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# Matching typed and untyped realizability

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## Abstract

Realizability interpretations of logics are given by saying what it means for computational objects of some kind to *realize* logical formulae. The computational objects in question might be drawn from an untyped universe of computation, such as a partial combinatory algebra, or they might be typed objects such as terms of a PCF-style programming language. In some instances, one can show that a particular untyped realizability interpretation matches a particular typed one, in the sense that they give the same set of realizable formulae. In this case, we have a very good fit indeed between the typed language and the untyped realizability model—we refer to this condition as *(constructive) logical full abstraction*.

We give some examples of this situation for a variety of extensions of PCF. Of particular interest are some models that are logically fully abstract for typed languages including *non-functional* features. Our results establish connections between what is computable in various programming languages and what is true inside various realizability toposes. We consider some examples of logical formulae to illustrate these ideas, in particular their application to exact real-number computability.

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## 1 Introduction

It is well-known that realizability models provide a good supply of denotational models for a range of functional programming languages. In the most familiar situation, one starts with a partial combinatory algebra  $A$ , and constructs the category  $\mathbf{Mod}(A)$  of *modest sets* over  $A$  (or equivalently the category  $\mathbf{PER}(A)$  of partial equivalence relations on  $A$ ). Since many familiar PCAs consist of *effective* objects of some kind (e.g.  $K_1$ ,  $\mathcal{P}\omega_{\text{re}}$ ,  $K_{2\text{re}}$ , or  $\Lambda^0/T$  for any  $\lambda$ -theory  $T$ ), the corresponding categories have a notion of computability built into them: all the morphisms are computable in some sense.

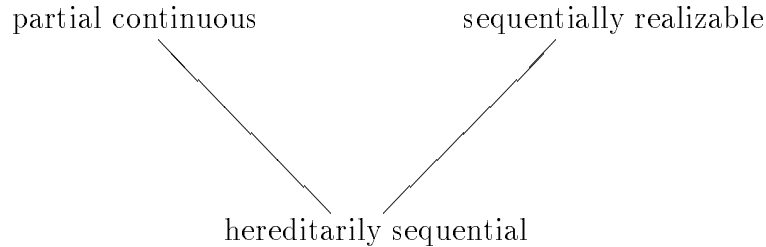
Interestingly, different PCAs embody different notions of computability. For example, we can often pick out an object of  $\mathbf{Mod}(A)$  playing the role of  $N_{\perp}$ , and then consider the finite types in  $\mathbf{Mod}(A)$  generated from  $N_{\perp}$  by exponentiation. Taking global elements of these objects (i.e. applying the functor  $\text{Hom}(1, -)$ ), we obtain a *finite type structure*, which we can think of

as the class of “computable” finite-type partial functionals relative to  $A$ . An interesting question is which PCAs give rise to which finite type structures.

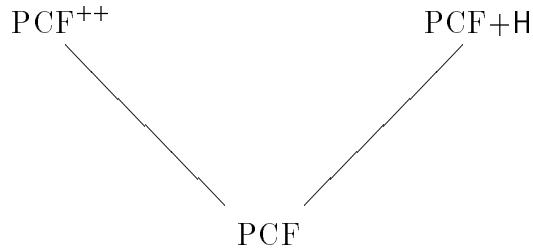
At present, it seems that there are essentially three different finite type structures that occur widely in nature, each of which comes in both a “full continuous” and an “effective” flavour. All six of these type structures have a number of different characterizations, and all have some claim to being mathematically natural objects of study. The three full type structures are:

- The *partial continuous* functionals: that is, the finite type structure arising from the familiar Scott domain model [31].
- The *hereditarily sequential* functionals of Nickau [20]: this coincides with the finite type structure arising from the fully abstract game models for PCF due to Abramsky, Hyland *et al* [1,7].
- The *strongly stable* functionals of Bucciarelli and Ehrhard [3]: these coincide with the *sequentially realizable* functionals of Longley [14].

Intuitively, the type structure of hereditarily sequential functionals is smaller than the other two (more precisely, it is a subquotient of each of the others):



Each of these type structures has a natural effective analogue. Rather remarkably, in each case one can find a programming language (with a decidable set of terms and an effective operational semantics) which defines precisely the functionals in the effective type structure:



Here  $PCF^{++}$  is the extension of PCF with **parallel-or** and **exists** operators as studied in [25]. For the functional  $H$ , see [14]. One can characterize the effective type structures as the closed term models for these programming languages.

For each of these six type structures there are known examples of PCAs giving rise to it:

- The partial continuous functionals arise from many “continuous PCAs” such as the Scott graph model  $\mathcal{P}\omega$  [30], the  $D_\infty$  models [29], Plotkin’s universal domain  $T^\omega$  [26], and Kleene’s second model  $K_2$  [10].
- The effective partial continuous functionals (corresponding to  $\text{PCF}^{++}$ ) arise from the effective analogues of each of the above PCAs, as well as from Kleene’s first model  $K_1$  [9].
- The hereditarily sequential functionals arise from various PCAs recently constructed by Abramsky (see [13]). They also from PCAs obtained by solving various recursive domain equations in known fully abstract models of PCF, such as categories of games or sequential domains (see [19]).
- The effective hereditarily sequential functionals (i.e. the PCF-definable functionals) arise from the effective analogues of any of these, and from the term models of certain impure  $\lambda$ -calculi (see [19]). Moreover, the *Longley-Phoa Conjecture* asserts that this type structure also arises from the pure term model  $\Lambda^0/T$  for any semi-sensible  $\lambda$ -theory  $T$  (see e.g. [12]).
- The sequentially realizable (SR) functionals arise from van Oosten’s combinatory algebra  $\mathcal{B}$  [23], and from the combinatory algebra  $\mathcal{A}$  constructed by Abramsky (see [13]). They also arise from the combinatory algebra  $\mathcal{B}_2$  described in [14].
- The effective SR functionals arise from the effective analogues of these.

All these PCAs yield realizability models that are fully abstract for the appropriate functional programming languages, and moreover, the effective ones even yield models that are *universal* (that is, every element of the model of appropriate type is denotable by a term of the language). Universality is already a strong criterion for goodness of fit between a language and a model. But since we have a choice of universal models for each of our three languages, it is natural to ask how they differ one from another, and in particular whether some are “better” than others in some sense. That is, can we find a stronger “goodness of fit” criterion than universality?

The purpose of the present paper is to introduce and study one such criterion, namely (*constructive*) *logical full abstraction*. This criterion asserts that the logic of realizability embodied by the PCA agrees with a notion of realizability derived from the programming language itself. We will see that this criterion does indeed introduce useful distinctions between PCAs that realize the same type structure, and will give examples of logically fully abstract models for each of our languages. Moreover, we will show that some of the above PCAs actually provide models that are logically fully abstract for *non-functional* extensions of PCF (in a sense we shall define). Finally, we will look at some examples of logical formulae that show up the differences between the various realizability interpretations, to illustrate how logical formulae can be used to express information about what is and is not computable in various kinds of programming language.

The notion of logical full abstraction (LFA) was first sketched in Chapter 8 of the author’s Ph.D. thesis [12], in both a *classical* and a (stronger) *constructive* version. The classical notion of LFA was further studied in [17]; the purpose of the present paper is to study the constructive notion in more detail.

## 2 Preliminary definitions

### 2.1 Realizability models

We first summarize some definitions concerning realizability models and fix some notation. The reader may consult [12] for more details and further background information. Note, however, that some of the definitions below are slightly refined versions of the ones given in [12].

**Definition 2.1 (PCA)** *A partial combinatory algebra (PCA) consists of a set  $A$  together with a partial binary operation  $\cdot : A \times A \rightarrow A$  (called application, and treated as left-associative) such that there exist elements  $k, s \in A$  satisfying*

$$k \cdot x \cdot y = x, \quad s \cdot x \cdot y \downarrow, \quad s \cdot x \cdot y \cdot z \succeq x \cdot z \cdot (y \cdot z)$$

for all  $x, y, z \in A$ .

Here the symbol  $\downarrow$  means “is defined”, and  $\succeq$  means “if the RHS is defined, so is the LHS and they are equal”. The above definition is thus slightly more general than the more usual definition of PCA in which we require  $\simeq$  in place of  $\succeq$ , but all the relevant theory works as usual. Moreover, the new definition seems to us to accord better with the spirit of the subject: we never care if a realizer for something does more than it is meant to! (To see that the new definition really is more general, consider the set of *solvable*  $\lambda$ -terms modulo  $\beta$ -equality, with the partial application operation introduced by ordinary application. However, we will not exploit this extra generality in this paper.)

We often abbreviate  $a \cdot b$  by  $ab$ , and write  $i$  for  $skk$  (note that  $ix = x$  for all  $x \in A$ ). In any PCA, one can define a *pairing* operation by  $\langle x, y \rangle = s(si(kx))(ky)$ . The corresponding *projections* are defined by  $fst = k$  and  $snd = ki$ ; note that  $fst\langle x, y \rangle = x$  and  $snd\langle x, y \rangle = y$ .

**Definition 2.2 (Modest sets)** *Let  $A$  be a PCA.*

(i) *A modest set  $X$  over  $A$  consists of an underlying set  $|X|$ , and for each  $x \in |X|$  an inhabited set  $\|x\| \subseteq A$  of realizers for  $x$ , such that if  $a \in \|x\|$  and  $a \in \|x'\|$  then  $x = x'$ . We sometimes write  $x \in X$  in place of  $x \in |X|$ .*

(ii) *A morphism  $f : X \rightarrow Y$  between modest sets is a function  $f : |X| \rightarrow |Y|$  for which there exists  $r \in A$  such that for all  $x \in |X|$  and  $a \in \|x\|$  we have  $r \cdot a \in \|f(x)\|$ . In this situation we say that  $r$  tracks  $f$ . We write  $\mathbf{Mod}(A)$  for the category of modest sets over  $A$ .*

The category  $\mathbf{Mod}(A)$  is cartesian-closed. Given modest sets  $X, Y$ , the exponential  $Y^X$  is constructed as follows:  $|Y^X|$  is the set of morphisms  $f :$

$X \rightarrow Y$ ; and  $\|f\|$  is the set of elements  $r \in A$  that track  $f$ .

$\mathbf{Mod}(A)$  also has a natural number object  $N$ . For any non-trivial PCA  $A$ , this may be constructed as follows: let  $|N|$  be the set  $\mathbb{N}$  of natural numbers, and let  $\|n\|$  be the singleton set  $\{\bar{n}\}$ , where  $\bar{n}$  is the *Curry numeral* for  $n$ :

$$\bar{0} = \langle ki, i \rangle, \quad \overline{n+1} = \langle k, \bar{n} \rangle$$

It is easy to see that  $\mathbf{Mod}(A)$  is equivalent to the well-known category  $\mathbf{PER}(A)$  of partial equivalence relations on  $A$ . In fact,  $\mathbf{Mod}(A)$  embeds as a full sub-CCC in the (standard) *realizability topos*  $\mathbf{RT}(A)$ , though the latter is more complicated to construct and we shall not need it here.

In order to interpret languages such as PCF in  $\mathbf{Mod}(A)$ , we want an object to play the role of  $N_\perp$ . We can obtain such an object if we have some extra structure on our PCA to capture the idea of non-termination. In [12,18] this extra structure took the form of a *divergence*; here we propose a slightly different notion.

**Definition 2.3** *Let  $A$  be a PCA. A non-termination set in  $A$  is a non-empty set  $E \subseteq A$  such that, for all  $a, b \in A$ , if  $a \in E$  then  $sab \in E$ . Any non-termination set  $E$  gives rise to a lift operation  $-_\perp$  on objects of  $\mathbf{Mod}(A)$  as follows: let  $|X_\perp| = |X| \sqcup \{\perp\}$ ; and take*

$$\begin{aligned} \|x\|_{X_\perp} &= \{\langle a, b \rangle \mid ai = i, bi \in \|x\|_X\} \quad (x \in |X|), \\ \|\perp\|_{X_\perp} &= \{\langle a, b \rangle \mid a \in E, b \in A\}. \end{aligned}$$

The lift operation  $-_\perp$  in fact extends to a monad on  $\mathbf{Mod}(A)$ , but here all we will need is the object  $N_\perp$ . For PCAs in which we have  $sxyz \simeq (xz)(yz)$ , the notion of non-termination set is related to that of divergence as follows:

- if  $E$  is a non-termination set, then  $\{ai \mid a \in E, ai \downarrow\}$  is a divergence giving rise to the same lift operation;
- if  $D$  is a divergence, then  $\{a \mid ai \downarrow \Rightarrow ai \in D\}$  is a non-termination set giving the same lift operation.

However, the definition of non-termination set is somewhat cleaner (if less intuitive!) than that of divergence. Moreover, non-termination sets work better with our more general definition of PCA, since for the lift functor arising from a divergence, the monad multiplication map may fail to be realizable. For the purposes of this paper, though, it does not matter much whether we work with non-terminating sets or divergences.

Let us say that a *choice of natural number domain* (or *choice of  $N_\perp$* ) in a cartesian-closed category  $\mathcal{C}$  is simply an object  $N_\perp$  of  $\mathcal{C}$  with a canonical identification  $|N_\perp| \cong \mathbb{N} \sqcup \{\perp\}$ . The natural number object in  $\mathbf{Mod}(A)$  together with a non-termination set gives rise to a choice of natural number domain, though we may on occasion be interested in choices of  $N_\perp$  not of this form. Technically, the choice of natural number domain is part of the data for a realizability model; however, in many cases of interest there is only one natural candidate for  $N_\perp$  that stands out, and so we shall not always bother

to mention it explicitly.

We can now interpret the *finite types* in any realizability model. The finite types are freely constructed from a single ground type  $\iota$  via the (right-associative) binary type constructors  $\times$  and  $\rightarrow$ .

**Definition 2.4 (Finite type structure)** *An (extensional) finite type structure (FTS)  $T$  consists of a set  $T^\sigma$  for each finite type  $\sigma$  such that  $T^\iota = \mathbb{N} \sqcup \{\perp\}$  and  $T^{\sigma \times \tau} = T^\sigma \times T^\tau$ , together with “application” functions  $\cdot_{\sigma\tau} : T^{\sigma \rightarrow \tau} \times T^\sigma \rightarrow T^\tau$  such that, for any  $f, g \in T^{\sigma \rightarrow \tau}$ , if  $f \cdot x = g \cdot x$  for all  $x \in T^\sigma$  then  $f = g$ .*

In any cartesian-closed category  $\mathcal{C}$  equipped with a choice of  $N_\perp$ , we have an interpretation  $\llbracket - \rrbracket$  of the finite types defined by

$$\llbracket \iota \rrbracket = N_\perp, \quad \llbracket \sigma \times \tau \rrbracket = \llbracket \sigma \rrbracket \times \llbracket \tau \rrbracket, \quad \llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \tau \rrbracket^{\llbracket \sigma \rrbracket}.$$

We hence obtain a finite type structure  $T(\mathcal{C}, N_\perp)$ , where  $T(\mathcal{C})^\sigma = \llbracket \llbracket \sigma \rrbracket \rrbracket$ , and the application operations are given by the evaluation morphisms in  $\mathcal{C}$ . In the case  $\mathcal{C} = \mathbf{Mod}(A)$ , we write this simply as  $T(A, N_\perp)$ , or  $T(A, E)$  if the choice of  $N_\perp$  arises from the non-termination set  $E$ . More loosely, we may write it as  $T(A)$  and refer to it as the *FTS over  $A$* .

## 2.2 Typed programming languages

Next we introduce some general notions concerning typed programming languages. By a *language*  $\mathcal{L}$  let us mean a family of sets  $\mathcal{L}_\sigma$  of *terms* of type  $\sigma$ , with the following closure properties:

- if  $M \in \mathcal{L}_{\sigma \times \tau}$  then  $\mathbf{fst}_{\sigma\tau} M \in \mathcal{L}_\sigma$  and  $\mathbf{snd}_{\sigma\tau} M \in \mathcal{L}_\tau$ ,
- if  $M \in \mathcal{L}_{\sigma \rightarrow \tau}$  and  $N \in \mathcal{L}_\sigma$  then  $MN \in \mathcal{L}_\tau$ .

We suppose that each term  $M$  has a set of *free variables*  $\mathbf{FV}(M)$ , such that

$$\begin{aligned} \mathbf{FV}(\mathbf{fst}_{\sigma\tau} M) &= \mathbf{FV}(\mathbf{snd}_{\sigma\tau} M) = \mathbf{FV}(M), \\ \mathbf{FV}(MN) &= \mathbf{FV}(M) \cup \mathbf{FV}(N). \end{aligned}$$

We write  $\mathcal{L}_\sigma^0$  for the set  $\{M \in \mathcal{L}_\sigma \mid \mathbf{FV}(M) = \emptyset\}$  of *closed* terms of type  $\sigma$ . If  $\Gamma$  is a finite non-repetitive list of variables in which all the free variables of  $M$  appear, we may say  $M$  is a *term in context*  $\Gamma$ . We also assume we have a notion of *substitution* for terms of  $\mathcal{L}$ , interacting with free variables in the expected way. Finally we suppose we are given an *evaluation* function  $\mathbf{Eval}_\mathcal{L} : \mathcal{L}_\iota^0 \rightarrow N_\perp$ .

A *translation*  $\theta$  from  $\mathcal{L}$  to  $\mathcal{L}'$  consists of a family of functions  $\theta_\sigma : \mathcal{L}_\sigma \rightarrow \mathcal{L}'_\sigma$  that preserve projections, application, and free variables, and such that for  $M \in \mathcal{L}_\iota^0$  we have  $\mathbf{Eval}_{\mathcal{L}'}(\theta_\iota M) = \mathbf{Eval}_\mathcal{L}(M)$ . If such a translation exists, we may think of  $\mathcal{L}$  as a sublanguage of  $\mathcal{L}'$ .

For any language  $\mathcal{L}$ , we can obtain a partial equivalence relation  $\approx_\sigma$  on each  $\mathcal{L}_\sigma^0$  as follows:

- $M \approx_\iota N$  iff  $\mathbf{Eval}_\mathcal{L}(M) = \mathbf{Eval}_\mathcal{L}(N)$ ,
- $M \approx_{\sigma \times \tau} N$  iff  $\mathbf{fst}_{\sigma\tau} M \approx_\sigma \mathbf{fst}_{\sigma\tau} N$  and  $\mathbf{snd}_{\sigma\tau} M \approx_\tau \mathbf{snd}_{\sigma\tau} N$ ,

- $M \approx_{\sigma \rightarrow \tau} N$  iff  $MP \approx_{\tau} NQ$  whenever  $P \approx_{\sigma} Q$ .

We may extend  $\approx_{\sigma}$  to open terms as follows: if  $M, N$  are terms in context  $x_1^{\sigma_1}, \dots, x_r^{\sigma_r}$ , then  $M \approx_{\sigma} N$  iff for all closed terms  $P_1, \dots, P_r, Q_1, \dots, Q_r$  such that  $P_i \approx_{\sigma_i} Q_i$  for each  $i$ , we have  $M[\vec{P}/\vec{x}] \approx_{\sigma} N[\vec{Q}/\vec{x}]$ . We say a term  $M : \sigma$  is *functional* if  $M \approx_{\sigma} M$ ; we say a language is *functional* if all its terms are functional. For any language  $\mathcal{L}$ , the sublanguage consisting of functional terms is a functional language, which we may call the *functional core* (or *Gandy hull*) of  $\mathcal{L}$ .

Given a *functional* language  $\mathcal{L}$  and a cartesian-closed category  $\mathcal{C}$  with choice of  $N_{\perp}$ , an *interpretation* of  $\mathcal{L}$  in  $(\mathcal{C}, N_{\perp})$  assigns to every term  $M \in \mathcal{L}_{\tau}$  in every context  $\Gamma = x_1^{\sigma_1}, \dots, x_r^{\sigma_r}$  a morphism  $\llbracket M \rrbracket_{\Gamma} : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_r \rrbracket \rightarrow \llbracket \tau \rrbracket$  in such a way that composition reflects substitution. Such an interpretation is *adequate* if for all  $M \in \mathcal{L}_i^0$  we have  $\llbracket M \rrbracket = \text{Eval}(M)$ ; it is *universal* if for any morphism  $f : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_r \rrbracket \rightarrow \llbracket \tau \rrbracket$  there is a term  $M \in \mathcal{L}_{\tau}$  in context  $\Gamma = x_1^{\sigma_1}, \dots, x_r^{\sigma_r}$  such that  $\llbracket M \rrbracket_{\Gamma} = f$ . If there is an adequate interpretation of  $\mathcal{L}$  in  $(\mathcal{C}, N_{\perp})$  we say that  $(\mathcal{C}, N_{\perp})$  is a *model* of  $\mathcal{L}$ .

In the case of a realizability model  $\mathbf{Mod}(A)$ , we will without comment identify morphisms  $1 \rightarrow \llbracket \sigma \rrbracket$  with elements of  $\llbracket \sigma \rrbracket$ . Furthermore, if  $\nu$  is a valuation assigning to each variable  $x_i^{\sigma_i} \in \Gamma$  an element  $\nu(x_i) \in \llbracket \sigma_i \rrbracket$ , and  $M : \sigma$  is a term in context  $\Gamma$ , we will write  $\llbracket M \rrbracket^{\nu}$  for the element  $\llbracket M \rrbracket_{\Gamma}(\nu(x_1), \dots, \nu(x_r))$  of  $\llbracket \sigma \rrbracket$ .

### 3 Untyped and typed realizability

Let  $\mathcal{L}$  be any functional language such that  $0, 1 \in \mathcal{L}_i^0$ . We will consider the class  $\mathbf{J}(\mathcal{L})$  of logical formulae given by the following grammar:

$$\phi ::= M =_{\sigma} N \mid P \downarrow \mid \phi_1 \wedge \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \exists x^{\sigma}. \phi_1 \mid \forall x^{\sigma}. \phi_1$$

where  $M, N : \sigma$  and  $P : \iota$  range over terms of  $\mathcal{L}$ , and  $x^{\sigma}$  ranges over variables of  $\mathcal{L}$ . Intuitively we have an equality predicate at each type  $\sigma$ , and a termination predicate at ground type; we will usually omit the subscript in equality formulae. We will write *true*, *false* for the formulae  $0 = 0$ ,  $0 = 1$  respectively, and  $\neg \phi$  as sugar for  $\phi \Rightarrow \text{false}$ . Note that we have omitted disjunction from the logic (see below); however, we may express disjunctions by translating  $\phi_1 \vee \phi_2$  to

$$\exists n^{\iota}. n \downarrow \wedge (n = 0 \Rightarrow \phi_1) \wedge ((\neg n = 0) \Rightarrow \phi_2).$$

#### 3.1 Untyped realizability

We recall the standard notion of untyped realizability for formulae of  $\mathbf{J}(\mathcal{L})$ . Suppose  $A$  is a PCA and  $E$  a non-termination set such that  $\mathcal{L}$  has an adequate interpretation  $\llbracket - \rrbracket$  in  $\mathbf{Mod}(A)$  as above. Then we may define a relation  $a \mathbf{r}^{\nu} \phi$  (read “ $a$  realizes  $\phi$  under  $\nu$ ”) between elements  $a \in A$ , valuations  $\nu$  and



formulae  $\phi \in \mathbf{J}(\mathcal{L})$  whose free variables are in  $\nu$  as follows:

- If  $\llbracket M \rrbracket^\nu = \llbracket N \rrbracket^\nu$ , then  $a \mathbf{r}^\nu M = N$  for any  $a \in A$ .
- If  $\llbracket P \rrbracket^\nu \in \mathbf{N}$ , then  $a \mathbf{r}^\nu P \downarrow$  for any  $a \in A$ .
- If  $fsta \mathbf{r}^\nu \phi$  and  $snda \mathbf{r}^\nu \psi$ , then  $a \mathbf{r}^\nu \phi \wedge \psi$ .
- If  $ab \mathbf{r}^\nu \psi$  whenever  $b \mathbf{r}^\nu \psi$ , then  $a \mathbf{r}^\nu \phi \Rightarrow \psi$ .
- If, for some  $e \in \llbracket \sigma \rrbracket$ ,  $fsta \in \llbracket e \rrbracket$  and  $snda \mathbf{r}^{\nu(x^\sigma \mapsto e)} \phi$ , then  $a \mathbf{r}^\nu \exists x^\sigma . \phi$ .
- If, for all  $e \in \llbracket \sigma \rrbracket$ , we have  $ab \mathbf{r}^{\nu(x^\sigma \mapsto e)} \phi$  whenever  $b \in \llbracket e \rrbracket$ , then  $a \mathbf{r}^\nu \forall x^\sigma . \phi$ .
- That's all.

We write just  $a \mathbf{r} \phi$  if  $a$  realizes  $\phi$  under the empty valuation. If there exists  $a \in A$  such that  $a \mathbf{r} \phi$ , we write  $(A, E) \models \phi$  (or just  $A \models \phi$ ), and say that  $\phi$  is *realizable in  $A$* . This notion of realizability is exactly the one arising from the *internal logic* of  $\mathbf{Mod}(A)$  (or of  $\mathbf{RT}(A)$ ); indeed, one can give an equivalent definition of the relation  $\models$  by exploiting the categorical structure of  $\mathbf{Mod}(A)$  (see [12, page 262]). However, the concrete definition in terms of realizers is perhaps easier to grasp, and is better suited to our present purposes.

It is interesting to note that, for the double-negation fragment of  $\mathbf{J}(\mathcal{L})$  (i.e. the image of the Gödel double-negation translation  $\phi \mapsto \phi^\circ$ ), the above interpretation agrees with a simple *classical* interpretation of logic in the finite type structure  $T(A)$ . That is, we have  $A \models \phi^\circ$  iff  $T(A) \models \phi$  (see [12, Chapter 8] for the easy definition of satisfaction in  $T(A)$ ). Semantically, this corresponds to the fact that passing from  $\mathbf{Mod}(A)$  or  $\mathbf{RT}(A)$  to the FTS corresponds to taking global elements; and the global elements functor  $\text{Hom}(1, -) : \mathbf{RT}(A) \rightarrow \mathbf{Set}$  is exactly the reflection from  $\mathbf{RT}(A)$  to its double-negation sheaf subtopos. What this means is that if two realizability models yield the same FTS, then the corresponding relations  $\models$  agree on the double-negation fragment of  $\mathbf{J}(\mathcal{L})$ . (In fact the converse also holds in the cases of interest: see [17].) However, they may well disagree on the rest of  $\mathbf{J}(\mathcal{L})$ : for example, the PCAs  $K_1$  and  $\mathcal{P}\omega_{\text{re}}$  give the same FTS but yield quite different realizability interpretations (see below). To summarize, the FTS only embodies information about the double-negation fragment of the internal logic.

It may be argued that this classical fragment of the logic is enough for many practical purposes (see for example [12, Chapter 9]). However, it is still natural to ask whether we can find a use for the finer distinctions between models given by their internal logic. This is the subject of the present paper.

Several variants of the above definitions are possible. In particular, one can define the Kreisel-style *modified realizability* relation  $a \mathbf{mr} \phi$ , giving rise to the satisfaction relation  $A \models_m \phi$ , though we will not give details here (see e.g. [24]).

### 3.2 Typed realizability

The above gives an interpretation for formulae of  $\mathbf{J}(\mathcal{L})$  relative to a particular model  $\mathbf{Mod}(A)$ , which we think of as a “semantic” model for  $\mathcal{L}$ . We now present an alternative, more “syntactic” notion of realizability, defined purely in terms of the typed programming language and without reference to any particular model. Our hope is that such an interpretation could be grasped relatively easily by a programmer without a background in denotational semantics.

The new definition of realizability is closely parallel to the one above, except that realizers are now terms of the typed programming language itself rather than elements of an untyped structure. Let  $\mathcal{L}$  be any language, and  $\mathcal{L}'$  its functional core. In order to obtain a pleasant logic in which the extensionality rule holds, *terms* will be drawn only from  $\mathcal{L}'$ , and variables are thought of as ranging only over  $\mathcal{L}'$ -terms. However, *realizers* for formulae are drawn from the whole of  $\mathcal{L}$  and may be non-functional programs.

Formally, we define a relation  $M \mathbf{R} \phi$  between closed terms  $M$  of  $\mathcal{L}$  and closed formulae  $\phi$  of  $\mathbf{J}(\mathcal{L}')$  inductively as follows:

- If  $N \approx_\sigma N'$ , then  $M \mathbf{R} (N =_\sigma N')$  for any  $M \in \mathcal{L}'_i$ .
- If  $P : \iota$  terminates, then  $M \mathbf{R} (P \downarrow)$  for any  $M \in \mathcal{L}'_i$ .
- If  $\text{fst}_{\sigma\tau} M \mathbf{R} \phi$  and  $\text{snd}_{\sigma\tau} M \mathbf{R} \psi$ , then  $M \mathbf{R} \phi \wedge \psi$ .
- If  $MN \mathbf{R} \psi$  whenever  $N \mathbf{R} \phi$ , then  $M \mathbf{R} \phi \Rightarrow \psi$ .
- If  $\text{fst}_{\sigma\tau} M : \sigma$  and  $\text{snd}_{\sigma\tau} M \mathbf{R} \phi[M/x^\sigma]$ , then  $M \mathbf{R} \exists x^\sigma. \phi$ .
- If  $MN \mathbf{R} \phi[N/x^\sigma]$  whenever  $N : \sigma$ , then  $M \mathbf{R} \forall x^\sigma. \phi$ .
- That’s all.

If there exists  $M$  such that  $M \mathbf{R} \phi$ , we write  $\mathcal{L} \models \phi$  and say that  $\phi$  is *realizable in  $\mathcal{L}$* . Note that any realizers for  $\phi$  must be of a type  $\tau(\phi)$  that can easily be read off from the structure of  $\phi$ ; we may think of  $\tau(\phi)$  as the type of “potential realizers” for  $\phi$ . (We can now see difficulty with disjunction: we would like the type of realizers for  $\phi \vee \psi$  to be a disjoint sum type, but such types are not honest computational datatypes since e.g. they do not have a bottom element. There may be a way round this, but we prefer to leave out disjunction altogether.)

It is easy to see that for the double-negation fragment of  $\mathbf{J}(\mathcal{L}')$  the typed realizability interpretation agrees with the *operational truth* interpretation  $\models_{\text{op}}$  defined in [12, Chapter 8]. That is, we have  $\mathcal{L} \models \phi^\circ$  iff  $\models_{\text{op}} \phi$ .

Note that if  $\mathcal{L}$  is itself functional, then  $\mathcal{L}' = \mathcal{L}$  and the relations  $\approx_\sigma$  coincide with *observational equivalence*; the definition of typed realizability thus admits a slightly simpler reading in this case. Examples of this special case will be considered in Section 4; other examples involving non-functional languages will be considered in Section 5.

Having given untyped and typed realizability interpretations for  $\mathbf{J}(\mathcal{L})$ , it

is natural to ask when they agree:

**Definition 3.1** *Let  $\mathcal{L}$  be a language with functional core  $\mathcal{L}'$  and  $A$  be a PCA such that  $\mathbf{Mod}(A)$  (with some choice of  $N_\perp$ ) is a model for  $\mathcal{L}'$ . We say this model is (constructively) logically fully abstract (LFA) for  $\mathcal{L}$  if, for all closed  $\phi \in \mathbf{J}(\mathcal{L}')$ , we have  $A \models \phi$  iff  $\mathcal{L} \models \phi$ .*

## 4 LFA models for functional languages

We now give some examples of LFA models for purely functional languages. The following easy result (partly folklore) describes a commonly occurring situation in which logical full abstraction holds.

**Proposition 4.1** *Suppose  $\mathcal{C}$  is a CCC giving a universal model for  $\mathcal{L}$  (for some choice of object  $N_\perp \in \mathcal{C}$ ), and suppose  $U$  is a universal object of  $\mathcal{C}$ . Let  $A$  be the combinatory algebra with underlying set  $\text{Hom}(1, U)$  obtained from some choice of retraction  $U^U \triangleleft U$ .*

(i) *If  $\mathcal{C}$  is well-pointed, then there is a full cartesian-closed embedding  $I : \mathcal{C} \rightarrow \mathbf{Mod}(A)$  into the projective objects of  $\mathbf{Mod}(A)$ .*

(ii) *More generally, if  $\mathcal{C}$  has a well-pointed cartesian-closed quotient  $\mathcal{C}/\approx$ , then there is a full cartesian-closed embedding  $I : \mathcal{C}/\approx \rightarrow \mathbf{Mod}(A)$ .*

*In either case, the induced interpretation of  $\mathcal{L}$  in  $\mathbf{Mod}(A)$  (with natural number domain  $I(N_\perp)$ ) is constructively LFA.*

In fact, in the above situation, the *modified* realizability interpretation of  $\mathbf{J}(\mathcal{L})$  over  $A$  is also LFA. In addition, it seems likely that a large supply of LFA models can be obtained using the notion of *extensional* realizability (see [22]), though we have not yet explored this in detail.

The above proposition represents a very pleasant situation and provides a cheap source of examples of LFA models; we will use it below to obtain LFA models of each of the three functional languages mentioned in the Introduction. It seems that there are other LFA models not of this form, but for these one has to work harder to prove logical full abstraction. (Of course, this might mean that the results obtained are more interesting!)

### 4.1 PCF and its extensions

First we recall the definition of call-by-name PCF. We include this here mainly to provide a basis for some of the less familiar extensions to PCF that we will define in the next section.

The types of PCF are the finite types defined above. For each type  $\sigma$  we have an infinite supply of variables of type  $\sigma$ , ranged over by  $x^\sigma, y^\sigma, z^\sigma$ . We

also have the following collection of constants:

$$\begin{aligned}
 0, 1, 2, \dots & : \iota, & \text{if} & : \iota \rightarrow \iota \rightarrow \iota \rightarrow \iota, \\
 \text{succ, pred} & : \iota \rightarrow \iota, & \mathbf{Y}_\sigma & : (\sigma \rightarrow \sigma) \rightarrow \sigma, \\
 \text{fst}_{\sigma\tau} & : (\sigma \times \tau) \rightarrow \sigma, & \text{snd}_{\sigma\tau} & : (\sigma \times \tau) \rightarrow \tau.
 \end{aligned}$$

The terms of PCF are built up from variables and constants as usual in the simply-typed  $\lambda$ -calculus:

- if  $M : \tau$ , then  $(\lambda x^\sigma.M) : \sigma \rightarrow \tau$ ;
- if  $M : \sigma$  and  $N : \tau$ , then  $\langle M, N \rangle : \sigma \times \tau$ ;
- if  $M : \sigma \rightarrow \tau$  and  $N : \sigma$ , then  $(MN) : \tau$ .

The *evaluation contexts*  $E[-]$  of PCF are defined inductively as follows: the identity context  $-$  is an evaluation context; and if  $E[-]$  is an evaluation context then so are  $\text{succ } E[-]$ ,  $\text{pred } E[-]$ ,  $\text{if } E[-]$ ,  $\text{fst}_{\sigma\tau} E[-]$ ,  $\text{snd}_{\sigma\tau} E[-]$  and  $E[-]N$  whenever these are well-typed. One then defines a one-step reduction relation  $\rightarrow$  on closed terms of the same type inductively as follows (here  $n$  ranges over the *numerals*  $0, 1, 2, \dots$ ):

- $(\lambda x^\sigma.M)N \rightarrow M[N/x^\sigma]$ ;
- $\text{succ } n \rightarrow (n + 1)$ ,  $\text{pred } (n + 1) \rightarrow n$ ,  $\text{pred } 0 \rightarrow 0$ ,  $\text{if } 0 \rightarrow (\lambda xy.x)$ ,  
 $\text{if } (n + 1) \rightarrow (\lambda xy.y)$ ,  $\mathbf{Y}_\sigma M \rightarrow M(\mathbf{Y}_\sigma M)$ ,  $\text{fst}_{\sigma\tau} \langle M, N \rangle \rightarrow M$ ,  
 $\text{snd}_{\sigma\tau} \langle M, N \rangle \rightarrow N$ ;
- if  $M \rightarrow M'$  and  $E[-]$  is an evaluation context such that  $E[M]$  is well-typed, then  $E[M] \rightarrow E[M']$ .

We write  $\rightarrow^*$  for the reflexive-transitive closure of  $\rightarrow$ . We say that a closed term  $M : \iota$  *terminates* if  $M \rightarrow^* n$  for some (necessarily unique) numeral  $n$ ; in this case, we set  $\text{Eval}(M) = n$ . If  $M$  does not terminate, then by convention we take  $\text{Eval}(M) = \perp$ .

The language  $\text{PCF}^{++}$  is defined in the same way as PCF except that we include two additional constants

$$\text{parallel-or} : \iota \rightarrow \iota \rightarrow \iota, \quad \text{exists} : (\iota \rightarrow \iota) \rightarrow \iota.$$

We will also consider the extension of PCF with a single constant

$$\mathbf{H} : ((\iota \rightarrow \iota) \rightarrow \iota) \rightarrow ((\iota \rightarrow \iota) \rightarrow \iota) \rightarrow \iota.$$

The above function  $\text{Eval}$  can be extended to yield an operationally defined evaluation relation for  $\text{PCF}^{++}$  [25], or for  $\text{PCF}+\mathbf{H}$  [14], though we will not give the details here.

It is shown in [18] that any realizability model is a model of PCF provided it satisfies a *completeness axiom*, which holds in most of the naturally occurring examples. Some natural realizability models are also models of  $\text{PCF}^{++}$  or  $\text{PCF}+\mathbf{H}$ .

## 4.2 Examples of LFA models

We now give some examples of LFA models for each of our three languages.

- For  $\text{PCF}^{++}$ : Recall from [12] that the PCA  $K_1$  (equipped with the non-termination set  $\{n \mid n \cdot 0 \uparrow\}$ ) gives rise to a universal model of  $\text{PCF}^{++}$ . Let  $\mathcal{C}$  be the full subcategory of  $\mathbf{Mod}(K_1)$  consisting of the retracts of the finite types. Then  $U = 2_{\perp}^N$  is a universal object in  $\mathcal{C}$  (by the “effective universality” of  $T^\omega$ —see [26]), and the corresponding combinatory algebra  $A$  is exactly  $T_{\text{re}}^\omega$ . Since we are in the situation of Proposition 4.1(i), the model  $\mathbf{Mod}(T_{\text{re}}^\omega)$  is LFA for  $\text{PCF}^{++}$  (as is the corresponding modified realizability model).

The PCA  $T_{\text{re}}^\omega$  is closely related to the Scott graph model  $\mathcal{P}\omega_{\text{re}}$ . Interestingly, the standard realizability model on  $\mathcal{P}\omega_{\text{re}}$  is not quite LFA for  $\text{PCF}^{++}$ : a counterexample (discussed in [12, page 263]) is the formula

$$\forall x'. \forall y'. \neg(x \downarrow \wedge y \downarrow) \Rightarrow \exists n'. (x \downarrow \Rightarrow n = 0) \wedge (y \downarrow \Rightarrow n = 1),$$

which is realizable in  $\mathcal{P}\omega_{\text{re}}$  but not in  $\text{PCF}^{++}$ . However, it appears that the *modified* realizability model over  $\mathcal{P}\omega_{\text{re}}$  is LFA, although this is not an instance of Proposition 4.1.

Note in passing that  $\mathbf{Mod}(K_1)$ , although a universal model of  $\text{PCF}^{++}$ , comes nowhere near being LFA for  $\text{PCF}^{++}$ . For instance, Church’s thesis is realizable in  $K_1$  but not in  $\text{PCF}^{++}$ :

$$\forall f^{t \rightarrow t}. \exists e'. \forall n'. “f(n) = e \cdot n”.$$

- For  $\text{PCF}+\text{H}$ : By analogy with the above, recall from [14] that the effective van Oosten algebra  $\mathcal{B}_{\text{re}}$  gives rise to a universal model for  $\text{PCF}+\text{H}$ . Let  $\mathcal{C}$  be the full subcategory of  $\mathbf{Mod}(\mathcal{B}_{\text{re}})$  consisting of retracts of finite types. It is shown in [14] that the object

$$U = N_{\perp}^{(N_{\perp}^N)}$$

is universal in  $\mathcal{C}$ , and it gives rise to the combinatory algebra  $\mathcal{B}_{2\text{re}}$ . Again we are in the situation of Proposition 4.1(i), and so the standard and modified realizability models over  $\mathcal{B}_{2\text{re}}$  are both LFA for  $\text{PCF}+\text{H}$ .

However, neither the standard nor the modified realizability model over  $\mathcal{B}_{\text{re}}$  is LFA for  $\text{PCF}+\text{H}$ .

- For  $\text{PCF}$ : The following construction is given by Marz, Rohr and Streicher in [19]. Let  $U$  be the canonical solution to some domain equation such as

$$U \cong \Sigma \oplus (U \circ \rightarrow U)_{\perp}$$

in a category  $\mathcal{S}$  of *sequential domains* (a fully abstract model of  $\text{PCF}$ ). It can be shown that all the  $\text{PCF}$  types, and also  $U^U$ , are syntactically definable retracts of  $U$  in the untyped  $\lambda$ -calculus  $\mathcal{L}$  corresponding to the above domain equation. Let  $L_U$  be the PCA of definable elements of  $U$  (this is a term model for  $\mathcal{L}$ ). By taking  $\mathcal{C}$  to be the category of definable retracts of  $U$  and definable morphisms between them, we see by Proposition 4.1(i) that the realizability model over  $L_U$  is LFA for  $\text{PCF}$ . (In particular it is universal—this establishes a variant of the Longley-Phoa conjecture.)

Similar results can be obtained by starting from a suitable intensional category  $\mathcal{G}$  of games and innocent strategies. However, unlike  $\mathcal{S}$ , the category  $\mathcal{G}$  is not well-pointed, so we are in the situation of Proposition 4.1(ii). The combinatory algebras thus obtained from  $\mathcal{S}$  and  $\mathcal{G}$  are very closely related: it seems likely that the former is a quotient of the latter.

It also seems plausible that the  $\lambda$ -term model  $\Lambda^0/T$  for any semi-sensible theory  $T$  yields an LFA model of PCF (this is a stronger claim than the Longley-Phoa conjecture). We have not yet considered whether Abramsky's recent constructions of combinatory algebras give LFA models for PCF.

### 4.3 A characterization of LFA models

In [12,17] a notion of *classical* logical full abstraction was introduced: the model  $\mathbf{Mod}(A)$  is classically LFA if for all (closed) formulae  $\phi$  we have

$$T(A) \models \phi \text{ iff } \models_{\mathbf{op}} \phi.$$

By our earlier remarks on double-negation formulae, this says precisely that for all closed formulae  $\phi$  we have

$$A \models \phi^\circ \text{ iff } \mathcal{L} \models \phi^\circ.$$

Hence constructive logical full abstraction implies classical logical full abstraction. We also know that classical logical full abstraction is equivalent to universality for models  $\mathbf{Mod}(A)$ . (This was proved in [17] for the languages PCF and PCF<sup>++</sup>, and with a trivial modification the same proof works for PCF+H.)

Since all three of our languages  $\mathcal{L}$  are functional, it is easy to see that all closed instances of the following schemata (the *axiom of choice* and the *independence of premiss* principle) are typed-realizable in each of them (for any finite types  $\sigma, \tau$ ):

$$\text{AC: } (\forall x^\sigma. \exists y^\tau. \phi[x, y]) \Rightarrow (\exists f^{\sigma \rightarrow \tau}. \forall x^\sigma. \phi[x, fx])$$

$$\text{IP: } \forall x^\sigma. ((\neg \phi[x]) \Rightarrow \exists y^\tau. \psi[x, y]) \Rightarrow \exists y^\tau. ((\neg \phi[x]) \Rightarrow \psi[x, y])$$

So in any PCA  $A$  which yields an LFA model of  $\mathcal{L}$ , these principles must be realizable. In fact, the above conditions together suffice for logical full abstraction:

**Theorem 4.2** *Let  $\mathcal{L}$  be one of our three purely functional languages. A realizability model  $(\mathbf{Mod}(A), N_\perp)$  is constructively LFA for  $\mathcal{L}$  iff it is a universal model for  $\mathcal{L}$  and all closed instances of AC and IP are realizable in  $A$ .*

**Proof.** The left-to-right implication is already clear from the above remarks. So suppose  $\mathbf{Mod}(A)$  is universal for  $\mathcal{L}$  and AC and IP are realizable in  $A$ . Call two formulae  $\phi, \phi'$  *equivalent* if the universal closure of  $\phi \Leftrightarrow \phi'$  is true under both the typed and untyped realizability interpretations. Any atomic formula  $\alpha$  is equivalent to  $\neg\neg\alpha$ ; hence if  $\phi$  is  $\exists$ -free then  $\phi$  is equivalent to  $\phi^\circ$ .

Starting with any closed formula  $\phi$ , we may transform it into an equivalent formula of the form  $\exists \vec{x}.\phi'$  where  $\phi'$  is  $\exists$ -free. This may be done by rewriting certain subformulae as follows:

$$\begin{aligned} \forall x^\sigma.\exists y^\tau.\psi[x, y] &\rightsquigarrow \exists f^{\sigma \rightarrow \tau}.\forall x^\sigma.\psi[x, fx] \\ (\exists x.\psi) \Rightarrow \theta &\rightsquigarrow \forall x.(\psi \Rightarrow \theta) && (x \notin \text{FV}(\theta)) \\ \psi \Rightarrow (\exists x.\theta) &\rightsquigarrow \exists x.(\psi \Rightarrow \theta) && (\psi \text{ } \exists\text{-free, } x \notin \text{FV}(\psi)) \\ (\exists x.\psi) \wedge \theta &\rightsquigarrow \exists x.(\psi \wedge \theta) && (x \notin \text{FV}(\theta)) \\ \psi \wedge (\exists x.\theta) &\rightsquigarrow \exists x.(\psi \wedge \theta) && (x \notin \text{FV}(\psi)) \end{aligned}$$

It is easy to see that by repeatedly performing these rewrites in any order (doing  $\alpha$ -conversions where necessary), we will eventually obtain a formula  $\exists \vec{x}.\phi'$  where  $\phi'$  is  $\exists$ -free. But both realizability relations are trivial for  $\phi'$ , and so by universality it is clear that  $A \models \exists \vec{x}.\phi'$  iff  $\mathcal{L} \models \exists \vec{x}.\phi'$ . Since  $\phi$  is equivalent to  $\exists \vec{x}.\phi'$ , we have  $A \models \phi$  iff  $\mathcal{L} \models \phi$ .  $\square$

The above theorem and its proof are strongly reminiscent of the characterization of (provable) modified realizability given in [32, Theorem 3.4.8]. Indeed, the same argument can be used to show that any universal *modified* realizability model for a functional language is logically fully abstract.

## 5 LFA models for non-functional languages

We now show how the notions of typed realizability and logical full abstraction can be extended to certain “impure” (i.e. non-functional) extensions of PCF. In doing so, we shall find a new use for some of the PCAs discarded above.

### 5.1 Conditions for logical full abstraction

We first give some general conditions which suffice for logical full abstraction. Intuitively, a model  $(\mathbf{Mod}(A), N_\perp)$  is LFA for a language  $\mathcal{L}$  if the typed language  $\mathcal{L}$  and the untyped structure  $A$  can be “simulated” sufficiently well in each other. The conditions we will give look rather cumbersome, but they are very useful for establishing particular instances of logical full abstraction.<sup>1</sup>

Firstly, define a *compilation* of  $\mathcal{L}$  to  $A$  (w.r.t.  $N_\perp$ ) to consist of

- a total relation  $\gamma$  from closed terms of  $\mathcal{L}$  to elements of  $A$ ,
- an element  $apply \in A$  such that

$$\gamma(M, a) \wedge \gamma(N, b) \implies \gamma(MN, apply \cdot a \cdot b),$$

- an element  $num \in A$  such that

$$M \in \mathcal{L}_i^0 \wedge \gamma(M, a) \implies num \cdot a \in \|\text{Eval}(M)\|_{N_\perp}.$$

<sup>1</sup> We now know how to formulate these conditions much more cleanly in terms of a certain 2-category in which both  $A$  and  $\mathcal{L}$  live: see Section 7 below and [15].

Secondly, define a *simulation* of  $A$  in  $\mathcal{L}$  to consist of

- a type  $\alpha$ ,
- a total relation  $\xi$  from  $A$  to  $\mathcal{L}_\alpha^0$ ,
- a term  $\mathbf{apply} : \alpha \times \alpha \rightarrow \alpha$  of  $\mathcal{L}$  such that

$$\xi(a, M) \wedge \xi(b, N) \wedge ab \downarrow \implies \xi(ab, \mathbf{apply} M N),$$

- a term  $\mathbf{num} : \alpha \rightarrow \iota$  such that for all  $x \in N_\perp$ ,

$$a \in \|x\| \wedge \xi(a, M) \implies \mathbf{Eval}(\mathbf{num} M) = x.$$

The following theorem now gives some sufficient conditions for logical full abstraction. It can be viewed as a generalization of Proposition 4.1.

**Theorem 5.1** *Suppose  $\mathcal{L}$  is a language,  $(\mathbf{Mod}(A), N_\perp)$  a realizability model, and the following conditions are satisfied:*

- (i) *There is a compilation  $(\gamma, \mathbf{apply}, \mathbf{num})$  of  $\mathcal{L}$  to  $A$  w.r.t.  $N_\perp$ .*
- (ii) *There is a simulation  $(\alpha, \xi, \mathbf{apply}, \mathbf{num})$  of  $A$  in  $\mathcal{L}$ .*
- (iii) *There is an element  $\mathbf{Code} \in A$  such that for any  $a \in A$  there is some  $M \in \mathcal{L}_\alpha^0$  such that  $\xi(a, M)$  and  $\gamma(M, \mathbf{Code} \cdot a)$ .*
- (iv) *For each type  $\sigma$  there is a term  $\mathbf{realizer}_\sigma : \sigma \rightarrow \alpha$  of  $\mathcal{L}$  such that for any  $M \in \mathcal{L}_\sigma^0$  there is some  $a \in A$  such that  $\gamma(M, a)$  and  $\xi(a, \mathbf{realizer}_\sigma M)$ .*

*Then  $(\mathbf{Mod}(A), N_\perp)$  is logically fully abstract for  $\mathcal{L}$ .*

**Proof (Sketch).** For each type  $\sigma$ , let  $\sim_\sigma$  be the PER on  $A$  corresponding to the modest set  $\llbracket \sigma \rrbracket$ , and let  $\approx_\sigma$  be the PER on  $\mathcal{L}_\sigma^0$  defined in Section 2.2. Write  $\sim_\sigma^\xi$  for the image of  $\sim_\sigma$  under  $\xi$ , defined by  $M \sim_\sigma^\xi N$  iff there exist  $a \sim_\sigma b$  such that  $\xi(a, M)$  and  $\xi(b, N)$ ; similarly write  $\approx_\sigma^\gamma$  for the image of  $\approx_\sigma$  under  $\gamma$ . One first verifies the following by simultaneous induction on  $\sigma$ :

- The relations  $\sim_\sigma$  and  $\approx_\sigma^\gamma$  are isomorphic PERs (that is, they correspond to isomorphic modest sets).
- The relations  $\approx_\sigma$  and  $\sim_\sigma^\xi$  are isomorphic PERs in an analogous “typed” sense.

For any closed formula  $\phi$ , let us write  $a \mathbf{R}^\gamma \phi$  if there exists  $M \mathbf{R} \phi$  such that  $\gamma(M, a)$ . Likewise, we write  $M \mathbf{r}^\xi \phi$  if there exists  $a \mathbf{r} \phi$  such that  $\xi(a, M)$ . One now proves the following for *all* formulae  $\phi$  by simultaneous induction on the structure of  $\phi$ :

- There are  $p, q \in A$  such that for all closed instances  $\phi'$  of  $\phi$  and all  $a, b \in A$ ,

$$a \mathbf{r} \phi' \implies pa \mathbf{R}^\gamma \phi', \quad b \mathbf{R}^\gamma \phi' \implies qb \mathbf{r} \phi'.$$

- There are  $P, Q \in \mathcal{L}^0$  such that for all closed instances  $\phi'$  of  $\phi$  and all  $M, N \in \mathcal{L}^0$ ,

$$M \mathbf{R} \phi' \implies PM \mathbf{r}^\xi \phi', \quad N \mathbf{r}^\xi \phi' \implies QN \mathbf{R} \phi'.$$

In the case of closed formulae  $\phi$ , it follows that  $A \models \phi$  iff  $\mathcal{L} \models \phi$ .  $\square$



This proof also shows that, in the above situation, the functional core  $\mathcal{L}'$  of  $\mathcal{L}$  has a universal interpretation in  $\mathbf{Mod}(A)$ . A fuller version of the above proof (in a cleaner setting) will appear in a future version of [15].

We now present three examples of non-functional languages and corresponding LFA models for them.

### 5.2 PCF+quote

Firstly, we extend PCF with a Lisp-style **quote** operator. We define the language PCF+**quote** in the same way as PCF except that we include a family of constants  $\mathbf{quote}_\sigma : \sigma \rightarrow \iota$ . Evaluation contexts for PCF+**quote** are defined exactly as for PCF. We then take  $\lceil - \rceil$  to be some effective Gödel-numbering of terms of PCF+**quote**, and include in the definition of one-step reduction all well-typed instances of

$$\mathbf{quote}_\sigma M \rightarrow \lceil M \rceil.$$

One might also consider adding Lisp-style **eval** operators with the property that  $\mathbf{eval}_\sigma \lceil M \rceil \rightarrow M$ , but in fact there is no need: such operators can be defined in PCF+**quote**. (The construction is not trivial, but it is a simple adaptation of the construction of the PCF *enumerators*  $E^\sigma$  in [17].)

The language PCF+**quote** is closely related to the model  $(\mathbf{Mod}(K_1), N_\perp)$  (with  $N_\perp$  given as usual by the non-termination set  $\{n \mid n \cdot 0 \uparrow\}$ ). Indeed, the four conditions of Theorem 5.1 are easily verified: the Gödel-numbering yields a compilation  $\gamma$ , and the operations  $\mathbf{quote}_\sigma$  give rise almost immediately to suitable terms  $\mathbf{realizer}_\sigma$ . Hence:

**Theorem 5.2** *The model  $(\mathbf{Mod}(K_1), N_\perp)$  is LFA for PCF+quote.*

Thus, realizability over PCF+**quote** yields exactly the logic of finite types over  $N_\perp$  in Hyland’s *effective topos* [6]. Note that the functional core of PCF+**quote** gives rise to the same type structure as PCF<sup>++</sup> (this follows from the universality of  $\mathbf{Mod}(K_1)$  for PCF<sup>++</sup>—see [12, Section 7.2]).

### 5.3 PCF+catch

Secondly, we consider a family of sequential programming languages which, in some sense, all embody the same computational power: PCF+**catch** [4], PCF+**call/cc**,  $\mu$ PCF [21], and a certain fragment of Standard ML admitting local uses of exceptions and references. It seems that these languages all admit good translations into each other, though we will not make this precise here (see [11] for a good indication of the state-of-the-art). For simplicity, we will choose the language PCF+**catch** (essentially the language SPCF of [4] without errors) as representative of this family of languages, but we believe that the result below would apply equally well to any of them.

The syntax of  $\text{PCF}+\text{catch}$  is defined as for  $\text{PCF}$  but with extra constants

$$\text{catch}_k : (\overbrace{\iota \rightarrow \cdots \rightarrow \iota}^k \rightarrow \iota) \rightarrow \iota$$

for  $k \geq 0$ . The evaluation contexts of  $\text{PCF}+\text{catch}$  are defined as for  $\text{PCF}$  with the following additional clause: if  $E[-]$  is an evaluation context then so is

$$\text{catch}_k(\lambda x_0 \dots x_{m-1}.E[-])$$

whenever  $0 \leq m \leq k$ . The one-step reduction relation is defined as for  $\text{PCF}$  with the following additional clauses:

- $\text{catch}_k(\lambda x_0 \dots x_{m-1}.E[x_i]) \rightarrow i$  whenever  $E[-]$  is an evaluation context and  $x_i$  is free in  $E[x_i]$ ;
- $\text{catch}_k(\lambda x_0 \dots x_{m-1}.n) \rightarrow m + n$ ;
- $\text{catch}_1(\text{succ}) \rightarrow 0$ ,  $\text{catch}_1(\text{pred}) \rightarrow 0$ ,  $\text{catch}_3(\text{if}) \rightarrow 0$ .

It follows from the universality of  $\text{PCF}+\text{catch}$  for effective sequential algorithms (see [8]) that the functional  $\mathbf{H}$  is definable in  $\text{PCF}+\text{catch}$  (see [14]). Thus we have a translation of  $\text{PCF}+\mathbf{H}$  into  $\text{PCF}+\text{catch}$ . (Indeed, the functional core of  $\text{PCF}+\text{catch}$  is equivalent to  $\text{PCF}+\mathbf{H}$ .) It is also easy to see that  $\text{PCF}+\text{catch}$  can be translated into  $\text{PCF}+\text{quote}$ .

A corresponding model is given by van Oosten's  $\mathcal{B}_{\text{re}}$ , with the evident choice of  $N_{\perp}$  arising from the non-termination set  $\{\lambda n.\perp\}$ :

**Theorem 5.3** *The model  $(\mathbf{Mod}(\mathcal{B}_{\text{re}}), N_{\perp})$  is LFA for  $\text{PCF}+\text{catch}$ .*

Once again, the proof uses Theorem 5.1. For condition (i), the necessary compilation is given essentially by the interpretation of  $\text{PCF}+\text{catch}$  in effective sequential algorithms (embedded in  $\mathcal{B}_{\text{re}}$  as retracts). Conditions (ii) and (iii) are easy, using the type  $\alpha = \iota \rightarrow \iota$ . Condition (iv) involves some cunning programming with  $\text{catch}$ ; the key lemma is the following:

**Lemma 5.4** *There is a closed term  $R : (\alpha \rightarrow \alpha) \rightarrow \alpha$  in  $\text{PCF}+\text{catch}$  such that, for any functional closed term  $M : \alpha \rightarrow \alpha$  of  $\text{PCF}+\text{catch}$ ,  $RM$  represents some realizer  $f$  for  $\llbracket M \rrbracket$  (in the sense that  $\xi(f, RM)$ ).*

#### 5.4 $\text{PCF}+\text{timeout}$

Finally, we briefly consider  $\text{PCF}$ -like languages extended with a “timeout” feature (essentially equivalent to the operator  $\mathbf{T}$  introduced by Escardó in [5]). The idea is to add an operator  $\text{timeout}$  which will try to evaluate an expression of ground type for a prescribed length of “time”. For simplicity, we define the *time* taken to evaluate  $P : \iota$  to be the number of recursion unfoldings (i.e. the number of reduction steps  $Y_{\sigma}M \rightarrow M(Y_{\sigma}M)$ ) involved in the reduction of  $P$  (this will be finite iff  $P$  terminates). The operator  $\text{timeout} : \iota \rightarrow \iota \rightarrow \iota$  will

then have the property that

$$\begin{aligned} \text{timeout } P \ k \ \rightarrow^* 0 & \quad \text{if } P \text{ does not terminate within time } k; \\ \text{timeout } P \ k \ \rightarrow^* n + 1 & \quad \text{if } P \text{ evaluates to } n \text{ within time } k. \end{aligned}$$

Recursion unfoldings give a reasonable way to measure time, because the fragment of PCF without  $\mathbf{Y}$  is normalizing, and so any infinite computation must contain infinitely many recursion unfoldings. This particular choice of how to measure time also fits well with the metric space interpretation of PCF discussed in [5]. However, we believe that for our purposes the precise way in which time is measured should not matter too much.

In an earlier version of this paper, we gave a formal definition of the language  $\text{PCF}+\text{timeout}$ , and claimed that it has an LFA model given by Kleene’s second model  $K_{2\text{re}}$ . Here we withdraw this claim with apologies: whilst it is possible to compile  $\text{PCF}+\text{timeout}$  to  $K_{2\text{re}}$ , the latter is powerful enough to simulate  $\text{catch}$  while the former is not.

We are now fairly confident, however, that the  $\text{catch}$  operator is all that is needed to repair our original proof. We hope that a proof of the following will appear elsewhere:

**Claim 5.5** *The model  $\mathbf{Mod}(K_{2\text{re}})$ , with a suitable choice of  $N_{\perp}$ , is LFA for the language  $\text{PCF}+\text{catch}+\text{timeout}$  (suitably defined).*

It remains an open question whether there exists a PCA giving rise to an LFA model for  $\text{PCF}+\text{timeout}$ .

### 5.5 Summary

The situation we have described so far is summarized by Figure 1, which shows the languages we have considered and the PCAs that give LFA models for them. The arrows here represent translations between the programming languages; it seems that no other translations are possible beyond those indicated. Note that not all these translations respect the functional core: e.g. the functional core of  $\text{PCF}+\text{catch}$  corresponds to  $\text{PCF}+\mathbf{H}$  while that of  $\text{PCF}+\text{catch}+\text{timeout}$  corresponds to  $\text{PCF}^{++}$ . This illustrates the non-functorial nature of the “extensional collapse” construction.

Although here we have concentrated on the connections between particular languages and particular PCAs, we believe the translations are also of interest. We view the above picture as representing various interesting notions of computability, ordered according to their computational strength in some sense. It is no accident that for each of the above translations there is a corresponding *applicative morphism* between the respective PCAs (see Section 7). We hope to study these translations more fully in a later paper.

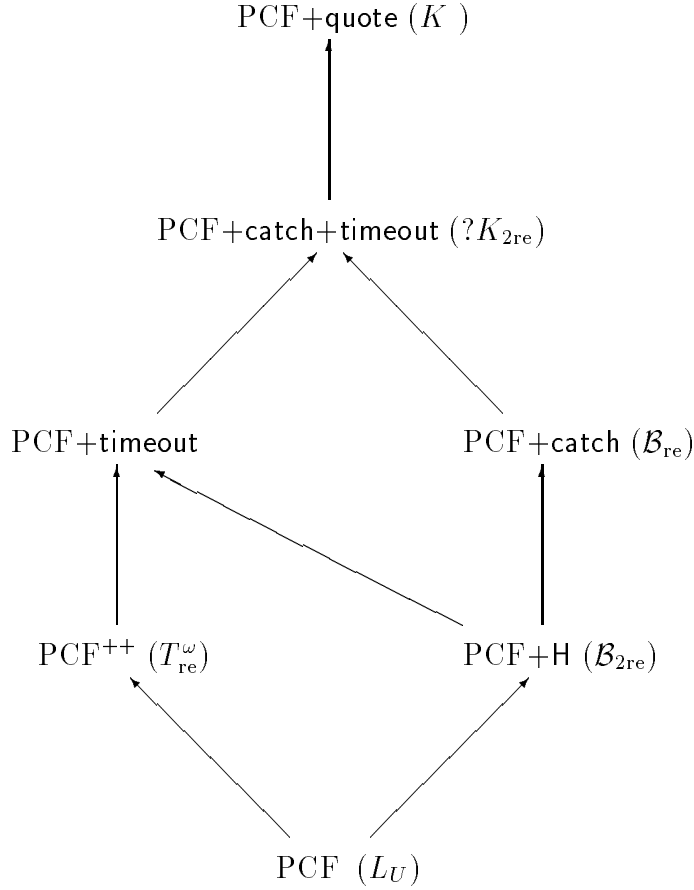


Fig. 1. Typed languages and their associated PCAs

## 6 Some logical examples

We have shown how both typed and untyped models of computation correspond to logical theories. These theories in some way capture the amount of computational power embodied by the models of computation. We now illustrate this with some particular examples of logical formulae, both to highlight the similarities and differences between our various notions of computability, and to demonstrate how logical formulae give a convenient way to summarize information about what is or is not computable in a certain setting. The two aspects of computability that seem to show up best are issues of *extensionality* (the difference between  $\forall x.\exists y$  and  $\exists f.\forall x$ ) and of *constructivity* (the difference between  $\neg\neg\exists x$  and  $\exists x$ ).

We begin with an assortment of simple examples, and then give some examples relating to exact real-number computability. We outline how, using our results, one can forge a link between real-number computability in various programming languages and real analysis inside various realizability toposes.

### 6.1 Simple examples

We have already mentioned a few examples of logical formulae: for instance, (certain instances of) the *axiom of choice* are realizable in all the purely functional languages but in none of the non-functional ones; and *Church's thesis* is realizable in  $K_1$  (hence in PCF+quote) but in none of the other settings. We now mention some further examples:

#### 6.1.1 Local moduli of continuity

Let us write **approx** for the PCF term

$$\lambda g^{\iota \rightarrow \iota}. \lambda n^{\iota}. (\lambda m^{\iota}. g(\text{if}(m \leq n) m \Omega))$$

where  $\leq$  is implemented as expected and  $\Omega$  is some diverging term. Since all computable type 2 functions are continuous, it is realizable in all our settings that

$$\models \forall F^2. \forall g^1. \neg \neg \exists n^0. F(\mathbf{approx} g n) = Fg$$

(where 0 stands for the type  $\iota$ , and  $i + 1$  stands for  $i \rightarrow \iota$ ). Moreover, in PCF+H and all the languages above it in Figure 1, one can actually compute a suitable modulus of continuity  $n$  from  $F$  and  $g$ , so in these settings the formula

$$\models \forall F^2. \forall g^1. \exists n^0. F(\mathbf{approx} g n) = Fg$$

is realizable. However, it is easy to see by monotonicity that this latter formula is not realizable in PCF or PCF<sup>++</sup>. Thus, this formula is internally true in all but two of the corresponding realizability toposes.

In PCF+H and PCF+catch, we even have that

$$\models \exists \Phi^{2 \rightarrow 1 \rightarrow 0}. \forall F^2. \forall g^1. F(\mathbf{approx} g (\Phi Fg)) = Fg.$$

However, this is not realizable in PCF+timeout or above, since in these languages there is no *extensional* way to compute a modulus of continuity. (This is related to the fact that the interpretation of type 2 in these languages includes parallel functions.)

#### 6.1.2 Uniform moduli of continuity

Classically, every continuous function from Cantor space  $2^{\mathbb{N}}$  to  $\mathbb{N}$  is uniformly continuous: this is essentially König's Lemma. The corresponding result fails in all our effective settings, because the notorious *Kleene tree* yields functions that are continuous on the effective analogue of Cantor space but not uniformly continuous there (see e.g. [2]). However, given a function which classically is uniformly continuous, we can effectively obtain a modulus of uniform continuity. That is, if we write  $\text{UnifMod}(F^2, n^0)$  for the formula

$$\forall g^1 h^1. (\forall m^0. m \leq n \Rightarrow gm = hm \leq 1) \Rightarrow (Fg \downarrow \wedge Fg = Fh)$$

then in all of our settings we have

$$\models \forall F^2. (\neg \neg \exists n^0. \text{UnifMod}(F, n)) \Rightarrow \exists n^0. \text{UnifMod}(F, n).$$

In PCF+**quote** (and for that matter in PCF<sup>++</sup> or PCF+**timeout**), a realizer can be easily constructed by means of a parallel search. In all our languages except PCF+**quote**, a realizer can be given using the remarkable Berger-Gandy definition of the fan functional in PCF (described e.g. in [27]), and so in fact we have the stronger formula:

$$\models \exists \Phi^3. \forall F^2. (\neg \neg \exists n^0. \text{UnifMod}(F, n)) \Rightarrow \text{UnifMod}(F, \Phi F).$$

However, this stronger version is *not* realizable in PCF+**quote** (at least with the above definition of UnifMod). Essentially this is because although we can obtain a uniform modulus of continuity by a parallel search, we can never be sure that we have found the smallest possible modulus.

### 6.1.3 Sequentiality indices

In the languages PCF, PCF+H and PCF+**catch** (but none of the others), every non-constant type 2 function has a sequentiality index, and so we have

$$\models \forall F^2. (\neg F(\lambda n^0. \Omega) \downarrow) \Rightarrow \neg \neg \exists n^0. \forall g^1. (Fg \downarrow) \Rightarrow (gn \downarrow).$$

Moreover, in PCF+**catch** (only), we can effectively compute a sequentiality index:

$$\models \forall F^2. (\neg F(\lambda n^0. \Omega) \downarrow) \Rightarrow \exists n^0. \forall g^1. (Fg \downarrow) \Rightarrow (gn \downarrow).$$

(Note that if  $F$  is everywhere undefined we might have  $n = \perp$ .) However, even in PCF+**catch**, there is no way to compute the sequentiality index extensionally in  $F$ , so the corresponding formula  $\exists \Phi^3. \forall F^2. \dots$  fails.

## 6.2 Real-number computability

Exact real-number computation provides an attractive application area for computation at higher types, so it is not surprising that the real numbers show up interesting differences between our various computational settings. This is an area of current joint research with Martín Escardó; we give here an informal sketch of some of our preliminary results.

Any standard realizability topos contains a *real number object*  $R$  (fortunately in such toposes the Cauchy and Dedekind reals always coincide). This means we can interpret formulae of real analysis (say in a language  $\mathcal{R}$  involving the types  $R$  and  $R \rightarrow R$ ) in the internal logic of any realizability topos. In general, different toposes will give rise to different flavours of real analysis, according to what formulae are true in them.

We can also represent real numbers using the finite types we have considered in this paper. The recursive reals (say in the interval  $[-1, 1]$ ) can be represented exactly by recursive infinite sequences of extended binary digits  $-1, 0, 1$ ; thus, arbitrary recursive reals can be represented e.g. by functions  $f$  of type 1 satisfying  $\forall n^0. fn \leq 2$  (in any of our languages  $\mathcal{L}$ ). Computable functions on these reals can then be represented by functions of type  $1 \rightarrow 1$  that behave extensionally on representations of reals.

It is easy to define predicates  $\text{Real}(x^1)$ ,  $\text{RealEq}(x^1, y^1)$ ,  $\text{RealFun}(f^{1 \rightarrow 1})$  and  $\text{RealFunEq}(f^{1 \rightarrow 1}, g^{1 \rightarrow 1})$ , meaning (respectively) that  $x$  represents a real number, that  $x, y$  represent the same real number, that  $f$  represents an (extensional) total function on the recursive reals, and that  $f, g$  represent the same real function. Using these predicates, it is easy to see how one can define a translation from the logic  $\mathcal{R}$  of the real number object to the logic  $\mathbf{J}$  (which we may take to be  $\mathbf{J}(\text{PCF})$ ) in such a way that, in any of our models, a closed formula  $\phi$  of  $\mathcal{R}$  is true iff its translation  $\hat{\phi}$  is. By logical full abstraction, it follows that  $\phi$  holds internally in one of our toposes iff  $\hat{\phi}$  is realizable in the corresponding typed programming language.

A simple example is given by the formula of  $\mathcal{R}$  asserting that *all functions on the reals are continuous*. This beautiful result holds in many constructive settings, and is sometimes known as the Kreisel-Lacombe-Shoenfield (KLS) theorem (see e.g. [2]):

$$\models \forall f : R \rightarrow R. \forall x : R. \forall \epsilon > 0. \exists \delta > 0. \forall y. |y - x| < \delta \Rightarrow |fy - fx| < \epsilon.$$

(We will feel free to sugar the syntax of  $\mathcal{R}$  as long as the meaning is evident.) The constructive force of this is that given  $f, x$  and  $\epsilon$  we can actually compute a  $\delta$  which works. Not surprisingly in view of the above results on local moduli of continuity, the translation of the this formula is realizable in  $\text{PCF}+\text{H}$  and above, but not in  $\text{PCF}$  or  $\text{PCF}^{++}$ . This corresponds to the fact that the KLS theorem holds in the realizability toposes over  $K_1, K_{2\text{re}}, \mathcal{B}_{\text{re}}$  and  $\mathcal{B}_{2\text{re}}$ , but not those over  $T_{\text{re}}^\omega$  or  $L_U$ .

A more shocking example is the following formula, which asserts that there is a semi-decidable subset of the reals that is not open in the usual topology:

$$\models \exists f : R \rightarrow \Sigma. \exists x : R. fx = \top \wedge \forall \epsilon > 0. \exists y. |y - x| < \epsilon \wedge fy = \perp.$$

It can be shown that this is realizable in  $\text{PCF}+\text{quote}$  (i.e. in  $K_1$ ), by a simple adaptation of the proof of Friedberg’s theorem (see e.g. [28, Section 15.3]). Mercifully, it is not realizable in any of the other settings!

Unfortunately, many of the formulae of  $\mathbf{J}$  that express interesting facts about real-number computability are not in the image of the translation from  $\mathcal{R}$ —that is, the language  $\mathcal{R}$  seems to be not as expressive as we would like. In particular, in  $\mathbf{J}$  we have the following useful formula  $\text{UnifCts}(f^{1 \rightarrow 1})$ , saying that a function  $f$  (representing, say, a function on  $I = [0, 1]$ ) is “uniformly continuous” in a sense analogous to that defined in Section 6.1.2:

$$\forall p^0. \exists n^0. \forall x^1 y^1. (\forall m^0. m \leq n \Rightarrow xm = ym \leq 2) \Rightarrow fx(p) \equiv fy(p)$$

(writing  $M \equiv N$  for  $M = N \wedge M \downarrow$ ). This condition is stronger than the usual  $\epsilon\delta$  definition of uniform continuity in real analysis, and is useful for excluding pathological functions with Kleene-tree-like behaviour. (Roughly speaking, it says that  $f$  would be total on the *classical* reals if we could apply it to them.) However, it seems that this property cannot be expressed in  $\mathcal{R}$ , since it is essentially a property of a *representation*  $f^{1 \rightarrow 1}$  of a real function rather than the real function itself. It would be pleasing if the above condition could be replaced by some reasonably clean mathematical condition involving the

object  $R$ , but at present we do not know whether this is possible.

Meanwhile, let us add the predicate  $\text{UnifCts}(f^{1 \rightarrow 1})$  to our language. (From now on, we will use a hybrid of  $\mathcal{R}$  and  $\mathbf{J}$  for our syntax, but officially we have in mind a corresponding formula of  $\mathbf{J}$ ). By analogy with the results of Section 6.1.2, the following formula holds in all of our settings:

$$\models \forall f : I \rightarrow R. (\neg \neg \text{UnifCts}(f)) \Rightarrow \text{UnifCts}(f)$$

There is an interesting class of formulae expressing the idea that (under various conditions) we can locate a zero of a function. One of the simplest examples is the following, which again holds in all our settings:

$$\models \forall f : I \rightarrow R. \text{UnifCts}(f) \Rightarrow (\neg \neg \exists ! x : I. fx = 0) \Rightarrow (\exists ! x : I. fx = 0).$$

The hypothesis that the zero is unique is essential here. However, one can also consider similar formulae with other hypotheses, and here it seems that interesting distinctions emerge between the different notions of computability.

Finally, we mention some formulae expressing the idea that we can compute (Riemann) integrals for some class of functions. Again, the simplest such formula holds in all our settings:

$$\models \forall f : I \rightarrow R. \text{UnifCts}(f) \Rightarrow \text{Integrable}(f)$$

However, differences emerge when we try to integrate (partial) functions with discontinuities. For instance, let us write  $\text{OneHole}(f^{1 \rightarrow 1})$  for the following formula saying that  $f$  represents a partial function  $I \rightarrow R$  which is undefined on at most one point  $x \in (0, 1)$

$$\exists x \in (0, 1). \forall y \in I. \neg \text{RealEq}(x, y) \Rightarrow \text{Real}(fy).$$

Now consider the following formula, which asserts in effect that there is a uniform algorithm for integrating all such functions:

$$\models \forall f. (\neg \neg \text{OneHole}(f)) \Rightarrow \text{Integrable}(f).$$

This formula is not realizable in PCF, but it is realizable in PCF+H. (The algorithm required is a simple adaptation of the integration algorithm described in [16].) In fact, for any  $k$  there is a formula asserting that all functions which are undefined on at most  $k$  points are integrable, and this is realizable in PCF+H. In PCF+catch one can do even better: we can integrate all functions that are undefined on only finitely many points without knowing a bound  $k$  in advance.

It would be interesting to undertake a more systematic investigation of these different flavours of real analysis, and perhaps for complex and functional analysis. It seems that there is a potentially large research field here waiting to be explored.

## 7 Further developments

Shortly after writing the original version of this paper, we discovered some definitions that allow us to clarify much of the above material considerably.



In essence, rather than considering our typed and untyped structures as living in two separate worlds, we are now able to subsume both these worlds in a single common setting. A preliminary account of these new ideas may be found in [15]; below we give only a very brief outline. More details will appear elsewhere.

The key observation is that the construction of realizability categories over PCAs can be generalized to a much wider class of structures, known as *partial combinatory type structures (PCTSs)*, which allow our realizers to have types. Indeed, for any PCTS  $A$ , we have a category  $\mathbf{Mod}(A)$  which is locally cartesian closed and regular. We may recover PCAs exactly as the PCTSs for which there is only one type. We also obtain PCTSs from the term models for each of the typed languages considered in this paper. Seen in this light, the untyped and typed realizability relations defined in Section 3 are both instances of the same definition.

There is a natural 2-category  $\mathbf{PCTS}$  consisting of PCTSs, *applicative morphisms* and *applicative transformations*. This expands the 2-category of PCAs considered in [12]. Note that translations between typed languages (as in Section 2.2) also provide examples of applicative morphisms. As in the case of PCAs, applicative morphisms between PCTSs correspond precisely to certain exact functors between the realizability categories.

In particular, two PCTSs  $A, B$  are equivalent in  $\mathbf{PCTS}$  iff the realizability categories on  $A, B$  are equivalent; in this situation we may say that  $A, B$  are *realizably equivalent*. Interestingly, one frequently finds that the term model for a certain typed language is realizably equivalent to a certain PCA: for example, the term model for  $\text{PCF} + \text{catch}$  is realizably equivalent to  $\mathcal{B}_{\text{re}}$ . Realizable equivalences of this kind certainly imply logical full abstraction; indeed, one can perhaps see realizable equivalence as a kind of ultimate “goodness of fit” criterion between a language and a model.

We also have instances of logical full abstraction that do not arise from realizable equivalences. Indeed, our Theorem 5.1 may now be seen much more simply as a special case of the following:

**Theorem 7.1** *Suppose  $A, B$  are PCTSs, and  $\gamma : A \rightarrow B$ ,  $\delta : B \rightarrow A$  are applicative morphisms “preserving  $N_{\perp}$ ” such that we have applicative transformations  $\text{id}_A \Rightarrow \delta\gamma$  and  $\text{id}_B \Rightarrow \gamma\delta$ . Then  $\mathbf{Mod}(A)$ ,  $\mathbf{Mod}(B)$  induce the same logical theory.*

For example, the PCA  $K_1$  and the language  $\text{PCF} + \text{quote}$  provide examples of PCTSs that satisfy these conditions but are not realizably equivalent. However, these PCTSs are certainly very close, in that there is an *applicative inclusion* from  $K_1$  to  $\text{PCF} + \text{quote}$ ; this justifies the intuition that they embody more or less the same notion of computability.

All of the correspondences between languages and models shown in Figure 1 are at least examples of the above theorem, except that we need to replace pure PCF by FPC. This explains why our translations between lan-

guages all give rise to applicative morphisms between PCAs.

Finally, we wish to mention a beautiful theorem discovered recently by Lietz and Streicher. For any PCTS  $A$ , in addition to the category  $\mathbf{Mod}(A)$  one may construct the larger categories  $\mathbf{Ass}(A)$  and  $\mathbf{RC}(A)$ , the latter being the standard realizability topos in the case of a PCA. We then have:

**Theorem 7.2** *For a PCTS  $A$ , the following are equivalent.*

- (i)  $A$  is equivalent (in  $\mathbf{PCTS}$ ) to a PCA.
- (ii)  $\mathbf{Ass}(A)$  contains a generic mono.
- (iii)  $\mathbf{RC}(A)$  is a topos.

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