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SINGULARITY CATEGORIES OF GENTLE ALGEBRAS

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ABSTRACT. We determine the singularity category of an arbitrary finite dimensional gentle algebra Λ . It is a finite product of n -cluster categories of type \mathbb{A}_1 . Equivalently, it may be described as the stable module category of a selfinjective gentle algebra. If Λ is a Jacobian algebra arising from a triangulation \mathcal{T} of an unpunctured marked Riemann surface, then the number of factors equals the number of inner triangles of \mathcal{T} .

1. INTRODUCTION

Singularity categories were introduced and studied by Buchweitz [8]. Recently, Orlov's global version [19] attracted a lot of interest in algebraic geometry and theoretical physics: in particular, its relation to Kontsevich's Homological Mirror Symmetry Conjecture.

For Iwanaga–Gorenstein rings, Buchweitz gave an equivalent description of singularity categories in terms of stable categories of Gorenstein projective modules (also known as maximal Cohen–Macaulay modules), see [8] and also Happel [15], Keller & Vossieck [17] and Rickard [22]. In particular, singularity categories of selfinjective algebras are equivalent to their stable module categories, which were thoroughly studied in representation theory. X.-W. Chen [11] described the singularity categories of artin algebras with radical square zero in terms of projective modules over certain von Neumann regular algebras. He shows that their underlying additive categories are semisimple abelian categories.

The aim of this note is to describe the singularity categories of another class of finite dimensional algebras - so called gentle algebras (Definition 2.1). As it turns out, their underlying additive categories are again semisimple. Examples of gentle algebras include tilted algebras of type \mathbb{A}_n [1] and $\tilde{\mathbb{A}}_n$ [3] and more generally all algebras which are derived equivalent to gentle algebras [24]. Moreover, algebras derived equivalent to \mathbb{A}_n -configurations of projective lines [9] and Jacobian algebras coming from triangulations of unpunctured marked surfaces are gentle. Furthermore, cluster tilted algebras of type \mathbb{A}_n and $\tilde{\mathbb{A}}_n$ are gentle - in fact, they arise from unpunctured marked discs and annuli [4].

Our proof combines Buchweitz' equivalence with the explicit classification of indecomposable modules over gentle algebras which follows (see e.g. [26, 10]) from work of Ringel [23], who builds on techniques developed by Gelfand & Ponomarev [13] in their study of indecomposable representations of the Lorentz group. More precisely, indecomposable modules are either *string* or *band* modules. Band modules are never submodules of projective modules - in particular, they cannot be Gorenstein projective (GP). We show that string modules are GP precisely if they are projective or left ideals generated by certain

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arrows. We complete our description of the singularity categories by proving that all non-trivial morphisms between indecomposable GPs factor over projectives.

2. DEFINITIONS AND MAIN RESULT

Let k be an algebraically closed field and let Q be a finite quiver with set of arrows Q_1 . We read elements in the path algebra kQ from right to left.

Definition 2.1. A *gentle algebra* is a finite dimensional k -algebra $\Lambda = kQ/I$ such that:

- (G1) At any vertex, there are at most two incoming and at most two outgoing arrows.
- (G2) I is a two-sided admissible ideal, which is generated by paths of length two.
- (G3) For each arrow $\beta \in Q_1$, there is at most one arrow $\alpha \in Q_1$ such that $0 \neq \alpha\beta \in I$ and at most one arrow $\gamma \in Q_1$ such that $0 \neq \beta\gamma \in I$.
- (G4) For each arrow $\beta \in Q_1$, there is at most one arrow $\alpha \in Q_1$ such that $\alpha\beta \notin I$ and at most one arrow $\gamma \in Q_1$ such that $\beta\gamma \notin I$.

Remark 2.2. It is well-known that gentle algebras can be described more conceptually as those finite dimensional algebras with special biserial repetitive algebra, see [3] and [21].

Example 2.3. An example of a gentle algebra $\Lambda = kQ/I$ is given by the quiver Q

$$\begin{array}{ccccccc}
 1 & \xrightarrow{a} & 2 & \xrightarrow{b} & 3 & \xrightarrow{c} & 4 \\
 d \uparrow & & e \uparrow & \searrow f & & \nearrow g & \uparrow h \\
 5 & \xleftarrow{i} & 6 & \xleftarrow{j} & 7 & \xleftarrow{k} & 8
 \end{array}$$

with two-sided ideal I generated by the paths ba , fe , jf , ej , kg , hk and gh .

Geiß & Reiten [12] have shown that gentle algebras are *Iwanaga–Gorenstein rings*, i.e. they have finite injective dimension as left and as right modules over themselves. For any Iwanaga–Gorenstein ring R , Zaks [27] has shown that $\text{inj. dim}_R R = d = \text{inj. dim } R_R$ holds. Following Buchweitz, we call d the *virtual dimension* of R - for commutative local Noetherian rings, it coincides with the *Krull dimension*. Inside the category $R\text{-mod}$ of all finite dimensional left R -modules, the full subcategory of *Gorenstein projective* R -modules

$$\text{GP}(R) = \{M \in R\text{-mod} \mid \text{Ext}_R^i(M, R) = 0 \text{ for all } i > 0\} \quad (2.1)$$

is of special interest. Let M and N be finite dimensional left R -modules. We list some well-known facts about Gorenstein projective R -modules, see e.g. Buchweitz [8].

- (GP1) A GP R -module is either projective or of infinite projective dimension.
- (GP2) M is GP if and only if $M \cong \Omega^d(N)$ for some N , where d is the virtual dimension. In particular, every GP module is a submodule of a projective module.
- (GP3) $\text{GP}(R)$ is a Frobenius category with $\text{proj GP}(R) = \text{proj } R$.

Moreover, the embedding $\text{GP}(R) \subseteq \mathcal{D}^b(\text{mod } R)$ induces a triangle equivalence (see [8])

$$\frac{\text{GP}(R)}{\text{proj } R} =: \underline{\text{GP}}(R) \longrightarrow \mathcal{D}_{sg}(R) := \frac{\mathcal{D}^b(\text{mod } R)}{K^b(\text{proj } R)}, \quad (2.2)$$

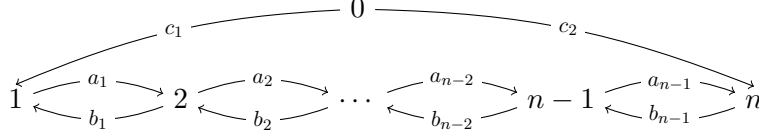
where the triangulated quotient category $\mathcal{D}_{sg}(R)$ is called the *singularity category* of R , see [8] and also [19]. The additive quotient category $\underline{\text{GP}}(R)$ is called the *stable category* of Gorenstein projective R -modules. It admits a triangulated structure by Happel's general

which is induced by

$$\mathcal{D}^b(\text{Coh } \mathbb{X}_n) \ni \mathcal{F} \longmapsto \left(\widehat{\mathcal{F}}_{s_1}, \dots, \widehat{\mathcal{F}}_{s_{n-1}} \right) \in \bigoplus_{i=1}^{n-1} \mathcal{O}_{nd} - \text{mod}, \quad (3.6)$$

where s_1, \dots, s_{n-1} denote the singular points of \mathbb{X}_n . For $1 \leq l \leq m \leq n$, let $\mathcal{O}_{[l,m]}$ be the structure sheaf of the subvariety $\bigcup_{k=l}^m C_k \subseteq \mathbb{X}_n$. Here, the C_i denote the irreducible components of \mathbb{X}_n as shown in (3.3). Then (3.6) maps $\mathcal{O}_{[1,i]}$ to $(\mathcal{O}_{nd}, \dots, \mathcal{O}_{nd}, k[[x]], 0, \dots, 0)$ and $\mathcal{O}_{[j,n]}$ to $(0, \dots, 0, k[[y]], \mathcal{O}_{nd}, \dots, \mathcal{O}_{nd})$, where $k[[x]]$ and $k[[y]]$ are located in the i -th and j -th place, respectively. In particular, the functor in (3.5) is essentially surjective. Therefore, the singularity category $\mathcal{D}_{sg}(\mathbb{X}_n)$ is idempotent complete.

We explain an alternative approach to obtain the equivalence (3.5), which uses and confirms Theorem 2.5. Burban [9] showed that $\mathcal{D}^b(\text{Coh } \mathbb{X}_n)$ has a tilting bundle with endomorphism algebra Λ_n given by the following quiver



with relations $a_i b_i = 0 = b_i a_i$ for all $1 \leq i \leq n-1$. Hence we have a triangle equivalence $\mathcal{D}^b(\text{Coh } \mathbb{X}_n) \rightarrow \mathcal{D}^b(\text{mod } - \Lambda_n)$ inducing an equivalence of triangulated categories

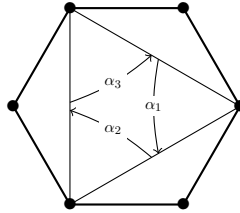
$$\mathcal{D}_{sg}(\mathbb{X}_n) \xrightarrow{\sim} \mathcal{D}_{sg}(\Lambda_n). \quad (3.7)$$

Since Λ_n is a gentle algebra, we can apply Theorem 2.5. $\mathcal{C}(\Lambda_n)$ consists of $n-1$ cycles of length two. Therefore $\mathcal{D}_{sg}(\Lambda_n)$ is equivalent to the right hand side of (3.4). In particular, we see again that the singularity category $\mathcal{D}_{sg}(\mathbb{X}_n)$ is idempotent complete.

Assem, Brüstle, Charbonneau-Jodoin & Plamondon [4] studied a class of gentle algebras $A(S, \mathcal{T})$ arising from triangulations \mathcal{T} of marked Riemann surfaces S without punctures. In particular, they show that the ‘inner triangles’ of \mathcal{T} are in bijection with the elements of $\mathcal{C}(A(S, \mathcal{T}))$, which in this case are all of length three. This has the following consequence.

Corollary 3.5. *In the notation above, the number of direct factors of the singularity category $\mathcal{D}_{sg}(A(S, \mathcal{T}))$ equals the number of inner triangles of \mathcal{T} .*

Example 3.6. A prototypical case is the hexagon S with six marked points on the boundary. We consider the following triangulation \mathcal{T} with exactly one inner triangle.



The corresponding gentle algebra $A(S, \mathcal{T})$ is a 3-cycle with relations $\alpha_2 \alpha_1 = 0$, $\alpha_3 \alpha_2 = 0$ and $\alpha_1 \alpha_3 = 0$. It is isomorphic to the *selfinjective* algebra I_3 defined in Remark 2.6. Hence

the singularity category $\mathcal{D}_{sg}(A(S, \mathcal{T}))$ is triangle equivalent to the stable module category $A(S, \mathcal{T}) - \underline{\text{mod}}$, by (2.2).

Remark 3.7. More generally, the algebras arising as Jacobian algebras from ideal triangulations of Riemann surfaces *with* punctures are often of infinite global dimension. It would be interesting to study their singularity categories and relate them to properties of the triangulation.

Example 3.8. In Example 2.3, the indecomposable non-projective GPs are given by:

$$\begin{aligned} c_1 & \begin{cases} R(e) = 2 - b \rightarrow 3 - c \rightarrow 4 - g \rightarrow 7 - j \rightarrow 6 - i \rightarrow 5 - d \rightarrow 1 - a \rightarrow 2 - f \rightarrow 7 - k \rightarrow 8; \\ R(f) = 7 - k \rightarrow 8; \\ R(j) = 6 - i \rightarrow 5 - d \rightarrow 1 - a \rightarrow 2 - f \rightarrow 7 - k \rightarrow 8; \end{cases} \\ c_2 & \begin{cases} R(g) = 7 - j \rightarrow 6 - i \rightarrow 5 - d \rightarrow 1 - a \rightarrow 2 - f \rightarrow 7 - k \rightarrow 8; & R(h) = 4; & R(k) = 8. \end{cases} \end{aligned}$$

They correspond to the two cycles $c_1 = jfe$ and $c_2 = kgh$ in $\mathcal{C}(\Lambda)$. Theorem 2.5 yields

$$\mathcal{D}_{sg}(\Lambda) \cong \frac{\mathcal{D}^b(k - \text{mod})}{[3]} \oplus \frac{\mathcal{D}^b(k - \text{mod})}{[3]}. \quad (3.8)$$

4. PROOF

We start with some background material on modules over gentle algebras $\Lambda = kQ/I$. A classification of indecomposable modules over gentle algebras can be deduced from work of Ringel [23] (see e.g. [26, 10]): they are either *string* or *band* modules $M(w)$, where w is a certain word in the alphabet $\{\alpha, \alpha^{-1} \mid \alpha \in Q_1\}$. Equivalently, one can consider certain quiver morphisms $\sigma: S \rightarrow Q$ (for strings) and $\beta: B \rightarrow Q$ (for bands), where S and B are of Dynkin types \mathbb{A}_n and $\tilde{\mathbb{A}}_n$, respectively. Then string and band modules are given as pushforwards $\sigma_*(M)$ and $\beta_*(R)$ of indecomposable kS -modules M and indecomposable *regular* kB -modules R , respectively (see e.g. [26]).

It follows from properties (G1), (G2) & (G4) in the Definition 2.1 of gentle algebras that the indecomposable projective Λ -modules are of the following form:

$$\begin{array}{c} \bullet \\ \alpha_1 \downarrow \\ \bullet \\ \vdots \\ \bullet \\ \alpha_l \downarrow \\ \bullet \end{array} \quad \text{or} \quad \begin{array}{c} \bullet \\ \beta_1 \swarrow \quad \searrow \gamma_1 \\ \bullet \quad \bullet \\ \vdots \quad \vdots \\ \bullet \quad \bullet \\ \beta_m \swarrow \quad \searrow \gamma_n \\ \bullet \quad \bullet \end{array} \quad (4.1)$$

They correspond to the words $\alpha_l \dots \alpha_1$ and $\beta_m \dots \beta_1 \gamma_1^{-1} \dots \gamma_n^{-1}$, respectively. The definition of quiver algebras kQ/I implies that the paths $\alpha_l \dots \alpha_1$, $\beta_m \dots \beta_1$ and $\gamma_n \dots \gamma_1$ appearing in (4.1) are *maximal*, e.g. there does not exist $\alpha \in Q_1$ such that $\alpha \alpha_l \notin I$, see for example [2].

It follows from (4.1) that the radical $\text{rad } P$ of an indecomposable projective Λ -module P has at most two indecomposable direct summands. Moreover, (4.1) yields the following result about submodules of projective modules.

Lemma 4.1. *Let $M = M(w)$ be an indecomposable Λ -module, such that w contains*

$$\alpha^{-1}\beta = \begin{array}{c} x \\ \alpha \searrow \quad \swarrow \beta \\ y \end{array} \quad (4.2)$$

with $\alpha \neq \beta$ as a subword. Then M is not a submodule of a projective Λ -module P .

Remark 4.2. In the picture (4.2), the letters x, y, z represent basis vectors of the module M . We do not exclude the case $x = z$. For example, the indecomposable injective module I_2 over the Kronecker quiver $1 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} 2$ is a *string* module of the form (4.2), with pairwise different basis vectors x, y, z . On the other hand, the indecomposable *band* modules

$$\begin{array}{c} 1 \\ \xrightarrow{\quad} \\ k \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{c} k \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} 1$$

with $\lambda \in k^*$ correspond to the same word $\alpha^{-1}\beta$ but we have to identify x and z in (4.2).

Throughout the proof, we use the properties (GP1) & (GP2) of Gorenstein projective modules over Iwanaga–Gorenstein rings, which are stated in Section 2.

4.1. Proof of part (a). Let $c \in \mathcal{C}(\Lambda)$ be a cycle, which we label as follows $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} n \xrightarrow{\alpha_n} 1$. Then there are short exact sequences

$$0 \rightarrow R(\alpha_i) \rightarrow P_i \rightarrow R(\alpha_{i-1}) \rightarrow 0, \quad (4.3)$$

for all $i = 1, \dots, n$, where we set $\alpha_0 = \alpha_n$. We give an illustration of this below.

$$\begin{array}{ccccccc} & & & & R(\alpha_{i-1}) & & \\ & & & & \underbrace{\hspace{10em}} & & \\ \dots & \xrightarrow{\alpha_{i-1}} & i & \xrightarrow{\beta_1} & \dots & \xrightarrow{\beta_n} & \\ & & \downarrow \alpha_i & & & & \\ \dots & \xleftarrow{\alpha_{i+1}} & i+1 & \xrightarrow{\gamma_1} & \dots & \xrightarrow{\gamma_m} & \\ & & \underbrace{\hspace{10em}} & & R(\alpha_i) & & \end{array}$$

Here $n = 0$ or $m = 0$ are allowed. In particular, (4.3) shows that for every $m \geq 0$ and every i there is a Λ -module X such that $R(\alpha_i)$ may be written as a m th-syzygy module $\Omega^m(X)$. Thus, $R(\alpha_i) \in \text{GP}(\Lambda)$ by (GP2). Since projective modules are GP by definition, this shows the inclusion ‘ \supseteq ’ in (a).

It remains to show that there are no further indecomposable Gorenstein projective modules. By property (GP2), we only have to consider submodules of projective modules. Using Lemma 4.1, we can exclude all modules which correspond to a word containing $\alpha^{-1}\beta$. In particular, band modules are not Gorenstein projective - the corresponding words are cyclic and always contain subwords of the form $\alpha^{-1}\beta$.

We claim that an indecomposable Gorenstein projective Λ -module M containing a subword of the form $\alpha\beta^{-1}$, with $\alpha \neq \beta$ is projective. We think of $\alpha\beta^{-1}$ as a ‘roof’

$$\begin{array}{c} \alpha \quad t \quad \beta \\ \swarrow \quad \searrow \\ s \quad \quad u \end{array}, \quad (4.4)$$

where s, t, u are basis vectors of M , such that $\alpha \cdot t = s$ and $\beta \cdot t = u$.

Let $U(t) \subset M$ be the submodule generated by $t \in M$. By (GP2), M is a submodule of some projective module P . Using this in conjunction with (4.4) and the properties (G1) & (G4), we see that $U(t) \cong P_{v(t)}$ is projective, where $v(t) \in Q_0$ is the vertex corresponding to t . If $U(t) \subsetneq M$, then M contains a subword of the form $\alpha^{-1}\beta$, with $\alpha \neq \beta$ (we note that this statement does not use the assumption that M is GP). By Lemma 4.1 this cannot happen. So we see that $M = U(t) \cong P_{v(t)}$ is indeed projective.

We have reduced the set of possible indecomposable GP Λ -modules to projective modules or directed strings $S = \beta_n \dots \beta_1$. We also allow S to consist of a single ‘lazy’ path e_i (this corresponds to a simple module). Let $M(S)$ be the corresponding GP Λ -module. It is contained in a projective module by (GP2). If $M(S)$ is not projective, then there exists an arrow α such that $\beta_n \dots \beta_1 \alpha \notin I$ and $\gamma \beta_n \dots \beta_1 \alpha \in I$ for every arrow $\gamma \in Q_1$. It follows that $M(S) = R(\alpha)$ is a direct summand of the radical of $P_{s(\alpha)}$.

Claim: If α does not lie on a cycle $c \in \mathcal{C}(\Lambda)$, then $R(\alpha)$ has finite projective dimension. If $R(\alpha)$ is not projective, then the situation locally looks as follows (we allow n to be zero)

$$\begin{array}{ccccccc} \dots & \xrightarrow{\alpha} & \sigma & \xrightarrow{\beta_1} & \dots & \xrightarrow{\beta_n} & \bullet \\ & \searrow^{\alpha_1} & & \underbrace{\hspace{4cm}} & & & \\ \dots & & & & & & R(\alpha) \end{array}$$

where $\alpha_1 \alpha \in I$. (There could be another arrow ending in σ . It is omitted from the picture since it does not affect our argument.) Moreover, α_1 cannot lie on a cycle in $\mathcal{C}(\Lambda)$, since this would contradict our assumption on α . As in (4.3), we have a short exact sequence

$$0 \rightarrow R(\alpha_1) \rightarrow P_\sigma \rightarrow R(\alpha) \rightarrow 0. \quad (4.5)$$

$R(\alpha_1)$ has the same properties as $R(\alpha)$, so we may repeat our argument. After finitely many steps, one of the occurring radical summands will be projective and the procedure stops. Indeed, otherwise we get a path $\dots \alpha_m \dots \alpha_1 \alpha$, such that every subpath of length two is contained in I . Since there are only finitely many arrows in Q , this path is a cycle. Contradiction. Hence $R(\alpha)$ has finite projective dimension.

Combining the claim with (GP1), we see that for arrows α , which do not lie on a cycle in $\mathcal{C}(\Lambda)$, $R(\alpha)$ is GP if and only if it is projective. Summing up, we have shown that indecomposable GP modules are either projective or direct summands $R(\alpha_i) = \Lambda \alpha_i$ of the radical of some indecomposable projective module $P_{s(\alpha_i)}$, where α_i is contained in a cycle $c \in \mathcal{C}(\Lambda)$. This proves part (a).

4.2. Proof of part (b). By Buchweitz’ equivalence (2.2), it suffices to describe the stable category $\underline{\text{GP}}(\Lambda)$. By part (a), the indecomposable objects in this category are precisely the radical summands $R(\alpha_i)$ for a cycle $c = \alpha_n \dots \alpha_1 \in \mathcal{C}(\Lambda)$ and (4.3) shows that $R(\alpha_i)[1] \cong R(\alpha_{i-1})$. In particular, $R(\alpha_i)[l(c)] \cong R(\alpha_i)$. We prove

$$\underline{\text{Hom}}_\Lambda(R(\alpha), R(\alpha')) \cong \delta_{\alpha\alpha'} \cdot k \quad (4.6)$$

below. This shows that the additive category $\underline{\text{GP}}(\Lambda)$ is equivalent to a semisimple abelian category and therefore itself semisimple abelian. It is well-known that a semisimple abelian category with autoequivalence [1] admits a unique triangulated structure with shift functor [1], see e.g. [11, Lemma 3.4.]. This completes the proof of part (b). The remaining part

of this subsection is concerned with the proof of (4.6). $R(\alpha)$ is given by a string of the following form (it starts in σ and we allow $n = 0$)

$$\cdots \xrightarrow{\alpha} \sigma \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} \bullet. \quad (4.7)$$

Here, α is on a cycle $c \in \mathcal{C}(\Lambda)$ and $\beta_1\alpha \notin I$, if $n \neq 0$. If there is a non-zero morphism of Λ -modules from $R(\alpha)$ to $R(\alpha')$, then the latter has to be a string of the following form

$$R(\alpha'): \quad \cdots \xrightarrow{\alpha'} \sigma' \xrightarrow{\beta'_1} \cdots \xrightarrow{\beta'_m} \sigma \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_k} \bullet, \quad (4.8)$$

where we allow $k = 0$ or $m = 0$. If both k and m are zero, then (G3) and the fact that $R(\alpha')$ is a submodule of an indecomposable projective Λ -module imply that there is only one arrow starting in σ (this arrow lies on the cycle c). Hence, $n = 0$ and therefore $R(\alpha) = R(\alpha')$. $k \neq 0$ and $m = 0$ imply $\alpha = \alpha'$ by (G4) – in particular, $R(\alpha) = R(\alpha')$.

We show that in both cases $\underline{\text{End}}_\Lambda(R(\alpha)) \cong k$ holds. For this, we claim that the simple module S_σ can appear (at most) twice as a composition factor of $R(\alpha)$. Indeed (G2) and the finite dimensionality of Λ imply that every arrow of Q appears at most once in the path defining $R(\alpha)$ and the arrow α itself does not appear at all. Now, using (G1) there is at most one arrow ending in σ which is different from α . This completes the proof of the claim. However, if S_σ occurs twice as a composition factor, then $R(\alpha)$ locally has the following form

$$\sigma \xrightarrow{\beta_1} \cdots \xrightarrow{\neq \alpha} \sigma \xrightarrow{\gamma} \cdots \rightarrow \bullet,$$

where $\gamma \neq \beta_1$ lies on the cycle c , because α lies on this cycle with full relations, see Example 3.8 for an illustration. In particular, this does not yield additional endomorphisms.

If $k \neq 0$ and $m \neq 0$, then it follows from (G4) that $\beta'_m = \alpha$. If $k = 0$, $m \neq 0$ and $\beta'_m \neq \alpha$, then there are two different arrows ending in σ . Since α is on a cycle there is an arrow $\gamma: \sigma \rightarrow \bullet$, such that $\gamma\alpha \in I$. It follows from (G3) that $\gamma\beta'_m \notin I$. Since $R(\alpha') = \Lambda\alpha'$ is a left ideal, the path starting in σ' has to be maximal. In particular, it does not end in σ . Contradiction. So we again have $\beta'_m = \alpha$.

In both cases our morphism factors over a projective module

$$R(\alpha) \rightarrow P_{s(\alpha)} \rightarrow R(\alpha') \quad (4.9)$$

and therefore $\underline{\text{Hom}}_\Lambda(R(\alpha), R(\alpha')) = 0$, see Example 2.4 for an illustration of this case. This completes the proof.

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REFERENCES

- [1] I. Assem, *Tilted algebras of type A_n* , Comm. Algebra **10** (1982), 2121–2139.
- [2] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory*, London Mathematical Society Student Texts **65** (Cambridge University Press, Cambridge, 2006).
- [3] I. Assem, A. Skowroński, *Iterated tilted algebras of type \tilde{A}_n* , Math. Z. **195** (1987), 269–290.
- [4] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, P.-G. Plamondon, *Gentle algebras arising from surface triangulations*, Algebra Number Theory **4** (2010), no. **2**, 201–229.
- [5] D. Avella-Alaminos, C. Geiß, *Combinatorial derived invariants for gentle algebras*, J. Pure Appl. Algebra **212** (2008), no. **1**, 228–243.
- [6] P. Balmer, M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra **236** (2001), no. **2**, 819–834.
- [7] A. B. Buan, D. Vatne, *Derived equivalence classification for cluster-tilted algebras of type A_n* , J. Algebra **319** (2008), no. **7**, 2723–2738.
- [8] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-Cohomology over Gorenstein rings*, Preprint 1987, available at <http://hdl.handle.net/1807/16682>.
- [9] I. Burban, *Derived categories of coherent sheaves on rational singular curves*, in Representations of finite dimensional algebras and related topics in Lie Theory and geometry. Proceedings from the ICRA X, Fields Inst. Commun. **40**, Amer. Math. Soc., Providence, RI (2004), 173–188.
- [10] M. C. R. Butler, C. M. Ringel *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), no. **1-2**, 145–179.
- [11] X.-W. Chen, *The singularity category of an algebra with radical square zero*. Doc. Math. **16** (2011), 921–936.
- [12] C. Geiß, I. Reiten, *Gentle algebras are Gorenstein*, in Representations of algebras and related topics, Fields Inst. Commun. **45**, Amer. Math. Soc., Providence, RI (2005), 129–133.
- [13] I. Gelfand, V. Ponomarev, *Indecomposable representations of the Lorentz group*, Uspekhi Mat. Nauk **23** (1968), 3–60. [English transl.: Russian Math. Surveys **23** (1968), 1–58]
- [14] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, LMS Lecture Notes Series, **119**, Cambridge University Press, (1988).
- [15] ———, *On Gorenstein algebras*, In Representation theory of finite groups and finite-dimensional algebras, Proc. Conf., Bielefeld/Ger. 1991, Prog. Math. **95** (1991), 389–404.
- [16] B. Keller, *On triangulated orbit categories*, Doc. Math., **10** (2005), 551–581.
- [17] B. Keller, D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. **6**, 225–228.
- [18] G. J. Murphy, *Derived equivalence classification of m -cluster tilted algebras of type A_n* , J. Algebra **323** (4) (2010) 920–965.
- [19] D. Orlov, *Triangulated categories of singularities and D -branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 240–262.
- [20] ———, *Formal completions and idempotent completions of triangulated categories of singularities*, Adv. Math. **226** (2011), no. **1**, 206–217.
- [21] Z. Pogorzaly, A. Skowroński, *Self-injective biserial standard algebras* J. Algebra, **138** (2) (1991), pp. 491–504.
- [22] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. **3**, 303–317.
- [23] C. M. Ringel, *The indecomposable representations of the dihedral 2-groups* Math. Ann. **214** (1975), 19–34.
- [24] J. Schröer, A. Zimmermann, *Stable endomorphism algebras of modules over special biserial algebras*, Math. Z. **244** (2003), no. **3**, 515–530.
- [25] H. Thomas, *Defining an m -cluster category*, J. Algebra, **318** (2007), no. **1**, 37–46.
- [26] B. Wald, J. Waschbüsch, *Tame biserial algebras*, J. Algebra **95** (1985), 480–500.
- [27] A. Zaks, *Injective dimension of semi-primary rings*, J. Algebra, **13** (1969), 73–86.

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