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# Scalar Theory as an example of AdS/QFT

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## Abstract

We take a look at the properties of scalar fields in anti–de–Sitter space in five dimensions. More specifically we define bulk–to–boundary and bulk–to–bulk propagators for a five–dimensional scalar field and use these to calculate four–point functions. Finally we take a look at two–point functions and three–point functions and numerically calculate masses, residues and decay constants for the fourdimensional particles coming from the five–dimensional scalar field.



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## Populärvetenskaplig beskrivning

Är hela världen, hela universum, endast ett hologram? Sent förra året (2013) publicerade en grupp fysiker sitt resultat från två simuleringar vilket kan vara början på att svara på frågan. Den första räknade ut observabler så som det radiella avståndet till händelsehorisonten, entropi med flera, baserat på strängteori. Den andra simuleringen beräknar en liknande sak fast i ett universum med lägre dimensioner, utan gravitation. Det visar sig att simuleringarnas resultat överlappar, vilket pekar på att de beskriver samma sak. Detta är ett exempel på den så kallade holografiska principen: All information om ett rum i $n$ dimensioner kan helt beskrivas genom att kolla på den $(n-1)$–dimensionella ytan av rummet där, enligt teorin, all information finns bevarad.

I slutet av år 1997 kom Juan Maldacena med det som nu kallas "AdS/CFT–korrespondensen" eller "Maldacena's förmodan" (Eng. Maldacena's conjecture) som använder den holografiska principen och relaterar strängteori, en teori i elva dimensioner, till en med färre dimensioner, exempelvis den teori vi använder för att beskriva kvarkar, vilken har endast fyra dimensioner. Detta betyder att man kan använda korrespondensen för att arbeta med samma teori som fysiker har arbetat med länge, och översätta resultaten till strängteori, ett relativt nytt område inom kvantfältteorin.

Korrespondensen ses därmed som en av de mest lyckade teorierna inom strängteori för tillfället. Denna avhandling tar en titt på grunderna i teorin, förklarar korrespondensen och använder den för att beräkna observabler av ett skalärt fält och jämföra den fem–dimensionella teorin med en liknande fyr–dimensionell teori.

# Contents

# 1 Introduction

In November 1997, J. Maldacena published a paper[1] proposing the anti–de–Sitter/ Conformal Field Theory correspondence in which he suggested the existence of a five-dimensional space known as the bulk corresponding to a four–dimensional (space–time) boundary in which all information is carried, analogous to how a three–dimensional hologram stores all information on a two–dimensional surface. The goal of this thesis is to study the difference between a five–dimensional theory, a theory which we will see have an infinite number of bound states, i.e, particles with a given mass $m_n$, and a four–dimensional theory for a simple scalar interaction; That is, we want to check how well a sum over poles (resonances) converges, or, in other words, we want to check if we can only use the first few bound states in order to get a decent approximation of the five–dimensional physics.

Using this construction it might be possible to perform calculations in higher-dimensional spaces and connect those with the observables (e.g., mass of mesons) in the four-dimensional space producing results that otherwise could be difficult to obtain.

In section 2, the metric of anti–de–Sitter space (AdS) is defined and the AdS/CFT–correspondence is explained. Using an action built on a five–dimensional scalar field we derive equations of motion and four–point functions. Here we also define propagators in the five–dimensional space. We also introduce Witten diagrams along with Witten rules – closely related to Feynman diagrams – as a way to graphically represent and calculate n–point correlation functions, also known as Green functions or propagators. In section 3 we derive the explicit expression for each propagator in terms of Bessel functions and use these to extract physical constants such as masses and decay constants. In section 4 numerical results obtained from the simplest theory of a scalar field in five–dimensional anti–de–Sitter space are presented in tables and depicted in plots.

# 2 The Correspondence

## 2.1 Definitions

In order to start working with the model, a few definitions must first be introduced. First of all we define the metric of the space that we will be working with. The metric of $AdS_5 \times S^5$ can be expressed via the invariant lengthd $(ds)^2$ as follows[2]:

$$ds^2 = \frac{x_5^2}{L^2}\left((dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\right) - \frac{L^2}{x_5^2}(dx^5)^2 - L^2 d\Omega^5. \tag{2.1}$$

As usual $x^0$ denotes the time coordinate, while $x^i$ denotes coordinates in space. The coordinate labeled $x_5$ denotes the added $5^{th}$ dimension, and $d\Omega_5$ is the corresponding solid angle of the five–dimensional hypersphere. The angular part is of no intrest in the following calculation and it will thus be neglected. With a substitution, $x_5 = \frac{L^2}{z}$, equation (2.1) becomes

$$ds^2 = \frac{L^2}{z^2}\left((dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dz)^2\right). \tag{2.2}$$

Using the metric tensor, $g_{MN}$, equation (2.2) can be expressed conveniently as before

$$ds^2 = g_{MN} dx^M dx^N, \tag{2.3}$$

with the metric tensor of AdS defined as

$$g_{MN} = \frac{L^2}{z^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.4}$$

The coefficient $\frac{L^2}{z^2} = a^2(z)$ is commonly referred to as the warp factor[3]. While at it, we define the same tensor without the warp factor; this is useful when explicitly writing out $z$–dependence:

$$\eta_{MN} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.5}$$

The determinant of the metric tensor is commonly used throughout as well:

$$g = \det(g_{MN}) = \frac{L^{10}}{z^{10}}. \tag{2.6}$$

From hereon the constant $L$ will be set to 1 in order to simplify expressions.

Finally, we introduce indicial conventions for various groupings of the dimensions, since it is often convenient to treat the four normal space–time components separately from the $z$–component:

- Capital Latin letters (M,N,...) denote all five dimensions; i.e., $(x^0, x^1, x^2, x^3, z)$
- Lower case Greek letters $(\mu, \nu, \ldots)$ denote the four normal spacetime dimensions; i.e., $(x^0, x^1, x^2, x^3)$

## 2.2 Formulation

As mentioned before, the goal is to make calculations of measurable observables easier than in normal conformal field theory. One can express operators of the 4–D theory in terms of fields in AdS space. From here, one can calculate the desired expectation value (e.g., mass, decay rates, etc.) and then transform the solution back into the 4–D theory.

To continue, we must assume that there exists an operator, denoted $\mathcal{O}(x^\mu)$, which is coupled to a field $\phi(x^\mu, z)$ living in the AdS space [2, 3, 4]. This field, known as the bulk field, has an explicit relation at the boundary:

$$\phi(x^\mu, 0) = z^{4-\Delta}\phi_0(x^\mu), \tag{2.7}$$

where $\Delta$ denotes the conformal dimension of $\mathcal{O}(x^\mu)$ and $\phi_0(x^\mu)$ is the source of the operator $\mathcal{O}(x^\mu)$, as defined in (2.9). We define $\mathcal{S}[\phi(x^\mu, z)]$ as the action of the field in all five dimensions and then solve this model with the boundary condition given by equation (2.7). Then we can define the generating functional

$$Z = \exp(\mathcal{S}[\phi(x^\mu, z)]), \tag{2.8}$$

and in turn write out the correspondence as[4]

$$Z = \left\langle \exp\left(\int \phi_0(x^\mu)\mathcal{O}(x^\mu)d^4x\right)\right\rangle_{\text{Field Theory}}. \tag{2.9}$$

The expectation value of the operators can now be computed via the functional derivative as

$$\frac{\delta^n Z}{\delta\phi_0(x_1^\mu)\ldots\delta\phi_0(x_n^\mu)} = \langle \mathcal{O}(x_1^\mu)\ldots\mathcal{O}(x_n^\mu)\rangle\,. \tag{2.10}$$

Equation (2.7) corresponds to the classical or tree level approximation in the AdS space, corresponding to the large $N_c$ limit of the conformal field theory [4]. We will not go beyond this approximation.

## 2.3 Free Theory and Two–point Function

So far, we only provided a lot of mathematical definitions. What follows will be an example of how to perform calculations for a scalar field, so that the reader can better understand the meaning of the correspondence. We focus on a five–dimensional scalar field in AdS space. We assume it comes from an underlying operator of the type $\bar{\psi}\psi$, which automatically gives us $\Delta = 3$ in the field theory in four dimensions. The field, denoted $\phi(z, x)$, has an action given as

$$\mathcal{S} = \int d^5x \frac{\sqrt{g}}{2} \left[ g^{MN} \partial_M \phi(x^\mu, z) \partial_N \phi(x^\mu, z) - m^2 \phi^2(x^\mu, z) \right], \tag{2.11}$$

where coupling constants and curvature of the AdS space have been normalized in such a way that calculations become as compact as possible. The five–dimensional mass $m^2$ is also fixed by [4], e.g, $m^2 = \Delta(\Delta - 4)$. Writing out the $z$–dependence of the determinant and the metric explicitly gives

$$\mathcal{S} = \int d^5x \frac{1}{2z^3} \left[ \eta^{MN} \partial_M \phi(x^\mu, z) \partial_N \phi(x^\mu, z) - \frac{m^2}{z^2} \phi^2(x^\mu, z) \right]. \tag{2.12}$$

Using the Euler-Lagrange equation for fields, we can use the action integral to calculate the equations of motion (abbreviated EOM) to be:

$$\eta^{MN} \partial_N \left( \frac{1}{z^3} \partial_M \phi(x^\mu, z) \right) + \frac{m^2}{z^5} \phi(x^\mu, z) = 0. \tag{2.13}$$

Note that, other than the dependence of the field itself, there are no explicit dependence of the EOM on $x^\mu$. Thus it is convenient to split the five–dimensional derivatives into the usual four dimensions plus the $z$–component. The resulting equation becomes

$$\left[ \frac{\partial_\mu \partial^\mu \phi(x^\mu, z)}{z^3} - \partial_z \left( \frac{1}{z^3} \partial_z \phi(x^\mu, z) \right) \right] + \frac{m^2}{z^5} \phi(x^\mu, z) = 0. \tag{2.14}$$

From here we can split the field into two parts, one dependent on $z$ and one on $x^\mu$. Then we can transform the equation to momentum space by using Fourier transformation (abbreviated FT):

$$\left[ \frac{-k^2 \phi_0(x^\mu) f_k(z)}{z^3} - \partial_z \left( \frac{1}{z^3} \partial_z (\phi_0(x^\mu) f_k(z)) \right) \right] + \frac{m^2}{z^5} \phi_0(x^\mu) f_k(z) = 0, \tag{2.15}$$

where $\phi_0(x^\mu)$ denotes the FT of the boundary field. Note that the EOM are satisfied for any boundary fields $\phi_0(x^\mu)$ and they can thus be factored out and discarded. The resulting equation thus becomes:

$$\left[ \frac{-k^2 f_k(z)}{z^3} - \partial_z \left( \frac{1}{z^3} \partial_z f_k(z) \right) \right] + \frac{m^2}{z^5} f_k(z) = 0. \tag{2.16}$$

After calculating the derivative (using the Leibniz rule), the final equation becomes

$$\partial_z \partial_z f_k(z) - \frac{3}{z}\partial_z f_k(z) + \left[k^2 - \frac{m^2}{z^2}\right] f_k(z) = 0. \tag{2.17}$$

Let us now return to the action integral, equation (2.12), and Fourier transform the integrand in the same way as in equation (2.15). The result is as follows:

$$\begin{aligned}\mathcal{S} = \frac{1}{2}\int d^4x \int dz \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4}\frac{1}{z^3}\Big[-k \cdot k' f_k(z) f_{k'}(z) - \\ -\partial_z f_k(z)\partial_z f_{k'}(z) - \frac{m^2}{z^2} f_k(z) f_{k'}(z)\Big]\tilde{\phi}_0(k)\tilde{\phi}_0(k')e^{-ix\cdot(k+k')}.\end{aligned} \tag{2.18}$$

We can use the exponential combined with the integral over $\mathrm{d}^4x$ to express the action integral in terms of a delta function relating $k$ and $k'$:

$$\begin{aligned}\mathcal{S} &= \frac{1}{2}\int_{L_0}^{L_1} dz \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4}\frac{1}{z^3}\Big[-k \cdot k' f_k(z) f_{k'}(z) - \\ &- \partial_z f_k(z)\partial_z f_{k'}(z) - \frac{m^2}{z^2} f_k(z) f_{k'}(z)\Big]\tilde{\phi}_0(k)\tilde{\phi}_0(k')\,(2\pi)^4\,\delta^{(4)}(k+k').\end{aligned} \tag{2.19}$$

The second term in the brackets can be integrated by parts with respect to $z$; i.e.:

$$\int_{L_0}^{L_1} \frac{dz}{z^3}\partial_z f_k(z)\partial_z\left(f_{k'}(z)\right) = \left(\frac{1}{z^3}\partial_z f_k(z) f_{k'}(z)\right)\Big|_{L_0}^{L_1} - \int_{L_0}^{L_1} f_{k'}(z)\partial_z\left(\frac{1}{z^3}\partial_z f_k(z)\right)dz. \tag{2.20}$$

Inserting this into equation (2.19) to gives

$$\begin{aligned}\mathcal{S} = \frac{1}{2}\int_{L_0}^{L_1} dz \int \frac{d^4k}{(2\pi)^4} \int d^4k' \frac{1}{z^3}\Big[f_{k'}(z)\partial_z\partial_z f_k(z) f_{k'}(z) - \frac{3}{z}\partial_z f_k(z) + \\ +k \cdot k' f_k(z) f_{k'}(z) - \frac{m^2}{z^2} f_k(z) f_{k'}(z)\Big]\tilde{\phi}_0(k)\tilde{\phi}_0(k')\delta^{(4)}(k+k') + \\ +\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \int d^4k' \tilde{\phi}_0(k)\tilde{\phi}_0(k')\delta^{(4)}(k+k')\frac{1}{z^3}\partial_z f_k(z) f_{k'}(z)\Big|_{L_0}^{L_1}.\end{aligned} \tag{2.21}$$

Note that the expression inside the brackets is exactly the EOM. Since the EOM equals 0,the integrand, and thus the entire first term vanish. The final expression for the action thus reduces to

$$\mathcal{S} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\int d^4k' \tilde{\phi}_0(k)\tilde{\phi}_0(k')\delta^{(4)}(k+k')\frac{1}{z^3}\partial_z f_k(z) f_{k'}(z)\Big|_{L_0}^{L_1}. \tag{2.22}$$

Now that we have a final expression for the action, we can apply the correspondence in order to obtain the expectation values of the observables. Since the EOM are expressed in momentum space, we first need to find the FT of expression for the expectation value:

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle = \int d^4x \int d^4x' \, \langle \mathcal{O}(x)\mathcal{O}(x')\rangle \, e^{ipx} e^{ip'x'}. \tag{2.23}$$

The correspondence states that we ought to take the functional derivative of $Z$, the exponential of the action. Since the functional derivative acts similarly to a normal derivative (more specifically, the chain rule works as with normal derivative) it will create an inner derivative of the action with respect to $\phi_0(x)$ and leave the exponent unchanged. By choosing the evaluation point carefully,, we can set the exponent to 0 ($e^0 = 1$) and thus ignore that factor. Expressing the expectation value explicitly in terms of action, we obtain

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle = \int d^4x \int d^4x' \frac{\delta^2 \mathcal{S}[\phi_0(x'')]}{\delta\phi_0(x)\delta\phi_0(x')} e^{ipx} e^{ip'x'}. \tag{2.24}$$

We can express the action as a functional of the field in momentum space, $\tilde{\phi}_0(k)$, the FT of $\phi_0(x)$. Using this formulation, we can ultimately get an expression of the observables' expectation values devoid of integrals[2]

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle = (2\pi)^4 \, \delta^{(4)}(p+p')\frac{\partial_z f_p(z) f_{p'}(z)}{z^3}\Big|_{L_0}^{L_1}. \tag{2.25}$$

We note that, due to the boundary conditions of $f_p(z)$, we need the field to dissappear at the IR boundary ($z = L_1$). This corresponds to either that the fields vanishes at infinity, i.e., $L_1 = \infty$, or that $L_1$ is a finite number but that we demand $\partial_z f_p(L_1) = 0$ in order to reduce interfarence of the boundary. Using this we can rewrite the equation as (2.26) with all $z$ evaluated at the UV cutoff, $z = L_0$

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle = -\,(2\pi)^4 \, \delta^{(4)}(p+p')\frac{\partial_z f_p(z) f_{p'}(z)}{z^3}\Big|_{L_0}. \tag{2.26}$$

## 2.4 Interactions

Let us continue with a more difficult action, consider the same action as before, equation (2.11), but with added terms corresponding to interactions: i.e., equation (2.27).

$$\mathcal{S} = \int d^5x\sqrt{g}\left[\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{m^2}{2}(\phi)^2 - \frac{b}{6}(\phi)^3 - \frac{\lambda}{24}(\phi)^4\right]. \tag{2.27}$$

The equations in this section will become very long, thus it is convenient to introduce shorthand notation; i.e., $\phi$'s dependence on $x^\mu$ and $z$ will not be written if it is not a special case such as a boundary limit.

As before we take the given action and use the Euler-Lagrange equations in order to obtain the EOM:

$$\frac{1}{\sqrt{g}}\partial_M(g^{MN}\sqrt{g}\partial_N\phi) + m^2\phi + \frac{b}{2}(\phi)^2 + \frac{\lambda}{6}(\phi)^3 = 0, \tag{2.28}$$

where we impose the boundary condition, $\phi(x^\mu, L_0) = L_0\phi_0(x^\mu)$, following from equation 2.7, for later use as well as $\partial_z\phi(x^\mu, L_1) = 0$ in order to make the theory as insensitive as possible to the IR boundary. While equation (2.28) looks nasty, we can simplify by recognizing $\frac{1}{\sqrt{g}}\partial_M(g^{MN}\sqrt{g}\partial_N) = \nabla^2$. Substituting this the equation becomes

$$\left(\nabla^2 + m^2\right)\phi = -\frac{b}{2}\phi^2 - \frac{\lambda}{6}\phi^3. \tag{2.29}$$

Unfortunately, there exist no nontrivial solution to the equation above, but we can find one by solving the equation iteratively.

### 2.4.1 Iterative Solution

Start by expanding $\phi(x^\mu, z)$ in terms of an unknown parameter $\epsilon$, which will be quivalent to powers of $\phi_0$:

$$\phi = \epsilon\varphi' + \epsilon^2\varphi'' + \epsilon^3\varphi''' + \cdots = \sum_{n=0}^{\infty}\epsilon^n\varphi^{(n)}. \tag{2.30}$$

Put this expression into the equation (2.29) to obtain

$$\left(\nabla^2 + m^2\right)\left(\epsilon\varphi' + \epsilon^2\varphi'' + \cdots\right) = -\frac{b}{2}\left(\epsilon\varphi' + \epsilon^2\varphi'' + \cdots\right)^2 - \frac{\lambda}{6}\left(\epsilon\varphi' + \epsilon^2\varphi'' + \cdots\right)^3. \tag{2.31}$$

We can now recognize the terms on the left–hand side with the terms on the right–hand side which contain the same power in $\epsilon$. This give rise to a system with infinite amounts of equations, but we will only consider the first few. Written out explicitly, the equations to solve is

$$
\begin{aligned}
\left(\nabla^2 + m^2\right)\varphi' &= 0, \\
\left(\nabla^2 + m^2\right)\varphi'' &= -\frac{b}{2}\left(\varphi'\right)^2, \\
\left(\nabla^2 + m^2\right)\varphi''' &= -b\varphi'\varphi'' - \frac{\lambda}{6}\left(\varphi'\right)^3,
\end{aligned}
\tag{2.32}
$$

with the following boundary conditions:

$$
\begin{aligned}
\varphi'(x^\mu, L_0) &= L_0\phi_0(x^\mu), \\
\varphi''(x^\mu, L_0) &= 0, \\
\varphi'''(x^\mu, L_0) &= 0,
\end{aligned}
\tag{2.33}
$$

as well as

$$
\begin{aligned}
\partial_z\varphi'(x^\mu, L_1) &= 0, \\
\partial_z\varphi''(x^\mu, L_1) &= 0, \\
\partial_z\varphi'''(x^\mu, L_1) &= 0.
\end{aligned}
\tag{2.34}
$$

Even though this is an infinite series of equations, we only need the action to order $(\phi_0)^4$ so the equations above are sufficient. The first iteration is solved by introducing a Green's function, $K(z, x, x')$. This function is in this context known as a bulk–to–boundary propagator and has the following properties:

$$
\begin{aligned}
\left(\nabla^2 + m^2\right)K(z, x, x') &= 0, \\
K(L_0, x, x') &= L_0\delta^{(4)}(x - x'), \\
\partial_z K(L_1, x, x') &= 0.
\end{aligned}
\tag{2.35}
$$

With this the solution to the first equation becomes

$$
\varphi' = \int d^4x' K(z, x, x')\phi_0(x^{\mu'}). \tag{2.36}
$$

It can easily be shown that this solution, together with the definitions of K, satisfies equation (2.32). For the second iteration, we use the result obtained in the previous equation and put that into equation (2.32). The second iteration is then solved by introducing a bulk–to–bulk propagator, $G(z, z', x, x')$, defined such that

$$
\begin{aligned}
\left(\nabla^2 + m^2\right)G(z, z', x, x') &= \frac{\delta(z - z')\delta^{(4)}(x - x')}{\sqrt{g}}, \\
G(L_0, z', x, x') &= 0, \\
\partial_z G(L_1, z', x, x') &= 0.
\end{aligned}
\tag{2.37}
$$

With these definitions, along with the boundary condition, we can write the solution to equation (2.32) as

$$\varphi'' = \int d^5x' \sqrt{g} G(z,z',x,x')(-\frac{b}{2}(\varphi')^2) = -\frac{b}{2}\int d^5x' \sqrt{g} G(z,z',x,x')$$
$$\int d^4x'' K(z',x',x'')\phi_0(x^{\mu''}) \int d^4x''' K(z',x',x''')\phi_0(x^{\mu'''}). \qquad (2.38)$$

Note that the second order solution contains two factors of $\phi_0(x^\mu)$. It is therefore said to be of order $\mathcal{O}(\phi_0^2)$. With the bulk–to–boundary propagator and bulk–to–bulk propagator defined all other iterations can be solved given enough time. e.g. the third iteration got the following solution:

$$\begin{aligned}
\varphi''' &= \frac{b^2}{2}\int d^5x' \sqrt{g} G(z,z',x,x') \int d^4x'' K(z',x',x'')\phi_0(x'') \times \\
&\times \int d^5x''' \sqrt{g} G(z',z''',x',x''') \int d^4x^{(4)} K(z''',x''',x^{(4)})\phi_0(x^{(4)}) \times \\
&\times \int d^4x^{(5)} K(z''',x''',x^{(5)})\phi_0(x^{(5)}) - \\
&- \frac{\lambda}{6}\int d^5x' \sqrt{g} G(z,z',x,x') \int d^4x'' K(z',x',x'')\phi_0(x'') \times \\
&\times \int d^4x''' K(z',x',x''')\phi_0(x''') \int d^4x^{(4)} K(z',x',x^{(4)})\phi_0(x^{(4)}). \qquad (2.39)
\end{aligned}$$

As consistent with previous equations, the third iteration is $\mathcal{O}(\phi_0^3)$. Even though we need the action to $\mathcal{O}(\phi_0^4)$ to construct the desired four–point function, there is no need to calculate the fourth iteration.

### 2.4.2 Relation Between G and K

What we do need though, is a connection between the two propagators, $G$ and $K$. This will allow us to simplify the so called boundary term as discussed later. Start by looking at the following expression:

$$\int d^5x \sqrt{g}[G(z,z',x,x')\left(\nabla^2+m^2\right)K(z,x,x'') - K(z,x,x'')\left(\nabla^2+m^2\right)G(z,z',x,x')]. \qquad (2.40)$$

From here we can do two things. First we can use the definitions of $G$ and $K$, expressing the terms containing $(\nabla^2+m^2)$ in a different way. Doing this we arrive at

$$\int d^5x \sqrt{g}\left[G(z,z',x,x')\,(0) - K(z,x,x'')\left(\frac{\delta(z-z')\delta^{(4)}(x-x')}{\sqrt{g}}\right)\right]. \qquad (2.41)$$

We see that the first term vanish due to the definition of $K$. For the second term, we can integrate away the delta functions, which in the end gives

$$\int d^5x \frac{-\sqrt{g}}{\sqrt{g}} K(z,x,x'')\delta(z-z')\delta^{(4)}(x-x') = -K(z',x',x''). \tag{2.42}$$

But we could just as well take equation (2.40) and rewrite it as

$$\int d^5x\sqrt{g}\Big[G(z,z',x,x')\nabla^2 K(z,x,x'') - K(z,x,x'')\nabla^2 G(z,z',x,x')\Big]. \tag{2.43}$$

From here we can use Green's second identity in order to obtain

$$\int d^4x\sqrt{\gamma}[n^M G(z,z',x,x')\partial_M K(z,x,x'') - K(z,x,x'')n^M\partial_M G(z,z',x,x')]\Big|_{L_0}^{L_1}, \tag{2.44}$$

where $\hat{n}^L$ denote the five–dimensional vector normal to the boundary and $\gamma$ denote the determinant of the metric of the boundary. By definition the derivatives of the fields vanish in the limit as $z$ tend to $L_1$. For the limit as $z$ tend to $L_0$, we can use the boundary conditions of $K$ and $G$, defined in (2.35) and (2.37), and obtain

$$\int d^4x\sqrt{\gamma}n^M\delta^{(4)}(x-x'')\partial_M G(L_0,z',x,x') = \sqrt{\gamma}n^M\partial_M G(L_0,z',x'',x'). \tag{2.45}$$

Since both (2.42) and (2.44) originates from the same equation, (2.40), we can equate them and finally obtain

$$-K(z',x',x'') = \sqrt{\gamma}n^M\partial_M G(L_0,z',x'',x'). \tag{2.46}$$

## 2.5 Three–point Function

The three–point function was derived in detail in [2]. We can derive it using the same techniques sas below for the four–point function, but for the sake of brevity we will only quote the results of the derivation:

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = -b\int d^4x\int dz\sqrt{g}K(z,x,x_1)K(z,x,x_2)K(z,x,x_3). \tag{2.47}$$

And the same equation transformed to momentum space:

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle = -b(2\pi)^4\delta^4(p_1+p_2+p_3)\int dz\sqrt{g}K_{p_1}(z)K_{p_2}(z)K_{p_3}(z). \tag{2.48}$$

## 2.6 Four–point Function

With the scalar field found in terms of bulk–to–boundary and bulk–to–bulk propagators, we are ready to calculate the Four–point function. Start integrating by parts the action (equation (2.27)) which gives

$$\mathcal{S} \;=\; \int d^4x \frac{\sqrt{\gamma}}{2}\phi n^L \partial_L \phi\Big|_{z=L_0} + \int d^5x\sqrt{g}\Big( -\frac{1}{2}\phi(\nabla^2+m^2)\phi - \frac{b}{6}\phi^3 - \frac{\lambda}{4!}\phi^4\Big). \quad (2.49)$$

Note that the expression above contains $(\nabla^2+m^2)\phi$ which, using equation (2.29), can be traded for $-\frac{b}{2}\phi^2 - \frac{\lambda}{6}\phi^3$; i.e.,

$$\mathcal{S} \;=\; \int d^4x \frac{\sqrt{\gamma}}{2}\phi n^L \partial_L \phi\Big|_{z=L_0} + \int d^5x\sqrt{g}\Big( -b\Big[\frac{1}{4}-\frac{1}{6}\Big]\phi^3 - \lambda\Big[\frac{1}{12}-\frac{1}{24}\Big]\phi^4\Big). \quad (2.50)$$

We can split the action in three different sectors, perform calculation on each part separately and in the end add together the sectors; i.e.,

$$\mathcal{S}_{Total} = \mathcal{S}_{\text{Boundary}} + \mathcal{S}_b + \mathcal{S}_\lambda. \quad (2.51)$$

Each sector of the action must be expanded to order $\mathcal{O}(\phi_0^4)$ in order to give meaningful results. For $S_{\text{Boundary}}$, this means that $\phi$ must be expanded to third order, $\varphi'''$, (Recall that $\phi(x^\mu, L_0) = L_0\phi_0(x^\mu)$ is given as boundary condition). Exchanging $\phi$ with equation (2.39) and using equation (2.46) results in

$$\begin{aligned}
\mathcal{S}_{\text{Boundary}} \;=\;& \frac{b^2}{4}\int d^4x\phi_0(x)\int d^4x'\int dz'\sqrt{g}K(z',x,x')\int d^4x'' K(z',x',x'')\phi_0(x'')\, \times \\
\times\;& \int d^5x'''\sqrt{g}G(z',z''',x',x''')\int d^4x^{(4)}K(z''',x''',x^{(4)})\phi_0(x^{(4)})\, \times \\
\times\;& \int d^4x^{(5)}K(z''',x''',x^{(5)})\phi_0(x^{(5)}) \\
-\;& \frac{\lambda}{12}\int d^4x\phi_0(x)\int dz'\sqrt{g}K(z',x,x')\int d^4x'' K(z',x',x'')\phi_0(x'') \\
\times\;& \int d^4x''' K(z',x',x''')\phi_0(x''')\int d^4x^{(4)}K(z',x',x^{(4)})\phi_0(x^{(4)}).
\end{aligned} \quad (2.52)$$

Apply exactly the same calculations on the remaining sectors; i.e., expand $\mathcal{S}_b$ to the second order, $\varphi''$, and $\mathcal{S}_\lambda$ to first order, $\varphi'$, to obtain

$$
\begin{aligned}
\mathcal{S}_b &= \frac{b^2}{8}\int d^4x \int dz\sqrt{g}\int d^4x' \int d^4x'' \int d^5x'''\sqrt{g}\int d^4x^{(4)}\int d^4x^{(5)} \times \\
&\times \; K(z,x,x')K(z,x,x'')G(z,z''',x,x''')K(z''',x''',x^{(4)})K(z''',x''',x^{(5)}) \times \\
&\times \; \phi_0(x')\phi_0(x'')\phi_0(x^{(4)})\phi_0(x^{(5)}) + \mathcal{O}(\phi_0^5)
\end{aligned} \tag{2.53}
$$

and

$$
\begin{aligned}
\mathcal{S}_\lambda &= \frac{1}{12}\int d^4x \int dz\sqrt{g}\int d^4x' K(z,x,x')\phi_0(x')\int d^4x'' K(z,x,x'')\phi_0(x'') \times \\
&\times \int d^4x''' K(z,x,x''')\phi_0(x''')\int d^4x^{(4)} K(z,x,x^{(4)})\phi_0(x^{(4)}).
\end{aligned} \tag{2.54}
$$

Note that all terms in the action with $\mathcal{O}(\phi_0^4)$ have either a factor $\lambda$ followed by four bulk–to–boundary propagators, or a factor $b^2$ followed by a bulk–to–bulk propagator and $2 \times 2$ bulk–to–boundary propagators. Adding together all sectors and rearranging coefficients gives the final action

$$
\begin{aligned}
\mathcal{S} &= \frac{b^2}{8}\int d^4x \int dz\sqrt{g}\int d^4x' \int d^4x'' \int d^5x'''\sqrt{g}\int d^4x^{(4)}\int d^4x^{(5)} \\
&\quad K(z,x,x')K(z,x,x'')G(z,z''',x,x''')K(z''',x''',x^{(4)})K(z''',x''',x^{(5)}) \times \\
&\times \; \phi_0(x')\phi_0(x'')\phi_0(x^{(4)})\phi_0(x^{(5)}) - \\
&- \; \frac{\lambda}{24}\int d^4x \int dz\sqrt{g}\int d^4x' \int d^4x'' \int d^4x''' \int d^4x^{(4)} \\
&\quad K(z,x,x')K(z,x,x'')K(z,x,x''')K(z,x,x^{(4)})\phi_0(x')\phi_0(x'')\phi_0(x''')\phi_0(x^{(4)}).
\end{aligned} \tag{2.55}
$$

Now that the action is expressed in terms of propagators and the boundary field, it is possible to evaluate the four–point function using the correspondence, or more precisely, equation (2.10) with four operators:

$$
\frac{\delta^4 Z}{\delta\phi_0(x_1)\delta\phi_0(x_2)\delta\phi_0(x_3)\delta\phi_0(x_4)} = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle\,. \tag{2.56}
$$

For the part of the action containing $\lambda$ the calculation is straightforward. There are four different fields in the expression and thus there are 4! ways of combining $x, x', x'''$ and $x^{(4)}$ with $x_1, \ldots, x_4$. The exact calculation can be found in section A.1, but essentially the factor $\frac{1}{24}$ is canceled, all fields $\phi_0$ are derived away and what is left (from the second term of equation (2.55)) becomes

$$
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = -\lambda\int d^4x \int dz\sqrt{g}K(z,x,x_1)K(z,x,x_2)K(z,x,x_3)K(z,x,x_4). \tag{2.57}
$$

The equation above can be interpreted as a Witten diagram depicted below.

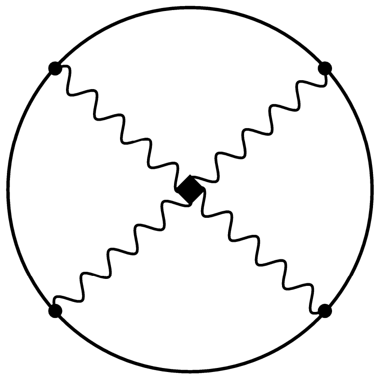

Figure 1: Equation (2.57) as a Witten diagram. The circle denotes the four–dimensional boundary. The wavy lines denote bulk–to–boundary propagators ($K$). The diamond-shaped dot denote a vertex with coupling strength $-\lambda$.

The first term of equation (2.55), the one including $b^2$, is more tricky since it contains a bulk–to–bulk propagator $G$. Because of this, we can not just assume all 4! combinations are equivalent to each other. Similar to the case with $\lambda$, the full derivation is listed in the appendix. The result from the calculations is

$$\begin{aligned}
& \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \\
= \; & b^2 \int d^4x \int dz\sqrt{g} \int d^4x' \int dz'\sqrt{g} \times \\
\times \; & K(z,x,x_1)K(z,x,x_2)G(z,z',x,x')K(z',x,x_4)K(z',x,x_5) + \\
+ \; & b^2 \int d^4x \int dz\sqrt{g} \int d^4x' \int dz'\sqrt{g} \times \\
\times \; & K(z,x,x_1)K(z,x,x_4)G(z,z',x,x''')K(z',x,x_2)K(z',x,x_5) + \\
+ \; & b^2 \int d^4x \int dz\sqrt{g} \int d^4x' \int dz'\sqrt{g} \times \\
\times \; & K(z,x,x_1)K(z,x,x_5)G(z,z',x,x')K(z',x,x_2)K(z',x,x_4).
\end{aligned} \tag{2.58}$$

Note that the (square roots of the) determinants of the metric, $g$, are functions of $z, z', \cdots$ and therefore cannot be extracted from the integrals. Yet again the result can be depicted via Witten diagrams (see below). Each of the three terms in the equation above correspond to similar looking albeit different diagrams.

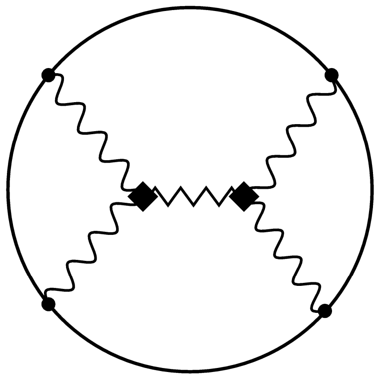

Figure 2: Equation (2.58) as a Witten diagram. The cirlce denote the four–dimensional boundary. The wavy lines denote bulk–to–boundary propagators ($K$) while the zigzag–line denotes bulk–to–bulk propagator ($G$). The diamond-shaped dots denote vertices with coupling strength $-b$.

In general we can extract Witten rules (analogous to Feynman rules for Feynman diagrams) as follows:

1. Propagators
   (a) For each line with one end in bulk and one in boundary: multiply by a bulk–to–boundary propagator ($K$)
   (b) For each line with both ends in bulk: multiply by a bulk–to–bulk propagator ($G$)
2. Vertecies
   (a) For each vertex that connects three lines: multiply by a factor $-b$
   (b) For each vertex that connects four lines: multiply by a factor $-\lambda$
3. Integrate over possible positions for vertices by adding a factor $\int d^4x \int dz \sqrt{g}$ for each vertex

Consider equation (2.57) (and (2.58)) which gives us the four–point function in position space. Due to conservation laws, physicists are often more interested in the momentum space (in this case the fifth dimension will remain in position space while the other four transform via FT to momentum space). Thus apply Fourier Transform and arrive at

$$
\begin{aligned}
\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle = & \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \times \\
\times \quad & e^{ip_1x_1}e^{ip_2x_2}e^{ip_3x_3}e^{ip_4x_4} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \\
= \quad & -\lambda \int d^4x \int dz\sqrt{g} \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \times \\
\times \quad & e^{ip_1x_1}e^{ip_2x_2}e^{ip_3x_3}e^{ip_4x_4} K(z,x,x_1)K(z,x,x_2)K(z,x,x_3)K(z,x,x_4).
\end{aligned}
\tag{2.59}
$$

From here we can use the fact that the bulk–to–boundary propagator is a function of the distance between the four–dimensional arguments,$(x_i - x)$ ; i.e.,

$$
\begin{aligned}
- \quad & \lambda \int d^4x \int dz\sqrt{g} \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 e^{ip_1x_1}e^{ip_2x_2}e^{ip_3x_3}e^{ip_4x_4} \times \\
\times \quad & K(z,x_1-x)K(z,x_2-x)K(z,x_3-x)K(z,x_4-x).
\end{aligned}
\tag{2.60}
$$

Now we can perform a suitable change of variables, $y_i = x_i - x$, in order to obtain

$$
\begin{aligned}
- \quad & \lambda \int d^4x \int dz\sqrt{g} \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4 e^{ip_1y_1}e^{ip_2y_2}e^{ip_3y_3}e^{ip_4y_4} \times \\
\times \quad & e^{ix(p_1+p_2+p_3+p_4)} K(z,y_1)K(z,y_2)K(z,y_3)K(z,y_4).
\end{aligned}
\tag{2.61}
$$

By making the following definitions:

$$
\delta^{(n)}(f(k)) = \int \frac{d^nx}{(2\pi)^n} e^{ixf(k)} \tag{2.62}
$$

and

$$
K_{p_i}(z) = \int d^4y_i e^{ip_iy_i} K(z,y_i), \tag{2.63}
$$

we can rewrite equation (2.61) as

$$
-\lambda \int dz\sqrt{g}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)K_{p_1}(z)K_{p_2}(z)K_{p_3}(z)K_{p_4}(z). \tag{2.64}
$$

In a similar way, we can rewrite equation (2.58) as

$$
\begin{aligned}
&\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle = b^2(2\pi)^4\delta^{(4)}(p_1+p_2+p_4+p_5) \\
\times\ & \Big[\int dz\sqrt{g}\int dz'\sqrt{g}K_{p_1}(z)K_{p_2}(z)G_{p_3}(z,z')K_{p_4}(z')K_{p_5}(z') + \\
+\ & \int dz\sqrt{g}\int dz'\sqrt{g}K_{p_1}(z)K_{p_4}(z)G_{p_3}(z,z')K_{p_2}(z')K_{p_5}(z') + \\
+\ & \int dz\sqrt{g}\int dz'\sqrt{g}K_{p_1}(z)K_{p_5}(z)G_{p_3}(z,z')K_{p_2}(z')K_{p_4}(z')\Big].
\end{aligned}
\tag{2.65}
$$

Considering the same Witten diagrams as before, we can construct Witten rules in momentum space:

1. Propagators
   (a) For each line with one end in bulk and one in boundary: multiply by a bulk–to–boundary propagator in momentum space ($K_p$)
   (b) For each line with both ends in bulk: multiply by a bulk–to–bulk propagator in momentum space ($G_p$)
2. Vertices
   (a) For each vertex that connects 3 lines: multiply by a factor $-b$
   (b) For each vertex that connects 4 lines: multiply by a factor $-\lambda$
3. For each vertex: multiply by a factor of $(2\pi)^4\delta^{(4)}(\Sigma_i p_i)$ because of conservation of momentum
4. Integrate over possible positions in $z$ for vertices by adding a factor $\int dz\sqrt{g}$ for each vertex

1q

## 2.7 Equations of Motion

As a final step, we need to find what equations the bulk–to–boundary (and bulk–to–bulk) propagators obey in momentum space. One way to do this is to solve equation (2.35) (or equation (2.37) for bulk–to–bulk propagator) in position space and use FT to convert result into momentum space. However, this is a long and tedious process and while it in the end gives the correct result, so does a possible shortcut. It is easier to first use FT on equation (2.35) in order to get the equation the bulk–to–boundary propagator in momentum space satisfies and then solve the equation from there. Recall what the Laplacian was replaced with in order to obtain

$$\left(\frac{1}{\sqrt{g}}\partial_M(g^{MN}\sqrt{g}\partial_N) + m^2\right) K(z,y) = 0. \tag{2.66}$$

Split the derivative into one $z$-dependent and one $x^\mu$-dependent part and insert the $z$-dependence from the metric explicitly to get

$$\left(z^2\partial_\mu\partial_\mu - z^5\partial_z(\frac{1}{z^3}\partial_z) + m^2\right) K(z,y) = 0. \tag{2.67}$$

Evaluate the second term in the parenthesis, which gives

$$\left(z^2\partial_\mu\partial_\mu - z^2\partial_z\partial_z + 3z\partial_z + m^2\right) K(z,y) = 0, \tag{2.68}$$

and after the FT the equation becomes

$$\left(-z^2p^2 - z^2\partial_z\partial_z + 3z\partial_z + m^2\right) K_p(z) = 0. \tag{2.69}$$

Since the right hand side is nonzero for the bulk–to–bulk propagator, the calculation naturally becomes a bit longer. FT of the right hand side gives

$$\begin{aligned} \int d^4y e^{ipy} z^5 \delta^{(4)}(y)\delta(z-z') &= z^5\delta(z-z') \int d^4y \delta^{(4)}(y) e^{ipy} \\ &= z^5\delta(z-z'). \end{aligned} \tag{2.70}$$

In the same way, the FT of the boundary conditions gives

$$K_p(z) = L_0, \tag{2.71}$$

$$G_p(z,z') = 0. \tag{2.72}$$

Divide equation (2.69) and (2.70) through by $z^2$ to obtain the equations

$$\left(-\partial_z\partial_z + \frac{3}{z}\partial_z - p^2 + \frac{m^2}{z^2}\right) K_p(z) = 0, \tag{2.73}$$

$$\left(-\partial_z\partial_z + \frac{3}{z}\partial_z - p^2 + \frac{m^2}{z^2}\right) G_p(z,z') = z^3\delta(z-z'). \tag{2.74}$$

# 3 Results

Let us gather together all the results we have derived so far. We note that $K_p(z)$, $G_p(z,z')$ as well as $\tilde{\phi}_p(z)$, the FT of $\phi(x^\mu, z)$, defined by

$$\tilde{\phi}_p(z) = \int d^4x e^{-ipx}\phi(x^\mu, z), \tag{3.1}$$

are all functions of $z$ which obeys the same equation, equation (2.73) albeit different boundary conditions. I.e., for $\tilde{\phi}_p(z)$ we need

$$\begin{aligned} \tilde{\phi}_p(z) = 0 \qquad & \text{at } z = L_0, \\ \partial_z\tilde{\phi}_p(z) = 0 \qquad & \text{at } z = L_1, \end{aligned} \tag{3.2}$$

for the bulk–to–boundary propagator $K_p(z)$ we need

$$\begin{aligned} K_p(z) = L_0 \qquad & \text{at } z = L_0, \\ \partial_z K_p(z) = 0 \qquad & \text{at } z = L_1, \end{aligned} \tag{3.3}$$

and for the bulk–to–bulk propagator $G_p(z,z')$ we need

$$\begin{aligned} G_p(z,z') = 0 \qquad & \text{at } z = L_0, \\ \partial_z G_p(z,z') = 0 \qquad & \text{at } z = L_1, \\ G_p(z_+,z') = G_p(z_-,z') \qquad & \text{at } z = z', \\ \partial_z G_p(z_+,z') - \partial_z G_p(z_-,z') = -(z')^3 \qquad & \text{at } z = z'. \end{aligned} \tag{3.4}$$

where we have made the following definitions:

$$\begin{aligned} z_- &= \lim_{\substack{z\to z' \\ z<z'}} z \\ z_+ &= \lim_{\substack{z\to z' \\ z>z'}} z \end{aligned} \tag{3.5}$$

The third condition from equation (3.4) comes from continuity at the point $z = z'$. The fourth condition is a bit trickier. Starting from equation (2.74), we can integrate both sides of the equation between $z' - \epsilon$ and $z' + \epsilon$, i.e.,

$$\int_{z'-\epsilon}^{z'+\epsilon} \left(-\partial_z\partial_z + \frac{3}{z}\partial_z - p^2 + \frac{m^2}{z^2}\right) G_p(z,z')dz = \int_{z'-\epsilon}^{z'+\epsilon} z^3\delta(z-z')dz. \tag{3.6}$$

Using a defining property of the delta function we can calculate the right side to be

$$\int_{z'-\epsilon}^{z'+\epsilon} z^3\delta(z-z')dz = (z')^3. \tag{3.7}$$

The left hand side is more complicated. We need to use partial integration on the first term in order to obtain

$$\int_{z'-\epsilon}^{z'+\epsilon} \left(-\partial_z\partial_z + \frac{3}{z}\partial_z - p^2 + \frac{m^2}{z^2}\right) G_p(z,z')dz = -\partial_z G_p(z,z')\Big|_{z=z_-}^{z=z_+} + \cdots \tag{3.8}$$

where, after some calculation, the remaining terms will be of order $\epsilon$ and vanish. Equating the left and right side gives

$$-\partial_z G_p(z,z')\Big|_{z=z_-}^{z=z_+} = (z')^3 \tag{3.9}$$

which, when evaluated at boundaries, gives the fourth condition.
We see that both of the differential equations given in (2.73) and (2.74) closely resembles the Bessel equation, but not quite. By a slight modification of respective propagator we hope to rewrite the equations as Bessels equation,

$$\left(z^2\partial_z^2 + z\partial_z + z^2 - \nu^2\right) y(z) = 0. \tag{3.10}$$

The equation is of second order in $\partial_z$, and thus two independent solutions, defined as $J_\nu(z)$ and $Y_\nu(z)$ exists. We can write the full solution as a superposition of the independent solutions, i.e.,

$$y(z) = AJ_\nu(z) + BY_\nu(z), \tag{3.11}$$

for some constants A and B. Back to (2.73), we try to make it look like (3.10) by rewriting the propagator as $K_p(z) = z^n f_p(pz)$ for some $n$. The derivatives are now acting on the product of $z^2$ and $f_p(pz)$ and must be evaluated according to the Leibnitz rule. The resulting differential equation becomes

$$z^{n-2}\left[z^2\partial_z\partial_z + (2n-3)z\partial_z + p^2z^2 + (n(n-1) - n - m^2)\right] f_p(pz). \tag{3.12}$$

From here it is obvious that $n = 2$ which gives

$$\left[z^2\partial_z\partial_z + z\partial_z + p^2z^2 - (m^2+4)\right] f_p(pz). \tag{3.13}$$

Recall that the five–dimensional mass is given by the equation $m^2 = \Delta(\Delta - 4)$.Since in this case $\Delta = 3$ we must set $m^2 + 4 = 1$. This leads to the solution to (3.13) as

$$f_p(pz) = AJ_1(pz) + BY_1(pz), \tag{3.14}$$

where A and B are given by the boundary conditions set by equations ((3.2) – (3.4)) and $p = \sqrt{p^2}$.

Before we need to find the value of the constants we make a short note here. The momentum $p$ in equation (3.12) is actually short notation for four–momentum (recall the four–dimensional FT made in equation (2.69)), which, using the metric of AdS space, may be negative. A direct implication of this is that the four–momentum squared may be negative. We can find those solutions by performing the substitution $\sqrt{p^2} \to \sqrt{-p^2}$, on equation (3.13).This gives us the so called modified Bessel equation:

$$\left[z^2\partial_z\partial_z + z\partial_z - (pz)^2 - (m^2+4)\right] f_p(pz). \tag{3.15}$$

The solution of the differential equation above can once again be written in terms of two linearly independent functions, most commonly known as the modified Bessel Functions. $m^2 + 4$ must still be equal to 1 since we are still dealing with the same object. Thus the solution is given by

$$f_p(pz) = CI_1(pz) + DK_1(pz). \tag{3.16}$$

where $C$ and $D$ are constants given by boundary conditions. Returning to (3.14), we can find the constants A and B by using the boundary conditions in order to obtain a system of equations. In order to evaluate one of the boundaries we need to know the derivative of $J_1(z)$, $Y_1(z)$ and so on. The derivatives of Bessel functions can be found in[5] to be

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k \left(z^\nu J_\nu(z)\right) = z^{\nu-k} J_{\nu-k}(z). \tag{3.17}$$

The equation above holds for $Y_\nu(z)$ and $I_\nu(z)$. In the case of $K_\nu(z)$ the identity is similar, but differ with a minus sign, i.e.

$$(-1)^k \left(\frac{1}{z}\frac{d}{dz}\right)^k \left(z^\nu K_\nu(z)\right) = z^{\nu-k} K_{\nu-k}(z). \tag{3.18}$$

Set $\nu = 1$ and $k = 1$ to obtain

$$\frac{1}{z}\frac{d}{dz}\bigg(zJ_1(z)\bigg) = J_0(z). \tag{3.19}$$

For us, the interesting case is given by equation (3.3). Expand the parenthesis, using the Leibniz rule, e.g.,

$$\partial_z(z[zJ_1(z)]) = [zJ_1(z)] + z[zJ_0(z)] \tag{3.20}$$

Using this we can set up an expression for the coefficients A and B:

$$\begin{aligned} A{L_0}^2J_1(pL_0) &+ B{L_0}^2Y_1(pL_0) = L_0, \\ A(L_1J_1(pL_1) + pL_1^2J_0(pL_1)) &+ B(L_1Y_1(pL_1) + pL_1^2Y_0(pL_1)) = 0. \end{aligned} \tag{3.21}$$

Using shorthand notation, $\eta_J = L_0^2J_1(pL_0)$ and $\xi_J = (L_1J_1(pL_1)+pL_1^2J_0(pL_1))$, where $\eta$ and $\xi$ are functions of four–momentum, and similar definitions made for the terms containing Bessel functions of second kind, i.e., $\eta_Y$ and $\xi_Y$, we can write the expression as:

$$\begin{aligned} A\eta_J + B\eta_Y &= 0 \text{ or } L_0, \\ A\xi_J + B\xi_Y &= 0. \end{aligned} \tag{3.22}$$

## 3.1 Bound States

In order for a bound state to appear we must demand that equation (3.2) are linearly dependent; Only then will we get an interesting, nonzero solution. If they are linearly dependent can be checked using a determinant, i.e.,

$$\begin{vmatrix} \eta_Y & \eta_J \\ \xi_Y & \xi_J \end{vmatrix} = 0. \tag{3.23}$$

The determinant evaluates to

$$\eta_Y\xi_J - \eta_J\xi_Y = 0. \tag{3.24}$$

Whenever the above equation is satisfied there is a bound state.

## 3.2 Bulk–to–boundary Propagator

We note that, solving for constants $A$ and $B$, (3.22), belonging to the solution of the bulk–to–boundary propagator, yields

$$A = \frac{-\xi_Y}{\eta_Y \xi_J - \eta_J \xi_Y}, \tag{3.25}$$

$$B = \frac{\xi_J}{\eta_Y \xi_J - \eta_J \xi_Y}. \tag{3.26}$$

Note that the poles correspond exactly to the bound states satisfying (3.24). These zeroes can be found numerically, which we will come to later. Using the newly found constants $A$ and $B$, we can write the bulk–to–bulk propagator as

$$K_p(z) = z^2 \frac{\xi_J Y_1(pz) - \xi_Y J_1(pz)}{\eta_Y \xi_J - \eta_J \xi_Y} \tag{3.27}$$

## 3.3 Two–point Function

We now wish to calculate the two–point–function of $K_p(z)$ using the expression given by (2.25), i.e.,

$$\Pi(p^2) = \langle \mathcal{O}(p)\mathcal{O}(p') \rangle = -\frac{\partial_z K_p(z) K_{p'}(z)}{z^3}\Big|_{L_0}. \tag{3.28}$$

We see that for discrete values of $p$ the denominator of $K_p(z)$ tends to zero and the propagator becomes infinite. This equates to the expression

$$\Pi(p^2) = \sum_{n=0}^{\infty} \frac{\mathrm{Res}(\Pi(p^2))_n}{p^2 - m_n^2}. \tag{3.29}$$

Where $\Pi(p^2)$ is the two–point function and $\mathrm{Res}(\Pi(p^2))_n$ is the residue of the $n^{\mathrm{th}}$ pole. I.e., every singularity of $K_p(z)$ corresponds to a particle with mass $m_n$. Equation (3.29) is known as the not subtracted dispersion relation.

While at it, we introduce the so called once subtracted dispersion relation, given by equation (3.30), as well as the twice subtracted dispersion relation, (3.31).

$$\Pi(p^2) = \Pi(0) + p^2 \sum_n \frac{\mathrm{Res}(\Pi(m_n^2))}{m_n^2 (p^2 - m_n^2)}. \tag{3.30}$$

,

$$\Pi(p^2) = \Pi(0) + p^2 \Pi'(0) + p^4 \sum_n \frac{\mathrm{Res}(\Pi(m_n^2))}{m_n^4 (p^2 - m_n^2)}.$$

How these formulas are derived can be found in detail in appendix (A.3). Take a closer look at (3.27), more specifically, the denominator,

$$\begin{aligned}\eta_Y\xi_J - \eta_J\xi_Y &= {L_0}^2 L_1\Big[Y_1(pL_0)[J_1(pL_1) + pL_1J_0(pL_1)] - \\ &- J_1(pL_0)[Y_1(pL_1) + pL_1Y_0(pL_1)]\Big]. \end{aligned} \tag{3.31}$$

The equation above becomes zero for discrete values of p. These zeroes equates to the poles of $K_p(z)$. Unfortunately, there exist no analytical solution, so in order to find the masses; the equation must be solved numerically. We are luckier when it comes to the residues – they can be found analytically. Looking at (3.29), we can multiply both sides by the denominator and obtain

$$\text{Res}(\Pi(p^2))_n = \lim_{p^2\to m_n^2} \Pi(p^2)(p^2 - m_n^2). \tag{3.32}$$

We know that $\Pi(p^2)$ is a function proportional to $K_p(z)$ which in turn we can write as fraction of functions, i.e.,

$$\text{Res}(\Pi(p^2))_n = \lim_{p^2\to m_n^2} \frac{f(p^2)}{g(p^2)}(p^2 - m_n^2). \tag{3.33}$$

Where $g(p^2)\Big|_{p^2=m_n^2} = 0$. We can Taylor expand $g(p^2)$ near the pole in order to obtain

$$\text{Res}(\Pi(p^2))_n = \lim_{p^2\to m_n^2} \frac{f(p^2)(p^2 - m_n^2)}{g(m_n^2) + (p^2 - m_n^2)g'(m_n^2) + \ldots}. \tag{3.34}$$

But $g(m_n^2) = 0$, so the expression simplifies, after truncating higher order terms, to

$$\text{Res}(\Pi(p^2))_n = \lim_{p^2\to m_n^2} \frac{f(p^2)}{g'(m_n^2)} = \frac{f(m_n^2)}{g'(m_n^2)}. \tag{3.35}$$

Once we obtained the residues we can use them to easily find the decay constant, which in standard field theories are defined by:

$$f = \langle 0|\mathcal{O}(p_n)|\phi_n\rangle = \sqrt{\text{Res}(\Pi(p^2))_n}. \tag{3.36}$$

## 3.4 Three–point Function

We can play the same game with the three–point function as with the two–point function. The difference being that there is much more calculation involved, including an integral over $z$ that must be evaluated (recall equation (2.48)). We can use the results, e.g., masses and decay constants, found numerically by simulating the two–point function in order to find new results, e.g., coupling constants $\lambda_{ijk}$. From equation (2.48) we got

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle = -b(2\pi)^4\delta^4(p_1+p_2+p_3)\int dz\sqrt{g}K_{p_1}(z)K_{p_2}(z)K_{p_3}(z). \quad (3.37)$$

where $p_n$ are momentum corresponding to the mass of the $\mathrm{n^{th}}$ particle. Using the fact that $K_p(z)$ can be rewritten as a sum over all $n$, i.e.,

$$K_p(z) = \sum_{n=0}^{\infty}\frac{\mathrm{Res}(K_{p_n}(z))}{p^2-m_n^2}, \quad (3.38)$$

we can then write the $z$-dependence of the integral as

$$\int dz\frac{1}{z^5}\,\mathrm{Res}\,(K_{p_1}(z))\,\mathrm{Res}\,(K_{p_2}(z))\,\mathrm{Res}\,(K_{p_3}(z)). \quad (3.39)$$

Just like in the case with the two–point function, we can rewrite the three–point function as a sum over all poles. This time though, we must multiply by the coupling constant that corresponds to the vertex in a Witten diagram. The resulting expression becomes

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\rangle = \sum_{i,j,k}^{\infty}\lambda_{ijk}\frac{f_i}{p_1^2-m_i^2}\frac{f_j}{p_2^2-m_j^2}\frac{f_k}{p_3^2-m_k^2}, \quad (3.40)$$

with $f$ defined by equation (3.36). Using this we can get the value of the coupling constant by calculating the integral (numerically) and divide by the appropriate decay factors, mathematically written as:

$$\lambda_{ijk} = \frac{1}{f_i f_j f_k}\int dz\frac{1}{z^5}\,\mathrm{Res}\,(K_{p_1}(z))\,\mathrm{Res}\,(K_{p_2}(z))\,\mathrm{Res}\,(K_{p_3}(z)). \quad (3.41)$$

Finally we would like to use $\lambda_{ijk}$ in order to find the form factor, $F(p_3^2)$, which in standard field theories are defined by

$$F(p_3^2) = \langle\phi_i|\mathcal{O}(p_3)|\phi_j\rangle = \sum_k\frac{\lambda_{ijk}f_k}{p_3^2-m_k^2} \quad (3.42)$$

# 4 Numerical Results

The results are obtained by computing the derived equations using numerical values. For the IR–cutoff, $L_1$, we used the value $L_1 = \frac{1}{0.2}\text{GeV}^{-1}$, a cutoff in the same region of energy as $\frac{1}{\Lambda_{QCD}}$. For the UV–cutoff, $L_0$, we used $L_0 = \frac{1}{10}\text{GeV}^{-1}$ as an upper bound.

| Root nr. | Mass $[GeV]$ | Residue | decay constant |
|---|---|---|---|
| 1 | 0.54734 | 0.09140 | 0.30233 |
| 2 | 1.14022 | 0.87315 | 0.93442 |
| 3 | 1.75792 | 3.14881 | 1.77449 |
| 4 | 2.38336 | 7.58235 | 2.75361 |
| 5 | 3.01242 | 14.68070 | 3.83154 |
| 6 | 3.64366 | 24.81338 | 4.98130 |
| 7 | 4.27638 | 38.23924 | 6.18379 |
| 8 | 4.91024 | 55.13265 | 7.42514 |
| 9 | 5.54497 | 75.60575 | 8.69516 |
| 10 | 6.18042 | 99.72618 | 9.98630 |

Table 1: Table showing results gathered from simulating two–point function using a bulk–to–boundary propagator.

Table 1 lists the first ten numerically found bound states together with their corresponding decay constant. Making variations in $L_0$ shows that the mass stays fairly constant while the residues, and thus the decay constants, changes greatly for the bound states close to $\frac{1}{L_0}$.

| State i | State j | State k | $\lambda_{ijk}$ |
|---|---|---|---|
| 1 | 1 | 1 | -0.2933 |
| 2 | 1 | 1 | 0.0100 |
| 2 | 2 | 1 | -0.2535 |
| 2 | 2 | 2 | -0.1650 |
| 3 | 1 | 1 | 0.0304 |
| 3 | 2 | 1 | 0.0348 |
| 3 | 2 | 2 | -0.1945 |
| 3 | 3 | 1 | -0.2445 |
| 3 | 3 | 2 | -0.1534 |
| 3 | 3 | 3 | -0.1578 |

Table 2: Table showing results gathered from simulating three–point function using bulk–to–boundary propagators.

Table 2 lists the coupling constants of the first few bound states. Since $\lambda_{ijk}$ is symmetric, all permutations of the three first bound states are found in the table.

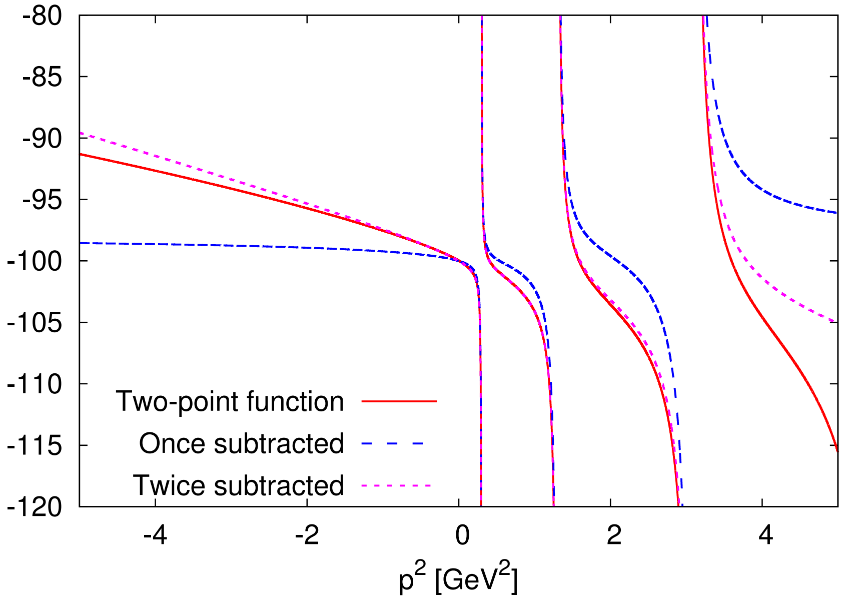


Figure 3: Plot showing the two–point function as well as the first dispersion relations calculated using first three poles.

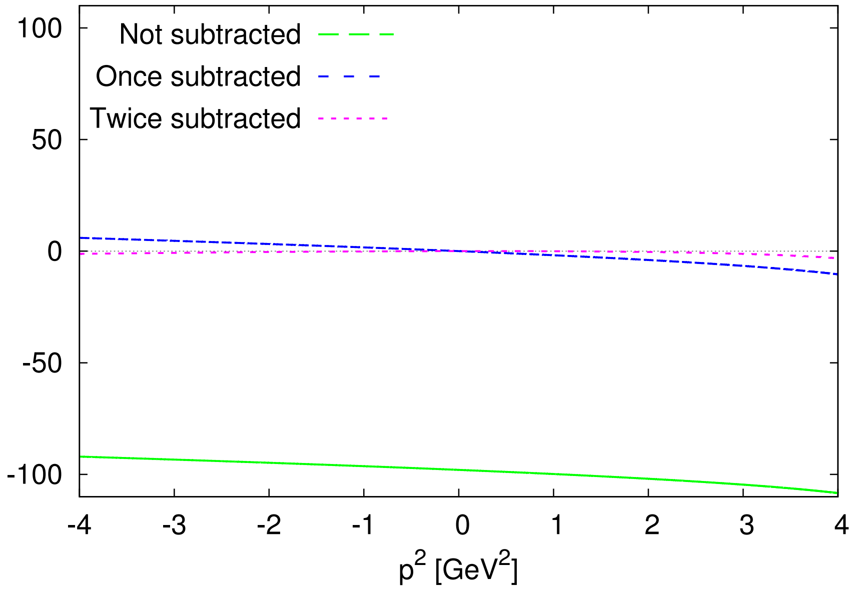


Figure 4: Plot showing the difference between the two–point function and the first dispersion relations calculated using the first three poles.

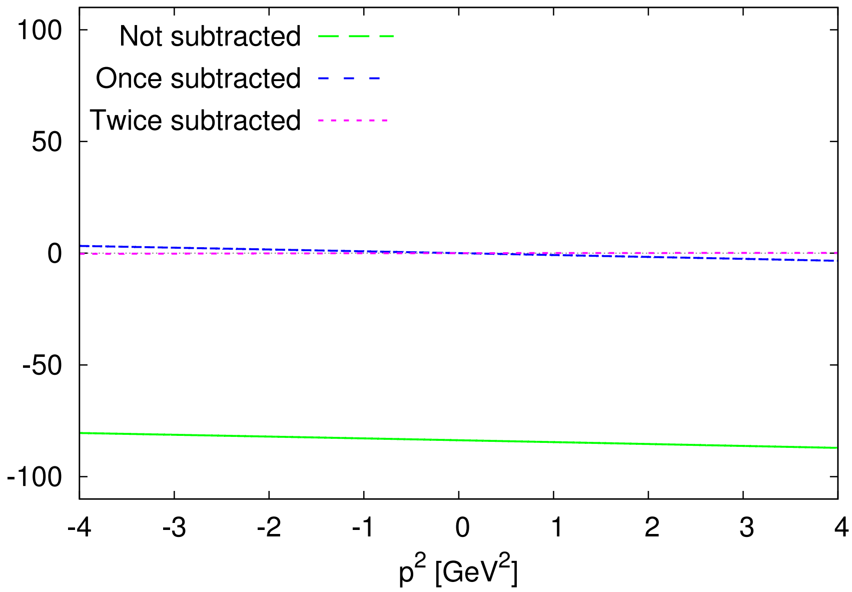


Figure 5: Plot showing the difference between the two–point function and the first dispersion relations calculated using the first ten poles.

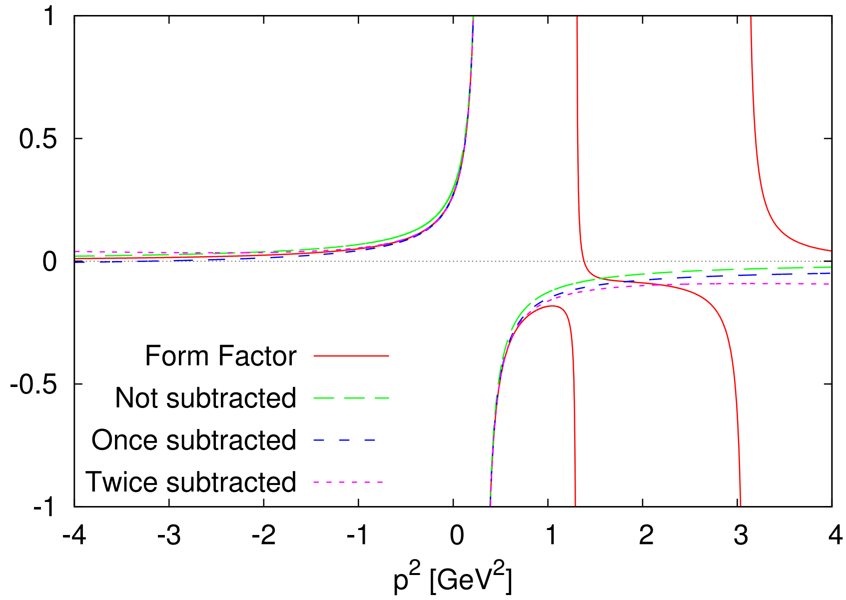


Figure 6: Plot showing the form factor $F(p_3^2)$ using equation (3.42) with $i = 0$ and $j = 0$.

In figure 3, we see that the twice subtracted dispersion relation is, as predicted, a better approximation to the two–point function compared to the once subtracted dispersion relation. We see that if only the three first bound states are considered the once subtracted dispersion relation is not a good approximation to use (too big error). The not subtracted dispersion relation is not plotted because it is simply not visible with the given $y$–axis, the not subtracted function's value lies around $y = 0$. As expected, both approximations fail to copy the two–point function shortly after the last pole is found. This is of course because the numerically calculated sum, equation (3.29), which is finite, has run out of terms.

In figure 4 the difference between a given approximation and the two–point function are plotted. Here we can see how the not subtracted dispersion relation fails to even come close to the two–point function. Yet again it is obvious that the twice subtracted dispersion relation is much better than the once subtracted. The failure of the not subtracted dispersion relation comes from the fact that the contour integral, see equation (A.22), is nonzero – in fact it contributes with an infinite constant since the integral diverges for $\Pi(k^2)$.

Figure 5 describes the same scenario as the previous figure. However, the number of bound states taken into consideration has grown, from 3, to 10. We note that while the green line (corresponding to the not subtracted relation) lies closer to the other lines it is still very far away. All lines in this plot are more flat compared to the corresponding lines in figure 4, which is to be expected – using more bound states in the approximation gives a better result.

Figure 6 shows the form factor as a function of momentum. We see that the not subtracted dispersion relation follows the curve very well this time, albeit not as good as neither the once subtracted nor the twice subtracted dispersion relation.

# 5 Conclusions

We have successfully derived the four–point function for a simple interacting scalar field in five–dimensional AdS space. We have also introduced Witten diagrams, a tool used to graphically represent $n$–point functions.

From our results deriving the two–point function we have found an explicit expression for a bulk–to–boundary propagator describing a scalar field in AdS space. Using this expression we have numerically found bound states as well as their corresponding decay factors.

From here we derived the not subtracted, once subtracted, and twice subtracted dispersion relations and used these to see how fast the sums to converge. These can be seen as a test of so called meson dominance of the two–point function. We see that the not subtracted dispersion relation does not fit the two–point function at all and the once subtracted dispersion relation, while it is better, does not give satisfying results when only a few bound states are considered.

We also took a look at the three-point function and derived the coupling constants for the first few bound states. Finally we used some of the coupling constants to find a form factor, $F(p_3^2)$. Here the not subtracted dispersion relation works very well, i.e., it follows the curve given by the form factor very well.

## 5.1 Outlook

This thesis concentrates to only look at a scalar field, $\phi(x^\mu, z)$, living in five-dimensional AdS space. As an obvious extension to this thesis one can derive the explicit expression for the bulk–to–bulk propagator, $G(x, x', z, z')$ and use this to numerically calculate the four–point function in a similar way the numerical results of the three–point function are obtained.

Another possible extension is that one can introduce vector fields, as done in [2, 3]. Using vector theory one may obtain theoretical results such as different meson masses, e.g., pion, kaon, etc. as well as the pion form factor [3].

# A Appendices

## A.1 Four–point Function; term containing $\lambda$

Start from second term in (2.55); I.e.,

$$\begin{aligned}\mathcal{S} &= -\frac{\lambda}{24}\int d^4x\int dz\sqrt{g}\int d^4x'\int d^4x''\int d^4x'''\int d^4x^{(4)}K(z,x,x')\times \\ &\times\ K(z,x,x'')K(z,x,x''')K(z,x,x^{(4)})\phi_0(x')\phi_0(x'')\phi_0(x''')\phi_0(x^{(4)}).\end{aligned} \tag{A.1}$$

Introduce the very short notation, $K(z,x,x^{(n)}) = K^n$, $\phi_0(x^{(n)}) = \phi^n$, as well as $\delta\phi_0(x^{(n)}) = \delta\phi_n$ and omitting the integrals. Also define a new notation for derivation, i.e.;

$$\frac{\delta(K^m\phi^n)}{\delta\phi_n} = K_n \tag{A.2}$$

With all definitions made we can start by writing out the action:

$$\mathcal{S} = -\frac{\lambda}{24}K^1K^2K^3K^4\phi^1\phi^2\phi^3\phi^4. \tag{A.3}$$

For the first derivative, we got four choices of K. This leads to four different terms:

$$\begin{aligned}\frac{\delta Z}{\delta\phi_1} &= -\frac{\lambda}{24}\Big[K_1K^2K^3K^4\phi^2\phi^3\phi^4 + K^1K_1K^3K^4\phi^1\phi^3\phi^4 + \\ &+\ K^1K^2K_1K^4\phi^1\phi^2\phi^4 + K^1K^2K^3K_1\phi^1\phi^2\phi^3\Big].\end{aligned} \tag{A.4}$$

For the second derivative, there are three choices left for each term generated by the first derivative, increasing the amount of terms to twelve.

$$\begin{aligned}&\frac{\delta^2 Z}{\delta\phi_1\delta\phi_2} = -\frac{\lambda}{24}\Big[ \\ &+K_1K_2K^3K^4\phi^3\phi^4 + K_1K^2K_2K^4\phi^2\phi^4 + K_1K^2K^3K_2\phi^2\phi^3 + \\ &+K_2K_1K^3K^4\phi^3\phi^4 + K^1K_1K_2K^4\phi^1\phi^4 + K^1K_1K^3K_2\phi^1\phi^3 + \\ &+K_2K^2K_1K^4\phi^2\phi^4 + K^1K_2K_1K^4\phi^1\phi^4 + K^1K^2K_1K_2\phi^1\phi^2 + \\ &+K_2K^2K^3K_1\phi^2\phi^3 + K^1K_2K^3K_1\phi^1\phi^3 + K^1K^2K_3K_1\phi^1\phi^2\Big].\end{aligned} \tag{A.5}$$

The pattern continues for the third derivative, yielding

$$
\begin{aligned}
&\frac{\delta^3 Z}{\delta\phi_1\delta\phi_2\delta\phi_3} = -\frac{\lambda}{24}\Bigg[ \\
&+K_1K_2K_3K^4\phi^4 + K_1K_3K_2K^4\phi^4 + K_1K_3K^3K_2\phi^3 + \\
&+K_1K_2K^3K_3\phi^3 + K_1K^2K_2K_3\phi^2 + K_1K^2K_3K_2\phi^2 + \\
&+K_2K_1K_3K^4\phi^4 + K_3K_1K_2K^4\phi^4 + K_3K_1K^3K_2\phi^3 + \\
&+K_2K_1K^3K_3\phi^3 + K^1K_1K_2K_3\phi^1 + K^1K_1K_3K_2\phi^1 + \\
&+K_2K_3K_1K^4\phi^4 + K_3K_2K_1K^4\phi^4 + K_1K^2K_1K_2\phi^2 + \\
&+K_2K^2K_1K_3\phi^2 + K^1K_2K_1K_3\phi^1 + K^1K_2K_1K_2\phi^1 + \\
&+K_2K_3K^3K_1\phi^3 + K_3K_2K^3K_1\phi^3 + K_1K^2K_3K_1\phi^2 + \\
&+K_2K^2K_3K_1\phi^2 + K^1K_2K_3K_1\phi^1 + K^1K_2K_3K_1\phi^1\Bigg].
\end{aligned}
\tag{A.6}
$$

Finally, in order to obtain the four–point function, we take the fourth derivative with respect to the last field left and thus obtain

$$
\begin{aligned}
&\frac{\delta^4 Z}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4} = -\frac{\lambda}{24}\Bigg[ \\
&+K_1K_2K_3K_4 + K_1K_3K_2K_4 + K_1K_3K_4K_2 + K_1K_2K_4K_3 + \\
&+K_1K_4K_2K_3 + K_1K_4K_3K_2 + K_2K_1K_3K_4 + K_3K_1K_2K_4 + \\
&+K_3K_1K_4K_2 + K_2K_1K_4K_3 + K_4K_1K_2K_3 + K_4K_1K_3K_2 + \\
&+K_2K_3K_1K_4 + K_3K_2K_1K_4 + K_1K_4K_3K_2 + K_2K_4K_1K_3 + \\
&+K_4K_2K_1K_3 + K_4K_3K_1K_2 + K_2K_3K_4K_1 + K_3K_2K_4K_1 + \\
&+K_2K_4K_3K_1 + K_2K_4K_3K_1 + K_4K_2K_3K_1 + K_4K_2K_3K_1\Bigg].
\end{aligned}
\tag{A.7}
$$

We see that all possible permutations of $K_1K_2K_3K_4$ appear in the final expression. Since the order of which the factors appear does not matter, there are effectively $4! = 24$ identical terms added together. Using this property, the factor $\frac{1}{24}$ cancels, and what is left is

$$
\frac{\delta^4 Z}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4} = -\lambda K_1K_2K_3K_4. \tag{A.8}
$$

Going back to full-length notation and recognizing the four–point function, the final result becomes

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = -\lambda\int d^4x\int dz\sqrt{g}K(z,x,x_1)K(z,x,x_2K(z,x,x_3)K(z,x,x_4). \tag{A.9}$$

## A.2 Four–point Function; term containing $b^2$

Start from second term in (2.55); I.e.,

$$\begin{aligned}\mathcal{S} &= \frac{b^2}{8}\int d^4x\int dz\sqrt{g}\int d^4x'\int d^4x''\int d^5x'''\sqrt{g}\int d^4x^{(4)}\int d^4x^{(5)}K(z,x,x')\times\\ &\times\ K(z,x,x'')G(z,z''',x,x''')K(z''',x''',x^{(4)})K(z''',x''',x^{(5)})\\ &\times\ \phi_0(x')\phi_0(x'')\phi_0(x^{(4)})\phi_0(x^{(5)}).\end{aligned} \tag{A.10}$$

Reintroduce the very short notation, $K(z,x,x^{(n)}) = K^n$, $G(z,z^{(n)},x,x^{(n)}) = G^n$, $\phi_0(x^{(n)}) = \phi^n$ as well as $\delta\phi_0(x^{(n)}) = \delta\phi_n$ and omitting the integrals. Rewrite the expression as

$$\mathcal{S} = \frac{b^2}{8}K^1K^2G^3K^4K^5\phi^1\phi^2\phi^4\phi^5. \tag{A.11}$$

Start by taking the first derivative

$$\begin{aligned}\frac{\delta Z}{\delta\phi_1} = \frac{b^2}{8}\Big[ &\quad K_1K^2G^3K^4K^5\phi^2\phi^4\phi^5 + K^1K_1G^3K^4K^5\phi^1\phi^4\phi^5 +\\ + &\quad K^1K^2G^3K_1K^5\phi^1\phi^2\phi^5 + K^1K^2G^3K^4K_1\phi^1\phi^2\phi^4\Big],\end{aligned} \tag{A.12}$$

followed by the second derivative

$$\begin{aligned}\frac{\delta^2 Z}{\delta\phi_1\delta\phi_2} = \frac{b^2}{8}\Big[ &\quad K_1K_2G^3K^4K^5\phi^4\phi^5 + K_1K^2G^3K_4K^5\phi^2\phi^5 + K_1K^2G^3K^4K_5\phi^2\phi^4 +\\ + &\quad K_2K_1G^3K^4K^5\phi^4\phi^5 + K^1K_1G^3K_2K^5\phi^1\phi^5 + K^1K_1G^3K^4K_2\phi^1\phi^4 +\\ + &\quad K_4K^2G^3K_1K^5\phi^2\phi^5 + K^1K_4G^3K_1K^5\phi^1\phi^5 + K^1K^2G^3K_1K_4\phi^1\phi^2 +\\ + &\quad K_5K^2G^3K^4K_1\phi^2\phi^4 + K^1K_5G^3K^4K_1\phi^1\phi^4 + K^1K^2G^3K_5K_1\phi^1\phi^2\Big].\end{aligned} \tag{A.13}$$

After the third derivative we obtain

$$\frac{\delta^3 Z}{\delta\phi_1\delta\phi_2\delta\phi_3} = \frac{b^2}{8}\Big[ \quad K_1K_2G^3K_4K^5\phi^5 + K_1K_4G^3K_2K^5\phi^5 + K_1K_4G^3K^4K_2\phi^4 +$$
$$+ \quad K_1K_2G^3K^4K_4\phi^4 + K_1K^2G^3K_2K_4\phi^2 + K_1K^2G^3K_4K_2\phi^2 +$$
$$+ \quad K_2K_1G^3K_4K^5\phi^5 + K_4K_1G^3K_2K^5\phi^5 + K_4K_1G^3K^4K_2\phi^4 +$$
$$+ \quad K_2K_1G^3K^4K_4\phi^4 + K^1K_1G^3K_2K_4\phi^1 + K^1K_1G^3K_4K_2\phi^1 +$$
$$+ \quad K_2K_4G^3K_1K^5\phi^5 + K_4K_2G^3K_1K^5\phi^5 + K_4K^2G^3K_1K_2\phi^2 +$$
$$+ \quad K_2K^2G^3K_1K_4\phi^2 + K^1K_2G^3K_1K_4\phi^1 + K^1K_4G^3K_1K_2\phi^1 +$$
$$+ \quad K_2K_4G^3K^4K_1\phi^4 + K_4K_2G^3K^4K_1\phi^4 + K_4K^2G^3K_2K_1\phi^2 +$$
$$+ \quad K_2K^2G^3K_4K_1\phi^2 + K^1K_2G^3K_4K_1\phi^1 + K^1K_4G^3K_2K_1\phi^1\Big]. \tag{A.14}$$

Finally the fourth derivative gives us the expression

$$\frac{\delta^4 Z}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4} = \frac{b^2}{8}\Big[ \quad K_1K_2G^3K_4K_5 + K_1K_4G^3K_2K_5 + K_1K_4G^3K_5K_2 +$$
$$+ \quad K_1K_2G^3K_5K_4 + K_1K_5G^3K_2K_4 + K_1K_5G^3K_4K_2 +$$
$$+ \quad K_2K_1G^3K_4K_5 + K_4K_1G^3K_2K_5 + K_4K_1G^3K_5K_2 +$$
$$+ \quad K_2K_1G^3K_5K_4 + K_5K_1G^3K_2K_4 + K_1K_1G^3K_4K_2 +$$
$$+ \quad K_2K_4G^3K_1K_5 + K_4K_2G^3K_1K_5 + K_4K_5G^3K_1K_2 +$$
$$+ \quad K_2K_5G^3K_1K_4 + K_5K_2G^3K_1K_4 + K_5K_4G^3K_1K_2 +$$
$$+ \quad K_2K_4G^3K_5K_1 + K_4K_2G^3K_5K_1 + K_4K_5G^3K_2K_1 +$$
$$+ \quad K_2K_5G^3K_4K_1 + K_5K_2G^3K_4K_1 + K_5K_4G^3K_2K_1\Big]. \tag{A.15}$$

As expected, all different permutations of $K_1K_2G^3K_4K_5$, with $G^3$ fixed, appears in the final expression. Since the order in which the factors appear does matter when it comes to G and K, but not between K and K, there are some symmetries that can be used. Essentially there exists three main types of terms in the equation above and we can classify them, looking at if $K_1$ and $K_2$, $K_1$ and $K_4$ or $K_1$ and $K_5$ are on the same side of $G^3$. Looking at the terms in the equation above, we see that there are eight cases when $K_1$ and $K_2$ are on the same side of $G^3$ (Note that it doesn't matter if they are to the left or to the right, nor does it matter if $K_1$ is before or after $K_2$). By symmetries, there exist 8 cases of the other types as well. This in the end cancels the factor $\frac{1}{8}$ in all cases and we are left with

$$\frac{\delta^4 Z}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4} = b^2[K_1K_2G^3K_4K_5 + K_1K_4G^3K_2K_5 + K_1K_5G^3K_2K_4]. \tag{A.16}$$

Going back to full length notation, recognizing the four–point function and relabel $d^5x'''$ as $d^5x'$, the final result becomes

$$\begin{aligned}
\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_4)\rangle &= \\
&= b^2\int d^4x\int dz\sqrt{g}\int d^5x'\sqrt{g}\,\times \\
&\times\ K(z,x,x_1)K(z,x,x_2)G(z,z',x,x')K(z',x,x_4)K(z',x,x_5)+ \\
&+\ b^2\int d^4x\int dz\sqrt{g}\int d^5x'\sqrt{g}\,\times \\
&\times\ K(z,x,x_1)K(z,x,x_4)G(z,z',x,x')K(z',x,x_2)K(z',x,x_5)+ \\
&+\ b^2\int d^4x\int dz\sqrt{g}\int d^5x'\sqrt{g}\,\times \\
&\times\ K(z,x,x_1)K(z,x,x_5)G(z,z',x,x')K(z',x,x_2)K(z',x,x_4).
\end{aligned} \tag{A.17}$$

## A.3 Derivation of the Dispersion Relation for the Two–point Function

First note that $\pi(k^2)$ can be plotted on the plane of complex $k^2$ with poles along the real axis. This means that we can use Cauchy's integral theorem to calculate the value of $\pi(k^2)$ in every point, excluding the poles. I.e.,

$$\Pi(k^2) = \frac{1}{2\pi i}\oint_C ds\frac{\Pi(s)}{s-k^2}. \tag{A.18}$$

This formula is valid for any curve, evaluated counter–clockwise, enclosing the point $k^2$ and not containing any of the singularities of $\Pi(k^2)$. Thus we can extend the border of the curve to infinity in all directions, but we have to be careful along the real axis since we can not pass through any poles. Recall that we are allowed to deform the curve in any way we want, so we can choose a curve with a branch cut of thickness $\epsilon$ to form what is called a keyhole contour. By letting $\epsilon$ tend to zero we get our original curve, now moved to infinity, plus all poles we encountered, evaluated clockwise, i.e

$$\Pi(k^2) = \frac{1}{2\pi i}\oint_{C_\infty} ds\frac{\Pi(s)}{s-k^2} + \sum_n \frac{-1}{2\pi i}\oint_{C_n} ds\frac{\Pi(s)}{s-k^2}. \tag{A.19}$$

Where the minus sign comes from evaluating the contour integrals clockwise. The sum is over all poles not enclosed by the original curve. We would like to express the integral inside the sum in a different way if possible. Note that the integrand can be rewritten as

$$\frac{\Pi(s)}{s-k^2} = \frac{\text{Res}(\Pi(s)_n)}{(s-m_n^2)(m_n^2-k^2)} + \ldots. \tag{A.20}$$

Where $\text{Res}(\pi(s)_n)$ denotes the residue. Inserting this into equation (A.19) gives

$$\Pi(k^2) = \frac{1}{2\pi i} \oint\limits_{C_\infty} ds \frac{\Pi(s)}{s-k^2} - \sum_n \frac{1}{2\pi i(m_n^2-k^2)} \oint\limits_{C_n} ds \frac{\text{Res}(\Pi(s)_n)}{(s-m_n^2)}. \tag{A.21}$$

We can use Cauchy's theorem to identify the integral in the sum as $2\pi i \, \text{Res}(\pi(m_n^2))$ which leads to

$$\Pi(k^2) = \frac{1}{2\pi i} \oint\limits_{C_\infty} ds \frac{\Pi(s)}{s-k^2} - \sum_n \frac{\text{Res}(\Pi(m_n^2))}{(m_n^2-k^2)}. \tag{A.22}$$

Equation (A.22) is known as the unsubtracted dispersion relation. The first term, the integral, is a constant whose value could be anything. The second term is a sum which for a certain $k^2$ converges proportional to $\frac{1}{m_n^2}$. Since the sum is evaluated numerically we can not have infinite amount of terms in it, even though there exist an infinite amount of poles. Therefore we would like the sum to converge as fast as possible. Introduce a new function, $\tilde{\pi}(s)$, defined as

$$\tilde{\Pi}(s) = \frac{\Pi(s) - \Pi(0)}{s}. \tag{A.23}$$

Note that there is no new pole in the limit as s tends to zero which can be easily verified, e.g., using l'Hôpital's rule. The corresponding residue for the new function is given by:

$$\text{Res}(\tilde{\Pi}(s))\Big|_{s=m_n^2} = \frac{\text{Res}(\Pi(s))}{m_n^2}. \tag{A.24}$$

Replace $\Pi(s)$ with $\tilde{\Pi}(s)$ in (A.22) to obtain:

$$\frac{\Pi(k^2) - \Pi(0)}{k^2} = \frac{1}{2\pi i} \oint\limits_{C_\infty} ds \frac{\Pi(s) - \Pi(0)}{s(s-k^2)} - \sum_n \frac{\text{Res}(\Pi(m_n^2))}{m_n^2(m_n^2-k^2)}. \tag{A.25}$$

Which can be rewritten so that $\Pi(k^2)$ is alone on the left side, i.e.

$$\Pi(k^2) = \Pi(0) + k^2 \left[ \frac{1}{2\pi i} \oint\limits_{C_\infty} ds \frac{\Pi(s) - \Pi(0)}{s(s-k^2)} - \sum_n \frac{\text{Res}(\Pi(m_n^2))}{m_n^2(m_n^2-k^2)} \right]. \tag{A.26}$$

(A.26) is known as the once subtracted dispersion relation. Yet again we have an integral which evaluates to some constant, but now the sum converges faster than before, proportional to $\frac{1}{m^4}$. We can play the same game by defining a new function, $\tilde{\tilde{\pi}}(s)$ defined as

$$\tilde{\tilde{\Pi}}(s) = \frac{\tilde{\Pi}(s) - \Pi'(0)}{s} = \frac{\Pi(s) - \Pi(0) - s\Pi'(0)}{s^2}. \tag{A.27}$$

Using l'Hôpital's rule shows that no new pole has been added. Following the exact same procedure as before leads to

$$\frac{\Pi(k^2) - \Pi(0) - k^2\Pi'(0)}{k^4} = \frac{1}{2\pi i} \oint\limits_{C_\infty} ds \frac{\Pi(s) - \Pi(0) - s\Pi'(0)}{s^2(s-k^2)} - \sum_n \frac{\mathrm{Res}(\Pi(m_n^2))}{m_n^4(m_n^2 - k^2)}. \tag{A.28}$$

Not surprisingly this is known as the twice subtracted dispersion relation.

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