Máster en Ingeniería de Sistemas e Informática
Trabajo Fin de Máster

Redes de Petri híbridas adaptativas: alcanzabilidad y ausencia de bloqueos

María Estíbaliz Fraca Santamaría

Director: Jorge Emilio Júlvez Bueno

Codirector: Manuel Silva Suárez

Departamento de Informática e Ingeniería de Sistemas
Escuela de Ingeniería y Arquitectura
Universidad de Zaragoza

Septiembre 2011

Escuela de Ingeniería y Arquitectura
Universidad Zaragoza

Departamento de Informática e Ingeniería de Sistemas
Universidad Zaragoza

Redes de Petri híbridas adaptativas: alcanzabilidad y ausencia de bloqueos

RESUMEN

Las redes de Petri (RdP) [3] constituyen un paradigma formal ampliamente aceptado para el modelado de sistemas de eventos discretos. No obstante, con poblaciones de gran tamaño, padecen del problema de la explosión de estados (crecimiento exponencial del tamaño del conjunto de estados alcanzables con respecto a la población inicial del sistema). Una manera de paliar este problema consiste en relajar la restricción de integralidad del formalismo y considerar redes de Petri continuas [5, 8]. Las redes de Petri continuas permiten abordar de manera eficiente el estudio de los sistemas mediante técnicas lineales de análisis.

Sin embargo, siendo las redes continuas una relajación de las discretas, no siempre preservan sus propiedades, como por ejemplo la ausencia de bloqueos [9]. En este Trabajo se introduce, formaliza y estudia un formalismo nuevo, denominado redes de Petri híbridas adaptativas (HAPN), basado en una relajación alternativa de la integralidad.

En una red de Petri discreta, continua o híbrida, las transiciones son definidas a priori como discretas o como continuas, lo que determina su modo de comportamiento en todo instante de tiempo [11]. Esta definición estática no permite adaptar el comportamiento del modelo a la carga, que varía dinámicamente. En cambio, el comportamiento de las transiciones de la red adaptativa es variable: una transición se comporta como *continua* si su carga de trabajo supera un umbral establecido inicialmente, en caso contrario se comporta como *discreta*. Dado que las inconsistencias entre las redes discretas y las continuas suelen darse cuando las poblaciones son pequeñas, se ha intentado que las redes adaptativas no presenten estos problemas, ya que en ese caso el comportamiento es discreto. Además, cuando las poblaciones son elevadas el comportamiento es continuo, por lo que las técnicas lineales son aplicables, evitando el problema de la explosión de estados.

En primer lugar, se ha definido formalmente el formalismo de redes de Petri adaptativas. En el ámbito de este Proyecto Fin de Máster, el formalismo no considera ninguna interpretación temporal. Tras estudiar diversas alternativas para determinar el comportamiento de las transiciones en función de su carga, la opción elegida consiste en establecer un umbral para la carga de trabajo de cada transición. Para toda carga inferior al umbral, el comportamiento de la transición es discreto, mientras que el comportamiento es continuo para cargas superiores.

A partir de la definición de las HAPNs, se ha caracterizado el conjunto de sus marcados alcanzables de las redes de Petri adaptativas. El conjunto global de marcados alcanzables no será, en general, convexo como lo es el de las redes continuas, pero es caracterizable como una unión de conjuntos convexos.

Por último, se estudia la ausencia de bloqueos, una propiedad básica y necesaria para que las acciones de un sistema tengan un comportamiento adecuado. Se intenta no tanto determinar si una red puede bloquearse, sino si la red adaptativa preserva la ausencia de bloqueos de la red discreta con misma estructura y marcado inicial.

En conclusión, se ha definido el formalismo de las HAPN, en el cual cada transición combina comportamientos discretos y continuos en función de la carga de trabajo, con el objetivo de realizar una fluidificación parcial de las RdP discretas que preserve algunas propiedades que las RdP completamente continuas no siempre preservan. Este formalismo incluye a las redes de Petri discretas, continuas e híbridas. Además, se han estudiado las propiedades de alcanzabilidad y ausencia de bloqueos del formalismo en relación a las Redes de Petri discretas.

Contents

List of Figures

Chapter 1

Introduction

This “Trabajo Fin de Máster” (TFM) introduces and studies **H**ybrid **A**daptive **P**etri **N**ets (HAPNs) [1], a formalism in the paradigm of Petri nets (PN) [3].

HAPNs combine discrete and continuous behaviours from the discrete and continuous Petri nets, and attempt to partially fluidify discrete PN models mantaining their relevant properties.

The results obtained in this work have been published in the proceedings of an international conference [1].

1.1 Context

Discrete event systems appear in many fields, for instance in manufacturing, logistics, computer networks, traffic systems, etc. Having suitable modeling formalisms and formal techniques for its design, development and implementation is essential to achieve correct and eficient systems behaviour.

Petri nets are a formal paradigm widely used for the modeling of discrete event systems, due to its powerful analysis and synthesis techniques and its direct graphical representation. However, as in most formalisms for discrete event systems, the set of reachable states grows exponentially with respect to the initial population of the system. Thus, many analysis techniques based on the exploration of the state space are inefficient for the analysis of high populated systems: this is the well known *state explosion problem*. It is a crucial drawback in the analysis of discrete event systems. An interesting technique to overcome this difficulty is to relax the original discrete model and deal with a continuous approximation. Such a relaxation aims at computationally more efficient analysis methods, at the price of losing some precision.

Unfortunately, the transformation to a continuous model may not always preserve important properties of the original discrete model. In the context of Petri nets (PNs), the transformation from discrete to continuous [6, 5, 8] does not preserve, in general, properties as deadlock-freeness, liveness, reversibility, etc [7].

1.2 Motivation

This TFM focuses on hybrid adaptive Petri nets [2], a Petri net based formalism in which the firing of transitions is partially relaxed. Transitions of HAPN can behave in two different modes: *continuous* and *discrete*. The continuous mode will be chosen when the transition workload is higher than a given threshold. It makes sense because, in general, the higher the workload the better the continuous approximation. Consequently, it also makes sense to switch to a discrete mode when the workload becomes low.

This way, a HAPN is able to *adapt* its behaviour to the net workload; it offers the possibility to represent more faithfully the discrete system and simplifies analysis techniques by behaving as continuous when the load is high. In contrast to [2], HAPNs will be defined and studied in the untimed framework. Notice that the introduction of time in a given PN system would produce a particular system trajectory that is also achievable in the untimed one. Thus, the results for some properties as deadlock-freness in the untimed framework can be almost straightforwardly applied on timed systems, in the form of necessary or sufficient conditions.

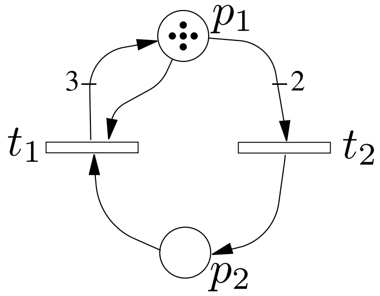


Figure 1.1: A Petri net system that deadlocks as continuous but it is deadlock-free as hybrid adaptive with appropriate thresholds.

Let us consider the PN system in Figure 1.1 [7] to introduce the behaviour of HAPNs. Let the initial marking of the system be $\boldsymbol{m_0} = (5, 0)$. If considered as a discrete system, it is deadlock-free: from the initial marking $\boldsymbol{m_0}$ only t_2 can fire, reaching $\boldsymbol{m_1} = (3, 1)$. From $\boldsymbol{m_1}$, both $\boldsymbol{m_0}$ and $\boldsymbol{m_2} = (1, 2)$ can be reached by firing t_1 and t_2 respectively. None of the reachable markings deadlocks the system, hence it is deadlock-free. This behaviour is represented in the reachability graph and reachability space in Figure 1.2. The arrows of the *reachability graph* on the right hand of the Figures 1.2, 1.4 and 1.3 are solid for the continuous firings and dotted for the discrete ones.

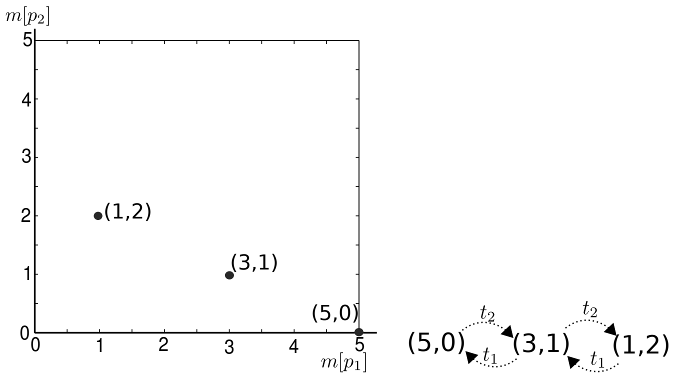


Figure 1.2: Reachability space of the Petri net in Figure 1.1 when considered discrete.

Consider now that the system is continuous [10], i.e., each transition can be fired in any non-negative real amount less than or equal to its enabling degree. As it will be explained in section 2.3, the enabling degree of t_2 at the marking $\boldsymbol{m_0}$ is 2.5. Therefore, t_2 can fire in any real amount in the interval $[0, 2.5]$. Figure 1.3 shows the reachability space of the continuous PN. The firing of t_2 in an amount lower than 2.5 produces positive markings in both places and both transitions are enabled. However, the firing of t_2 in 2.5 from $\boldsymbol{m_0}$ leads to $(0, 2.5)$ where no transition is enabled and the system deadlocks. Consequently, deadlock-freeness is not preserved by the continuous PN.

Let us finally assume that the net system is hybrid adaptive. For these systems, a transition t_i can have two different firing modes: *continuous* and *discrete*. It behaves as continuous when its enabling

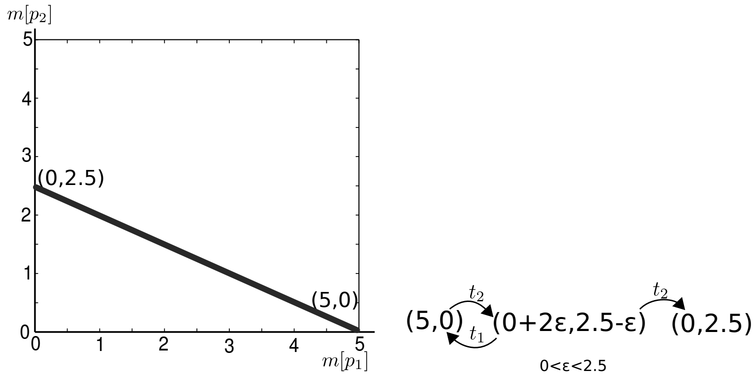


Figure 1.3: Reachability space of the Petri net in Figure 1.1 when considered continuous.

degree is higher than a given threshold μ_i. Otherwise, t_i behaves as discrete.

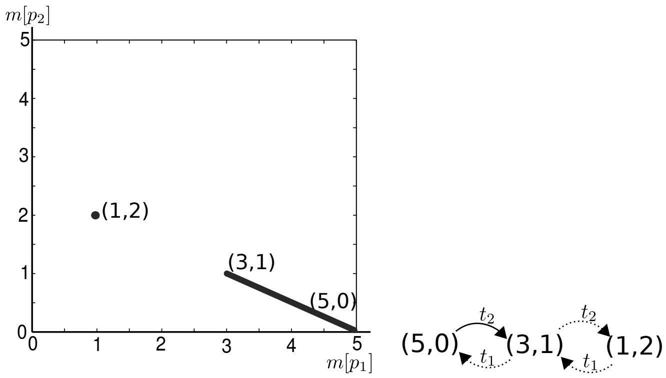


Figure 1.4: Reachability space of the Petri net in Figure 1.1 when considered hybrid adaptive.

When a discrete system is considered as hybrid adaptive, appropiate thresholds have to be defined. Let us define $\mu_1 = 1$ for t_1 and $\mu_2 = 1.5$ for t_2 for the system of Figure 1.1. At the initial marking $\boldsymbol{m_0} = (5,0)$, t_1 is not enabled, and t_2 behaves as continuous, and it can fire in real amounts while it remains continuous. If t_2 is fired in an amount of 1, $\boldsymbol{m_1} = (3,1)$ is reached. At $\boldsymbol{m_1}$, both t_1 and t_2 are enabled as discrete. The firing of $t_1(t_2)$ from $\boldsymbol{m_1}$ leads to $\boldsymbol{m_0}(\boldsymbol{m_2} = (1,2))$. At $\boldsymbol{m_2}$ both transitions are discrete but only t_1 is enabled, whose firing leads to $\boldsymbol{m_1}$. Hence, although the adaptive system still keeps some continuous behaviour, it preserves the deadlock-freeness property of the discrete system. Figure 1.4 shows the reachability space of the HAPN.

In summary, deadlock-freeness property of a discrete system might not be preserved by the continuous aproximation; nevertheless, it could be preserved by the *hybrid adaptive* approximation.

1.3 Objectives and scope

The aim of this TFM is the definition and study of a new formalism: the Hybrid Adaptive Petri net (HAPN). In this formalism, each transition combinates discrete and continuous behaviours, depending on the *workload*, what results in a partial fluidification of the PN. The main goal of HAPN is to reserve some properties of the discrete nets that may not be preserved by the usual continuous aproximation, while avoiding the state explosion problem.

The first objective is the mathematical definition of the HAPN formalism. The second is to study its reachability space. And the third is to study the conditions needed by the HAPNs to preserve the deadlock-freeness property of the equivalent discrete PNs.

1.4 Document organization

The rest of the document is organised as follows: First, some previous concepts are presented in Chapter 2. In Chapter 3, HAPNs are formally defined. Chapter 4 studies the reachability space of HAPNs and relates it to those of the discrete and continous Petri nets. Chapter 5 presents some results about deadlock-freeness in HAPNs. Finally, conclusions and future work are presented in Chapter 6.

Chapter 2

Preliminary Concepts

2.1 Petri nets

Petri nets (PN) [3] are a mathematical formalism to model discrete event systems. They allow to easily model concurrency, sincronization, mutual exclusion and conflicts.

PNs have a graphical representation given by a bipartited graph, where nodes can be places, represented with circles; or transitions, represented by bars or boxes. The arcs between nodes can be directed from a place to a transition or from a transition to a place. Each place can have a certain number of tokens. Usually, places describe states of the system, while transitions represent events that modify the system state.

In discrete Petri nets, the tokens of a place are a natural number and the firing of the transitions is discrete. The fluidification of discrete Petri nets gives the continuous PN formalism, where firings of transitions are continuous and the tokens contained in a place can have positive real values. Both Petri net formalisms are presented formally in the next sections.

The PN considered here are autonomous, i.e., they do not have a time interpretation.

2.2 Discrete Petri nets

The discrete Petri nets [4, 7] are Petri nets whose transitions fire discrete tokens. They can be defined as follows:

Definition 1 *A discrete PN is a tuple* $\mathcal{N}_D = \langle P, T, Pre, Post \rangle$ *where:*

- $P = \{p_1, p_2, ..., p_n\}$ *and* $T = \{t_1, t_2, ..., t_m\}$ *are disjoint and finite sets of places and transitions.*
- Pre *and* $Post$ *are* $|P| \times |T|$ *sized, natural valued, incidence matrices.*

$Post[p, t] = \omega$ means that there is an *arc* from t to p with *weight* (or *multicipliplity*) ω. While $Post[p, t] = 0$ indicates no *arc* from t to p. In the same way, $Pre[p, t]$ indicates if there is an *arc* from p to t and its *weight*. Given a place (or transition) $v \in P$ (or T), its *preset*, ${}^\bullet v$, is defined as the set of its input transitions (or places), and its *postset* $v^\bullet$ as the set of its output transitions (or places).

A *marking* m of a discrete Petri net is defined as a $|P|$ sized, natural valued, vector: $\boldsymbol{m} \in (\mathbb{N}_{\geq 0})^{|P|}$. Given a Petri net and a marking, the Petri net system can be defined:

Definition 2 *A discrete Petri net system is a tuple* $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$*, where* $\boldsymbol{m_0} \in (\mathbb{N}_{\geq 0})^{|P|}$ *is the initial marking.*

A transition $t_i \in T$ is enabled at a marking m if and only if (iff) for every $p \in {}^\bullet t_i$, $m[p] \geq Pre[p, t_i]$. The enabling degree is calculated as follows:

$enab(t_i, m) = min_{p \in {}^\bullet t_i} \lfloor \frac{m[p]}{Pre[p,t_i]} \rfloor$

An enabled transition t_i can be fired any natural amount less than or equal to the enabling degree. The firing of a transition t in a certain amount $\alpha \leq$ enab(t, $\boldsymbol{m}$) leads to a new marking $\boldsymbol{m'}$, and it is denoted as $\boldsymbol{m} \xrightarrow{\alpha t} \boldsymbol{m'}$. It holds $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free). Hence, as in discrete systems, $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves, where $\boldsymbol{\sigma}$ is the firing count vector of the fired sequence. Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. As in discrete systems, when $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*.

The set of all the reachable markings for a given system $\langle N, m_0 \rangle$ is denoted as RS(N, m_0):

Definition 3 *RS (N, m_0) =* $\{\ m |\ a\ finite\ fireable\ sequence\ \sigma\ =\ \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}\ exists\ such\ that\ m_0 \overset{\alpha_1 t_{\gamma_1}}{\longrightarrow}\ m_1 \overset{\alpha_2 t_{\gamma_2}}{\longrightarrow}\ m_2 \cdots \overset{\alpha_k t_{\gamma_k}}{\longrightarrow}\ m_k = m\ \ where\ t_{a_i} \in T\ \}$

The set of reachable markings is used to define liveness and deadlock-freeness properties. Let $\langle \mathcal{N}, m_0 \rangle$ be a discrete system:

- $\langle \mathcal{N}, m_0 \rangle$ deadlocks iff a marking $m \in$ RS ($\mathcal{N}$,m_0) exists such that $\forall t \in T$ is not enabled.
- $\langle \mathcal{N}, m_0 \rangle$ is live iff for every transition t and for any marking $m \in$ RS ($\mathcal{N}$,m_0) there exists $m' \in$ RS(N, m) such that t is enabled.
- $\mathcal{N}$ is structurally live (deadlock-free) iff $\exists m_0$ such that $\langle \mathcal{N}, m_0 \rangle$ is live (deadlock-free).

As it will be shown in the next Chapter, a discrete Petri net is equivalent to an hybrid adaptive Petri net where $\forall i,\ \ \mu_i = \infty$, or μ_i is high enough.

2.3 Continuous Petri nets

The continuous Petri nets [6, 5] are the fluidification of discrete Petri nets: the firing of the transitions can be in $\mathbb{R}_{>0}$, not only in $\mathbb{N}$. Similarly to discrete PN, continuos Petri nets can be defined:

Definition 4 *A Continuous PN is a tuple* $\mathcal{N}_C$ = $\langle P, T, Pre, Post \rangle$ *where:*

- $P = \{p_1, p_2, ..., p_n\}$ *and* $T = \{t_1, t_2, ..., t_m\}$ *are disjoint and finite sets of places and transitions.*
- Pre *and* $Post$ *are* $|P| \times |T|$ *sized, natural valued, incidence matrices.*

As in continuous PN, $Post[p, t] = \omega$ means that there is an *arc* from t to p with *weight* (or *multiciplity*) ω. While $Post[p, t] = 0$ indicates no *arc* from t to p. In the same way, $Pre[p, t]$ indicates if there is an *arc* from p to t and its *weight*. Given a place (or transition) $v \in P$ (or T), its *preset*, ${}^\bullet v$, is defined as the set of its input transitions (or places), and its *postset* $v^\bullet$ as the set of its output transitions (or places).

A *marking* m in a continuous Petri net is defined as a $|P|$ sized, real valued, vector: $\boldsymbol{m} \in (\mathbb{R}_{\geq 0})^{|P|}$ Continuous Petri net systems can be defined as follows:

Definition 5 *A continuous Petri net system is a tuple* $\langle \mathcal{N}_C, \boldsymbol{m_0} \rangle$, *where* $\boldsymbol{m_0} \in (\mathbb{R}_{\geq 0})^{|P|}$ *is the initial marking.*

Unlike discrete nets, a transition $t_i \in T$ in a continuous net is enabled at a marking m iff for every $p \in^{\bullet} t_i$, $m[p] > 0$. The enabling degree is calculated as follows:

$$enab(t_i, m) = min_{p \in {}^{\bullet}t_i}\{\frac{m[p]}{Pre[p, t_i]}\}$$

A transition t_i can be fired in a certain amount α :

$0 < \alpha \leq enab(t_i)$ with $\alpha \in \mathbb{R}$

The firing of t_i in the amount α leads to a new marking **m'**, and it is denoted as **m** $\xrightarrow{\alpha t_i}$ **m'**.

The set of all the reachable markings for a given system $\langle N, m_0 \rangle$ is denoted as RS(N, m_0):

Definition 6 *RS (*N, m_0*) = { m |* $a\ finite\ fireable\ sequence\ \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}\ exists\ such\ that\ m_0 \xrightarrow{\alpha_1 t_{\gamma_1}} m_1 \xrightarrow{\alpha_2 t_{\gamma_2}} m_2 \cdots \xrightarrow{\alpha_k t_{\gamma_k}} m_k = m\ \ where\ t_{a_i} \in T\ and\ \alpha_i \in \mathbb{R}^+\}$

Definition 7 *Let* $\langle \mathcal{N}_C, \boldsymbol{m_0} \rangle$ *be a continuous system. A marking* $m \in (\mathbb{R} + \cup\{0\})^{|P|}$ *is lim-reachable, iff a sequence of reachable markings* $\{m_i\}_{i \geq 1}$ *exists such that*

$$m_0 \xrightarrow{\sigma_1} m_1 \xrightarrow{\sigma_2} m_2 \ldots\ m_{i-1} \xrightarrow{\sigma_i} m_i \ldots$$

and $\lim_{i \to \infty} \boldsymbol{m_i} = \boldsymbol{m}$. *The lim-reachable set is the set of lim-reachable markings, and it will be denoted lim-RS(*$\mathcal{N}C$*,*$\boldsymbol{m_0}$*) [8].*

As it will be shown in the next Chapter, a continuous Petri net is equivalent to a hybrid adaptive Petri net where $\forall i,\ \ \mu_i = 0$.

2.4 Hybrid Petri nets

The hybrid Petri nets [5] are partially fluidified Petri nets in which each of the transitions is defined as continuous or as discrete, such that each transition will behave always discrete or always continuous, but not both. Hybrid PNs are defined as follows:

Definition 8 *A hybrid PN is a tuple* $\mathcal{N}_C$ = $\langle P, T, Pre, Post \rangle$ *where:*

- $P = \{p_1, p_2, ..., p_n\}$ *and* $T = \{t_1, t_2, ..., t_m\}$ *are disjoint and finite sets of places and transitions.*
- Pre *and* $Post$ *are* $|P| \times |T|$ *sized, natural valued, incidence matrices.*
- *The set of transitions* T *is partitioned in two sets,* T^c *and* T^d *, where* T^c *contains the set of continuous transitions and* T^d *the set of discrete ones.*

In contrast to [5, 11], the set of places P can be partitioned in two sets, here no explicit partition is considered, being the marking of a place a natural or real number depending on the firings of its intput and output transition. A *marking* m in a hybrid Petri net is defined as a $|P|$ sized, real valued, vector: $\boldsymbol{m} \in (\mathbb{R}_{\geq 0})^{|P|}$. The hybrid Petri net system can be also defined as follows.

Definition 9 *A hybrid Petri net system is a tuple* $\langle \mathcal{N}_H, \boldsymbol{m_0} \rangle$*, where* $\boldsymbol{m_0} \in (\mathbb{R}_{\geq 0})^{|P|}$ *is the initial marking.*

In the same way that HAPN include discrete and continuous Petri net, also hybrid adaptive Petri nets are included in the HAPN formalism. To define an hybrid Petri net, an infinite threshold should be associated to each discrete transition, $\forall i \in T^d, \ \mu_i = \infty$ and a threshold equal to 0 to each continuous transition: $\forall j \in T^c, \ \ \mu_j = 0$.

2.5 Some net subclasses

Typically, Petri net subclasses are defined by imposing some constraints on the structure of the net. The following ones are among the most usual net subclasses:

Definition 10 *(Some Petri net subclasses).*

- Ordinary Petri nets are those nets whose arc weights are 1, i.e., $\forall p \in P \ \ \forall t \in T, Pre[p,t] \in \{0,1\}$ and $Post[p,t] \in \{0,1\}$.
- Choice free Petri nets are PN where each place has at most one output transition, i.e., $\forall p \ |p^\bullet| \leq 1$.
- State machines (SM) are ordinary Petri nets where each transition has one input and one output place, i.e., $\forall t, |{}^\bullet t| = |t^\bullet| = 1$.
- Marked graphs (MG) are ordinary Petri nets where each place has one input and one output transition, i.e., $\forall p |{}^\bullet p| = |p^\bullet| = 1$.
- Join free (JF) nets are Petri nets in which each transition has at most one input place, i.e., $\forall t \in T, |{}^\bullet t| \leq 1)$.
- Choice free (CF) nets are Petri nets in which each place has at most one output transition, i.e., $forallp, |p^\bullet| \leq 1$.
- Free choice (FC) nets are ordinary Petri nets in which conflicts are always equal, i.e., $\forall t, t'$, if ${}^\bullet t \cap {}^\bullet t' =$, then ${}^\bullet t = {}^\bullet t'$.
- Equal Conflict (EQ) nets are Petri nets in which conflicts are always

Chapter 3
Hybrid adaptive Petri nets

This Chapter introduces the formalism of hybrid adaptive Petri nets, which consists on a partial fluidification of the firing of transitions.

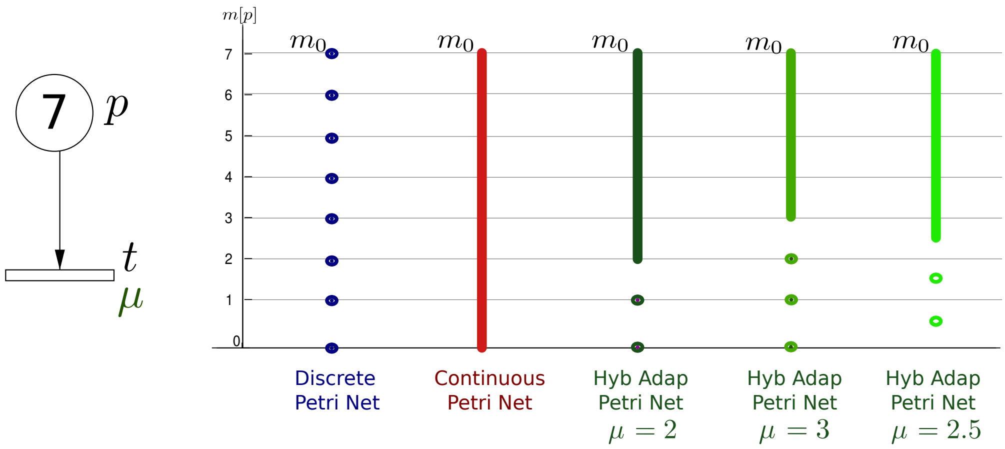
3.1 Formal definition

Hybrid adaptive PNs are a relaxation of discrete PNs, such that a threshold is associated to each transition, which determines the behaviour mode.

The following example illustrates the behaviour of an adaptive transition, explaining the behaviour of a PN with just one place and one transition.

Example 11 *Figure 3.1 (b) explains the behaviour of the Petri net in Figure 3.1 (a), from the initial marking* $m_0 = (7)$*. Firstly, the possible markings reachable from the PN when it is a discrete Petri net are shown in blue color. The net starts with the initial marking* $m[p] = 7$*, and it decreases with the discrete firings of* t *until it reaches* $m[p] = 0$*.*

Secondly, the red line of the Figure represents the possible reachable markings of the net when it is a continuous PN. The marking of place p *can decrease in any amount by the firing of the continuous transition* t*. As explained in the section 2.3, the marking* $m[p] = 0$ *will be reached just in the limit.*



(a) (b)

Figure 3.1: Example of a Petri net system and the possible markings of the place p.

Finally, the net is considered as HAPN. Different values of the threshold μ *are considered, and the reachable markings for each* μ *are sketched in green color in the Figure 3.1. Notice that transition* t *has an associated threshold* μ*. When the marking of* p*, is bigger than* μ*, the firings of* t *are continuous.*

Otherwise, the firings are discrete. For example, when the threshold is $\mu = 2.5$, the firings are continuous from $m[p] = 7$ to $m[p] = 2.5$, and it is discrete for $m[p] \leq 2.5$. In this example, half token will remain in p.

The formal definition of the HAPNs is inspired in the definition of discrete Petri nets, adding the thresholds that are asociated to each transitions. The hybrid adaptive Petri nets are defined mathematically below.

Definition 12 *A HAPN is a tuple $\mathcal{N}_A = \langle P, T, \mathbf{Pre}, \mathbf{Post}, \boldsymbol{\mu} \rangle$ where:*

- *$P = \{p_1, p_2, ..., p_n\}$ and $T = \{t_1, t_2, ..., t_m\}$ are disjoint and finite sets of places and transitions.*
- *$\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, incidence matrices.*
- *$\boldsymbol{\mu} \in (\mathbb{R}_{\geq 0} \cup \infty)^{|T|}$ is the vector of thresholds.*

Given a place (transition) $v \in P(T)$, its *preset*, ${}^\bullet v$, is defined as the set of its input transitions (places), and its *postset* $v^\bullet$ as the set of its output transitions (places).

Definition 13 *A HAPN system is a tuple $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$, where $\boldsymbol{m_0} \in (\mathbb{R}_{\geq 0})^{|P|}$ is the initial marking.*

A threshold μ is associated with each transition t. When the marking of ${}^\bullet t$ is above the threshold, t behaves in continuous mode (C); and otherwise it behaves in discrete mode (D)

As in continuous PNs, the enabling degree of t_i at m is defined as:

$$enab(t_i, m) = min_{p \in {}^\bullet t_i} \left\{ \frac{m[p]}{Pre[p, t_i]} \right\} \tag{3.1}$$

The threshold μ_i of a transition t_i determines the values of the enabling degree for which the transition behaves in continuous (C) or in discrete (D) mode:

$$mode(t_i, m) = \begin{cases} C & \text{if } enab(t_i, m) > \mu_i \\ D & \text{otherwise} \end{cases} \tag{3.2}$$

If a transition t_i is in *continuous* mode then $enab(t_i, m) > \mu_i$ what implies that t_i is enabled as continuous. On the other hand, if t_i is in *discrete* mode then it is enabled iff $enab(t_i, m) \geq 1$. This two conditions together imply that t_i is enabled (either as discrete or continuous) iff the following expression is true:

$$(mode(t_i, m) = C) \vee (mode(t_i, m) = D \wedge enab(t_i, m) \geq 1)$$

This expression is equivalent to:

$$(enab(t_i, m) > \mu_i) \vee (enab(t_i, m) \leq \mu_i \wedge enab(t_i, m) \geq 1)$$

what simplifies to:

$$enab(t_i, m) > \mu_i \vee enab(t_i, m) \geq 1, \text{ with } \mu \in \mathbb{R}_{\leq 0}$$

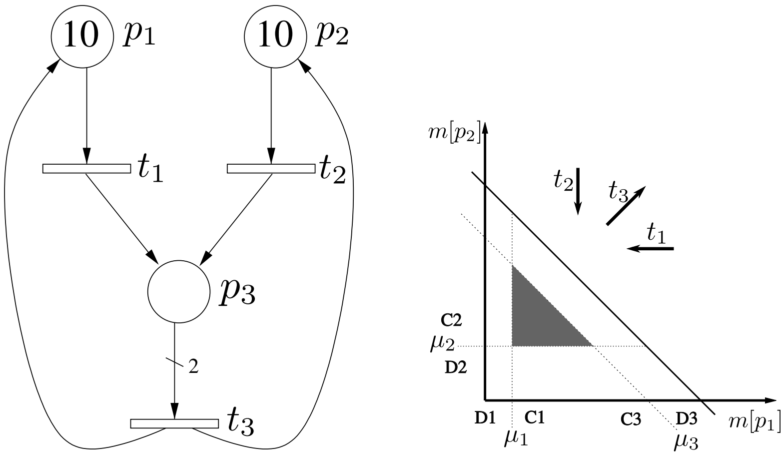
A transition t_i that is enabled can fire. The admissible firing amounts depend on its mode. If $mode(t_i, m) = C$, t_i can fire in any real amount $\alpha \in \mathbb{R}_{\geq 0}$ that does not make the enabling degree cross the threshold μ_i, i.e., $0 < \alpha \leq enab(t_i, m) - \mu_i$. If $mode(t_i, m) = D$, t_i can fire as a usual discrete transition in any natural amount $\alpha \in \mathbb{N}$ such that $0 < \alpha \leq enab(t_i, m)$.

As in discrete or continuous PN, the firing of a transition t in a certain amount $\alpha \leq$ enab(t, $\boldsymbol{m}$) leads to a new marking $\boldsymbol{m}'$, and it is denoted as $\boldsymbol{m} \xrightarrow{\alpha t} \boldsymbol{m}'$.

It holds $\boldsymbol{m}' = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free). Hence, as in discrete systems, $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves, where $\boldsymbol{\sigma}$ is the firing count vector of the fired sequence.

Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. As in discrete systems, when $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*.

The following example illustrates which is the behaviour mode of each transition of a given HAPN



	t_1	t_2	t_3
A	D	C	D
B	D	C	C
C	C	C	C
D	C	C	D
E	D	D	C
F	C	D	C
G	C	D	D

(a) (b) (c)

Figure 3.2: An hybrid adaptive Petri net (a) and the behaviour of its transitions (b) and (c).

Example 14 *Figure 3.2 (b) illustrates the behaviour of the transitions of the HAPN in Figure 3.2 (a), with any $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. Notice that the three arrows (t_1, t_2, t_3) of Figure 3.2 (b) indicate the "direction" in which the marking "moves" when t_1, t_2 or t_3 are fired. The Figure shows the regions in which t_1, t_2 and t_3 behave as discrete (regions D_1, D_2, D_3) or continuous (C_1, C_2, C_3). For example, t_3 behaves as continuous (C_3) below the dotted line corresponding to μ_3 and discrete above the line (D_3). In the triangular grey region of the center of Figure 3.2 (b), the PN behaves as continuous, and in the other regions, it has a partially discrete behaviour (some transitions behave as discrete and some as continuous).*

Figure 3.2 (c) summarizes the behaviour of each one of the transitions in the different areas identified in the Figure 3.2 (b). For example, in the area A, t_1 and t_3 behave as discrete while t_2 behaves as continuous.

Finally, notice that if $\boldsymbol{\mu} = \mathbf{0}$, all transitions will behave as continuous, and if $\boldsymbol{\mu} = \infty$ all transition will behave as discrete. Hence, the HAPN formalism includes both the continuous and discrete PN formalisms.

3.2 Reachability and liveness definitions

The set of all the reachable markings of a given HAPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is denoted as reachability space, RS($\mathcal{N}$, $\boldsymbol{m_0}$), and it is defined as follows.

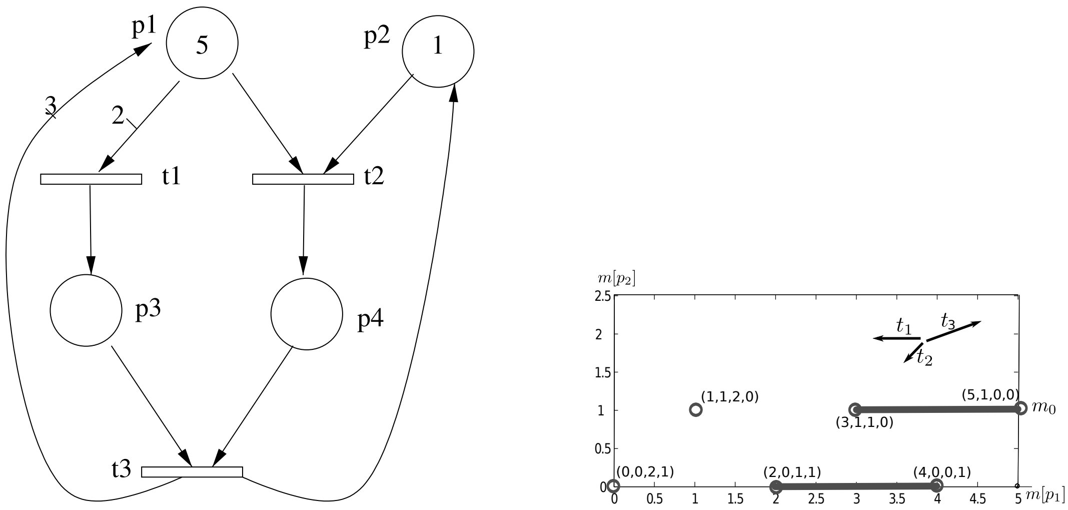
Definition 15 *RS($\mathcal{N}, \boldsymbol{m_0}$) =* $\{\boldsymbol{m} \mid \exists\, \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ *such that* $\boldsymbol{m_0} \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ *where* $\alpha_i \in \mathbb{R}^+$ *if* $mode(t_{\gamma_i}, m_{i-1}) = C,$ *and* $\alpha_i \in \mathbb{N}^+$ *if* $mode(t_{\gamma_i}, m_{i-1}) = D\}$

Liveness and deadlock-freeness properties are defined in a similar way to those of discrete systems.

Definition 16 *Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a HAPN system.*

- *$\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ deadlocks iff a marking $\boldsymbol{m} \in$ RS ($\mathcal{N}$, $\boldsymbol{m_0}$) exists such that $\forall t \in T$, t is not enabled.*
- *$\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is live iff for every transition t and for any marking $\boldsymbol{m} \in$ RS ($\mathcal{N}$, $\boldsymbol{m_0}$) there exists $\boldsymbol{m}$' $\in$ RS($\mathcal{N}$, $\boldsymbol{m}$) such that t is enabled at $\boldsymbol{m}$'.*
- *$\mathcal{N}$ is structurally live (deadlock-free) iff $\exists\, \boldsymbol{m_0}$ such that $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is live (deadlock-free).*

The example below illustrates the concepts defined in this Chapter.



(a) (b)

Figure 3.3: Example of a live hybrid adaptive Petri net (a) and its reachability space (b).

Example 17 *The Petri net of Figure 3.3 can be defined mathematically as follows.*

$\mathcal{N}_A = \langle P, T, Pre, Post, \boldsymbol{\mu} \rangle$, *where*

$P = \{p_1, p_2, p_3, p4\}$

$T = \{t_1, t_2, t_3\}$

$$Pre = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Post = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\boldsymbol{\mu} = \{\mu_1 = 1.5, \mu_2 = 1, \mu_3 = 1\}$

And the HAPN system is the tuple $\langle \mathcal{N}, m_0 = (5,1,0,0) \rangle$.

In the initial marking $m_0 = (5,1,0,0)$, *the transition* t_1 *is enabled as continuous (mode(*t_1,m_0*) = C), and enab(*t_1, m_0*) = 2.5 . The transition can be fired any real amount* α *such that* $0 < \alpha \leq$ *enab(*t_1, m_0*) -* μ_1*. It is* $0 < \alpha \leq 1$, *where* $\alpha \in \mathbb{R}$.

Moreover, transition t_2 *is enabled as discrete (mode(*t_2,m_0*) = D), and enab(*t_2, m_0*) = 1. Since it is in discrete mode, it can be fired any natural amount* α *such that* $0 < \alpha \leq$ *enab(*t_2, m_0*). It is* $0 < \alpha \leq 1$, *where* $\alpha \in \mathbb{N}$.

The reachability space of $\langle \mathcal{N}_A, m_0 = (5,1,0,0) \rangle$ *is defined as: RS(*$\mathcal{N}_A$, $\boldsymbol{m_0}$*) =* $\{\boldsymbol{m} \mid \exists\, \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ $such\ that\ \boldsymbol{m_0} \overset{\alpha_1 t_{\gamma_1}}{\longrightarrow} \boldsymbol{m_1} \overset{\alpha_2 t_{\gamma_2}}{\longrightarrow} \boldsymbol{m_2} \ldots \overset{\alpha_k t_{\gamma_k}}{\longrightarrow} \boldsymbol{m_k} = \boldsymbol{m}\ where\ \alpha_i \in \mathbb{R}^+\ if\ mode(t_{\gamma_i}, m_{i-1})$ *= C,* $and\ \alpha_i \in \mathbb{N}^+\ if\ mode(t_{\gamma_i}, m_{i-1}) = D\}$

From m_0, *transition* t_1 *can be fired, and the reachable markings are all the possible markings between (5,1,0,0) (the initial marking) and (3,1,1,0) (given by the maximal firing). The set of all these markings form a straight line in the* $\mathbb{R}^{|P|}$ *space. From all the markings of this "straight line", transition* t_2 *can be fired from* m_0, *resulting another straight line in the Reachability Space: the line from (4,0,0,1) to (2,0,1,1). Finally, (1,1,2,0) and (0,0,2,1) are reachable from (3,1,1,0) and (2,1,1,1) respectively when* t_1 *is fired as discrete an amount of 1.*

It can be observed that $m[p_3]$ *and* $m[p_4]$ *are linearly dependent if* $m[p_1]$ *and* $m[p_2]$*:* $m[p_3] = 4 - m[p_1] + m[p_2]$ *and* $m[p_4] = 1 - m[p_2]$*. Because of that, the reachablity space can be represented just with the axes* $m[p_1]$ *and* $m[p_2]$, *as it can be observed in Figure 3.3(b).*

Regarding to the deadlock-freeness property, the HAPN system of this example is deadlockfree becasuse none of the reachable markings is a deadlock. It is also liven because from any of the reachable markings, there exists a reachable marking from which any transition can also be fired. If the marking (0,1,2.5,0) would be reachable then the system would be not live (and not deadlock-free).

Chapter 4

Reachability Space of HAPNs

In this Chapter, the reachability space (RS) of HAPN systems is studied and compared to the RS of discrete and continuous systems. In the first section, RS of discrete, continuous and hybrid adaptive PN are compared. In the second one, a method to calculate the reachability space of HAPN is presented.

The following definitions will be used in the rest of the document: $\mathcal{N}_D$ denotes a discrete Petri net with a given structure $\langle P, T, \mathbf{Pre}, \mathbf{Post}\rangle$, $\mathcal{N}_C$ denotes the continuous net with the same structure, and $\mathcal{N}_A$ denotes the hybrid adaptive Petri net with the same structure and an arbitrary $\boldsymbol{\mu}$. In order to compare the reachability spaces, the same initial marking $\boldsymbol{m_0} \in \mathbb{N}^{|P|}$ is considered for all three types of Petri nets (discrete, continuous or adaptive).

For simplicity, it was decided to start the study the RS in the HAPN by considerig ordinary PNs; the subclass of Petri nets in which all the arc weights are equal to 1. Notice that although ordinary PNs are a subclass of general PNs, any non-ordinary Petri net can be converted to an ordinary PN [3]. It will be proved that, under rather general conditions, the RS of a HAPN $\mathcal{N}_A$ contains the RS of $\mathcal{N}_D$, and that the RS of $\mathcal{N}_C$ contains the RS of $\mathcal{N}_A$. This is a straightforward consequence of the fact that, in contrast to continuous nets, HAPNs are a partial relaxation of discrete nets.

4.1 Reachability space of discrete and hybrid adaptive PN

Theorem 18 *RS($\mathcal{N}_D$, $\boldsymbol{m_0}$) $\subseteq$ RS($\mathcal{N}_A$, $\boldsymbol{m_0}$) for any ordinary HAPN $\mathcal{N}_A$ with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.*

Proof Let $\boldsymbol{m} \in$ RS($\mathcal{N}_D$, $\boldsymbol{m_0}$). Then, there exists $\sigma_d = t_{\gamma_1} \dots t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{1t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{1t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{1t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ in $\langle \mathcal{N}_D, \boldsymbol{m_0}\rangle$. We will prove that there exists a sequence $\sigma_a = \beta_1 t_{\gamma_1} \dots \beta_k t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{\beta_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\beta_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\beta_k t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ in $\langle \mathcal{N}_A, \boldsymbol{m_0}\rangle$.

Let us start with t_{γ_1}, and let us check if β_1 = 1 can be chosen. Two cases must be considered.

a) $enab(t_{\gamma_1}, \boldsymbol{m_0}) \leq \mu_{t_{\gamma_1}}$. From the definition of HAPN, t_{γ_1} behaves as discrete, i. e., $mode(t_{\gamma_1}, \boldsymbol{m_0}) = D$.

   Given that t_{γ_1} is enabled in $\langle N_D, \boldsymbol{m_0}\rangle$, it holds that $enab(t_{\gamma_1}, \boldsymbol{m_0}) = min_{p \in {}^\bullet t_{\gamma_1}} \{m_0[p]\} \geq 1$. Hence, it is also enabled in $\langle \mathcal{N}_A, \boldsymbol{m_0}\rangle$ in the same amount.

   Therefore, $\beta_1 = 1$ can be chosen, and the same $\boldsymbol{m_1}$ of the discrete system is reached.

b) $enab(t_{\gamma_1}, \boldsymbol{m_0}) > \mu_{t_{\gamma_1}}$. From the definition of HAPN, t_{γ_1} behaves as continuous, i. e., $mode(t_{\gamma_1}, \boldsymbol{m_0}) = C$.

Since $\mu_{t_{\gamma_1}} \in \mathbb{N}$ and $enab(t_{\gamma_1}, \boldsymbol{m_0}) > \mu_{t_{\gamma_1}}$, it holds that $enab(t_{\gamma_1}, \boldsymbol{m_0}) - \mu_{t_{\gamma_1}} \geq 1$. Therefore, $\beta_1 = 1 \leq enab(t_{\gamma_1})$ - $\mu_{t_{\gamma_1}}$ can be chosen and $\boldsymbol{m_1}$ is reached.

The same reasoning can be applied to the rest of the transitions in the sequence $t_{\gamma_2} \ldots t_{\gamma_k}$. ■

However, if non ordinary PNs or non natural thresholds are considered, RS($\mathcal{N}_D, \boldsymbol{m_0}$) is in general not contained in RS($\mathcal{N}_A, \boldsymbol{m_0}$). Let us show both cases through examples.

- When non natural thresholds, $\boldsymbol{\mu} \notin \mathbb{N}^{|T|}$, are considered, RS($\mathcal{N}_D, \boldsymbol{m_0}$) is in general not contained in RS($\mathcal{N}_A, \boldsymbol{m_0}$) for ordinary HAPN. Let us show it with the following example. Consider the net of the Figure 4.1 as discrete, $\mathcal{N}_D$, with the initial marking $\boldsymbol{m_0}$ = (3,4). Both t_1 and t_2 can be fired until the place p_1 is empty (when enabling degree is 0). Its reachability space RS($\mathcal{N}_D$, $\boldsymbol{m_0}$) is represented in Figure 4.2 (a). Let us consider now the net as adaptive, with $\boldsymbol{\mu} = (1.5, 1.5)$ Thus, t_1 can fire as continuous while $m[p_1] > 1.5$. And t_2 can fire as continuous while $m[p_1] > 1.5$ and $m[p_2] > 1.5$. When $m[p_1] = 1.5$, t_1 changes from continuos to discrete, and it can fire a discrete amount. Analogously, t_2 changes to discrete and can fire as discrete when $m[p_1] = 1.5$. Its reachability space is shown in Figure 4.2 (c). Notice that RS($\mathcal{N}_D$, $\boldsymbol{m_0}$) contains some markings that are not reachable in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. For example, the marking $\boldsymbol{m_2}$ = (1, 4) $\in$ RS($\mathcal{N}_D$, $\boldsymbol{m_0}$), but $\boldsymbol{m_2} \notin$ RS($\mathcal{N}_A$, $\boldsymbol{m_0}$).

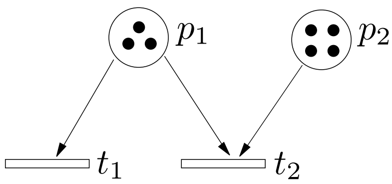


Figure 4.1: A net whose reachability space as discrete is not contained in the reachability space as adaptive with with $\boldsymbol{\mu} = (1.5, 1.5)$, see Figure 4.2.

- If non-ordinary PN are considered, RS($\mathcal{N}_D$) is in general no contained in RS($\mathcal{N}_A$), with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. This can be shown through an example. The reachability space of the HAPN in Figure

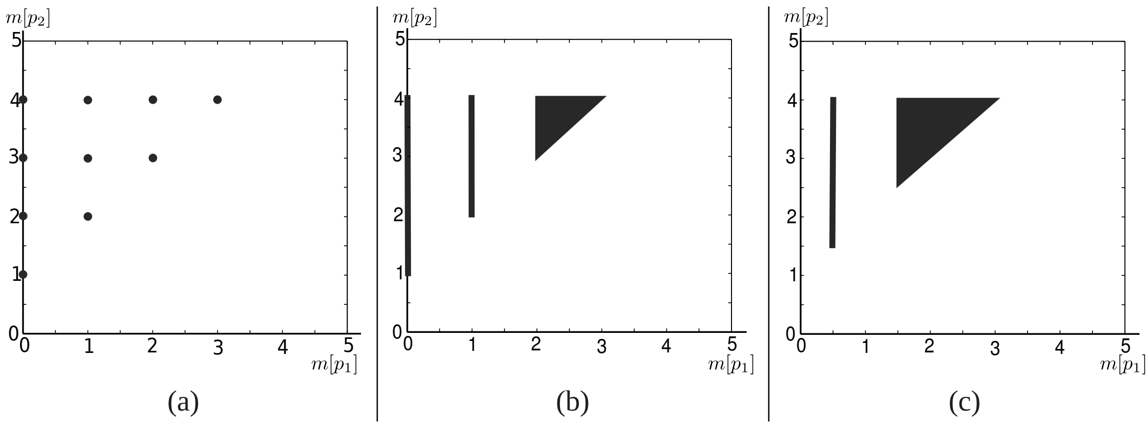


Figure 4.2: Reachability space of the Petri Net of Figure 4.1 behaving as Discrete (a), HAPN with $\boldsymbol{\mu} = (2, 2)$ (b) or HAPN with $\boldsymbol{\mu} = (1.5, 1.5)$ (c).

1.1 with $\boldsymbol{\mu} = (1, 1)$ is shown in Figure 4.3. Transition t_2 is enabled as continuous from marking $(5, 0)$ to $(2, 1.5)$, where it changes to discrete. If t_2 is fired as discrete (from $(2, 1.5)$), $(0, 2.5)$ is reached. In $(0, 2.5)$ none of the transitions are enabled (and the net deadlocks). Transition t_1 is enabled as continuous from $(2, 1.5)$ to $(3, 1)$, where it is enabled as discrete. When t_1 is fired as discrete from $(3, 1)$, $(5, 0)$ is reached and t_1 becomes not enabled.

The marking $\boldsymbol{m} = (1, 2)$ is reachable in the discrete Petri net, but not in the adaptive one with $\forall \mu, \mu = 1$. Therefore, RS($\mathcal{N}_D, \boldsymbol{m_0}$) is not, in general, included in RS($\mathcal{N}_A, \boldsymbol{m_0}$) with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$ for non ordinary HAPNs.

On the other hand, it is straightforward to prove that, given that HAPNs allow real-valued markings, the RS of $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is not, in general, included in RS($\mathcal{N}_D, \boldsymbol{m_0}$). Nonetheless, if $\boldsymbol{\mu} = \infty$, the HAPN always behaves as discrete and its RS is trivially identical to that of the discrete PN.

4.2 Reachability space of continuous and hybrid adaptive PN

Let us now compare the RS of the HAPN to the RS of its associated continuous PN.

Theorem 19 *RS($\mathcal{N}_A, \boldsymbol{m_0}$) $\subseteq$ RS($\mathcal{N}_C, \boldsymbol{m_0}$) with $\boldsymbol{\mu} \in \mathbb{R}^{|T|}_{\geq 0}$.*

Proof Let $\boldsymbol{m} \in$ RS($\mathcal{N}_A, \boldsymbol{m_0}$). Therefore, there exists $\sigma_a = \beta_1 t_{\gamma_1} \ldots \beta_k t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{\beta_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\beta_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\beta_3 t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ *where* $\beta_i \in \mathbb{R}^+$ *if mode* $(t_{\gamma_i}, m_{i-1}) = C$ *and* $\beta_i \in \mathbb{N}^+$ *if mode*$(t_{\gamma_i}, m_{i-1}) = D$

For any of the β_i of σ_a, if $mode(t_{\gamma_i}, \boldsymbol{m_{i-1}}) = C$, then t_{γ_i} will be also enabled in $\langle \mathcal{N}_C, \boldsymbol{m_{i-1}}, \rangle$ and the same $\beta_i \in T$ can be chosen. If $mode(t_{\gamma_i}, m_{i-1}) = D$, then t_{γ_i} will be also enabled in $\langle \mathcal{N}_C, \boldsymbol{m_{i-1}}, \rangle$ and also the same $\beta_i \in \mathbb{N}^+$ can be chosen because $\beta_i \in \mathbb{R}$. Consequently, the same firing sequence σ_a of the HAPN system can be chosen in the continuous system and the same marking $\boldsymbol{m}$ is obtained.
■

The following Corollary is straightforwardly obtained from Theorems 18 and 19.

Corollary 20 *RS($\mathcal{N}_D, \boldsymbol{m_0}$) $\subseteq$ RS($\mathcal{N}_A, \boldsymbol{m_0}$) $\subseteq$ RS($\mathcal{N}_C, \boldsymbol{m_0}$) for ordinary nets with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.*

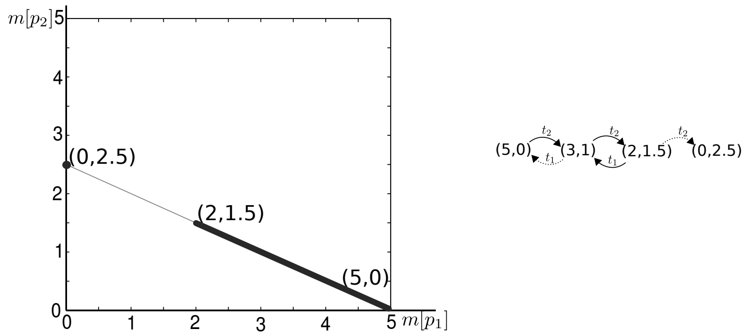


Figure 4.3: Reachability space and reachability graph of the Petri Net of the Figure 1.1 behaving as HAPN with $\boldsymbol{\mu} = (1, 1)$.

Furthermore, let us show through an example that the RS of the continuous system is, in general, not contained in the RS of the HAPN system, i.e., RS($\mathcal{N}_C, \boldsymbol{m_0}$) $\not\subseteq$ RS($\mathcal{N}_A, \boldsymbol{m_0}$) with $\boldsymbol{\mu} \in \mathbb{R}^{|T|}$. In the

PN system of Figure 1.1 (with $\boldsymbol{\mu}$ = (1.5, 1.5)), the marking $\boldsymbol{m}$ = (0.5, 2) is included in RS($\mathcal{N}_C, \boldsymbol{m_0}$), but cannot be reached by the HAPN, i.e., it is not included in RS($\mathcal{N}_A, \boldsymbol{m_0}$). Both spaces are trivially equal if all the transitions of the HAPN always behave as continuous, i.e., when $\boldsymbol{\mu} = 0$.

4.3 An algorithm to obtain the reachability space of HAPN

After some general considerations, this section provides an algorithm to compute the set of reachable markings of a HAPN.

As known, the reachability space (RS) of a discrete Petri net system is the union of all the markings which are reachable from the initial marking, which are points in the $\mathbb{N}^{|P|}$ space. On the other hand, the RS of a continuous PN system is a convex set in $\mathbb{R}^{|P|}$ [10].

Considering a general HAPN, its reachability space contains some convex sets due to the continuous firing of transitions. However, the RS is not a unique convex set because of the discrete firings, which induce a "leap" in the RS, and some intermediate markings are not reachable, leading to several reachable sets. For example, the RS in Figure 4.3 is constituted by two convex sets: a convex set (from (5,0) to (2,1.5)) due to the possible continuous firings from m_0, and another set (the point (0,2.5)) due to the discrete firing of a transition (the firing of t_2 from (2,1.5)). In conclusion, the RS of a HAPN can represented as the union of one or several convex sets. The maximum number of convex sets will be bounded by the values of the thresholds of the HAPN.

In this section, an algorithm to compute those convex sets that constitute the RS of a HAPN is proposed. The algorithm consists on a recursive procedure that, given a marking or a set of markings (denominated *region*), it calculates the markings which are reachable from it, considering both discrete and continuous firings of transitions. This recursive procedure is named *explore*, and its input will be a *region*, where a *region* is defined as a set of markings such that it is a convex set, and in which all the transitions remain in the same behaviour mode (each transition is D or C in the *region*, but not both).

Each execution of the *explore* procedure calculates the possible different markings that are reachable from the region R: the markings reachable just with continuous firings of the transitions whose mode in R is C, and the markings reachable with the discrete firing of each of the transitions whose mode in R is D. When a new region is obtained, the algorithm checks if it was already included in the RS, and if it was not included before, it is also explored with a recursive invocation to the *explore* procedure.

The *explore* procedure starts calculating set of possible markings obtained from R due to continuous firings. Due to the modes of the transitions, any amount $\sigma(t) \in \mathbb{R}^+$ can be fired from any marking $\in R$ if mode(t,R) = C. The mathematical formula is presented below.

continuousReachMarkings ($\mathcal{N}$, R) = $\{\boldsymbol{m} |\ \boldsymbol{m} = \boldsymbol{m_0} + C\sigma$
$\wedge\ \sigma(t) = 0$ if mode(t,R) = D
$\wedge\ \sigma(t) \geq 0$ if mode(t,R) = C
$\wedge\ \boldsymbol{m_0} \in R$
$\wedge\ \forall t$, mode(t,m) = mode(t,R) $\}$

Notice that there exist some markings that are reachable with continuous firings from R, but are exactly the ones that makes the mode change (from C to D or vice versa) . Those markings do not belong to the continuousReachMarkings region because the *region* is defined as a set of markings where the mode of the transitions do not change. Those markings are just at the border of a continuous region (they are reachable from continuous firings but its mode is discrete), and they are considerated in a special region in this procedure: the *frontier*. As it will be later explained in the Example 21, given a frontier the continous firings from the frontier should be also considered to calculate all the reachable markings. The *frontier* reachable markings and the continuous reachable markings from the frontier are obtained as follows:

frontierReachMarkings ($\mathcal{N}$, R, t_f) = $\{\boldsymbol{m} |\ \boldsymbol{m} = \boldsymbol{m_0} + C\sigma$
$\wedge\ \sigma(t) = 0$ if mode(t,R) = D
$\wedge\ \sigma(t) \geq 0$ if mode(t,R) = C
$\wedge$ enab(t_f, m) = μ_f
$\wedge\ \forall t \neq t_f$, mode($t$, m) = mode(t, R) $\}$

contReachMarkingsFromFrontier ($\mathcal{N}$, R, t_f) = $\{\boldsymbol{m} |\ \boldsymbol{m} = \boldsymbol{m_0} + C\sigma$
$\wedge\ \sigma(t) \geq 0$ if t = t_f
$\wedge\ \sigma(t) = 0$ if $t \neq t_f \wedge$ mode(t,R) = D
$\wedge\ \sigma(t)\mathbb{R}_{\geq 0}$ if $t \neq t_f \wedge$ mode(t,R) = C
$\wedge\ \boldsymbol{m_0} \in R$
$\wedge$ mode(t_f,m) = C
$\wedge\ \forall t$, mode(t,m) = mode(t,R) $\}$

When exploring a region, also discrete firings are considered. Giving a certain *region*, each of the transitions which are enabled in discrete mode can be fired an amount $\sigma = 1$, obtaning a new set of values of the reachability space. This function is defined for each one of the transitions which are in discrete mode.

discreteReachMarkings ($\mathcal{N}$, R, t_d) = $\{\boldsymbol{m} |\ \boldsymbol{m} = \boldsymbol{m_0} + C\sigma$
$\wedge\ \sigma(t_f) = 1$
$\wedge\ \sigma(t) = 0$ if $t \neq t_f$
$\wedge\ \boldsymbol{m_0} \in R\}$

However, each of the sets obtained by the "discreteReachMarkings" method may not be strictly *regions* where each transition has a unique behaviour mode. In that case, the function partition(d, t) partitions a region in sub-regions, such that $\forall subReg$, $\forall t$, mode(t,$subReg$) is homogeneous (or C or D, but not both).

The a possible execution of the algorithm to obtain the reachability space of a HAPN system is explained in the following example.

Example 21 *This example shows the RS of the Petri net system of Figure 4.4 when discrete, continuous and hybrid adaptive. And it illustrate the execution of the proposed algorithm.*

Firstly, let consider the reachability space of the system when it is discrete or continuous. Figure 4.5 shows the RS of the PN system in Figure 4.4 when it is discrete. In this example, it is enough to represent $m[p_1]$ *and* $m[p_2]$ *axes because* $m[p_3]$ *is linearly dependent on* $m[p_1]$ *and* $m[p_2]$*; more*

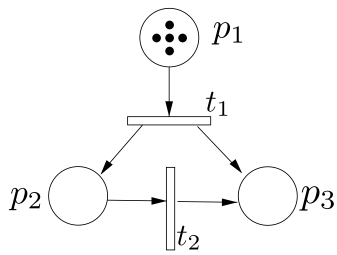


Figure 4.4: Example of a Petri net

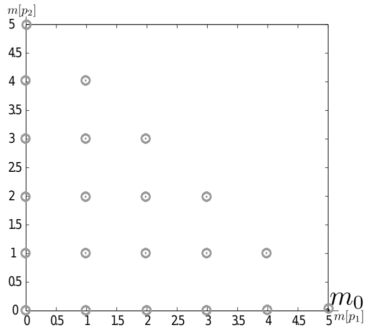


Figure 4.5: Reachability space of the net of Fig. 4.4 when discrete

precisely $m[p_3] = 2\, m[p_1] - m[p_2]$. *If the PN system is considered continous, its RS is the convex set represented in Figure 4.6.*

Considering the Petri net system to be HAPN, with $\boldsymbol{\mu}$ *= (1,1), its Reachability space is represented in Figure 4.7. And it has been calculated as follows.*

The initial marking $\boldsymbol{m_0}$ *= (5,0,0) is considered as the initial region, which is a point in* $\mathbb{R}^{|P|}$. *Given this region, we "explore" it in a recursive way. From* m_0*, the continuousReachMarkings(*m_0*) are calculated, and the region* $R1$ *is obtained (see Figure 4.7). This region contains the markings which are reachable from* $\boldsymbol{m_0}$ *when* t_1 *is fired as continuous (mode (*t_1, m_0*) = C), while* t_2 *remains discrete (mode (*t_2, m_0*) = D). We can observe that the region* $R1$ *is a convex set.*

The region $R1$ *contains the reachable markings in which* t_1 *is continuous and* t_2 *remains discrete; however it does not contains the marking* $F_1 = (4, 1, 1)$*, because mode(*t_1, F_1*) = D. As explained, it is a point just at the border: when enab(*t_2*) is exactly* μ_2. *This region is considered a frontier, represented as* $F1$ *on the Figure. From the frontier a new region (*R_2*) is obtained considering all the posible continuous firings from* $F1$ *(continuousReachMarkingsFromFrontier).* R_2 *is a new region,*

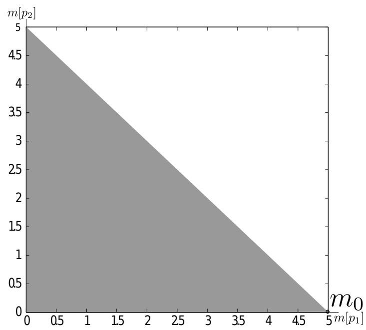


Figure 4.6: Reachability space of the net of Fig. 4.4 when continuous

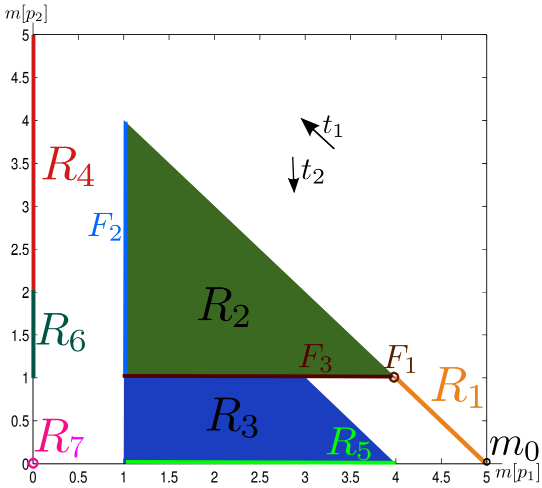


Figure 4.7: Reachability space of the net of Fig. 4.4 when HAPN

which is also explored, with an invocation to explore. No possible discrete firings are fireable from m_0 *or* R_1 *because none transition is enabled in discrete mode.*

When exploring R_2*, two new frontiers are obtained:* F_2 *and* F_3*, which are also explored. rom the discrete firing of* t_1 *from* F_2 *(discreteReachMarkings(*$\mathcal{N}$*,* F_2*,* t_1*)), region* R_4 *is obtained. While* R_5 *is obtained by the discrete firing of transition* t_2 *from* F_3 *(discreteReachMarkings(*$\mathcal{N}$*,* F_3*,* t_2*)). Each new region obtained is explored recursivelly until all the regions have been calculated.*

The resulting reachability space is the mathematical union of all the obtained convex sets.

The algorithm which has been proposed and explained in this section is described below.

Algorithm 1 Reach Space

Require: HAPN (N), initial marking ($\boldsymbol{m_0}$)

Ensure: Reachability space ($reachSpace$)

1: $initialRegion$:= { $\boldsymbol{m_0}$ }
2: $reachSpace$:=
3: $reachSpace$:= explore ($\mathcal{N}$, $reachSpace$, $initialRegion$)
4: **return** $reachSpace$

Where `explore` is the recursive function which explores the reachability space of $\mathcal{N}$ from a certain *region*, and it is showed below.

Algorithm 2 explore

Require: HAPN ($\mathcal{N}$), set of markingSet (RS), markingSet (Reg)
Ensure: set of markingSet (RS)

```
if Reg ∉ RS then
    RS := RS ∪ Reg
    cont := continuousReachMarkings (N, Reg)
    RS := RS ∪ cont
    for all t ∈ T such that mode(t, Reg) = C do
        f := frontierReachMarkings (N, cont, t)
        RS := RS ∪ explore(f))
    end for
    for all t_i ∈ T such that (∀m ∈ Reg enab(t_i, Reg) = μ_i) do
        c := continuousReachMarkingsFromFrontier (N, Reg, t_i)
        RS := RS ∪ explore(c)
    end for
    for all t ∈ T such that mode(t, Reg) = D ∧ t is enabled do
        d := discreteReachMarkings (N, Reg, t)
        for all subReg ∈ partition (d, t) do
            RS := RS ∪ explore (subReg)
        end for
    end for
end if
return RS
```

Chapter 5

Deadlock-freeness in HAPNs

This Chapter studies the deadlock-freeness property of HAPNs, and relates it to deadlock-freeness of the equivalent discrete PNs. Although for arbitrary $\boldsymbol{\mu}$ deadlock-freeness of the discrete PN is, in general, not preserved by the HAPN, it is shown that the appropriate selection of $\boldsymbol{\mu}$ can preserve the property for a large class of nets.

5.1 Preliminary results

Let us first show, by considering the net in Figure 1.1, that:

$\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free $\not\Rightarrow$ $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is deadlock-free.

The system in Figure 5.1 with $\boldsymbol{m_0} = (5, 0)$ is deadlock-free if considered as discrete. However, if considered as HAPN with $\boldsymbol{\mu} = (1, 1)$ it deadlocks after firing t_2 as continuous in an amount of 1.5, and again t_2 as discrete.

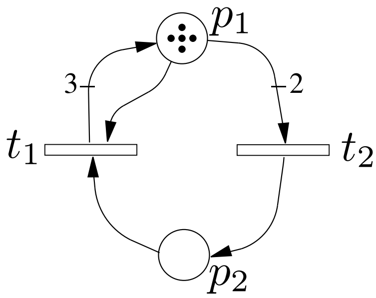


Figure 5.1: A Petri net system that deadlocks as continuous but it is deadlock-free as hybrid adaptive with appropriate thresholds.

Furthermore, in general, deadlock-freeness of a HAPN system does not guarantee deadlock-freeness of the equivalent discrete system:

$\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is deadlock-free $\not\Rightarrow$ $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free.

The system in the Figure 5.1 with $\boldsymbol{m_0} = (4, 0)$ deadlocks as discrete. If considered as HAPN, it is deadlock-free with $\boldsymbol{m_0} = (4, 0)$ and $\boldsymbol{\mu} = (1.5, 1.5)$ because t_2 commutes from continuous to discrete when $m[p_1] = 3$, and $m[p_1]$ never empties.

5.2 Deadlock-freeness in ordinary, deadlock-free nets

Although the deadlock-freness property of discrete systems is not preserved in general by HAPNs with arbitrary $\boldsymbol{\mu}$, it will be proved that for choice free nets with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$ deadlock-freness of the

HAPN system is necessary and sufficient for deadlock-freeness of the discrete system. As previously defined in Section 2.5, choice free nets are a subnet of PN, such that each place of a choice free PN has at most one output transition: $\forall p\ |p^\bullet| \leq 1$.

Let us first prove that it is a sufficient condition.

Theorem 22 *Let $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ be an ordinary deadlock-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. Then, the discrete system $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free.*

Proof Let us assume that the discrete $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ deadlocks at a marking $\boldsymbol{m}$. According to Theorem 18, marking $\boldsymbol{m}$ can be reached by $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. Given that the net is ordinary, for every transition t, there exists $p \in {}^\bullet t$ such that $\boldsymbol{m}[p] = 0$, i.e., $\boldsymbol{m}$ is a deadlock for $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. ■

For the necessary condition, two technical lemmas are introduced before stating the final result. The first one states that if a sequence σ is fireable in the adaptive system, its *ceil sequence* $\lceil \sigma \rceil$ is also fireable in the discrete one.

Definition 23 *Let $\sigma = \alpha_1 t_{\gamma_1}\ \alpha_2 t_{\gamma_2} \ldots \alpha_k t_{\gamma_k}$ be a firing sequence of a given HAPN $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. The ceil sequence, $\lceil \sigma \rceil$ of σ is defined as: $\lceil \sigma \rceil = \alpha'_1 t_{\gamma_1}\ \alpha'_2 t_{\gamma_2}\ \ldots\ \alpha'_k t_{\gamma_k}$ where*

$$\alpha'_i = \left\lceil \sum_{1 \leq j \leq i | t_{\gamma_i} = t_{\gamma_j}} \alpha_j \right\rceil - \sum_{1 \leq j < i | t_{\gamma_i} = t_{\gamma_j}} \alpha'_j$$

For example, for the sequence $\sigma_1 = 0.1\ t_1\ 0.8\ t_2\ 0.1\ t_1\ 0.2\ t_1\ 0.8\ t_2$ in the HAPN of Figure 3.2 (a), the ceil sequence $\lceil \sigma_1 \rceil$ is defined as $\lceil \sigma_1 \rceil = 1\ t_1\ 1\ t_2\ 0\ t_1\ 0\ t_1\ 1\ t_2$.

Lemma 24 *Let $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ be an ordinary choice-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. If σ is a fireable sequence in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ then $\lceil \sigma \rceil$ is fireable in $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$.*

Proof Let us assume without loss of generality that $\sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ and $0 < \alpha_j \leq 1$ for every $j \in \{1, \ldots, k\}$. Induction on the length of σ: $|\sigma| = k$.

- Base case ($|\sigma| = 1$). Let $\sigma = \alpha_1 t_{\gamma_1}$, then $\forall p \in {}^\bullet t_{\gamma_1}, \boldsymbol{m_0}[p] \geq \alpha_1$ and given that $\boldsymbol{m_0}[p] \in \mathbb{N}$, it holds that $\boldsymbol{m_0}[p] \geq \lceil \alpha_1 \rceil$. Thus $\lceil \sigma \rceil = \lceil \alpha_1 \rceil t_{\gamma_1}$ can be fired in $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$.

- Inductive step. Assume that the Lemma holds for $|\sigma| = k$. Let us consider the $k+1$ firing, i.e., $t_{\gamma_{k+1}}$ fires in α_{k+1}. Two cases can occur:

  a) $\alpha'_{k+1} = 0$. In this case, the Lemma trivially holds.

  b) $\alpha'_{k+1} = 1$. Let $\boldsymbol{m_i}$ and $\boldsymbol{\sigma}_i$ ($\boldsymbol{m'_i}$ and $\boldsymbol{\sigma}'_i$) be the marking and firing count vector obtained just after the firing of t_{γ_i} in an amount α_i (α'_i). If $t_{\gamma_{k+1}}$ fires in the HAPN system, it means that $\boldsymbol{m_k}[p] > 0$ for every $p \in {}^\bullet t$. Notice that, by definition of ceil sequence, after the k^{th} firing the following inequalities are satisfied: $\boldsymbol{\sigma}'_k[t] \geq \boldsymbol{\sigma}_k[t]$ and $\boldsymbol{\sigma}'_k[t^q] \geq \boldsymbol{\sigma}_k[t^q]$ for every $t^q \in {}^\bullet({}^\bullet t)$. Given that the net is choice-free, for every place p it holds that $|{}^\bullet p| = 0$ or $|{}^\bullet p| \geq |p^\bullet| = 1$. If for $p \in {}^\bullet t$, it holds that $|{}^\bullet p| \geq |p^\bullet| = 1$, then the previous inequalities ensure $\boldsymbol{m'_k}[p] \geq 1$. If p has no input transitions, then it must hold that $\boldsymbol{\sigma}'_{k+1}[t] \leq \boldsymbol{m_0}[p]$. Therefore $t_{\gamma_{k+1}}$ can fire from $\boldsymbol{m'_k}$ an amount of 1.

The second lemma states that if a certain sequence σ deadlocks a HAPN, then its firing count vector is in the naturals.

Lemma 25 *Let $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ be an ordinary choice-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. If σ is a fireable sequence $\boldsymbol{m_0} \xrightarrow{\sigma} \boldsymbol{m}$, such that $\langle \mathcal{N}_A, \boldsymbol{m}_0 \rangle$ deadlocks at $\boldsymbol{m}$, then $\boldsymbol{\sigma} \in (\mathbb{N} \cup \{0\})^{|T|}$, where $\boldsymbol{\sigma}$ is the firing count vector of σ.*

Proof Let us first prove that if $\boldsymbol{m}$ is a deadlock marking then for every transition t there exists $p \in {}^\bullet t$ such that $\boldsymbol{m}[p] = 0$. Notice that just after the last firing of t in the sequence σ, which is necessarily discrete firing given that $\boldsymbol{\mu} \in \mathbb{N}^{|P|}$, at least one place $p \in {}^\bullet t$ becomes empty. Assume that after such a firing, a transition $t' \in {}^\bullet p$ fires. If the firing of t' is discrete then t would become enabled again; if it is continuous then t' is sufficiently enabled to fire also as discrete what would enable t. Hence, after the last firing of t, no transition $t' \in {}^\bullet p$ can fire and p remains empty.

Assume that $\boldsymbol{\sigma}[t] > 0$ is not a natural number and that $\boldsymbol{m}[p] = 0$ for a given $p \in {}^\bullet t$. Then, there exists $t' \in {}^\bullet p$ such that $\boldsymbol{\sigma}[t']$ is not a natural number and $\boldsymbol{\sigma}[t'] \leq \boldsymbol{\sigma}[t] - \boldsymbol{m_0}[p]$. Notice that there also exists $p' \in {}^\bullet t'$ such that $\boldsymbol{m}[p'] = 0$, hence $t'' \in {}^\bullet p'$ exists such that $\boldsymbol{\sigma}[t'']$ is not a natural number and $\boldsymbol{\sigma}[t''] \leq \boldsymbol{\sigma}[t'] - \boldsymbol{m_0}[p'] \leq \boldsymbol{\sigma}[t] - \boldsymbol{m_0}[p] - \boldsymbol{m_0}[p']$. This reasoning can be repeated until a transition t^* is found such that it deadlocked with $\boldsymbol{\sigma}[t^*] < 1$. Contradiction since natural thresholds do not allow $\boldsymbol{\sigma}[t^*]$ to be less than 1. ■

Therefore, because of Lemmas 24 and 25, if a deadlock marking $\boldsymbol{m}$ is reachable in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ when σ is fired, the same deadlock marking $\boldsymbol{m'}$ is reachable in $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$, when $\lceil \sigma \rceil$ is fired. Thus, if $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free, then $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is deadlock-free too.

Theorem 26 *Let $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ be an ordinary choice-free and deadlock-free discrete system. Then, the HAPN system $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is deadlock-free for any $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.*

The following Corollary is straightforwardly obtained from Theorems 22 and 26.

Corollary 27 *Let $\mathcal{N}$ be an ordinary choice-free net. $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free iff $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is deadlock-free with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.*

Chapter 6

Conclusions and future work

This Chapter puts forward some conclusions obtained in this “Trabajo Fin de Máster”, and it proposes some future work.

6.1 Conclusions

As most formalisms for discrete event systems, Petri nets suffer from the state explosion problem. Such a problem renders enumerative analysis techniques unfeasible for large systems. The hybrid adaptive Petri nets considered here aim at alleviating the state explosion problem by partially relaxing the firing of transitions. More precisely, a transition can fire in real amounts when its load is *higher* than a given threshold, and it is forced to fire in discrete amounts when its *load* is lower than that threshold. This partial relaxation offers a chance of preserving important properties of discrete event systems, as deadlock-freeness, that are not always retained by fully continuous approximations.

This work focused on the reachability space and the deadlock-freeness property of hybrid adaptive nets. A general algorithm was proposed for the characterization of the reachablity space of any HAPN. Furthermore, an inclusion relationship was proved for the reachability spaces of the discrete, hybrid adaptive and continuous nets; for a rather general class of nets,. With respect to deadlock-freeness, although this property is not preserved in general for arbitrary real thresholds, it was shown that it is necessary and sufficient for deadlock-freeness of choice-free nets with arbitrary natural thresholds.

It has been shown that the HAPN is a general formalism that includes the used and known PN formalisms of discrete, continuous and discrete PN. Due to its high generality, HAPN has a very powerful modeling capability. However, developing analysis techniques for such a general formalism can be costly. Hybrid Adaptive analysis techniques involucre discrete, continuous and hybrid PN techniques, maintaining or increasing its complexity.

6.2 Future Work

In this TFM, the formalism of HAPN has been defined and some preliminar results about the Reachability Space, the relations among the Reachability Spaces of hybrid adaptive, discrete and continuous PN, and the conditions to preserve the liveness property are presented. However, the liveness property and the reachability space comparison have been done in just two classes of PN, the ordinary PN and the choice-free PN. The future work will be to study the properties of more general classes of HAPN, and the time interpretation of HAPN:

- Preserving properties: Given a certain property, for example deadlock-freeness, it would be very useful to obtain a general method to calculate an adequate threshold vector $\boldsymbol{\mu}$ such that it the discrete PN was deadlock-free, the HAPN is also deadlock-free. In this case, any discrete PN would be aproximated by a partially fluidified hybrid adaptive PN.

- Relations among the Reachability Spaces of the HAPN, discrete, continuous and hybrid Petri nets of more general subclases of Petri nets can be studied in more depth.
- Time interpretation: After defining and studying in depth the autonomous HAPN, a certain firing semantics can be defined. The time interpretation allows the study of certain properties such as performance and the simulation of the nets. When simultaing Petri nets, the inconsistencies between continuous and discrete PN are bigger when the workload is low. Therefore, the simulation of HAPN instead of continuous PN could approximate better the behaviour of the discrete one.
- Modeling of a real system using the HAPN. Through a real case study, the characteristics and potential of HAPN would be pointed out.

Bibliography

[1] E. Fraca, J. Júlvez, C. Mahulea, M. Silva. On reachability and deadlock-freeness of Hybrid Adaptive Petri nets. *In Proc. of 18th IFAC World Congress, 2011*

[2] H. Yang, C. Lin and Q. Li. Hybrid simulation of biochemical systems using hybrid adaptive Petri nets. *VALUETOOLS '09: Proceedings of the Fourth International ICST Conference on Performance Evaluation Methodologies and Tools*. 2009.

[3] M. Silva. Las Redes de Petri: en la Automática y la Informática *AC, 1985*

[4] T. Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 77(4):541–580, 1989.

[5] R. David and H. Alla. Discrete, Continuous and Hybrid Petri Nets. *Springer*. 2004. (Revised 2^{nd} edition, 2010), Berlin.

[6] R. David and H. Alla. Continuous Petri nets. In *Proc. of the 8th European Workshop on Application and Theory of Petri Nets*, pages 275–294, Zaragoza, Spain, 1987.

[7] L. Recalde, E. Teruel, and M. Silva. Autonomous continuous P/T systems. In J. Kleijn S. Donatelli, editor, *Application and Theory of Petri Nets 1999*, volume 1639 of *Lecture Notes in Computer Science*, pages 107–126. Springer, 1999.

[8] M. Silva and L. Recalde. Petri nets and integrality relaxations: A view of continuous Petri net models. *IEEE Trans. on Systems, Man, and Cybernetics*, 32(4):314–327, 2002.

[9] M. Silva, L. Recalde. On fluidification of Petri net models: from discrete to hybrid and continuous models. *Annual Reviews in Control*, Vol. 28(2): 253-266. 2004.

[10] J. Júlvez, L. Recalde and M. Silva. On reachability in autonomous continuous Petri net systems. 24^{th} *International Conference on Application and Theory of Petri Nets (ICATPN 2003), Lecture Notes in Computer Science, Springer* 2003

[11] F. Balduzzi, A. Giua, G. Menga. First-Order Hybrid Petri Nets: a Model for Optimization and Control. *IEEE Trans. on Robotics and Automation*, Vol. 16 (4):382-399, 2000.