe that with respect to $\{z\,\omega^1, |z|^2\,\omega^2, z|z|^2\,\omega^3\}$, where $z \neq 0$ satisfies $\bar{z}\,|C| = z\,C$, the equations (3.1) reduce to (3.13).

Finally it can be seen, by a similar argument to the first part of the proof of Lemma 3.1.7, the non-equivalence of the different complex structures defined in (3.11), (3.12) and (3.13). □

The following result provides a classification of abelian structures in the 3-step case in a slightly more straightforward way than the one given in [4].

Corollary 3.1.17. *Let* J *be an abelian structure on a six-dimensional nilpotent Lie algebra* $\mathfrak{g}$ *given by* (3.11) *or* (3.12)*. Then,* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_{15}$*, except for* $c = 1$ *in which case* $\mathfrak{g} \cong \mathfrak{h}_9$.

Proof. For the equations (3.12), let us consider a basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by $\omega^1 = -e^1 + i\,e^2$, $\omega^2 = 2e^3 + 2i\,e^4$ and $\omega^3 = 2e^5 + 2(c+1)i\,e^6$. Then, e^1, e^2, e^3 are closed, $de^4 = e^{12}$, $de^5 = (c-1)(e^{13} + e^{42})$ and $de^6 = e^{14} + e^{23}$. Thus, if $c \neq 1$ then the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$; otherwise, $\mathfrak{g} \cong \mathfrak{h}_9$. Finally, if $\{e^1, \ldots, e^6\}$ is a basis of 1-forms satisfying the structure equations of $\mathfrak{h}_{15}$, then the $(1,0)$-basis:

$$\omega^1 = e^1 + ie^2, \quad \omega^2 = 2e^3 - 2ie^4, \quad \omega^3 = -2e^5 + 2ie^6,$$

defines the complex structure given by (3.11). □

Remark 3.1.18. Notice that the family (3.13) includes the case $\mathfrak{h}_7$ precisely for $\rho = B = 1$ and $c = 0$ as it is shown in Proposition 3.1.15.

Next we determine the Lie algebras underlying the complex equations (3.13) in the remaining cases. They all have first Betti number equal to 3 and are nilpotent in step 3. Also notice that the dimension of their center is at least 2.

Proposition 3.1.19. *Let* J *be a nilpotent complex structure on a* 3*-step nilpotent Lie algebra* $\mathfrak{g}$ *given by* (3.13)*. Then* $\mathfrak{g}$ *has* 3*-dimensional center if and only if* $|B| = 1$*,* $B \neq 1$ *and* $c = 0$*. In such case* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_{16}$.

Proof. Let Z_1, Z_2, Z_3 be the dual basis of $\omega_1, \omega_2, \omega_3$. Then, $\mathfrak{Re}\,(Z_3)$ and $\mathfrak{Im}\,(Z_3)$ are central elements. Let $T = \lambda_1 Z_1 + \bar{\lambda}_1 \bar{Z}_1 + \lambda_2 Z_2 + \bar{\lambda}_2 \bar{Z}_2$ be another non-zero element in the center of $\mathfrak{g}$, where $(\lambda_1, \lambda_2) \in \mathbb{C}^2 - \{(0,0)\}$. It follows from (3.13) that

$$0 = [T, Z_1] = \bar{\lambda}_1 Z_2 - \bar{\lambda}_1 \bar{Z}_2 - (\lambda_2 - B\bar{\lambda}_2) Z_3 - c\bar{\lambda}_2 \bar{Z}_3,$$

which implies $\lambda_1 = 0$, $c\lambda_2 = 0$ and $\lambda_2 = B\bar{\lambda}_2$. Therefore, $c = 0$ and $|B| = 1$ in order the center to be 3-dimensional, because otherwise the equation $\lambda_2 = B\bar{\lambda}_2$ would have trivial solution. Moreover, $B \neq 1$ because $\mathfrak{g}$ is nilpotent in step 3.

Finally, since $|B| = 1$ and $B \neq 1$, let us consider the basis $\{e^1, \ldots, e^6\}$ for $\mathfrak{g}^*$ given by: $e^1 + i\,e^2 = i(B-1)\omega^1$, $e^3 = \omega^2 + \omega^{\bar{2}}$, $e^4 = \frac{1-\mathfrak{Re}\,B}{1-B} i(\omega^2 + B\,\omega^{\bar{2}})$, $e^5 + i\,e^6 = (1 - \mathfrak{Re}\,B)\omega^3$. Then, we can write the differential of ω^3 in the form

$$d\omega^3 = \omega^1 \wedge (\omega^2 + B\,\omega^{\bar{2}}) = \left(\frac{i(B-1)}{1-\mathfrak{Re}\,B}\omega^1\right) \wedge \left(\frac{1-\mathfrak{Re}\,B}{1-B} i(\omega^2 + B\,\omega^{\bar{2}})\right),$$

which implies that e^1, e^2, e^3 are closed, $de^4 = e^{12}$, $de^5 = e^{14}$ and $de^6 = e^{24}$, i.e. $\mathfrak{g} \cong \mathfrak{h}_{16}$. □

Next we establish the conditions for the coefficients B and c in terms of the dimension of $\mathfrak{g}^2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$.

Lemma 3.1.20. *Let J be a complex structure on a 3-step nilpotent Lie algebra $\mathfrak{g}$ given by* (3.13). *Then:*

(i) *If $c = |B-1| \neq 0$, then* $\dim \mathfrak{g}^2 = 1$.

(ii) *If $c \neq |B-1|$, then* $\dim \mathfrak{g}^2 = 2$.

Proof. From (3.13) we have that

$$\mathfrak{g}^2 = [Z_2 - \bar{Z}_2, \mathfrak{g}] = \langle (1-B)Z_3 + c\,\bar{Z}_3,\; c\,Z_3 + (1-\bar{B})\bar{Z}_3 \rangle.$$

It is clear that $\dim \mathfrak{g}^2 = 2$ if and only if $(1-B)(1-\bar{B}) - c^2 \neq 0$. □

Notice that if $c = |B-1| \neq 0$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$. Since the case $c = 0 \neq |B-1|$, $|B| = 1$ corresponds to $\mathfrak{g} \cong \mathfrak{h}_{16}$ by Proposition 3.1.19, we conclude that for $c \neq |B-1|$ and $(c, |B|) \neq (0,1)$ the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$.

In order to distinguish the underlying Lie algebras, we use the following argument for $\mathfrak{g} = \mathfrak{h}_k$, $10 \leq k \leq 15$. Let $\alpha(\mathfrak{g})$ be the number of linearly independent elements τ in $\wedge^2\mathfrak{g}^*$ such that $\tau \in d(\mathfrak{g}^*)$ and $\tau \wedge \tau = 0$. This number can be identified with the number of linearly independent exact 2-forms which are decomposable, that is, $\alpha(\mathfrak{h}_k) = 3$ for $k = 10, 12, 13$, $\alpha(\mathfrak{h}_k) = 2$ for $k = 11, 14$ and $\alpha(\mathfrak{h}_k) = 1$ for $k = 15$.

If τ is any exact element in $\wedge^2\mathfrak{g}^*$ then $\tau = \mu\,d\omega^2 + \bar{\mu}\,d\omega^{\bar{2}} + \nu\,d\omega^3 + \bar{\nu}\,d\omega^{\bar{3}}$, for some $\mu, \nu \in \mathbb{C}$, and by (3.13) we have

$$\tau = (\mu - \bar{\mu})\,\omega^{1\bar{1}} + \nu\,\omega^{12} + (\nu B - \bar{\nu} c)\,\omega^{1\bar{2}} + (\nu c - \bar{\nu}\bar{B})\,\omega^{2\bar{1}} + \bar{\nu}\,\omega^{\bar{1}\bar{2}}.$$

A direct calculation shows that

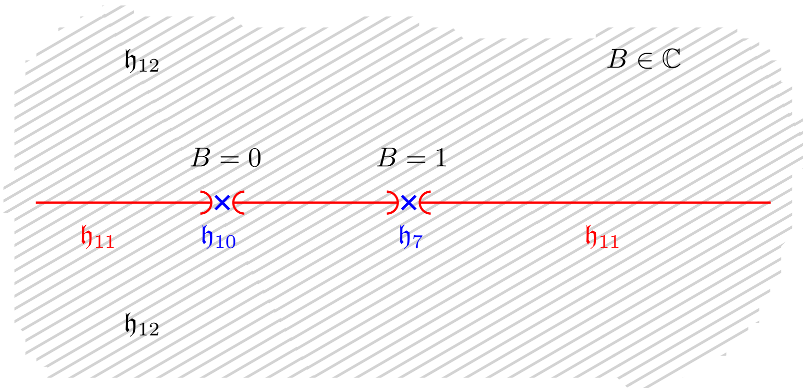
$$\tau \wedge \tau = 2\left(|\nu|^2(1-|B|^2-c^2) + c\left(\nu^2 B + \bar{\nu}^2 \bar{B}\right)\right)\omega^{12\bar{1}\bar{2}}.$$

Thus, if we denote $p = \mathfrak{Re}\ \nu$ and $q = \mathfrak{Im}\ \nu$, then $\tau \wedge \tau = 0$ if and only if

$$(3.14)\qquad \left(1-|B|^2-c^2+2c\,\mathfrak{Re}\ B\right)p^2 - (4c\,\mathfrak{Im}\ B)\,pq + \left(1-|B|^2-c^2-2c\,\mathfrak{Re}\ B\right)q^2 = 0.$$

Observe that the trivial solution $p = q = 0$ corresponds to $\tau = 2i\,\mathfrak{Im}\ \mu\,\omega^{1\bar{1}}$, according to the fact that $\alpha(\mathfrak{g}) \geq 1$.

Figure 3.3: Complex structures satisfying (3.13).



Proposition 3.1.21. *Let J be a complex structure on a 3-step nilpotent Lie algebra $\mathfrak{g}$ given by* (3.13) *with $c = |B-1| \neq 0$. Then:*

(i) $\mathfrak{g} \cong \mathfrak{h}_{10}$ *if and only if* $B = 0$*;*

(ii) $\mathfrak{g} \cong \mathfrak{h}_{11}$ *if and only if* $B \in \mathbb{R} - \{0, 1\}$*;*

(iii) $\mathfrak{g} \cong \mathfrak{h}_{12}$ *if and only if* $\mathfrak{Im}\, B \neq 0$.

In particular, all the complex structures on $\mathfrak{h}_{10}$ are equivalent.

Proof. Since $c = |B-1| \neq 0$, it follows from Lemma 3.1.20 that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$.

Firstly, $\mathfrak{g} \cong \mathfrak{h}_{10}$ if and only if the coefficients in equation (3.14) vanish. In fact, for $\mathfrak{h}_{10}$ we have by Theorem 2.1.25 that $\nu\, d\omega^3 + \bar{\nu}\, d\omega^{\bar{3}} \in \langle e^{12}, e^{13}, e^{14}\rangle$ for any $\nu \in \mathbb{C}$ so any pair $(p, q) \in \mathbb{R}^2$ solves the equation (3.14), which implies the vanishing of its coefficients. Conversely, if the coefficients $1-|B|^2-c^2+2c\,\mathfrak{Re}\ B$, $c\,\mathfrak{Im}\ B$ and $1-|B|^2-c^2-2c\,\mathfrak{Re}\ B$ are all zero then necessarily $B = 0$ and $c = 1$, that is, $d\omega^1 = 0$, $d\omega^2 = \omega^{1\bar{1}}$ and $d\omega^3 = (\omega^1 - \omega^{\bar{1}}) \wedge \omega^2$, and therefore the Lie algebra is isomorphic to $\mathfrak{h}_{10}$.

On the other hand, notice that if $c = |B-1| \neq 0$ and $(B, c) \neq (0, 1)$ then (3.14) is a second degree equation in p or q. Since its discriminant is a positive multiple of $(\mathfrak{Im}\ B)^2$, if $\mathfrak{Im}\ B \neq 0$ then we get two independent solutions and $\alpha(\mathfrak{g}) = 3$, that is, $\mathfrak{g} \cong \mathfrak{h}_{12}$. Finally, for $\mathfrak{Im}\ B = 0$ the equation (3.14) provides one solution and $\alpha(\mathfrak{g}) = 2$, so $\mathfrak{g} \cong \mathfrak{h}_{11}$. □

Proposition 3.1.22. *Let J be a complex structure on a 3-step nilpotent Lie algebra $\mathfrak{g}$ given by (3.13) with $c \neq |B-1|$ such that $(c, |B|) \neq (0,1)$ and define $\Delta(B,c) := c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2 < 0$. Then:*

(i) $\mathfrak{g} \cong \mathfrak{h}_{13}$ *if and only if* $\Delta(B,c) < 0$*;*

(ii) $\mathfrak{g} \cong \mathfrak{h}_{14}$ *if and only if* $\Delta(B,c) = 0$*;*

(iii) $\mathfrak{g} \cong \mathfrak{h}_{15}$ *if and only if* $\Delta(B,c) > 0$*.*

Proof. Since $c \neq |B-1|$ and $(c, |B|) \neq (0,1)$, it follows from Lemma 3.1.20 and Proposition 3.1.19 that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$.

Notice that the condition $(c, |B|) \neq (0,1)$ implies that the coefficients of p^2 and q^2 in equation (3.14) cannot be both zero, so (3.14) is always a second degree equation. Let

$$\Delta(B,c) := c^4 - 2(|B|^2+1)c^2 + (|B|^2-1)^2.$$

Since the discriminant as a second degree equation in p is equal to $-4q^2\Delta(B,c)$ and the discriminant as a second degree equation in q equals $-4p^2\Delta(B,c)$, the number of independent solutions of equation (3.14) depends on the sign of $\Delta(B,c)$. Thus, for $\Delta(B,c) < 0$ there exist two such solutions and thus $\mathfrak{g} \cong \mathfrak{h}_{13}$, for $\Delta(B,c) = 0$ there exists only one such solution and $\mathfrak{g} \cong \mathfrak{h}_{14}$, and finally for $\Delta(B,c) > 0$ there is no solution and $\alpha(\mathfrak{g}) = 1$, which implies that $\mathfrak{g} \cong \mathfrak{h}_{15}$. □

3.1.2 Classification of complex structures

In this section we aim to have a complete description of the complex structures on six-dimensional nilpotent Lie algebras up to equivalence.

Firstly, we deal with the classification of abelian structures J on 6-dimensional nilpotent Lie algebras obtained by Andrada, Barberis and Dotti in [4]. In the 3-step case we use directly the equations given in Lemma 3.1.16 and Corollary 3.1.17, but in the 2-step case we have written the complex structure equations of any abelian J in a form that fits in our Proposition 3.1.5. More precisely, in the 2-step case we first consider the following reduction of the equations (3.2) of any abelian complex structure.

Corollary 3.1.23. *If J is abelian and $\mathfrak{g}$ is a 2-step six-dimensional nilpotent Lie algebra, then there is a basis $\{\omega^j\}_{j=1}^3$ for $(\mathfrak{g}^*)^{1,0}$ satisfying one of the following equations:*

(i) $d\omega^1 = d\omega^2 = d\omega^3 = 0$*;*

(ii) $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} + D\,\omega^{2\bar{2}}$*, with* $D \in \mathbb{C},\ |D| = 1,\ \mathfrak{Im}\, D \geq 0$*;*

(iii) $d\omega^1 = d\omega^2 = 0,\ d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\,\omega^{2\bar{2}}$*, with* $D \in \mathbb{C},\ \mathfrak{Im}\, D \geq 0$*.*

Proof. Suppose $\rho = 0$ in (3.2). If in addition $\lambda = 0$, then in terms of the basis $\{\sqrt{|D|}\ \omega^1,\ |D|\,\omega^2,\ |D|\,\omega^3\}$ we obtain (i) or (ii), whereas if $\lambda \neq 0$ then we get equations (iii) with respect to $\{\omega^1,\ \lambda\,\omega^2,\ \omega^3\}$. □

Next we illustrate how to rewrite the complex structure equations of any abelian J on the Lie algebra $\mathfrak{h}_5$ in a form that fits in our Corollary 3.1.23. By [4, Theorem 3.5] there is, up to isomorphism, one family J_t, $t \in (0, 1]$, of abelian structures given by

$$J_t e^1 = e^3, \quad J_t e^2 = e^4, \quad J_t e^5 = \frac{1}{t}\, e^6.$$

With respect to the (1,0)-basis $\{\sigma^1 = e^1 - i\, e^3, \sigma^2 = e^2 - i\, e^4, \sigma^3 = -2i\, e^5 - \frac{2}{t}\, e^6\}$, the complex structure equations for J_t are

$$d\sigma^1 = d\sigma^2 = 0, \quad d\sigma^3 = \sigma^{1\bar{1}} - \frac{i}{t}\sigma^{1\bar{2}} - \frac{i}{t}\sigma^{2\bar{1}} - \sigma^{2\bar{2}}.$$

Now, by [95, Lemma 11] there exists a (1,0)-basis $\{\omega^j\}_{j=1}^3$ satisfying

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\, \omega^{2\bar{2}},$$

with $D = \frac{1-t^2}{4}$. Notice that $D \in [0, \frac{1}{4})$ because $t \in (0, 1]$.

Now, we summarize all the results concerning the complex structures on six-dimensional nilpotent Lie algebras. Firstly, the parallelizable complex structures are modelled by:

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\omega^{12},$$

with $\rho = 0$ or 1, and the Lie algebras are the abelian $\mathfrak{h}_1$ (for $\rho = 0$) and $\mathfrak{h}_5$ (for $\rho = 1$), where the latter case corresponds to the Iwasawa manifold. The remaining complex structures in dimension 6 are parametrized, up to equivalence, by the following three families:

$$\textbf{Family I:} \quad d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}},$$

where $\rho = 0$ or 1, $D \in \mathbb{C}$ with $\mathfrak{Im}\, D \geq 0$ and $\lambda \in \mathbb{R}^{\geq 0}$. The complex structure is abelian if and only if $\rho = 0$. The Lie algebra is 2-step nilpotent with first Betti number ≥ 4, i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_8$.

$$\textbf{Family II:} \quad d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \rho\omega^{12} + B\omega^{1\bar{2}} + c\omega^{2\bar{1}},$$

where $\rho = 0$ or 1, $B \in \mathbb{C}$ and $c \in \mathbb{R}^{\geq 0}$. Moreover $(\rho, B, \mathbb{C}) \neq (0, 0, 0)$. The complex structure is abelian if and only if $\rho = 0$. The Lie algebra is isomorphic to $\mathfrak{h}_7$ or $\mathfrak{h}_9, \ldots, \mathfrak{h}_{16}$. Finally, we recall that the non-nilpotent complex structures on six-dimensional NLAs are classified by Ugarte and Villacampa [96].

$$\textbf{Family III:} \quad d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = \epsilon\omega^{1\bar{1}} \pm (\omega^{1\bar{2}} - \omega^{2\bar{1}}),$$

where $\epsilon = 0$ or 1. The corresponding Lie algebras are $\mathfrak{h}_{19}^-$ (for $\epsilon = 0$) and $\mathfrak{h}_{26}^+$ (for $\epsilon = 1$).

The classification up to equivalence of complex structures on six-dimensional nilpotent Lie algebras is summarized in Table 3.1.

Lie algebra	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	**complex structure**	**conditions** $(\lambda \geq 0,\ c \geq 0,\ B \in \mathbb{C})$ $\Delta(B,c) := c^4 - 2(\lvert B\rvert^2+1)c^2 + (\lvert B\rvert^2-1)^2$	**Frölicher type**
$\mathfrak{h}_1 = (0,0,0,0,0,0)$	6	15	20	$J_1 := (0,0,0)$,		$E_1 \cong E_\infty$
$\mathfrak{h}_2 = (0,0,0,0,12,34)$	4	8	10	$J_1^D := (0,0,\omega^{1\bar{1}} + D\omega^{2\bar{2}})$,	$\mathrm{Im}D = 1$	$E_1 \ncong E_2 \cong E_\infty$
				$J_2^D := (0,0,\omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}})$,	$\mathrm{Im}D > 0$	$E_1 \cong E_\infty$
$\mathfrak{h}_3 = (0,0,0,0,0,12+34)$	5	9	10	$J^\pm := (0,0,\omega^{1\bar{1}} \pm \omega^{2\bar{2}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_4 = (0,0,0,0,12,14+23)$	4	8	10	$J_1 := (0,0,\omega^{1\bar{1}} + \omega^{1\bar{2}} + \frac{1}{4}\omega^{2\bar{2}})$,		$E_1 \ncong E_2 \cong E_\infty$
				$J_2^D := (0,0,\omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}})$	$D \in \mathbb{R} \setminus \{0\}$	$E_1 \cong E_\infty$
$\mathfrak{h}_5 = (0,0,0,0,13+42,14+23)$	4	8	10	$J_1^D := (0,0,\omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}})$,	$D \in [0,\frac{1}{4})$	$E_1 \ncong E_2 \cong E_\infty$
				$J_2 := (0,0,\omega^{12})$,		$E_1 \ncong E_2 \cong E_\infty$
				$J_3^{(\lambda,D)} := (0,0,\omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}})$,	$(\lambda,D) \in \{(0,x+iy) \in \mathbb{R}\times\mathbb{C}\ :\ y \geq 0, 4y^2 < 1+4x\}$	$E_1 \cong E_\infty$ $(D \neq 0)$
					$\cup\{(\lambda,iy) \in \mathbb{R}\times\mathbb{C}\ :\ 0 < \lambda^2 < \frac{1}{2},\ 0 \leq y < \frac{\lambda^2}{2}\}$	$E_1 \ncong E_2 \cong E_\infty$ $(D = 0)$
					$\cup\{(\lambda,iy) \in \mathbb{R}\times\mathbb{C}\ :\ \frac{1}{2} \leq \lambda^2 < 1,\ 0 \leq y < \frac{1-\lambda^2}{2}\}$	
					$\cup\{(\lambda,iy) \in \mathbb{R}\times\mathbb{C}\ :\ \lambda^2 > 1,\ 0 \leq y < \frac{\lambda^2-1}{2}\}$	
$\mathfrak{h}_6 = (0,0,0,0,12,13)$	4	9	12	$J := (0,0,\omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_7 = (0,0,0,12,13,23)$	3	8	12	$J := (0,\omega^{1\bar{1}},\omega^{12} + \omega^{1\bar{2}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_8 = (0,0,0,0,0,12)$	5	11	14	$J := (0,0,\omega^{1\bar{1}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_9 = (0,0,0,0,12,14+25)$	4	7	8	$J := (0,\omega^{1\bar{1}},\omega^{1\bar{2}} + \omega^{2\bar{1}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_{10} = (0,0,0,12,13,14)$	3	6	8	$J := (0,\omega^{1\bar{1}},\omega^{12} + \omega^{2\bar{1}})$,		$E_1 \cong E_\infty$
$\mathfrak{h}_{11} = (0,0,0,12,13,14+23)$	3	6	8	$J^B := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}} + \lvert B-1\rvert\omega^{2\bar{1}})$,	$B \in \mathbb{R} \setminus \{0,1\}$	$E_1 \cong E_\infty$
$\mathfrak{h}_{12} = (0,0,0,12,13,24)$	3	6	8	$J^B := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}} + \lvert B-1\rvert\omega^{2\bar{1}})$,	$\mathrm{Im}B \neq 0$	$E_1 \cong E_\infty$
$\mathfrak{h}_{13} = (0,0,0,12,13+14,24)$	3	5	6	$J^{(B,c)} := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}} + c\omega^{2\bar{1}})$,	$c \neq \lvert B-1\rvert,\quad (c,\lvert B\rvert) \neq (0,1),\quad \Delta(B,c) < 0$	$E_1 \cong E_2 \ncong E_3 \cong E_\infty$
$\mathfrak{h}_{14} = (0,0,0,12,14,13+42)$	3	5	6	$J^{(B,c)} := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}} + c\omega^{2\bar{1}})$,	$c \neq \lvert B-1\rvert,\quad (c,\lvert B\rvert) \neq (0,1),\quad \Delta(B,c) = 0$	$E_1 \cong E_2 \ncong E_3 \cong E_\infty$
$\mathfrak{h}_{15} = (0,0,0,12,13+42,14+23)$	3	5	6	$J_1 := (0,\omega^{1\bar{1}},\omega^{2\bar{1}})$,		$E_1 \ncong E_2 \ncong E_3 \cong E_\infty$
				$J_2^c := (0,\omega^{1\bar{1}},\omega^{1\bar{2}} + c\omega^{2\bar{1}})$,	$c \neq 1$	$E_1 \ncong E_2 \cong E_\infty$ $(c = 0)$
						$E_1 \ncong E_2 \ncong E_3 \cong E_\infty$ $(c \neq 0)$
				$J_3^{(B,c)} := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}} + c\omega^{2\bar{1}})$,	$c \neq \lvert B-1\rvert,\quad (c,\lvert B\rvert) \neq (0,1),\quad \Delta(B,c) > 0$	$E_1 \cong E_2 \ncong E_3 \cong E_\infty$
$\mathfrak{h}_{16} = (0,0,0,12,14,24)$	3	5	6	$J^B := (0,\omega^{1\bar{1}},\omega^{12} + B\omega^{1\bar{2}})$,	$\lvert B\rvert = 1,\quad B \neq 1$	$E_1 \ncong E_2 \cong E_\infty$
$\mathfrak{h}_{19}^- = (0,0,0,12,23,14-35)$	3	5	6	$J^\pm := (0,\omega^{13} + \omega^{1\bar{3}}, \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}))$,		$E_1 \cong E_\infty$
$\mathfrak{h}_{26}^+ = (0,0,12,13,23,14+25)$	2	4	6	$J^\pm := (0,\omega^{13} + \omega^{1\bar{3}}, i\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}))$,		$E_1 \ncong E_2 \cong E_\infty$

Table 3.1: Complex structures on 6-dimensional nilpotent Lie algebras.

3.2 Complex structures on solvmanifolds

In this section we classify, up to equivalence, the complex structures having non-zero closed (3,0)-form on the Lie algebras of Theorem 2.2.14. We divide the study according to the Lie algebra is decomposable or not.

3.2.1 The decomposable case

According to Theorem 2.2.14, the decomposable Lie algebras are $\mathfrak{g}_1, \mathfrak{g}_2^\alpha$ with $\alpha \geq 0$ and $\mathfrak{g}_3$.

The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2^\alpha$

We start studying the $5 \oplus 1$ decomposable Lie algebras obtained in Theorem 2.2.14. The specific form of the expressions of the almost-complex structures J_ρ^* with $\rho, J_\rho^*\rho \in Z^3(\mathfrak{g})$ enables us to state the following lemma:

Lemma 3.2.1. *Let J be any complex structure on $\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ or $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$, $\alpha \geq 0$, with closed volume $(3,0)$-form, then there is a non-zero closed $(1,0)$-form.*

Proof. Let $\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$. Any $\rho \in Z^3(\mathfrak{g}_1)$ is given by

$$\begin{aligned}\rho \ &= a_1 e^{125} + a_2 e^{126} + a_3 e^{135} + a_4 e^{136} + a_6 e^{156} + a_8 e^{245} + a_9 e^{246} \\ &\quad + a_{10} e^{256} + a_{11} e^{345} + a_{12} e^{346} + a_{13} e^{356} + a_{14} e^{456},\end{aligned}$$

for $a_1, \ldots, a_{14} \in \mathbb{R}$. We use the equation (2.19) to compute the endomorphisms on $\mathfrak{g}_1$ corresponding to $\rho \in Z^3(\mathfrak{g}_1)$. When we compute the images of e^5, e^6 by J_ρ^* we find that the subspace spanned by e^5, e^6 is J_ρ^*-invariant:

$$\begin{aligned} J_\rho^* e^5 \ &= \tfrac{1}{\sqrt{|\lambda(\rho)|}}((a_1a_{12} + a_{11}a_2 - a_4a_8 - a_3a_9)e^5 + 2(a_{12}a_2 - a_4a_9)e^6), \\ J_\rho^* e^6 \ &= \tfrac{1}{\sqrt{|\lambda(\rho)|}}(-2(a_1a_{11} - a_3a_8)e^5 - (a_1a_{12} + a_{11}a_2 - a_4a_8 - a_3a_9)e^6). \end{aligned}$$

This holds for the particular case of $\lambda(\rho) < 0$ and therefore, the (1,0)-form $\eta = e^5 - iJ_\rho^* e^5$ is closed for any almost-complex structure J_ρ^* on $\mathfrak{g}_1$ with $\rho \in Z^3(\mathfrak{g}_1)$.

The same situation appears in $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} = (\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, -\alpha e^{35} + e^{45}, -e^{35} - \alpha e^{45}, 0, 0)$ with $\alpha \geq 0$ because the subspace spanned by e^5, e^6 is found to be J_ρ^*-invariant for all the endomorphisms defined by equation (2.19) with $\rho \in Z^3(\mathfrak{g}_2^\alpha)$. □

Lemma 3.2.2. *Let J be any complex structure on $\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ or $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$, $\alpha \geq 0$, with closed volume $(3,0)$-form. Then, there is a $(1,0)$-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the following reduced equations*

(3.15)
$$\begin{cases} d\omega^1 = A\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}),\\ d\omega^2 = -A\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}),\\ d\omega^3 = 0,\end{cases}$$

where $A = \cos\theta + i\sin\theta$*,* $\theta \in [0,\pi)$*.*

Proof. Using Lemma 3.2.1 let us consider a basis of $(1,0)$-forms $\{\eta^1, \eta^2, \eta^3\}$ such that $\eta^3 = e^5 - iJ^*_\rho e^5$ is closed. The structure equations of $\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$ force the differential of any 1-form to be a multiple of $e^5 = \frac{1}{2}(\eta^3 + \eta^{\bar{3}})$, so there exist $A, B, C, D, E, F \in \mathbb{C}$ such that

$$\begin{cases} d\eta^1 = (A\,\eta^1 + B\,\eta^2 + E\,\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}),\\ d\eta^2 = (C\,\eta^1 + D\,\eta^2 + F\,\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}),\\ d\eta^3 = 0.\end{cases}$$

Moreover, since $d(\eta^{123}) = 0$ necessarily $D = -A$.

Let us consider the non-zero 1-form $\tau^1 = A\,\eta^1 + B\,\eta^2 + E\,\eta^3$. Notice that

$$d\tau^1 = \left((A^2 + BC)\eta^1 + (AE + BF)\eta^3\right) \wedge \left(\eta^3 + \eta^{\bar{3}}\right),$$

which implies that $A^2 + BC \neq 0$ because otherwise $d\tau^1$ would be a multiple of e^{56}. Then, with respect to the new (1,0)-basis $\{\tau^1, \tau^2, \tau^3\}$ given by

$$\tau^1 = A\,\eta^1 + B\,\eta^2 + E\,\eta^3, \quad \tau^1 = C\,\eta^1 - A\,\eta^2 + F\,\eta^3, \quad \tau^3 = \eta^3,$$

the complex structure equations are

(3.16)
$$\begin{cases} d\tau^1 = (A\,\tau^1 + B\,\tau^2) \wedge (\tau^3 + \tau^{\bar{3}}),\\ d\tau^2 = (C\,\tau^1 - A\,\tau^2) \wedge (\tau^3 + \tau^{\bar{3}}),\\ d\tau^3 = 0.\end{cases}$$

Now we distinguish two cases:

• If $B \neq 0$ then we consider the new basis $\{\omega^1, \omega^2, \omega^3\}$ given by

$$\omega^1 = \left(A + \sqrt{A^2 + BC}\right)\tau^1 + B\,\tau^2, \quad \omega^2 = \left(A - \sqrt{A^2 + BC}\right)\tau^1 + B\,\tau^2, \quad \omega^3 = \left|\sqrt{A^2 + BC}\right|\tau^3.$$

With respect to this basis, the equations (3.16) reduce to

$$d\omega^1 = \frac{\sqrt{A^2 + BC}}{\left|\sqrt{A^2 + BC}\right|}\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -\frac{\sqrt{A^2 + BC}}{\left|\sqrt{A^2 + BC}\right|}\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = 0,$$

that is, the equations are of the form (3.15).
• If $C \neq 0$ then with respect to the basis $\{\omega^1, \omega^2, \omega^3\}$ given by

$$\omega^1 = C\,\tau^1 - \left(A - \sqrt{A^2+BC}\right)\tau^2, \quad \omega^2 = C\,\tau^1 - \left(A + \sqrt{A^2+BC}\right)\tau^2, \quad \omega^3 = \left|\sqrt{A^2+BC}\right|\tau^3,$$

the equations (3.16) again reduce to equations of the form (3.15).

Finally, notice that in the equations (3.15) one can change the sign of A by changing the sign of ω^3, so we can suppose that $A = \cos\theta + i\sin\theta$ with angle $\theta \in [0, \pi)$. □

Proposition 3.2.3. *Up to isomorphism, there is only one complex structure with closed $(3,0)$-form on the Lie algebras $\mathfrak{g}_1 = A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ and $\mathfrak{g}_2^0 = A_{5,17}^{0,0,1} \oplus \mathbb{R}$, whereas $\mathfrak{g}_2^\alpha = A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ has two such complex structures for any $\alpha > 0$. More concretely, the complex structures are:*

$$(\mathfrak{g}_1, J)\colon\ d\omega^1 = \omega^1 \wedge (\omega^3 + \omega^{\bar 3}), \quad d\omega^2 = -\omega^2 \wedge (\omega^3 + \omega^{\bar 3}), \quad d\omega^3 = 0; \tag{3.17}$$

$$(\mathfrak{g}_2^0, J)\colon\ d\omega^1 = i\,\omega^1 \wedge (\omega^3 + \omega^{\bar 3}), \quad d\omega^2 = -i\,\omega^2 \wedge (\omega^3 + \omega^{\bar 3}), \quad d\omega^3 = 0; \tag{3.18}$$

$$(\mathfrak{g}_2^{\alpha=\frac{\cos\theta}{\sin\theta}}, J^\pm)\colon \begin{cases} d\omega^1 = (\pm\cos\theta + i\sin\theta)\,\omega^1 \wedge (\omega^3 + \omega^{\bar 3}), \\ d\omega^2 = -(\pm\cos\theta + i\sin\theta)\,\omega^2 \wedge (\omega^3 + \omega^{\bar 3}), \\ d\omega^3 = 0, \end{cases} \tag{3.19}$$

where $\theta \in (0, \pi/2)$.

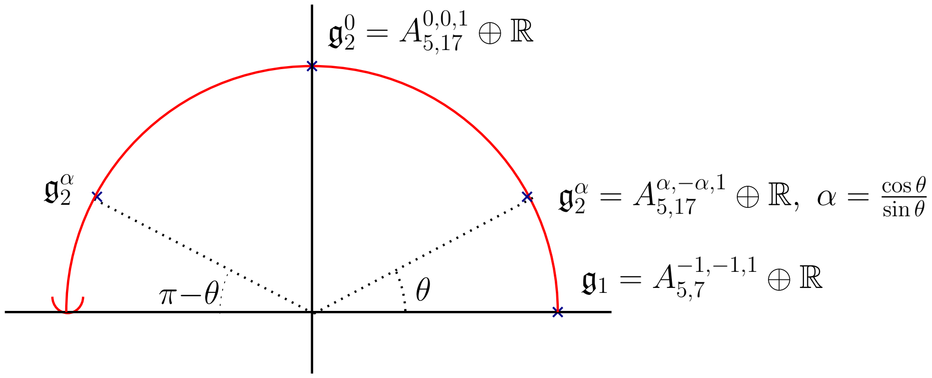
Proof. A real Lie algebra underlying the equations (3.15) is isomorphic to $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ for some $\alpha \geq 0$. In fact, in terms of the real basis $\{\beta^1, \dots, \beta^6\}$ given by $\omega^1 = \beta^1 + i\beta^2$, $\omega^2 = \beta^3 + i\beta^4$ and $\omega^3 = \frac{1}{2}(\beta^5 + i\beta^6)$, we have

$$d\beta^1 = \cos\theta\,\beta^{15} - \sin\theta\,\beta^{25}, \quad d\beta^3 = -\cos\theta\,\beta^{35} + \sin\theta\,\beta^{45}, \quad d\beta^5 = 0,$$

$$d\beta^2 = \sin\theta\,\beta^{15} + \cos\theta\,\beta^{25}, \quad d\beta^4 = -\sin\theta\,\beta^{35} - \cos\theta\,\beta^{45}, \quad d\beta^6 = 0.$$

In particular:
• If $\theta = 0$ then taking $e^1 = \beta^1$, $e^2 = \beta^4$, $e^3 = \beta^3$, $e^4 = \beta^2$, $e^5 = \beta^5$ and $e^6 = \beta^6$ the resulting structure equations are precisely those of the Lie algebra $\mathfrak{g}_1$.
• If $\theta \in (0, \pi)$ then $\sin\theta \neq 0$ and taking $e^1 = \beta^1$, $e^2 = -\beta^2$, $e^3 = \beta^3$, $e^4 = \beta^4$, $e^5 = \sin\theta\,\beta^5$ and $e^6 = \beta^6$ we get the structure equations of $\mathfrak{g}_2^{|\alpha|}$ with $\alpha = -\frac{\cos\theta}{\sin\theta}$. Notice that α takes any real value when θ varies in $(0, \pi)$, and if $\theta \neq \frac{\pi}{2}$ then θ and $\pi - \theta$ correspond to two complex structures on the same Lie algebra. By a standard argument one can prove that these two complex structures are non-equivalent. □

Figure 3.4: Complex structures satisfying (3.15) according to Proposition 3.2.3.



The Lie algebra $\mathfrak{g}_3$

Let us consider now the Lie algebra $\mathfrak{g}_3 = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$. Any non-zero closed 3-form $\rho \in Z^3(\mathfrak{g}_3)$ is given by

$$\begin{aligned}\rho \;\; = \;\; & a_1 e^{123} + a_2 e^{124} + a_3 e^{134} + a_4 e^{145} + a_5 e^{146} + a_6 e^{156} + a_7 e^{234} + a_8(e^{136} - e^{245}) + \\ & a_9(e^{135} - e^{246}) + a_{10}(e^{126} + e^{345}) + a_{11}(e^{125} + e^{346}) + a_{12} e^{456},\end{aligned}$$

for $a_1, \ldots, a_{12} \in \mathbb{R}$. By imposing the closedness of $J_\rho^* \rho$ together with the condition $\mathrm{tr}(J_\rho^{*2}) < 0$, one can arrive by a long computation to an explicit description of the complex structure J_ρ^*, which allows us to prove that the family $\{e^1, e^2, e^3, J_\rho^* e^1, J_\rho^* e^2, J_\rho^* e^3\}$ of 1-forms of $\mathfrak{g}_3$ is always linearly independent (for further details see Appendix A). In conclusion, the forms

$$\omega^1 = e^1 - iJ_\rho^* e^1, \quad \omega^2 = e^2 - iJ_\rho^* e^2, \quad \omega^3 = e^3 - iJ_\rho^* e^3,$$

constitute a (1,0)-basis for the complex structure J_ρ^*, and with respect to this basis the complex structure equations have the form

$$(3.20) \qquad \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} + b\,\omega^{1\bar{1}} + fi\,\omega^{1\bar{2}} - fi\,\omega^{2\bar{1}} - (\frac{1}{2} + gi)\omega^{1\bar{3}} + gi\,\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + c\,\omega^{1\bar{1}} + (\frac{1}{2} + hi)\omega^{1\bar{2}} - hi\,\omega^{2\bar{1}} - fi\,\omega^{1\bar{3}} + fi\,\omega^{3\bar{1}}, \end{cases}$$

where the coefficients b, c, f, g, h are real and satisfy $4gh = 4f^2 - 1$ (see Appendix A, Lemma A.0.7).

Proposition 3.2.4. *Up to isomorphism, the complex structures with closed* $(3,0)$*-form on the Lie algebra* $\mathfrak{g}_3 = \mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$ *are given by*

$$(3.21) \qquad (\mathfrak{g}_3, J^x)\colon \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} - (\frac{1}{2} + xi)\omega^{1\bar{3}} + xi\,\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\,\omega^{2\bar{1}}, \end{cases}$$

where $x \in \mathbb{R}^{\geq 0}$.

Proof. Observe first that with respect to the (1,0)-basis $\{\omega^1, \omega^2 + 2c\,\omega^1, \omega^3 - 2b\,\omega^1\}$, the complex structure equations express again as in (3.20) but with $b = c = 0$, that is to say, one can suppose that the coefficients b and c both vanish.

Let us prove next that we can also suppose the coefficient f to be zero. To see this, let $\{\omega^1, \omega^2, \omega^3\}$ be a (1,0)-basis satisfying (3.20) with $b = c = 0$ and $f \neq 0$, and let us consider the (1,0)-basis $\{\eta^1, \eta^2, \eta^3\}$ given by

$$\eta^1 = \omega^1, \qquad \eta^2 = \omega^2 - \tfrac{g-h-\sqrt{1+(g+h)^2}}{2f}\,\omega^3, \qquad \eta^3 = \tfrac{g-h-\sqrt{1+(g+h)^2}}{2f}\omega^2 + \omega^3.$$

A direct calculation shows that with respect to $\{\eta^1, \eta^2, \eta^3\}$ the corresponding coefficient f vanishes. Finally, since $4gh = -1$ we are led to the reduced equations (3.21), where we have written x instead of g.

To conclude the proof, let J^x and $J^{x'}$ be two complex structures corresponding to $x, x' \in \mathbb{R}$. It is easy to see that the structures are equivalent if and only if $xx' = -\frac{1}{4}$. This represents an hyperbola in the (x, x')-plane, so the equivalence class is given by one of the branches of the hyperbola, that is, we can take $x > 0$. □

3.2.2 The indecomposable case

Next we classify the complex structures with closed volume form on the six-dimensional indecomposable non-nilpotent solvable unimodular Lie algebras. According to Theorem 2.2.14 they are the Lie algebras $\mathfrak{g}_4, \dots, \mathfrak{g}_9$.

The Lie algebras $\mathfrak{g}_4, \mathfrak{g}_5, \mathfrak{g}_6$ and $\mathfrak{g}_7$

Lemma 3.2.5. *Let* J *be any complex structure on* $\mathfrak{g}_k$ $(4 \leq k \leq 7)$ *with closed* $(3,0)$*-form. Then, there is a* $(1,0)$*-basis* $\{\omega^1, \omega^2, \omega^3\}$ *such that*

$$(3.22) \qquad \begin{cases} d\omega^1 = A\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^2 = -A\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \\ d\omega^3 = G_{11}\,\omega^{1\bar{1}} + G_{12}\,\omega^{1\bar{2}} + \overline{G}_{12}\,\omega^{2\bar{1}} + G_{22}\,\omega^{2\bar{2}}, \end{cases}$$

where $A, G_{12} \in \mathbb{C}$ *and* $G_{11}, G_{22} \in \mathbb{R}$, *with* $(G_{11}, G_{12}, G_{22}) \neq (0,0,0)$, *satisfy*

$$(3.23) \qquad |A| = 1, \quad (A + \overline{A})G_{11} = 0, \quad (A + \overline{A})G_{22} = 0, \quad (A - \overline{A})G_{12} = 0.$$

Proof. Let us consider first the Lie algebra $\mathfrak{g}_4$ with structure equations given as in Theorem 2.2.14. Any element $\rho \in Z^3(\mathfrak{g}_4)$ is given by

$$\begin{aligned}\rho &= a_1e^{123} + a_2e^{126} + a_3(e^{125} - e^{134}) + a_4(e^{124} + e^{135}) + a_5e^{136}a_6(e^{156} + e^{234})+ \\ &\quad a_7(e^{146} - e^{235}) + a_8e^{236} + a_9e^{246} + a_{10}e^{256} + a_{11}e^{346} + a_{12}e^{356} + a_{13}e^{456},\end{aligned}$$

for $a_1, \ldots, a_{13} \in \mathbb{R}$. A direct calculation shows that if $a_3^2 + a_4^2 = 0$ then there do not exist closed 3-forms ρ satisfying the conditions $d(J_\rho^*\rho) = 0$ and $\lambda(\rho) < 0$.

Suppose that $a_3^2 + a_4^2 \neq 0$. Then, an element $\rho \in Z^3(\mathfrak{g}_4)$ satisfies the condition $d(J_\rho^*\rho) = 0$ if and only if

$$a_{10} = \frac{a_3(a_6^2 - a_7^2) + 2a_4a_6a_7 - a_{11}(a_3^2 + a_4^2)}{a_3^2 + a_4^2},\ a_{12} = \frac{2a_3a_6a_7 - a_4(a_6^2 - a_7^2) + a_9(a_3^2 + a_4^2)}{a_3^2 + a_4^2},$$

and $a_{13} = 0$. Moreover, under these relations one has that $\lambda(\rho) = -4(a_3a_9 - a_4a_{11} + a_6a_7)^2 \leq 0$.

Let $\rho \in Z^3(\mathfrak{g}_4)$ be such that $\lambda(\rho) < 0$ and $d(J_\rho^*\rho) = 0$. A direct calculation shows that $\tilde{J}_\rho^*e^6$ is given by

$$\tilde{J}_\rho^*e^6 = 2(a_3^2 + a_4^2)e^1 + 2(a_3a_6 + a_4a_7)e^2 + 2(a_3a_7 - a_4a_6)e^3 + 2(a_3a_{11} + a_4a_9 + a_7^2)e^6.$$

Therefore, the coefficient of e^1 in $J_\rho^*e^6$ is non-zero for any ρ. A similar computation for the Lie algebras $\mathfrak{g}_5$, $\mathfrak{g}_6$ and $\mathfrak{g}_7$ shows that for any complex structure J_ρ with closed $(3,0)$-form, we also have that

$$J_\rho^*e^6 = c_1e^1 + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6,$$

where the coefficient c_1 is non-zero.

Let us consider the $(1,0)$-form $\eta^3 = e^6 - iJ_\rho^*e^6$. From the structure equations of $\mathfrak{g}_k$ $(4 \leq k \leq 7)$ in Theorem 2.2.14, it follows that

$$\begin{cases} d\eta^3 = ic_1e^{23} - i\alpha \wedge e^6, & \text{if} \quad \mathfrak{g} = \mathfrak{g}_4, \\ d\eta^3 = ic_1(e^{24} + e^{35}) - i\alpha \wedge e^6, & \text{if} \quad \mathfrak{g} = \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_7, \end{cases}$$

where α is a 1-form. Since $c_1 \neq 0$ we can write the 2-forms e^{23} and $e^{24} + e^{35}$ as

$$\begin{cases} e^{23} = -\frac{i}{c_1}d\eta^3 + \frac{1}{c_1}\alpha \wedge e^6, & \text{if} \quad \mathfrak{g} = \mathfrak{g}_4, \\ e^{24} + e^{35} = -\frac{i}{c_1}d\eta^3 + \frac{1}{c_1}\alpha \wedge e^6, & \text{if} \quad \mathfrak{g} = \mathfrak{g}_5, \mathfrak{g}_6, \mathfrak{g}_7. \end{cases} \tag{3.24}$$

Now, let η^1, η^2 be such that $\{\eta^1, \eta^2, \eta^3\}$ is a basis of $(1,0)$-forms. Since e^6 is closed and $\eta^3 + \eta^{\bar{3}} = 2e^6$, the integrability of the complex structure implies that $d\eta^3$ has no component of type $(2,0)$ and

$$d\eta^3 = G_{11}\,\eta^{1\bar{1}} + G_{12}\,\eta^{1\bar{2}} + G_{13}\,\eta^{1\bar{3}} + \overline{G}_{12}\,\eta^{2\bar{1}} + G_{22}\,\eta^{2\bar{2}} + G_{23}\,\eta^{2\bar{3}} + \overline{G}_{13}\,\eta^{3\bar{1}} + \overline{G}_{23}\,\eta^{3\bar{2}} + G_{33}\,\eta^{3\bar{3}},$$

for some $G_{11}, G_{22}, G_{33} \in \mathbb{R}$ and $G_{12}, G_{13}, G_{23} \in \mathbb{C}$.

From the structure of the Lie algebras $\mathfrak{g}_k$ $(4 \leq k \leq 7)$, the relation (3.24) and taking into account that $d\eta^3$ is of type (1,1), it follows that there exist $\lambda, \mu \in \mathbb{C}$ such that

$$
(3.25) \qquad \begin{cases} d\eta^1 = \lambda\, d\eta^3 + (A\,\eta^1 + B\,\eta^2 + E\,\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\[2mm] d\eta^2 = \mu\, d\eta^3 + (C\,\eta^1 + D\,\eta^2 + F\,\eta^3) \wedge (\eta^3 + \eta^{\bar{3}}), \\[2mm] d\eta^3 = \ \ G_{11}\,\eta^{1\bar{1}} + G_{12}\,\eta^{1\bar{2}} + G_{13}\,\eta^{1\bar{3}} + \overline{G}_{12}\,\eta^{2\bar{1}} + G_{22}\,\eta^{2\bar{2}} + G_{23}\,\eta^{2\bar{3}} \\[2mm] \qquad\quad + \overline{G}_{13}\,\eta^{3\bar{1}} + \overline{G}_{23}\,\eta^{3\bar{2}} + G_{33}\,\eta^{3\bar{3}}, \end{cases}
$$

for some $A, B, C, D, E, F \in \mathbb{C}$.

Now, we prove that these complex equations can be reduced to equations of the form (3.22). Notice first that with respect to the (1,0)-basis $\{\eta^1 - \lambda\,\eta^3, \eta^2 - \mu\,\eta^3, \eta^3\}$ we get complex equations of the form (3.25) with $\lambda = \mu = 0$. So, without loss of generality we can suppose $\lambda = \mu = 0$. Moreover, the coefficients E and F also vanish. In fact, suppose for example that $E \neq 0$ (the case $F \neq 0$ is similar). Using (3.25) with $\lambda = \mu = 0$, the condition $d(d\eta^1) = 0$ is equivalent to

$$
EG_{11} = EG_{12} = EG_{13} = EG_{22} = EG_{23} = 0,
$$

so $E \neq 0$ implies $d\eta^3 = G_{33}\,\eta^{3\bar{3}} = G_{33}\,\eta^3 \wedge (\eta^3 + \eta^{\bar{3}})$. But this is a contradiction with the structure of the Lie algebras $\mathfrak{g}_k$ $(4 \leq k \leq 7)$, because $d(\mathfrak{g}_k^*)$ would be annihilated by the real 1-form $\eta^3 + \eta^{\bar{3}}$.

From now on, we suppose that $\lambda = \mu = E = F = 0$ in the equations (3.25). A direct calculation shows that

$$
d\eta^{123} = \overline{G}_{13}\,\eta^{123\bar{1}} + \overline{G}_{23}\,\eta^{123\bar{2}} + (A + D + G_{33})\eta^{123\bar{3}},
$$

so η^{123} is closed if and only if $G_{13} = G_{23} = 0$ and $D = -A - G_{33}$. Moreover, the unimodularity of the Lie algebras $\mathfrak{g}_k$ $(4 \leq k \leq 7)$ implies that $G_{33} = 0$. In fact, taking the real basis $\{f^1, \ldots, f^6\}$ of $\mathfrak{g}_k^*$ given by

$$
\eta^1 = f^2 + i\,f^3, \qquad \eta^2 = f^4 + i\,f^5, \qquad \eta^3 = f^6 + i\,f^1,
$$

we get that the trace of ad_{f_6} is zero if and only if $G_{33} = -2\,\mathfrak{Re}\,A - 2\,\mathfrak{Re}\,D$, which implies, using that $G_{33} = -A - D$, that the coefficient $G_{33} = 0$.

Summing up, we have proved the existence of a $(1,0)$-basis $\{\eta^1, \eta^2, \eta^3\}$ satisfying the reduced complex equations

$$
(3.26) \qquad \begin{cases} d\eta^1 = (A\,\eta^1 + B\,\eta^2) \wedge (\eta^3 + \eta^{\bar{3}}), \\[2mm] d\eta^2 = (C\,\eta^1 - A\,\eta^2) \wedge (\eta^3 + \eta^{\bar{3}}), \\[2mm] d\eta^3 = \ \ G_{11}\,\eta^{1\bar{1}} + G_{12}\,\eta^{1\bar{2}} + \overline{G}_{12}\,\eta^{2\bar{1}} + G_{22}\,\eta^{2\bar{2}}, \end{cases}
$$

where $A, B, C, G_{12} \in \mathbb{C}$ and $G_{11}, G_{22} \in \mathbb{R}$.

Notice that $A^2 + BC \neq 0$ because otherwise the (1,0)-form $A\,\eta^1 + B\,\eta^2$ would be closed, but this is a contradiction to $b_1(\mathfrak{g}_k) = 1$, for $4 \leq k \leq 7$. Therefore, arguing as in the proof of Lemma 3.2.2 we can suppose that $B = C = 0$ and $|A| = 1$ in (3.26). Finally, the condition $d(d\eta^3) = 0$ is satisfied if and only if $(A + \overline{A})G_{11} = (A + \overline{A})G_{22} = (A - \overline{A})G_{12} = 0$. $\square$

As a consequence of the previous lemma, we have the following classification of complex structures on $\mathfrak{g}_k$, for $4 \leq k \leq 7$.

Proposition 3.2.6. *Up to isomorphism there is only one complex structure J with closed $(3,0)$-form on the Lie algebras $\mathfrak{g}_5 = A^{0,1,1}_{6,82}$ and $\mathfrak{g}_6 = A^{0,1,1}_{6,88}$, and two such complex structures on the Lie algebras $\mathfrak{g}_4 = A^{0,0,1}_{6,37}$ and $\mathfrak{g}_7 = B^1_{6,6}$. More concretely, the complex structures are given by:*

(3.27) $\quad (\mathfrak{g}_4, J^\pm)\colon d\omega^1 = i\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^2 = -i\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^3 = \pm\,\omega^{1\bar{1}};$

(3.28) $\quad (\mathfrak{g}_5, J)\colon d\omega^1 = \omega^1 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^2 = -\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^3 = \omega^{1\bar{2}} + \omega^{2\bar{1}};$

(3.29) $\quad (\mathfrak{g}_6, J)\colon d\omega^1 = i\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^2 = -i\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^3 = \omega^{1\bar{1}} + \omega^{2\bar{2}};$

(3.30) $\quad (\mathfrak{g}_7, J^\pm)\colon d\omega^1 = i\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^2 = -i\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}),\ d\omega^3 = \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}}).$

Proof. First notice that in the equations (3.22), after changing the sign of ω^3 if necessary, we can always suppose that $A = \cos\theta + i\sin\theta$ with angle $\theta \in [0, \pi)$. We have the following cases:

• If $\cos\theta \neq 0$, then (3.23) implies $G_{11} = G_{22} = 0$ and $\sin\theta\, G_{12} = 0$, so $\sin\theta = 0$ because $(G_{11}, G_{12}, G_{22}) \neq (0,0,0)$ is satisfied if and only if $G_{12} \neq 0$. Therefore, in this case $A = 1$ and, moreover, we can normalize the coefficient G_{12} (it suffices to consider $G_{12}\,\omega^1$ instead of ω^1). So the complex structure equations take the form (3.28), and in terms of the real basis $\{e^1, \ldots, e^6\}$ defined by $\omega^1 = e^2 - ie^3$, $\omega^2 = e^5 + ie^4$ and $\omega^3 = \frac{1}{2}e^6 - 2ie^1$, one has

$$de^1 = e^{24} + e^{35}, \quad de^2 = e^{26}, \quad de^3 = e^{36}, \quad de^4 = -e^{46}, \quad de^5 = -e^{56}, \quad de^6 = 0,$$

that is, the underlying Lie algebra is $\mathfrak{g}_5$.

• If $\cos\theta = 0$, then (3.23) implies that $A = i$ and $G_{12} = 0$. Therefore, the complex structure equations become

$$d\omega^1 = i\,\omega^1 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^2 = -i\,\omega^2 \wedge (\omega^3 + \omega^{\bar{3}}), \quad d\omega^3 = G_{11}\,\omega^{1\bar{1}} + G_{22}\,\omega^{2\bar{2}},$$

where $(G_{11}, G_{22}) \neq (0,0)$. We have the following possibilities:

- When $G_{22} = 0$ we can suppose that $G_{11} = \pm 1$ (it suffices to consider $\sqrt{|G_{11}|}\,\omega^1$ instead of ω^1), and then the complex structure equations reduce to (3.27). In terms of the real basis $\{e^1, \ldots, e^6\}$ given by $\omega^1 = e^2 - ie^3$, $\omega^2 = e^4 + ie^5$ and $\omega^3 = -\frac{1}{2}e^6 \pm 2ie^1$, we arrive at

$$de^1 = e^{23}, \quad de^2 = -e^{36}, \quad de^3 = e^{26}, \quad de^4 = -e^{56}, \quad de^5 = e^{46}, \quad de^6 = 0,$$

that is, the underlying Lie algebra is $\mathfrak{g}_4$. A standard argument allows to conclude that the two complex structures in (3.27) are non-isomorphic.

- The case $G_{11} = 0$ easily reduces to the previous case, so it does not produce any non-isomorphic complex structure.

- Finally, if $G_{11} \neq 0$ and $G_{22} \neq 0$ then we can suppose $G_{11} = \pm 1$ and $G_{22} = \pm 1$ (it suffices to consider $\sqrt{|G_{kk}|}\,\omega^k$ instead of ω^k for $k = 1, 2$). It is clear that the case $G_{11} = G_{22} = -1$ is equivalent to $G_{11} = G_{22} = 1$, so it remains to study the following three cases: $(G_{11}, G_{22}) = (1, 1), (1, -1), (-1, 1)$. In terms of the real basis $\{\beta^1, \ldots, \beta^6\}$ defined by $\omega^1 = \beta^2 + i\beta^4$, $\omega^2 = \beta^3 + i\beta^5$ and $\omega^3 = \frac{1}{2}\beta^6 + 2i\beta^1$, one has

$$d\beta^1 = -G_{11}\,\beta^{24} - G_{22}\,\beta^{35},\ d\beta^2 = -\beta^{46},\ d\beta^3 = \beta^{56},\ d\beta^4 = \beta^{26},\ d\beta^5 = -\beta^{36},\ d\beta^6 = 0.$$

When $(G_{11}, G_{22}) = (1, 1)$, taking the basis $e^1 = -2\beta^1$, $e^2 = \beta^2 + \beta^3$, $e^3 = -\beta^4 + \beta^5$, $e^4 = \beta^4 + \beta^5$, $e^5 = \beta^2 - \beta^3$ and $e^6 = -\beta^6$, the real structure equations are

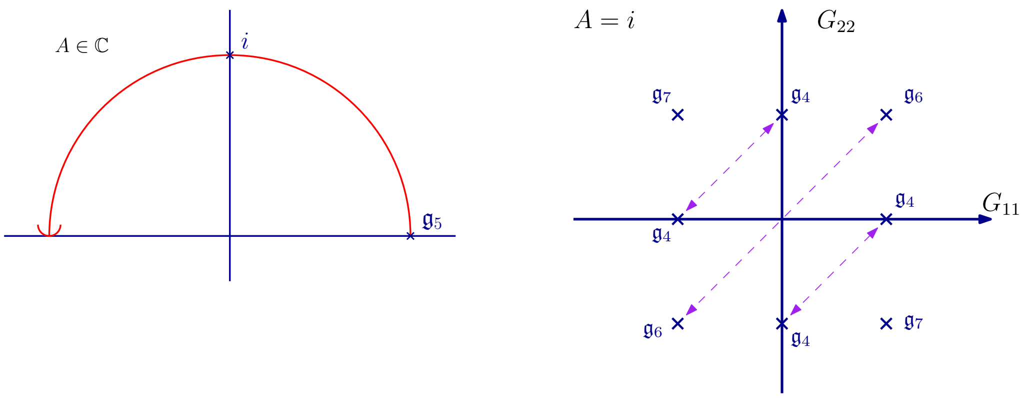
$$de^1 = e^{24} + e^{35},\ de^2 = -e^{36},\ de^3 = e^{26},\ de^4 = -e^{56},\ de^5 = e^{46},\ de^6 = 0,$$

so the underlying Lie algebra is $\mathfrak{g}_6$ and the complex structure is given by (3.29). The cases $(G_{11}, G_{22}) = (1, -1)$ and $(G_{11}, G_{22}) = (-1, 1)$ both correspond to the same Lie algebra (in fact, a change in the sign of β^1 gives an isomorphism), so we suppose next that $(G_{11}, G_{22}) = (1, -1)$, i.e.

$$d\beta^1 = -\beta^{24} + \beta^{35},\ d\beta^2 = -\beta^{46},\ d\beta^3 = \beta^{56},\ d\beta^4 = \beta^{26},\ d\beta^5 = -\beta^{36},\ d\beta^6 = 0.$$

Taking $e^1 = -\beta^1$, $e^3 = -\beta^3$ and $e^6 = -\beta^6$, we conclude that $\mathfrak{g}_7$ is the underlying Lie algebra. Therefore, the complex structures on $\mathfrak{g}_7$ are given by (3.27), and it can be proved that they are non-isomorphic. $\square$

Figure 3.5: Complex structures satisfying (3.22) according to Proposition 3.2.6.



The Lie algebra $\mathfrak{g}_8$

Proposition 3.2.7. *Let J be any complex structure on $\mathfrak{g}_8$ with closed volume $(3,0)$-form. Then, there is a $(1,0)$-basis $\{\omega^1,\omega^2,\omega^3\}$ satisfying one of the following reduced equations:*

$$(\mathfrak{g}_8, J')\colon\ d\omega^1 = 2i\,\omega^{13} + \omega^{3\bar3},\ \ d\omega^2 = -2i\,\omega^{23},\ \ d\omega^3 = 0; \tag{3.31}$$

$$(\mathfrak{g}_8, J'')\colon\ d\omega^1 = 2i\,\omega^{13} + \omega^{3\bar3},\ \ d\omega^2 = -2i\,\omega^{23} + \omega^{3\bar3},\ \ d\omega^3 = 0; \tag{3.32}$$

$$(\mathfrak{g}_8, J^A)\colon \begin{cases} d\omega^1 = -(A-i)\omega^{13} - (A+i)\omega^{1\bar3}, \\ d\omega^2 = (A-i)\omega^{23} + (A+i)\omega^{2\bar3}, \\ d\omega^3 = 0, \end{cases} \tag{3.33}$$

where $A \in \mathbb{C}$ with $\mathfrak{Im}\, A \neq 0$. Moreover, the complex structures above are non-equivalent.

Proof. With respect to the structure equations of $\mathfrak{g}_8$ given in Theorem 2.2.14, any closed 3-form $\rho \in Z^3(\mathfrak{g}_8)$ is given by

$$\begin{aligned}\rho &= a_1\, e^{126} + a_2\, e^{135} + a_3\, e^{145} + a_4\, e^{156} + a_5\, e^{235} + a_6(e^{146} + e^{236}) \\ &\quad + a_7\, e^{245} + a_8(e^{136} - e^{246}) + a_9\, e^{256} + a_{10}\, e^{346} + a_{11}\, e^{356} + a_{12}\, e^{456},\end{aligned}$$

where $a_1,\dots,a_{12} \in \mathbb{R}$. A direct calculation shows that such a ρ satisfies the conditions $d(J^*_\rho \rho) = 0$ and $\lambda(\rho) < 0$ if and only if $a_1 = 0$, $a_2 = -a_7$, $a_3 = a_5$, $a_{10} = 0$ and $a_6a_7 - a_5a_8 \neq 0$. Moreover, in this case $\lambda(\rho) = -4(a_6a_7 - a_5a_8)^2$.

The associated complex structures J^*_ρ express in terms of the real basis $\{e^1,\dots,e^6\}$ as

$$\begin{aligned}
J^*_\rho e^1 &= e^2 + \tfrac{a_5a_{12}-a_7a_{11}}{a_6a_7-a_5a_8}e^5 + \tfrac{a_6a_{12}-a_8a_{11}}{a_6a_7-a_5a_8}e^6, &\quad J^*_\rho e^4 &= -e^3 - \tfrac{a_4a_5+a_7a_9}{a_6a_7-a_5a_8}e^5 - \tfrac{a_4a_6+a_8a_9}{a_6a_7-a_5a_8}e^6,\\
J^*_\rho e^2 &= -e^1 + \tfrac{a_5a_{11}+a_7a_{12}}{a_6a_7-a_5a_8}e^5 + \tfrac{a_6a_{11}+a_8a_{12}}{a_6a_7-a_5a_8}e^6, &\quad J^*_\rho e^5 &= \tfrac{a_5a_6+a_7a_8}{a_6a_7-a_5a_8}e^5 + \tfrac{a_6^2+a_8^2}{a_6a_7-a_5a_8}e^6,\\
J^*_\rho e^3 &= e^4 + \tfrac{a_4a_7-a_5a_9}{a_6a_7-a_5a_8}e^5 + \tfrac{a_4a_8-a_6a_9}{a_6a_7-a_5a_8}e^6, &\quad J^*_\rho e^6 &= -\tfrac{a_5^2+a_7^2}{a_6a_7-a_5a_8}e^5 - \tfrac{a_5a_6+a_7a_8}{a_6a_7-a_5a_8}e^6.
\end{aligned}$$

Let us consider the basis of (1,0)-forms $\{\omega^1,\omega^2,\omega^3\}$ given by

$$\begin{aligned}
\omega^1 &= e^1 - iJ^*_\rho e^1 = e^1 - i\left(e^2 + k_1e^5 + k_2e^6\right),\\
\omega^2 &= e^3 - iJ^*_\rho e^3 = e^3 - i\left(e^4 + k_3e^5 + k_4e^6\right),\\
\omega^3 &= \tfrac{1}{2c}(e^5 - iJ^*_\rho e^5) = \tfrac{1}{2c}e^5 - i\left(\tfrac{b}{2c}e^5 + \tfrac{1}{2}e^6\right),
\end{aligned}$$

where $k_1 = \frac{a_5a_{12}-a_7a_{11}}{a_6a_7-a_5a_8}$, $k_2 = \frac{a_6a_{12}-a_8a_{11}}{a_6a_7-a_5a_8}$, $k_3 = \frac{a_4a_7-a_5a_9}{a_6a_7-a_5a_8}$, $k_4 = \frac{a_4a_8-a_6a_9}{a_6a_7-a_5a_8}$, $b = \frac{a_5a_6+a_7a_8}{a_6a_7-a_5a_8}$ and $c = \frac{a_6^2+a_8^2}{a_6a_7-a_5a_8}$. Notice that $c \neq 0$, and $-2(a_6 + ia_8)\omega^{123} = \rho + i\,J^*_\rho\rho$.

With respect to this basis, the complex structure equations are

$$
(3.34)\qquad \begin{cases} d\omega^1 = -(A-i)\omega^{13} - (A+i)\omega^{1\bar{3}} + B\,\omega^{3\bar{3}}, \\ d\omega^2 = (A-i)\omega^{23} + (A+i)\omega^{2\bar{3}} + C\,\omega^{3\bar{3}}, \\ d\omega^3 = 0, \end{cases}
$$

where $A = b+ic$, $B = 2c(k_1+ik_2)$ and $C = -2c(k_3+ik_4)$. Notice that $\mathfrak{Im}\,A = c \neq 0$. Now, we reduce the complex equations (3.34) as follows:

• If $A \neq -i$, then with respect to the (1,0)-basis $\{\eta^1, \eta^2, \eta^3\}$ given by

$$
\eta^1 = -(A+i)\omega^1 + B\omega^3, \quad \eta^2 = (A+i)\omega^2 + C\omega^3, \quad \eta^3 = \omega^3,
$$

the complex structure equations are of the form (3.33).

• If $A = -i$, the equations (3.34) reduce to

$$
J_{(B,C)}\colon\ d\omega^1 = 2i\omega^{13} + B\omega^{3\bar{3}}, \quad d\omega^2 = -2i\omega^{23} + C\omega^{3\bar{3}}, \quad d\omega^3 = 0.
$$

Notice that the structures $J_{(B,C)}$ and $J_{(C,B)}$ are equivalent, since it suffices to consider the change of basis $\eta^1 = \omega^2$, $\eta^2 = \omega^1$, $\eta^3 = -\omega^3$. Now:

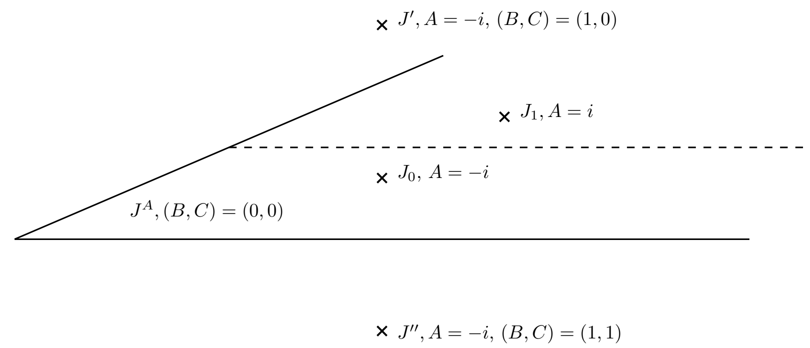
- if $B = C = 0$ then the complex equations are of the form (3.33) with $A = -i$;

- if only one of the coefficients B, C is non-zero, for instance B, then taking $\frac{1}{B}\omega^1$ instead of ω^1, we arrive at the complex equations (3.31);

- finally, if $B, C \neq 0$ then we can normalize both coefficients and the corresponding complex equations are (3.32).

It is straightforward to check that the complex structures given in equations (3.31)–(3.33) are non-equivalent. □

Figure 3.6: Complex structures on $\mathfrak{g}_8$ according to Proposition 3.2.7.



Remark 3.2.8. Note that on $\mathfrak{g}_8$ there exists a unique bi-invariant complex structure and a unique abelian complex structure corresponding to the values $A=-i$ and $A=i$ in (3.33). We denote them by J_0 and J_1 respectively. It is worth noticing, that the bi-invariant complex structure J_0 corresponds to the complex structure equations (2.11) of the Nakamura manifold [67]. Hence, as a consequence of Proposition 3.2.7, we have found an infinitely non-isomorphic family of invariant complex structures on the real six-dimensional solvmanifold underlying the complex parallelizable Nakamura manifold. The abelian structure J_1 was also studied in [4].

Remark 3.2.9. The proof of Theorem 2.1.32 is based on an invariant deformation of a solvmanifold $(M=G/\Gamma, J)$, where $(\mathfrak{g}, J)$ is isomorphic to $(\mathfrak{g}_8, J_1)$. Hence, we see that the Lie algebra $\mathfrak{g}_8$ admits complex structures with no closed $(3,0)$-form. Moreover, there are integrable complex structures J^*_ρ on $\mathfrak{g}_8$ which do not come from a closed 3-form ρ. For example, the 3-form $\rho = e^{123} + \frac{1}{\sqrt{2}}(-e^{135}+e^{146}+e^{236}+e^{245}) \notin Z^3(\mathfrak{g}_8)$ defines the following integrable complex structure:

$$\begin{aligned}
&J^*_\rho e^1 = e^2, && J^*_\rho e^2 = -e^1, && J^*_\rho e^3 = e^4,\\
&J^*_\rho e^4 = -e^3, && J^*_\rho e^5 = \sqrt{2}e^1 + e^6, && J^*_\rho e^6 = -\sqrt{2}e^2 - e^5.
\end{aligned}$$

If we consider the (1,0)-basis $\omega^1 = e^1 - ie^2$, $\omega^2 = e^3 - ie^4$ and $\omega^3 = e^5 - i(\sqrt{2}e^1 + e^6)$ then the complex structure equations are:

$$\begin{cases}
d\omega^1 = -\frac{1}{\sqrt{2}}\omega^{1\bar{1}} - i\omega^{1\bar{3}},\\
d\omega^2 = -\frac{1}{\sqrt{2}}\omega^{12} + \frac{1}{\sqrt{2}}\omega^{2\bar{1}} + i\omega^{2\bar{3}},\\
d\omega^3 = -\frac{1}{\sqrt{2}}\omega^{1\bar{3}} - \frac{1}{\sqrt{2}}\omega^{3\bar{1}}.
\end{cases}$$

This fact reveals the rich complex geometry on the Lie algebra $\mathfrak{g}_8$ in a form similar to the nilpotent Lie algebra $\mathfrak{h}_5$ underlying the Iwasawa manifold.

The Lie algebra $\mathfrak{g}_9$

Let us consider now the Lie algebra $\mathfrak{g}_9 = B^1_{6,4} = (e^{45}, e^{15}+e^{36}, e^{14}-e^{26}+e^{56}, -e^{56}, e^{46}, 0)$. If J is a complex structure with a closed complex volume form then we can find a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations:

$$\begin{cases}
d\omega^1 = -c^2\omega^{1\bar{1}} - c\,\omega^{3\bar{1}} - c\,\omega^{1\bar{3}} - \omega^{3\bar{3}},\\
d\omega^2 = (E+\frac{1}{2})c\,\omega^{1\bar{1}} - \frac{i}{2}\omega^{2\bar{1}} + E\omega^{3\bar{1}} + \frac{1}{2}\omega^{1\bar{3}} + \frac{i}{2}\omega^{12} - E\omega^{13},\\
d\omega^3 = (c^2+\frac{i}{2})c\,\omega^{1\bar{1}} + (c^2+\frac{i}{2})\omega^{3\bar{1}} + c^2\omega^{1\bar{3}} + c\,\omega^{3\bar{3}} - \frac{i}{2}\omega^{13},
\end{cases} \tag{3.35}$$

where the coefficient c is real and $E \in \mathbb{C}$ (for further details see Appendix A, Lemma A.0.11). The next proposition shows that the coefficients c and E in (3.35) can be reduced to zero.

Proposition 3.2.10. *Up to isomorphism, there is only one complex structure with closed* $(3,0)$*-form on the Lie algebra* $\mathfrak{g}_9 = B^1_{6,4}$*, whose complex equations are*

$$(\mathfrak{g}_9, J)\colon\ d\omega^1 = -\omega^{3\bar{3}},\quad d\omega^2 = \frac{i}{2}\omega^{12} + \frac{1}{2}\omega^{1\bar{3}} - \frac{i}{2}\omega^{2\bar{1}},\quad d\omega^3 = -\frac{i}{2}\omega^{13} + \frac{i}{2}\omega^{3\bar{1}}. \tag{3.36}$$

Proof. Now, let $\{\omega^1, \omega^2, \omega^3\}$ be a (1,0)-basis satisfying (3.35), and consider the new (1,0)-basis

$$\{\sigma^1 = \omega^1,\quad \sigma^2 = icE\omega^1 + \omega^2 + iE\omega^3,\quad \sigma^3 = c\omega^1 + \omega^3\}.$$

A direct calculation shows that this basis satisfies equations (3.35) with $c = 0$ and $E = 0$, that is, the complex equations can always be reduced to (3.36). In particular, all the complex structures are equivalent. $\square$

The results of this section are summarized in Table 3.2.

Lie algebra	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$	**complex structure**	**Frölicher type**
$\mathfrak{g}_1 = (25, -15, 45, -35, 0, 0)$	2	5	8	$J := (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), 0),$	$E_1 \cong E_\infty$
$\mathfrak{g}_2^0 = (15, -25, -35, 45, 0, 0)$	2	5	8	$J := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), 0),$	$E_1 \cong E_\infty$
$\mathfrak{g}_2^{\alpha>0} = (\alpha 15 + 25, -15 + \alpha 25, -\alpha 35 + 45, -35 - \alpha 45, 0, 0)$	2	3	4	$J^\pm := ((\pm\cos\theta + i\sin\theta)(\omega^{13} + \omega^{1\bar{3}}), (\mp\cos\theta - i\sin\theta)(\omega^{23} + \omega^{2\bar{3}}), 0),$ $\theta \in (0, \frac{\pi}{2}),\ \alpha = \frac{\cos\theta}{\sin\theta}$	$E_1 \cong E_\infty$
$\mathfrak{g}_3 = (0, -13, 12, 0, -46, -45)$	2	3	4	$J^x := (0, -\frac{1}{2}\omega^{13} - (\frac{1}{2} + ix)\omega^{1\bar{3}} + ix\omega^{3\bar{1}}, \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\omega^{2\bar{1}})$ $x > 0$	$E_1 \cong E_\infty$
$\mathfrak{g}_4 = (23, -36, 26, -56, 46, 0)$	1	3	6	$J^\pm := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm\omega^{1\bar{1}}),$	$E_1 \cong E_\infty$
$\mathfrak{g}_5 = (24 + 35, 26, 36, -46, -56, 0)$	1	3	6	$J := (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{2}} + \omega^{2\bar{1}}),$	$E_1 \cong E_\infty$
$\mathfrak{g}_6 = (24 + 35, -36, 26, -56, 46, 0)$	1	3	6	$J := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{1}} + \omega^{2\bar{2}}),$	$E_1 \cong E_\infty$
$\mathfrak{g}_7 = (24 + 35, 46, 56, -26, -36, 0)$	1	3	6	$J^\pm := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}})),$	$E_1 \cong E_\infty$
$\mathfrak{g}_8 = (16 - 25, 15 + 26, -36 + 45, -35 - 46, 0, 0)$	2	3	4	$J^A := (-(A - i)\omega^{13} - (A + i)\omega^{1\bar{3}}, (A - i)\omega^{23} + (A + i)\omega^{2\bar{3}}, 0),\ \mathfrak{Im}\,(A) \neq 0$ $J' := (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23}, 0)$ $J'' := (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23} + \omega^{3\bar{3}}, 0)$	$E_1 \cong E_\infty\ A \neq \pm i$ $E_1 \ncong E_2 \cong E_\infty\ A = \pm i$ $E_1 \ncong E_2 \cong E_\infty$ $E_1 \ncong E_2 \cong E_\infty$
$\mathfrak{g}_9 = (45, 15 + 36, 14 - 26 + 56, -56, 46, 0)$	1	1	2	$J := (-\omega^{3\bar{3}}, \frac{1}{2}(i\omega^{12} - i\omega^{2\bar{1}} + \omega^{1\bar{3}}), \frac{i}{2}(-\omega^{13} + \omega^{3\bar{1}})),$	$E_1 \cong E_\infty$

Table 3.2: Complex structures with non-zero closed complex volume form $\Psi \in \wedge^{3,0}\mathfrak{g}^*$ on 6-dimensional unimodular solvable Lie algebras.

Chapter 4

Complex cohomologies and the $\partial\bar{\partial}$-lemma

Let $M = G/\Gamma$ be a six-dimensional solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle. We make use of the classification of the complex structures obtained in Chapter 3 in order to study the behaviour of some important complex invariants as the Dolbeault cohomology and, more generally, the Frölicher spectral sequence $\{E_r(M)\}_{r\geq 1}$. The validity of the $\partial\bar{\partial}$-lemma is also considered. Concretely, Section 4.1 is devoted to give a general description of the behaviour of the Frölicher spectral sequence of nilmanifolds. As a consequence, we provide an example based on a complex nilmanifold answering to a question posed by Angella and Tomassini [10] concerning the existence of a compact complex manifold satisfying $E_1(M) \cong E_\infty(M)$ and the symmetry of the Hodge numbers $h^{q,p}_{\bar{\partial}}(M) = h^{p,q}_{\bar{\partial}}(M)$ for every $p, q \in \mathbb{N}$, but not the $\partial\bar{\partial}$-lemma. Section 4.2 deals with the computation of the complex cohomologies of (non-nilpotent) solvmanifolds. We firstly compute the Frölicher sequence for the Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ and, as a consequence, we provide a partial result about the behaviour of the Frölicher sequence for solvmanifolds. We consider the results of Kasuya and Angella [51, 7] for computing the Dolbeault and Bott-Chern cohomology of complex solvmanifolds of splitting type. The complex structures with holomorphically trivial canonical bundle which are of splitting type have underlying Lie algebras isomorphic to $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha}$ with $\alpha \geq 0$ or $\mathfrak{g}_8$. In addition, we compute some lattices on the corresponding Lie groups which are compatible with the splitting and we finish the section providing several examples of solvmanifolds with different cohomological behaviour from the point of view of the Frölicher sequence and the $\partial\bar{\partial}$-lemma.

The study of complex invariants under holomorphic deformations is considered in Section 4.3. This section is divided into two parts, the first devoted to the Frölicher sequence and the second to the $\partial\bar{\partial}$-lemma. We provide an invariant deformation of a complex nilmanifold with underlying Lie algebra isomorphic to $\mathfrak{h}_4$ endowed with the abelian complex structure, showing the non-closedness of the E_1-degeneration property of the Frölicher spectral sequence. In addition, a family of complex structures on the nilpotent Lie algebra $\mathfrak{h}_{15}$ shows that for $r \geq 2$ the dimension of the terms $E_r^{\bullet,\bullet}(J_t)$ in general is neither upper nor lower semi-continuous function of t, in contrast to the upper semi-continuity of the dimensions of the first step terms $E_1^{\bullet,\bullet}$ proved by Kodaira and

Spencer [53]. Moreover, this family J_t allows us to prove that the degeneration of the Frölicher sequence at the second step is not an open property under deformations. As regards the $\partial\bar\partial$-lemma, we recall the recent results obtained by Angella and Kasuya [8] used to prove the non-closedness of the $\partial\bar\partial$-lemma. Making use of these techniques we construct a countable family of complex solvmanifolds $\{(M_k, J_k)\}_{k\in\mathbb{Z}}$ with underlying Lie algebra isomorphic to $\mathfrak{g}_8$ which do not satisfy the $\partial\bar\partial$-lemma but admitting an invariant holomorphic deformation $\{(M_k, J_{k,t})\}_{t\in\Delta}$ satisfying the $\partial\bar\partial$-lemma for any $t \in \Delta^*$. This family contains the one obtained by Angella and Kasuya in [8] to prove the non-closedness of the $\partial\bar\partial$-lemma.

4.1 The Frölicher spectral sequence on nilmanifolds

Let $M = G/\Gamma$ be a six-dimensional nilmanifold endowed with an invariant complex structure J, and let $\mathfrak{g}$ be the Lie algebra of G. Rollenske [80, Section 4.2] proves that if $\mathfrak{g} \ncong \mathfrak{h}_7$ then the natural inclusion of differential graded algebras $\left(\wedge^{\bullet,\bullet}\mathfrak{g}^*, \bar\partial\right) \hookrightarrow (\wedge^{\bullet,\bullet}M, \bar\partial)$ induces an isomorphism

$$H^{\bullet,\bullet}_{\bar\partial}(\mathfrak{g}) \longrightarrow H^{\bullet,\bullet}_{\bar\partial}(M),$$

between the Lie-algebra Dolbeault cohomology of $(\mathfrak{g}, J)$ and the Dolbeault cohomology of (M, J). Thus, an inductive argument [23, Theorem 4.2] implies that the natural map $\iota\colon E_r^{p,q}(\mathfrak{g}) \longrightarrow E_r^{p,q}(M)$ is also an isomorphism, and therefore $E_r^{p,q}(M) \cong E_r^{p,q}(\mathfrak{g})$ for any p, q and any $r \geq 1$, whenever $\mathfrak{g} \ncong \mathfrak{h}_7$. Using these results and the classification of invariant complex structures up to equivalence obtained for nilmanifolds, we show the general behaviour of the Frölicher sequence in dimension 6.

For the study of the degeneration of the Frölicher sequence at the first step, it is sufficient to study the Dolbeault cohomology in relation to the (de Rham) cohomology of the Lie algebra. Moreover, using the Serre duality in Dolbeault cohomology for Lie algebras proved in [79], namely $H^{p,q}_{\bar\partial}(\mathfrak{g}) = H^{n-p,n-q}_{\bar\partial}(\mathfrak{g})$, it suffices to study the spaces $H^{p,q}_{\bar\partial}$ for $(p,q) = (1,0), (0,1), (2,0), (1,1), (0,2), (3,0)$ and $(2,1)$. In what follows we use the notation:

$$\delta_A^B = \begin{cases} 1 & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases}$$

Proposition 4.1.1. *Let us consider a six-dimensional nilpotent Lie algebra $\mathfrak{g}$ with a complex structure J in the family*

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \rho\,\omega^{12} + \omega^{1\bar1} + \lambda\,\omega^{1\bar2} + D\,\omega^{2\bar2}, \tag{4.1}$$

with $\rho = 0, 1$ and $\lambda \in \mathbb{R}$, $D \in \mathbb{C}$ with $\lambda, \mathfrak{Im}\, D \geq 0$. Then:

- *(i) If $\mathfrak{g} \cong \mathfrak{h}_3$, $\mathfrak{h}_6$ or $\mathfrak{h}_8$, then the Frölicher sequence degenerates at the first step for any J.*
- *(ii) If $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$, then $E_1 \cong E_\infty$ if and only if J is not abelian. Moreover, any abelian complex structure on $\mathfrak{h}_2$ or $\mathfrak{h}_4$ satisfies $E_1 \ncong E_2 \cong E_\infty$.*

(iii) A complex structure J on $\mathfrak{g} \cong \mathfrak{h}_5$ satisfies $E_1 \cong E_\infty$ if and only if $\rho D \neq 0$. Moreover, if $\rho = 0$ or $D = 0$ then $E_1 \not\cong E_2 \cong E_\infty$.

Proof. It is direct to see that the Dolbeault groups $H^{p,q}_{\bar\partial}$ for $(p,q) = (1,0), (2,0), (3,0), (0,1)$ and $(0,2)$ of any complex structure given by the structure equations (4.1) are:

$$H^{1,0}_{\bar\partial} = \mathbb{C}\langle[\omega^1],[\omega^2]\rangle, \qquad H^{2,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{12}], \delta_0^D[\omega^{13}]\rangle, \qquad H^{3,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{123}]\rangle,$$

$$H^{0,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1}],[\omega^{\bar2}],(1-\rho)[\omega^{\bar3}]\rangle, \qquad H^{0,2}_{\bar\partial} = \mathbb{C}\langle(1-\rho)[\omega^{\bar1\bar2}],[\omega^{\bar1\bar3}],[\omega^{\bar2\bar3}]\rangle.$$

For $\rho = 0$, the Dolbeault groups $H^{1,1}_{\bar\partial}$ and $H^{2,1}_{\bar\partial}$ for any complex structure in the family are:

$$H^{1,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{1\bar2}],[\omega^{2\bar1}],[\omega^{2\bar2}],[\omega^{1\bar3}],[\omega^{2\bar3}],\delta_0^D[\omega^{3\bar1}+\lambda\omega^{3\bar2}]\rangle,$$

$$H^{2,1}_{\bar\partial} = \mathbb{C}\langle\delta_0^D[\omega^{12\bar2}],[\omega^{12\bar3}],[\omega^{13\bar1}-D\omega^{23\bar2}],[\omega^{23\bar1}+\lambda\omega^{23\bar2}],[\omega^{13\bar2}],\delta_0^D[\omega^{13\bar3}]\rangle,$$

whereas for $\rho = 1$ these Dolbeault cohomology groups are:

$$H^{1,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{1\bar2}],[\omega^{2\bar1}],[\omega^{2\bar2}],[\omega^{1\bar3}+\omega^{3\bar2}],[D\omega^{2\bar3}-\omega^{3\bar1}-\lambda\omega^{3\bar2}]\rangle,$$

$$H^{2,1}_{\bar\partial} = \mathbb{C}\langle\delta_0^D[\omega^{12\bar2}],[D\omega^{12\bar3}-\omega^{13\bar1}],[\lambda\omega^{12\bar3}+\omega^{23\bar1}],[\omega^{12\bar3}-\omega^{23\bar2}],[\omega^{13\bar2}]\rangle.$$

(i) The Lie algebras $\mathfrak{h}_3$, $\mathfrak{h}_6$ and $\mathfrak{h}_8$

If $\mathfrak{g} \cong \mathfrak{h}_3$ then we have that $\rho = \lambda = 0$ and $D = \pm 1$ (see Table 3.1), and counting the dimension of the Dolbeault groups we get

$$\dim E_1^{|1|} = 5 = b_1(\mathfrak{h}_3), \quad \dim E_1^{|2|} = 9 = b_2(\mathfrak{h}_3), \quad \dim E_1^{|3|} = 10 = b_3(\mathfrak{h}_3).$$

Hence $E_1 \cong E_\infty$ for the two complex structures on $\mathfrak{h}_3$. It follows from Table 3.1 that if $\mathfrak{g} \cong \mathfrak{h}_6$ then it admits only a complex structure corresponding to $\rho = \lambda = 1$ and $D = 0$, therefore

$$\dim E_1^{|1|} = 4 = b_1(\mathfrak{h}_6), \quad \dim E_1^{|2|} = 9 = b_2(\mathfrak{h}_6), \quad \dim E_1^{|3|} = 12 = b_3(\mathfrak{h}_6).$$

Similarly if $\mathfrak{g} \cong \mathfrak{h}_8$ then it admits only a complex structure which corresponds to $\rho = \lambda = D = 0$ and

$$\dim E_1^{|1|} = 5 = b_1(\mathfrak{h}_8), \quad \dim E_1^{|2|} = 11 = b_2(\mathfrak{h}_8), \quad \dim E_1^{|3|} = 14 = b_3(\mathfrak{h}_8).$$

Therefore, for any complex structure on $\mathfrak{h}_6$ and $\mathfrak{h}_8$ we get $E_1 \cong E_\infty$ and the proof of (i) is complete.

(ii) The Lie algebras $\mathfrak{h}_2$ and $\mathfrak{h}_4$

In order to prove (ii), first we notice that if J is non-abelian on $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$ then from Table 3.1 the coefficient $D \neq 0$, and counting the dimension of the Dolbeault groups above we get

$$\dim E_1^{|1|} = 4 = b_1(\mathfrak{g}), \quad \dim E_1^{|2|} = 8 = b_2(\mathfrak{g}), \quad \dim E_1^{|3|} = 10 = b_3(\mathfrak{g});$$

therefore, $E_1 \cong E_\infty$ for any non-abelian complex structure on $\mathfrak{h}_2$ or $\mathfrak{h}_4$.

Let us suppose now that J is abelian ($\rho = 0$) on $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$. Since from Table 3.1 the coefficient $D \neq 0$ again, counting dimensions we get that $E_1 \ncong E_\infty$. More precisely,

$$\begin{aligned} &\dim E_1^{|1|} = 5 > 4 = b_1(\mathfrak{g}), \quad \dim E_1^{|2|} = 9 > 8 = b_2(\mathfrak{g}), \quad \dim E_1^{|3|} = 10 = b_3(\mathfrak{g}), \\ &\dim E_1^{|4|} = 9 > 8 = b_4(\mathfrak{g}), \quad \dim E_1^{|5|} = 5 > 4 = b_5(\mathfrak{g}). \end{aligned}$$

Therefore $E_1^{|3|} \cong E_\infty^{|3|}$ and we must compute $\dim E_2^{|1|}$, $\dim E_2^{|2|}$, $\dim E_2^{|4|}$ and $\dim E_2^{|5|}$. Next we show that the map $E_1^{0,1} \xrightarrow{d_1} E_1^{1,1}$ is non-zero. For the class $[\omega^{\bar{3}}] \in E_1^{0,1}$, we have

$$d_1([\omega^{\bar{3}}]) = [\partial\omega^{\bar{3}}] = [\omega^{1\bar{1}} + \lambda\,\omega^{2\bar{1}} + \bar{D}\,\omega^{2\bar{2}}].$$

Since $\bar{\partial}(\mathfrak{g}^{(1,0)}) = \langle \bar{\partial}\omega^3 = \omega^{1\bar{1}} + \lambda\,\omega^{1\bar{2}} + D\,\omega^{2\bar{2}} \rangle$, we conclude that $d_1([\omega^{\bar{3}}]) = 0$ if and only if $\lambda = 0$ and $D \in \mathbb{R}$, but this corresponds to $\mathfrak{g} \cong \mathfrak{h}_3$. Therefore, $\dim E_2^{0,1} = \dim \operatorname{Ker} d_1 \leq \dim E_1^{0,1} - 1$ and so $\dim E_2^{|1|} \leq \dim E_1^{|1|} - 1 = 4 = b_1(\mathfrak{g})$, which implies $E_2^{|1|} \cong E_\infty^{|1|}$. Moreover, $\dim E_2^{1,1} \leq \dim E_1^{1,1} - \dim \operatorname{Im} d_1$ and so $E_2^{|2|} \cong E_\infty^{|2|}$ because $\dim E_2^{|2|} \leq \dim E_1^{|2|} - 1 = 8 = b_2(\mathfrak{g})$.

A similar argument can be applied to prove that the map $E_1^{2,2} \xrightarrow{d_1} E_1^{3,2}$ is also non-zero. In fact,

$$d_1([\omega^{13\bar{1}\bar{3}} - \lambda\,\omega^{13\bar{2}\bar{3}} - D\,\omega^{23\bar{2}\bar{3}}]) = (\lambda^2 - D + \bar{D})[\omega^{123\bar{1}\bar{2}}]$$

is zero if and only if $\lambda = 0$ and D is a non-zero real number, but this corresponds to $\mathfrak{g} \cong \mathfrak{h}_3$. Arguing as above allows us to conclude that $E_2^{|k|} \cong E_\infty^{|k|}$ also for $k = 4, 5$, which completes the proof of (ii).

(iii) The Lie algebra $\mathfrak{h}_5$

If $\mathfrak{g} \cong \mathfrak{h}_5$ we divide the study of the behaviour of the Frölicher sequence into abelian and non-abelian complex structures. The relevant Betti numbers of $\mathfrak{h}_5$ are $b_1(\mathfrak{h}_5) = 4$, $b_2(\mathfrak{h}_5) = 8$ and $b_3(\mathfrak{h}_5) = 10$.

If J is a non-abelian complex structure then $\rho = 1$ and the complex parameter D may assume the values $D = 0$ or $D \neq 0$.

• If $D=0$ then

$$\dim E_1^{|1|}=4=b_1(\mathfrak{h}_5),\qquad \dim E_1^{|2|}=9>8=b_2(\mathfrak{h}_5),\quad \dim E_1^{|3|}=12>10=b_3(\mathfrak{h}_5),$$
$$\dim E_1^{|4|}=9>8=b_4(\mathfrak{h}_5),\quad \dim E_1^{|5|}=4=b_5(\mathfrak{h}_5).$$

Hence $E_1^{|k|}\cong E_\infty^{|k|}$ for $k=1,5$. The map $E_1^{1,1}\xrightarrow{d_1}E_1^{2,1}$ is not zero as for instance $d_1([\omega^{1\bar3}+\omega^{3\bar2}])=(1-\lambda^2)[\omega^{12\bar2}]$ which is non-zero because from Table 3.1 the coefficient λ^2 cannot be equal to 1. Hence, $\dim E_2^{|2|}\le\dim E_1^{|2|}-1=8=b_2(\mathfrak{h}_5)$. Similarly the map $E_1^{1,2}\xrightarrow{d_1}E_1^{2,2}$ is not zero because $d_1([\omega^{1\bar1\bar3}+\lambda\omega^{3\bar2\bar3}])=[\omega^{12\bar1\bar3}+\lambda\omega^{12\bar2\bar3}+\lambda^2\omega^{23\bar1\bar2}]$ which is non-zero in $E_1^{2,2}$ because $\lambda^2\ne1$ and $\bar\partial(\wedge^{2,1}\mathfrak{g}^*)=\langle\omega^{12\bar1\bar2},\omega^{12\bar1\bar3}+\lambda\omega^{12\bar2\bar3}+\omega^{23\bar1\bar2}\rangle$. In conclusion $E_2\cong E_\infty$.

• if $D\ne0$ then $\dim E_1^{|1|}=4=b_1(\mathfrak{h}_5)$, $\dim E_1^{|2|}=8=b_2(\mathfrak{h}_5)$, $\dim E_1^{|3|}=10=b_3(\mathfrak{h}_5)$, that is, $E_1\cong E_\infty$.

Let us consider now if J is an abelian complex structure on $\mathfrak{h}_5$ ($\rho=0$, $\lambda=1$ and $D\in[0,\frac14)$ in (4.1):

• if $D=0$ counting dimensions we get:

$$\dim E_1^{|1|}=5>b_1(\mathfrak{h}_5),\quad \dim E_1^{|2|}=11>b_2(\mathfrak{h}_5),\quad \dim E_1^{|3|}=14>b_3(\mathfrak{h}_5),$$
$$\dim E_1^{|4|}=11>b_4(\mathfrak{h}_5),\quad \dim E_1^{|5|}=5>b_5(\mathfrak{h}_5).$$

We consider the following non-zero maps d_1, where we specify one cohomology class and its corresponding non-zero image in each case:

$$\begin{array}{ccccccc}
E_1^{0,1} & \xrightarrow{d_1} & E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} & \xrightarrow{d_1} & E_1^{3,1}\\
[\omega^{\bar3}] & \mapsto & [\omega^{2\bar1}]-[\omega^{1\bar2}] & & & & \\
 & & [\omega^{1\bar3}] & \mapsto & -[\omega^{12\bar2}] & & \\
 & & & & [\omega^{13\bar3}] & \mapsto & [\omega^{123\bar1}].
\end{array}$$

Similarly, the following homomorphisms

$$E_1^{0,2}\xrightarrow{d_1}E_1^{1,2}\xrightarrow{d_1}E_1^{2,2}\xrightarrow{d_1}E_1^{3,2}$$

are non-zero (take for instance the classes $[\omega^{\bar2\bar3}]$, $[\omega^{3\bar1\bar3}+\omega^{3\bar2\bar3}]$ and $[\omega^{23\bar1\bar3}+\omega^{23\bar2\bar3}]$). Since $E_2^{p,q}\cong\operatorname{Ker}d_1/\operatorname{Im}d_1$, counting the dimensions we get

$$\begin{array}{ll}
\dim E_2^{|1|}\le\dim E_1^{|1|}-1=4=b_1(\mathfrak{h}_5), & \dim E_2^{|2|}\le\dim E_1^{|2|}-3=8=b_2(\mathfrak{h}_5),\\
\dim E_2^{|3|}\le\dim E_1^{|3|}-4=10=b_3(\mathfrak{h}_5), & \dim E_2^{|4|}\le\dim E_1^{|4|}-3=8=b_4(\mathfrak{h}_5),\\
\dim E_2^{|5|}\le\dim E_1^{|5|}-1=4=b_5(\mathfrak{h}_5). &
\end{array}$$

This implies that $E_2 \cong E_\infty$ because $\dim E_2^{|k|} = b_k(\mathfrak{h}_5)$ for all k.
• if $D \neq 0$, we get:

$$\dim E_1^{|1|} = 5 > b_1(\mathfrak{h}_5), \quad \dim E_1^{|2|} = 9 > b_2(\mathfrak{h}_5), \quad \dim E_1^{|3|} = 10 = b_3(\mathfrak{h}_5),$$
$$\dim E_1^{|4|} = 5 > b_4(\mathfrak{h}_5), \quad \dim E_1^{|5|} = 9 > b_5(\mathfrak{h}_5).$$

It is enough to see that the maps $E_1^{0,1} \xrightarrow{d_1} E_1^{1,1}$ and $E_1^{2,2} \xrightarrow{d_1} E_1^{3,2}$ are not zero. In the first case it is clear because $d_1([\omega^{\bar3}]) = [\partial\omega^{\bar3}] = [\omega^{1\bar1} + \omega^{2\bar1}] \neq 0$. For the second case $d_1([\omega^{13\bar2\bar3}]) = [\partial\omega^{13\bar2\bar3}] = [-\omega^{123\bar1\bar2}] \neq 0$. Therefore $E_1 \ncong E_2 \cong E_\infty$.

□

Proposition 4.1.2. *Let us consider a six-dimensional Lie algebra $\mathfrak{g}$ with a complex structure J in the family*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar1}, \quad d\omega^3 = \rho\,\omega^{12} + B\,\omega^{1\bar2} + c\,\omega^{2\bar1}, \tag{4.2}$$

where $\rho = 0, 1$ and $B \in \mathbb{C}$, $c \geq 0$ with $(\rho, B, c) \neq (0, 0, 0)$. Then:

(i) If $\mathfrak{g} \cong \mathfrak{h}_7$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$, then the Frölicher sequence degenerates at the first step for any J.

(ii) Any complex structure on $\mathfrak{h}_{16}$ satisfies $E_1 \ncong E_2 \cong E_\infty$.

(iii) Any complex structure on $\mathfrak{h}_{13}$ or $\mathfrak{h}_{14}$ satisfies $E_1 \cong E_2 \ncong E_3 \cong E_\infty$.

(iv) On $\mathfrak{h}_{15}$ we have:

(iv.1) $E_1 \ncong E_2 \cong E_\infty$, if $c = 0$ and $B \neq \rho$;
(iv.2) $E_1 \cong E_2 \ncong E_3 \cong E_\infty$, if $\rho = 1$ and $|B - 1| \neq c \neq 0$;
(iv.3) $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$, if $\rho = 0$ and $|B| \neq c \neq 0$.

Proof. We follow a similar argument as in the proof of Proposition 4.1.1 studying the spaces $H^{p,q}_{\bar\partial}$ for $(p,q) = (1,0), (0,1), (2,0), (1,1), (0,2), (3,0)$ and $(2,1)$. For $\rho = 1$, these Dolbeault groups for any complex structure in the family are:

$$\begin{aligned}
&H^{1,0}_{\bar\partial} = \mathbb{C}\langle[\omega^1], \delta_0^B\delta_0^c[\omega^3]\rangle, \qquad H^{2,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{12}], \delta_0^c[\omega^{13}]\rangle, \qquad H^{3,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{123}]\rangle,\\
&H^{0,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1}], [\omega^{\bar2}]\rangle, \qquad H^{0,2}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}]\rangle,\\
&H^{1,1}_{\bar\partial} = \mathbb{C}\langle(Bc + \delta_0^B)[\omega^{1\bar2}], [\omega^{1\bar3} + \omega^{2\bar2}], [B\omega^{1\bar3} - \omega^{3\bar1}], \delta_0^c[\omega^{2\bar1}], \delta_0^c[\omega^{3\bar2}]\rangle,\\
&H^{2,1}_{\bar\partial} = \mathbb{C}\langle\delta_0^c[\omega^{12\bar1}], [\omega^{12\bar2}], [c\,\omega^{12\bar3} + \omega^{13\bar2}], [B\omega^{12\bar3} + \omega^{23\bar1}], \delta_0^c[\omega^{13\bar3} + \omega^{23\bar2}]\rangle.
\end{aligned} \tag{4.3}$$

Notice that the coefficient $Bc + \delta_0^B$ is non-zero except for $B \neq 0$ and $c = 0$. A first consequence is that $\dim E_1^{|2|} \geq 6$ in any case, which implies that $E_1 \ncong E_\infty$ for any complex structure on $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$, or $\mathfrak{h}_{16}$, as well as for any non-abelian complex structure on $\mathfrak{h}_{15}$, because these Lie algebras have $b_2 = 5$.

For the abelian complex structures ($\rho = 0$) since $(B, c) \neq (0, 0)$ we get

$$(4.4)\quad \begin{aligned}
&H^{1,0}_{\bar\partial} = \mathbb{C}\langle[\omega^1]\rangle, \qquad H^{2,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{12}], \delta^c_0[\omega^{13}]\rangle, \qquad H^{3,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{123}]\rangle,\\
&H^{0,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1}], [\omega^{\bar2}], [\omega^{\bar3}]\rangle, \qquad H^{0,2}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1\bar2}], [\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}]\rangle,\\
&H^{1,1}_{\bar\partial} = \mathbb{C}\langle(1-\delta^c_0)[\omega^{1\bar2}], [\omega^{1\bar3}], \delta^c_0[\omega^{2\bar1}], [B\omega^{2\bar2} + \omega^{3\bar1}], \delta^c_0[\omega^{3\bar2}]\rangle,\\
&H^{2,1}_{\bar\partial} = \mathbb{C}\langle\delta^c_0[\omega^{12\bar1}], [\omega^{12\bar2}], [\omega^{12\bar3}], [B\omega^{13\bar2} - c\omega^{23\bar1}], \delta^c_0[\omega^{13\bar3}]\rangle.
\end{aligned}$$

(i) The Lie algebras $\mathfrak{h}_7$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$

For the Lie algebra $\mathfrak{g} \cong \mathfrak{h}_7$ there is only one complex structure defined by $\rho = 1$ and $(B, c) = (1, 0)$ in (4.2) which implies

$$\dim E_1^{|1|} = 3 = b_1(\mathfrak{h}_7), \quad \dim E_1^{|2|} = 8 = b_2(\mathfrak{h}_7), \quad \dim E_1^{|3|} = 12 = b_3(\mathfrak{h}_7).$$

If $\mathfrak{g} \cong \mathfrak{h}_{10}$ it admits also a unique complex structure given by $\rho = 1$ and $(B, c) = (0, 1)$, which implies

$$\dim E_1^{|1|} = 3 = b_1(\mathfrak{h}_{10}), \quad \dim E_1^{|2|} = 6 = b_2(\mathfrak{h}_{10}), \quad \dim E_1^{|3|} = 8 = b_3(\mathfrak{h}_{10}).$$

For the Lie algebras $\mathfrak{g} \cong \mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$ we have that $\rho = 1$, $B \neq 0$ and $c = |B - 1| \neq 0$ because $B \neq 1$ in both cases. Thus,

$$\dim E_1^{|1|} = 3 = b_1(\mathfrak{g}), \quad \dim E_1^{|2|} = 6 = b_2(\mathfrak{g}), \quad \dim E_1^{|3|} = 8 = b_3(\mathfrak{g}).$$

Therefore, $E_1 \cong E_\infty$ for any complex structure on $\mathfrak{h}_7$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$ or $\mathfrak{h}_{12}$.

Finally, it remains to study the abelian case $\rho = 0$, whose corresponding Lie algebra is $\mathfrak{h}_9$ or $\mathfrak{h}_{15}$. Now from Table 3.1 we get $\mathfrak{h}_9$ for $B = c = 1$ and it is easy to see that in this case $\dim E_1^{|k|} = b_k(\mathfrak{h}_9)$ for any k, and so $E_1 \cong E_\infty$ for any complex structure on $\mathfrak{h}_9$. This concludes the proof of (i).

(ii) The Lie algebra $\mathfrak{h}_{16}$

If $\mathfrak{g} \cong \mathfrak{h}_{16}$ we know from Table 3.1 that the complex structures are defined by the values $c = 0$, $\rho = 1$, $|B| = 1$ and $B \neq 1$. From (4.3) the dimensions of $E_1^{|k|}$ for any complex structure on $\mathfrak{h}_{16}$ are

$$\begin{aligned}
&\dim E_1^{|1|} = 3 = b_1(\mathfrak{h}_{16}), \quad \dim E_1^{|2|} = 8 > 5 = b_2(\mathfrak{h}_{16}), \quad \dim E_1^{|3|} = 12 > 6 = b_3(\mathfrak{h}_{16}),\\
&\dim E_1^{|4|} = 8 > 5 = b_4(\mathfrak{h}_{16}), \quad \dim E_1^{|5|} = 3 = b_5(\mathfrak{h}_{16}).
\end{aligned}$$

Hence $E_1 \ncong E_\infty$. We consider the following non-zero maps d_1, where we specify some cohomology classes having linearly independent images in each case:

$$\begin{array}{llllll}
E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} & & \xrightarrow{d_1} & E_1^{3,1} \\
[\omega^{1\bar3}+\omega^{2\bar2}] & \mapsto & (\bar B-1)[\omega^{12\bar1}] & & & \\
[\omega^{3\bar2}] & \mapsto & (1-B)[\omega^{12\bar2}] & & & \\
 & & & [\omega^{13\bar3}+\omega^{23\bar2}] & \mapsto & (\bar B-1)[\omega^{123\bar1}].
\end{array}$$

Similarly,

$$\begin{array}{llllll}
E_1^{0,2} & \xrightarrow{d_1} & E_1^{1,2} & & \xrightarrow{d_1} & E_1^{2,2} \\
[\omega^{\bar2\bar3}] & \mapsto & (\bar B-1)[\omega^{1\bar1\bar3}] & & & \\
 & & [B\omega^{2\bar2\bar3}+\omega^{3\bar1\bar3}] & \mapsto & (1-B)[\omega^{12\bar1\bar3}] & \\
 & & [\omega^{3\bar2\bar3}] & \mapsto & (1-B)[\omega^{12\bar2\bar3}]+(1-\bar B)[\omega^{23\bar1\bar2}]. &
\end{array}$$

Now, counting the dimensions for $E_2^{|k|}$ we get that:

$$\begin{array}{l}
\dim E_2^{|2|} \le \dim E_1^{|2|} - 3 = 5 = b_2(\mathfrak{h}_{16}), \quad \dim E_2^{|3|} \le \dim E_1^{|3|} - 6 = 6 = b_3(\mathfrak{h}_{16}), \\
\dim E_2^{|4|} \le \dim E_1^{|4|} - 3 = 5 = b_4(\mathfrak{h}_{16}).
\end{array}$$

This implies that $E_2 \cong E_\infty$ because necessarily $\dim E_2^{|k|} = b_k(\mathfrak{h}_{16})$ for all k.

(iii) The Lie algebras $\mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$

Now, we prove (iii) and (iv.2) as they belong to the same structure of complex equations given by Proposition 3.1.22 (see Table 3.1). They correspond to the complex structures $J^{(B,c)}$ of the Lie algebras $\mathfrak{h}_{13}$ and $\mathfrak{h}_{14}$ together with the complex structures $J_3^{(B,c)}$ of the Lie algebra $\mathfrak{h}_{15}$ for $\rho = 1$ and $|B-1| \neq c \neq 0$ in (4.2). As $\dim E_1^{|1|} = 3 = b_1(\mathfrak{g})$, being $\mathfrak{g} \cong \mathfrak{h}_{13}$, $\mathfrak{h}_{14}$ or $\mathfrak{h}_{15}$, we get that $E_1^{|1|} \cong E_\infty^{|1|}$. We consider the following non-zero d_2 map:

$$\begin{array}{lll}
E_2^{0,2} & \xrightarrow{d_2} & E_2^{2,1} \\
[\omega^{\bar2\bar3}] & \mapsto & \left[\partial\left(\omega^{2\bar3} + \frac{1-\bar B}{c}\,\omega^{3\bar2}\right)\right] = \frac{|B-1|^2-c^2}{c}\,[\omega^{12\bar2}].
\end{array}$$

It is easy to check that $[\omega^{12\bar2}]$ defines a non-zero class in $E_2^{2,1}$, because $\omega^{12\bar2} \neq \bar\partial\beta_{2,0} + \partial\gamma_{1,1}$ for any $\beta_{2,0}$ and any $\bar\partial$-closed $\gamma_{1,1}$. Hence,

$$b_2(\mathfrak{g}) \le \dim E_3^{|2|} \le \dim E_2^{|2|} - 1 \le \dim E_1^{|2|} - 1 = 6 - 1 = 5 = b_2(\mathfrak{g})$$

and we conclude that $E_\infty^{|2|} \cong E_3^{|2|} \ncong E_2^{|2|} \cong E_1^{|2|}$.

We can also consider

$$\begin{array}{ccc} E_2^{1,2} & \xrightarrow{d_2} & E_2^{3,1} \\ [\omega^{3\bar{1}\bar{3}} + B\omega^{2\bar{2}\bar{3}}] & \mapsto & \left[\partial\left(\frac{1-B}{c}\,\omega^{13\bar{3}} + \frac{1-B-c^2}{c}\,\omega^{23\bar{2}}\right)\right] = \frac{c^2-|B-1|^2}{c}\,[\omega^{123\bar{1}}], \end{array}$$

where the class $[\omega^{123\bar{1}}]$ is non-zero in $E_2^{3,1}$ because $\omega^{123\bar{1}} \neq \bar{\partial}\beta_{3,0} + \partial\gamma_{2,1}$ for any $\beta_{3,0}$ and any $\bar{\partial}$-closed $\gamma_{2,1}$. Thus,

$$b_3(\mathfrak{g}) \leq \dim E_3^{|3|} \leq \dim E_2^{|3|} - 2 \leq \dim E_1^{|3|} - 2 = 8 - 2 = 6 = b_3(\mathfrak{g})$$

and we conclude that $E_\infty^{|3|} \cong E_3^{|3|} \ncong E_2^{|3|} \cong E_1^{|3|}$. By the same argument,

$$b_4(\mathfrak{g}) \leq \dim E_3^{|4|} \leq \dim E_2^{|4|} - 1 \leq \dim E_1^{|4|} - 1 = 6 - 1 = 5 = b_4(\mathfrak{g})$$

and therefore $E_\infty^{|4|} \cong E_3^{|4|} \ncong E_2^{|4|} \cong E_1^{|4|}$. Summing up all the information, we conclude that $E_1 \cong E_2 \ncong E_3 \cong E_\infty$.

(iv) The Lie algebra $\mathfrak{h}_{15}$

Finally we prove the cases (iv.1) and (iv.3). In order to prove (iv.1) we need to study independently the abelian and the non-abelian complex structures with $c = 0$ and $B \neq \rho$ on $\mathfrak{h}_{15}$. We start with the abelian ones. In this case, from (4.4) it follows that the dimensions of $E_1^{|k|}$ are

$$\dim E_1^{|1|} = 4 > 3 = b_1(\mathfrak{h}_{15}), \quad \dim E_1^{|2|} = 9 > 5 = b_2(\mathfrak{h}_{15}), \quad \dim E_1^{|3|} = 12 > 6 = b_3(\mathfrak{h}_{15}).$$

We consider the following non-zero maps d_1, specifying again cohomology classes having linearly independent images in each case:

$$\begin{array}{ccccccc} E_1^{0,1} & \xrightarrow{d_1} & E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} & \xrightarrow{d_1} & E_1^{3,1} \\ [\omega^{\bar{3}}] & \mapsto & -[\omega^{2\bar{1}}] & & & & \\ & & [\omega^{1\bar{3}}] & \mapsto & [\omega^{12\bar{1}}] & & \\ & & [\omega^{3\bar{2}}] & \mapsto & -[\omega^{12\bar{2}}] & & \\ & & & & [\omega^{13\bar{3}}] & \mapsto & [\omega^{123\bar{1}}], \end{array}$$

and similarly,

$$
\begin{array}{ccccccc}
E_1^{0,2} & \xrightarrow{d_1} & E_1^{1,2} & & \xrightarrow{d_1} & E_1^{2,2} & & \xrightarrow{d_1} & E_1^{3,2} \\
[\omega^{\bar{2}\bar{3}}] & \mapsto & [\omega^{2\bar{1}\bar{2}}] & & & & & & \\
 & & [\omega^{2\bar{2}\bar{3}} + \omega^{3\bar{1}\bar{3}}] & & \mapsto & -[\omega^{12\bar{1}\bar{3}}] & & & \\
 & & & [\omega^{3\bar{2}\bar{3}}] & \mapsto & -[\omega^{13\bar{1}\bar{3}}] + [\omega^{23\bar{1}\bar{2}}] & & & \\
 & & & & & & [\omega^{13\bar{2}\bar{3}}] & \mapsto & -[\omega^{123\bar{1}\bar{2}}].
\end{array}
$$

Counting the dimensions for $E_2^{|k|}$ we get that

$$
\begin{aligned}
&\dim E_2^{|1|} \le \dim E_1^{|1|} - 1 = 3 = b_1(\mathfrak{h}_{15}), \quad \dim E_2^{|2|} \le \dim E_1^{|2|} - 4 = 5 = b_2(\mathfrak{h}_{15}),\\
&\dim E_2^{|3|} \le \dim E_1^{|3|} - 6 = 6 = b_3(\mathfrak{h}_{15}), \quad \dim E_2^{|4|} \le \dim E_1^{|4|} - 4 = 5 = b_4(\mathfrak{h}_{15}),\\
&\dim E_2^{|5|} \le \dim E_1^{|5|} - 1 = 3 = b_5(\mathfrak{h}_{15}).
\end{aligned}
$$

This implies that $E_2 \cong E_\infty$ because necessarily $\dim E_2^{|k|} = b_k(\mathfrak{h}_{15})$ for all k.

If $\rho = 1$ and $c = 0$, then $B \neq 1$ and $\dim E_1^{|1|} = b_1(\mathfrak{h}_{15}) + \delta_0^B$. So $E_1^{|1|} \cong E_\infty^{|1|}$ when $B \neq 0$. For $B = 0$ we can consider the two following maps

$$
\begin{array}{ccc}
E_1^{1,0} & \xrightarrow{d_1} & E_1^{2,0} \\
[\omega^{3}] & \mapsto & [\omega^{12}] \neq 0,
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_1^{1,3} & \xrightarrow{d_1} & E_1^{2,3} \\
[\omega^{3\bar{1}\bar{2}\bar{3}}] & \mapsto & [\omega^{12\bar{1}\bar{2}\bar{3}}] \neq 0,
\end{array}
$$

to conclude that $\dim E_2^{|1|} \le \dim E_1^{|1|} - 1 = 3 = b_1(\mathfrak{h}_{15})$ and $\dim E_2^{|5|} \le \dim E_1^{|5|} - 1 = 3 = b_1(\mathfrak{h}_{15})$, and therefore, $E_2^{|k|} \cong E_\infty^{|k|}$ if $k = 1$ or $k = 5$.

Now, for $B \neq 1$ we have that $\dim E_1^{|2|} = 8 + \delta_0^B > 5 = b_2(\mathfrak{h}_{15})$, $\dim E_1^{|3|} = 12 > 6 = b_3(\mathfrak{h}_{15})$, $\dim E_1^{|4|} = 8 + \delta_0^B > 5 = b_4(\mathfrak{h}_{15})$. In order to conclude that $E_2 \cong E_\infty$ it suffices to consider the following non-zero maps d_1:

$$
\begin{array}{ccccc}
E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} & \xrightarrow{d_1} & E_1^{3,1} \\
[\omega^{1\bar{3}} + \omega^{2\bar{2}}] & \mapsto & (\bar{B}-1)[\omega^{12\bar{1}}] & & \\
[\omega^{3\bar{2}}] & \mapsto & (1-B)[\omega^{12\bar{2}}] & & \\
 & & [\omega^{13\bar{3}} + \omega^{23\bar{2}}] & \mapsto & (\bar{B}-1)[\omega^{123\bar{1}}]
\end{array}
$$

and

$$\begin{array}{ccccccc}
E_1^{0,2} & \xrightarrow{d_1} & E_1^{1,2} & & \xrightarrow{d_1} & E_1^{2,2} \\
[\omega^{\bar{2}\bar{3}}] & \mapsto & (\bar{B}-1)[\omega^{1\bar{1}\bar{3}}] & & & \\
& & [B\omega^{2\bar{2}\bar{3}}+\omega^{3\bar{1}\bar{3}}] & \mapsto & (1-B)[\omega^{12\bar{1}\bar{3}}] & \\
& & & [\omega^{3\bar{2}\bar{3}}] & \mapsto & [\omega^{12\bar{2}\bar{3}}-\omega^{13\bar{1}\bar{3}}+\bar{B}\omega^{23\bar{1}\bar{2}}].
\end{array}$$

For the last case (iv.3), we first observe that $\dim E_1^{|1|} = 4 > 3 = b_1(\mathfrak{h}_{15})$, but $d_1([\omega^{\bar{3}}]) = -c[\omega^{1\bar{2}}] - \bar{B}[\omega^{2\bar{1}}]$. Since this class is zero if and only if $c\,\omega^{1\bar{2}} + \bar{B}\omega^{2\bar{1}} \in \bar{\partial}(\wedge^{1,0}\mathfrak{g}^*) = \langle \omega^{1\bar{1}}, B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}} \rangle$, i.e. $|B| = c$, we have that the map $d_1\colon E_1^{0,1} \longrightarrow E_1^{1,1}$ is non-zero. Therefore, $\dim E_2^{|1|} \le \dim E_1^{|1|} - 1 = 3$, i.e. $E_1^{|1|} \ncong E_2^{|1|} \cong E_\infty^{|1|}$. We also have the following non-zero d_2 map:

$$\begin{array}{ccc}
E_2^{0,2} & \xrightarrow{d_2} & E_2^{2,1} \\
[\omega^{\bar{2}\bar{3}}] & \mapsto & \left[\partial(\omega^{2\bar{3}} - \frac{\bar{B}}{c}\omega^{3\bar{2}})\right] = \frac{|B|^2-c^2}{c}\,[\omega^{12\bar{2}}] \neq 0.
\end{array}$$

We deduce that

$$b_2(\mathfrak{h}_{15}) \le \dim E_3^{|2|} \le \dim E_2^{|2|} - 1 \le \dim E_1^{|2|} - 2 = 7 - 2 = 5 = b_2(\mathfrak{h}_{15})$$

and we conclude that $E_\infty^{|2|} \cong E_3^{|2|} \ncong E_2^{|2|} \ncong E_1^{|2|}$.

The map

$$\begin{array}{ccc}
E_2^{1,2} & \xrightarrow{d_2} & E_2^{3,1} \\
[\omega^{3\bar{1}\bar{3}} + B\omega^{2\bar{2}\bar{3}}] & \mapsto & \left[\partial\left(\frac{-B}{c}\omega^{13\bar{3}} - c\,\omega^{23\bar{2}}\right)\right] = \frac{c^2-|B|^2}{c}\,[\omega^{123\bar{1}}] \neq 0
\end{array}$$

is also non-zero, which implies that

$$b_3(\mathfrak{h}_{15}) \le \dim E_3^{|3|} \le \dim E_2^{|3|} - 2 \le \dim E_1^{|3|} - 2 = 8 - 2 = 6 = b_3(\mathfrak{h}_{15}),$$

and we conclude that $E_\infty^{|3|} \cong E_3^{|3|} \ncong E_2^{|3|} \cong E_1^{|3|}$. We also have,

$$b_4(\mathfrak{h}_{15}) \le \dim E_3^{|4|} \le \dim E_2^{|4|} - 1 \le \dim E_1^{|4|} - 2 = 7 - 2 = 5 = b_4(\mathfrak{h}_{15}),$$

and therefore $E_\infty^{|4|} \cong E_3^{|4|} \ncong E_2^{|4|} \ncong E_1^{|4|}$. Consequently, in case (iv.3) we get $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$. □

Proposition 4.1.3. *Let us consider a six-dimensional nilpotent Lie algebra* $\mathfrak{g}$ *with a complex structure* J *in the family*

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{13} + \omega^{1\bar{3}}, \quad d\omega^3 = i\epsilon\,\omega^{1\bar{1}} \pm i(\omega^{1\bar{2}} - \omega^{2\bar{1}}), \tag{4.5}$$

with $\epsilon = 0, 1$. *Then:*

(i) The Frölicher sequence degenerates at the first step for any J on $\mathfrak{h}_{19}^-$ $(\epsilon = 0)$.

(ii) Any complex structure on $\mathfrak{h}_{26}^+$ $(\epsilon = 1)$ satisfies $E_1 \ncong E_2 \cong E_\infty$.

Proof. It is easy to see that

$$H^{1,0}_{\bar\partial} = \mathbb{C}\langle[\omega^1]\rangle, \qquad H^{2,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{12}]\rangle, \qquad H^{1,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{1\bar1}],[\omega^{2\bar3}]\rangle,$$
$$H^{0,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1}],[\omega^{\bar3}]\rangle, \qquad H^{0,2}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1\bar2}],[\omega^{\bar2\bar3}]\rangle, \quad H^{3,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{123}]\rangle,$$
$$H^{2,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{12\bar3}],[\omega^{13\bar1}]\rangle.$$

This implies that $\dim E_1^{|1|} = 3$, $\dim E_1^{|2|} = 5$ and $\dim E_1^{|3|} = 6$. Since $b_1(\mathfrak{h}_{19}^-) = 3$, $b_2(\mathfrak{h}_{19}^-) = 5$ and $b_3(\mathfrak{h}_{19}^-) = 6$, we conclude that the Frölicher sequence degenerates at the first step for any J on $\mathfrak{h}_{19}^-$.

Next we suppose $\mathfrak{g} \cong \mathfrak{h}_{26}^+$. In this case $\dim E_1^{|1|} = 3 > 2 = b_1(\mathfrak{h}_{26}^+)$, and so any J on $\mathfrak{h}_{26}^+$ satisfies $E_1 \ncong E_\infty$. Moreover,

$$\dim E_1^{|1|} = 3 > 2 = b_1(\mathfrak{g}), \quad \dim E_1^{|2|} = 5 > 4 = b_2(\mathfrak{g}), \quad \dim E_1^{|3|} = 6 = b_3(\mathfrak{g}),$$
$$\dim E_1^{|4|} = 5 > 4 = b_4(\mathfrak{g}), \quad \dim E_1^{|5|} = 3 > 2 = b_5(\mathfrak{g}).$$

Since the maps
$$E_1^{0,1} \xrightarrow{d_1} E_1^{1,1}, \qquad E_1^{2,2} \xrightarrow{d_1} E_1^{3,2},$$
are non-zero (take for instance $[\omega^{\bar3}]$ and $[\omega^{23\bar2\bar3} \pm \omega^{23\bar1\bar3}]$), it follows that $E_2 \cong E_\infty$. □

Once we have computed the behaviour of the Frölicher spectral sequence for all the complex structures on 6-dimensional nilpotent Lie algebras we state the following theorem describing the behaviour of the Frölicher spectral sequence for invariant complex structures on nilmanifolds. The result (including the Frölicher behaviour for $(\mathfrak{h}_7, J)$) is summarized in the Table 3.1.

Theorem 4.1.4. *Let $M = G/\Gamma$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that the underlying Lie algebra $\mathfrak{g} \ncong \mathfrak{h}_7$. Then the Frölicher spectral sequence $\{E_r(M)\}_{r\geq1}$ behaves as follows:*

(a) If $\mathfrak{g} \cong \mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_6$, $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ or $\mathfrak{h}_{19}^-$, then $E_1(M) \cong E_\infty(M)$ for any J.

(b) If $\mathfrak{g} \cong \mathfrak{h}_2$ or $\mathfrak{h}_4$, then $E_1(M) \cong E_\infty(M)$ if and only if J is non-abelian; moreover, any abelian complex structure on $\mathfrak{h}_2$ or $\mathfrak{h}_4$ satisfies $E_1(M) \ncong E_2(M) \cong E_\infty(M)$.

(c) If $\mathfrak{g} \cong \mathfrak{h}_5$ and J is a complex structure on $\mathfrak{h}_5$ given in Table 3.1, then:

(c.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ when J is complex-parallelizable;

(c.2) $E_1(M) \cong E_\infty(M)$ if and only if J is not complex-parallelizable and $\rho D \neq 0$; moreover, $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ when $\rho D = 0$.

(d) If $\mathfrak{g} \cong \mathfrak{h}_{16}$ *or* $\mathfrak{h}_{26}^{+}$, *then* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *for any* J.

(e) If $\mathfrak{g} \cong \mathfrak{h}_{13}$ *or* $\mathfrak{h}_{14}$, *then* $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$ *for any* J.

(f) If $\mathfrak{g} \cong \mathfrak{h}_{15}$ *and* J *is a complex structure on* $\mathfrak{h}_{15}$ *given in Table 3.1, then:*

(f.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$, *when* $c = 0$ *and* $|B - \rho| \neq 0$;
(f.2) $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$, *when* $\rho = 1$ *and* $|B - 1| \neq c \neq 0$;
(f.3) $E_1(M) \ncong E_2(M) \ncong E_3(M) \cong E_\infty(M)$, *when* $\rho = 0$ *and* $|B| \neq c \neq 0$.

As a consequence of the previous study we face the following problem posed by Angella and Tomassini in [10] (see Remark 1.3.6): to construct a compact complex manifold M such that $E_1(M) \cong E_\infty(M)$ and $h^{p,q}_{\bar\partial}(M) = h^{q,p}_{\bar\partial}(M)$ for every $p, q \in \mathbb{N}$ but for which the $\partial\bar\partial$-lemma does not hold. Since nilmanifolds do not satisfy the $\partial\bar\partial$-lemma (as they are not formal unless they are complex tori), the following result provides a solution.

Proposition 4.1.5. *Let* J *be any invariant complex structure on a nilmanifold* M *with underlying Lie algebra isomorphic to* $\mathfrak{h}_6$. *Then* $E_1(M) \cong E_\infty(M)$ *and the Hodge numbers satisfy*

$$
\begin{gathered}
h^{0,0}_{\bar\partial}(M) = 1,\\
h^{1,0}_{\bar\partial}(M) = 2, \quad h^{0,1}_{\bar\partial}(M) = 2,\\
h^{2,0}_{\bar\partial}(M) = 2, \quad h^{1,1}_{\bar\partial}(M) = 5, \quad h^{0,2}_{\bar\partial}(M) = 2,\\
h^{3,0}_{\bar\partial}(M) = 1, \quad h^{2,1}_{\bar\partial}(M) = 5, \quad h^{1,2}_{\bar\partial}(M) = 5, \quad h^{0,3}_{\bar\partial}(M) = 1,\\
h^{3,1}_{\bar\partial}(M) = 2, \quad h^{2,2}_{\bar\partial}(M) = 5, \quad h^{1,3}_{\bar\partial}(M) = 2,\\
h^{3,2}_{\bar\partial}(M) = 2, \quad h^{2,3}_{\bar\partial}(M) = 2,\\
h^{3,3}_{\bar\partial}(M) = 1.
\end{gathered}
$$

Proof. Any complex structure J on $\mathfrak{h}_6$ is equivalent to the complex structure given in Table 3.1, that is, $\rho = \lambda = 1$ and $D = 0$. Its Dolbeault cohomology groups $H^{p,q}_{\bar\partial}$ for $(p,q) = (1,0), (0,1), (2,0), (1,1), (0,2), (3,0)$ and $(2,1)$ are

$$
\begin{aligned}
&H^{1,0}_{\bar\partial} = \mathbb{C}\langle[\omega^1], [\omega^2]\rangle, \qquad H^{2,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{12}], [\omega^{13}]\rangle, \qquad H^{3,0}_{\bar\partial} = \mathbb{C}\langle[\omega^{123}]\rangle,\\
&H^{0,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1}], [\omega^{\bar2}]\rangle, \qquad H^{0,2}_{\bar\partial} = \mathbb{C}\langle[\omega^{\bar1\bar3}], [\omega^{\bar2\bar3}]\rangle,\\
&H^{1,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{1\bar2}], [\omega^{2\bar1}], [\omega^{2\bar2}], [\omega^{1\bar3} + \omega^{3\bar2}], [\omega^{3\bar1} + \omega^{3\bar2}]\rangle,\\
&H^{2,1}_{\bar\partial} = \mathbb{C}\langle[\omega^{12\bar2}], [\omega^{13\bar1}], [\omega^{12\bar3} + \omega^{23\bar1}], [\omega^{12\bar3} - \omega^{23\bar2}], [\omega^{13\bar2}]\rangle.
\end{aligned}
$$

By Serre duality we get the above Hodge diamond which is symmetric. Moreover,

$$
\dim E_1^{|1|} = 4 = b_1(\mathfrak{h}_6), \quad \dim E_1^{|2|} = 9 = b_2(\mathfrak{h}_6), \quad \dim E_1^{|3|} = 12 = b_3(\mathfrak{h}_6),
$$

so the Frölicher spectral sequence degenerates at the first step. $\square$

4.2 The Frölicher sequence and the $\partial\bar{\partial}$-lemma on solvmanifolds

From now on we consider a (non-nilpotent) solvmanifold $(M = G/\Gamma, J)$ endowed with an invariant complex structure with holomorphically trivial canonical bundle. As we mentioned before, the choice of the lattice can contribute to the values of the Hodge numbers of the complex solvmanifold, or more in general, to the behaviour of the Frölicher sequence. However, we prove in the following proposition that the non-degeneracy at the first step of the Frölicher sequence is enough to be proved at the level of the Lie algebra.

Proposition 4.2.1. *Let $M = G/\Gamma$ be a compact manifold endowed with an invariant complex structure J. If $(\mathfrak{g}, J)$ satisfies that $E_1 \not\cong E_\infty$, then $E_1(M) \not\cong E_\infty(M)$.*

Proof. It follows from the symmetrization process (see Lemma 2.1.27). Let $[\alpha] \in E_1^{p,q}(\mathfrak{g})$ such that $\partial\alpha \notin \bar{\partial} \wedge^{p+1,q-1} \mathfrak{g}^*$, then α defines a left-invariant (p, q)-form on M. If there exists $\psi \in \wedge^{p+1,q-1}M$ such that $\partial\alpha = \bar{\partial}\psi$ then by the symmetrization process $(\partial\alpha)_\nu = \partial\alpha_\nu = \partial\alpha = (\bar{\partial}\psi)_\nu = \bar{\partial}\psi_\nu$ contradicting the hypothesis. □

Hence, Section 4.2.1 starts by computing the Frölicher spectral sequence for the underlying real Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ endowed with the complex structures described in Propositions 3.2.3, 3.2.4, 3.2.6, 3.2.7 and 3.2.10 included in Table 3.2. As a consequence of Proposition 4.2.1, we provide a partial result of the behaviour of the Frölicher spectral sequence of the corresponding solvmanifolds.

4.2.1 Behaviour at the invariant level

We compute the Frölicher sequence at the level of the Lie algebra dividing the study according to the different families of complex structures found in the classification up to equivalence done in Section 3.2.

Lemma 4.2.2. *If J is a complex structure with closed complex volume (3,0)-form on the Lie algebras $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$, then $E_1 \cong E_\infty$.*

Proof. The Betti numbers of the Lie algebra $\mathfrak{g}_1$ coincide with the ones of $\mathfrak{g}_2^0$, namely, $b_1(\mathfrak{g}_1) = 2$, $b_2(\mathfrak{g}_1) = 5$ and $b_3(\mathfrak{g}_1) = 8$, whereas the Betti numbers of the Lie algebras $\mathfrak{g}_2^\alpha$ with $\alpha > 0$ are $b_1(\mathfrak{g}_2^\alpha) = 2$, $b_2(\mathfrak{g}_2^\alpha) = 3$ and $b_3(\mathfrak{g}_2^\alpha) = 4$. Lemma 3.2.2 states that every complex structure with closed complex volume (3,0)-form on the Lie algebras $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ are described by a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the structure equations (3.15). A direct computation shows that:

$$H^{1,0}_{\bar{\partial}} = \mathbb{C}\langle[\omega^3]\rangle, \qquad H^{2,0}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{12}]\rangle, \qquad H^{3,0}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{123}]\rangle,$$

$$H^{0,1}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{\bar{3}}]\rangle, \qquad H^{0,2}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{\bar{1}\bar{2}}]\rangle,$$

$$H^{1,1}_{\bar{\partial}} = \mathbb{C}\langle\delta^i_A[\omega^{1\bar{1}}],\, \delta^1_A[\omega^{1\bar{2}}],\, \delta^1_A[\omega^{2\bar{1}}],\, \delta^i_A[\omega^{2\bar{2}}],\, [\omega^{3\bar{3}}]\rangle,$$

$$H^{2,1}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{12\bar{3}}],\, \delta^i_A[\omega^{13\bar{1}}],\, \delta^1_A[\omega^{13\bar{2}}],\, \delta^1_A[\omega^{23\bar{1}}],\, \delta^i_A[\omega^{23\bar{2}}]\rangle.$$

for any $A \in \mathbb{C}$. If $A = 1$ or i, i.e. if $\mathfrak{g} \cong \mathfrak{g}_1$ or $\mathfrak{g}_2^0$:

$$\dim E_1^{|1|} = 2 = b_2(\mathfrak{g}_1), \quad \dim E_1^{|2|} = 5 = b_2(\mathfrak{g}_1), \quad \dim E_1^{|3|} = 8 = b_3(\mathfrak{g}_1)$$

and therefore $E_1 \cong E_\infty$. For the structures given by $A \neq 1, i$:

$$\dim E_1^{|1|} = 2 = b_1(\mathfrak{g}_2^\alpha), \quad \dim E_1^{|2|} = 3 = b_2(\mathfrak{g}_2^\alpha), \quad \dim E_1^{|3|} = 4 = b_3(\mathfrak{g}_2^\alpha)$$

obtaining again $E_1 \cong E_\infty$. □

Lemma 4.2.3. *If J is a complex structure with closed complex volume (3,0)-form on the Lie algebra $\mathfrak{g}_3$ then $E_1 \cong E_\infty$.*

Proof. By Proposition 3.2.4 every complex structure on $\mathfrak{g}_3$ with complex closed volume (3,0)-form is described by a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations (3.21). The Dolbeault cohomology groups are:

$$H^{1,0}_{\bar{\partial}} = \langle[\omega^1]\rangle, \quad H^{2,0}_{\bar{\partial}} = \{0\}, \qquad H^{1,1}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{1\bar{1}}], [\tfrac{i}{2x}\omega^{2\bar{3}} + \omega^{3\bar{2}}]\rangle,$$
$$H^{0,1}_{\bar{\partial}} = \langle[\omega^{\bar{1}}]\rangle, \quad H^{0,2}_{\bar{\partial}} = \langle[\omega^{\bar{2}\bar{3}}]\rangle, \quad H^{2,1}_{\bar{\partial}} = \mathbb{C}\langle[\tfrac{i}{2x}\omega^{12\bar{3}} + \omega^{13\bar{2}}]\rangle, \quad H^{3,0}_{\bar{\partial}} = \langle[\omega^{123}]\rangle.$$

Clearly we have

$$\dim E_1^{|1|} = 2 = b_1(\mathfrak{g}_3), \quad \dim E_1^{|2|} = 3 = b_2(\mathfrak{g}_3), \quad \dim E_1^{|3|} = 4 = b_3(\mathfrak{g}_3),$$

therefore $E_1 \cong E_\infty$. □

Lemma 4.2.4. *If J is a complex structure with closed complex volume (3,0)-form on the Lie algebras $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$ or $\mathfrak{g}_7$ then $E_1 \cong E_\infty$.*

Proof. The Betti numbers of the four Lie algebras coincide and we have $b_1(\mathfrak{g}_i) = 1$, $b_2(\mathfrak{g}_i) = 3$ and $b_3(\mathfrak{g}_i) = 6$ for $i = 4, 5, 6, 7$. By Lemma 3.2.5 the complex structures on the Lie algebras $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$ or $\mathfrak{g}_7$ with closed volume (3,0)-form may be written in terms of a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations (3.22). It is direct to check that

$$H^{1,0}_{\bar{\partial}} = \{0\}, \qquad H^{0,1}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{\bar{3}}]\rangle, \qquad H^{2,0}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{12}]\rangle,$$
$$H^{1,1}_{\bar{\partial}} = \mathbb{C}\langle\delta^1_A[\omega^{1\bar{2}}], \delta^i_A[\omega^{2\bar{2}}]\rangle, \quad H^{0,2}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{\bar{1}\bar{2}}]\rangle, \qquad H^{3,0}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{123}]\rangle,$$
$$H^{2,1}_{\bar{\partial}} = \mathbb{C}\langle[\omega^{12\bar{3}}], \delta^1_A[\omega^{13\bar{2}} - \omega^{23\bar{1}}], \delta^i_A[G_{11}\omega^{13\bar{1}} - G_{22}\omega^{23\bar{2}}]\rangle.$$

Thus, counting dimensions we have

$$\dim E_1^{|1|} = 1 = b_1(\mathfrak{g}_i), \quad \dim E_1^{|2|} = 3 = b_2(\mathfrak{g}_i), \quad \dim E_1^{|3|} = 6 = b_3(\mathfrak{g}_i),$$

for $i = 4, 5, 6, 7$ and then $E_1 \cong E_\infty$ for the complex structures given by (3.22).

□

Lemma 4.2.5. *Let J be a complex structure with closed complex volume (3,0)-form on the Lie algebra $\mathfrak{g}_8$, then $E_1 \cong E_\infty$ if J satisfies* (3.33) *with $A \neq \pm i$ and $E_1 \ncong E_2 \cong E_\infty$ in other cases.*

Proof. We have seen in Proposition 3.2.7 that the complex structures on $\mathfrak{g}_8$ with a closed complex volume (3,0)-form may be represented by a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations:

$$d\omega^1 = -(A-i)\omega^{13} - (A+i)\omega^{1\bar3} + B\omega^{3\bar3},\ d\omega^2 = (A-i)\omega^{23} + (A+i)\omega^{2\bar3} + C\omega^{3\bar3},\ d\omega^3 = 0.,$$

where $A = -i$ and $(B, C) \in \{(0,0), (1,0), (1,1)\}$ or $A \neq -i$, $\mathfrak{Im}\, A \neq 0$ and $B = C = 0$. The Dolbeault cohomology groups are:

$$\begin{aligned}
&H^{1,0}_{\bar\partial} = \mathbb{C}\langle \delta_A^{-i}\delta_B^0[\omega^1], \delta_A^{-i}[C\omega^1 - \omega^2], [\omega^3]\rangle, \quad H^{0,1}_{\bar\partial} = \mathbb{C}\langle \delta_A^{i}[\omega^1], \delta_A^{i}[\omega^2], [\omega^3]\rangle,\\
&H^{2,0}_{\bar\partial} = \mathbb{C}\langle \delta_B^0\delta_C^0[\omega^{12}], \delta_A^{-i}[\omega^{13}], \delta_A^{-i}[\omega^{23}]\rangle, \quad H^{0,2}_{\bar\partial} = \mathbb{C}\langle [\omega^{\bar1\bar2}], \delta_A^{i}[\omega^{\bar1\bar3}], \delta_A^{i}[\omega^{\bar2\bar3}]\rangle,\\
&H^{1,1}_{\bar\partial} = \mathbb{C}\langle \delta_A^{-i}[\omega^{1\bar3}], \delta_A^{-i}[\omega^{2\bar3}], \delta_A^{i}[\omega^{3\bar1}], \delta_A^{i}[\omega^{3\bar2}], \delta_B^0\delta_C^0[\omega^{3\bar3}]\rangle,\\
&H^{2,1}_{\bar\partial} = \mathbb{C}\langle \delta_A^{i}[\omega^{12\bar1}], \delta_A^{i}[\omega^{12\bar2}], [\omega^{12\bar3}], \delta_A^{-i}[\omega^{13\bar3}], \delta_B^0\delta_C^0\delta_A^{-i}[\omega^{23\bar3}]\rangle, \quad H^{3,0}_{\bar\partial} = \mathbb{C}\langle [\omega^{123}]\rangle
\end{aligned}$$

For the complex structures J^A satisfying (3.33) with $A \neq \pm i$ we get

$$\dim E_1^{|1|} = 2 = b_1(\mathfrak{g}_8), \quad \dim E_1^{|2|} = 3 = b_2(\mathfrak{g}_8), \quad \dim E_1^{|3|} = 4 = b_3(\mathfrak{g}_8),$$

therefore $E_1 \cong E_\infty$. For the parallelizable J_0 and the abelian J_1 structure corresponding to (3.33) with $A \pm i$ we get:

$$\begin{aligned}
&\dim E_1^{|1|} = 4 > 2 = b_1(\mathfrak{g}_8), \quad \dim E_1^{|2|} = 7 > 3 = b_2(\mathfrak{g}_8), \quad \dim E_1^{|3|} = 8 > 4 = b_3(\mathfrak{g}_8),\\
&\dim E_1^{|4|} = 7 > 3 = b_4(\mathfrak{g}_8), \quad \dim E_1^{|2|} = 4 > 2 = b_5(\mathfrak{g}_8).
\end{aligned}$$

Finally, for the structures J' and J'' given by equations (3.31) and (3.32) we get:

$$\begin{aligned}
&\dim E_1^{|1|} = 3 > b_1(\mathfrak{g}_8), \quad \dim E_1^{|2|} = 5 > b_2(\mathfrak{g}_8), \quad \dim E_1^{|3|} = 6 > b_3(\mathfrak{g}_8),\\
&\dim E_1^{|4|} = 5 > b_4(\mathfrak{g}_8), \quad \dim E_1^{|2|} = 3 > b_5(\mathfrak{g}_8).
\end{aligned}$$

Therefore we must study the Frölicher sequence at the second step $E_2^{p,q}$ if $A = i$ or

$A = -i$. If $A = i$ then we have:

$$\begin{array}{ccccccc}
E_1^{0,1} & \xrightarrow{d_1} & E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} & \xrightarrow{d_1} & E_1^{3,1} \\
[\omega^{\bar{1}}] & \mapsto & 2i[\omega^{3\bar{1}}] & & & & \\
[\omega^{\bar{2}}] & \mapsto & 2i[\omega^{3\bar{2}}] & & & & \\
 & & [\omega^{1\bar{1}}] & \mapsto & 2i[\omega^{13\bar{1}}] & & \\
 & & [\omega^{1\bar{2}}] & \mapsto & -2i[\omega^{13\bar{2}}] & & \\
 & & & & [\omega^{12\bar{1}}] & \mapsto & -2i[\omega^{123\bar{1}}] \\
 & & & & [\omega^{12\bar{2}}] & \mapsto & 2i[\omega^{123\bar{2}}].
\end{array}$$

Similarly, the homomorphism $E_1^{2,2} \xrightarrow{d_1} E_1^{3,2}$ is non-zero because $d_1[\omega^{12\bar{1}\bar{3}}] = [\bar{\partial}\omega^{12\bar{1}\bar{3}}] = -2i[\omega^{123\bar{1}\bar{3}}] \neq 0$ and $d_1[\omega^{12\bar{2}\bar{3}}] = [\bar{\partial}\omega^{12\bar{2}\bar{3}}] = 2i[\omega^{123\bar{2}\bar{3}}] \neq 0$. Since $E_2^{p,q} \cong \operatorname{Ker} d_1/\operatorname{Im} d_1$, counting the dimensions we get

$$\begin{array}{ll}
\dim E_2^{|1|} \leq \dim E_1^{|1|} - 2 = 2 = b_1(\mathfrak{g}_8), & \dim E_2^{|2|} \leq \dim E_1^{|2|} - 4 = 3 = b_2(\mathfrak{g}_8), \\
\dim E_2^{|3|} \leq \dim E_1^{|3|} - 4 = 4 = b_3(\mathfrak{g}_8), & \dim E_2^{|4|} \leq \dim E_1^{|4|} - 4 = 3 = b_4(\mathfrak{g}_8), \\
\dim E_2^{|5|} \leq \dim E_1^{|5|} - 2 = 2 = b_5(\mathfrak{g}_8). &
\end{array}$$

This implies that $E_2 \cong E_\infty$ because $\dim E_2^{|k|} = b_k(\mathfrak{g}_8)$ for all k if $A = i$.

If $A = -i$ and $(B, C) = (0, 0)$ then we have the following non-zero homomorphisms:

$$\begin{array}{ccccccc}
E_1^{1,0} & \xrightarrow{d_1} & E_1^{2,0} & \qquad & E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} \\
[\omega^1] & \mapsto & 2i[\omega^{13}] & & [\omega^{1\bar{3}}] & \mapsto & 2i[\omega^{13\bar{3}}] \\
[\omega^2] & \mapsto & -2i[\omega^{23}], & & [\omega^{2\bar{3}}] & \mapsto & -2i[\omega^{23\bar{3}}],
\end{array}$$

$$\begin{array}{ccccccc}
E_1^{1,2} & \xrightarrow{d_1} & E_1^{2,2} & \qquad & E_1^{1,3} & \xrightarrow{d_1} & E_1^{2,3} \\
[\omega^{1\bar{1}\bar{2}}] & \mapsto & 2i[\omega^{13\bar{1}\bar{2}}] & & [\omega^{1\bar{1}\bar{2}\bar{3}}] & \mapsto & 2i[\omega^{13\bar{1}\bar{2}\bar{3}}] \\
[\omega^{2\bar{1}\bar{2}}] & \mapsto & -2i[\omega^{23\bar{1}\bar{2}}], & & [\omega^{2\bar{1}\bar{2}\bar{3}}] & \mapsto & -2i[\omega^{23\bar{1}\bar{2}\bar{3}}].
\end{array}$$

Since $E_2^{p,q} \cong \operatorname{Ker} d_1/\operatorname{Im} d_1$, counting the dimensions we get

$$\begin{array}{ll}
\dim E_2^{|1|} \leq \dim E_1^{|1|} - 2 = 2 = b_1(\mathfrak{g}_8), & \dim E_2^{|2|} \leq \dim E_1^{|2|} - 4 = 3 = b_2(\mathfrak{g}_8), \\
\dim E_2^{|3|} \leq \dim E_1^{|3|} - 4 = 4 = b_3(\mathfrak{g}_8), & \dim E_2^{|4|} \leq \dim E_1^{|4|} - 4 = 3 = b_4(\mathfrak{g}_8), \\
\dim E_2^{|5|} \leq \dim E_1^{|5|} - 2 = 2 = b_5(\mathfrak{g}_8). &
\end{array}$$

This implies that $E_2 \cong E_\infty$ because $\dim E_2^{|k|} = b_k(\mathfrak{g}_8)$ for all k if $A = -i$ and $(B, C) = (0, 0)$.

Finally, if $A = -i$ and $(B, C) \neq (0, 0)$ then the following homomorphism:

$$\begin{array}{ccccccc} E_1^{1,0} & \xrightarrow{d_1} & E_1^{2,0} & \qquad & E_1^{1,1} & \xrightarrow{d_1} & E_1^{2,1} \\ [C\omega^1 - B\omega^2] & \mapsto & 2i[C\omega^{13} + B\omega^{23}], & & [\omega^{1\bar{3}}] & \mapsto & 2i[\omega^{13\bar{3}}], \end{array}$$

$$\begin{array}{ccccccc} E_1^{1,2} & \xrightarrow{d_1} & E_1^{2,2} & \qquad & E_1^{1,3} & \xrightarrow{d_1} & E_1^{2,3} \\ [C\omega^{1\bar{1}\bar{2}} - B\omega^{1\bar{1}\bar{2}}] & \mapsto & 2i[C\omega^{13\bar{1}\bar{2}} + B\omega^{23\bar{1}\bar{2}}], & & [\omega^{1\bar{1}\bar{2}\bar{3}}] & \mapsto & 2i[\omega^{13\bar{1}\bar{2}\bar{3}}]. \end{array}$$

are non-zero. Finally, we get

$$\begin{array}{ll} \dim E_2^{|1|} \leq \dim E_1^{|1|} - 1 = 2 = b_1(\mathfrak{g}_8), & \dim E_2^{|2|} \leq \dim E_1^{|2|} - 2 = 3 = b_2(\mathfrak{g}_8), \\ \dim E_2^{|3|} \leq \dim E_1^{|3|} - 2 = 4 = b_3(\mathfrak{g}_8), & \dim E_2^{|4|} \leq \dim E_1^{|4|} - 2 = 3 = b_4(\mathfrak{g}_8), \\ \dim E_2^{|5|} \leq \dim E_1^{|5|} - 1 = 2 = b_5(\mathfrak{g}_8). & \end{array}$$

This implies that $E_2 \cong E_\infty$ because $\dim E_2^{|k|} = b_k(\mathfrak{g}_8)$ for all k if $A = -i$ and $(B, C) \neq (0, 0)$ and this concludes the proof. $\square$

Lemma 4.2.6. *If J is a complex structure with closed complex volume (3,0)-form on the Lie algebra $\mathfrak{g}_9$,then $E_1 \cong E_\infty$.*

Proof. Proposition 3.2.10 states that there is only one complex structure on $\mathfrak{g}_9$ with closed complex volume (3,0)-form satisfying the complex structure equations (3.36). It is direct to check that

$$\begin{array}{llll} H_{\bar{\partial}}^{1,0} = \{0\}, & H_{\bar{\partial}}^{2,0} = \{0\}, & H_{\bar{\partial}}^{1,1} = \mathbb{C}\langle[\omega^{\bar{1}}]\rangle, & \\ H_{\bar{\partial}}^{0,1} = \{0\}, & H_{\bar{\partial}}^{0,2} = \mathbb{C}\langle[\omega^{\bar{2}\bar{3}}]\rangle, & H_{\bar{\partial}}^{3,0} = \mathbb{C}\langle[\omega^{123}]\rangle, & H_{\bar{\partial}}^{2,1} = \{0\}. \end{array}$$

Clearly we get $\dim E_1^{|1|} = 1 = b_1(\mathfrak{g}_9)$, $\dim E_1^{|2|} = 1 = b_2(\mathfrak{g}_9)$, $\dim E_1^{|3|} = 2 = b_3(\mathfrak{g}_9)$, hence $E_1 \cong E_\infty$. $\square$

We summarize the results obtained in Lemmas 4.2.2, 4.2.3, 4.2.4, 4.2.5 and 4.2.6 in the following Proposition:

Proposition 4.2.7. *Let $(\mathfrak{g}, J)$ be a six-dimensional non-nilpotent solvable unimodular Lie algebra endowed with a complex structure with a closed volume $(3,0)$-form.*

- *If $\mathfrak{g} \cong \mathfrak{g}_i$ for $i \neq 8$, then $E_1 \cong E_\infty$.*
- *If $\mathfrak{g} = \mathfrak{g}_8$, then $E_1 \cong E_\infty$ if J satisfies (3.33) with $A \neq \pm i$ and $E_1 \ncong E_2 \cong E_\infty$ in other case.*

As a direct consequence of Propositions 4.2.7 and 4.2.1 we state the following:

Corollary 4.2.8. *Let $M = G/\Gamma$ be a solvmanifold with underlying real Lie algebra $\mathfrak{g}_8$ endowed with an invariant complex structure satisfying* (3.31), (3.32) *or* (3.33) *with $A = \pm i$ then $E_1(M) \ncong E_\infty(M)$. In particular the complex solvmanifolds (M, J) do not satisfy the $\partial\bar{\partial}$-lemma.*

4.2.2 Complex structures of splitting type

The works of Angella and Kasuya [51, 7] provide some results concerning the computation of the Dolbeault and Bott-Chern cohomologies of certain solvmanifolds $M = G/\Gamma$ endowed with an invariant complex structure of *splitting type*.

Definition 4.2.9 (Kasuya [51, Assumption 1.1])**.** *A solvmanifold $M = G/\Gamma$ endowed with an invariant complex structure J is said to be of* splitting type *if G is a semi-direct product $\mathbb{C}^n \ltimes_\varphi N$ such that:*

1. *N is a connected simply-connected $2m$-dimensional nilpotent Lie group endowed with a left-invariant complex structure J_N;*
2. *for any $t \in \mathbb{C}^n$, it holds that $\varphi(t) \in \mathrm{GL}(N)$ is a holomorphic automorphism of N with respect to J_N;*
3. *φ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ associated to N;*
4. *G has a lattice Γ which can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes_\varphi \Gamma_N$, where $\Gamma_{\mathbb{C}^n}$ and Γ_N are lattices of $\mathbb{C}^n$ and N respectively, and it holds $\varphi(t)\left(\Gamma_N\right) \subseteq \Gamma_N$ for any $t \in \Gamma_{\mathbb{C}^n}$;*
5. *the inclusion $\left(\wedge^{\bullet,\bullet}\mathfrak{n}^*, \bar{\partial}\right) \hookrightarrow \left(\wedge^{\bullet,\bullet}N/\Gamma_N, \bar{\partial}\right)$ induces an isomorphism*
$$H^{\bullet,\bullet}_{\bar{\partial}}(\mathfrak{n}) \overset{\cong}{\to} H^{\bullet,\bullet}_{\bar{\partial}}(N/\Gamma_N) .$$

Kasuya [51] proves that the computation of the Dolbeault cohomology of a compact complex solvmanifold endowed with an invariant complex structure of splitting type can be done by means of a finite-dimensional sub-complex $\left(B^{\bullet,\bullet}_\Gamma, \bar{\partial}\right)$ of the complex of differential forms $(\wedge^{\bullet,\bullet}M, \bar{\partial})$. Analogously, Angella and Kasuya [7] show that the Bott-Chern cohomology is computable by means of a finite-dimensional double sub-complex $\left(C^{\bullet,\bullet}_\Gamma, \partial, \bar{\partial}\right)$ of the double complex $(\wedge^{\bullet,\bullet}M, \partial, \bar{\partial})$. In both cases, the operators ∂ and $\bar{\partial}$ in the complexes $\left(B^{\bullet,\bullet}_\Gamma, \bar{\partial}\right)$ and $\left(C^{\bullet,\bullet}_\Gamma, \partial, \bar{\partial}\right)$ are the restriction of the differential operators ∂ and $\bar{\partial}$ induced by the complex structure on the solvmanifold to the spaces $B^{\bullet,\bullet}_\Gamma, C^{\bullet,\bullet}_\Gamma \subset \wedge^{\bullet,\bullet}M$. Next, we sketch the basic tools to obtain the spaces $B^{\bullet,\bullet}_\Gamma$ and $C^{\bullet,\bullet}_\Gamma$ stating the main theorems concerning the computation of the Dolbeault and the Bott-Chern cohomology.

Let $G = \mathbb{C}^n \ltimes_\varphi N$ be a Lie group endowed with a left-invariant complex structure of splitting type. Consider the standard basis $\{X_1, \ldots, X_n\}$ of $\mathbb{C}^n$ and a left-invariant

$(1,0)$-basis $\{Y_1, \dots, Y_m\} \subset \mathfrak{n}^{1,0}$ for the complex structure J_N of N such that the induced action φ on $\mathfrak{n}^{1,0}$ is represented in this basis by the diagonal matrix

$$\varphi = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix},$$

for $\alpha_1, \dots, \alpha_m$ characters of $\mathbb{C}^n$ (recall that $\alpha\colon \mathbb{C}^n \to \mathbb{C}^*$ is a character of $\mathbb{C}^n$ if α is a homomorphism between the abelian groups $(\mathbb{C}^n, +)$ and $(\mathbb{C}^*, \cdot)$).

Let $\left\{x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m\right\}$ be the basis of $(\mathfrak{g}^*)^{1,0}$ which is dual to the basis of $\mathfrak{g}^{(1,0)}$ given by

$$\{X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m\}.$$

By using [51, Lemma 2.2], it turns out that for any $j \in \{1, \dots, m\}$, there exist unique unitary characters $\beta_j, \gamma_j\colon \mathbb{C}^n \to \mathbb{C}^*$ such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \gamma_j^{-1}$ are holomorphic. Hence, Kasuya states the following:

Theorem 4.2.10 (Kasuya [51, Corollary 4.2])**.** *Let $M = G/\Gamma$ be a solvmanifold endowed with an invariant complex structure of splitting type and $B_\Gamma^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet} M$ be the finite-dimensional subspace given by*

(4.6)

$$\begin{aligned} B_\Gamma^{p,q} \;:=\; & \mathbb{C}\langle x_I \wedge \left(\alpha_J^{-1} \beta_J\right) y_J \wedge \bar{x}_K \wedge \left(\bar{\alpha}_L^{-1} \gamma_L\right) \bar{y}_L \mid |I| + |J| = p \text{ and } |K| + |L| = q \\ & \text{such that } (\beta_J \gamma_L)|_\Gamma = 1\rangle, \end{aligned}$$

for $(p,q) \in \mathbb{N}^2$. Then, the inclusion $(B_\Gamma^{\bullet,\bullet}, \bar{\partial}) \hookrightarrow (\wedge^{\bullet,\bullet} M, \bar{\partial})$ induces a cohomology isomorphism

$$H_{\bar{\partial}}^{\bullet,\bullet}(B_\Gamma^{\bullet,\bullet}) \cong H_{\bar{\partial}}^{\bullet,\bullet}(M).$$

Remark 4.2.11. In (4.6) we shorten $\alpha_I := \alpha_{i_1} \cdot \dots \cdot \alpha_{i_k}$ for a multi-index $I = (i_1, \dots, i_k)$.

Angella and Kasuya [7] extend the previous technique to the computation of the Bott-Chern cohomology of a solvmanifold.

Theorem 4.2.12 (Angella and Kasuya [7, Theorem 2.16])**.** *Let $M = G/\Gamma$ be a solvmanifold endowed with an invariant complex structure of splitting type and $C_\Gamma^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet} M$ be the finite-dimensional subspace given by*

$$C_\Gamma^{\bullet,\bullet} \;:=\; B_\Gamma^{\bullet,\bullet} + \bar{B}_\Gamma^{\bullet,\bullet}, \tag{4.7}$$

where $\bar{B}_\Gamma^{\bullet,\bullet} := \left\{\bar{\omega} \in \wedge^{\bullet,\bullet} M \mid \omega \in B_\Gamma^{\bullet,\bullet}\right\}$ and $B_\Gamma^{\bullet,\bullet}$ defined by (4.6). Then, the inclusion $(C_\Gamma^{\bullet,\bullet}, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet,\bullet} M, \partial, \bar{\partial})$ induces a cohomology isomorphism

$$H_{BC}^{\bullet,\bullet}(C_\Gamma^{\bullet,\bullet}) \cong H_{BC}^{\bullet,\bullet}(M).$$

Lemma 4.2.13. *Let $(M = G/\Gamma, J)$ be a complex solvmanifold of splitting type and $(B_\Gamma^{\bullet,\bullet}, \partial, \bar\partial)$ the complex defined by* (4.6)*. If $\partial|_{B_\Gamma^{\bullet,\bullet}} = \bar\partial|_{B_\Gamma^{\bullet,\bullet}} = 0$ and $B_\Gamma^{q,p} = \overline{B_\Gamma^{p,q}}$ for all $p, q \in \mathbb{N}$ then (M, J) satisfies the $\partial\bar\partial$-lemma.*

Proof. If the complex $(B_\Gamma^{\bullet,\bullet}, \partial, \bar\partial)$ satisfies $B_\Gamma^{q,p} = \overline{B_\Gamma^{p,q}}$ for all $p, q \in \mathbb{N}$ then $C_\Gamma^{\bullet,\bullet} = B_\Gamma^{\bullet,\bullet}$. Furthermore, the condition $\partial|_{B_\Gamma^{\bullet,\bullet}} = \bar\partial|_{B_\Gamma^{\bullet,\bullet}} = 0$ forces the natural isomorphisms

$$H_{\mathrm{BC}}^{\bullet,\bullet}(M) \cong H_{\mathrm{BC}}^{\bullet,\bullet}(C_\Gamma) = C_\Gamma^{\bullet,\bullet} = B_\Gamma^{\bullet,\bullet} = H_{\bar\partial}^{\bullet,\bullet}(B_\Gamma) \cong H_{\bar\partial}^{\bullet,\bullet}(M).$$

Hence, (M, J) satisfies the $\partial\bar\partial$-lemma. □

We would like to use the previous Theorems 4.2.10 and 4.2.12 in order to study the cohomology of solvmanifolds $M = G_k/\Gamma$ with underlying real Lie algebras $\mathfrak{g}_k$ endowed with the invariant complex structures with holomorphically trivial canonical bundle (see Table 3.2). However, in order to do this, it is necessary to know which of these complex manifolds are of splitting type. The following lemma states a simple cohomological obstruction to the existence of a complex structure of splitting type:

Lemma 4.2.14. *If $(M = G/\Gamma, J)$ is a solvmanifold endowed with an invariant complex structure of splitting type and $G = \mathbb{C}^n \ltimes_\varphi N$ then $b_1(\mathfrak{g}) \geq 2n$. In particular, $b_1(\mathfrak{g}) \geq 2$.*

Proof. As a consequence of Definition 4.2.9, if $M = G/\Gamma$ is of splitting type the connected and simply-connected Lie group G endowed with the left-invariant structure must admit a semi-direct product decomposition such that $G = \mathbb{C}^n \ltimes_\varphi N$. But this implies that $b_1(\mathfrak{g}) \geq 2n$. □

The Lie algebras $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, $\mathfrak{g}_7$ or $\mathfrak{g}_9$ have $b_1(\mathfrak{g}) = 1$, hence by Lemma 4.2.14 we have that if $M = G/\Gamma$ is a solvmanifold with underlying real Lie algebra $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, $\mathfrak{g}_7$ or $\mathfrak{g}_9$ endowed with an invariant complex structure J, then (M, J) is not of splitting type.

Proposition 4.2.15. *Let G be a Lie group with underlying real Lie algebra $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$ or $\mathfrak{g}_8$ endowed with a left-invariant complex structure J defined by* (3.15) *or* (3.33)*. Then $G = \mathbb{C}^2 \rtimes_{\varphi_A} \mathbb{C}$ where $\varphi_A \colon \mathbb{C} \to GL(\mathbb{C}^2)$ is defined by the diagonal matrix:*

$$\varphi_A(z) := \begin{pmatrix} \alpha_1(z) & 0 \\ 0 & \alpha_2(z) \end{pmatrix} \tag{4.8}$$

and $\alpha_1, \alpha_2 \colon \mathbb{C} \to \mathbb{C}^*$ are characters such that $\alpha_2 = \alpha_1^{-1}$ and

$$\alpha_1(z) := \begin{cases} e^{A(z+\bar z)} & \text{if } J \text{ satisfies (3.15)} \\ e^{-(A-i)z-(A+i)\bar z} & \text{if } J \text{ satisfies (3.33)} \end{cases} \qquad \text{for any } z \in \mathbb{C}.$$

Proof. We provide the proof for the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$. Let $\{\omega^1, \omega^2, \omega^3\}$ be a left-invariant basis of $(1,0)$-forms satisfying the complex structure equations (3.15), namely $d\omega^1 = A\omega^1 \wedge (\omega^3 + \omega^{\bar 3})$, $d\omega^2 = -A\omega^2 \wedge (\omega^3 + \omega^{\bar 3})$, $d\omega^3 = 0$ with $A = e^{i\theta}$ and

$\theta \in [0, \pi)$. We can take complex coordinates $(z_1, z_2, z_3) \in \mathbb{C}^3$ on G and integrate these equations obtaining the expressions of the left-invariant $(1,0)$-forms:

$$\omega^1 = e^{-A(z_3+\bar{z}_3)}dz_1, \quad \omega^2 = e^{A(z_3+\bar{z}_3)}dz_2, \quad \omega^3 = dz_3.$$

The invariance of the $(1,0)$-forms $L_g^*(\omega^j) = \omega^j$ by an element $g \in G$ with coordinates $(a_1, a_2, a_3) \in \mathbb{C}^3$ determines the multiplication law of the group

$$(a_1, a_2, a_3) \cdot (z_1, z_2, z_3) = (e^{A(a_3+\bar{a}_3)}z_1 + a_1, e^{-A(a_3+\bar{a}_3)}z_2 + a_2, z_3 + a_3) \tag{4.9}$$

and hence we can give a matrix representation of the corresponding Lie group endowed with the complex structure as

$$G_A = \left\{ \begin{pmatrix} e^{A(z_3+\bar{z}_3)} & 0 & 0 & z_1 \\ 0 & e^{-A(z_3+\bar{z}_3)} & 0 & z_2 \\ 0 & 0 & 1 & z_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}. \tag{4.10}$$

By the coordinate expression of the multiplication given by (4.9) it is clear that $G_A = \mathbb{C}^2 \rtimes_{\varphi_A} \mathbb{C}$ and by the matrix representation (4.10) the complex manifolds G_A are biholomorphic to $\mathbb{C}^3$. The action $\varphi_A \colon \mathbb{C} \to \mathrm{GL}(\mathbb{C}^2)$ may be represented by the following matrix:

$$\varphi_A(z_3) = \begin{pmatrix} e^{A(z_3+\bar{z}_3)} & 0 \\ 0 & e^{-A(z_3+\bar{z}_3)} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Similarly, every complex structure on $\mathfrak{g}_8$ with closed volume form satisfying equations (3.33) is a semi-direct product $G_A = \mathbb{C}^2 \rtimes_{\varphi_A} \mathbb{C}$ where the action $\varphi_A \colon \mathbb{C} \to \mathrm{GL}(\mathbb{C}^2)$ is:

$$\varphi_A(z_3) = \begin{pmatrix} e^{-(A-i)z_3-(A+i)\bar{z}_3} & 0 \\ 0 & e^{(A-i)z_3+(A+i)\bar{z}_3} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

□

Remark 4.2.16. The complex structure on $\mathfrak{g}_1$ given by (3.17) and the parallelizable complex structure J_0 on $\mathfrak{g}_8$ given by (3.33) for $A = -i$ correspond to the complex structures studied by Kasuya in [51, Examples 1 and 2]. On the other hand, it can be proved that the complex structures on $\mathfrak{g}_3$ defined by (3.21) and the complex structures J' and J'' on $\mathfrak{g}_8$ defined by (3.31) and (3.32) are not of splitting type.

From now on, we consider $G = \mathbb{C} \ltimes_\varphi \mathbb{C}^2$ the connected and simply-connected Lie group with underlying real Lie algebra $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$ or $\mathfrak{g}_8$ endowed with an invariant complex structure satisfying (3.15) or (3.33). In order to construct suitable solvmanifolds $M = G/\Gamma$ for using Theorems 4.2.10 and 4.2.12, we must consider lattices $\Gamma = \Gamma' \ltimes_\varphi \Gamma''$ on G where Γ'' is a lattice of $\mathbb{C}^2$ and Γ' is a lattice of $\mathbb{C}$ compatible with the splitting. This implies that the matrices $\varphi(z_3)|_{\Gamma'}$ must be in the conjugation class of a regular integer matrix for any $z_3 \in \mathbb{C}$.

Lemma 4.2.17. *A matrix* $M_f = \begin{pmatrix} e^f & 0 \\ 0 & e^{-f} \end{pmatrix} \in GL(2,\mathbb{C})$ *with* $f \in \mathbb{C}$ *is in the class of conjugation of an integer matrix if and only if* $f = \log(\frac{n+\sqrt{n^2-4}}{2})$ *with* $n \in \mathbb{Z}$.

Proof. As the matrix M_f is diagonal then it is conjugated with an integer matrix if and only if the characteristic polynomial

$$p_{M_f}(\lambda) = \lambda^2 + (e^f + e^{-f})\lambda + 1 \in \mathbb{Z}[\lambda]$$

and this holds if and only if $e^f + e^{-f} = n \in \mathbb{Z}$. Solving this equation we get $f_\pm = \log(\frac{n\pm\sqrt{n^2-4}}{2})$ but $f_- = -f_+$ and it is direct to check that the matrices M_f and M_{-f} are conjugated. □

Remark 4.2.18. Notice that for $n = 2$ then $\varphi(f) = \mathrm{Id}_{\mathbb{C}^2}$, giving rise to the abelian complex Lie group $G = \mathbb{C}^3$, therefore we exclude this case in the following. In order to make easier the computations of lattices, we show the values of the function $f(n) := \log(\frac{n+\sqrt{n^2-4}}{2}) \in \mathbb{C}$ for $n \in \mathbb{Z}$ in the next table:

n	≤ -3	-2	-1	0	1	2	≥ 3
$\log(\frac{n+\sqrt{n^2-4}}{2})$	$\log(\lvert\frac{n+\sqrt{n^2-4}}{2}\rvert) + i\pi$	$i\pi$	$\frac{2i\pi}{3}$	$\frac{i\pi}{2}$	$\frac{i\pi}{3}$	0	$\log(\frac{n+\sqrt{n^2-4}}{2})$

Table 4.1: Values of the function $f(n)$.

Results on $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$

Proposition 4.2.15 states that the connected and simply-connected Lie groups G with underlying real Lie algebra $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$ endowed with a left-invariant complex structure described by (3.15) may be written as a semi-direct product $\mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$, where the holomorphic action φ_A is described by a diagonal matrix (4.8). The characters $\alpha_j\colon \mathbb{C} \to \mathbb{C}^*$ required to construct the double complex $(B_\Gamma^{\bullet,\bullet}, \bar{\partial})$ are

$$\alpha_1(z_3) = e^{A(z_3+\bar{z}_3)}, \qquad \alpha_2(z_3) = e^{-A(z_3+\bar{z}_3)}, \tag{4.11}$$

for $A = e^{i\theta}$ with $\theta \in (0,\pi)$. In particular, by considering a set $\{z_1, z_2\}$ of local coordinates on $\mathbb{C}^2$ and z_3 a local coordinate on $\mathbb{C}$, we have a basis of left-invariant $(1,0)$-forms:

$$\omega^1 = \alpha_1^{-1} dz_1 = e^{-A(z_3+\bar{z}_3)} dz_1, \qquad \omega^2 = \alpha_2^{-1} dz_2 = e^{A(z_3+\bar{z}_3)} dz_2, \qquad \omega^3 = dz_3.$$

The unitary characters $\beta_1, \beta_2, \gamma_1, \gamma_2\colon \mathbb{C} \to \mathbb{C}^*$ satisfying that $\alpha_1\beta_1^{-1}$, $\alpha_2\beta_2^{-1}$, $\bar{\alpha}_1\gamma_1^{-1}$, $\bar{\alpha}_2\gamma_2^{-1}$ are holomorphic are the following:

$$\begin{aligned} \beta_1(z_3) &= e^{-\bar{A}z_3 + A\bar{z}_3}, & \beta_2(z_3) &= \beta_1(z_3)^{-1} = e^{\bar{A}z_3 - A\bar{z}_3}, \\ \gamma_1(z_3) &= e^{-Az_3 + \bar{A}\bar{z}_3}, & \gamma_2(z_3) &= \gamma_1(z_3)^{-1} = e^{Az_3 - \bar{A}\bar{z}_3}. \end{aligned} \tag{4.12}$$

According to (4.6), the generators of the complex $B_\Gamma^{\bullet,\bullet} = \wedge^{\bullet,\bullet}\langle\varphi^1, \varphi^2, \varphi^3, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3\rangle$ are

$$\begin{cases} \varphi^1 = \beta_1\omega^1 = e^{-(A+\bar{A})z_3}dz_1, \\ \varphi^2 = \beta_2\omega^2 = e^{(A+\bar{A})z_3}dz_2, \\ \varphi^3 = \omega^3 = dz_3, \end{cases} \qquad \begin{cases} \tilde{\varphi}^1 = \gamma_1\omega^{\bar{1}} = e^{-(A+\bar{A})z_3}d\bar{z}_1, \\ \tilde{\varphi}^2 = \gamma_2\omega^{\bar{2}} = e^{(A+\bar{A})z_3}d\bar{z}_2, \\ \tilde{\varphi}^3 = \omega^{\bar{3}} = d\bar{z}_3. \end{cases}$$

where $\varphi^1, \varphi^2, \varphi^3$ have bidegree $(1,0)$, $\tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3$ have bidegree $(0,1)$ and, in addition, one of the following conditions

$$\beta_1|_\Gamma = 1, \quad \gamma_1|_\Gamma = 1, \quad (\beta_1\gamma_1)|_\Gamma = 1, \quad (\beta_1\gamma_1^{-1})|_\Gamma = 1,$$

concerning the compatibility of the complex structure with the lattice $\Gamma \subset G$ must be satisfied. The complex structure equations expressed in the co-frame $\{\varphi^1, \varphi^2, \varphi^3, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3\}$ are:

$$\begin{cases} d\varphi^1 = (A+\bar{A})\varphi^{13}, \\ d\varphi^2 = -(A+\bar{A})\varphi^{23}, \\ d\varphi^3 = 0, \end{cases} \qquad \begin{cases} d\tilde{\varphi}^1 = -(A+\bar{A})\varphi^{3\bar{1}}, \\ d\tilde{\varphi}^2 = (A+\bar{A})\varphi^{3\bar{2}}, \\ d\tilde{\varphi}^3 = 0 \end{cases} \tag{4.13}$$

Now, we are concerned to find lattices $\Gamma := \Gamma' \ltimes_{\varphi_A} \Gamma''$, where $\Gamma' \subset \mathbb{C}$ and $\Gamma'' \subset \mathbb{C}^2$ are lattices of $\mathbb{C}$ and $\mathbb{C}^2$ respectively, compatible with the semi-direct product $G = \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$. As we mentioned before, this means that the restriction to Γ' of φ_A must be in the conjugation class of an integer matrix.

Lemma 4.2.19. *Let $x_3, b \in \mathbb{R}$ be such that $0 \neq b \in \mathbb{R}$ and*

1. $x_3 \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$ *for* $A = e^{i\frac{\pi}{2}}$.
2. $x_3(n) = \frac{\pi}{2\mathfrak{Im}\, A_n}$ *for* $A_n = e^{i\theta_n}$ *such that* $\theta_n \in (0,\pi)$ *satisfies*

$$\tan\theta_n = \frac{\pi}{\log(|\frac{n+\sqrt{n^2-4}}{2}|)} \quad \text{for } n \leq -3. \tag{4.14}$$

If $\varphi_A \colon \mathbb{C} \to GL(\mathbb{C}^2)$ is described by (4.8) and (4.11), then the lattice $\Gamma' = x_3\mathbb{Z} \oplus ib\mathbb{Z}$ satisfies that $\varphi_A|_\Gamma$ is in the conjugation class of an integer matrix.

Proof. Suppose $0 \neq z_3 \in \Gamma' \subset \mathbb{C}$ and Γ' is lattice such that $\varphi_A|_\Gamma$ is in the conjugation class of an integer matrix. By Lemma 4.2.17, $z_3 = x_3 + iy_3$ satisfies that $e^{i\theta}(z_3 + \bar{z}_3) = \log(n + \sqrt{n^2-4})$ with $\theta \in (0,\pi)$ holds, namely,

$$2x_3\sin\theta = \operatorname{Arg}\left(\frac{n+\sqrt{n^2-4}}{2}\right), \quad 2x_3\cos\theta = \log\left|\frac{n+\sqrt{n^2-4}}{2}\right|. \tag{4.15}$$

Hence $x_3 \neq 0$ (recall that we excluded the value $n = 2$, see Remark 4.2.18) and we can consider $y_3 = 0$.

(i) If $A = e^{i\frac{\pi}{2}}$ then substituting $\theta = \frac{\pi}{2}$ in (4.15) and observing Table 4.1 yield the solutions $x_3 \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$ for $n \in \{-2, -1, 0, 1\}$.

(ii) If $A = e^{i\theta}$ with $\theta \in (0, \pi)$ and $\theta \neq \frac{\pi}{2}$, then, as $x_3 \neq 0$, the results in Table 4.1 force that $n \leq -3$. Dividing both equations in (4.15) we obtain that the complex structures on G satisfying equations (3.19) admitting a lattice compatible with the splitting are given by the unique angle $\theta \in (0, \pi)$ satisfying (4.14). Finally, substituting in (4.15) we get $x_3(n) = \frac{\pi}{2\mathfrak{Im}\, A_n}$. □

Remark 4.2.20. Recall that the complex structure satisfying (3.19) for the parameter θ determines the underlying real Lie algebra by the expression $\alpha = |\frac{\cos\theta}{\sin\theta}|$, hence the Lie groups $G = \mathbb{C} \ltimes_{\varphi_{A_n}} \mathbb{C}^2$ with $A_n = e^{i\theta_n}$ and θ_n defined by (4.14) have underlying real Lie algebra $\mathfrak{g}_2^{\alpha_n}$ with $\alpha_n = \frac{1}{\pi}|\log(|\frac{n+\sqrt{n^2-4}}{2}|)|$ for $n \leq -3$.

In addition, if $\theta_n \in (0, \pi)$ with $\theta \neq \frac{\pi}{2}$ is a solution of (4.14) then it defines a complex structure on $\mathfrak{g}_2^{\alpha_n}$ satisfying (3.19). The other non-equivalent complex structure on $\mathfrak{g}_2^{\alpha_n}$ is represented by $\pi - \theta_n$. Notice that as $\sin(\pi - \theta_n) = \sin\theta_n$ then, fixed $0 \neq b \in \mathbb{R}$, the same lattice $\Gamma' = x_3(n)\mathbb{Z} \oplus ib\mathbb{Z}$ is compatible with both non-equivalent complex structures.

Once we have computed the lattices compatible with the splitting structures we can use Theorem 4.2.10 in order to compute cohomologies of the corresponding complex solvmanifolds.

Proposition 4.2.21. *Let $G = \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$ be a Lie group endowed with an invariant complex structure of splitting type where φ_A is described by* (4.8) *and* (4.11) *for $A = e^{i\frac{\pi}{2}}$ and $\Gamma = \Gamma' \ltimes_{\varphi_A} \Gamma''$ is a lattice of G compatible with the splitting, where Γ' is a lattice of $\mathbb{C}$ according to Lemma 4.2.19. Then, the complex solvmanifold $(M = G/\Gamma, J)$ satisfies the $\partial\bar{\partial}$-lemma.*

Moreover if $\Gamma' = \frac{\pi}{2}\mathbb{Z} \oplus i\mathbb{Z}$ then the inclusion $H^\bullet(\mathfrak{g}_2^{0*}) \hookrightarrow H^\bullet(M)$ is an isomorphism although $\mathfrak{g}_2^0$ is not completely solvable.

Proof. Let $\Gamma = \Gamma' \ltimes_\varphi \Gamma''$ be a lattice compatible with the splitting. By Lemma 4.2.19, $\Gamma' = x_3\mathbb{Z} \oplus ib\mathbb{Z}$ where $x_3 \in \{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}\}$ and $0 \neq b \in \mathbb{R}$. Then for the unitary characters given by (4.12) we have

$$\beta_1|_\Gamma \neq 1, \quad \gamma_1|_\Gamma \neq 1, \quad (\beta_1\gamma_1)|_\Gamma = 1,$$

for any x_3 and $(\beta_1\gamma_1^{-1})|_\Gamma = 1$ if and only if $x_3 = \frac{\pi}{2}$. The results of the computation of the double complex $B_\Gamma^{\bullet,\bullet}$ are summarized in Table 4.2. Moreover, the complex structure on $\mathfrak{g}_2^0$ corresponds to $A = i$ in equations (4.13) and we get the hypothesis of Lemma 4.2.13, therefore the complex solvmanifold $M = G/\Gamma$ satisfies the $\partial\bar{\partial}$-lemma property for every lattice compatible with the splitting.

Particularly, for the lattice $\Gamma' = \frac{\pi}{2}\mathbb{Z} \oplus ib\mathbb{Z}$ with $0 \neq b \in \mathbb{R}$ we find that $b_k(M) = b_k(\mathfrak{g}_2^0)$ for $k = 0, \ldots, 6$ and therefore the inclusion $H^\bullet(\mathfrak{g}_2^{0*}) \hookrightarrow H^\bullet(M)$ is an isomorphism. Nevertheless, the eigenvalues of the endomorphism $\mathrm{ad}_{e_5} \colon \mathfrak{g}_2^0 \to \mathfrak{g}_2^0$ are $\lambda_1 = -\lambda_2 = i$, $\lambda_3 = 0$, hence $\mathfrak{g}_2^0$ is not completely solvable. □

Proposition 4.2.22. *Let $G = \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$ be a Lie group endowed with an invariant complex structure of splitting type, where φ_A is described by* (4.8) *and* (4.11) *and $A_n = e^{i\theta_n}$ for some $n \leq -3$, where $\theta_n \in (0, \pi)$ satisfies* (4.14) *and $\Gamma = \Gamma' \ltimes_{\varphi_{A_n}} \Gamma''$ is a lattice of G compatible with the splitting where Γ' is a lattice of $\mathbb{C}$ according to Lemma 4.2.19. Then $\Gamma' = \frac{\pi}{2\mathfrak{Im}\, A_n}\mathbb{Z} \oplus ib\mathbb{Z}$ and if $(M = G/\Gamma, J)$ is the complex solvmanifold, then*

- *$E_1(M) \not\cong E_\infty(M)$ for $b = k\frac{\pi}{2\mathfrak{Re}\, A_n}$ with $0 \neq k \in \mathbb{Z}$.*
- *(M, J) satisfies the $\partial\bar{\partial}$-lemma for $b \neq k\frac{\pi}{2\mathfrak{Re}\, A_n}$ with $0 \neq k \in \mathbb{Z}$.*

Proof. Let $\Gamma = \Gamma' \ltimes_{\varphi_{A_n}} \Gamma''$ be a lattice compatible with the splitting. By Lemma 4.2.19, $\Gamma' = \frac{\pi}{2\mathfrak{Im}\, A_n}\mathbb{Z} \oplus ib\mathbb{Z}$ with $0 \neq b \in \mathbb{R}$. Then for the unitary characters given by (4.12) we have

$$\begin{cases} (\beta_1\gamma_1^{-1})|_\Gamma = 1 & \text{for any } b \in \mathbb{R}, \\ (\beta_1\gamma_1)|_\Gamma = 1 & \text{if and only if } b = k\frac{\pi}{2\mathfrak{Re}\, A_n} \text{ with } k \in \mathbb{Z}, \\ \beta_1|_\Gamma = \gamma_1|_\Gamma = 1 & \text{if and only if } b = (2k+1)\frac{\pi}{2\mathfrak{Re}\, A_n} \text{ with } k \in \mathbb{Z} \end{cases}$$

The results of the computation of the complex $B_\Gamma^{\bullet,\bullet}$ can be found in Table 4.3. For the lattices $\Gamma' = \frac{\pi}{2\mathfrak{Im}\, A_n}\mathbb{Z} \oplus ib\mathbb{Z}$ with $b \neq k\frac{\pi}{2\mathfrak{Re}\, A_n}$ for any $k \in \mathbb{Z}$, the complex $B_\Gamma^{\bullet,\bullet}$ satisfies the hypothesis of Lemma 4.2.13 and hence the corresponding complex solvmanifolds satisfy the $\partial\bar{\partial}$-lemma. □

	$B_\Gamma^{\bullet,\bullet}(\Gamma' = \frac{\pi}{2}\mathbb{Z} \oplus ib\mathbb{Z},\, 0 \neq b \in \mathbb{R})$	$\mathbf{h}_{\bar{\partial}}^{\bullet,\bullet}$	$\mathbf{b}_\bullet$	$B_\Gamma^{\bullet,\bullet}(\Gamma' = x_3\mathbb{Z} \oplus ib\mathbb{Z},\, x_3 \neq \frac{\pi}{2},\, 0 \neq b \in \mathbb{R})$	$\mathbf{h}_{\bar{\partial}}^{\bullet,\bullet}$	$\mathbf{b}_\bullet$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{3} \right\rangle$	**1**	2	$\mathbb{C}\left\langle \varphi^{3} \right\rangle$	**1**	2
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\bar{3}} \right\rangle$	**1**	
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	**1**	
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}}, \varphi^{1\tilde{2}}, \varphi^{2\tilde{1}}, \varphi^{2\tilde{2}}, \varphi^{3\bar{3}} \right\rangle$	**5**	7	$\mathbb{C}\left\langle \varphi^{1\tilde{1}}, \varphi^{2\tilde{2}}, \varphi^{3\bar{3}} \right\rangle$	**3**	5
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	**1**	
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{12\bar{3}}, \varphi^{13\tilde{1}}, \varphi^{13\tilde{2}}, \varphi^{23\tilde{1}}, \varphi^{23\tilde{2}} \right\rangle$	**5**	12	$\mathbb{C}\left\langle \varphi^{12\bar{3}}, \varphi^{13\tilde{1}}, \varphi^{23\tilde{2}} \right\rangle$	**3**	8
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\bar{3}}, \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{2\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}} \right\rangle$	**5**		$\mathbb{C}\left\langle \varphi^{1\tilde{1}\bar{3}}, \varphi^{2\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}} \right\rangle$	**3**	
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{123\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{123\bar{3}} \right\rangle$	**1**	
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{13\tilde{1}\bar{3}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\bar{3}}, \varphi^{23\tilde{2}\bar{3}} \right\rangle$	**5**	7	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{13\tilde{1}\bar{3}}, \varphi^{23\tilde{2}\bar{3}} \right\rangle$	**3**	5
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}} \right\rangle$	**1**	2	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}} \right\rangle$	**1**	2
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	**1**	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	**1**

Table 4.2: The double complex $B_\Gamma^{\bullet,\bullet}$ for computing the Dolbeault cohomology of the complex solvmanifolds $(M = G/\Gamma, J)$, with underlying Lie algebra $\mathfrak{g}_2^0$, described in Proposition 4.2.21.

	$B_\Gamma^{\bullet,\bullet}(\Gamma' = \frac{i\pi}{2\mathfrak{Im}\,A_n}\mathbb{Z} \oplus \frac{i(2k-1)\pi}{2\mathfrak{Re}\,A_n}\mathbb{Z})$	$\mathbf{h}_{\bar\partial}^{\bullet,\bullet}$	$\mathbf{b}_\bullet$	$B_\Gamma^{\bullet,\bullet}(\Gamma' = \frac{i\pi}{2\mathfrak{Im}\,A_n}\mathbb{Z} \oplus \frac{2ik\pi}{2\mathfrak{Re}\,A_n}\mathbb{Z})$	$\mathbf{h}_{\bar\partial}^{\bullet,\bullet}$	$\mathbf{b}_\bullet$	$B_\Gamma^{\bullet,\bullet}(\Gamma' = \frac{i\pi}{2\mathfrak{Im}\,A_n}\mathbb{Z} \oplus ib\mathbb{Z},\ b \neq \frac{k\pi}{2\mathfrak{Re}\,A_n})$	$\mathbf{h}_{\bar\partial}^{\bullet,\bullet}$	$\mathbf{b}_\bullet$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^1, \varphi^2, \varphi^3 \right\rangle$	$\mathbf{3}$	2	$\mathbb{C}\left\langle \varphi^3 \right\rangle$	$\mathbf{1}$	2	$\mathbb{C}\left\langle \varphi^3 \right\rangle$	$\mathbf{1}$	2
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \tilde\varphi^1, \tilde\varphi^2, \varphi^{\bar3} \right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle \varphi^{\bar3} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{\bar3} \right\rangle$	$\mathbf{1}$	
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{12}, \varphi^{13}, \varphi^{23} \right\rangle$	$\mathbf{3}$	5	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	$\mathbf{1}$	5	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	$\mathbf{1}$	5
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{1\tilde1}, \varphi^{1\tilde2}, \varphi^{1\bar3}, \varphi^{2\tilde1}, \varphi^{2\tilde2}, \varphi^{2\bar3}, \varphi^{3\tilde1}, \varphi^{3\tilde2}, \varphi^{3\bar3} \right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle \varphi^{1\tilde1}, \varphi^{1\tilde2}, \varphi^{2\tilde1}, \varphi^{2\tilde2}, \varphi^{3\bar3} \right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle \varphi^{1\tilde2}, \varphi^{2\tilde1}, \varphi^{3\bar3} \right\rangle$	$\mathbf{3}$	
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2}, \varphi^{\tilde1\bar3}, \varphi^{\tilde2\bar3} \right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2} \right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	$\mathbf{1}$	8	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	$\mathbf{1}$	8	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	$\mathbf{1}$	8
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{12\tilde1}, \varphi^{12\tilde2}, \varphi^{12\bar3}, \varphi^{13\tilde1}, \varphi^{13\tilde2}, \varphi^{13\bar3}, \varphi^{23\tilde1}, \varphi^{23\tilde2}, \varphi^{23\bar3} \right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle \varphi^{12\bar3}, \varphi^{13\tilde1}, \varphi^{13\tilde2}, \varphi^{23\tilde1}, \varphi^{23\tilde2} \right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle \varphi^{12\bar3}, \varphi^{13\tilde2}, \varphi^{23\tilde1} \right\rangle$	$\mathbf{3}$	
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{1\tilde1\tilde2}, \varphi^{1\tilde1\bar3}, \varphi^{1\tilde2\bar3}, \varphi^{2\tilde1\tilde2}, \varphi^{2\tilde1\bar3}, \varphi^{2\tilde2\bar3}, \varphi^{3\tilde1\tilde2}, \varphi^{3\tilde1\bar3}, \varphi^{3\tilde2\bar3} \right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle \varphi^{1\tilde1\bar3}, \varphi^{1\tilde2\bar3}, \varphi^{2\tilde1\bar3}, \varphi^{2\tilde2\bar3}, \varphi^{3\tilde1\tilde2} \right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle \varphi^{1\tilde2\bar3}, \varphi^{2\tilde1\bar3}, \varphi^{3\tilde1\tilde2} \right\rangle$	$\mathbf{3}$	
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{123\tilde1}, \varphi^{123\tilde2}, \varphi^{123\bar3} \right\rangle$	$\mathbf{3}$	5	$\mathbb{C}\left\langle \varphi^{123\bar3} \right\rangle$	$\mathbf{1}$	5	$\mathbb{C}\left\langle \varphi^{123\bar3} \right\rangle$	$\mathbf{1}$	5
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2}, \varphi^{12\tilde1\bar3}, \varphi^{12\tilde2\bar3}, \varphi^{13\tilde1\tilde2}, \varphi^{13\tilde1\bar3}, \varphi^{13\tilde2\bar3}, \varphi^{23\tilde1\tilde2}, \varphi^{23\tilde1\bar3}, \varphi^{23\tilde2\bar3} \right\rangle$	$\mathbf{9}$		$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2}, \varphi^{13\tilde1\bar3}, \varphi^{13\tilde2\bar3}, \varphi^{23\tilde1\bar3}, \varphi^{23\tilde2\bar3} \right\rangle$	$\mathbf{5}$		$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2}, \varphi^{13\tilde2\bar3}, \varphi^{23\tilde1\bar3} \right\rangle$	$\mathbf{3}$	
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{1\tilde1\tilde2\bar3}, \varphi^{2\tilde1\tilde2\bar3}, \varphi^{3\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle \varphi^{3\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{3\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2}, \varphi^{123\tilde1\bar3}, \varphi^{123\tilde2\bar3} \right\rangle$	$\mathbf{3}$	2	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2} \right\rangle$	$\mathbf{1}$	2	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2} \right\rangle$	$\mathbf{1}$	2
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2\bar3}, \varphi^{13\tilde1\tilde2\bar3}, \varphi^{23\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{3}$		$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$		$\mathbb{C}\left\langle \varphi^{12\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	$\mathbf{1}$	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	$\mathbf{1}$	$\mathbb{C}\left\langle \varphi^{123\tilde1\tilde2\bar3} \right\rangle$	$\mathbf{1}$	$\mathbf{1}$

Table 4.3: The double complex $B_\Gamma^{\bullet,\bullet}$ for computing the Dolbeault cohomology of the complex solvmanifolds $(M = G/\Gamma, J)$, with underlying Lie algebra $\mathfrak{g}_2^{\alpha_n}$ with $\alpha_n = \frac{1}{\pi}|\log(|\frac{n+\sqrt{n^2-4}}{2}|)|$, described in Proposition 4.2.22.

Results on $\mathfrak{g}_8$

Proposition 4.2.15 states that the connected and simply-connected Lie groups G with underlying real Lie algebra $\mathfrak{g}_8$ endowed with a left-invariant complex structure described by (3.33) may be written as a semi-direct product $\mathbb{C} \ltimes_\varphi \mathbb{C}^2$, where the action φ is described by a diagonal matrix of the form (4.8). Now, the characters $\alpha_1, \alpha_2 \colon \mathbb{C} \to \mathbb{C}^*$ required to construct the double complex $(B_\Gamma^{\bullet,\bullet}, \bar{\partial})$ are

$$\alpha_1(z_3) = e^{-(A-i)z_3-(A+i)\bar{z}_3}, \qquad \alpha_2(z_3) = e^{(A-i)z_3+(A+i)\bar{z}_3}, \tag{4.16}$$

where $A \in \mathbb{C}$ such that $\mathfrak{Im}\, A \neq 0$. In particular, by considering a set $\{z_1, z_2\}$ of local coordinates on $\mathbb{C}^2$ and z_3 a local coordinate on $\mathbb{C}$, we have the following basis of left-invariant $(1,0)$-forms:

$$\omega^1 = \alpha_1^{-1} dz_1 = e^{(A-i)z_3+(A+i)\bar{z}_3} dz_1, \quad \omega^2 = \alpha_2^{-1} dz_2 = e^{-(A-i)z_3-(A+i)\bar{z}_3} dz_2, \quad \omega^3 = dz_3.$$

The unitary characters $\beta_1, \beta_2, \gamma_1, \gamma_2 \colon \mathbb{C} \to \mathbb{C}^*$ satisfying that $\alpha_1\beta_1^{-1}$, $\alpha_2\beta_2^{-1}$, $\bar{\alpha}_1\gamma_1^{-1}$, $\bar{\alpha}_2\gamma_2^{-1}$ are holomorphic are in this case:

$$\begin{aligned} &\beta_1(z_3) = e^{(\bar{A}-i)z_3-(A+i)\bar{z}_3}, && \beta_2(z_3) = \beta_1(z_3)^{-1} = e^{-(\bar{A}-i)z_3+(A+i)\bar{z}_3}, \\ &\gamma_1(z_3) = e^{(A-i)z_3-(\bar{A}+i)\bar{z}_3}, && \gamma_2(z_3) = \gamma_1(z_3)^{-1} = e^{-(A-i)z_3+(\bar{A}+i)\bar{z}_3}, \end{aligned} \tag{4.17}$$

and the generators of the complex $B_\Gamma^{\bullet,\bullet} = \wedge^{\bullet,\bullet}\langle \varphi^1, \varphi^2, \varphi^3, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3 \rangle$ are:

$$\begin{cases} \varphi^1 = \beta_1\omega^1 = e^{(A+\bar{A}-2i)z_3} dz_1, \\ \varphi^2 = \beta_2\omega^2 = e^{-(A+\bar{A}-2i)z_3} dz_2, \\ \varphi^3 = dz_3, \end{cases} \qquad \begin{cases} \tilde{\varphi}^1 = \gamma_1\omega^{\bar{1}} = e^{(A+\bar{A}-2i)z_3} d\bar{z}_1, \\ \tilde{\varphi}^2 = \gamma_1\omega^{\bar{2}} = e^{-(A+\bar{A}-2i)z_3} d\bar{z}_2, \\ \tilde{\varphi}^3 = d\bar{z}_3, \end{cases}$$

where $\varphi^1, \varphi^2, \varphi^3$ have bidegree $(1,0)$ and $\tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3$ have bidegree $(0,1)$. The complex structure equations expressed in the co-frame $\{\varphi^1, \varphi^2, \varphi^3, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3\}$ are:

$$\begin{cases} d\varphi^1 = (A+\bar{A}-2i)\varphi^{13}, \\ d\varphi^2 = -(A+\bar{A}-2i)\varphi^{23}, \\ d\varphi^3 = 0, \end{cases} \quad \begin{cases} d\tilde{\varphi}^1 = -(A+\bar{A}-2i)\varphi^{3\tilde{1}}, \\ d\tilde{\varphi}^2 = (A+\bar{A}-2i)\varphi^{3\tilde{2}}, \\ d\tilde{\varphi}^3 = 0. \end{cases} \tag{4.18}$$

Lemma 4.2.23. *Let $A \in \mathbb{C}$ be such that $\mathfrak{Im}\, A \neq 0$ and $F_A \colon \mathbb{Z} \to \mathbb{C}$ be the function defined by:*
(4.19)

$$F_A(n) := \frac{1}{2\mathfrak{Im}\, A} Arg\left(\frac{n+\sqrt{n^2-4}}{2}\right) + \frac{i}{2}(\log\left|\frac{n+\sqrt{n^2-4}}{2}\right| - \frac{\mathfrak{Re}\, A}{\mathfrak{Im}\, A} Arg\left(\frac{n+\sqrt{n^2-4}}{2}\right)).$$

If $\varphi_A \colon \mathbb{C} \to GL(\mathbb{C}^2)$ is described by (4.8) *and* (4.16) *then the lattice $\Gamma' := F(n)\mathbb{Z} \oplus F(n')\mathbb{Z}$ of $\mathbb{C}$ with $n \neq n' \in \mathbb{Z}$ satisfies that $\varphi_A|_\Gamma$ is in the conjugation class of an integer matrix.*

Proof. Let $\Gamma = \Gamma' \ltimes_{\varphi_A} \Gamma'' \subset \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$, where Γ' is a lattice of $\mathbb{C}$ compatible with the splitting. This means that the restriction of φ_A to Γ' must be in the conjugation class of an integer matrix. By using Lemma 4.2.17, $z_3 = x_3 + iy_3$ satisfies that $-(A-i)z_3 - (A+i)\bar{z}_3 = \log(n + \sqrt{n^2-4})$ holds. The previous condition gives rise to the equations

$$2\mathfrak{Re}\, A x_3 + 2y_3 = \log\left|\frac{n+\sqrt{n^2-4}}{2}\right|, \quad 2x_3 \mathfrak{Im}\, A = \operatorname{Arg}\left(\frac{n+\sqrt{n^2-4}}{2}\right).$$

Solving the previous system we conclude that

$$x_3 = \frac{1}{2\mathfrak{Im}\, A}\operatorname{Arg}\left(\frac{n+\sqrt{n^2-4}}{2}\right), \; y_3 = \frac{1}{2}(\log\left|\frac{n+\sqrt{n^2-4}}{2}\right| - \frac{\mathfrak{Re}\, A}{\mathfrak{Im}\, A}\operatorname{Arg}\left(\frac{n+\sqrt{n^2-4}}{2}\right)),$$

hence we can define a function $F_A\colon \mathbb{Z} \to \mathbb{C}$ by $F_A(n) := x_3 + iy_3$. □

Remark 4.2.24. In order to obtain different lattices of Lie group $G = \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$ with φ_A described by (4.8) and (4.16), we find convenient to tabulate the different values of the map $F_A\colon \mathbb{Z} \to \mathbb{C}$ defined by (4.19) in the following table:

n	$F_A(n)$
≤ -3	$\frac{\pi}{2\mathfrak{Im}\, A} + \frac{i}{2}(\log\lvert\frac{n+\sqrt{n^2-4}}{2}\rvert - \frac{\mathfrak{Re}\, A}{\mathfrak{Im}\, A}\pi)$
-2	$\frac{\pi}{2\mathfrak{Im}\, A}(1 - i\mathfrak{Re}\, A)$
-1	$\frac{\pi}{3\mathfrak{Im}\, A}(1 - i\mathfrak{Re}\, A)$
0	$\frac{\pi}{4\mathfrak{Im}\, A}(1 - i\mathfrak{Re}\, A)$
2	0
1	$\frac{\pi}{6\mathfrak{Im}\, A}(1 - i\mathfrak{Re}\, A)$
≥ 3	$\frac{i}{2}\log(\frac{n+\sqrt{n^2-4}}{2})$

Table 4.4: Values of the function $F_A(n)$ defined by (4.19).

Proposition 4.2.25. *Let $G = \mathbb{C} \ltimes_{\varphi_A} \mathbb{C}^2$ be a Lie group endowed with an invariant complex structure of splitting type where φ_A is described by* (4.8) *and* (4.16) *and $\Gamma = \Gamma' \ltimes_{\varphi_A} \Gamma''$ is a lattice of G compatible with the splitting, where $\Gamma' = F_A(-2)\mathbb{Z} \oplus F_A(n)\mathbb{Z}$ for some $n \geq 3$. The complex solvmanifold $(M = G/\Gamma, J)$ satisfies the $\partial\bar{\partial}$-lemma if and only if $A \neq \frac{i}{k} \in \mathbb{C}$ for $0 \neq k \in \mathbb{Z}$.*

Proof. Let $\Gamma = \Gamma' \ltimes_{\varphi_A} \Gamma''$ be a lattice compatible with the splitting, where $\Gamma' = \frac{\pi}{2\mathfrak{Im}\, A}\mathbb{Z} \oplus \frac{i}{2}\log(\frac{n+\sqrt{n^2-4}}{2})\mathbb{Z}$ then we have

$$\begin{cases} (\beta_1\gamma_1)|_\Gamma = 1, & \text{for any } A \in \mathbb{C}, \\ (\beta_1\gamma_1^{-1})|_\Gamma = 1, & \text{if and only if } A = \frac{i}{k} \text{ with } 0 \neq k \in \mathbb{Z}, \\ \beta_1|_\Gamma = \gamma_1|_\Gamma = 1, & \text{if and only if } A = \frac{i}{2k+1} \text{ with } k \in \mathbb{Z}. \end{cases}$$

The computation of the double complex $B_\Gamma^{\bullet,\bullet}$ for the solvmanifolds $M = G/\Gamma$ can be found in Table 4.5. If $A = \frac{i}{k}$ for $0 \neq k \in \mathbb{Z}$ then $E_1(M) \ncong E_\infty(M)$ and in particular M does not satisfy the $\partial\bar{\partial}$-lemma, whereas for $A \neq \frac{i}{k}$ the hypothesis of Lemma 4.2.13 are satisfied and hence all the corresponding complex solvmanifolds satisfy the $\partial\bar{\partial}$-lemma. $\square$

	$B_{\Gamma}^{\bullet,\bullet}(A = \frac{i}{2k+1},\, k \in \mathbb{Z})$	$\mathbf{h}_{\bar{\partial}}^{\bullet,\bullet}$	$\mathbf{b}_{\bullet}$	$B_{\Gamma}^{\bullet,\bullet}(A = \frac{i}{2k},\, 0 \neq k \in \mathbb{Z})$	$\mathbf{h}_{\bar{\partial}}^{\bullet,\bullet}$	$\mathbf{b}_{\bullet}$	$B_{\Gamma}^{\bullet,\bullet}(A \neq \frac{i}{k},\, 0 \neq k \in \mathbb{Z})$	$\mathbf{h}_{\bar{\partial}}^{\bullet,\bullet}$	$\mathbf{b}_{\bullet}$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{1}, \varphi^{2}, \varphi^{3} \right\rangle$	**3**	2	$\mathbb{C}\left\langle \varphi^{3} \right\rangle$	**1**	2	$\mathbb{C}\left\langle \varphi^{3} \right\rangle$	**1**	2
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \tilde{\varphi}^{1}, \tilde{\varphi}^{2}, \varphi^{\bar{3}} \right\rangle$	**3**		$\mathbb{C}\left\langle \varphi^{\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\bar{3}} \right\rangle$	**1**	
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{12}, \varphi^{13}, \varphi^{23} \right\rangle$	**3**	5	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	**1**	5	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	**1**	5
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}}, \varphi^{1\tilde{2}}, \varphi^{1\bar{3}}, \varphi^{2\tilde{1}}, \varphi^{2\tilde{2}}, \varphi^{2\bar{3}}, \varphi^{3\tilde{1}}, \varphi^{3\tilde{2}}, \varphi^{3\bar{3}} \right\rangle$	**9**		$\mathbb{C}\left\langle \varphi^{1\tilde{1}}, \varphi^{1\tilde{2}}, \varphi^{2\tilde{1}}, \varphi^{2\tilde{2}}, \varphi^{3\bar{3}} \right\rangle$	**5**		$\mathbb{C}\left\langle \varphi^{1\tilde{2}}, \varphi^{2\tilde{1}}, \varphi^{3\bar{3}} \right\rangle$	**3**	
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}}, \varphi^{\tilde{1}\bar{3}}, \varphi^{\tilde{2}\bar{3}} \right\rangle$	**3**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	**1**	8	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	**1**	8	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	**1**	8
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}}, \varphi^{12\tilde{2}}, \varphi^{12\bar{3}}, \varphi^{13\tilde{1}}, \varphi^{13\tilde{2}}, \varphi^{13\bar{3}}, \varphi^{23\tilde{1}}, \varphi^{23\tilde{2}}, \varphi^{23\bar{3}} \right\rangle$	**9**		$\mathbb{C}\left\langle \varphi^{12\bar{3}}, \varphi^{13\tilde{1}}, \varphi^{13\tilde{2}}, \varphi^{23\tilde{1}}, \varphi^{23\tilde{2}} \right\rangle$	**5**		$\mathbb{C}\left\langle \varphi^{12\bar{3}}, \varphi^{13\tilde{2}}, \varphi^{23\tilde{1}} \right\rangle$	**3**	
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\tilde{2}}, \varphi^{1\tilde{1}\bar{3}}, \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\tilde{2}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{2\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}}, \varphi^{3\tilde{1}\bar{3}}, \varphi^{3\tilde{2}\bar{3}} \right\rangle$	**9**		$\mathbb{C}\left\langle \varphi^{1\tilde{1}\bar{3}}, \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{2\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}} \right\rangle$	**5**		$\mathbb{C}\left\langle \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}} \right\rangle$	**3**	
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}}, \varphi^{123\tilde{2}}, \varphi^{123\bar{3}} \right\rangle$	**3**	5	$\mathbb{C}\left\langle \varphi^{123\bar{3}} \right\rangle$	**1**	5	$\mathbb{C}\left\langle \varphi^{123\bar{3}} \right\rangle$	**1**	5
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{12\tilde{1}\bar{3}}, \varphi^{12\tilde{2}\bar{3}}, \varphi^{13\tilde{1}\tilde{2}}, \varphi^{13\tilde{1}\bar{3}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\tilde{2}}, \varphi^{23\tilde{1}\bar{3}}, \varphi^{23\tilde{2}\bar{3}} \right\rangle$	**9**		$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{13\tilde{1}\bar{3}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\bar{3}}, \varphi^{23\tilde{2}\bar{3}} \right\rangle$	**5**		$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\bar{3}} \right\rangle$	**3**	
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**3**		$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}}, \varphi^{123\tilde{1}\bar{3}}, \varphi^{123\tilde{2}\bar{3}} \right\rangle$	**3**	2	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}} \right\rangle$	**1**	2	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}} \right\rangle$	**1**	2
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}}, \varphi^{13\tilde{1}\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**3**		$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**		$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	**1**	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	**1**	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	**1**	**1**

Table 4.5: The double complex $B_{\Gamma}^{\bullet,\bullet}$ for computing the Dolbeault cohomology of the complex solvmanifolds $(M = G/\Gamma, J)$, with underlying Lie algebra $\mathfrak{g}_8$, described in Proposition 4.2.25.

4.3 Cohomological invariants under holomorphic deformations

This section is devoted to show the behaviour of several cohomological properties of complex manifolds under holomorphic deformations. More in detail, we are especially concerned with the degeneracy of the Frölicher sequence as well as the validity of the $\partial\bar{\partial}$-lemma property. We consider in all cases that $M = G/\Gamma$ is a solvmanifold endowed with an invariant complex structure J with holomorphically trivial canonical bundle.

From now on, we mean by an *invariant deformation* a holomorphic deformation given by a family of invariant sections $\{\Psi(t)\}_{t\in\Delta} \subset H^{0,1}_{\bar{\partial}}(\mathfrak{g}^*_{\mathbb{C}}; \mathfrak{g}^{(1,0)})$.

4.3.1 Degeneration of the Frölicher sequence

Although the Kuranishi space provides a tool to understand the local moduli space of complex structures of a complex manifold it is nearly always difficult to compute. Nevertheless, nilmanifolds endowed with an invariant complex structure are a class of compact complex manifolds for which the Kuranishi space may be computed.

On the other hand, Maclaughlin, Pedersen, Poon and Salamon [60, Theorem 4.3] prove that the deformations arising from a nilmanifold $M = G/\Gamma$ with G a 2-step nilpotent Lie group endowed with an invariant abelian complex structure are invariant. This result is generalized by Console, Fino and Poon [21] for nilmanifolds endowed with an invariant abelian complex structure in general. Both [60, 21] also show that the Kuranishi space of an invariant abelian complex nilmanifold is often smooth. More recently, Rollenske proves the following:

Theorem 4.3.1 (Rollenske [79, Theorem]). *Let $M = G/\Gamma$ be a nilmanifold endowed with an invariant complex structure J such that $H^{1,\bullet}_{\bar{\partial}}(\mathfrak{g}) \to H^{1,\bullet}_{\bar{\partial}}(M)$ is an isomorphism. Then all small deformations of the complex structure J are also invariant complex structures. More precisely, the Kuranishi family contains only invariant complex structures.*

As regards parallelizable complex structures, Rollenske [78, Theorem 4.5] shows that the Kuranishi space of a complex parallelizable nilmanifold is cut out by polynomial equations but is frequently singular and reducible.

Now, we make use of the description of the Frölicher sequence on nilmanifolds obtained in Theorem 4.1 in order to find some interesting behaviours under holomorphic deformations. The following result shows that there are many complex nilmanifolds $M = G/\Gamma$ for which the Frölicher spectral sequence is stable under small deformations of the complex structure.

Proposition 4.3.2. *Let $M = G/\Gamma$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J, and let $\mathfrak{g}$ be the Lie algebra of G. If*

$$\mathfrak{g} \cong \mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_6, \mathfrak{h}_8, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{16}, \mathfrak{h}_{19}^- \text{ or } \mathfrak{h}_{26}^+,$$

then $\dim E_r^{p,q}(M, J)$ *is stable under small deformations of J for any p, q and any $r \geq 1$.*

Proof. All the Lie algebras of the list satisfy, by Theorem 4.3.1, that the inclusion $\iota\colon H^{0,1}_{\bar{\partial}}(\mathfrak{g}) \to H^{0,1}_{\bar{\partial}}(M)$ is an isomorphism. Hence, all small deformations of the complex structure J are again invariant complex structures. Proceeding as in the proof of Theorem 4.1.4, it can be proved that if $\mathfrak{g} \not\cong \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or $\mathfrak{h}_{15}$, then $\dim E_r^{p,q}(M)$ does not depend on the invariant complex structure on M for any p, q and any $r \geq 1$, so it is stable under small deformation of J. □

Although the non-closedness of the E_1-degeneration of Frölicher spectral sequences was proven by Eastwood and Singer [27] (see Theorem 1.4.19), we show a counterexample to the closedness of the $E_1(M) \cong E_\infty(M)$ property by means of an invariant holomorphic deformation of a nilmanifold endowed with a complex structure.

Corollary 4.3.3. *Let $(M = G/\Gamma, J_1)$ be a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with its abelian complex structure J_1. Then, there is a holomorphic family of compact complex manifolds $\{M_t := (M, I_t)\}_{t\in\Delta}$, with $I_0 = J_1$ and $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that $E_1(M_t) \cong E_\infty(M_t)$ for each $t \in \Delta^*$, but $E_1(M_0) \not\cong E_\infty(M_0)$.*

Proof. Let us consider the structure equations of the abelian complex structure J_1 on $\mathfrak{h}_4$ as

$$d\eta^1 = d\eta^2 = 0, \quad d\eta^3 = \frac{i}{2}\eta^{1\bar{1}} + \frac{1}{2}\eta^{1\bar{2}} + \frac{1}{2}\eta^{2\bar{1}}.$$

According to Theorem 4.1.4, the complex nilmanifold (M, J_1) satisfies $E_1(M, J_1) \not\cong E_2(M, J_1) \cong E_\infty(M, J_1)$ (see also Table 3.1). The Kuranishi space is studied by Maclaughlin, Pedersen, Poon and Salamon [60, Example8]. Hence if $\{X_1, X_2, X_3\} \subset \mathfrak{h}_{4\mathbb{C}}$ is a $(1,0)$-basis dual to $\{\eta^1, \eta^2, \eta^3\}$, we consider the invariant holomorphic deformation given by the direction $\Psi(t) := tX_1 \otimes (\eta^{\bar{1}} - i\eta^{\bar{2}})$ for each $t \in \mathbb{C}$ such that $|t| < 1$. The deformation I_t is described by a basis $(1,0)$-basis $\{\mu_t^1, \mu_t^2, \mu_t^3\}$ given by

$$\mu_t^1 := \eta^1 + t\eta^{\bar{1}} - it\eta^{\bar{2}}, \quad \mu_t^2 := \eta^2, \quad \mu_t^3 := \eta^3.$$

Notice that this corresponds to $\Phi_1^1 = t$, $\Phi_2^1 = -it$ and $\Phi_1^2 = \Phi_2^2 = \Phi_1^3 = \Phi_2^3 = \Phi_3^3 = 0$ in the parameter space for J_1 obtained in [60, Example 8]. A direct calculation shows that

$$d\mu_t^1 = d\mu_t^2 = 0, \quad d\mu_t^3 = \frac{1}{2(1-|t|^2)}\left(2\bar{t}\mu_t^{12} + i\mu_t^{1\bar{1}} + \mu_t^{1\bar{2}} + \mu_t^{2\bar{1}} - i|t|^2\mu_t^{2\bar{2}}\right), \tag{4.20}$$

hence, the equations define a complex structure I_t on M for each $t \in \Delta$. In particular, I_t correspond to non-abelian complex structures on $\mathfrak{h}_4$ for all $t \in \Delta^*$ and by Theorem 4.1.4 (see Table 3.1) the complex manifolds satisfies $E_1(M_t) \cong E_\infty(M_t)$ for any $t \in \Delta^*$. □

The Lie algebra $\mathfrak{h}_{15}$ has a rich complex geometry with respect to the Frölicher sequence (see Table 3.1). We construct in the following example a differentiable family $\{J_t\}_{t\in\mathbb{R}}$ along which the three cases in (f) of Theorem 4.1.4 are realized.

Example 4.3.4. On $\mathfrak{h}_{15}$, let us consider the following family of complex structures

$$\begin{aligned} J_t e^1 &= -\sqrt{\frac{3(3-\sin t)(7+3\sin t)}{(5+\sin t)(11-\sin t)}}\, e^2, \\ J_t e^3 &= \sqrt{\frac{3(3-\sin t)(11-\sin t)}{(5+\sin t)(7+3\sin t)}}\, e^4, \\ J_t e^5 &= -\sqrt{\frac{(11-\sin t)(7+3\sin t)}{3(3-\sin t)(5+\sin t)}}\, e^6, \end{aligned}$$

where $t \in \mathbb{R}$. Let

$$\begin{aligned} 4\,\omega^1 &= \sqrt{(11-\sin t)(5+\sin t)}\, e^1 + i\,\sqrt{3(3-\sin t)(7+3\sin t)}\, e^2, \\ 8\,\omega^2 &= (5+\sin t)(7+3\sin t)\, e^3 - i\,\sqrt{3(5+\sin t)(3-\sin t)(11-\sin t)(7+3\sin t)}\, e^4, \end{aligned}$$

and

$$\begin{aligned} 128\,\omega^3 = (5+\sin t)(7+3\sin t)\Big(&3(3-\sin t)\sqrt{(11-\sin t)(5+\sin t)}\, e^5 \\ &+ i\,(11-\sin t)\sqrt{3(3-\sin t)(7+3\sin t)}\, e^6\Big)\,. \end{aligned}$$

Then, $\{\omega^1, \omega^2, \omega^3\}$ is a $(1,0)$-basis for J_t satisfying

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \frac{1-\sin t}{2}\,\omega^{12} + 2\,\omega^{1\bar{2}} + \frac{1+\sin t}{4}\,\omega^{2\bar{1}}.$$

It is clear that the complex structure J_t is abelian if and only if $t = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$. For any $t \neq \frac{4k+1}{2}\pi$ the complex structure equations can be written as

$$d\omega^1 = 0, \quad d\omega^2 = \omega^{1\bar{1}}, \quad d\omega^3 = \omega^{12} + \frac{4}{1-\sin t}\,\omega^{1\bar{2}} + \frac{1+\sin t}{2(1-\sin t)}\,\omega^{2\bar{1}}.$$

Thus, concerning the Frölicher spectral sequence for the family $\{J_t\}_{t\in\mathbb{R}}$, by Theorem 4.1.4 (f) we get:

- $E_1 \ncong E_2 \ncong E_3 \cong E_\infty$, for $t = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$;
- $E_1 \ncong E_2 \cong E_\infty$, for $t = \frac{4k-1}{2}\pi$, $k \in \mathbb{Z}$;
- $E_1 \cong E_2 \ncong E_3 \cong E_\infty$, for any other value of t.

As a consequence of this example, in the following result we show that for $r \geq 2$ the dimension of the term $E_r^{\bullet,\bullet}(J_t)$ in general is neither upper nor lower semi-continuous function of t.

Corollary 4.3.5. *Let M be a nilmanifold with underlying Lie algebra $\mathfrak{h}_{15}$ endowed with the invariant complex structures J_t given in Example 4.3.4. Then,*

$$\dim E_2^{0,2}(J_{\frac{\pi}{2}}) = 3 > 2 = \dim E_2^{0,2}(J_t), \quad \dim E_2^{1,1}(J_{\frac{\pi}{2}}) = 2 < 3 = \dim E_2^{1,1}(J_t),$$

and

$$\dim E_3^{0,2}(J_{\frac{\pi}{2}}) = 2 > 1 = \dim E_3^{0,2}(J_t), \quad \dim E_3^{1,1}(J_{\frac{\pi}{2}}) = 2 < 3 = \dim E_3^{1,1}(J_t),$$

for any $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Therefore, the dimensions of the terms $E_2^{1,1}(J_t)$ and $E_3^{1,1}(J_t)$ are not upper semi-continuous functions of t, and the dimensions of the terms $E_2^{0,2}(J_t)$ and $E_3^{0,2}(J_t)$ are not lower semi-continuous functions of t.

Proof. It follows directly from the proof of Theorem 4.1.4 taking into account that for $t = \frac{\pi}{2}$ the complex structure lies in case (f.3) and for any $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$ the structures J_t lie in case (f.2). □

This behaviour of the dimensions of the Frölicher terms $E_r^{\bullet,\bullet}(M)$ for $r \geq 2$ is in deep contrast with the case $r = 1$, as Kodaira and Spencer [53] show the upper semicontinuity of the Hodge numbers, namely $\dim H_{\bar{\partial}}^{\bullet,\bullet}(J_t)$, with respect to t along a deformation. Finally, we state the following:

Corollary 4.3.6. *The property of* the Frölicher spectral sequence degenerating at E_2 *is not deformation open.*

Proof. The family J_t given in Example 4.3.4 satisfies $E_2(J_{-\frac{\pi}{2}}) \cong E_\infty(J_{-\frac{\pi}{2}})$, because $J_{-\frac{\pi}{2}}$ is in case (f.1) of Theorem 4.1.4, but $E_2(J_t) \ncong E_\infty(J_t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. □

4.3.2 The $\partial\bar{\partial}$-lemma

As we mentioned in Chapter 1, Angella and Kasuya [8] use a deformation of the Nakamura manifold to show the non-closedness of the $\partial\bar{\partial}$-lemma property (see Theorem 1.4.21). It turns out that the underlying complex structure on the Nakamura manifold belongs to the family $\{J^A\}_{A\in\mathbb{C}\setminus\mathbb{R}}$ of complex structures on $\mathfrak{g}_8$ described by (3.33) taking the value of the parameter $A = -i$ (see Remark 3.2.9). On the other hand, Proposition 4.2.25 provides a countable family of complex structures of splitting type on $\mathfrak{g}_8$ together with lattices compatible with the splitting such that the corresponding complex solvmanifolds do not satisfy the $\partial\bar{\partial}$-lemma property. The aim of this section is to provide a suitable holomorphic deformation for any of these complex solvmanifolds satisfying the $\partial\bar{\partial}$-lemma for all the parameters of the deformations.

To achieve this end, we firstly present the results obtained by Angella and Kasuya [8] to compute the Dolbeault and the Bott-Chern cohomologies of the deformed complex solvmanifolds of a complex solvmanifold of splitting type. We consider in this paragraph a solvmanifold $M = G/\Gamma$ endowed with a complex structure J of splitting type and $(B_\Gamma^{\bullet,\bullet}, \bar{\partial})$ and $(C_\Gamma^{\bullet,\bullet}, \partial, \bar{\partial})$ the finite-dimensional differential sub-complexes defined by the expressions (4.6) and (4.7). Let $\{J_t\}_{t\in\Delta}$ be a holomorphic deformation of J and denote by ∂_t and $\bar\partial_t$ the complex differential operators induced by the complex structure J_t.

Angella and Kasuya provide conditions in the following theorems in order that suitable deformations $(B_t^{\bullet,\bullet},\bar\partial_t)$ and $(C_t^{\bullet,\bullet},\partial_t,\bar\partial_t)$ for $t\in\Delta$ of the complexes $(B_\Gamma^{\bullet,\bullet},\bar\partial)$ and $(C_\Gamma^{\bullet,\bullet},\partial,\bar\partial)$ still allow to compute the Dolbeault and the Bott-Chern cohomologies of the deformed complex solvmanifolds (M,J_t).

Theorem 4.3.7 (Angella and Kasuya [8, Theorem 1.1])**.** *Let (X,J) be a compact complex manifold, and consider deformations $\{J_t\}_{t\in\Delta}$ such that $J_0=J$. We suppose that we have a family $\left\{C_t^{\bullet,\bullet}=\langle\phi_j^{\bullet,\bullet}(t)\rangle_j\right\}_{t\in\Delta}$ of sub-vector spaces of $(\wedge_{J_t}^{\bullet,\bullet}X,\partial_t,\bar\partial_t)$ parametrized by $t\in\Delta$ such that:*

(1) for each $t\in\Delta$, it holds that $(C_t^{\bullet,\bullet},\bar\partial_t)$ is a sub-complex of $(\wedge_{J_t}^{\bullet,\bullet}X,\bar\partial_t)$;

(2) $\phi_j^{\bullet,\bullet}(t)$ is smooth on $X\times\Delta$, for any j;

(3) the inclusion $C_0^{\bullet,\bullet}\subset\wedge_J^{\bullet,\bullet}X$ induces the cohomology isomorphism

$$H_{\bar\partial_0}^{\bullet,\bullet}(C_0^{\bullet,\bullet})\cong H_{\bar\partial}^{\bullet,\bullet}(X)\ ;$$

(4) there exits a smooth family $\{g_t\}_{t\in\Delta}$ of J_t-Hermitian metrics such that $\bar*_{g_t}(C_t^{\bullet,\bullet})\subseteq C_t^{n-\bullet,n-\bullet}$, where we denote by $\bar*_{g_t}$ the anti-$\mathbb{C}$-linear Hodge-$*$-operator of g_t, and by $2n$ the real dimension of X.

Then, for sufficiently small t, the inclusion $C_t^{\bullet,\bullet}\subset\wedge_{J_t}^{\bullet,\bullet}(X)$ induces the cohomology isomorphism

$$H_{\bar\partial_t}^{\bullet,\bullet}(C_t^{\bullet,\bullet})\cong H_{\bar\partial_t}^{\bullet,\bullet}(X)\ .$$

Theorem 4.3.8 (Angella and Kasuya [8, Theorem 1.2])**.** *Let (X,J) be a compact complex manifold, and consider deformations $\{J_t\}_{t\in\Delta}$ such that $J_0=J$. We suppose that we have a family $\left\{C_t^{\bullet,\bullet}=\langle\phi_j^{\bullet,\bullet}(t)\rangle_j\right\}_{t\in\Delta}$ of sub-vector spaces of $(\wedge_{J_t}^{\bullet,\bullet}X,\partial_t,\bar\partial_t)$ parametrized by $t\in\Delta$ such that:*

(1) for each $t\in\Delta$, it holds that $(C_t^{\bullet,\bullet},\partial_t,\bar\partial_t)$ is a sub-double-complex of $(\wedge_{J_t}^{\bullet,\bullet}X,\partial_t,\bar\partial_t)$;

(2) $\phi_j^{\bullet,\bullet}(t)$ is smooth on $X\times\Delta$, for any j;

(3) the inclusion $C_0^{\bullet,\bullet}\subset\wedge_J^{\bullet,\bullet}X$ induces the Bott-Chern cohomology isomorphism

$$H_{BC}^{\bullet,\bullet}(C_0^{\bullet,\bullet})\cong H_{BC}^{\bullet,\bullet}(X)\ ;$$

(4) there exits a smooth family $\{g_t\}_{t\in\Delta}$ of J_t-Hermitian metrics such that $\bar*_{g_t}(C_t^{\bullet,\bullet})\subseteq C_t^{n-\bullet,n-\bullet}$, where we denote by $\bar*_{g_t}$ the anti-$\mathbb{C}$-linear Hodge-$*$-operator of g_t, and by $2n$ the real dimension of X.

Then, for sufficiently small t, the inclusion $C_t^{\bullet,\bullet} \subset \wedge_{J_t}^{\bullet,\bullet} X$ induces the Bott-Chern cohomology isomorphism

$$H_{BC}^{\bullet,\bullet}(C_t^{\bullet,\bullet}) \cong H_{BC}^{\bullet,\bullet}(X)\ .$$

Now, we consider the connected and simply-connected Lie group G with underlying real Lie algebra $\mathfrak{g}_8$ and the family $\{J_k\}_{0\neq k\in\mathbb{Z}}$ of non-isomorphic left-invariant complex structures where $J_k := J^{A_k}$ satisfies the equations (3.33) for the values $A_k = \frac{i}{2k+1}$ with $k \in \mathbb{Z}$.

Looking at Table 4.4, we associate to the Lie group G endowed with a fixed complex structure J_k the lattice $\Gamma_k = \Gamma'_k \ltimes_{\varphi_{A_k}} \Gamma''$ on G compatible with the splitting where Γ'_k is the lattice of $\mathbb{C}$ given by

$$\Gamma'_k = \frac{k\pi}{2}\mathbb{Z} \oplus i\log\big(\frac{n+\sqrt{n^2-4}}{2}\big)\mathbb{Z} \tag{4.21}$$

for some natural number $n \geq 3$. Hence we obtain a countable family of compact complex manifolds $\{(M_k := G/\Gamma_k, J_k)\}_{k\in\mathbb{Z}}$. We show in the next proposition that in this family is possible to find a countable subfamily of examples of the non-closedness of the $\partial\bar\partial$-lemma property under small deformations.

Proposition 4.3.9. *Every compact complex solvmanifold of the family $\{(M_k, J_k)\}_{k\in\mathbb{Z}}$ with underlying real Lie algebra $\mathfrak{g}_8$ does not satisfy the $\partial\bar\partial$-lemma, but admits a small holomorphic deformation satisfying the $\partial\bar\partial$-lemma.*

Proof. Notice that by Proposition 4.2.25, none of these complex solvmanifolds satisfy the $\partial\bar\partial$-lemma.

Now, for any fixed $A_k = \frac{i}{2k+1}$, we consider an open disc $\Delta(0,\varepsilon_k) \subset \mathbb{C}$ for $\varepsilon_k > 0$ small enough, and the family $\{J_{k,t}\}_{t\in\Delta(0,\varepsilon_k)}$ of holomorphic deformations of J_k in the invariant direction

$$\Psi_k(t) := t\, X_3^k \otimes \bar\omega_k^3 \in (\mathfrak{g}_8)_{J_k}^{1,0} \otimes \wedge_{J_k}^{0,1}\mathfrak{g}_8^*, \tag{4.22}$$

where $\{\omega_k^1, \omega_k^2, \omega_k^3\}$ constitute a $(1,0)$-basis of J_k satisfying the complex structure equations (3.33) and $\{X_1^k, X_2^k, X_3^k\} \subset (\mathfrak{g}_8)_{J_k}^{1,0}$ denotes the dual basis. The holomorphic deformation (4.22) corresponds to consider the following basis of invariant $(1,0)$-forms with respect to the complex structure $J_{k,t}$:

$$\omega_{k,t}^1 := \omega_k^1\ ,\quad \omega_{k,t}^2 := \omega_k^2\ ,\quad \omega_{k,t}^3 := \omega_k^3 - t\,\bar\omega_k^3$$

With respect to such co-frame, the structure equations are

$$\begin{cases} d\omega_{k,t}^1 &= -\frac{(A_k-i)+(A_k+i)\,\bar t}{1-|t|^2}\,\omega_{k,t}^1\wedge\omega_{k,t}^3 - \frac{(A_k+i)+(A_k-i)\,t}{1-|t|^2}\,\omega_{k,t}^1\wedge\bar\omega_{k,t}^3,\\ d\omega_{k,t}^2 &= \frac{(A_k-i)+(A_k+i)\,\bar t}{1-|t|^2}\,\omega_{k,t}^2\wedge\omega_{k,t}^3 + \frac{(A_k+i)+(A_k-i)\,t}{1-|t|^2}\,\omega_{k,t}^2\wedge\bar\omega_{k,t}^3,\\ d\omega_{k,t}^3 &= 0. \end{cases}$$

Hence, we get that the complex solvmanifolds $(M_k, J_{k,t})$ are of splitting type for any $t \in \Delta(0, \epsilon_k)$, where the action $\varphi_{k,t} \colon \mathbb{C} \to \mathrm{GL}(\mathbb{C}^2)$ is described by a diagonal matrix of the form (4.8) and the characters $\alpha^1_{k,t}, \alpha^2_{k,t} \colon \mathbb{C} \to \mathbb{C}^*$ required to construct the complex $(B^{\bullet,\bullet}_t, \bar\partial_t)$ are:

$$\alpha^1_{k,t}(z_3) := e^{-\frac{(A_k - i)+(A_k+i)\,\bar t}{1-|t|^2} z_3 - \frac{(A_k+i)+(A_k-i)\,t}{1-|t|^2}\bar z_3}, \quad \alpha^2_{k,t}(z_3) = \alpha^1_{k,t}(z_3)^{-1}.$$

Since the unitary characters $\beta^1_{k,t}, \beta^2_{k,t}, \gamma^1_{k,t}, \gamma^2_{k,t} \colon \mathbb{C} \to \mathbb{C}^*$ satisfying that $\alpha^1_{k,t}(\beta^1_{k,t})^{-1}$, $\alpha^2_{k,t}(\beta^2_{k,t})^{-1}$, $\bar\alpha^1_{k,t}(\gamma^1_{k,t})^{-1}$, $\bar\alpha^2_{k,t}(\gamma^2_{k,t})^{-1}$ are holomorphic are unique, we can define the generators of the complex $B^{\bullet,\bullet}_t = \wedge^{\bullet,\bullet}\langle \varphi^1_{k,t}, \varphi^2_{k,t}, \varphi^3_{k,t}, \tilde\varphi^1_{k,t}, \tilde\varphi^2_{k,t}, \tilde\varphi^3_{k,t}\rangle$ of Theorem 4.3.7 by:

$$\begin{cases} \varphi^1_{k,t} \;:=\; \beta^1_{k,t}\,\omega^1_{k,t} \;=\; e^{-2iz_3}\,dz^1, \\[1ex] \varphi^2_{k,t} \;:=\; \beta^2_k\,\omega^2_{k,t} \;=\; e^{2iz_3}\,dz^2, \\[1ex] \varphi^3_{k,t} \;:=\; \omega^3_{k,t} \;=\; dz^3 - t\,d\bar z^3, \end{cases} \qquad \begin{cases} \tilde\varphi^1_{k,t} \;:=\; \gamma^1_k\,\bar\omega^1_{k,t} \;=\; e^{-2iz_3}\,d\bar z^1, \\[1ex] \tilde\varphi^2_{k,t} \;:=\; \gamma^2_k\,\bar\omega^2_{k,t} \;=\; e^{2iz_3}\,d\bar z^2, \\[1ex] \tilde\varphi^3_{k,t} \;:=\; \bar\varphi^3_{k,t} \;=\; d\bar z^3 - \bar t\,dz^3, \end{cases}$$

where $\varphi^1_{k,t}, \varphi^2_{k,t}, \varphi^3_{k,t}$ have bidegree $(1,0)$ and $\tilde\varphi^1_{k,t}, \tilde\varphi^2_{k,t}, \tilde\varphi^3_{k,t}$ have bidegree $(0,1)$ for the complex structure $J_{k,t}$. Consider the bi-differential bi-graded complex $C^{\bullet,\bullet}_t$ of Theorem 4.3.8 defined by:

$$C^{\bullet,\bullet}_t := B^{\bullet,\bullet}_t + \overline{B^{\bullet,\bullet}_t},$$

where we take into account the following identities: $\bar\varphi^3_{k,t} = \tilde\varphi^3_{k,t}$, $\tilde\varphi^1_{k,t} \wedge \tilde\varphi^2_{k,t} = \bar\varphi^1_{k,t} \wedge \bar\varphi^2_{k,t}$, $\varphi^1_{k,t} \wedge \bar{\tilde\varphi}^1_{k,t} = 0$, $\varphi^2_{k,t} \wedge \bar{\tilde\varphi}^2_{k,t} = 0$, $\varphi^1_{k,t} \wedge \tilde\varphi^2_{k,t} = \bar{\tilde\varphi}^1_{k,t} \wedge \bar\varphi^2_{k,t}$, $\varphi^2_{k,t} \wedge \tilde\varphi^1_{k,t} = \bar{\tilde\varphi}^2_{k,t} \wedge \bar\varphi^1_{k,t}$, $\varphi^1_{k,t} \wedge \bar\varphi^1_{k,t} = \bar{\tilde\varphi}^1_{k,t} \wedge \tilde\varphi^1_{k,t}$, $\varphi^2_{k,t} \wedge \bar\varphi^2_{k,t} = \bar{\tilde\varphi}^2_{k,t} \wedge \tilde\varphi^2_{k,t}$, as explicitly is described in Table 4.7. The complex structure equations of $J_{k,t}$ expressed in this basis are:

$$\begin{cases} d\varphi^1_{k,t} \;=\; \frac{2\,i}{1-|t|^2}\,\varphi^1_{k,t} \wedge \varphi^3_{k,t} - \frac{2\,t\,i}{1-|t|^2}\,\varphi^1_{k,t} \wedge \bar\varphi^3_{k,t}, \\ d\varphi^2_{k,t} \;=\; -\frac{2\,i}{1-|t|^2}\,\varphi^2_{k,t} \wedge \varphi^3_{k,t} + \frac{2\,t\,i}{1-|t|^2}\,\varphi^2_{k,t} \wedge \bar\varphi^3_{k,t}, \\ d\varphi^3_{k,t} \;=\; 0, \\ d\bar{\tilde\varphi}^1_{k,t} \;=\; -\frac{2\,i}{1-|t|^2}\,\bar{\tilde\varphi}^1_{k,t} \wedge \bar\varphi^3_{k,t} + \frac{2\,\bar t\,i}{1-|t|^2}\,\bar{\tilde\varphi}^1_{k,t} \wedge \varphi^3_{k,t}, \\ d\bar{\tilde\varphi}^2_{A_k,t} \;=\; \frac{2\,i}{1-|t|^2}\,\bar{\tilde\varphi}^2_{k,t} \wedge \bar\varphi^3_{k,t} - \frac{2\,\bar t\,i}{1-|t|^2}\,\bar{\tilde\varphi}^2_{k,t} \wedge \varphi^3_{k,t}, \end{cases} \quad \begin{cases} d\tilde\varphi^1_{k,t} \;=\; -\frac{2\,i}{1-|t|^2}\,\varphi^3_{k,t} \wedge \tilde\varphi^1_{k,t} - \frac{2\,t\,i}{1-|t|^2}\,\tilde\varphi^1_{k,t} \wedge \bar\varphi^3_{k,t}, \\ d\tilde\varphi^2_{k,t} \;=\; \frac{2\,i}{1-|t|^2}\,\varphi^3_{k,t} \wedge \tilde\varphi^2_{k,t} + \frac{2\,t\,i}{1-|t|^2}\,\tilde\varphi^2_{k,t} \wedge \bar\varphi^3_{k,t}, \\ d\tilde\varphi^3_{k,t} \;=\; 0, \\ d\bar\varphi^1_{k,t} \;=\; -\frac{2\,i}{1-|t|^2}\,\bar\varphi^1_{k,t} \wedge \bar\varphi^3_{k,t} - \frac{2\,\bar t\,i}{1-|t|^2}\,\varphi^3_{k,t} \wedge \bar\varphi^1_{k,t}, \\ d\bar\varphi^2_{k,t} \;=\; \frac{2\,i}{1-|t|^2}\,\bar\varphi^2_{k,t} \wedge \bar\varphi^3_{k,t} + \frac{2\,\bar t\,i}{1-|t|^2}\,\varphi^3_{k,t} \wedge \bar\varphi^2_{k,t}. \end{cases}.$$

Finally, taking the Hermitian metric on $(M_k, J_{A_k,t})$ defined by:

$$g_{k,t} := \varphi^1_{k,t} \odot \tilde\varphi^1_{k,t} + \varphi^2_{k,t} \odot \tilde\varphi^2_{k,t} + \varphi^3_{k,t} \odot \bar\varphi^3_{k,t}.$$

By Theorems 4.3.7, 4.3.8, the complexes $(B^{\bullet,\bullet}_t, \bar\partial_t)$ and $(C^{\bullet,\bullet}_t, \partial_t, \bar\partial_t)$ allow to compute the Dolbeault cohomology and the Bott-Chern cohomology of the compact complex manifolds $(M_k, J_{k,t})$ for any $t \in \Delta(0, \epsilon_k)$. The computations of the spaces $B^{\bullet,\bullet}_t$ and $C^{\bullet,\bullet}_t$ can be found in the Tables 4.6 and 4.7, whereas the cohomology groups $H^{\bullet,\bullet}_{\bar\partial}(B_t)$ and $H^{\bullet,\bullet}_{BC}(C_t)$ are tabulated in the Tables 4.8 and 4.9. Finally the dimensions of the cohomology groups are summarized in Table 4.10. Observing Table 4.10 it is direct to check that for $t \neq 0$ the inequality of Angella and Tomassini (1.7) vanishes for any $k = 0, \ldots, 6$ and hence the compact complex manifolds $(M_k, J_{k,t})$ satisfy the $\partial\bar{\partial}$-lemma for any $t \in \Delta^*(0, \varepsilon_k)$. □

Remark 4.3.10. The parallelizable Nakamura manifold together with its small deformation given by Angella and Kasuya in [8] corresponds to the family of holomorphic deformations $\{(M_{-1}, J_{-i,t})\}_{t\in\Delta(0,\epsilon_{-1})}$.

On the other hand, Proposition 4.2.25 states that the compact complex manifolds of the family $\{(M := G/\Gamma_k, J_k)\}_{0\neq k\in\mathbb{Z}}$ for $A = \frac{i}{k}$ do not satisfy the $\partial\bar{\partial}$-lemma, hence it is reasonable that the result obtained in Proposition 4.3.9 can be extended to this family.

	$B_t^{\bullet,\bullet}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{1},\ \varphi_t^{2},\ \varphi_t^{3}\right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{\bar{1}},\ \varphi_t^{\bar{2}},\ \varphi_t^{\bar{3}}\right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{12},\ \varphi_t^{13},\ \varphi_t^{23}\right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{1\bar{1}},\ \varphi_t^{1\bar{2}},\ \varphi_t^{1\bar{3}},\ \varphi_t^{2\bar{1}},\ \varphi_t^{2\bar{2}},\ \varphi_t^{2\bar{3}},\ \varphi_t^{3\bar{1}},\ \varphi_t^{3\bar{2}},\ \varphi_t^{3\bar{3}}\right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{\bar{1}\bar{2}},\ \varphi_t^{\bar{1}\bar{3}},\ \varphi_t^{\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{123}\right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{12\bar{1}},\ \varphi_t^{12\bar{2}},\ \varphi_t^{12\bar{3}},\ \varphi_t^{13\bar{1}},\ \varphi_t^{13\bar{2}},\ \varphi_t^{13\bar{3}},\ \varphi_t^{23\bar{1}},\ \varphi_t^{23\bar{2}},\ \varphi_t^{23\bar{3}}\right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{1\bar{1}\bar{2}},\ \varphi_t^{1\bar{1}\bar{3}},\ \varphi_t^{1\bar{2}\bar{3}},\ \varphi_t^{2\bar{1}\bar{2}},\ \varphi_t^{2\bar{1}\bar{3}},\ \varphi_t^{2\bar{2}\bar{3}},\ \varphi_t^{3\bar{1}\bar{2}},\ \varphi_t^{3\bar{1}\bar{3}},\ \varphi_t^{3\bar{2}\bar{3}}\right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{\bar{1}\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{123\bar{1}},\ \varphi_t^{123\bar{2}},\ \varphi_t^{123\bar{3}}\right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{12\bar{1}\bar{2}},\ \varphi_t^{12\bar{1}\bar{3}},\ \varphi_t^{12\bar{2}\bar{3}},\ \varphi_t^{13\bar{1}\bar{2}},\ \varphi_t^{13\bar{1}\bar{3}},\ \varphi_t^{13\bar{2}\bar{3}},\ \varphi_t^{23\bar{1}\bar{2}},\ \varphi_t^{23\bar{1}\bar{3}},\ \varphi_t^{23\bar{2}\bar{3}}\right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{1\bar{1}\bar{2}\bar{3}},\ \varphi_t^{2\bar{1}\bar{2}\bar{3}},\ \varphi_t^{3\bar{1}\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{123\bar{1}\bar{2}},\ \varphi_t^{123\bar{1}\bar{3}},\ \varphi_t^{123\bar{2}\bar{3}}\right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{12\bar{1}\bar{2}\bar{3}},\ \varphi_t^{13\bar{1}\bar{2}\bar{3}},\ \varphi_t^{23\bar{1}\bar{2}\bar{3}}\right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{123\bar{1}\bar{2}\bar{3}}\right\rangle$

Table 4.6: The space $B_t^{\bullet,\bullet}$ for computing the Dolbeault cohomology of the complex solvmanifolds $(M_k := G/\Gamma_k, J_{k,t})$ where $\{J_{k,t}\}_{t\in\Delta(0,\varepsilon_k)\subset\mathbb{C}}$ is the holomorphic deformation defined by the direction $\Psi_k(t) := tX_3^k \otimes \bar{\omega}_k^3 \in (\mathfrak{g}_8)^{1,0}_{J_k} \otimes \wedge^{0,1}_{J_k}\mathfrak{g}_8^*$ of the complex structure $J_k := J^{A_k}$ satisfying the equations (3.33) for the value $A_k = \frac{i}{2k+1}$ for some $k \in \mathbb{Z}$. The lattice Γ_k of G is the semi-direct product of a lattice Γ'_A in $\mathbb{C}$ given by (4.21) and a lattice Γ'' in $\mathbb{C}^2$.

	$C_t^{\bullet,\bullet}$
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{1},\ \varphi_t^{2},\ \varphi_t^{3},\ \varphi_t^{\bar{\tilde{1}}},\ \varphi_t^{\bar{\tilde{2}}} \right\rangle$
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}},\ \varphi_t^{\tilde{2}},\ \varphi_t^{\bar{3}},\ \varphi_t^{\bar{1}},\ \varphi_t^{\bar{2}} \right\rangle$
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{12},\ \varphi_t^{13},\ \varphi_t^{23},\ \varphi_t^{\bar{\tilde{1}}3},\ \varphi_t^{\bar{\tilde{2}}3} \right\rangle$
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}},\ \varphi_t^{1\tilde{2}},\ \varphi_t^{1\bar{3}},\ \varphi_t^{2\tilde{1}},\ \varphi_t^{2\tilde{2}},\ \varphi_t^{2\bar{3}},\ \varphi_t^{3\tilde{1}},\ \varphi_t^{3\tilde{2}},\ \varphi_t^{3\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}},\ \varphi_t^{3\bar{1}},\ \varphi_t^{\bar{\tilde{2}}\bar{2}},\ \varphi_t^{3\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{3}} \right\rangle$
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}},\ \varphi_t^{\tilde{1}\bar{3}},\ \varphi_t^{\tilde{2}\bar{3}},\ \varphi_t^{\bar{1}\bar{3}},\ \varphi_t^{\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi_t^{123} \right\rangle$
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}},\ \varphi_t^{12\tilde{2}},\ \varphi_t^{12\bar{3}},\ \varphi_t^{13\tilde{1}},\ \varphi_t^{13\tilde{2}},\ \varphi_t^{13\bar{3}},\ \varphi_t^{23\tilde{1}},\ \varphi_t^{23\tilde{2}},\ \varphi_t^{23\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{1}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}3\bar{2}},\ \varphi_t^{\bar{\tilde{1}}3\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{3}} \right\rangle$
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}},\ \varphi_t^{1\tilde{1}\bar{3}},\ \varphi_t^{1\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}},\ \varphi_t^{2\tilde{1}\bar{3}},\ \varphi_t^{2\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}},\ \varphi_t^{3\tilde{1}\bar{3}},\ \varphi_t^{3\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}\bar{3}},\ \varphi_t^{3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{2}\bar{3}},\ \varphi_t^{3\bar{2}\bar{3}} \right\rangle$
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}},\ \varphi_t^{123\tilde{2}},\ \varphi_t^{123\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{1}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{2}} \right\rangle$
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}},\ \varphi_t^{12\tilde{1}\bar{3}},\ \varphi_t^{12\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}},\ \varphi_t^{13\tilde{1}\bar{3}},\ \varphi_t^{13\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}},\ \varphi_t^{23\tilde{1}\bar{3}},\ \varphi_t^{23\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{2}}3\bar{1}\bar{2}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{2}\bar{3}} \right\rangle$
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{1\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{2\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{3\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{1}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}},\ \varphi_t^{123\tilde{1}\bar{3}},\ \varphi_t^{123\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{1}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}\bar{\tilde{2}}3\bar{2}\bar{3}} \right\rangle$
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{12\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{13\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{23\tilde{1}\tilde{2}\bar{3}},\ \varphi_t^{\bar{\tilde{1}}3\bar{1}\bar{2}\bar{3}},\ \varphi_t^{\bar{\tilde{2}}3\bar{1}\bar{2}\bar{3}} \right\rangle$
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi_t^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$

Table 4.7: The space $C_t^{\bullet,\bullet}$ for computing the Bott-Chern cohomology of the complex solvmanifolds $(M_k := G/\Gamma_k, J_{k,t})$ where $\{J_{k,t}\}_{t\in\Delta(0,\varepsilon_k)\subset\mathbb{C}}$ is the holomorphic deformation defined by the direction $\Psi_k(t) := tX_3^k \otimes \bar{\omega}_k^3 \in (\mathfrak{g}_8)^{1,0}_{J_k} \otimes \wedge^{0,1}_{J_k}\mathfrak{g}_8^*$ of the complex structure $J_k := J^{A_k}$ satisfying the equations (3.33) for the value $A_k = \frac{i}{2k+1}$ for some $k \in \mathbb{Z}$. The lattice Γ_k of G is the semi-direct product of a lattice Γ'_A in $\mathbb{C}$ given by (4.21) and a lattice Γ'' in $\mathbb{C}^2$.

$H^{\bullet,\bullet}_{\bar\partial}(B_t)$	$t = 0$	dim	$t \neq 0$	dim
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1 \right\rangle$	1	$\mathbb{C}\left\langle 1 \right\rangle$	1
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{1}, \varphi^{2}, \varphi^{3} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{3} \right\rangle$	1
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}}, \varphi^{\tilde{2}}, \varphi^{\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{\bar{3}} \right\rangle$	1
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{12}, \varphi^{13}, \varphi^{23} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{12} \right\rangle$	1
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}}, \varphi^{1\tilde{2}}, \varphi^{1\bar{3}}, \varphi^{2\tilde{1}}, \varphi^{2\tilde{2}}, \varphi^{2\bar{3}}, \varphi^{3\tilde{1}}, \varphi^{3\tilde{2}}, \varphi^{3\bar{3}} \right\rangle$	9	$\mathbb{C}\left\langle \varphi^{1\tilde{2}}, \varphi^{2\tilde{1}}, \varphi^{3\bar{3}} \right\rangle$	3
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}}, \varphi^{\tilde{1}\bar{3}}, \varphi^{\tilde{2}\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}} \right\rangle$	1
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	1	$\mathbb{C}\left\langle \varphi^{123} \right\rangle$	1
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}}, \varphi^{12\tilde{2}}, \varphi^{12\bar{3}}, \varphi^{13\tilde{1}}, \varphi^{13\tilde{2}}, \varphi^{13\bar{3}}, \varphi^{23\tilde{1}}, \varphi^{23\tilde{2}}, \varphi^{23\bar{3}} \right\rangle$	9	$\mathbb{C}\left\langle \varphi^{12\bar{3}}, \varphi^{13\tilde{2}}, \varphi^{23\tilde{1}} \right\rangle$	3
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\tilde{2}}, \varphi^{1\tilde{1}\bar{3}}, \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\tilde{2}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{2\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}}, \varphi^{3\tilde{1}\bar{3}}, \varphi^{3\tilde{2}\bar{3}} \right\rangle$	9	$\mathbb{C}\left\langle \varphi^{1\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}} \right\rangle$	3
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}}, \varphi^{123\tilde{2}}, \varphi^{123\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{123\bar{3}} \right\rangle$	1
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{12\tilde{1}\bar{3}}, \varphi^{12\tilde{2}\bar{3}}, \varphi^{13\tilde{1}\tilde{2}}, \varphi^{13\tilde{1}\bar{3}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\tilde{2}}, \varphi^{23\tilde{1}\bar{3}}, \varphi^{23\tilde{2}\bar{3}} \right\rangle$	9	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}}, \varphi^{13\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\bar{3}} \right\rangle$	3
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\tilde{2}\bar{3}}, \varphi^{2\tilde{1}\tilde{2}\bar{3}}, \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}}, \varphi^{123\tilde{1}\bar{3}}, \varphi^{123\tilde{2}\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}} \right\rangle$	1
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}}, \varphi^{13\tilde{1}\tilde{2}\bar{3}}, \varphi^{23\tilde{1}\tilde{2}\bar{3}} \right\rangle$	3	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}} \right\rangle$	1

Table 4.8: The harmonic representatives of the Dolbeault cohomology of the complex solvmanifolds $(M_k := G/\Gamma_k, J_{k,t})$ where $\{J_{k,t}\}_{t\in\Delta(0,\varepsilon_k)\subset\mathbb{C}}$ is the holomorphic deformation defined by the direction $\Psi_k(t) := tX_3^k \otimes \bar\omega_k^3 \in (\mathfrak{g}_8)^{1,0}_{J_k} \otimes \wedge^{0,1}_{J_k}\mathfrak{g}_8^*$ of the complex structure $J_k := J^{A_k}$ satisfying the equations (3.33) for the value $A_k = \frac{i}{2k+1}$ for some $k \in \mathbb{Z}$. The lattice Γ_k of G is the semi-direct product of a lattice Γ'_A in $\mathbb{C}$ given by (4.21) and a lattice Γ'' in $\mathbb{C}^2$.

$H^{\bullet,\bullet}_{\mathrm{BC}}(C_t)$	$t=0$	dim	$t\neq 0$	dim
$(\mathbf{0},\mathbf{0})$	$\mathbb{C}\left\langle 1\right\rangle$	1	$\mathbb{C}\left\langle 1\right\rangle$	1
$(\mathbf{1},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{3}\right\rangle$	1	$\mathbb{C}\left\langle \varphi^{3}\right\rangle$	1
$(\mathbf{0},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{\bar{3}}\right\rangle$	1	$\mathbb{C}\left\langle \varphi^{\bar{3}}\right\rangle$	1
$(\mathbf{2},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{12},\, \varphi^{13},\, \varphi^{23}\right\rangle$	3	$\mathbb{C}\left\langle \varphi^{12}\right\rangle$	1
$(\mathbf{1},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{1\tilde{2}},\, \varphi^{2\tilde{1}},\, \varphi^{3\tilde{1}},\, \varphi^{3\tilde{2}},\, \varphi^{3\bar{3}},\, \varphi^{\bar{\tilde{1}}\bar{3}},\, \varphi^{\bar{\tilde{2}}\bar{3}}\right\rangle$	7	$\mathbb{C}\left\langle \varphi^{1\tilde{2}},\, \varphi^{2\tilde{1}},\, \varphi^{3\bar{3}}\right\rangle$	3
$(\mathbf{0},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}},\, \varphi^{\tilde{1}\bar{3}},\, \varphi^{\tilde{2}\bar{3}}\right\rangle$	3	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}}\right\rangle$	1
$(\mathbf{3},\mathbf{0})$	$\mathbb{C}\left\langle \varphi^{123}\right\rangle$	1	$\mathbb{C}\left\langle \varphi^{123}\right\rangle$	1
$(\mathbf{2},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{12\bar{3}},\, \varphi^{13\tilde{1}},\, \varphi^{13\tilde{2}},\, \varphi^{13\bar{3}},\, \varphi^{23\tilde{1}},\, \varphi^{23\tilde{2}},\, \varphi^{23\bar{3}},\, \varphi^{\bar{\tilde{1}}3\bar{3}},\, \varphi^{\bar{\tilde{2}}3\bar{3}}\right\rangle$	9	$\mathbb{C}\left\langle \varphi^{12\bar{3}},\, \varphi^{13\tilde{2}},\, \varphi 23\tilde{1}\right\rangle$	3
$(\mathbf{1},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{1\tilde{2}\bar{3}},\, \varphi^{2\tilde{1}\bar{3}},\, \varphi^{3\tilde{1}\tilde{2}},\, \varphi^{3\tilde{1}\bar{3}},\, \varphi^{3\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{1}}\tilde{1}\bar{3}},\, \varphi^{3\tilde{1}\bar{3}},\, \varphi^{\bar{\tilde{2}}\tilde{2}\bar{3}},\, \varphi^{3\tilde{2}\bar{3}}\right\rangle$	9	$\mathbb{C}\left\langle \varphi^{1\tilde{2}\bar{3}},\, \varphi^{2\tilde{1}\bar{3}},\, \varphi^{3\tilde{1}\tilde{2}}\right\rangle$	3
$(\mathbf{0},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1	$\mathbb{C}\left\langle \varphi^{\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1
$(\mathbf{3},\mathbf{1})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}},\, \varphi^{123\tilde{2}},\, \varphi^{123\bar{3}}\right\rangle$	3	$\mathbb{C}\left\langle \varphi^{123\bar{3}}\right\rangle$	1
$(\mathbf{2},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}},\, \varphi^{13\tilde{1}\tilde{2}},\, \varphi^{13\tilde{1}\bar{3}},\, \varphi^{13\tilde{2}\bar{3}},\, \varphi^{23\tilde{1}\tilde{2}},\, \varphi^{23\tilde{1}\bar{3}},\, \varphi^{23\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{1}}\bar{\tilde{2}}\tilde{1}\bar{3}},\, \varphi^{\bar{\tilde{1}}\bar{\tilde{2}}\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{1}}3\tilde{1}\bar{3}},\, \varphi^{\bar{\tilde{2}}3\tilde{2}\bar{3}}\right\rangle$	11	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}},\, \varphi^{13\tilde{2}\bar{3}},\, \varphi^{23\tilde{1}\bar{3}}\right\rangle$	3
$(\mathbf{1},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{1\tilde{1}\tilde{2}\bar{3}},\, \varphi^{2\tilde{1}\tilde{2}\bar{3}},\, \varphi^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	3	$\mathbb{C}\left\langle \varphi^{3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1
$(\mathbf{3},\mathbf{2})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}},\, \varphi^{123\tilde{1}\bar{3}},\, \varphi^{123\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{1}}23\tilde{1}\bar{3}},\, \varphi^{\bar{\tilde{1}}23\tilde{2}\bar{3}}\right\rangle$	5	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}}\right\rangle$	1
$(\mathbf{2},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}},\, \varphi^{13\tilde{1}\tilde{2}\bar{3}},\, \varphi^{23\tilde{1}\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{1}}3\tilde{1}\tilde{2}\bar{3}},\, \varphi^{\bar{\tilde{2}}3\tilde{1}\tilde{2}\bar{3}}\right\rangle$	5	$\mathbb{C}\left\langle \varphi^{12\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1
$(\mathbf{3},\mathbf{3})$	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1	$\mathbb{C}\left\langle \varphi^{123\tilde{1}\tilde{2}\bar{3}}\right\rangle$	1

Table 4.9: The harmonic representatives of the Bott-Chern cohomology of the complex solvmanifolds $(M_k := G/\Gamma_k, J_{k,t})$ where $\{J_{k,t}\}_{t\in\Delta(0,\varepsilon_k)\subset\mathbb{C}}$ is the holomorphic deformation defined by the direction $\Psi_k(t) := tX_3^k \otimes \bar{\varphi}_k^3 \in (\mathfrak{g}_8)^{1,0}_{J_k} \otimes \wedge^{0,1}_{J_k}\mathfrak{g}_8^*$ of the complex structure $J_k := J^{A_k}$ satisfying the equations (3.33) for the value $A_k = \frac{i}{2k+1}$ for some $k \in \mathbb{Z}$. The lattice Γ_k of G is the semi-direct product of a lattice Γ'_A in $\mathbb{C}$ given by (4.21) and a lattice Γ'' in $\mathbb{C}^2$.

$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\sharp_{J_{A,t}}}$		$t=0$		$t\neq 0$	
	dR	$\bar\partial$	BC	$\bar\partial$	BC
$(\mathbf{0},\mathbf{0})$	1	1	1	1	1
$(\mathbf{1},\mathbf{0})$	2	3	1	1	1
$(\mathbf{0},\mathbf{1})$		3	1	1	1
$(\mathbf{2},\mathbf{0})$	5	3	3	1	1
$(\mathbf{1},\mathbf{1})$		9	7	3	3
$(\mathbf{0},\mathbf{2})$		3	3	1	1
$(\mathbf{3},\mathbf{0})$	8	1	1	1	1
$(\mathbf{2},\mathbf{1})$		9	9	3	3
$(\mathbf{1},\mathbf{2})$		9	9	3	3
$(\mathbf{0},\mathbf{3})$		1	1	1	1
$(\mathbf{3},\mathbf{1})$	5	3	3	1	1
$(\mathbf{2},\mathbf{2})$		9	11	3	3
$(\mathbf{1},\mathbf{3})$		3	3	1	1
$(\mathbf{3},\mathbf{2})$	2	3	5	1	1
$(\mathbf{2},\mathbf{3})$		3	5	1	1
$(\mathbf{3},\mathbf{3})$	1	1	1	1	1

Table 4.10: Summary of the dimensions of the cohomologies of complex solvmanifolds $(M := G/\Gamma_k, J_k)$, where $J_k := J^{A_k}$ is the complex structure satisfying the equations (3.33) for the value $A_k = \frac{i}{2k+1}$ for some $k \in \mathbb{Z}$, and of its small deformations given by the direction $\Psi_k(t) := tX_3^k \otimes \bar\omega_k^3 \in (\mathfrak{g}_8)^{1,0}_{J_k} \otimes \wedge^{0,1}_{J_k} \mathfrak{g}_8^*$. The lattice Γ_k of G is the semi-direct product of a lattice Γ'_A in $\mathbb{C}$ given by (4.21) and a lattice Γ'' in $\mathbb{C}^2$.

Chapter 5

Special Hermitian geometry on solvmanifolds

We study in this chapter the existence and the behaviour under holomorphic deformations of several special Hermitian metrics on complex solvmanifolds endowed with an invariant complex structure with holomorphically trivial canonical bundle. Section 5.1 deals with the problem of existence of balanced, strongly Gauduchon, strong Kähler with torsion (briefly SKT) and generalized Gauduchon Hermitian metrics on this type of complex solvmanifolds. The existence of Kähler, balanced or SKT [31, 95] metrics on a compact manifold of the form $M = G/\Gamma$ reduces to the level of the Lie algebra. In addition, the existence of SKT [33, 95], balanced [95] and invariant 1-st generalized Gauduchon [35] metrics on 6-dimensional complex nilmanifolds (M, J) with J invariant has already been studied. Hence, we restrict our investigation to the existence of these special metrics on the Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ compatible with the complex structures with a closed form of bidegree $(3, 0)$ obtained in Chapter 3. It follows that the 6-dimensional solvmanifolds with an invariant complex structure admitting compatible Calabi-Yau metrics have underlying Lie algebra isomorphic to $\mathbb{R}^6$ or $\mathfrak{g}_2^0$. As regards the strongly Gauduchon metrics, motivated by [74], we provide new examples of compact complex manifolds admitting such metrics but that are not balanced and on which the $\partial\bar{\partial}$-lemma does not hold. Finally, using the results obtained in this section and in Chapter 4, we present new examples based on the complex structures on the Lie algebra $\mathfrak{h}_5$ showing that the property $E_1(M) \cong E_\infty(M)$ and strongly Gauduchon property are unrelated. The results of this section are summarized in Tables 5.1 and 5.2.

Section 5.2 deals with the behaviour of Hermitian metrics under holomorphic deformations, particularly with the problems of closedness of the balanced and the strongly Gauduchon properties [74]. By means of a holomorphic deformation on a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with its abelian complex structure, we demonstrate that the strongly Gauduchon and the balanced properties are not closed under holomorphic deformations. Popovici [72, Proposition 4.1] shows that if a holomorphic deformation $\{(M, J_t)\}_{t\in\Delta}$ satisfies the $\partial\bar{\partial}$-lemma for any $t \in \Delta^*$, then the central limit admits a strongly Gauduchon metric. We conclude the chapter presenting a family of compact complex manifolds $\{(M, J_t)\}_{t\in\Delta}$ such that (M, J_t) satisfies the $\partial\bar{\partial}$-lemma and admits balanced metrics for any $t \neq 0$, but the central limit neither satisfies the

$\partial\bar{\partial}$-lemma nor admits balanced metrics.

5.1 Existence of special Hermitian metrics

We study the existence of special Hermitian metrics on 6-dimensional solvmanifolds $(M = G/\Gamma, J)$ endowed with an invariant complex structure J with holomorphically trivial canonical bundle. Recall that by Theorem 2.3.7, the underlying real Lie algebras $\mathfrak{g}$ are isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}, \mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$ if $\mathfrak{g}$ is nilpotent and $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha\geq 0}, \mathfrak{g}_3 \ldots, \mathfrak{g}_8$ or $\mathfrak{g}_9$ if $\mathfrak{g}$ is not nilpotent.

Any Hermitian metric g on the Lie algebra $\mathfrak{g}$ passes to a Hermitian structure on the solvmanifold M. Hence, our strategy to check the existence of special Hermitian metrics on these solvmanifolds consists on starting with the classification of pairs $(\mathfrak{g}, J)$ obtained in Chapter 3 and then finding the J-Hermitian structures F on $\mathfrak{g}$ satisfying the required conditions. The positive-definiteness of the metric g and the compatibility between the metric and the complex structure J is equivalent to $F \in \wedge^{1,1}\mathfrak{g}^*$, together with the existence of some invariant (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ for the complex structure J such that the fundamental 2-form F may be represented by:

$$(5.1) \qquad 2\,F = i\,(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}}) + u\omega^{1\bar{2}} - \bar{u}\omega^{2\bar{1}} + v\omega^{2\bar{3}} - \bar{v}\omega^{3\bar{2}} + z\omega^{1\bar{3}} - \bar{z}\omega^{3\bar{1}},$$

where the coefficients r^2, s^2, t^2 are non-zero real numbers and u, v, $z \in \mathbb{C}$ satisfy $r^2s^2 > |u|^2$, $s^2t^2 > |v|^2$, $r^2t^2 > |z|^2$ and $r^2s^2t^2 + 2\mathfrak{Re}\,(i\bar{u}\bar{v}z) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2$.

5.1.1 Strong Kähler with torsion geometry

The existence of SKT metrics on six-dimensional nilmanifolds $M = G/\Gamma$ admitting invariant complex structures was firstly studied by Fino, Parton and Salamon [33]. In fact, they prove that such a complex nilmanifold admits an invariant SKT metric if and only if the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or $\mathfrak{h}_8$. By using the symmetrization process, Ugarte [95] shows that the same classification is valid if the invariance of the metric is not required. We include the previous results in the statement of the following classification theorem:

Theorem 5.1.1. *Let $M = G/\Gamma$ be a six-dimensional solvmanifold admitting invariant complex structures with holomorphically trivial canonical bundle. M has an SKT metric if and only if $\mathfrak{g} \cong \mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or $\mathfrak{h}_8$ if $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \cong \mathfrak{g}_2^0$ or $\mathfrak{g}_4$ if $\mathfrak{g}$ is not nilpotent.*

Proof. Let F be a J-Hermitian metric given by (5.1). We firstly study the existence of SKT metrics on $\mathfrak{g}_1$ and $\mathfrak{g}_2^\alpha$. The equations (3.15) parametrize all the complex structures J on $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$, so we get

$$(5.2)\ \ \partial\bar{\partial}F = -2ir^2(\mathfrak{Re}\,A)^2\omega^{13\bar{1}\bar{3}} + 2u(\mathfrak{Im}\,A)^2\omega^{13\bar{2}\bar{3}} - 2\bar{u}(\mathfrak{Im}\,A)^2\omega^{23\bar{1}\bar{3}} - 2is^2(\mathfrak{Re}\,A)^2\omega^{23\bar{2}\bar{3}}.$$

Thus, $\partial\bar{\partial}F = 0$ implies $\mathfrak{Re}\,A = 0$, and necessarily $A = i$. In this case F is SKT if and only if $u = 0$. By Proposition 3.2.3 the corresponding Lie algebra is $\mathfrak{g}_2^0$.

For the Lie algebra $\mathfrak{g}_3$, by Proposition 3.2.4 any J on $\mathfrak{g}_3$ is equivalent to one complex structure J_x given by (3.21). The (3,3)-form $\partial\bar\partial F \wedge F$ is the following

$$\partial\bar\partial F \wedge F \;= \tfrac{1+4x^2}{16x^2}\left(4x^2s^4+t^4\right)\omega^{123\bar1\bar2\bar3}. \tag{5.3}$$

Since this form is never zero, there is no SKT metric on $\mathfrak{g}_3$.

For the Lie algebras $\mathfrak{g}_k$ ($4 \le k \le 7$), using the equations (3.22), which parametrize all the complex structures J on $\mathfrak{g}_k$, we get

$$\begin{aligned}\partial\bar\partial F \;&= it^2(G_{11}G_{22}-|G_{12}|^2)\omega^{12\bar1\bar2} - 2ir^2(\mathfrak{Re}\,A)^2\omega^{13\bar1\bar3} - 2is^2(\mathfrak{Re}\,A)^2\omega^{23\bar2\bar3}\\ &\qquad + 2u(\mathfrak{Im}\,A)^2\omega^{13\bar2\bar3} - 2\bar u(\mathfrak{Im}\,A)^2\omega^{23\bar1\bar3}.\end{aligned} \tag{5.4}$$

Thus, $\partial\bar\partial F = 0$ implies $\mathfrak{Re}\,A = 0$, and from the conditions given for equations (3.22) we have $G_{12}=0$. Now, $\partial\bar\partial F = 0$ also implies $G_{11}G_{22}=0$, so from Proposition 3.2.6 it follows that only $\mathfrak{g}_4$ admits SKT structures: in fact, a generic F given by (5.1) is SKT if and only if $u=0$.

For the study of SKT metrics on $\mathfrak{g}_8$, instead of using the complex structure equations (3.31), (3.32) and (3.33), we use the equations (3.34) obtained in the proof of Proposition 3.2.7. A direct calculation shows that

$$\partial\bar\partial F \wedge F = 2\left(r^2s^2(1+\mathfrak{Re}(A)^2)+|u|^2\mathfrak{Im}(A)^2\right)\omega^{123\bar1\bar2\bar3}, \tag{5.5}$$

and in particular, this form does not depend on the complex coefficients B, C in (3.34). The form $\partial\bar\partial F \wedge F$ never vanishes, so there is no SKT metric on $\mathfrak{g}_8$.

Finally, for the Lie algebra $\mathfrak{g}_9$, from the complex equations (3.36) in Proposition 3.2.10 it follows

$$\partial\bar\partial F \wedge F \;= \left(|v|^2+\tfrac{s^4}{8}\right)\omega^{123\bar1\bar2\bar3} \neq 0, \tag{5.6}$$

so the Lie algebra $\mathfrak{g}_9$ does not admit SKT metrics. □

Remark 5.1.2. In the previous theorem we have proved that any complex structure with non-trivial closed (3,0)-form on $\mathfrak{g}_2^0$ or $\mathfrak{g}_4$ admits SKT metrics. Moreover, a generic metric F given by (5.1) satisfies the SKT condition with respect to the complex equations (3.18) for $(\mathfrak{g}_2^0, J)$, or (3.27) for $(\mathfrak{g}_4, J_\pm)$, if and only if $u=0$. Therefore, in both cases the SKT metrics are given by

$$2\,F = i\,(r^2\omega^{1\bar1}+s^2\omega^{2\bar2}+t^2\omega^{3\bar3}) + v\omega^{2\bar3} - \bar v\omega^{3\bar2} + z\omega^{1\bar3} - \bar z\omega^{3\bar1}, \tag{5.7}$$

where the coefficients r^2, s^2, t^2 are non-zero real numbers and $v,\, z \in \mathbb{C}$ satisfy $r^2s^2t^2 > r^2|v|^2 + s^2|z|^2$. Whereas it was already known that the Lie algebra $\mathfrak{g}_2^0$ admits SKT metrics, a solvmanifold based on $\mathfrak{g}_4$ provides, as far as we know, a new example of 6-dimensional compact SKT manifold.

We recall that a complex structure J on a symplectic manifold (M,ω) is said to be *tamed* by the symplectic form ω if $\omega(X, JX) > 0$ for any non-zero vector field X on M.

The pair (ω, J) is also called *Hermitian-symplectic* structure in [88]. By [29, Proposition 2.1] the existence of a Hermitian-symplectic structure on a complex manifold (M, J) is equivalent to the existence of a J-compatible SKT metric g whose fundamental form satisfies $\partial F = \bar{\partial}\beta$ for some ∂-closed $(2,0)$-form β. As a consequence of Theorem 5.1.1, we have that a six-dimensional solvmanifold $(M = G/\Gamma, J)$ with J invariant and holomorphically trivial canonical bundle, has a symplectic form ω taming J if and only if $\mathfrak{g} \cong \mathfrak{g}_2^0$ and (J, ω) is a Kähler structure. By [29], if (M, J) admits a non-invariant symplectic form taming J, then there exists an invariant one. So we can immediately exclude the solvmanifolds with $\mathfrak{g} \cong \mathfrak{g}_4$ since $\mathfrak{g}_4$ does not admit any symplectic form. For the solvmanifolds with $\mathfrak{g} \cong \mathfrak{g}_2^0$ by a direct computation we have that $\partial F = \bar{\partial}\beta$, for some ∂-closed $(2,0)$-form β, if and only if $dF = 0$.

The space of SKT metrics on $\mathfrak{g}_2^0$ and $\mathfrak{g}_6$ is parametrized by (5.7). It is immediate to check that there is no Kähler metric on $\mathfrak{g}_6$ compatible with the complex structure (3.29). However, the fundamental form $F = \frac{i}{2}(r^2\omega^{1\bar{1}} + s^2\omega^{2\bar{2}} + t^2\omega^{3\bar{3}})$ with $rst \neq 0$ defines a Kähler metric on $(\mathfrak{g}_2^0, J)$ with J satisfying (3.18). The symmetrization process allows us to state the following:

Theorem 5.1.3. *Let $M = G/\Gamma$ be a six-dimensional solvmanifold endowed with an invariant complex structure. Then M admits a Calabi-Yau metric if and only if the underlying real Lie algebra is isomorphic to $\mathbb{R}^6$ or $\mathfrak{g}_2^0$.*

Proof. Recall that a Calabi-Yau metric is a Kähler metric compatible with a complex structure with holomorphically trivial canonical bundle. By the symmetrization process the existence of a Kähler structure on M implies the existence of an invariant one. Furthermore, by Proposition 2.1.31 the existence of a closed complex volume form also implies the existence of an invariant one. Therefore, the only possible underlying Lie algebras are $\mathbb{R}^6$ and $\mathfrak{g}_2^0$ and this concludes the proof. □

5.1.2 Generalized Gauduchon structures

As we mentioned in Section 1.3, the study of generalized Gauduchon structures in the class of six-dimensional manifolds (or complex dimension three) reduces uniquely to the class of 1-st generalized Gauduchon metrics. Fu, Wang and Wu [39] prove the following general result concerning the sign of the invariant $\gamma_1(F)$ for three-dimensional compact complex manifolds:

Theorem 5.1.4 (Fu, Wang and Wu [39, Theorem 6])**.** *On any compact 3-dimensional complex manifold there exists a Hermitian metric g such that its fundamental 2-form F has $\gamma_1(F) > 0$.*

Therefore, if one finds a Hermitian metric $\tilde{F}$ such that $\gamma_1(\tilde{F}) < 0$ then the smooth variation of the invariant γ_1 in the space of Hermitian metrics implies the existence of a 1-st generalized Gauduchon metric on M.

As regards complex manifolds of the form $(M = G/\Gamma, J)$ with J an invariant complex structure, it is remarkable that the symmetrization process cannot be applied to the 1-st

Gauduchon condition in order to reduce the problem to the Lie algebra level. In the realm of six-dimensional nilmanifolds endowed with invariant complex structures, Fino and Ugarte prove the following:

Theorem 5.1.5 (Fino and Ugarte [35, Propositions 3.3, 3.5])**.** *Let $(M = G/\Gamma, J)$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J. An invariant J-Hermitian metric is 1-st Gauduchon if and only if it is SKT. Moreover, an invariant J-Hermitian metric has $\gamma_1 < 0$ if and only if $\mathfrak{g} \cong \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4$ or $\mathfrak{h}_5$.*

Now, we study the existence of 1-st Gauduchon metrics and the sign of γ_1 for the non-nilpotent solvmanifolds with an invariant complex structure with holomorphically trivial canonical bundle:

Theorem 5.1.6. *Let $M = G/\Gamma$ be a 6-dimensional non-nilpotent solvmanifold and denote by $\mathfrak{g}$ the Lie algebra of G. Let J be an invariant complex structure with holomorphically trivial canonical bundle, and F an invariant J-Hermitian metric on M. Then:*

(i) *If $\mathfrak{g} \cong \mathfrak{g}_1, \mathfrak{g}_2^{\alpha}(\alpha > 0), \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7, \mathfrak{g}_8$ or $\mathfrak{g}_9$, then $\gamma_1(F) > 0$ for any (J, F).*

(ii) *If $\mathfrak{g} \cong \mathfrak{g}_2^0$ or $\mathfrak{g}_4$, then $\gamma_1(F) \geq 0$ for any (J, F); moreover, an invariant Hermitian metric is 1-st Gauduchon if and only if it is SKT.*

(iii) *If $\mathfrak{g} \cong \mathfrak{g}_6$ then there exist invariant Hermitian metrics such that $\gamma_1(F) > 0$, $= 0$ or < 0; in particular, there are invariant 1-st Gauduchon metrics which are not SKT.*

Proof. Let F be an invariant J-Hermitian metric given by (5.1). Then,

$$F^3 = -\frac{3}{4}\det(F)\,\omega^{123\bar{1}\bar{2}\bar{3}}, \quad \text{where} \quad \det(F) = \begin{vmatrix} i\,r^2 & u & z \\ -\bar{u} & i\,s^2 & v \\ -\bar{z} & -\bar{v} & i\,t^2 \end{vmatrix}.$$

Notice that the conditions required by the coefficients of the fundamental form imply $i\det(F) > 0$. Now, if

$$\partial\bar{\partial}F \wedge F = \mu\,\omega^{123\bar{1}\bar{2}\bar{3}},$$

then $\frac{i}{2}\partial\bar{\partial}F \wedge F = \frac{2\mu}{3i\det(F)}F^3$, which implies that

$$\gamma_1(F) > 0, = 0 \text{ or } < 0 \quad \text{if and only if} \quad \mu > 0, = 0 \text{ or } < 0.$$

In what follows we will compute μ for any triple $(\mathfrak{g}, J, F)$ and study its possible signs.

For the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2^{\alpha}$, from (5.2) it follows that

$$\partial\bar{\partial}F \wedge F \;= 2\left(r^2 s^2 (\mathfrak{Re}\,A)^2 + |u|^2 (\mathfrak{Im}\,A)^2\right)\omega^{123\bar{1}\bar{2}\bar{3}}.$$

Therefore, $\gamma_1(F) \geq 0$ for any F. Moreover, $\gamma_1(F) \geq 0$ if and only if $\mathfrak{Re}\, A = 0$ and $u = 0$, which precisely corresponds an to SKT metric on $\mathfrak{g}_2^0$.

From (5.3), (5.5) and (5.6) it follows that $\gamma_1 > 0$ for any (J, F) on $\mathfrak{g}_3$, $\mathfrak{g}_8$ and $\mathfrak{g}_9$.

For the Lie algebras $\mathfrak{g}_k$ $(4 \leq k \leq 7)$, using (5.4) we get

$$2\partial\bar{\partial}F \wedge F \;= \left[4r^2s^2(\mathfrak{Re}\, A)^2 + 4|u|^2(\mathfrak{Im}\, A)^2 - t^4(G_{11}G_{22} - |G_{12}|^2)\right]\omega^{123\bar{1}\bar{2}\bar{3}}.$$

Let us firstly consider $\mathfrak{g}_4$. By (3.27) we can take $A = i$, $G_{11} = \pm 1$ and $G_{12} = G_{22} = 0$, so $2\partial\bar{\partial}F \wedge F = 4|u|^2\omega^{123\bar{1}\bar{2}\bar{3}}$. This implies that $\gamma_1 \geq 0$, and it is equal to zero if and only if the structure is SKT. This completes the proof of (i).

For the Lie algebra $\mathfrak{g}_5$, by (3.28) we have that $A = G_{12} = 1$ and $G_{11} = G_{22} = 0$, so $2\partial\bar{\partial}F \wedge F = (4r^2s^2 + t^4)\omega^{123\bar{1}\bar{2}\bar{3}}$ and $\gamma_1 > 0$.

Similarly, using (3.30), for $\mathfrak{g}_7$ we can take $A = i$, $G_{12} = 0$ and $(G_{11}, G_{22}) = (-1, 1)$ or $(1, -1)$. Therefore, $2\partial\bar{\partial}F \wedge F = (t^4 + 4|u|^2)\omega^{123\bar{1}\bar{2}\bar{3}}$ and thus $\gamma_1 > 0$. This completes the proof of (ii).

Finally, to prove (iii), by (3.29) we consider $A = i$, $G_{12} = 0$ and $G_{11} = G_{22} = 1$. Since $2\partial\bar{\partial}F \wedge F = (4|u|^2 - t^4)\omega^{123\bar{1}\bar{2}\bar{3}}$, we conclude that on $\mathfrak{g}_6$ there exist Hermitian metrics such that $\gamma_1 > 0$, $= 0$ or < 0, depending on the sign of $4|u|^2 - t^4$. □

Remark 5.1.7. On the other hand, it is worthy to remark that on the solvmanifold $M = G/\Gamma$ with Lie algebra $\mathfrak{g} \cong \mathfrak{g}_6$ there exist invariant 1-st Gauduchon metrics, although M does not admit any SKT metric. In fact, with respect to the complex equations (3.29), any invariant Hermitian metric F given by (5.1) with $|u| = \frac{t^2}{2}$ is 1-st Gauduchon. However, there is no SKT metric by Theorem 5.1.1. This is in deep contrast with the nilpotent case, because any invariant 1-st Gauduchon metric on a 6-nilmanifold is necessarily SKT (see Theorem 5.1.5).

5.1.3 Balanced metrics

Given an homogeneous space $M = G/\Gamma$ endowed with an invariant complex structure J, Fino and Grantcharov [31] state, using the symmetrization process, that the existence of compatible balanced metrics on (M, J) can be reduced to the existence of invariant metrics. Balanced geometry has been studied also under the point of view of its relation with other properties on the complex manifold. For instance, it turns out [56] that the balanced, the Frölicher degeneration and the $\mathcal{C}^\infty$-pure and full properties are unrelated. The classification of six-dimensional nilmanifolds admitting balanced Hermitian metrics is obtained by Ugarte [95]. We include his result in the following theorem, where in particular new examples of balanced solvmanifolds are found.

Theorem 5.1.8. *Let $M = G/\Gamma$ be a six-dimensional solvmanifold admitting invariant complex structures with holomorphically trivial canonical bundle. M admits a balanced metric if and only if $\mathfrak{g} \cong \mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$ if $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \cong \mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha \geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ or $\mathfrak{g}_8$ if $\mathfrak{g}$ is not nilpotent.*

Moreover, the solvable and non-nilpotent Lie algebras endowed with J with non-zero closed $(3,0)$-form admit balanced metrics, except for the complex structures which are isomorphic to (3.31) or (3.32) on $\mathfrak{g}_8$.

Proof. Since a J-Hermitian metric F given by (5.1) is balanced if and only if $\partial F^2 = 0$, next we compute the (3,2)-form ∂F^2 for each Lie algebra $\mathfrak{g}$.

For the existence of balanced metrics on $\mathfrak{g}_1$ and $\mathfrak{g}_2^\alpha$, from the complex structure equations (3.15) it follows

$$2\,\partial F^2 = (ir^2 z + \bar{u}v)\bar{A}\,\omega^{123\bar{1}\bar{3}} + (is^2 v - uz)\bar{A}\,\omega^{123\bar{2}\bar{3}}. \tag{5.8}$$

Since A is non-zero, this form vanishes if and only if $is^2 v - uz = 0$ and $ir^2 z + \bar{u}v = 0$. Now, $r^2 s^2 - |u|^2 > 0$ implies that these conditions are equivalent to $v = z = 0$.

For the Lie algebra $\mathfrak{g}_3$, a direct calculation using the complex equations (3.21) shows that

$$\begin{aligned} 2\,\partial F^2 = &-\tfrac{1}{2x}\left(t^2\,\mathfrak{Re}\,u + \mathfrak{Im}\,(\bar{v}z) - x(it^2 u + \bar{v}z)\right)\omega^{123\bar{1}\bar{2}} + \\ &2x\left(s^2\,\mathfrak{Re}\,z - \mathfrak{Im}\,(uv) + \tfrac{is^2 z - uv}{4x}\right)\omega^{123\bar{1}\bar{3}}. \end{aligned} \tag{5.9}$$

Thus, the form F^2 is closed if and only if

$$\begin{cases} it^2 u + \bar{v}z = (t^2\mathfrak{Re}\,u + \mathfrak{Im}\,(\bar{v}z))/x, \\ is^2 z - uv = -4x(s^2\mathfrak{Re}\,z - \mathfrak{Im}\,(uv)). \end{cases}$$

Notice that since x is real, we have that both $it^2 u + \bar{v}z$ and $is^2 z - uv$ are also real numbers. But this implies that $t^2\mathfrak{Re}\,u + \mathfrak{Im}\,(\bar{v}z) = 0$ and $s^2\mathfrak{Re}\,z - \mathfrak{Im}\,(uv) = 0$, so the system above is homogeneous. Finally, since $s^2 t^2 - |v|^2 > 0$ necessarily $u = z = 0$.

For the Lie algebras $\mathfrak{g}_k$ $(4 \le k \le 7)$, from equations (3.22) we have

$$\begin{aligned} 2\,\partial F^2 \;= &\left[(s^2t^2 - |v|^2)G_{11} + (r^2t^2 - |z|^2)G_{22} + (v\bar{z} - it^2\bar{u})G_{12} + (\bar{v}z + it^2 u)\overline{G}_{12}\right]\omega^{123\bar{1}\bar{2}} + \\ &(ir^2 v + \bar{u}z)\overline{A}\,\omega^{123\bar{1}\bar{3}} + (is^2 z - uv)\overline{A}\,\omega^{123\bar{2}\bar{3}}. \end{aligned} \tag{5.10}$$

Since A is non-zero and $r^2 s^2 - |u|^2 > 0$, the coefficients of $\omega^{123\bar{1}\bar{3}}$ and $\omega^{123\bar{2}\bar{3}}$ vanish if and only if $v = z = 0$. The latter conditions reduce the expression of the form (5.10) to

$$2\,\partial F^2 = t^2\left(s^2\,G_{11} + r^2\,G_{22} - i\bar{u}\,G_{12} + iu\,\overline{G}_{12}\right)\omega^{123\bar{1}\bar{2}}.$$

Now, we can use the complex classification given in Proposition 3.2.6 to conclude that the only possibilities to get a closed form F^2 are, either $G_{12} = 0$ and $(G_{11}, G_{22}) = (1, -1), (-1, 1)$, or $G_{11} = G_{22} = 0$ and $G_{12} = 1$. The first case corresponds to $\mathfrak{g}_7$ and the coefficients r^2 and s^2 in the metric must be equal, whereas the second case corresponds to $\mathfrak{g}_5$ with metric coefficient $u \in \mathbb{R}$.

For the study of balanced Hermitian metrics on $\mathfrak{g}_8$, by the complex equations (3.34), a direct calculation shows that

$$\begin{aligned} 2\,\partial F^2 = &-\left[(ir^2 v + \bar{u}z)(\overline{A} - i) + (r^2 s^2 - |u|^2)\overline{C}\,\right]\omega^{123\bar{1}\bar{3}} + \\ &\left[(uv - is^2 z)(\overline{A} - i) + (r^2 s^2 - |u|^2)\overline{B}\,\right]\omega^{123\bar{2}\bar{3}}. \end{aligned} \tag{5.11}$$

Since $r^2s^2 - |u|^2 \neq 0$, the structure (J, F) is balanced if and only if

$$B = -\frac{is^2\bar{z} + \bar{u}\bar{v}}{r^2s^2 - |u|^2}(A + i), \qquad C = \frac{ir^2\bar{v} - u\bar{z}}{r^2s^2 - |u|^2}(A + i).$$

It follows from Proposition 3.2.7 that the complex structures (3.31) and (3.32) do not admit balanced metrics, because $A = -i$ but B is not zero. However, any complex structure in the family (3.33) has balanced Hermitian metrics because $B = C = 0$. In fact, if $A \neq -i$ then the metric (5.1) is balanced if and only if $v = z = 0$, and for $A = -i$ (i.e. the complex structure is bi-invariant) any metric is balanced.

In the case of the Lie algebra $\mathfrak{g}_9$, from the complex equations (3.36) it follows

(5.12)
$$4\partial F^2 = \big(i\,\bar{u}\bar{v} - s^2\bar{z}\big)\,\omega^{123\bar{1}\bar{2}} - \big(i\,v\bar{z} + t^2\bar{u} - uv + i\,s^2 z\big)\,\omega^{123\bar{1}\bar{3}} + 2(|u|^2 - r^2s^2)\omega^{123\bar{2}\bar{3}},$$

which implies that the component of ∂F^2 in $\omega^{123\bar{2}\bar{3}}$ is non-zero, so there are not balanced Hermitian metrics.

Finally, notice that for the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$, $\mathfrak{g}_3$, $\mathfrak{g}_5$ and $\mathfrak{g}_7$ we have proved above that any complex structure J admits balanced Hermitian metrics. However, for the Lie algebra $\mathfrak{g}_8$, a complex structure J admits balanced metric if and only if it is isomorphic to one in the family (3.33). □

5.1.4 Strongly Gauduchon metrics

The symmetrization process is valid for the balanced and the SKT condition. We see in the following proposition that it also holds for the strongly Gauduchon condition:

Proposition 5.1.9. *$(M = G/\Gamma, J)$ has an sG metric if and only if it has an invariant one.*

Proof. Suppose that $(\mathfrak{g}, J)$ does not admit any strongly Gauduchon metric and let $F \in \wedge^{1,1}\mathfrak{g}^*$ be a fundamental form, that is, there is no $u \in \wedge^{n,n-2}\mathfrak{g}^*$ satisfying $\bar{\partial}F^{n-1} = \partial u$. F defines a fundamental 2-form $\tilde{F} \in \wedge^{1,1}M$ and if $\tilde{u} \in \wedge^{n,n-2}M$ satisfies $\bar{\partial}\tilde{F}^{n-1} = \partial\tilde{u}$, then the form $\tilde{u}_\nu$ would satisfy the equation $\bar{\partial}F^{n-1} = \partial\tilde{u}_\nu$. This contradicts the fact that $(\mathfrak{g}, J)$ does not admit any strongly Gauduchon metric. □

Therefore, the existence of sG metrics on $(M = G/\Gamma, J)$ is reduced to the existence at the level of Lie algebra $\mathfrak{g}$ of G. Recall that for complex surfaces the existence of balanced metrics is equivalent to the existence of sG metrics. In higher dimensions, there exist compact complex manifolds having sG metrics but not admitting any balanced metric and on which the $\partial\bar{\partial}$-lemma does not hold [76, Theorem 1.10]. If the complex structure is abelian then both classes of Hermitian metrics are the same, as we show in the following result.

Corollary 5.1.10. *Let F be an invariant Hermitian structure on $(M = G/\Gamma, J)$. If J is abelian, then F is sG if and only if it is balanced.*

Proof. If F is a strongly Gauduchon metric, then there is a form $u \in \wedge^{n,n-2}M$ such that $\partial F^{n-1} = \bar{\partial} u$. This is equivalent to $\bar{\partial} F^{n-1} = \partial v$ with $v \in \wedge^{n-2,n}M$ because F is a real 2-form. If J is abelian, then the complex differential operators $\partial\colon \wedge^{\bullet,\bullet}\mathfrak{g}^* \to \wedge^{\bullet+1,\bullet}\mathfrak{g}^*$ are identically zero. Thus, the balanced condition yields to $dF^{n-1} = \bar{\partial} F^{n-1} = 0$ and therefore, F is balanced. □

Now we suppose that $(M = G/\Gamma, J)$ is a nilmanifold endowed with an invariant complex structure. Next we prove that the nilmanifolds admitting an sG metric are the same as those admitting a balanced metric, although any complex structure on such nilmanifolds admits sG metrics.

Proposition 5.1.11. *Let $M = G/\Gamma$ be a six-dimensional nilmanifold admitting invariant complex structures. M admits an sG metric if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$.*

Proof. By Proposition 5.1.9 it suffices to study the invariant case. Let us start with the non-nilpotent case, that is, the complex structures corresponding to the nilpotent Lie algebras $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ (see Appendix, Table 3.1). Using the calculations in the proof of [95, Proposition 25] we have

$$4\partial F \wedge F = \big(i\epsilon\,(s^2t^2 - |v|^2) \pm (t^2 u + t^2 \bar{u} + iv\bar{z} - i\bar{v}z)\big)\, \omega^{123\bar{1}\bar{2}} + \big(uv - is^2 z\big)\, \omega^{123\bar{1}\bar{3}}.$$

As $\bar{\partial}(\wedge^{3,1}\mathfrak{g}^*) = \langle \omega^{123\bar{1}\bar{3}} \rangle$, if the Hermitian structure (J, F) is sG then

$$\mp\, i\epsilon\,(s^2t^2 - |v|^2) = t^2(u + \bar{u}) + i(v\bar{z} - \bar{v}z).$$

Since the left-hand side is purely imaginary and the right-hand side is real, we get that $\epsilon = 0$ and therefore, $\mathfrak{g} \cong \mathfrak{h}_{19}^-$.

For the nilpotent case, let us consider the general complex equations (3.1). Now, the fundamental 2-form of any J-Hermitian metric is given also by (5.1). Using again [95, Proposition 25], we get

$$\begin{aligned} 4\partial F \wedge F = &\big((1-\epsilon)\bar{A}(s^2t^2 - |v|^2) + \bar{B}(it^2 u + \bar{v}z) - \bar{C}(it^2\bar{u} - v\bar{z}) \\ &+ (1-\epsilon)\bar{D}(r^2t^2 - |z|^2)\big)\, \omega^{123\bar{1}\bar{2}} - \epsilon(s^2t^2 - |v|^2)\, \omega^{123\bar{1}\bar{3}}. \end{aligned}$$

Since $\bar{\partial}(\wedge^{3,1}\mathfrak{g}^*) = \langle \rho\, \omega^{123\bar{1}\bar{2}} \rangle$, if the Hermitian structure (J, F) is sG then $\epsilon = 0$, i.e. $\mathfrak{g} \cong \mathfrak{h}_i$ for $i = 1, \ldots, 6$. Moreover, if in addition $\rho = 1$, then any J-Hermitian structure is sG.

In conclusion, if there exists an sG metric then $\mathfrak{g} \cong \mathfrak{h}_1, \ldots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$. The converse follows directly from [95, Theorem 26] because these Lie algebras admit balanced Hermitian metrics. □

Remark 5.1.12. From the proof of the previous proposition it follows that on $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ and $\mathfrak{h}_6$, if J is a non-abelian nilpotent complex structure, then any invariant J-Hermitian metric is sG. This is in contrast with $\mathfrak{h}_{19}^-$, where for any complex structure the space of balanced metrics is strictly contained in the space of sG metrics, and moreover there are

Hermitian metrics which are not sG. For instance, consider a Hermitian metric on $\mathfrak{h}_{19}^-$ given by

$$\Omega = \frac{i}{2}\,\omega^{1\bar{1}} + (u^2+z^2+1)i\,\omega^{2\bar{2}} + (u^2+z^2+1)i\,\omega^{3\bar{3}} + \frac{u}{2}(\omega^{1\bar{2}} - \omega^{2\bar{1}}) + \frac{z}{2}(\omega^{1\bar{3}} - \omega^{3\bar{1}}),$$

that is, in (5.1) we take $r=1$, $v=0$, u and z real and $s^2=t^2=2(u^2+z^2+1)$:

- if $u=z=0$, then the metric is balanced;
- if $u=0$ and $z\neq 0$, then the metric is sG but not balanced;
- if $u\neq 0$, then the metric is not sG.

Notice that this indicates a contrast between the sG and SKT geometries, since by [33] the existence of an SKT structure on a 6-dimensional nilpotent Lie algebra depends only on the complex structure.

With the previous result we state a theorem of existence of strongly Gauduchon metrics in the class of six-dimensional solvmanifolds endowed with an invariant complex structure with holomorphically trivial canonical bundle.

Theorem 5.1.13. *Let $M=G/\Gamma$ be a six-dimensional solvmanifold admitting invariant complex structures with holomorphically trivial canonical bundle. M has a strongly Gauduchon metric if and only if $\mathfrak{g}\cong\mathfrak{h}_1,\ldots,\mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$ if $\mathfrak{g}$ is nilpotent and $\mathfrak{g}\cong\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ or $\mathfrak{g}_8$ if $\mathfrak{g}$ is not nilpotent.*

Moreover, if $\mathfrak{g}\cong\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$ or $\mathfrak{g}_8$ endowed with J with non-zero closed $(3,0)$-form, then any Hermitian metric is sG.

Proof. Since balanced condition implies the sG condition, by Theorem 5.1.8 we know that if $\mathfrak{g}\cong\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ or $\mathfrak{g}_8$, then there exist sG metrics. Moreover, any J on the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$ and $\mathfrak{g}_7$ admits sG metrics. We prove next that there are not sG metrics on $\mathfrak{g}_4$, $\mathfrak{g}_6$ and $\mathfrak{g}_9$.

From (3.22) we have $\bar\partial(\wedge^{3,1}\mathfrak{g}^*)=\langle\omega^{123\bar{1}\bar{3}},\omega^{123\bar{2}\bar{3}}\rangle$, and by (5.10) the (3,2)-form ∂F^2 is a combination of $\omega^{123\bar{1}\bar{2}}$, $\omega^{123\bar{1}\bar{3}}$ and $\omega^{123\bar{2}\bar{3}}$. Hence, the existence of sG metric is equivalent to the vanishing of the coefficient of $\omega^{123\bar{1}\bar{2}}$ in ∂F^2. By (3.27), the Lie algebra $\mathfrak{g}_4$ corresponds to $A=i$, $G_{11}=\pm 1$ and $G_{12}=G_{22}=0$, so the coefficient of $\omega^{123\bar{1}\bar{2}}$ is equal to $\pm(s^2t^2-|v|^2)$, which is never zero. On the other hand, by (3.29) the Lie algebra $\mathfrak{g}_6$ corresponds to $A=i$, $G_{11}=G_{22}=1$ and $G_{12}=0$, and the coefficient of $\omega^{123\bar{1}\bar{2}}$ is $(s^2t^2-|v|^2)+(r^2t^2-|z|^2)$, which is strictly positive. In conclusion, there do not exist sG metrics for $\mathfrak{g}_4$ or $\mathfrak{g}_6$.

For the Lie algebra $\mathfrak{g}_9$, equations (3.36) imply

$$\bar\partial\omega^{123\bar{1}}=0,\qquad \bar\partial\omega^{123\bar{2}}=(i/2)\,\omega^{123\bar{1}\bar{2}},\qquad \bar\partial\omega^{123\bar{3}}=-(i/2)\,\omega^{123\bar{1}\bar{3}},$$

therefore $\bar\partial\wedge^{3,1}\mathfrak{g}^*=\langle\omega^{123\bar{1}\bar{2}},\omega^{123\bar{1}\bar{3}}\rangle$. By (5.12) we have that the component of ∂F^2 in $\omega^{123\bar{2}\bar{3}}$ is nonzero, so $\partial F^2\notin\bar\partial\wedge^{3,1}\mathfrak{g}^*$ and F is never sG. Thus, there do not exist sG metrics for $\mathfrak{g}_9$.

To finish the proof, it remains to see that any pair (J, F) on $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$, $\mathfrak{g}_3$ and $\mathfrak{g}_8$ is sG. By Proposition 3.2.3 and (5.8), a direct calculation implies $\partial F^2 \in \bar\partial(\wedge^{3,1}\mathfrak{g}^*)$, so any (J, F) on $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ is sG. For $\mathfrak{g}_3$ (resp. $\mathfrak{g}_8$) Analogously, we also have $\partial F^2 \in \bar\partial(\wedge^{3,1}\mathfrak{g}^*)$ for any Hermitian structure (J, F), by Proposition 3.2.4 and (5.9) (resp. Proposition 3.2.7 and (5.11)). □

Remark 5.1.14. The results concerning the existence of special Hermitian metrics on the solvable and non-nilpotent Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$ are summarized in Table 5.2.

Motivated by [76, Theorem 1.10], we are concerned to find examples of "pure"' strongly Gauduchon manifolds, that is, compact complex manifolds admitting a compatible strongly Gauduchon metric but no balanced metric.

Proposition 5.1.15. *Let $M = G/\Gamma$ be a 6-dimensional nilmanifold with an invariant complex structure J such that M does not admit balanced metrics. If (M, J) has a sG metric, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$.*

Moreover, according to the classification in Table 3.1, such a J is given by: $x+y^2 \geq \frac{1}{4}$ on $\mathfrak{h}_2$; $x \geq \frac{1}{4}$ on $\mathfrak{h}_4$; and $\lambda = 0, y \neq 0$ or $\lambda = y = 0, x \geq 0$ on $\mathfrak{h}_5$.

Proof. Any complex structure on $\mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$ admits balanced metrics. From [97], we have that only $\mathfrak{h}_3$ and $\mathfrak{h}_5$ have abelian complex structures J admitting balanced metrics. In fact, any such J on $\mathfrak{h}_5$ admits balanced Hermitian metrics, whereas for $\mathfrak{h}_3$ the complex structure must be equivalent to the choice of $(-)$-sign in Table 3.1. From Corollary 5.1.10, it remains to study the non-abelian nilpotent complex structures J on $\mathfrak{h}_2$, $\mathfrak{h}_4$ and $\mathfrak{h}_5$. Since any such J admits sG metrics by Remark 5.1.12, next we show which of them do not admit balanced metrics.

In the three cases the complex equations are of the form

$$d\omega^1 = d\omega^2 = 0, \qquad d\omega^3 = \omega^{12} + \omega^{1\bar1} + \lambda\,\omega^{1\bar2} + D\,\omega^{2\bar2}. \tag{5.13}$$

A similar argument as in the proof of [97, Proposition 2.3] shows that, up to equivalence, the fundamental 2-form of any J-Hermitian metric is given by

$$2\,F = i\,(\omega^{1\bar1} + s^2\,\omega^{2\bar2} + t^2\,\omega^{3\bar3}) + u\,\omega^{1\bar2} - \bar u\,\omega^{2\bar1},$$

where $s^2 > |u|^2$ and $t^2 > 0$.

If $D = x + iy$ and $u = u_1 + iu_2$, the balanced condition is

$$s^2 + x + iy = u_2\lambda + iu_1\lambda. \tag{5.14}$$

We distinguish several cases depending on the values of λ.

If $\lambda \neq 0$, then F is balanced if and only if $u_1 = y/\lambda$ and $u_2 = (s^2 + x)/\lambda$. The condition $s^2 > |u|^2$ is equivalent to $s^4 + (2x - \lambda^2)s^2 + x^2 + y^2 < 0$, and it is easy to see that a non-zero s satisfying this condition exists if and only if

$$\lambda^4 - 4x\lambda^2 - 4y^2 > 0. \tag{5.15}$$

From Table 3.1, we get that any J on $\mathfrak{h}_2$ such that $x+y^2 \geq \frac{1}{4}$ has no balanced metrics. Similarly, for $\mathfrak{h}_4$ any J such that $x \geq \frac{1}{4}$ does not admit balanced metrics.

For $\mathfrak{h}_5$ and $\lambda \neq 0$, we have that $x=0$ by Table 3.1. Thus, there are no balanced metrics if and only if $\lambda^4 \leq 4y^2$. Since $y \geq 0$, this is equivalent to $\lambda^2 \leq 2y$. But from Table 3.1, we get that this cannot happen. Therefore, for $\lambda \neq 0$ the complex structures admit balanced metrics.

Finally, in the case of $\mathfrak{h}_5$ with $\lambda = 0$ we get that the balanced condition (5.14) reduces to $y=0$ and $s^2=-x>0$. From Table 3.1 we have that $0<1+4x$, i.e. $x \in (-\frac{1}{4}, \infty)$. Therefore, if $y \neq 0$ or $y=0, x \geq 0$, then there are no balanced metrics. □

Remark 5.1.16. As a consequence of the proof of Proposition 5.1.15, we show in Table 5.1 the complex structures J, up to equivalence, on the solvable Lie algebras $\mathfrak{h}_1, \dots, \mathfrak{h}_6, \mathfrak{h}_{19}^-$ that admit balanced Hermitian metrics.

As pointed out by Popovici [76, Theorem 1.11], the degeneration of the Frölicher sequence at E_1 and the existence of sG metrics are unrelated. From the study of the sG geometry above and from Theorem 4.1.4 we get:

Theorem 5.1.17. *Let $M = G/\Gamma$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J. If there exists an sG metric, then the Frölicher spectral sequence degenerates at the second level, i.e. $E_2(M) \cong E_\infty(M)$. Moreover, if there exists an sG metric and $\mathfrak{g} \not\cong \mathfrak{h}_5$, then $E_1(M) \cong E_\infty(M)$.*

Proof. By Proposition 5.1.11, the Lie algebra $\mathfrak{g}$ underlying $M = G/\Gamma$ must be isomorphic to $\mathfrak{h}_1, \dots, \mathfrak{h}_6$ or $\mathfrak{h}_{19}^-$, so Theorem 4.1.4 implies that the Frölicher sequence degenerates at the second level. The last assertion follows directly by taking into account Corollary 5.1.10 and Table 5.1 below. □

Concerning other relations among cohomological properties of compact complex manifolds, it is proved in [56] that there is no relation between cohomological decomposition at the first stage (in the sense of [9, 59]) and degeneration of the Frölicher sequence at the first step, as well as that the cohomological decomposition at the first stage and the existence of balanced Hermitian metric are also unrelated. Moreover, in [56] the balanced Hermitian structures (J, F) on nilmanifolds of dimension 6 for which the map $\mathcal{L}\colon H^1_{\mathrm{dR}}(M) \longrightarrow H^5_{\mathrm{dR}}(M)$ given by the cup product by $[F^2]$ is an isomorphism are studied. This has applications to a result of Angella and Tomassini [9] in the context of semi-Kähler geometry.

$\mathfrak{g}$	Abelian structures	Non-Abelian structures
$\mathfrak{h}_1$	$d\omega^2=0,\ d\omega^3=0$	—
$\mathfrak{h}_2$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}+(x+iy)\,\omega^{2\bar{2}}$, $y>0,\quad x+y^2<\frac{1}{4}$
$\mathfrak{h}_3$	$d\omega^2=0,\ d\omega^3=\omega^{1\bar{1}}-\omega^{2\bar{2}}$	—
$\mathfrak{h}_4$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}+x\,\omega^{2\bar{2}}$, $x<\frac{1}{4},\ x\neq 0$
$\mathfrak{h}_5$	$d\omega^2=0$, $d\omega^3=\omega^{1\bar{1}}+\omega^{1\bar{2}}+x\,\omega^{2\bar{2}}$, $0\leq x<\frac{1}{4}$	$d\omega^2=0,\ d\omega^3=\omega^{12}$
		$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\lambda\,\omega^{1\bar{2}}+(x+iy)\,\omega^{2\bar{2}}$, with (λ,x,y) satisfying one of: • $\lambda=y=0,\ x\in\left(-\frac{1}{4},0\right)$; • $0<\lambda^2<\frac{1}{2},\ 0\leq y<\frac{\lambda^2}{2},\ x=0$; • $\frac{1}{2}\leq\lambda^2<1,\ 0\leq y<\frac{1-\lambda^2}{2},\ x=0$; • $\lambda^2>1,\ 0\leq y<\frac{\lambda^2-1}{2},\ x=0$.
$\mathfrak{h}_6$	—	$d\omega^2=0,\ d\omega^3=\omega^{12}+\omega^{1\bar{1}}+\omega^{1\bar{2}}$
$\mathfrak{h}_{19}^-$	—	$d\omega^2=\omega^{13}+\omega^{1\bar{3}},\ d\omega^3=\pm i(\omega^{1\bar{2}}-\omega^{2\bar{1}})$

Table 5.1: Classification of complex structures on nilpotent Lie algebras admitting balanced metrics.

5.2 Balanced and strongly Gauduchon metrics under holomorphic deformations

In this section we study some properties related to the existence of balanced and sG metrics under deformation of the complex structure. Let Δ be an open disc around the origin in $\mathbb{C}$. In what follows, we denote by $\{(M, J_t)\}_{t\in\Delta}$ a holomorphic family of compact complex manifolds. As we mentioned in Section 1.4.2, the strongly Gauduchon and the balanced properties of compact complex manifolds are conjectured to be closed under holomorphic deformations [76, Conjectures 1.21,1.23]. We provide in this section a counterexample to both conjectures.

We consider a nilmanifold $M = G/\Gamma$ with underlying real Lie algebra isomorphic to $\mathfrak{h}_4$. Recall that by Proposition 5.1.15 the abelian complex structure J_0 on M does not admit sG metrics. Thus, it is sufficient to deform holomorphically J_0 in such a way that J_t admits balanced metric for any $t \neq 0$, taking into account our existence result of balanced metrics summarized in Table 5.1.

Theorem 5.2.1. *Let $(M = G/\Gamma, J_1)$ be a nilmanifold with underlying Lie algebra $\mathfrak{h}_4$ endowed with its abelian complex structure J_1. Then, there is a holomorphic family of compact complex manifolds $\{M_a := (M, I_a)\}_{a\in\Delta}$, with $I_0 = J_1$ and $\Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, such that M_a has a balanced metric for each $a \in \Delta\backslash\{0\}$.*

Proof. Let us consider the holomorphic deformation $\{J_a\}_{a\in\Delta}$ of the pair $(\mathfrak{h}_4, J_1)$ given in Corollary 4.3.3. Recall that J_a is described by a $(1,0)$-basis satisfying the complex structure equations (4.20):

$$d\mu_a^1 = d\mu_a^2 = 0, \quad d\mu_a^3 = \frac{1}{2(1-|a|^2)}\left(2\bar{a}\mu_a^{12} + i\mu_a^{1\bar{1}} + \mu_a^{1\bar{2}} + \mu_a^{2\bar{1}} - i|a|^2\mu_a^{2\bar{2}}\right).$$

By Corollary 5.1.10, if $a = 0$ then the complex nilmanifold (M, J_1) does not admit sG metrics because J_1 is abelian and J_1 does not have compatible balanced metrics (see Table 5.1).

For any $a \in \mathbb{C}$ such that $0 < |a| < 1$ the complex structure is nilpotent but not abelian. In this case we can normalize the coefficient of μ_a^{12} by taking $\frac{1-|a|^2}{\bar{a}}\mu_a^3$ instead of μ_a^3, so we can suppose that the complex structure equations are

$$d\mu_a^1 = d\mu_a^2 = 0, \quad d\mu_a^3 = \mu_a^{12} + \frac{i}{2\bar{a}}\mu_a^{1\bar{1}} + \frac{1}{2\bar{a}}(\mu_a^{1\bar{2}} + \mu_a^{2\bar{1}}) - \frac{ia}{2}\mu_a^{2\bar{2}}.$$

With respect to the (1,0)-basis $\{\omega_a^1 = \mu_a^1 - i\mu_a^2, \omega_a^2 = -2\bar{a}i\,\mu_a^2, \omega_a^3 = -2\bar{a}i\,\mu_a^3\}$, the structure equations for J_a become

$$d\omega_a^1 = d\omega_a^2 = 0, \quad d\omega_a^3 = \omega_a^{12} + \omega_a^{1\bar{1}} - \frac{1}{a}\,\omega_a^{1\bar{2}} + \frac{1-|a|^2}{4|a|^2}\,\omega_a^{2\bar{2}}.$$

Now, as in the proof of Proposition 3.1.5 we can suppose that the coefficient of $\omega_a^{1\bar{2}}$ is equal to $1/|a|$.

In conclusion, for any $a \in \mathbb{C}$ such that $0 < |a| < 1$ there exists a (1,0)-basis for which the complex equations are of the form (5.13), with $\lambda = \frac{1}{|a|}$ and $D = \frac{1-|a|^2}{4|a|^2}$. Taking $x = \mathfrak{Re}\, D = \frac{1-|t|^2}{4|t|^2}$ and $y = \mathfrak{Im}\, D = 0$, one has $4x + \rho - \lambda^2 = 0$ according to Proposition 3.1.5 (ii.2). Now, following the proof of Proposition 5.1.15, since $\lambda \neq 0$ the complex structure J_t admits a balanced metric if and only if (5.15) is satisfied. But the latter condition reads

$$\lambda^2(\lambda^2 - 4x) = \frac{1}{|a|^2} > 0,$$

so there exists a balanced Hermitian metric for each $a \in \mathbb{C}$ such that $0 < |a| < 1$. □

Remark 5.2.2. It is worth giving a closer look at the failure of the sG property at $a = 0$. Let $\{J_a\}_{a\in\Delta}$ be the family of complex structures given by (4.20) for any $a \in \Delta = \{a \in \mathbb{C} \mid |a| < 1\}$, and let us consider the real 2-form F compatible with J_a given by

$$2F = ir^2\, \mu^{1\bar{1}} + is^2\, \mu^{2\bar{2}} + it^2\, \mu^{3\bar{3}},$$

where $r, s, t \in \mathbb{R}$. Since

$$4F \wedge dF = \frac{it^2}{2(1-|a|^2)}(s^2 - |a|^2 r^2)(\mu^{12\bar{1}\bar{2}\bar{3}} - \mu^{123\bar{1}\bar{2}}),$$

the 4-form F^2 is closed if and only if $s^2 = |a|^2 r^2$, i.e. if and only if F is given by

$$2F = ir^2\, \mu^{1\bar{1}} + i|a|^2 r^2\, \mu^{2\bar{2}} + it^2\, \mu^{3\bar{3}}.$$

This defines a balanced J_a-Hermitian metric for any $r, t > 0$ and for any $0 < |a| < 1$. However, in the "central limit" $a = 0$ the form becomes degenerate, that is, the underlying metric is not positive definite.

Although the sG property is not closed, Popovici has proved that the existence of sG metrics in the central limit is guaranteed under strong additional conditions concerning the $\partial\bar{\partial}$-lemma.

Proposition 5.2.3 (Popovici [72, Proposition 4.1])**.** *If the $\partial\bar{\partial}$-lemma holds on (M, J_a) for every $a \in \Delta \setminus \{0\}$, then (M, J_0) has an sG metric.*

An interesting problem is if the conclusion in the above proposition holds under weaker conditions than the $\partial\bar{\partial}$-lemma. Latorre, Ugarte and Villacampa [57, Corollary 4.5] prove that the vanishing of some complex invariants, which are closely related to the $\partial\bar{\partial}$-lemma, is not sufficient to ensure the existence of an sG metric in the central limit.

Another problem related to Proposition 5.2.3 is if the central limit admits a Hermitian metric, stronger than sG, under the $\partial\bar{\partial}$-lemma condition. Our aim is now to construct a holomorphic family of compact complex manifolds $(M, J_t)_{t\in\Delta}$ such that (M, J_t) satisfies the $\partial\bar{\partial}$-lemma and admits balanced metric for any $t \neq 0$, but the central limit neither satisfies the $\partial\bar{\partial}$-lemma nor admits balanced metric. The construction is based on the balanced Hermitian geometry of $\mathfrak{g}_8$ studied in Theorem 5.1.8, which is the real Lie algebra underlying the Nakamura manifold.

We recall that the $\partial\bar\partial$-lemma property is open and non-closed under holomorphic deformations. The non-closedness of the $\partial\bar\partial$-lemma property is proved by Angella and Kasuya [8] by means of a suitable deformation (M, I_t) of the holomorphically parallelizable Nakamura manifold (M, I_0) (notice that (M, I_0) has balanced metrics). We will use their result on the $\partial\bar\partial$-lemma for (M, I_t), $t \neq 0$, as a key ingredient in the proof of the following result.

Theorem 5.2.4. *There exists a solvmanifold M with a holomorphic family of complex structures J_a, $a \in \Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, such that (M, J_a) satisfies the $\partial\bar\partial$-lemma and admits balanced metric for any $a \neq 0$, but the central limit (M, J_0) neither satisfies the $\partial\bar\partial$-lemma nor admits balanced metrics.*

Proof. Let J' be the complex structure on the Lie algebra $\mathfrak{g}_8$ defined by the $(1,0)$-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying (3.31) in Proposition 3.2.7. By Theorem 5.1.8, any complex solvmanifold $(M = G/\Gamma, J')$ with underlying Lie algebra $\mathfrak{g}_8$ does not admit balanced metrics.

For each $a \in \mathbb{C}$ such that $|a| < 1$, we consider the complex structure J_a on M defined by the (1,0)-basis

$$\Phi_a^1 := \omega^1, \quad \Phi_a^2 := \omega^2, \quad \Phi_a^3 := \omega^3 + a\,\omega^{\bar 3}.$$

A simple computation shows that the complex structure equations are

$$\left\{\begin{array}{l} d\Phi_a^1 = \frac{2i}{1-|a|^2}\Phi_a^{13} - \frac{2ia}{1-|a|^2}\Phi_a^{1\bar 3} + \frac{1}{1-|a|^2}\Phi_a^{3\bar 3}, \\ d\Phi_a^2 = -\frac{2i}{1-|a|^2}\Phi_a^{23} + \frac{2ia}{1-|a|^2}\Phi_a^{2\bar 3}, \\ d\Phi_a^3 = 0. \end{array}\right. \tag{5.16}$$

Using these equations, the (2,3)-form $\bar\partial F^2$ for a generic metric (5.1) with respect to the basis $\{\Phi_a^1, \Phi_a^2, \Phi_a^3\}$ reads as

$$2\,\bar\partial F^2 = \left(\frac{2ia(ir^2\bar v - u\bar z)}{1-|t|^2}\right)\Phi_a^{13\bar 1\bar 2\bar 3} + \left(\frac{2ia(is^2\bar z + \bar u\bar v)}{1-|a|^2} + (r^2s^2 - |u|^2)\right)\Phi_a^{23\bar 1\bar 2\bar 3}.$$

Suppose that $a \neq 0$ with $|a| < 1$. If $u = v = 0$, then the balanced condition reduces to solve

$$\frac{2a\bar z}{1-|t|^2} = r^2, \qquad \text{with } r^2t^2 > |z|^2.$$

Thus, taking $z = (1-|a|^2)r^2/(2\bar a)$, the condition $r^2t^2 > |z|^2$ is satisfied for any t such that $t^2 > \frac{(1-|a|^2)^2r^2}{4|a|^2}$.

Therefore, we have proved that for any J_a, $a \in \Delta^*$, the structures

$$2\,F = i\,(r^2\Phi_a^{1\bar 1} + s^2\Phi_a^{2\bar 2} + t^2\Phi_a^{3\bar 3}) + \frac{(1-|a|^2)r^2}{2\bar a}\Phi_a^{1\bar 3} - \frac{(1-|a|^2)r^2}{2a}\Phi_a^{3\bar 1}, \tag{5.17}$$

with $r, s \neq 0$ and $t^2 > \frac{(1-|a|^2)^2 r^2}{4|a|^2}$, are balanced.

Notice that the previous argument is valid for the quotient M of any lattice in the simply-connected Lie group G associated to $\mathfrak{g} = \mathfrak{g}_8$. However, to ensure the $\partial\bar{\partial}$-lemma for the complex structures J_a with $a \neq 0$ we need to consider the lattice Γ considered by Angella and Kasuya in [8]. In fact, in [8] the authors consider the holomorphically parallelizable Nakamura manifold $X = (G/\Gamma, I_0)$, whose complex structure I_0 precisely corresponds to the complex structure J_{-i} in our family (3.33) (see Proposition 3.2.7). If $\{\eta^1, \eta^2, \eta^3\}$ is a $(1,0)$-basis satisfying (3.33) for $A = -i$, they consider a invariant small deformation I_t given by the direction

$$\Psi_t := t\frac{\partial}{\partial z_3} \otimes d\bar{z}_3 \in H^{0,1}(M; \Theta_M), \tag{5.18}$$

where z_3 is a complex coordinate such that $\eta^3 = dz_3$. By [8, Proposition 4.1] one has that $X_t = (G/\Gamma_8, I_t)$ satisfies the $\partial\bar{\partial}$-lemma for any $t \neq 0$. Since $I_0 = J_{-i}$, the deformation I_t defined by (5.18) can be expressed in terms of the following (1,0)-basis

$$\Upsilon_t^1 := \eta^1, \quad \Upsilon_t^2 := \eta^2, \quad \Upsilon_t^3 := \eta^3 - t\,\eta^{\bar{3}},$$

and the structure equations for I_t are

$$\begin{cases} d\Upsilon_t^1 = \frac{2i}{1-|t|^2}\Upsilon_t^{13} + \frac{2it}{1-|t|^2}\Upsilon_t^{1\bar{3}}, \\ d\Upsilon_t^2 = -\frac{2i}{1-|t|^2}\Upsilon_t^{23} - \frac{2it}{1-|t|^2}\Upsilon_t^{2\bar{3}}, \\ d\Upsilon_t^3 = 0. \end{cases} \tag{5.19}$$

On the other hand, it is easy to see that for any $a \neq 0$ the equations (5.16) express with respect to the (1,0)-basis $\{\Theta_a^1 = \Phi_a^1 + \frac{i}{2a}\Phi_a^3, \Theta_a^2 = \Phi_a^2, \Theta_a^3 = \Phi_a^3\}$ as

$$\begin{cases} d\Theta_a^1 = \frac{2i}{1-|a|^2}\Theta_a^{13} - \frac{2ia}{1-|a|^2}\Theta_a^{1\bar{3}}, \\ d\Theta_a^2 = -\frac{2i}{1-|a|^2}\Theta_a^{23} + \frac{2ia}{1-|a|^2}\Theta_a^{2\bar{3}}, \\ d\Theta_a^3 = 0. \end{cases} \tag{5.20}$$

Now, from (5.19) and (5.20) we conclude that for $a \neq 0$ the complex structure J_a is precisely the complex structure I_t with $t = -a$. Therefore, for any $a \neq 0$ the compact complex manifold $(M, J_a) = (G/\Gamma, J_a)$ satisfies the $\partial\bar{\partial}$-lemma because $X_{t=-a}$ does by [8, Proposition 4.1].

Finally, by Corollary 4.2.8, the central limit of $\{J_a\}_{a\in\Delta}$ or $\{I_t\}_{t\in\Delta}$ satisfies $E_1(M) \ncong E_\infty(M)$. Hence, the $\partial\bar{\partial}$-lemma does not hold. □

Remark 5.2.5. Notice that (5.17) defines balanced metrics for any $a \neq 0$ in the complex deformation (M, J_a). However, the central limit of any metric (5.17) does not exist; actually, J' does not admit any balanced metric by Theorem 5.1.8.

We can construct another deformation where the central limit is the complex structure J'' given by (3.32). It turns out that this deformation has the same behaviour as for the deformation of the complex structure J given by (3.31) constructed in Theorem 5.2.4. Therefore, the complex structures J' and J'' given by (3.31) and (3.32), respectively, are the central limits of complex structures that satisfy the $\partial\bar{\partial}$-lemma. Notice that this is consistent with Proposition 5.2.3, because by our Theorem 5.1.13 both complex structures admit sG metric.

Lie algebra	Complex structure	sign of $\gamma_1(\mathbf{F})$	SKT metrics	Balanced metrics	sG metrics	G metrics
$\mathfrak{g}_1$	$J := (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), 0)$	> 0 for any (J, F)	$\nexists$	$\exists$ for any J	any (J, F)	any (J, F)
$\mathfrak{g}_2^0$	$J := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), 0)$	≥ 0 for any (J, F)	$\exists$ for any J	$\exists$ for any J	any (J, F)	any (J, F)
$\mathfrak{g}_2^\alpha$ with $\alpha > 0$	$J^\pm := ((\pm\cos\theta + i\sin\theta)(\omega^{13} + \omega^{1\bar{3}}), (\mp\cos\theta - i\sin\theta)(\omega^{23} + \omega^{2\bar{3}}), 0)$ $\theta \in (0, \frac{\pi}{2}),\ \alpha = \lvert\frac{\cos\theta}{\sin\theta}\rvert$	> 0 for any (J, F)	$\nexists$	$\exists$ for any J	any (J, F)	any (J, F)
$\mathfrak{g}_3$	$J^g := (0, -\frac{1}{2}\omega^{13} - (\frac{1}{2} + ig)\omega^{1\bar{3}} + ig\omega^{3\bar{1}}, \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4g})\omega^{1\bar{2}} + \frac{i}{4g}\omega^{2\bar{1}})$ $g \in \mathbb{R},\ g > 0$	> 0 for any (J, F)	$\nexists$	$\exists$ for any J	any (J, F)	any (J, F)
$\mathfrak{g}_4$	$J^\pm := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm\omega^{1\bar{1}})$	≥ 0 for any (J, F)	$\exists$ for any J	$\nexists$	$\nexists$	any (J, F)
$\mathfrak{g}_5$	$J := (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{2}} + \omega^{2\bar{1}})$	> 0 for any (J, F)	$\nexists$	$\exists$ for any J	$\exists$ for any J	any (J, F)
$\mathfrak{g}_6$	$J := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{1}} + \omega^{2\bar{2}})$	$\exists F$ s.t. $\begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$	$\nexists$	$\nexists$	$\nexists$	any (J, F)
$\mathfrak{g}_7$	$J^\pm := (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}}))$	> 0 for any (J, F)	$\nexists$	$\exists$ for any J	$\exists$ for any J	any (J, F)
$\mathfrak{g}_8$	$J_1^A := (-(A - i)\omega^{13} - (A + i)\omega^{1\bar{3}}, (A - i)\omega^{23} + (A + i)\omega^{2\bar{3}}, 0)$, $J' := (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23}, 0)$, $J'' = (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23} + \omega^{3\bar{3}}, 0)$	> 0 for any (J, F)	$\nexists$	$\exists$ for any $J \neq J', J''$	any (J, F)	any (J, F)
$\mathfrak{g}_9$	$J := (-\omega^{3\bar{3}}, \frac{1}{2}(i\omega^{12} - i\omega^{2\bar{1}} + \omega^{1\bar{3}}), \frac{i}{2}(-\omega^{13} + \omega^{3\bar{1}}))$	> 0 for any (J, F)	$\nexists$	$\nexists$	$\nexists$	any (J, F)

Table 5.2: Special Hermitian metrics on six-dimensional unimodular solvable Lie algebras with a complex structure with non-zero closed complex volume form $\Psi \in \wedge^{3,0}\mathfrak{g}^*$.

Appendices

Appendix A
Complex structures on solvable Lie algebras

In this section we show several results related to the description of the invariant complex geometry with holomorphically trivial canonical bundle on six-dimensional solvmanifolds. We show the specific procedure followed with complex structures on the Lie algebras $\mathfrak{g}_3$ and $\mathfrak{g}_9$ to arrive to the expressions (3.20), in the case of $\mathfrak{g}_3$, and (3.35), in the case of $\mathfrak{g}_9$. These equations are the starting point to obtain the classification up to equivalence of complex structures with closed complex volume form for these Lie algebras, summarized in the expressions (3.21) and (3.36), respectively. In both cases, we have proceeded following in the next steps:

i) We define an orientation on the vector space underlying the Lie algebra fixing the volume form $\nu := e^{123456}$, where $\{e^1, \ldots, e^6\}$ denotes the basis of 1-forms in which the Lie algebra is expressed. Then, we compute the space of closed 3-forms denoted by $Z^3(\mathfrak{g})$ and, taking an arbitrary $\rho \in Z^3(\mathfrak{g})$, we obtain the endomorphism $\tilde{J}^*_\rho : \mathfrak{g}^* \to \mathfrak{g}^*$ defined by the formula (2.20). At this point, we focus our attention in the real 4-form $d(\tilde{J}^*_\rho \rho)$ and the real number $\frac{1}{6}tr(\tilde{J}^{*2}_\rho)$ and using Lemma 2.2.8 we look for all the closed 3-forms $\rho \in Z^3(\mathfrak{g})$ satisfying both the conditions $d(\tilde{J}^*_\rho \rho) = 0$ and $\frac{1}{6}tr(\tilde{J}^{*2}_\rho) < 0$.

ii) Once we have computed the space of integrable complex structures admitting a (3,0)-form $\{J : \mathfrak{g}^* \to \mathfrak{g}^* \mid J^2 = -\mathrm{Id}_\mathfrak{g},\ d(\wedge^{3,0}\mathfrak{g}^*) = 0\}$ we look for a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ on $\mathfrak{g}^*_\mathbb{C}$ and compute the complex structure equations in this basis.

iii) We conclude finding new $(1,0)$-bases obtained by means of successive Lie algebra automorphisms compatible with the complex structure equations until we arrive to the final reduced expressions of the complex structure equations.

The Lie algebra $\mathfrak{g}_3$.

The Lie algebra $\mathfrak{g}_3 := \mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$ is the decomposable Lie algebra described by a basis $\{e^1, \ldots, e^6\}$ of $\mathfrak{g}_3^*$ satisfying the structure equations $de^1 = de^4 = 0$, $de^2 = -e^{13}$, $de^3 = e^{12}$, $de^5 = -e^{46}$, $de^6 = -e^{45}$. We provide several results in order to arrive to the complex structure equations (3.20).

Lemma A.0.6. *There are almost complex structures admitting a closed complex volume form on the Lie algebra* $\mathfrak{g}_3$.

Proof. In the proof we look for all the complex structures admitting a closed complex volume form. Any closed 3-form $\rho \in Z^3(\mathfrak{g}_3)$ is given by:

$$\begin{aligned}\rho \;=\;& a_1e^{123} + a_2e^{124} + a_3e^{134} + a_4e^{145} + a_5e^{146} + a_6e^{156} + a_7e^{234} + a_8(e^{136} - e^{245}) + \\ & a_9(e^{135} - e^{246}) + a_{10}(e^{126} + e^{345}) + a_{11}(e^{125} + e^{346}) + a_{12}e^{456},\end{aligned}$$

where $a_1, \ldots, a_{12} \in \mathbb{R}$. When we compute the exterior derivative of $\tilde{J}^*_\rho$ we obtain:

$$\begin{aligned}\tfrac{1}{2}d(\tilde{J}^*_\rho\rho) \;=\;& \\ &(a_{10}^3 - a_{10}a_{11}^2 - a_1a_{10}a_{12} - a_{10}a_6a_7 + a_{10}a_8^2 - a_1a_6a_9 + a_{12}a_7a_9 - 2a_{11}a_8a_9 + a_{10}a_9^2)e^{1245} + \\ &(a_{10}^2a_{11} - a_{11}^3 - a_1a_{11}a_{12} - a_{11}a_6a_7 - a_1a_6a_8 + a_{12}a_7a_8 - a_{11}a_8^2 + 2a_{10}a_8a_9 - a_{11}a_9^2)e^{1246} - \\ &(a_1a_{11}a_6 - a_{11}a_{12}a_7 + a_{10}^2a_8 + a_{11}^2a_8 - a_1a_{12}a_8 - a_6a_7a_8 + a_8^3 - 2a_{10}a_{11}a_9 - a_8a_9^2)e^{1345} - \\ &(a_1a_{10}a_6 - a_{10}a_{12}a_7 + 2a_{10}a_{11}a_8 - a_{10}^2a_9 - a_{11}^2a_9 - a_1a_{12}a_9 - a_6a_7a_9 + a_8^2a_9 - a_9^3)e^{1346},\end{aligned}$$

whereas the value of the trace is:

$$\begin{aligned}\tfrac{1}{6}tr(\tilde{J}^{*2}_\rho) =\;& a_{10}^4 + a_{11}^4 + a_1^2a_{12}^2 - 2a_1a_{12}a_6a_7 + a_6^2a_7^2 + 4a_{11}(a_1a_6 - a_{12}a_7)a_8 - 2a_1a_{12}a_8^2 - \\ & 2a_6a_7a_8^2 + a_8^4 - 4a_{10}(a_1a_6 - a_{12}a_7 + 2a_{11}a_8)a_9 + 2a_1a_{12}a_9^2 + 2a_6a_7a_9^2 - 2a_8^2a_9^2 + a_9^4 - \\ & 2a_{10}^2(a_{11}^2 + a_1a_{12} + a_6a_7 - a_8^2 - a_9^2) + 2a_{11}^2(a_1a_{12} + a_6a_7 + a_8^2 + a_9^2).\end{aligned}$$

Now, we have that $d(\tilde{J}^*_\rho\rho) = 0$ and $tr(\tilde{J}^{*2}_\rho) < 0$ if and only if the coefficients $a_1, \ldots, a_{12}$ belong to one of the following four families:

I) $a_{10}a_{12} + a_6a_9 \neq 0$ and $a_1 = \frac{(a_{10}^3 - a_{10}a_{11}^2 - a_{10}a_6a_7 + a_{10}a_8^2 + a_{12}a_7a_9 - 2a_{11}a_8a_9 + a_{10}a_9^2)}{a_{10}a_{12} + a_6a_9}$, $a_7 = \frac{(a_{10}^2a_6 - a_{11}^2a_6 + 2a_{11}a_{12}a_8 + a_6a_8^2 - 2a_{10}a_{12}a_9 - a_6a_9^2)}{a_6^2 + a_{12}^2}$. In particular, if $a_8 = \frac{1}{2}$, $a_{10} = a_{12} = 0$ and $a_{11} = 1$ then $tr(\tilde{J}^{*2}_\rho) = -1$;

II) $a_{11}a_6 - a_{12}a_8 \neq 0$, $a_{12} \neq 0$ and $a_9 = a_{10} = 0$, $a_1 = \frac{-(a_{11}^2a_{12} + 2a_{11}a_6a_8 - a_{12}a_8^2)}{a_6^2 + a_{12}^2}$, $a_7 = \frac{(-a_{11}^3 - a_1a_{11}a_{12} - a_1a_6a_8 - a_{11}a_8^2)}{a_{11}a_6 - a_{12}a_8}$. In particular if $a_6 = 0$, $a_8 = \frac{1}{2}$ and $a_{11} = a_{12} = 1$ then $tr(\tilde{J}^{*2}_\rho) = -1$;

III) $a_9 = a_{12} = 0$, $a_6, a_{10} \neq 0$ and $a_1 = -\frac{2a_{11}a_8}{a_6}$, $a_7 = \frac{(a_{10}^2 - a_{11}^2 + a_8^2)}{a_6}$. If $a_8 = \frac{1}{2}$ and $a_{11} = 1$ is clear that $tr(\tilde{J}^{*2}_\rho) = -1$;

IV) $a_{12} = 0$, $a_6 \neq 0$, $a_{10} = 0$, $a_{11} \neq 0$, $a_8^{\pm} = \frac{-a_1a_6 \pm \sqrt{-4a_{11}^4 + a_1^2a_6^2 - 4a_{11}^2a_6a_7}}{2a_{11}}$, $a_7 = \frac{-4a_{11}^4 + a_1^2a_6^2}{4a_{11}^2a_6}$. If $a_1 = a_6 = 1$ then $tr(\tilde{J}^{*2}_\rho) = -1$. □

Lemma A.0.7. *Let $J : \mathfrak{g}_3 \to \mathfrak{g}_3$ be a complex structure admitting a closed complex volume form, then there is a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations:*

$$\text{(A.1)} \qquad \begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} + b\,\omega^{1\bar{1}} + fi\,\omega^{1\bar{2}} - fi\,\omega^{2\bar{1}} - (\frac{1}{2} + gi)\omega^{1\bar{3}} + gi\,\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + c\,\omega^{1\bar{1}} + (\frac{1}{2} + hi)\omega^{1\bar{2}} - hi\,\omega^{2\bar{1}} - fi\,\omega^{1\bar{3}} + fi\,\omega^{3\bar{1}}, \end{cases}$$

where the coefficients b, c, f, g, h are real and satisfy $4gh = 4f^2 - 1$.

Proof. Firstly, we prove that the (1,0)-forms:

$$\omega^1 = e^1 - iJ_\rho^* e^1, \quad \omega^2 = e^2 - iJ_\rho^* e^2, \quad \omega^3 = e^3 - iJ_\rho^* e^3$$

are linearly independent for any $J : \mathfrak{g}_3 \to \mathfrak{g}_3$ admitting a closed (3,0)-form. When we compute ω^{123} expressed as (complex) linear combination of the basis $\{e^{123}, \ldots, e^{456}\}$ and we get the coefficient multiplying the element e^{456} (namely the contraction $\iota_{e_6}\iota_{e_5}\iota_{e_4}(\omega^{123})$) we obtain:

- for family I: $\iota_{e_6}\iota_{e_5}\iota_{e_4}(\omega^{123}) = -2i(a_{12} - ia_6)^2$, which never is cancelled as $a_6^2 + a_{12}^2 \neq 0$.
- for family II: $\iota_{e_6}\iota_{e_5}\iota_{e_4}(\omega^{123}) = 2i(a_{12}^2 + a_6^2)$, which never is cancelled as $a_{12} \neq 0$.
- for family III: $\iota_{e_6}\iota_{e_5}\iota_{e_4}(\omega^{123}) = -2ia_6^2$, which never is cancelled as $a_6 \neq 0$.
- for family IV: $\iota_{e_6}\iota_{e_5}\iota_{e_4}(\omega^{123}) = -\frac{2i}{a_1^2}$, which never is cancelled.

Now, we show the complex equations obtained for the family III. For instance, substituting the values obtained for family III (see the proof of Lemma A.0.6) we get $tr(\tilde{J}_\rho^{*2}) = -4(a_8 a_{11})^2$. If we substitute a_8 by $\frac{1}{2a_{11}}$ we get a whole family of integrable complex structures. Moreover, the complex structure equations for all them expressed in terms of the (1,0)-basis $\{\omega^1 = e^1 - iJ_\rho^* e^1, \omega^2 = e^2 - iJ_\rho^* e^2, \omega^3 = e^3 - iJ_\rho^* e^3\}$ are:

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\frac{1}{2}\omega^{13} + b\omega^{1\bar{1}} + i\frac{a_{10}}{2a_{11}}(-\omega^{1\bar{2}} + \omega^{2\bar{1}}) - (\frac{1}{2} + i(a_{10}^2 - a_{11}^2))\omega^{1\bar{3}} + i(a_{10}^2 - a_{11}^2)\omega^{3\bar{1}}, \\ d\omega^3 = \frac{1}{2}\omega^{12} + c\omega^{1\bar{1}} + (\frac{1}{2} + \frac{i}{4a_{11}^2})\omega^{1\bar{2}} - \frac{i}{4a_{11}^2}\omega^{2\bar{1}} + i\frac{a_{10}}{2a_{11}}(\omega^{1\bar{3}} - \omega^{3\bar{1}}). \end{cases}$$

Comparing the former equations with (A.1) we observe for this family that $f = -\frac{a_{10}}{2a_{11}}$, $g = a_{10}^2 - a_{11}^2$, $h = \frac{4}{a_{11}^2}$ and the coefficients b and c are real (we omit the explicit expressions of b and c because of their length). We can identify the same structure of the complex structure equations for the other families I, II and IV identifying the coefficients b, c, f, g, h. Finally, if we consider a Lie algebra $\mathfrak{g}$ equipped with a complex structure and a (1,0)-basis satisfying the complex equations (A.1) then $d(\omega^{123}) = 0$ and the relation $4gh = 4f^2 - 1$ turns out from imposing the Jacobi identity. $\square$

The Lie algebra $\mathfrak{g}_9$.

The Lie algebra $\mathfrak{g}_9 := B^1_{6,4}$ is the indecomposable Lie algebra described by a basis $\{e^1, \ldots, e^6\}$ of $\mathfrak{g}_9^*$ satisfying the structure equations $de^1 = e^{45}$, $de^2 = e^{15} + e^{36}$, $de^3 = e^{14} - e^{26} + e^{56}$, $de^4 = -e^{56}$, $de^5 = e^{46}$, $de^6 = 0$.

Remark A.0.8. In the lists of the six-dimensional indecomposable solvable Lie algebras [37] we find that the Lie algebras $B^{\pm 1}_{6,4} := (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} \pm e^{56}, -e^{56}, e^{46}, 0)$ admit an almost complex structure with a closed $(3,0)$-form. Actually, $B^{-1}_{6,4}$ and $B^1_{6,4}$ are isomorphic. If $\{\tilde{e}^1, \ldots, \tilde{e}^6\}$ is a basis of $B^{-1}_{6,4}$ then the basis $\{e^1, \ldots, e^6\}$ given by $e^1 = -\tilde{e}^1$, $e^6 = -\tilde{e}^6$ and $e^j = \tilde{e}^j$ for $j = 2, 3, 4, 5$ yields the Lie algebra $B^1_{6,4}$.

We provide several results in order to arrive to the complex structure equations (3.35).

Lemma A.0.9. *There are almost complex structures admitting a closed complex volume form on the Lie algebra $\mathfrak{g}_9$.*

Proof. As we have done with the Lie algebra $\mathfrak{g}_3$ in the proof of Lemma A.0.6 we aim to cover all the complex structures admitting a closed complex volume form. Any closed 3-form $\rho \in Z^3(\mathfrak{g}_9)$ is given by:

$$\rho = a_1(e^{124} - e^{135}) + a_2 e^{145} + a_3 e^{146} + a_4 e^{156} + a_5(e^{136} - e^{245}) + a_6(e^{125} + e^{134} - e^{246}) + a_7 e^{256} + a_8(e^{126} + e^{345}) + a_9 e^{346} + a_{10}(e^{125} + e^{134} + e^{356}) + a_{11} e^{456},$$

where $a_1, \ldots, a_{11} \in \mathbb{R}$. When we compute the exterior derivative of $\tilde{J}^*_\rho$ we obtain:

$$\tfrac{1}{2} d(\tilde{J}^*_\rho \rho) = -(-a_{10}a_5^2 + a_5^2 a_6 + a_1^2 a_7 + a_{10}^2 a_7 - 2a_{10}a_6 a_7 + a_6^2 a_7 - 2a_1 a_5 a_8 + a_{10}a_8^2 - a_6 a_8^2 - a_1^2 a_9 - a_{10}^2 a_9 + 2a_{10}a_6 a_9 - a_6^2 a_9)e^{1246} + (a_1^2 a_{10} + a_{10}^3 + a_1 a_5^2 + a_1^2 a_6 - a_{10}^2 a_6 - a_{10}a_6^2 + a_6^3 - 2a_{10}a_5 a_8 + 2a_5 a_6 a_8 - a_1 a_8^2)e^{1256} - (a_1^2 a_{10} + a_{10}^3 + a_1 a_5^2 + a_1^2 a_6 - a_{10}^2 a_6 - a_{10}a_6^2 + a_6^3 - 2a_{10}a_5 a_8 + 2a_5 a_6 a_8 - a_1 a_8^2)e^{1346} - (-a_{10}a_5^2 + a_5^2 a_6 + a_1^2 a_7 + a_{10}^2 a_7 - 2a_{10}a_6 a_7 + a_6^2 a_7 - 2a_1 a_5 a_8 + a_{10}a_8^2 - a_6 a_8^2 - a_1^2 a_9 - a_{10}^2 a_9 + 2a_{10}a_6 a_9 - a_6^2 a_9)e^{1356} + (-a_1 a_{10}^2 - a_5^2 a_6 - a_1 a_6^2 - a_{10}^2 a_7 + 2a_{10}a_6 a_7 - a_6^2 a_7 + a_1 a_5 a_8 + a_5 a_7 a_8 + a_6 a_8^2 + a_1^2 a_9 + 2a_1 a_7 a_9 - a_5 a_8 a_9)e^{1456} + (a_{10}^2 a_5 - a_5 a_6^2 + a_1 a_5 a_7 + a_1 a_{10} a_8 - a_5^2 a_8 + a_1 a_6 a_8 - a_{10} a_7 a_8 + a_6 a_7 a_8 - a_8^3 - a_1 a_5 a_9 + a_{10} a_8 a_9 - a_6 a_8 a_9)e^{2456} + (a_1 a_{10} a_5 + a_5^3 + a_1 a_5 a_6 - a_{10} a_5 a_7 + a_5 a_6 a_7 - a_{10}^2 a_8 + a_6^2 a_8 - a_1 a_7 a_8 + a_5 a_8^2 + a_{10} a_5 a_9 - a_5 a_6 a_9 + a_1 a_8 a_9)e^{3456},$$

and the corresponding trace is:

$$\tfrac{1}{6}tr(\tilde{J}_\rho^{*2}) = a_5^4 + 4a_{10}^3a_6 - 8a_{10}^2a_6^2 + 4a_{10}a_6^3 + a_{10}^2a_7^2 - 2a_{10}a_6a_7^2 + a_6^2a_7^2 + 4a_5(-a_{10}^2 + a_6^2)a_8 +$$
$$2a_{10}a_7a_8^2 - 2a_6a_7a_8^2 + a_8^4 - 2a_{10}^2a_7a_9 + 4a_{10}a_6a_7a_9 - 2a_6^2a_7a_9 - 2a_{10}a_8^2a_9 + 2a_6a_8^2a_9 + a_{10}^2a_9^2 -$$
$$2a_{10}a_6a_9^2 + a_6^2a_9^2 + a_1^2(a_{10}^2 + 2a_{10}a_6 + a_6^2 - 4a_7a_9) + 2a_5^2(a_8^2 + a_6(a_7 - a_9) + a_{10}(-a_7 + a_9)) +$$
$$2a_1(a_5^2a_6 + 2a_5a_8(-a_7 + a_9) + a_{10}^2(a_7 + a_9) + a_{10}(a_5^2 - a_8^2 - 2a_6(a_7 + a_9)) +$$
$$a_6(-a_8^2 + a_6(a_7 + a_9)).$$

A straightforward computation shows that the coefficient in $d(\tilde{J}_\rho^*\rho)$ multiplying e^{1246} vanishes if and only if $a_1 = 0$ and $a_{10} = a_6$ then we get $\mathrm{tr}(\tilde{J}_\rho^{*2}) = (a_5^2 + a_8^2)^2 \geq 0$. So we have necessarily that $a_1^2 + (a_{10} - a_6)^2 \neq 0$ and we can perform the substitution:

$$a_9 = \frac{(-a_{10}a_5^2 + a_5^2a_6 + a_1^2a_7 + a_{10}^2a_7 - 2a_{10}a_6a_7 + a_6^2a_7 - 2a_1a_5a_8 + a_{10}a_8^2 - a_6a_8^2)}{a_1^2 + (a_{10} - a_6)^2}.$$

We find that $d(\tilde{J}_\rho^*\rho) = 0$ and $\mathrm{tr}(\tilde{J}_\rho^{*2}) < 0$ if and only if the coefficients $a_1 \dots, a_{11}$ belong to one of the following families:

I) $a_1 = a_8 = 0$, $a_{10} = -a_6 \neq 0$, $a_9 = -a_7$ and $a_7 = -a_6a_5^2$. In particular, if $a_6 = \frac{1}{2}$ then $\mathrm{t}r(\tilde{J}_\rho^{*2}) = -1$;

II) $a_1 = 0$, $a_8 \neq 0$, $a_5 = \frac{1+2a_6(a_{10}-a_6)}{2a_8}, a_7 = \frac{4a_6+(a_{10}-a_6)(1+4(a_6^2-a_8^4))}{8a_8^2}, a_9 = -a_7$. In particular, if $a_{10} = a_6 + 1$ then $\mathrm{tr}(\tilde{J}_\rho^{*2}) = -1$;

III) $a_1 \neq 0$ and $a_7 = a_7^+$, $a_8 = a_8^+$. In particular, if $a_6 = \sqrt{1 - a_1^2} + a_{10}$ and $a_1 > 0$ then $\mathrm{t}r(\tilde{J}_\rho^{*2}) = -1$;

IV) $a_1 \neq 0$ and $a_7 = a_7^-$, $a_8 = a_8^+$. In particular, if $a_6 = \sqrt{1 - a_1^2} + a_{10}$ and $a_1 < 0$ then $\mathrm{t}r(\tilde{J}_\rho^{*2}) = -1$;

V) $a_1 \neq 0$ and $a_7 = a_7^+$, $a_8 = a_8^-$. In particular, if $a_6 = \sqrt{1 - a_1^2} + a_{10}$ and $a_1 < 0$ then $\mathrm{t}r(\tilde{J}_\rho^{*2}) = -1$;

VI) $a_1 \neq 0$ and $a_7 = a_7^-$, $a_8 = a_8^-$. In particular, if $a_6 = \sqrt{1 - a_1^2} + a_{10}$ and $a_1 > 0$ then $\mathrm{t}r(\tilde{J}_\rho^{*2}) = -1$,

where

$$a_8^\pm = \tfrac{1}{2a_1}\Big(- 2a_{10}a_5 + 2a_5a_6 \pm \big((-2a_{10}a_5 + 2a_5a_6)^2 + 4a_1(a_1^2a_{10} + a_{10}^3 + a_1a_5^2 +$$
$$a_1^2a_6 - a_{10}^2a_6 - a_{10}a_6^2 + a_6^3)\big)^{1/2}\Big),$$

and

$$
\begin{aligned}
a^{\epsilon}_{7_{\pm}} \;=\; & -\frac{1}{4a_1^4}\Big(a_1^5 + a_1^3 a_{10}^2 + 4a_1^2 a_{10} a_5^2 + 2a_1^3 a_{10} a_6 - 4a_1^2 a_5^2 a_6 - 3a_1^3 a_6^2 - \\
& \epsilon\Big[4a_1^2 a_5\sqrt{(a_1 a_{10} + a_5^2 + a_1 a_6)(a_1^2 + a_{10}^2 - 2a_{10}a_6 + a_6^2)} \pm \\
& \Big(\big(-a_1^5 - a_1^3 a_{10}^2 - 4a_1^2 a_{10} a_5^2 - 2a_1^3 a_{10} a_6 + 4a_1^2 a_5^2 a_6 + 3a_1^3 a_6^2 + \\
& 4a_1^2 a_5\sqrt{(a_1 a_{10} + a_5^2 + a_1 a_6)(a_1^2 + a_{10}^2 - 2a_{10}a_6 + a_6^2)}\big)^2 + \\
& 8a_1^4(-3a_1^3 a_{10} a_5^2 - 3a_1 a_{10}^3 a_5^2 - 2a_1^2 a_5^4 - 4a_{10}^2 a_5^4 - a_1^4 a_{10} a_6 - \\
& a_1^2 a_{10}^3 a_6 - a_1^3 a_5^2 a_6 + a_1 a_{10}^2 a_5^2 a_6 + 8a_{10} a_5^4 a_6 + a_1^4 a_6^2 + a_1^2 a_{10}^2 a_6^2 + \\
& 7a_1 a_{10} a_5^2 a_6^2 - 4a_5^4 a_6^2 + a_1^2 a_{10} a_6^3 - 5a_1 a_5^2 a_6^3 - a_1^2 a_6^4 + (a_1^3 a_5 + \\
& a_1 a_{10}^2 a_5 + 4a_{10} a_5^3 + 2a_1 a_{10} a_5 a_6 - 4a_5^3 a_6 - 3a_1 a_5 a_6^2) \\
& \sqrt{(a_1 a_{10} + a_5^2 + a_1 a_6)(a_1^2 + a_{10}^2 - 2a_{10}a_6 + a_6^2)}\Big)^{1/2}\Big]\Big),
\end{aligned}
$$

where $\epsilon = \pm 1$ and $a^{+}_{7_{\pm}}$ denotes the two solutions corresponding to a_8^{+} (analogously for $a^{-}_{7_{\pm}}$). □

Lemma A.0.10. *Let $J : \mathfrak{g}_9 \to \mathfrak{g}_9$ be a complex structure admitting a closed complex volume form, then there is a (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ satisfying the complex structure equations:*

$$
\text{(A.2)}\qquad
\begin{cases}
d\omega^1 = -c^2\omega^{1\bar{1}} - c\,\omega^{3\bar{1}} - c\,\omega^{1\bar{3}} - \omega^{3\bar{3}}, \\[4pt]
d\omega^2 = D\,\omega^{1\bar{1}} - \frac{i}{2}\omega^{2\bar{1}} + E\omega^{3\bar{1}} + F\omega^{1\bar{3}} + \frac{i}{2}\omega^{12} + H\omega^{13} + G\omega^{3\bar{3}}, \\[4pt]
d\omega^3 = cK\,\omega^{1\bar{1}} + K\omega^{3\bar{1}} + c^2\omega^{1\bar{3}} + c\,\omega^{3\bar{3}} - \frac{i}{2}\omega^{13},
\end{cases}
$$

where the coefficient c is real, the coefficients D, E, F, G, H, K are complex and $K = c^2 + \frac{i}{2}$, $F = cG + \frac{1}{2}$, $H = F - E - \frac{1}{2}$ and $G = -i(c(1 + 2E) - 2D)$.

Proof. Firstly, we show that the (1,0) forms

$$\omega^1 = e^6 - iJ^*_{\rho}e^6, \quad \omega^2 = e^2 - iJ^*_{\rho}e^2, \quad \omega^3 = e^4 - iJ^*_{\rho}e^4$$

are linearly independent for any $J : \mathfrak{g}_9 \to \mathfrak{g}_9$ admitting a closed complex volume form. Proceeding in the same manner as in the proof of Lemma A.0.7 we compute the 3-form ω^{123} for the twelve families obtained in Lemma A.0.9 and express it as a linear combination of the basis $\{e^{123}, \ldots, e^{456}\}$. It turns out that the coefficient multiplying e^{124} is $2i$ in all cases, namely $\iota_{e_4}\iota_{e_2}\iota_{e_1}(\omega^{123}) = 2i$, and therefore the (1,0)-forms $\omega^1, \omega^2, \omega^3$ are linearly independent constituting then a (1,0)-basis for $\mathfrak{g}^*_{9\mathbb{C}}$.

When we compute the complex structure equations for the family I expressed in the (1,0)-basis $\{\omega^1, \omega^2, \omega^3\}$ we get:

$$\begin{cases} d\omega^1 &= -a_5^2\omega^{1\bar{1}} - a_5\omega^{1\bar{3}} - a_5\omega^{3\bar{1}} - \omega^{3\bar{3}}, \\ d\omega^2 &= \frac{i}{2}\omega^{12} + \frac{1}{4}(-1 - 4a_{11} + 4a_4a_5 + 2a_2a_5^2 - i(2a_2 - a_5^2))\omega^{13} + \frac{1}{4}(-2ia_4 + \\ & \quad 2a_5 + 4a_{11}a_5 - ia_5^3 + 2a_2a_5^3)\omega^{1\bar{1}} + \frac{1}{2}(1 + 2a_4a_5 - ia_5^2 + 2a_2a_5^2)\omega^{1\bar{3}} \\ & \quad -\frac{i}{2}\omega^{2\bar{1}} + \frac{1}{4}(1 + 4a_{11} + 2a_2a_5^2 + i(2a_2 - a_5^2))\omega^{3\bar{1}} + \frac{1}{2}(2a_4 + 2a_2a_5 - ia_5)\omega^{3\bar{3}}, \\ d\omega^3 &= -\frac{i}{2}\omega^{13} - \frac{1}{2}ia_5(-1 + (1+i)a_5)(1 + (1+i)a_5)\omega^{1\bar{1}} + a_5^5\omega^{1\bar{3}} - \\ & \quad \frac{1}{2}ia_5(-1 + (1+i)a_5)(1 + (1+i)a_5)\omega^{3\bar{1}} + a_5\omega^{3\bar{3}}. \end{cases}$$

Comparing with the equations (A.2) we identify the coefficients

$$\begin{aligned} &D = \tfrac{1}{4}(-2ia_4 + 2a_5 + 4a_{11}a_5 - ia_5^3 + 2a_2a_5^3), && E = i(2a_2 - a_5^2), \\ &F = \tfrac{1}{2}(1 + 2a_4a_5 - ia_5^2 + 2a_2a_5^2), && G = \tfrac{1}{2}(2a_4 + 2a_2a_5 - ia_5), \\ &H = -\tfrac{i}{2}, && K = \tfrac{1}{2}ia_5(-1 + (1+i)a_5)(1 + (1+i)a_5), \end{aligned}$$

and $c = a_5$. It is direct to check that $F = cG + \frac{1}{2}$, $H = F - E - \frac{1}{2}$ and $G = -i(c(1 + 2E) - 2D)$ whereas the condition $K = c^2 + \frac{i}{2}$ arises from imposing $d(\omega^{123}) = 0$. □

Lemma A.0.11. *Let $\mathfrak{g}$ be a Lie algebra equipped with a complex structure satisfying the complex equations* (A.2) *then there is a (1,0)-basis satisfying the equations:*

$$\text{(A.3)} \qquad \begin{cases} d\omega^1 = -c^2\omega^{1\bar{1}} - c\,\omega^{3\bar{1}} - c\,\omega^{1\bar{3}} - \omega^{3\bar{3}}, \\ d\omega^2 = (E + \frac{1}{2})c\,\omega^{1\bar{1}} - \frac{i}{2}\omega^{2\bar{1}} + E\omega^{3\bar{1}} + \frac{1}{2}\omega^{1\bar{3}} + \frac{i}{2}\omega^{12} - E\omega^{13}, \\ d\omega^3 = (c^2 + \frac{i}{2})c\,\omega^{1\bar{1}} + (c^2 + \frac{i}{2})\omega^{3\bar{1}} + c^2\omega^{1\bar{3}} + c\,\omega^{3\bar{3}} - \frac{i}{2}\omega^{13}. \end{cases}$$

The coefficient c is real and $E \in \mathbb{C}$.

Proof. Let $\{\omega^1, \omega^2, \omega^3\}$ be a (1,0)-basis satisfying the complex structure equations (A.2). Consider the (1,0)-basis $\{\eta^1, \eta^2, \eta^3\}$ given by $\eta^1 = \omega^1$, $\eta^2 = \omega^2 + G\omega^1$ and $\eta^3 = \omega^3$. Then the complex structure equations in the basis are the same as (A.2) but with $G = 0$ and hence by the relations between the complex coefficients we have $F = \frac{1}{2}$ and $H = -E$. Now, after imposing the Jacobi identity we get that $D = c(E + \frac{1}{2})$. □

Appendix B

Tables of solvable Lie algebras and complex geometry

In this section we include the Lie algebras studied to obtain the classification Theorem 2.2.14 of six-dimensional unimodular (non-nilpotent) solvable Lie algebras admitting a complex structure with a closed complex volume form. As the existence of a lattice in the corresponding Lie group implies the unimodularity of $\mathfrak{g}$, that is $b_6(\mathfrak{g}) = 1$, and according to the obstruction given by Lemma 2.2.3, we have center our study in those solvable Lie algebras $\mathfrak{g}$ with $b_3(\mathfrak{g}) \geq 2$.

As regards the descomposable Lie algebras, we follow the lists and notations of low dimensional Lie algebras contained in [84] and [36]. The $3 \oplus 3$ case is the product of two three-dimensional unimodular solvable Lie algebras (notice that in this case, b_3 is always ≥ 2 by (2.21)). The $4 \oplus 2$ case is the product by $\mathbb{R}^2$ of a four-dimensional unimodular solvable Lie algebra $\mathfrak{h}$ with $b_1(\mathfrak{h})+2b_2(\mathfrak{h})+b_3(\mathfrak{h}) \geq 2$. Finally, the $5\oplus 1$ case is the product by $\mathbb{R}$ of a five-dimensional unimodular solvable Lie algebra $\mathfrak{h}$ with $b_2(\mathfrak{h})+b_3(\mathfrak{h}) \geq 2$. The results concerning the decomposable case are included in Table B.1.

Table B.2 contains the six-dimensional solvable unimodular indecomposable Lie algebras. The Lie algebras labeled by $N_{6,18}^{0,-1,-1}$ and $N_{6,20}^{-1,-1}$ arise from the classification by Turkowski [91], and they are the only six-dimensional unimodular solvable Lie algebras with nilradical of dimension 4 and $b_3 \geq 2$. The other solvable Lie algebras of Table B.2 are taken from the lists of [37]. We also include in Table B.2 the column "$\lambda(\rho) \geq 0$" in which the symbol ✓ means that any closed 3-form ρ on the Lie algebra satisfies $\lambda(\rho) \geq 0$, in particular, ρ does not give rise to an almost complex structure (a similar study was done in [36] for any decomposable Lie algebra).

Finally, we recall that the existence of a closed complex volume form is equivalent to the existence of a complex structure when $\mathfrak{g}$ is nilpotent (see Corollary 2.1.17). However, we decide to include them in these tables to underline that we recover with the method used in Section 2.2.2 of this work the classification of Salamon [82] of six-dimensional nilpotent Lie algebras admitting complex structure. We denote the 34 types of 6-dimensional nilpotent Lie algebras by $\mathfrak{n}_k$ with $k = 1, \ldots, 34$.

Lie algebra	**Structure equations**	**closed volume form**
$\mathfrak{n}_1 = \mathfrak{h}_1$	$(0,0,0,0,0,0)$	$(e^1 - ie^2) \wedge (e^3 - ie^4) \wedge (e^5 - ie^6)$
$\mathfrak{n}_2 = \mathfrak{h}_8$	$(0,0,0,0,0,12)$	$(e^1 - ie^2) \wedge (e^3 - ie^4) \wedge (e^5 + 2ie^6)$
$\mathfrak{n}_3 = \mathfrak{h}_3$	$(0,0,0,0,0,12+34)$	$(e^1 + ie^3) \wedge (e^2 + ie^4) \wedge (e^5 + 2ie^6)$
$\mathfrak{n}_4 = \mathfrak{h}_6$	$(0,0,0,0,12,13)$	$(e^1 - i(e^2 + 2e^4)) \wedge (e^3 - ie^4) \wedge (e^5 - 2ie^6)$
$\mathfrak{n}_5 = \mathfrak{h}_2$	$(0,0,0,0,12,34)$	$(e^2 + 2ie^1) \wedge (e^3 - ie^4) \wedge (e^5 - ie^6)$
$\mathfrak{n}_8 = \mathfrak{h}_9$	$(0,0,0,0,12,14+25)$	$(e^1 - ie^2) \wedge (e^4 - ie^5) \wedge (e^6 - 2ie^3)$
$\mathfrak{n}_9$	$(0,0,0,0,12,15)$	–
$\mathfrak{n}_{18} = \mathfrak{h}_{16}$	$(0,0,0,12,14,24)$	$(e^1 - ie^2) \wedge (e^4 + 2ie^3) \wedge (e^5 - ie^6)$
$\mathfrak{n}_{22}$	$(0,0,0,12,14,15)$	–
$\mathfrak{n}_{23}$	$(0,0,0,12,14,15+24)$	–
$\mathfrak{e}(2) \oplus \mathfrak{e}(2)$	$(0,-e^{13},e^{12},0,-e^{46},e^{45})$	–
$\mathfrak{e}(2) \oplus \mathfrak{e}(1,1)$	$(0,-e^{13},e^{12},0,-e^{46},-e^{45})$	$(e^1 - ie^4) \wedge (e^2 - 2i(e^2 - e^6)) \wedge (e^3 + i(\frac{e^3}{2} + e^5))$
$\mathfrak{e}(2) \oplus \mathfrak{h}_3$	$(0,-e^{13},e^{12},0,0,e^{45})$	–
$\mathfrak{e}(2) \oplus \mathbb{R}^3$	$(0,-e^{13},e^{12},0,0,0)$	–
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$(0,-e^{13},-e^{12},0,-e^{46},-e^{45})$	–
$\mathfrak{e}(1,1) \oplus \mathfrak{h}_3$	$(0,-e^{13},-e^{12},0,0,e^{45})$	–
$\mathfrak{e}(1,1) \oplus \mathbb{R}^3$	$(0,-e^{13},-e^{12},0,0,0)$	–
$A_{4,2}^{-2} \oplus \mathbb{R}^2$	$(-2e^{14}, e^{24}+e^{34}, e^{34},0,0,0)$	–
$A_{4,5}^{\alpha,-1-\alpha} \oplus \mathbb{R}^2$ $-1 < \alpha \leq -\frac{1}{2}$	$(e^{14}, \alpha e^{24}, -(1+\alpha)e^{34},0,0,0)$	–
$A_{4,6}^{\alpha,-\frac{\alpha}{2}} \oplus \mathbb{R}^2$ $\alpha > 0$	$(\alpha e^{14}, -\frac{\alpha}{2}e^{24}+e^{34}, -e^{24}-\frac{\alpha}{2}e^{34},0,0,0)$	–
$A_{4,8} \oplus \mathbb{R}^2$	$(e^{23}, e^{24}, -e^{34},0,0,0)$	–
$A_{4,10} \oplus \mathbb{R}^2$	$(e^{23}, e^{34}, -e^{24},0,0,0)$	–
$A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$	$(e^{15}, -e^{25}, -e^{35}, e^{45},0,0)$	$(e^1 - ie^4) \wedge (e^2 - ie^3) \wedge (e^5 - ie^6)$
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$ $0 < \beta < 1$	$(e^{15}, -e^{25}, \beta e^{35}, -\beta e^{45},0,0)$	–
$A_{5,8}^{-1} \oplus \mathbb{R}$	$(e^{25}, 0, e^{35}, -e^{45},0,0)$	–
$A_{5,9}^{-1,-1} \oplus \mathbb{R}$	$(e^{15}+e^{25}, e^{25}, -e^{35}, -e^{45},0,0)$	–
$A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$ $\gamma > 0$	$(e^{15}, -e^{25}, \gamma e^{45}, -\gamma e^{35},0,0)$	–

Table B.1: Six-dimensional decomposable unimodular solvable Lie algebras with $b_3 \geq 2$.

Lie algebra	Structure equations	closed volume form
$A^0_{5,14}\oplus\mathbb{R}$	$(e^{25},0,e^{45},-e^{35},0,0)$	$-$
$A^{-1}_{5,15}\oplus\mathbb{R}$	$(e^{15}+e^{25},e^{25},-e^{35}+e^{45},-e^{45},0,0)$	$-$
$A^{0,0,\gamma}_{5,17}\oplus\mathbb{R}$ $0<\gamma<1$	$(e^{25},-e^{15},\gamma e^{45},-\gamma e^{35},0,0)$	$-$
$A^{\alpha,-\alpha,1}_{5,17}\oplus\mathbb{R}$ $\alpha\geq 1$	$(\alpha e^{15}+e^{25},-e^{15}+\alpha e^{25},$ $-\alpha e^{35}+e^{45},-e^{35}-\alpha e^{45},0,0)$	$(e^1-ie^2)\wedge(e^4-ie^3)\wedge(e^6-ie^5)$
$A^0_{5,18}\oplus\mathbb{R}$	$(e^{25}+e^{35},-e^{15}+e^{45},e^{45},-e^{35},0,0)$	$-$
$A^{-1,2}_{5,19}\oplus\mathbb{R}$	$(-e^{15}+e^{23},e^{25},-2e^{35},2e^{45},0,0)$	$-$
$A^{1,-2}_{5,19}\oplus\mathbb{R}$	$(e^{15}+e^{23},e^{25},0,-2e^{45},0,0)$	$-$
$A^0_{5,20}\oplus\mathbb{R}$	$(e^{23}+e^{45},e^{25},-e^{35},0,0,0)$	$-$
$A^{0,\pm1}_{5,26}\oplus\mathbb{R}$	$(e^{23}\pm e^{45},-e^{35},e^{25},0,0,0)$	$-$
$A^{-1,-1}_{5,33}\oplus\mathbb{R}$	$(e^{14},e^{25},-e^{34}-e^{35},0,0,0)$	$-$
$A^{0,-2}_{5,35}\oplus\mathbb{R}$	$(-2e^{14},e^{24}+e^{35},-e^{25}+e^{34},0,0,0)$	$-$

Table B.1: Six-dimensional decomposable unimodular solvable Lie algebras with $b_3\geq 2$ (continued).

Lie algebra	Structure equations	closed volume form	$\lambda(\rho)\geq 0$
$\mathfrak{n}_6=\mathfrak{h}_4$	$(0,0,0,0,e^{12},e^{14}+e^{23})$	$(e^1-ie^2)\wedge(e^3+ie^4)\wedge(e^5+2ie^6)$	$-$
$\mathfrak{n}_7=\mathfrak{h}_5$	$(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23})$	$(e^1-2ie^2)\wedge(e^3-ie^4)\wedge(e^5-ie^6)$	$-$
$\mathfrak{n}_{10}$	$(0,0,0,0,e^{12},e^{15}+e^{34})$	$-$	$-$
$\mathfrak{n}_{11}=\mathfrak{h}_7$	$(0,0,0,e^{12},e^{13},e^{23})$	$(e^2+ie^3)\wedge(e^2+2ie^5)\wedge(e^4-ie^6)$	$-$
$\mathfrak{n}_{12}=\mathfrak{h}_{10}$	$(0,0,0,e^{12},e^{13},e^{14})$	$(e^2+2ie^1)\wedge(e^3+ie^4)\wedge(e^5+ie^6)$	$-$
$\mathfrak{n}_{13}=\mathfrak{h}_{12}$	$(0,0,0,e^{12},e^{13},e^{24})$	$(e^1+i(e^1+e^2))\wedge(e^3+ie^4)\wedge(e^5+i(e^5+e^6))$	$-$
$\mathfrak{n}_{14}=\mathfrak{h}_{11}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{23})$	$(e^2-2ie^1)\wedge(e^3-ie^4)\wedge(e^5+ie^6)$	$-$
$\mathfrak{n}_{15}=\mathfrak{h}_{15}$	$(0,0,0,e^{12},e^{13}+e^{14},e^{24})$	$(e^2+3ie^1)\wedge(e^3-i\frac{2}{3}e^4)\wedge(e^5-ie^6)$	$-$
$\mathfrak{n}_{16}=\mathfrak{h}_{14}$	$(0,0,0,e^{12},e^{14},e^{13}-e^{24})$	$(e^1+ie^2)\wedge(e^3+ie^4)\wedge(e^6+2ie^5)$	$-$
$\mathfrak{n}_{17}=\mathfrak{h}_{13}$	$(0,0,0,e^{12},e^{13}+e^{14},e^{24})$	$(e^1+ie^2)\wedge(e^3+ie^4)\wedge(e^6+2ie^5)$	$-$

Table B.2: Six-dimensional indecomposable unimodular solvable Lie algebras with $b_3\geq\ 2$.

Lie algebra	**Structure equations**	**closed volume form**	$\lambda(\rho) \geq 0$
$\mathfrak{n}_{19} = \mathfrak{h}_{19^-}$	$(0,0,0,e^{12},e^{23},e^{14}-e^{35})$	$(e^1-ie^3)\wedge(e^2-2ie^6)\wedge(e^4+ie^5)$	–
$\mathfrak{n}_{20}$	$(0,0,0,e^{12},e^{23},e^{14}+e^{35})$	–	–
$\mathfrak{n}_{21}$	$(0,0,0,e^{12},e^{13},e^{14}+e^{35})$	–	–
$\mathfrak{n}_{24}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23}+e^{24})$	–	–
$\mathfrak{n}_{25}$	$(0,0,0,e^{12},e^{14},e^{15}+e^{23})$	–	–
$\mathfrak{n}_{26}$	$(0,0,0,e^{12},e^{14}-e^{23},e^{15}+e^{34})$	–	–
$\mathfrak{n}_{27} = \mathfrak{h}_{26^+}$	$(0,0,e^{12},e^{13},e^{23},e^{14}+e^{25})$	$(e^1-ie^2)\wedge(e^3-2ie^6)\wedge(e^4-ie^5)$	–
$\mathfrak{n}_{28}$	$(0,0,e^{12},e^{13},e^{23},e^{14}-e^{25})$	–	–
$\mathfrak{n}_{29}$	$(0,0,e^{12},e^{13},e^{23},e^{14})$	–	–
$\mathfrak{n}_{30}$	$(0,0,e^{12},e^{13},e^{14},e^{15}+e^{23})$	–	–
$\mathfrak{n}_{31}$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24})$	–	–
$\mathfrak{n}_{32}$	$(0,0,e^{12},e^{13},e^{14},e^{15})$	–	–
$\mathfrak{n}_{33}$	$(0,0,e^{12},e^{13},e^{14},-e^{25}+e^{34})$	–	–
$\mathfrak{n}_{34}$	$(0,0,e^{12},e^{13},e^{14}+e^{23},-e^{25}+e^{34})$	–	–
$N_{6,18}^{0,-1,-1}$	$(e^{16}-e^{25},e^{15}+e^{26},-e^{36}+e^{45},-e^{35}-e^{46},0,0)$	$(e^1+ie^2)\wedge(e^3+ie^4)\wedge(e^5-ie^6)$	–
$N_{6,20}^{-1,-1}$	$(-e^{56},-e^{25}-e^{26},-e^{36},e^{45},0,0)$	–	–
$A_{6,13}^{-1,b,-2b+1}$ $b \notin \{-1,0,\frac{1}{2},1,2\}$	$((b-1)e^{16}+e^{23},-e^{26},be^{36},e^{46},(1-2b)e^{56},0)$	–	–
$A_{6,13}^{a,-2a,2a-1}$ $a \notin \{-1,0,\frac{1}{3},\frac{1}{2}\}$	$(-ae^{16}+e^{23},ae^{26},-2ae^{36},e^{46},(2a-1)e^{56},0)$	–	✓
$A_{6,13}^{a,-a,-1}$ $a>0,\ a\neq 1$	$(e^{23},ae^{26},-ae^{36},e^{46},-e^{56},0)$	–	✓
$A_{6,13}^{a,b,c}$ $(a,b,c)\in\{(0,-1,1),(-1,1,-1),$ $(-1,-1,3),(-1,2,-3)\}$	$((a+b)e^{16}+e^{23},ae^{26},be^{36},e^{46},ce^{56},0)$	–	–
$A_{6,14}^{\frac{1}{3},-\frac{2}{3}}$	$(-\frac{1}{3}e^{16}+e^{23}+e^{56},\frac{1}{3}e^{26},-\frac{2}{3}e^{36},e^{46},-\frac{1}{3}e^{56},0)$	–	✓
$A_{6,14}^{-1,\frac{2}{3}}$	$(-\frac{1}{3}e^{16}+e^{23}+e^{56},-e^{26},\frac{2}{3}e^{36},e^{46},-\frac{1}{3}e^{56},0)$	–	–
$A_{6,15}^{-1}$	$(e^{23},e^{26},-e^{36},e^{26}+e^{46},e^{36}-e^{56},0)$	–	–
$A_{6,17}^{0,-\frac{1}{2}}$	$(-\frac{1}{2}e^{16}+e^{23},-\frac{1}{2}e^{26},0,e^{36},e^{56},0)$	–	–
$A_{6,18}^{a,b}$ $(a,b)\in\{(-\frac{1}{2},-2),(-2,1)\}$	$((a+1)e^{16}+e^{23},ae^{26},e^{36},e^{36}+e^{46},be^{56},0)$	–	✓

Table B.2: Six-dimensional indecomposable unimodular solvable Lie algebras with $b_3 \geq 2$ (continued).

Lie algebra	Structure equations	closed volume form	$\lambda(\rho) \geq 0$
$A_{6,18}^{a,b}$ $(a,b) \in \{(-1,-1),(-3,3)\}$	$((a+1)e^{16}+e^{23}, ae^{26}, e^{36}, e^{36}+e^{46}, be^{56}, 0)$	–	–
$A_{6,21}^{a,b}$ $(a,b) \in \{(0,-1),(-1,3),(-\frac{1}{3},\frac{1}{3})\}$	$(2ae^{16}+e^{23}, ae^{26}, e^{26}+ae^{36}, e^{46}, be^{56}, 0)$	–	–
$A_{6,25}^{a,b}$ $(a,b) \in \{(0,-1),(-\frac{1}{2},-\frac{1}{2})\}$	$((b+1)e^{16}+e^{23}, e^{26}, be^{36}, ae^{46}, e^{46}+ae^{56}, 0)$	–	✓
$A_{6,25}^{-1,0}$	$(e^{16}+e^{23}, e^{26}, 0, -e^{46}, e^{46}-e^{56}, 0)$	–	–
$A_{6,26}^{-1}$	$(e^{23}+e^{56}, e^{26}, -e^{36}, 0, e^{46}, 0)$	–	–
$A_{6,32}^{0,b,-b}$ $b>0$	$(e^{23}, -e^{36}, e^{26}, be^{46}, -be^{56}, 0)$	–	✓
$A_{6,34}^{0,0,\epsilon}$ $\epsilon \in \{0,1\}$	$(e^{23}+\epsilon e^{56}, -e^{36}, e^{26}, 0, e^{46}, 0)$	–	✓
$A_{6,35}^{a,b,c}$ $a>0,\ (b,c) \in \{(-2a,a),(-a,0)\}$	$((a+b)e^{16}+e^{23}, ae^{26}, be^{36}, ce^{46}-e^{56}, e^{46}+ce^{56}, 0)$	–	✓
$A_{6,36}^{0,0}$	$(e^{23}, 0, e^{26}, -e^{56}, e^{46}, 0)$	–	–
$A_{6,37}^{0,0,c}$ $c>0,\ c \neq 1$	$(e^{23}, -e^{36}, e^{26}, -ce^{56}, ce^{46}, 0)$	–	✓
$A_{6,37}^{0,0,1}$	$(e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0)$	$(e^1+ie^6) \wedge (e^2-ie^3) \wedge (e^4+ie^5)$	–
$A_{6,39}^{a,b}$ $(a,b) \in \{(-1,-1),(-\frac{5}{2},-\frac{1}{2}),$ $(5,-3),(2,-2)\}$	$((b+1)e^{16}+e^{45}, e^{15}+(b+2)e^{26}, ae^{36}, be^{46}, e^{56}, 0)$	–	–
$A_{6,41}^{-1}$	$(e^{45}, e^{15}+e^{26}, -e^{36}+e^{46}, -e^{46}, e^{56}, 0)$	–	–
$A_{6,54}^{a,b}$ $(a,b) \in \{(0,-1),(-1,-\frac{3}{2}),(2,0)\}$	$(e^{16}+e^{35}, be^{26}+e^{45}, (1-a)e^{36}, (b-a)e^{46}, ae^{56}, 0)$	–	–
$A_{6,63}^{-1}$	$(e^{16}+e^{35}, -e^{26}+e^{45}+e^{46}, e^{36}, -e^{46}, 0, 0)$	–	–
$A_{6,70}^{0,0}$	$(-e^{26}+e^{35}, e^{16}+e^{45}, -e^{46}, e^{36}, 0, 0)$	–	–
$A_{6,76}^{-1}$	$(-e^{16}+e^{25}, e^{45}, e^{24}+e^{36}, e^{46}, -e^{56}, 0)$	–	–
$A_{6,78}$	$(-e^{16}+e^{25}, e^{45}, e^{24}+e^{36}+e^{46}, e^{46}, -e^{56}, 0)$	–	–
$B_{6,3}^{0}$	$(e^{45}, e^{15}+e^{36}, e^{14}-e^{26}, -e^{56}, e^{46}, 0)$	–	–
$A_{6,82}^{0,1,b}$ $0 \leq b < 1$	$(e^{24}+e^{35}, e^{26}, be^{36}, -e^{46}, -be^{56}, 0)$	–	–
$B_{6,4}^{1}$	$(e^{45}, e^{15}+e^{36}, e^{14}-e^{26}+e^{56}, -e^{56}, e^{46}, 0)$	$(e^1-i\frac{e^6}{2}) \wedge (e^2+ie^3) \wedge (e^4-ie^5)$	–

Table B.2: Six-dimensional indecomposable unimodular solvable Lie algebras with $b_3 \geq 2$ (continued).

$A_{6,82}^{0,1,1}$	$(e^{24}+e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0)$	$(e^1 - i\frac{e^6}{2}) \wedge (e^2 - ie^3) \wedge (e^4 - ie^5)$	$-$
$A_{6,83}^{0,1}$	$(e^{24}+e^{35}, e^{26}, e^{26}+e^{36}, -e^{46}-e^{56}, -e^{56}, 0)$	$-$	$-$
$A_{6,88}^{0,1,b}$ $b>0$	$(e^{24}+e^{35}, e^{26}-be^{36}, be^{26}+e^{36}, -e^{46}-be^{56}, be^{46}-e^{56}, 0)$	$-$	$-$
$A_{6,88}^{0,0,1}$	$(e^{24}+e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0)$	$(e^1 + i\frac{e^6}{2}) \wedge (e^2 + ie^4) \wedge (e^3 + ie^5)$	$-$
$A_{6,89}^{0,1,b}$ $b \in \mathbb{R}$	$(e^{24}+e^{35}, be^{26}, -e^{56}, -be^{46}, e^{36}, 0)$	$-$	$-$
$A_{6,90}^{0,\pm 1}$	$(e^{24}+e^{35}, e^{46}, \pm e^{56}, 0, \mp e^{36}, 0)$	$-$	$-$
$A_{6,93}^{0,1}$	$(e^{24}+e^{35}, -e^{56}, -e^{46}-e^{56}, e^{26}+e^{36}, e^{26}, 0)$	$-$	$-$
$B_{6,6}^{a}$ $-1<a<1,\ a\neq 0$	$(e^{24}+e^{35}, e^{46}, ae^{56}, -e^{26}, -ae^{36}, 0)$	$-$	$-$
$B_{6,6}^{1}$	$(e^{24}+e^{35}, e^{46}, ae^{56}, -e^{26}, -ae^{36}, 0)$	$(e^1 - i\frac{e^6}{2}) \wedge (e^2 + ie^4) \wedge (e^3 - ie^5)$	$-$
$A_{6,94}^{-2}$	$(e^{25}+e^{34}, -e^{26}+e^{35}, -2e^{36}, 2e^{46}, e^{56}, 0)$	$-$	$-$

Table B.2: Six-dimensional indecomposable unimodular solvable Lie algebras with $b_3 \geq 2$ (continued).

Conclusions

In this section we enumerate the most important results obtained in the thesis.

Chapter 1 is devoted to introduce the subject of the thesis in the broader context of complex geometry. We recall the most relevant notions and results about complex manifolds, with special attention to the several cohomologies which can be defined on them, such as de Rham, Dolbeault, Aeppli [1] and Bott-Chern [14], as well as some special Hermitian metrics considered in the work: balanced [62], strongly Gauduchon [74], strong Kähler with torsion [33] or generalized Gauduchon [39]. We also recall the results of the theory of holomorphic deformations that we have needed for our work.

The goal of Chapter 2 is to classify the solvable Lie algebras $\mathfrak{g}$ of dimension 6 underlying the complex solvmanifolds $(M = G/\Gamma, J)$, where J is an invariant complex structure with holomorphically trivial canonical bundle. By using the symmetrization process [11], we obtain the first important result of the thesis reducing the previous problem to the study of the existence of certain complex geometry on the underlying Lie algebras:

Result 1. *A $2n$-dimensional compact manifold $M = G/\Gamma$ endowed with an invariant complex structure J has holomorphically trivial canonical bundle if and only if it admits an invariant non-vanishing closed section $\Psi \in \wedge^{n,0}M$.*

When the Lie algebra is nilpotent and of dimension 6, the posed problem at the beginning of the chapter was solved by Salamon [82], finding that the Lie algebra is isomorphic to one in the list $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. For this reason, we focus our attention in finding the (non-nilpotent) solvable and unimodular six-dimensional Lie algebras admitting a complex structure with a non-zero closed $(3,0)$-form. After presenting and adapting the formalism of stable forms introduced by Hitchin [48] to our problem, we get the following result, which includes the one obtained by Salamon:

Result 2. *If a solvable and unimodular Lie algebra $\mathfrak{g}$ of dimension 6 admits a complex structure with a non-zero closed $(3,0)$-form, then it is isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$ if $\mathfrak{g}$ is nilpotent, or $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3, \ldots, \mathfrak{g}_8$ or $\mathfrak{g}_9$ if $\mathfrak{g}$ is not nilpotent, where:*

$\mathfrak{g}_1 := A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0)$,

$\mathfrak{g}_2^{\alpha} := A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} = (\alpha e^{15}+e^{25}, -e^{15}+\alpha e^{25}, -\alpha e^{35}+e^{45}, -e^{35}-\alpha e^{45}, 0, 0)$, $\alpha \geq 0$,

$\mathfrak{g}_3 := \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45})$,

$\mathfrak{g}_4 := A_{6,37}^{0,0,1} = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$

$\mathfrak{g}_5 := A_{6,82}^{0,1,1} = (e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0),$

$\mathfrak{g}_6 := A_{6,88}^{0,0,1} = (e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$

$\mathfrak{g}_7 := B_{6,6}^{1} = (e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0),$

$\mathfrak{g}_8 := N_{6,18}^{0,-1,-1} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0),$

$\mathfrak{g}_9 := B_{6,4}^{1} = (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0).$

Once we know which Lie algebras can be endowed with the required complex geometry, we determine which of the corresponding connected and simply-connected Lie groups admit lattices giving rise to solvmanifolds. The answer to this question turns out to be positive and well-known when the Lie group is nilpotent with underlying Lie algebra isomorphic to $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$ or $\mathfrak{h}_{26}^+$, because all of them admit a rational structure [61]. For the solvable Lie groups with Lie algebra isomorphic to $\mathfrak{g}_1, \mathfrak{g}_2^{\alpha\geq 0}, \mathfrak{g}_3 \ldots, \mathfrak{g}_8$ or $\mathfrak{g}_9$ we get the following:

Result 3. *For any $k \neq 2$, the connected and simply-connected Lie group G_k with underlying Lie algebra $\mathfrak{g}_k$ admits a lattice. For $k = 2$ there exists a countable family $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{\geq 0}$, including the case $\alpha = 0$, such that the connected and simply-connected Lie group with Lie algebra isomorphic to $\mathfrak{g}_2^{\alpha_n}$ admits a lattice.*

We classify in Chapter 3 the complex structures with a closed $(3,0)$-form on the solvable Lie algebras $\mathfrak{g}$ up to equivalence of complex structures on Lie algebras. We have divided the study depending on the nilpotency of the Lie algebra. For nilpotent Lie algebras of dimension 6, there are several partial results for this problem when the complex structure is abelian [4] or non-nilpotent [96]. We have obtained the following result concerning the remaining case, that is, the complex structure is nilpotent and non-abelian (and non complex-parallelizable):

Result 4. *If $\mathfrak{g}$ is a nilpotent Lie algebra of dimension 6 admitting a nilpotent complex structure J, which is neither abelian nor complex-parallelizable, then there exists a basis $\{\omega^1, \omega^2, \omega^3\}$ of $(1,0)$-forms satisfying one of the following reduced complex structure equations:*

$(\mathfrak{h}_2, J^D) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}}),\ \mathfrak{Im}\, D > 0,$

$(\mathfrak{h}_4, J^D) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}}),\ D \in \mathbb{R}\backslash\{0\},$

$(\mathfrak{h}_5, J^{\lambda,D}) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}})$, *where* $\mathfrak{Re}\, D = 0$ *and*

- $0 \leq 2\mathfrak{Im}\, D < \lambda^2$ *con* $0 < \lambda^2 < \frac{1}{2}$;
- $0 \leq 2\mathfrak{Im}\, D < |1 - \lambda^2|$ *con* $\frac{1}{2} \leq \lambda^2$,

$(\mathfrak{h}_6, J) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}})$,

$(\mathfrak{h}_7, J) : (0, \omega^{1\bar{1}}, \omega^{12} + \omega^{1\bar{2}})$,

$(\mathfrak{h}_{10}, J) : (0, \omega^{1\bar{1}}, \omega^{12} + \omega^{2\bar{1}})$,

$(\mathfrak{h}_{11}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + |B-1|\omega^{2\bar{1}})$, $B \in \mathbb{R} \setminus \{0, 1\}$,

$(\mathfrak{h}_{12}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + |B-1|\omega^{2\bar{1}})$, $\mathfrak{Im}\, B \neq 0$,

$(\mathfrak{h}_{13}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B-1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) < 0$,

$(\mathfrak{h}_{14}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B-1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) = 0$,

$(\mathfrak{h}_{15}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B-1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) > 0$,

$(\mathfrak{h}_{16}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}})$ *with* $|B| = 1$ *and* $B \neq 1$,

where $\lambda, c \geq 0$, $B, D \in \mathbb{C}$ *and* $\Delta(B, c) := c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2$.

It is remarkable that the latter result, together with [4, 96], provides a complete description of the complex geometry on nilpotent Lie algebras of dimension 6. The second part of the chapter deals with classifying the complex structures with a non-vanishing closed $(3,0)$-form on the solvable Lie algebras of the list $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. We have obtained the following result:

Result 5. *If* $\mathfrak{g}$ *is a (non-nilpotent) solvable unimodular Lie algebra of dimension 6 admitting a complex structure* J *with a non-vanishing closed* $(3,0)$*-form then there exists a basis* $\{\omega^1, \omega^2, \omega^3\}$ *of* $(1,0)$*-forms satisfying one of the following reduced complex structure equations:*

$(\mathfrak{g}_1, J) : (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), 0)$,

$(\mathfrak{g}_2^0, J) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), 0)$,

$(\mathfrak{g}_2^{\alpha>0}, J^{\pm}) : ((\pm\cos\theta + i\sin\theta)(\omega^{13} + \omega^{1\bar{3}}), (\mp\cos\theta - i\sin\theta)(\omega^{23} + \omega^{2\bar{3}}), 0)$, $\alpha = \frac{\cos\theta}{\sin\theta}$,

$(\mathfrak{g}_3, J^x) : (0, -\frac{1}{2}\omega^{13} - (\frac{1}{2} + ix)\omega^{1\bar{3}} + ix\omega^{3\bar{1}}, \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\omega^{2\bar{1}})$,

$(\mathfrak{g}_4, J^{\pm}) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm\omega^{1\bar{1}})$,

$(\mathfrak{g}_5, J) : (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{2}} + \omega^{2\bar{1}})$,

$(\mathfrak{g}_6, J) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{1}} + \omega^{2\bar{2}})$,

$(\mathfrak{g}_7, J^{\pm}) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}}))$,

$(\mathfrak{g}_8, J^A) : (-(A-i)\omega^{13} - (A+i)\omega^{1\bar{3}}, (A-i)\omega^{23} + (A+i)\omega^{2\bar{3}}, 0)$, $\mathfrak{Im}\,(A) \neq 0$,

$(\mathfrak{g}_8, J') : (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23}, 0)$,

$(\mathfrak{g}_8, J'') : (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23} + \omega^{3\bar{3}}, 0),$

$(\mathfrak{g}_9, J) : (-\omega^{3\bar{3}}, \frac{1}{2}(i\omega^{12} - i\omega^{2\bar{1}} + \omega^{1\bar{3}}), \frac{i}{2}(-\omega^{13} + \omega^{3\bar{1}})),$

where $\theta \in (0, \frac{\pi}{2})$*,* $x > 0$*,* $A \in \mathbb{C}$ *with* $\mathfrak{Im}\,(A) \neq 0$*.*

The goal of Chapter 4 is to study some complex invariants, such as the Frölicher spectral sequence and the $\partial\bar{\partial}$-lemma, on complex solvmanifolds endowed with invariant complex structures with holomorphically trivial canonical bundle. We have divided the study depending on the Lie algebra is nilpotent or not. By the results in [80, 23], the Dolbeault cohomology of a complex nilmanifold can be computed at the level of the Lie algebra (whenever the Lie algebra is not isomorphic to $\mathfrak{h}_7$, in such case it is not known whether this is true or not). As a consequence, and by using the classification of complex structures obtained in Chapter 3, we provide a general picture of the behaviour of the Frölicher spectral sequence for nilmanifolds of dimension 6 endowed with an invariant complex structure:

Result 6. *Let* $M = G/\Gamma$ *be a nilmanifold of dimension 6 endowed with an invariant complex structure* J *such that the underlying Lie algebra* $\mathfrak{g} \ncong \mathfrak{h}_7$*. The Frölicher spectral sequence* $\{E_r(M)\}_{r\geq 1}$ *has the following behaviour:*

(a) If $\mathfrak{g} \cong \mathfrak{h}_1, \mathfrak{h}_3, \mathfrak{h}_6, \mathfrak{h}_8, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}$ *or* $\mathfrak{h}_{19}^-$*, then* $E_1(M) \cong E_\infty(M)$ *for any* J*.*

(b) If $\mathfrak{g} \cong \mathfrak{h}_2$ *or* $\mathfrak{h}_4$*, then* $E_1(M) \cong E_\infty(M)$ *if and only if* J *is non abelian; in addition, any abelian complex structure on* $\mathfrak{h}_2$ *or* $\mathfrak{h}_4$ *satisfies* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$*.*

(c) If $\mathfrak{g} \cong \mathfrak{h}_5$ *and* J *is a complex structure on* $\mathfrak{h}_5$ *given in Result 4, then:*

(c.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *when* J *is parallelizable;*

(c.2) $E_1(M) \cong E_\infty(M)$ *if and only if* J *is not complex parallelizable and* $\rho D \neq 0$*; in addition,* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *when* $\rho D = 0$*.*

(d) If $\mathfrak{g} \cong \mathfrak{h}_{16}$ *or* $\mathfrak{h}_{26}^+$*, then* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *for any* J*.*

(e) If $\mathfrak{g} \cong \mathfrak{h}_{13}$ *or* $\mathfrak{h}_{14}$*, then* $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$ *for any* J*.*

(f) If $\mathfrak{g} \cong \mathfrak{h}_{15}$ *and* J *is a complex structure on* $\mathfrak{h}_{15}$ *given in Result 4, then:*

(f.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$*, when* $c = 0$ *and* $|B - \rho| \neq 0$*;*

(f.2) $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$*, when* $\rho = 1$ *and* $|B - 1| \neq c \neq 0$*;*

(f.3) $E_1(M) \ncong E_2(M) \ncong E_3(M) \cong E_\infty(M)$*, when* $\rho = 0$ *and* $|B| \neq c \neq 0$*.*

As for the de Rham cohomology, the Dolbeault cohomology of a complex solvmanifold is not computable only with the data of the complex structure on the Lie algebra. However, the computation of the Frölicher spectral sequence on the Lie algebra allows us to state that if $E_1(\mathfrak{g}) \ncong E_\infty(\mathfrak{g})$ then $E_1(M) \ncong E_\infty(M)$. From this fact we extract some partial results about certain complex solvmanifolds with underlying Lie algebra isomorphic to $\mathfrak{g}_8$. In addition, thanks to some previous works of Kasuya and Angella [51, 7], it is possible to compute the Dolbeault and the Bott-Chern cohomologies of a solvmanifold endowed with a complex structure of splitting type by means of certain finite-dimensional differential complexes. By a complex structure of splitting type [51, Assumption 1.1] on a solvmanifold $M = G/\Gamma$ we mean a complex structure coming from a left-invariant complex structure on the Lie group such that it can be conceived as a semidirect product $G = \mathbb{C} \ltimes_\varphi N$, N being nilpotent, and such that the lattice $\Gamma \subset G$ presents some compatibility with the splitting. Therefore, in order to apply these results to the complex structures with holomorphically trivial canonical bundle we must firstly identify which of these complex structures are of splitting type. Secondly, we must find lattices compatible with the splittings for the corresponding Lie groups. We summarize the results in the following:

Result 7. *A complex solvmanifold $(M = G/\Gamma, J)$, where J is invariant with holomorphically trivial canonical bundle, is of splitting type if $(\mathfrak{g}, J)$ is equivalent to $(\mathfrak{g}_1, J)$, $(\mathfrak{g}_2^0, J)$, $(\mathfrak{g}_2^\alpha, J^\pm)$ with $\alpha > 0$ or $(\mathfrak{g}_8, J^A)$ with $A \in \mathbb{C}$, $\mathfrak{Im}\, A \neq 0$. In addition:*

(a) if $(\mathfrak{g}, J)$ is isomorphic to $(\mathfrak{g}_2^0, J)$, then the corresponding Lie group admits lattices compatible with the splitting such that the complex solvmanifolds satisfy $E_1(M) \ncong E_\infty(M)$, and other lattices such that the complex solvmanifolds satisfy the $\partial\bar\partial$-lemma.

(b) if $(\mathfrak{g}, J)$ is isomorphic to $(\mathfrak{g}_2^\alpha, J^\pm)$ with $\alpha > 0$, then for a countable family of α's the corresponding Lie groups admit lattices compatible with the non-equivalent complex structures of splitting type $J^\pm$. For some lattices the complex solvmanifolds satisfy $E_1(M) \ncong E_\infty(M)$ and for other lattices the complex solvmanifolds satisfy the $\partial\bar\partial$-lemma.

(c) if $(\mathfrak{g}, J)$ is isomorphic to $(\mathfrak{g}_8, J^A)$, where $A \in \mathbb{C}$ with $\mathfrak{Im}\, A \neq 0$, then the corresponding Lie group admits lattices compatible with the splitting. In addition, there exists a family of lattices Γ_A compatible with the splitting of the Lie groups with complex structure J^A such that the complex solvmanifolds satisfy the $\partial\bar\partial$-lemma if and only if $A \neq \frac{i}{k}$ with integer $k \neq 0$.

Moreover, any complex solvmanifold (M, J) with underlying Lie algebra $\mathfrak{g}_8$ and J equivalent to J^{-i}, J^i, J' or J'' does not satisfy the $\partial\bar\partial$-lemma.

The last part of Chapter 4 is devoted to the study of these complex invariants under holomorphic deformations. As regards the Frölicher spectral sequence, Kodaira and Spencer [53] proved the upper semi-continuity of the Hodge numbers, $\dim E_1^{\bullet,\bullet}(M)$, in a holomorphic deformation, which implies that the property $E_1(M) \cong E_\infty(M)$ is open under deformations. In contrast, Eastwood and Singer [27] prove the non-closedness of the property $E_1(M) \cong E_\infty(M)$. The complex geometry on the nilpotent Lie algebras $\mathfrak{h}_4$ and $\mathfrak{h}_{15}$ together with the description of the Frölicher spectral sequence for complex nilmanifolds allows us to build two examples presenting interesting behaviours. The first provides another example of the result of Eastwood and Singer by using only an invariant holomorphic deformation. The second shows that the behaviour of the property $E_2(M) \cong E_\infty(M)$ is different from the one proved by Kodaira and Spencer for the property $E_1(M) \cong E_\infty(M)$:

Result 8. *A nilmanifold $(M = G/\Gamma, J_1)$, with $\mathfrak{g}$ isomorphic to $\mathfrak{h}_4$ and J_1 its abelian structure, admits an invariant holomorphic deformation $\{M_t := (M, I_t)\}_{t\in\Delta}$, with $I_0 = J_1$, such that $E_1(M_t) \cong E_\infty(M_t)$ for any $t \in \Delta^*$, whereas $E_1(M_0) \not\cong E_\infty(M_0)$.*

In addition, there exists a family of complex nilmanifolds $\{M_t := (M, J_t)\}_{t\in\mathbb{R}}$, where $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$, such that the dimensions $\dim E_2^{0,2}(M_t)$, $\dim E_3^{0,2}(M_t)$ are not lower semi-continuous and the dimensions $\dim E_2^{1,1}(M_t)$, $\dim E_3^{1,1}(M_t)$ are not upper semi-continuous. Moreover, the property $E_2(M) \cong E_\infty(M)$ is not open under holomorphic deformations.

As regards the $\partial\bar\partial$-lemma, it is known that it is open under holomorphic deformations [98, 101, 10]. More recently, Angella and Kasuya [8] have proved that the $\partial\bar\partial$-lemma is not closed under holomorphic deformations by means of an invariant deformation of the Nakamura manifold, which corresponds in our classification to a solvmanifold endowed with an invariant complex structure such that $(\mathfrak{g}, J)$ is isomorphic to $(\mathfrak{g}_8, J^{-i})$. By using the study of cohomology for complex solvmanifolds done in this chapter, we obtain a countable family of examples extending the previous result due to Angella and Kasuya:

Result 9. *There is a countable family of complex solvmanifolds of splitting type $\{(M_k := G/\Gamma_k, J_k)\}_{k\in\mathbb{Z}}$ not satisfying the $\partial\bar\partial$-lemma, where (G, J_k) is a Lie group endowed with an invariant complex structure such that $(\mathfrak{g}, J_k)$ is isomorphic to $(\mathfrak{g}_8, J^{A_k})$ with $A_k = \frac{i}{2k+1}$ and $\Gamma_k \subset G$ is a lattice compatible with the splitting induced by J_k. In addition, for any $k \in \mathbb{Z}$ there exists a holomorphic deformation $\{(M_k, J_{k,t})\}_{t\in\Delta}$ satisfying the $\partial\bar\partial$-lemma for any $t \in \Delta^*$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$.*

Chapter 5 deals with Hermitian structures on six-dimensional solvmanifolds compatible with the invariant complex structures that trivialize the holomorphic canonical bundle. There are known some results concerning the existence of strong Kähler with torsion [33], invariant 1-generalized Gauduchon [35] and balanced [95] metrics for the case of nilmanifolds. Hence, we have centered our attention in the existence of these special metrics for the complex structures on the solvable Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. In the case of strongly Gauduchon metrics we have performed this study for the previous Lie algebras together with the nilpotent Lie algebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. The results obtained are included in the following:

Result 10. *Let $M = G/\Gamma$ be a six-dimensional solvmanifold admitting invariant complex structures with holomorphically trivial canonical bundle, then M admits a:*

(a) Kähler metric (and therefore Calabi-Yau) if and only if $\mathfrak{g}$ is isomorphic to $\mathbb{R}^6$, $\mathfrak{g}_2^0$;

(b) strong Kähler with torsion metric if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_8$, $\mathfrak{g}_2^0$, $\mathfrak{g}_4$;

(c) *invariant 1-generalized Gauduchon metric if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_8$, $\mathfrak{g}_2^0$, $\mathfrak{g}_4$, $\mathfrak{g}_6$;

(d) *balanced metric if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$, $\mathfrak{h}_{19}^-$, $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$, $\mathfrak{g}_8$;

(e) *strongly Gauduchon metric if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$, $\mathfrak{h}_{19}^-$, $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$, $\mathfrak{g}_8$.

The detailed study of the existence of these metrics on the solvable Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$, *endowed with a complex structure with non-zero closed* $(3,0)$*-form, is summed up in the following table:*

	Metrics			
	SKT	**1st-Gauduchon (invariant)**	**Balanced**	**sG**
$\mathfrak{g}_1$	$\nexists$	$\nexists$	$\exists$ *for any* J	*any* (J, F)
$\mathfrak{g}_2^0$	$\exists$ *for any* J	$\exists$ *for any* J	$\exists$ *for any* J	*any* (J, F)
$\mathfrak{g}_2^{\alpha>0}$	$\nexists$	$\nexists$	$\exists$ *for any* J	*any* (J, F)
$\mathfrak{g}_3$	$\nexists$	$\nexists$	$\exists$ *for any* J	*any* (J, F)
$\mathfrak{g}_4$	$\exists$ *for any* J	$\exists$ *for any* J	$\nexists$	$\nexists$
$\mathfrak{g}_5$	$\nexists$	$\nexists$	$\exists$ *for any* J	$\exists$ *for any* J
$\mathfrak{g}_6$	$\nexists$	$\exists$ *for any* J	$\nexists$	$\nexists$
$\mathfrak{g}_7$	$\nexists$	$\nexists$	$\exists$ *for any* J	$\exists$ *for any* J
$\mathfrak{g}_8$	$\nexists$	$\nexists$	$\exists$ *for any* $J \neq J', J''$	*any* (J, F)
$\mathfrak{g}_9$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

The balanced condition trivially implies the strongly Gauduchon condition. Furthermore, Popovici [72] proves that a compact complex manifold satisfying the $\partial\bar{\partial}$-lemma admits strongly Gauduchon metrics compatible with the complex structure. However, there are "pure" strongly Gauduchon manifolds in the sense that the $\partial\bar{\partial}$-lemma does not hold on them and they do not admit balanced metrics. We give a complete description in the class of six-dimensional nilmanifolds of the invariant complex structures admitting strongly Gauduchon metrics but no balanced metrics (recall that except the tori, nilmanifolds do not satisfy the $\partial\bar{\partial}$-lemma because they are not formal [44]):

Result 11. *Let $M = G/\Gamma$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J such that M does not admit balanced metrics. If (M, J) admits a strongly Gauduchon metric then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2$, $\mathfrak{h}_4$ or $\mathfrak{h}_5$. Moreover, J is non-abelian and given by: $\mathfrak{Re}\, D + (\mathfrak{Im}\, D)^2 \geq \frac{1}{4}$ on $\mathfrak{h}_2$; $\mathfrak{Re}\, D \geq \frac{1}{4}$ on $\mathfrak{h}_4$; and $\lambda = 0, \mathfrak{Im}\, D \neq 0$ or $\lambda = \mathfrak{Im}\, D = 0, \mathfrak{Re}\, D \geq 0$ on $\mathfrak{h}_5$, according to the description of the complex structures provided in Result 4.*

Finally, we have studied the balanced and the strongly Gauduchon properties under holomorphic deformations. Alessandrini and Bassanelli proved that the balanced property is not open under deformations [3] and Popovici [73] proves that the strongly Gauduchon property is open. It was conjectured that both properties are closed [76, Conjecture 1.21, Conjecture 1.23]. By means of the study performed for the previous result about the existence of balanced and strongly Gauduchon metrics on six-dimensional nilmanifolds, we have constructed a counterexample to both conjectures:

Result 12. *A nilmanifold $(M = G/\Gamma, J_1)$, with $\mathfrak{g}$ isomorphic to $\mathfrak{h}_4$ and J_1 its abelian structure, admits an invariant holomorphic deformation $\{M_t := (M, I_t)\}_{t\in\Delta}$, with $I_0 = J_1$, such that M_t are balanced for all $t \in \Delta^*$, but M_0 is not strongly Gauduchon.*

As a consequence, the balanced and the strongly Gauduchon properties are not closed under holomorphic deformations.

Although we mentioned previously that the $\partial\bar{\partial}$-lemma property is not closed under holomorphic deformations, Popovici [72] proves that a holomorphic deformation $\{(M, J_t)\}_{t\in\Delta}$ satisfying the $\partial\bar{\partial}$-lemma for any $t \in \Delta^*$ implies the existence of a compatible strongly Gauduchon metric in the central limit (recall that the strongly Gauduchon property is a necessary condition to the validity of the $\partial\bar{\partial}$-lemma). In the following result we show, by means of a holomorphic deformation of a complex solvmanifold, that the result by Popovici is optimal in the sense that we cannot expect to find (in general) balanced metrics in the central limit.

Result 13. *There exists a holomorphic deformation $\{(M = G/\Gamma, J_t)\}_{t\in\Delta}$ of a solvmanifold , where $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_8$ and J_t are invariant with holomorphically trivial canonical bundle, satisfying the $\partial\bar{\partial}$-lemma and admitting balanced metrics for any $t \in \Delta^*$, but the central limit (M, J_0) does not admit balanced metrics nor the $\partial\bar{\partial}$-lemma holds.*

Conclusiones

En esta sección se muestran los resultados más importantes obtenidos en la tesis.

El Capítulo 1 es un capítulo introductorio que tiene como fin situar el objeto de estudio de la tesis en el contexto más amplio de la geometría compleja general. Se introducen las nociones y resultados más relevantes sobre variedades complejas, prestando especial atención a las diversas cohomologías que se les pueden asociar, tales como de Rham, Dolbeault, Aeppli [1] y Bott-Chern [14], así como algunas métricas Hermitianas especiales que son consideradas en la memoria: equilibradas [62], fuertemente Gauduchon [74], Kähler con torsión [33] o generalizadas Gauduchon [39]. También son presentados los aspectos más necesarios para el resto del trabajo relativos a la teoría de deformaciones holomorfas.

El objetivo del Capítulo 2 es clasificar las álgebras de Lie resolubles $\mathfrak{g}$ de dimensión 6 subyacentes a las solvariedades complejas $(M = G/\Gamma, J)$, siendo J una estructura compleja invariante con fibrado canónico holomórficamente trivial. Mediante el proceso de simetrización [11] se obtiene el primer resultado importante de la tesis, que reduce el anterior problema a un estudio sobre la existencia de cierta geometría compleja sobre las álgebras de Lie subyacentes:

Resultado 1. *Una variedad compacta $M = G/\Gamma$ de dimensión $2n$ dotada de una estructura compleja invariante J tiene fibrado canónico holomórficamente trivial si y sólo si admite una sección invariante $\Psi \in \wedge^{n,0}M$ cerrada que no se anula en ningún punto.*

Cuando el álgebra de Lie es nilpotente y de dimensión 6, el problema planteado en el objetivo del capítulo ya había sido resuelto por Salamon [82], encontrando que el álgebra de Lie debe ser isomorfa a una de la lista $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. Por ello nos hemos centrado en encontrar las álgebras de Lie unimodulares resolubles y no nilpotentes de dimensión 6 que admiten una estructura compleja con una forma de tipo $(3, 0)$ cerrada y no nula. Tras presentar y adaptar el formalismo de formas estables de Hitchin [48] a nuestro problema, se obtiene el siguiente resultado que incluye el resultado obtenido por Salamon:

Resultado 2. *Si un álgebra de Lie unimodular y resoluble $\mathfrak{g}$ de dimensión 6 admite una estructura compleja con una forma cerrada no nula de tipo $(3, 0)$, entonces es isomorfa a $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$ ó $\mathfrak{h}_{26}^+$ si $\mathfrak{g}$ es nilpotente, o a $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3, \ldots, \mathfrak{g}_8$ ó $\mathfrak{g}_9$ si $\mathfrak{g}$ no es nilpotente, siendo:*

$$\mathfrak{g}_1 := A_{5,7}^{-1,-1,1} \oplus \mathbb{R} = (e^{15}, -e^{25}, -e^{35}, e^{45}, 0, 0),$$

$$\mathfrak{g}_2^{\alpha} := A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} = (\alpha e^{15}+e^{25}, -e^{15}+\alpha e^{25}, -\alpha e^{35}+e^{45}, -e^{35}-\alpha e^{45}, 0, 0),\ \alpha \geq 0,$$

$$\mathfrak{g}_3 := \mathfrak{e}(2) \oplus \mathfrak{e}(1,1) = (0, -e^{13}, e^{12}, 0, -e^{46}, -e^{45}),$$

$$\mathfrak{g}_4 := A_{6,37}^{0,0,1} = (e^{23}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$

$$\mathfrak{g}_5 := A_{6,82}^{0,1,1} = (e^{24} + e^{35}, e^{26}, e^{36}, -e^{46}, -e^{56}, 0),$$

$$\mathfrak{g}_6 := A_{6,88}^{0,0,1} = (e^{24} + e^{35}, -e^{36}, e^{26}, -e^{56}, e^{46}, 0),$$

$$\mathfrak{g}_7 := B_{6,6}^{1} = (e^{24} + e^{35}, e^{46}, e^{56}, -e^{26}, -e^{36}, 0),$$

$$\mathfrak{g}_8 := N_{6,18}^{0,-1,-1} = (e^{16} - e^{25}, e^{15} + e^{26}, -e^{36} + e^{45}, -e^{35} - e^{46}, 0, 0),$$

$$\mathfrak{g}_9 := B_{6,4}^{1} = (e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + e^{56}, -e^{56}, e^{46}, 0).$$

Una vez conocidas qué álgebras de Lie pueden soportar la geometría compleja buscada, ha sido preciso determinar cuáles de los correspondientes grupos de Lie conexos y simplemente conexos admiten lattices que dan lugar a solvariedades. La respuesta a esta pregunta es afirmativa y ya conocida para los grupos de Lie nilpotentes con álgebra de Lie subyacente isomorfa a $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$, $\mathfrak{h}_{19}^-$ ó $\mathfrak{h}_{26}^+$, ya que todas ellas admiten una estructura racional [61]. Para los grupos de Lie resolubles con álgebra de Lie subyacente isomorfa a $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha \geq 0}$, $\mathfrak{g}_3, \ldots, \mathfrak{g}_8$ ó $\mathfrak{g}_9$ se ha obtenido lo siguiente:

Resultado 3. *Para cada $k \neq 2$, el grupo de Lie conexo y simplemente conexo G_k con álgebra de Lie subyacente $\mathfrak{g}_k$ admite lattice. Para $k = 2$ existe una familia numerable $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^{\geq 0}$, que incluye $\alpha = 0$, de manera que el grupo de Lie conexo y simplemente conexo con álgebra de Lie subyacente $\mathfrak{g}_2^{\alpha_n}$ admite lattice.*

En el Capítulo 3 se han clasificado las estructuras complejas con forma de tipo $(3,0)$ cerrada no nula sobre álgebras de Lie $\mathfrak{g}$ resolubles salvo equivalencia de estructuras complejas sobre el álgebra de Lie. Este estudio se ha dividido según $\mathfrak{g}$ sea nilpotente o no. Para el caso de las álgebras de Lie nilpotentes se conocían resultados parciales a este problema cuando la estructura compleja es abeliana [4] o no nilpotente [96]. Hemos obtenido el siguiente resultado relativo al caso pendiente, que es cuando la estructura compleja es nilpotente y no abeliana (y no compleja paralelizable):

Resultado 4. *Si $\mathfrak{g}$ es un álgebra de Lie nilpotente de dimensión 6 que admite una estructura compleja J nilpotente, que no es abeliana ni compleja paralelizable, entonces existe una base $\{\omega^1, \omega^2, \omega^3\}$ de formas de tipo $(1,0)$ cumpliendo alguna de las ecuaciones de estructura complejas reducidas siguientes:*

$$(\mathfrak{h}_2, J^D) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}}),\ \mathfrak{Im}\, D > 0,$$

$$(\mathfrak{h}_4, J^D) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}} + D\omega^{2\bar{2}}),\ D \in \mathbb{R}\backslash\{0\},$$

$(\mathfrak{h}_5, J^{\lambda,D}) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}})$, *siendo* $\mathfrak{Re}\, D = 0$ *y*

- $0 \leq 2\mathfrak{Im}\, D < \lambda^2$ *con* $0 < \lambda^2 < \frac{1}{2}$*;*
- $0 \leq 2\mathfrak{Im}\, D < |1 - \lambda^2|$ *con* $\frac{1}{2} \leq \lambda^2$*,*

$(\mathfrak{h}_5, J^{\lambda,D}) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \lambda\omega^{1\bar{2}} + D\omega^{2\bar{2}})$, $4(\mathfrak{Im}\, D)^2 < 1 + 4\mathfrak{Re}\, D$,

$(\mathfrak{h}_6, J) : (0, 0, \omega^{12} + \omega^{1\bar{1}} + \omega^{1\bar{2}})$,

$(\mathfrak{h}_7, J) : (0, \omega^{1\bar{1}}, \omega^{12} + \omega^{1\bar{2}})$,

$(\mathfrak{h}_{10}, J) : (0, \omega^{1\bar{1}}, \omega^{12} + \omega^{2\bar{1}})$,

$(\mathfrak{h}_{11}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + |B - 1|\omega^{2\bar{1}})$, $B \in \mathbb{R} \setminus \{0, 1\}$,

$(\mathfrak{h}_{12}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + |B - 1|\omega^{2\bar{1}})$, $\mathfrak{Im}\, B \neq 0$,

$(\mathfrak{h}_{13}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B - 1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) < 0$,

$(\mathfrak{h}_{14}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B - 1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) = 0$,

$(\mathfrak{h}_{15}, J^{B,c}) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}} + c\,\omega^{2\bar{1}})$, $c \neq |B - 1|$, $(c, |B|) \neq (0, 1)$, $\Delta(B, c) > 0$,

$(\mathfrak{h}_{16}, J^B) : (0, \omega^{1\bar{1}}, \omega^{12} + B\omega^{1\bar{2}})$ *con* $|B| = 1$ *y* $B \neq 1$,

donde $\lambda, c \geq 0$*,* $B, D \in \mathbb{C}$ *y* $\Delta(B, c) := c^4 - 2(|B|^2 + 1)c^2 + (|B|^2 - 1)^2$.

Destacamos que el anterior resultado, junto con los mencionados [4, 96], completa una descripción general de la geometría compleja sobre las álgebras de Lie nilpotentes de dimensión 6. La segunda parte del capítulo se ha dedicado a clasificar las estructuras complejas con una forma de tipo $(3, 0)$ cerrada sobre las álgebras de Lie resolubles de la lista $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. El resultado obtenido es el siguiente:

Resultado 5. *Si* $\mathfrak{g}$ *es un álgebra de Lie resoluble (no nilpotente) y unimodular de dimensión 6 que admite una estructura compleja con una forma cerrada no nula de tipo* $(3, 0)$ *entonces existe una base* $\{\omega^1, \omega^2, \omega^3\}$ *de formas de tipo* $(1, 0)$ *cumpliendo alguna de las ecuaciones de estructura complejas reducidas siguientes:*

$(\mathfrak{g}_1, J) : (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), 0)$,

$(\mathfrak{g}_2^0, J) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), 0)$,

$(\mathfrak{g}_2^{\alpha>0}, J^{\pm}) : ((\pm\cos\theta + i\sin\theta)(\omega^{13} + \omega^{1\bar{3}}), (\mp\cos\theta - i\sin\theta)(\omega^{23} + \omega^{2\bar{3}}), 0)$, $\alpha = \frac{\cos\theta}{\sin\theta}$,

$(\mathfrak{g}_3, J^x) : (0, -\frac{1}{2}\omega^{13} - (\frac{1}{2} + ix)\omega^{1\bar{3}} + ix\omega^{3\bar{1}}, \frac{1}{2}\omega^{12} + (\frac{1}{2} - \frac{i}{4x})\omega^{1\bar{2}} + \frac{i}{4x}\omega^{2\bar{1}})$,

$(\mathfrak{g}_4, J^{\pm}) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm\omega^{1\bar{1}})$,

$(\mathfrak{g}_5, J) : (\omega^{13} + \omega^{1\bar{3}}, -(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{2}} + \omega^{2\bar{1}})$,

$(\mathfrak{g}_6, J) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \omega^{1\bar{1}} + \omega^{2\bar{2}}),$

$(\mathfrak{g}_7, J^{\pm}) : (i(\omega^{13} + \omega^{1\bar{3}}), -i(\omega^{23} + \omega^{2\bar{3}}), \pm(\omega^{1\bar{1}} - \omega^{2\bar{2}})),$

$(\mathfrak{g}_8, J^{A}) : (-(A - i)\omega^{13} - (A + i)\omega^{1\bar{3}}, (A - i)\omega^{23} + (A + i)\omega^{2\bar{3}}, 0),$

$(\mathfrak{g}_8, J') : (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23}, 0),$

$(\mathfrak{g}_8, J'') : (2i\omega^{13} + \omega^{3\bar{3}}, -2i\omega^{23} + \omega^{3\bar{3}}, 0),$

$(\mathfrak{g}_9, J) : (-\omega^{3\bar{3}}, \frac{1}{2}(i\omega^{12} - i\omega^{2\bar{1}} + \omega^{1\bar{3}}), \frac{i}{2}(-\omega^{13} + \omega^{3\bar{1}})),$

donde $\theta \in (0, \frac{\pi}{2})$, $x > 0$, $A \in \mathbb{C}$ *con* $\mathfrak{Im}\,(A) \neq 0$.

El objetivo del Capítulo 4 es estudiar algunos invariantes complejos, tales como la sucesión espectral de Frölicher y el $\partial\bar{\partial}$-lema, sobre solvariedades complejas dotadas de estructura compleja invariante con fibrado canónico holomorfo trivial. Este estudio se ha dividido según el álgebra de Lie sea nilpotente o no. Por los resultados de [80, 23] la cohomología de Dolbeault de una nilvariedad compleja puede ser calculada a nivel del álgebra de Lie (exceptuando si el álgebra de Lie es isomorfa a $\mathfrak{h}_7$, en cuyo caso se desconoce si esto último es cierto o no). Como consecuencia de estos resultados, y usando la clasificación de estructuras complejas obtenida en el Capítulo 3, proporcionamos una descripción general del comportamiento de la sucesión espectral de Frölicher para nilvariedades de dimensión 6 dotadas de una estructura compleja invariante:

Resultado 6. *Sea* $M = G/\Gamma$ *una nilvariedad de dimensión 6 dotada de una estructura compleja invariante* J *tal que el álgebra de Lie subyacente* $\mathfrak{g}$ *no es isomorfa a* $\mathfrak{h}_7$*. La sucesión espectral de Frölicher* $\{E_r(M)\}_{r\geq 1}$ *se comporta como sigue:*

(a) Si $\mathfrak{g} \cong \mathfrak{h}_1$, $\mathfrak{h}_3$, $\mathfrak{h}_6$, $\mathfrak{h}_8$, $\mathfrak{h}_9$, $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{12}$ *ó* $\mathfrak{h}_{19}^-$*, entonces* $E_1(M) \cong E_\infty(M)$ *para toda* J.

(b) Si $\mathfrak{g} \cong \mathfrak{h}_2$ *ó* $\mathfrak{h}_4$*, entonces* $E_1(M) \cong E_\infty(M)$ *si y sólo si* J *es no abeliana; además, cualquier estructura compleja abeliana sobre* $\mathfrak{h}_2$ *ó* $\mathfrak{h}_4$ *cumple* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$.

(c) Si $\mathfrak{g} \cong \mathfrak{h}_5$ *y* J *es una estructura compleja sobre* $\mathfrak{h}_5$ *dada en el resultado 4 entonces:*

  (c.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *cuando* J *es compleja paralelizable;*

  (c.2) $E_1(M) \cong E_\infty(M)$ *si y sólo si* J *no es compleja paralelizable y* $\rho D \neq 0$*; además,* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *cuando* $\rho D = 0$.

(d) Si $\mathfrak{g} \cong \mathfrak{h}_{16}$ *ó* $\mathfrak{h}_{26}^+$*, entonces* $E_1(M) \ncong E_2(M) \cong E_\infty(M)$ *para toda* J.

(e) Si $\mathfrak{g} \cong \mathfrak{h}_{13}$ *ó* $\mathfrak{h}_{14}$*, entonces* $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$ *para toda* J.

(f) Si $\mathfrak{g} \cong \mathfrak{h}_{15}$ *y* J *es una estructura compleja sobre* $\mathfrak{h}_{15}$ *dada en el resultado 4 entonces:*

  (f.1) $E_1(M) \ncong E_2(M) \cong E_\infty(M)$*, cuando* $c = 0$ *y* $|B - \rho| \neq 0$*;*

(f.2) $E_1(M) \cong E_2(M) \ncong E_3(M) \cong E_\infty(M)$, *cuando* $\rho = 1$ *y* $|B-1| \neq c \neq 0$;

(f.3) $E_1(M) \ncong E_2(M) \ncong E_3(M) \cong E_\infty(M)$, *cuando* $\rho = 0$ *y* $|B| \neq c \neq 0$.

En general, al igual que sucede con la cohomología de de Rham, la coholomogía de Dolbeault de una solvariedad con una estructura compleja invariante no es computable sólo con los datos de la estructura en el álgebra de Lie. Sin embargo, el cálculo de la sucesión espectral de Frölicher a nivel del álgebra de Lie permite afirmar que si $E_1(\mathfrak{g}) \ncong E_\infty(\mathfrak{g})$ entonces $E_1(M) \ncong E_\infty(M)$. De esto obtenemos resultados parciales para algunas solvariedades complejas con $\mathfrak{g}$ isomorfa a $\mathfrak{g}_8$. A su vez, gracias a trabajos previos de Kasuya y Angella [51, 7], es posible determinar las cohomologías de Dolbeault y de Bott-Chern de una solvariedad dotada de una estructura compleja de tipo splitting por medio de ciertos complejos diferenciales de dimensión finita. Por una estructura compleja de tipo splitting [51, Assumption 1.1] sobre una solvariedad $M = G/\Gamma$ se entiende una estructura compleja que procede de una invariante por la izquierda sobre el grupo de Lie G de manera que éste se puede presentar como un producto semidirecto $G = \mathbb{C} \ltimes_\varphi N$, siendo N nilpotente, y tal que el lattice $\Gamma \subset G$ escogido presenta cierta compatibilidad con el splitting. Por tanto, para aplicar estos resultados a las estructuras complejas que dan lugar a un fibrado canónico holomorfo trivial es preciso identificar en primer lugar cuáles de estas estructuras complejas son a su vez de tipo splitting. Se deben encontrar en segundo lugar lattices compatibles con los splittings para los grupos de Lie correspondientes. En el siguiente resultado se sintetiza lo obtenido en el capítulo en relación a lo anterior:

Resultado 7. *Una solvariedad compleja* $(M = G/\Gamma, J)$*, siendo* J *invariante con fibrado canónico holomórficamente trivial, es de tipo splitting si* $(\mathfrak{g}, J)$ *es equivalente a* $(\mathfrak{g}_1, J)$, $(\mathfrak{g}_2^0, J)$, $(\mathfrak{g}_2^\alpha, J^\pm)$ *con* $\alpha > 0$ *ó* $(\mathfrak{g}_8, J^A)$ *con* $A \in \mathbb{C}$, $\mathfrak{Im}\, A \neq 0$. *Además:*

(a) si $(\mathfrak{g}, J)$ *es isomorfa a* $(\mathfrak{g}_2^0, J)$*, entonces el grupo de Lie correspondiente admite algunos lattices compatibles con el splitting tales que las solvariedades complejas cumplen* $E_1(M) \ncong E_\infty(M)$*, y otros lattices tales que las solvariedades complejas cumplen el* $\partial\bar{\partial}$*-lema.*

(b) si $(\mathfrak{g}, J)$ *es isomorfa a* $(\mathfrak{g}_2^\alpha, J^\pm)$ *para un* $\alpha > 0$*, entonces, para una familia numerable de* α*'s, los grupos de Lie correspondientes admiten lattices compatibles con las dos estructuras no equivalentes de tipo splitting* $J^\pm$*. Para unos lattices las solvariedades complejas cumplen* $E_1(M) \ncong E_\infty(M)$ *y para otros lattices las solvariedades complejas cumplen el* $\partial\bar{\partial}$*-lema.*

(c) si $(\mathfrak{g}, J)$ *es isomorfa a* $(\mathfrak{g}_8, J^A)$ *con* $A \in \mathbb{C}$, $\mathfrak{Im}\, A \neq 0$*, entonces el grupo de Lie correspondiente admite lattices compatibles con el splitting. Además, existe una familia de lattices* Γ_A *compatibles con el splitting de los grupos de Lie con estructura compleja* J^A *de manera que las solvariedades complejas cumplen el* $\partial\bar{\partial}$*-lema si y sólo si* $A \neq \frac{i}{k}$ *con* $k \neq 0$ *entero.*

Finalmente, toda solvariedad compleja $(M = G/\Gamma, J)$ *con álgebra de Lie subyacente* $\mathfrak{g}_8$ *y* J *equivalente a* J^{-i}, J^i, J' *ó* J'' *no satisface el* $\partial\bar{\partial}$*-lema.*

La última parte del Capítulo 4 está dedicada al estudio de estos invariantes complejos bajo deformaciones holomorfas. En relación a la sucesión espectral de Frölicher, Kodaira y Spencer [53] prueban la semicontinuidad de los números de Hodge, $\dim E_1^{\bullet,\bullet}(M)$, a lo largo de una deformación holomorfa, de donde se sigue que la propiedad $E_1(M) \cong E_\infty(M)$ es abierta por deformaciones. Eastwood y Singer [27] prueban que por contra, la propiedad $E_1(M) \cong E_\infty(M)$ no es cerrada por deformaciones. La geometría compleja sobre las álgebras de Lie nilpotentes $\mathfrak{h}_4$ y $\mathfrak{h}_{15}$ junto con la descripción de la sucesión espectral de Frölicher para nilvariedades complejas construídas a partir de ellas, nos permite obtener dos ejemplos que presentan comportamientos interesantes. El primero proporciona otro ejemplo del resultado de Eastwood y Singer mediante una deformación invariante y el segundo muestra un comportamiento de la propiedad $E_2(M) \cong E_\infty(M)$ diferente al hallado por Kodaira y Spencer para la propiedad $E_1(M) \cong E_\infty(M)$:

Resultado 8. *Una nilvariedad $(M = G/\Gamma, J_1)$, con $\mathfrak{g}$ isomorfa a $\mathfrak{h}_4$ y J_1 su estructura abeliana, admite una deformación holomorfa invariante $\{M_t := (M, I_t)\}_{t\in\Delta}$, con $I_0 = J_1$, tal que $E_1(M_t) \cong E_\infty(M_t)$ para todo $t \in \Delta^*$, mientras que $E_1(M_0) \ncong E_\infty(M_0)$.*

Además, existe una familia de nilvariedades complejas $\{M_t := (M, J_t)\}_{t\in\mathbb{R}}$, con $\mathfrak{g}$ isomorfa a $\mathfrak{h}_{15}$, tal que las dimensiones $\dim E_2^{0,2}(M_t)$, $\dim E_3^{0,2}(M_t)$ no son monótonamente crecientes y las dimensiones $\dim E_2^{1,1}(M_t)$, $\dim E_3^{1,1}(M_t)$ no son monótonamente decrecientes. Más aún, la propiedad $E_2(M) \cong E_\infty(M)$ no es abierta por deformaciones.

En relación al $\partial\bar{\partial}$-lema, es conocido que se trata de una propiedad abierta por deformaciones [98, 101, 10]. Más recientemente, Angella y Kasuya [8] han demostrado que el $\partial\bar{\partial}$-lema no es una propiedad cerrada bajo deformaciones holomorfas usando una deformación invariante de la variedad de Nakamura, que en nuestra clasificación se corresponde con una solvariedad dotada de una estructura compleja invariante tal que $(\mathfrak{g}, J)$ es isomorfa a $(\mathfrak{g}_8, J^{-i})$. Usando el estudio realizado para la cohomología de solvariedades complejas en este capítulo, obtenemos una familia infinita numerable de ejemplos que extienden el resultado de Angella y Kasuya:

Resultado 9. *Existe una familia numerable de solvariedades complejas de tipo splitting $\{(M_k := G/\Gamma_k, J_k)\}_{k\in\mathbb{Z}}$ que no cumplen el $\partial\bar{\partial}$-lema, siendo (G, J_k) un grupo de Lie con una estructura compleja invariante de manera que $(\mathfrak{g}, J_k)$ es isomorfa a $(\mathfrak{g}_8, J^{A_k})$ con $A_k = \frac{i}{2k+1}$ y $\Gamma_k \subset G$ un lattice compatible con el splitting inducido por J_k. Además, para cada $k \in \mathbb{Z}$ existe una deformación holomorfa $\{(M_k, J_{k,t})\}_{t\in\Delta}$ tal que cumple el $\partial\bar{\partial}$-lema para todo $t \in \Delta^*$, siendo $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$.*

En el Capítulo 5 nos hemos centrado en el estudio de la geometría Hermitiana sobre solvariedades de dimensión 6 compatible con las estructuras complejas invariantes que trivializan el fibrado canónico holomorfo. Ya eran conocidos resultados de existencia de métricas Kähler con torsión [33], invariantes 1-Gauduchon generalizadas [35] y equilibradas [95] para el caso de las nilvariedades. Por esta razón hemos estudiado la existencia de estas métricas para las estructuras complejas sobre las álgebras de Lie resolubles $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$. Para el caso de las métricas fuertemente Gauduchon el estudio se ha realizado para las álgebras de Lie anteriores junto con las nilpoltentes $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}$,

$\mathfrak{h}_{19}^-$, $\mathfrak{h}_{26}^+$. Los resultados obtenidos, junto con los ya conocidos para nilvariedades, los agrupamos en el siguiente:

Resultado 10. *Una solvariedad $M = G/\Gamma$ de dimensión 6 que admite estructuras complejas invariantes con fibrado canónico holomórficamente trivial, admite una métrica:*

(a) Kähler (y por tanto Calabi-Yau) si y sólo si $\mathfrak{g}$ es isomorfa a $\mathbb{R}^6$ ó $\mathfrak{g}_2^0$;

(b) Kähler con torsión si y sólo si $\mathfrak{g}$ es isomorfa a $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_8$, $\mathfrak{g}_2^0$ ó $\mathfrak{g}_4$;

(c) 1-Gauduchon generalizada invariante si y sólo si $\mathfrak{g}$ es isomorfa a $\mathfrak{h}_2$, $\mathfrak{h}_4$, $\mathfrak{h}_5$, $\mathfrak{h}_8$, $\mathfrak{g}_2^0$, $\mathfrak{g}_4$ ó $\mathfrak{g}_6$;

(d) equilibrada si y sólo si $\mathfrak{g}$ es isomorfa a $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$, $\mathfrak{h}_{19}^-$, $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ ó $\mathfrak{g}_8$;

(e) fuertemente Gauduchon si y sólo si $\mathfrak{g}$ es isomorfa a $\mathfrak{h}_1, \ldots, \mathfrak{h}_6$, $\mathfrak{h}_{19}^-$, $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha\geq 0}$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ ó $\mathfrak{g}_8$.

El estudio pormenorizado de existencia de estas métricas sobre las álgebras de Lie resolubles $\mathfrak{g}_1, \ldots, \mathfrak{g}_9$, dotadas con una estructura compleja admitiendo una forma cerrada no nula de tipo $(3,0)$, se resume en la siguiente tabla:

	Métricas			
	SKT	**1-Gauduchon (invariantes)**	**Equilibradas**	**sG**
$\mathfrak{g}_1$	$\nexists$	$\nexists$	$\exists$ para cualquier J	cualquier (J,F)
$\mathfrak{g}_2^0$	$\exists$ para cualquier J	$\exists$ para cualquier J	$\exists$ para cualquier J	cualquier (J,F)
$\mathfrak{g}_2^{\alpha>0}$	$\nexists$	$\nexists$	$\exists$ para cualquier J	cualquier (J,F)
$\mathfrak{g}_3$	$\nexists$	$\nexists$	$\exists$ para cualquier J	cualquier (J,F)
$\mathfrak{g}_4$	$\exists$ para cualquier J	$\exists$ para cualquier J	$\nexists$	$\nexists$
$\mathfrak{g}_5$	$\nexists$	$\nexists$	$\exists$ para cualquier J	$\exists$ para cualquier J
$\mathfrak{g}_6$	$\nexists$	$\exists$ para cualquier J	$\nexists$	$\nexists$
$\mathfrak{g}_7$	$\nexists$	$\nexists$	$\exists$ para cualquier J	$\exists$ para cualquier J
$\mathfrak{g}_8$	$\nexists$	$\nexists$	$\exists$ para cualquier $J \neq J', J''$	cualquier (J,F)
$\mathfrak{g}_9$	$\nexists$	$\nexists$	$\nexists$	$\nexists$

La condición equilibrada implica trivialmente la condición fuertemente Gauduchon. Popovici [72] a su vez prueba que una variedad compacta compleja que cumple el $\partial\bar{\partial}$-lema admite métricas fuertemente Gauduchon compatibles con la estructura compleja. Sin embargo, existen variedades complejas que son fuertemente Gauduchon "puras" en el sentido de que ni cumplen el $\partial\bar{\partial}$-lema ni son equilibradas. Sobre la clase de las nilvariedades de dimensión 6 (que, salvo los toros complejos, no cumplen el $\partial\bar{\partial}$-lema por no ser formales [44]) damos una descripción de aquellas estructuras complejas invariantes que admiten métricas fuertemente Gauduchon compatibles sin admitir métricas equilibradas:

Resultado 11. *Sea $M = G/\Gamma$ una nilvariedad de dimensión 6 dotada de una estructura compleja invariante J tal que (M, J) no admite métricas equilibradas. Si (M, J) admite métricas fuertemente Gauduchon entonces $\mathfrak{g}$ es isomorfa a $\mathfrak{h}_2$, $\mathfrak{h}_4$ ó $\mathfrak{h}_5$.*

Además, tal J es no abeliana y dada por: $\mathfrak{Re}\, D + (\mathfrak{Im}\, D)^2 \geq \frac{1}{4}$ en $\mathfrak{h}_2$; $\mathfrak{Re}\, D \geq \frac{1}{4}$ en $\mathfrak{h}_4$; y $\lambda = 0, \mathfrak{Im}\, D \neq 0$ ó $\lambda = \mathfrak{Im}\, D = 0, \mathfrak{Re}\, D \geq 0$ en $\mathfrak{h}_5$, según la descripción de las estructuras complejas proporcionada en el Resultado 4.

Finalmente nos hemos centrado en el estudio de las propiedades equilibrada y fuertemente Gauduchon en relación a la teoría de deformaciones. Alessandrini y Bassanelli [3] prueban que la propiedad equilibrada no es abierta por deformaciones mientras que Popovici [73] prueba que la propiedad fuertemente Gauduchon sí lo es. Sin embargo, se conjeturó que estas propiedades eran cerradas por deformaciones holomorfas [76, Conjetura 1.21, Conjetura 1.23]. Mediante el estudio realizado en el resultado anterior sobre la existencia de geometría equilibrada y fuertemente Gauduchon sobre nilvariedades de dimensión 6, hemos construído un contraejemplo a ambas conjeturas que recogemos en el siguiente resultado:

Resultado 12. *Una nilvariedad $(M = G/\Gamma, J_1)$, con $\mathfrak{g}$ isomorfa a $\mathfrak{h}_4$ y J_1 su estructura abeliana, admite una deformación holomorfa invariante $\{M_t := (M, I_t)\}_{t\in\Delta}$, con $I_0 = J_1$, tal que las M_t son equilibradas para todo $t \in \Delta^*$, mientras que M_0 no es fuertemente Gauduchon.*

Como consecuencia, las propiedades equilibrada y fuertemente Gauduchon no son cerradas bajo deformaciones holomorfas.

Aunque, como ya se ha mencionado antes, la propiedad del $\partial\bar{\partial}$-lema no es cerrada por deformaciones, Popovici [72] prueba que una deformación holomorfa $\{(M, J_t)\}_{t\in\Delta}$ cumpliendo el $\partial\bar{\partial}$-lema en todo $t \in \Delta^*$ implica que en el límite central existe una métrica fuertemente Gauduchon, que como hemos dicho, es una propiedad necesaria para que la variedad compacta compleja cumpla el $\partial\bar{\partial}$-lema. En el siguiente resultado mostramos, mediante una deformación holomorfa de una solvariedad compleja, que el resultado de Popovici es óptimo, en el sentido de que no se puede esperar (en general) encontrar métricas equilibradas en el límite central:

Resultado 13. *Existe una deformación holomorfa invariante $\{M_t := (G/\Gamma, J_t)\}_{t\in\Delta}$ de una solvariedad, siendo $\mathfrak{g}$ isomorfa a $\mathfrak{g}_8$ y J_t con fibrado canónico holomorfo trivial, tal que M_t cumple el $\partial\bar{\partial}$-lema y es equilibrada en todo $t \in \Delta^*$ mientras que M_0 no cumple el $\partial\bar{\partial}$-lema ni es equilibrada.*

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