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A General and Intuitive Envelope Theorem*

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Abstract

We present an envelope theorem for establishing first-order conditions in decision problems involving continuous and discrete choices. Our theorem accommodates general dynamic programming problems, even with unbounded marginal utilities. And, unlike classical envelope theorems that focus only on differentiating value functions, we accommodate other endogenous functions such as default probabilities and interest rates. Our main technical ingredient is how we establish the differentiability of a function at a point: we sandwich the function between two differentiable functions from above and below. Our theory is widely applicable. In unsecured credit models, neither interest rates nor continuation values are globally differentiable. Nevertheless, we establish an Euler equation involving marginal prices and values. In adjustment cost models, we show that first-order conditions apply universally, even if optimal policies are not (S,s). Finally, we incorporate indivisible choices into a classic dynamic insurance analysis.

Keywords: First-order conditions, discrete choice, unsecured credit, adjustment costs, informal insurance arrangements.

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1 Introduction

First-order conditions are one of the most important tools for studying economic trade-offs. However, they seem inapplicable to many important markets. Our leading application is set in Arellano's (2008) model of unsecured credit markets, where borrowers choose between defaulting on their loans and honouring them.[1] The presence of this discrete choice causes the continuation value to be a non-differentiable function of the debt choice. But this is not the only problem. When a borrower takes on more debt, he has a higher incentive to default in the future. This default risk will be reflected in the interest rate. The borrower's intertemporal trade-off therefore involves two endogenous functions of debt: the interest rate and the continuation value. Both functions' derivatives should appear in first-order conditions, but neither derivative exists globally.

Previous techniques are inapplicable. Existing envelope theorems,[2] perturbation and variational methods do not accommodate non-differentiable interest rates. Moreover, previous envelope theorems do not simultaneously accommodate the discrete default choice and the Inada condition involving unbounded marginal utilities.

We present an envelope theorem that addresses these limitations. Consider an optimisation objective that is constructed using standard operations[3] from a collection of functions which are either known to be differentiable, or have "differentiable lower support functions" (explained below). We establish that at optimal choices, (i) this objective is differentiable, (ii) the constituent functions are differentiable, and (iii) the first-order condition holds.

In our leading application of unsecured credit markets, the objective is constructed out of a differentiable utility function and two globally non-differentiable endogenous functions, namely the continuation value and the interest rate. Small debts trade at the risk-free rate. But at a particular debt size – the risk-free limit – the interest rate increases with a kink. When the borrower chooses this kink, first-order conditions do not apply. At any other optimal debt choice, our theorem establishes that the endogenous functions are differentiable. The theorem also implies a first-order condition holds: the marginal benefit of taking on debt includes the marginal interest rate, and the marginal cost is the marginal continuation value of owing debt. This first-order condition has been used in

[1] This model is in the tradition of Eaton and Gersovitz (1981). See also Bulow and Rogoff (1989), Kehoe and Levine (1993), Cole and Kehoe (2000), Kletzer and Wright (2000), and Cooley, Marimon and Quadrini (2004). Recent work includes Aguiar and Gopinath (2006), Chatterjee, Corbae, Nakajima and Ríos-Rull (2007), Hopenhayn and Werning (2008), Aguiar, Amador and Gopinath (2009), Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012), and Aguiar, Amador, Farhi and Gopinath (2013).

[2] We discuss the important envelope theorems of Mirman and Zilcha (1975), Benveniste and Scheinkman (1979), Dechert and Nishimura (1983), Amir, Mirman and Perkins (1991), Milgrom and Segal (2002), and Cotter and Park (2006) below.

[3] The standard operations we consider are addition, multiplication, upper envelopes (maximum), and function composition.

previous work, but without establishing that the relevant derivatives exist.[4] We provide the missing link. This allows us to derive a new economic conclusion: while borrowers might optimally choose to exhaust their risk-free limits, it is not optimal to exhaust their (risky) overall limits.

Our envelope theorem involves a novel proof technique which simplifies the logic from previous envelope theorems.[5] The main ingredient is our Differentiable Sandwich Lemma. It establishes that any function F is differentiable at any point $\bar{c}$ where it is sandwiched between two differentiable functions from above and below. Specifically, the lemma applies if the two functions, which we call differentiable upper and lower support functions U and L, satisfy (i) $U(\bar{c}) = F(\bar{c}) = L(\bar{c})$, (ii) $U(c) \geq F(c) \geq L(c)$ for all c, and (iii) L and U are differentiable. We do not require any other conditions on F, such as continuity. Technically speaking, the rest of the paper is devoted to constructing appropriate upper and lower support functions. Since our focus is on differentiability at optimal choices, there is a straightforward way to a construct differentiable upper support function for any objective function: the constant function that passes through the maxima. For the differentiable lower support functions, we provide several generalisations of Benveniste and Scheinkman's (1979) construction in our applications. With these two constructions in hand, the Differentiable Sandwich Lemma establishes differentiability of objective functions at optimal choices. Finally, our Reverse Calculus Lemma establishes that if an objective function is differentiable, then all of its constituent functions (in particular the continuation value and any other endogenous function) are differentiable as well.

We present three additional applications. In our second application, firms have a fixed cost of adjusting their capital stock (or labour force, prices, etc.) in response to shocks. Open questions in the literature (discussed below) are: under which conditions are (i) optimal policies two-sided (S,s), i.e. based on cut-offs such that adjustment occurs only after sufficiently good and sufficiently bad shocks, (ii) such cut-offs differentiable functions, and (iii) optimal adjustments determined by first-order conditions. Most of the literature simply assumes that these endogenous properties are satisfied without analysing for which model parameters this is the case. We develop a general model of adjustment costs that nests most of the previous models.[6] We then show that the third criterion is always satisfied, regardless of the first two criteria (on which we do not make any progress). Specifically, we show that the value function is differentiable at optimal adjustment choices, even though it is not globally differentiable and even if the optimal adjustment policy is

[4] As we explain below, Aguiar and Gopinath (2006), Hatchondo and Martinez (2009), and Arellano and Ramanarayanan (2012) discuss such a heuristic first-order condition.

[5] The proofs of the previous envelope theorems we review are based on directional derivatives – which may not exist (see Section 2).

[6] See Khan and Thomas (2008a), Leahy (2008), and Caplin and Leahy (2010). We will discuss in particular Bar-Ilan (1990), Caballero and Engel (1999), Cooper and Haltiwanger (2006), Gertler and Leahy (2008), Khan and Thomas (2008b), and Elsby and Michaels (2012).

not (S,s).

In the third application of our envelope theorem, villagers in an agrarian economy insure each other against risks such as agricultural output and health shocks. Morten (2012) observes that in rural India, informal insurance arrangements include temporary migration to cities.[7] Her seemingly simple extension of Ligon, Thomas and Worrall (2002) is complicated because the indivisible migration choice leads the households' value functions to be neither concave nor differentiable. Nevertheless, we show that they are differentiable at optimal choices, and the analysis of Ligon *et al.* (2002) generalises in a straightforward way. Our new economic result is that even with both continuous and discrete choices, households perfectly insure against all shocks unless they are so big that autarky constraints bind. This means that the lumpy migration decision is smoothed out by reallocating the divisible consumption good.

These applications indicate our envelope theorem is widely applicable, and is especially useful when trade-offs involve endogenous functions or discrete choices. The unsecured credit market analysis can potentially be adapted to any problem involving cut-off policies, such as whether to accept an offer or to exercise an option. For example, in stochastic bargaining games, higher offers are more likely to be accepted. Our envelope theorem might be used to characterise optimal offers with first-order conditions involving marginal acceptance probabilities. Our theorem might also be applied to problems involving discrete choices, such as deciding between work, study, and vacation; migrating or staying; which candidate to vote for; how many children to have; and whether to get married.[8]

Our envelope theorem is related to several classical theorems. As our fourth application, we provide an elementary proof of the most important envelope theorem for recursive macroeconomics, the Benveniste and Scheinkman (1979) theorem. This theorem applies to smooth dynamic programming problems in which the value function is concave. In fact, if the value function is not concave, then the decision maker is locally risk-loving and can attain a strictly higher pay-off with a suitable lottery. Therefore, the theorem is applicable to dynamic programming problems that accommodate lotteries, even if the primitives are not concave.[9] However, the theorem only applies to value functions, but not to other endogenous functions. Their proof involves a lemma from convex analysis. Our proof is based on the Differentiable Sandwich Lemma. We construct the top half of the sandwich using the supporting hyperplane theorem, and retain their construction

[7] The model we study is in the tradition of Thomas and Worrall (1988) and Kocherlakota (1996), which only have continuous choices. Other important papers in this literature include Townsend (1994), Attanasio and Ríos-Rull (2000), and Krueger and Perri (2006).

[8] Rust (2008) and Aguirregabiria and Mira (2010) cite many more examples. These are typically numerical papers in the tradition of Rust (1987). Our theorem has been applied by Fella (2013) and Iskhakov, Rust and Schjerning (2012) in the design of numerical dynamic programming algorithms.

[9] Rogerson (1988) studies a model in which labour is indivisible. This leads workers to play lotteries over work obligations (or equivalently, over wealth).

for the bottom half.

For smooth but non-convex problems, Dechert and Nishimura (1983) supply an envelope theorem in the context of a growth model. Milgrom and Segal (2002, Corollary 2) accommodate discrete choices. Their result is a special case of ours, but their proof is based on directional derivatives which do not exist in general. They therefore impose superfluous conditions of equidifferentiability and bounded derivatives which we drop without imposing any new conditions. These conditions are difficult to meet in the presence of Inada conditions that require unbounded marginal utility.

To our knowledge, Santos (1991) is the only envelope theorem to depart from studying value functions. He provides sufficient conditions for the policy function to be differentiable via twice-differentiability of the value function. Our theorem is a more drastic departure, as it potentially applies to any endogenous function that might need to be differentiated in a first-order condition.

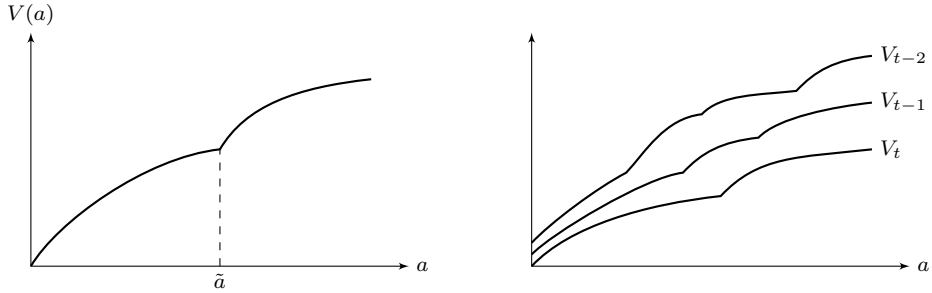
This paper is organized as follows: Section 2 surveys the threats to applying first-order conditions. Section 3 presents three lemmas and our envelope theorem. In Section 4, we apply the envelope theorem to study four applications. We conclude in Section 5. The appendix presents a technical discussion about the relationship of our technique to Fréchet subderivatives.

2 Threats to First-Order Conditions

This section surveys the threats to the validity of first-order conditions through a series of examples. All envelope theorems must either assume these threats away, or provide a reason why they do not occur.

Some envelope theorems, such as Benveniste and Scheinkman (1979), have difficulties accommodating value functions that are neither concave nor globally differentiable. The first example illustrates how such value functions arise when there are discrete choices. Suppose a worker chooses whether to work ($h = 1$) or relax ($h = 0$) based on his savings level a. He is paid a wage w and his utility is given by $u(a + wh, h)$. His value of savings, depicted in Figure 1a, may be written as $V(a) = \max_{h\in\{0,1\}} u(a + wh, h)$. At savings level $\tilde{a}$, the worker is indifferent between working and relaxing. At this indifference point, there is a discontinuous increase in the marginal value of saving, which is sometimes called a downward kink. This means the value function is neither differentiable nor concave.

The problem becomes more severe in the finite horizon version of the model depicted in Figure 1b. The worker chooses whether to work in each period. He has many possible sequences of discrete choices, each of which leads to a kink in the first-period value function. In other words, kinks from tomorrow's value function can back-propagate to kinks in today's value function. If any of these kinks are optimal choices, then first-order conditions will not be satisfied.



(a) A downward kink in a value function

(b) Backward propagation of downward kinks

Figure 1

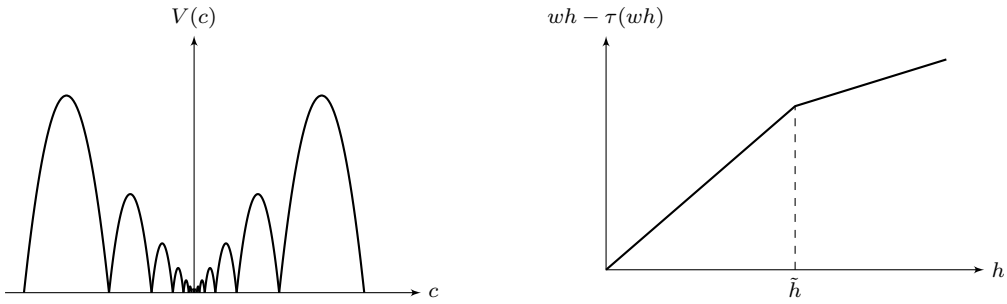
The proofs of all previous envelope theorems (discussed below) are based on directional derivatives. But in infinite horizon problems with discrete choices, the value function's directional derivatives may fail to exist. This possibility is illustrated abstractly in the "bouncing ball" function depicted in Figure 2a, which is the upper envelope of a countable set of parabolas.[10] This function has directional derivatives everywhere except at $c = 0$. In particular, the right directional derivative $V_1(0+)$ does not exist because the slope oscillates between 0 and $(\sqrt{2}-1)^2$. To avoid this problem, previous envelope theorems have imposed strong conditions on primitives, such as concavity or bounded derivatives, to ensure that directional derivatives exist. However, all of these conditions rule out studying models with discrete choices and unbounded marginal utilities of consumption.

In the first example, the value function was not differentiable because of a downward kink. Could upward kinks, such as discontinuous jumps downwards in a marginal value, also threaten first-order conditions? Consider the following example where the worker pays a progressive wage tax $\tau(wh)$ on his labour income wh. The tax is piecewise-linear, with a jump at income $\tilde{I}$ so that the after-tax labour income, $wh - \tau(wh)$ has an upward kink at $\tilde{h} = \tilde{I}/w$, depicted in Figure 2b. His value function may be written as $V(a) = \max_{h \geq 0} u(a + wh - \tau(wh), h)$. The choice corresponding with the upward kink, $\tilde{h}$ is attractive in the sense that the marginal benefit of working a bit beyond this level is low. The kink is therefore an optimal labour choice at some states.[11] This means upward kinks are another threat to the applicability of first-order conditions.

[10] The set of parabolas is $\{v(\cdot, d)\}_{d \in D}$ where

$$v(c,d) = -\frac{1}{|d|}(c-d)\left(c - \frac{d}{2}\right) \quad \text{and} \quad D = \left\{\frac{s}{2^n} : s \in \{-1,1\}, n \in \mathbb{N}\right\}.$$

[11] But it turns out that this value function V does not inherit any kinks from the budget constraint.



(a) The bouncing ball function: an abstract illustration of infinite horizon problems

(b) An upward kink in a budget constraint

Figure 2

To summarise, establishing first-order conditions requires that the relevant endogenous functions be differentiable at optimal choices. Discrete choices lead to downward kinks in value functions. In problems with multiple periods, these kinks may multiply. In infinite horizon problems, even directional derivatives may not exist. More generally, endogenous functions might exhibit both upward and downward kinks, both of which threaten the validity of first-order conditions. The following section will establish conditions under which all endogenous functions are differentiable at interior optimal choices.

3 Envelope Theorem

This section presents a method for verifying that first-order conditions are satisfied at optimal choices – even when the objective includes endogenous functions. First, we present and illustrate three lemmas which are the main steps in the method. Then, we introduce the method in the context of a simple example. Finally, we prove a theorem establishing that this method applies to a wide class of optimisation problems.

3.1 Differentiable Sandwich Lemma

Our first lemma is a general tool for establishing the differentiability of functions, and is depicted in Figure 3. Specifically, we establish that a function F is differentiable at $\bar{c}$ if it is sandwiched between two differentiable functions, from above and below. Figure 3 illustrates two examples of differentiable sandwiches. The second example is pathological; the sandwiched function is discontinuous in every open neighbourhood of the sandwich point. Nevertheless, in both examples the sandwiched functions are differentiable at the sandwich point.

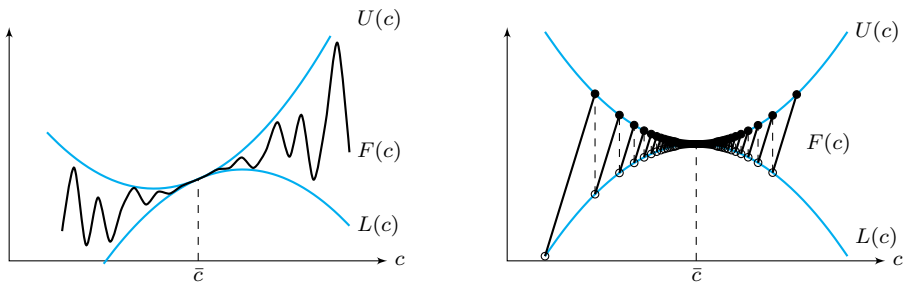


Figure 3: Differentiable Sandwich Lemma

Definition 1. *We say that $F : C \to \mathbb{R}$ is **differentiably sandwiched** between the lower and upper support functions $L, U : C \to \mathbb{R}$ at $\bar{c} \in C$ if*

*(i) L is a **differentiable lower support function** of F at $\bar{c}$, i.e. L is differentiable, $L(c) \leq F(c)$ for all $c \in C$, and $L(\bar{c}) = F(\bar{c})$, and*

*(ii) U is a **differentiable upper support function** of F at $\bar{c}$, i.e. U is differentiable, $U(c) \geq F(c)$ for all $c \in C$, and $U(\bar{c}) = F(\bar{c})$.*

Before stating the lemma, we need to be precise about what a derivative is. Since we would like to accommodate many continuous choices (such as asset portfolio choices), we use the standard multidimensional definition of differentiability. This definition is different from its one-dimensional counterpart to ensure that the chain rule and other calculus identities are valid.

Definition 2. *A function $F : C \to \mathbb{R}$ with domain $C \subseteq \mathbb{R}^n$ is **differentiable** at $c \in \operatorname{int}(C)$ if there is some row vector m with $m^\top \in \mathbb{R}^n$ such that*

$$\lim_{\Delta c \to 0} \frac{F(c + \Delta c) - F(c) - m\,\Delta c}{\|\Delta c\|} = 0. \tag{1}$$

m is called the derivative of F at c, and may be written as $F'(c)$.

In fact, this definition is almost identical to the case where the domain is a subset of a Banach space $(X, \|\cdot\|)$, and all our results and proofs in this paper generalize without amendment, as discussed in the appendix.[12]

Lemma 1 (Differentiable Sandwich Lemma). *If F is differentiably sandwiched between L and U at $\bar{c}$ then F is differentiable at $\bar{c}$ with $F'(\bar{c}) = L'(\bar{c}) = U'(\bar{c})$.*

[12] In Banach spaces, the derivative m is called a "Fréchet derivative" and lies in the topological dual space $X^* = \{m : X \to \mathbb{R}$ such that m is linear and continuous$\}$. For our purposes, it is unnecessary to define a topology on X^* because all limits are taken in $(X, \|\cdot\|)$ and $\mathbb{R}$.

Proof. The difference function $d(c) = U(c) - L(c)$ is minimized at $\bar{c}$. Therefore, $d'(\bar{c}) = 0$ and we conclude $L'(\bar{c}) = U'(\bar{c})$.

Let $m = L'(\bar{c}) = U'(\bar{c})$. For all Δc,

$$\frac{L(\bar{c} + \Delta c) - F(\bar{c}) - m\,\Delta c}{\|\Delta c\|} \leq \frac{F(\bar{c} + \Delta c) - F(\bar{c}) - m\,\Delta c}{\|\Delta c\|} \leq \frac{U(\bar{c} + \Delta c) - F(\bar{c}) - m\,\Delta c}{\|\Delta c\|}. \quad (2)$$

Consider the limits as $\Delta c \to 0$. Since $L'(\bar{c}) = U'(\bar{c}) = m$, the limits of the first and last fractions are 0. By Gauss' Squeeze Theorem, we conclude that the limit in the middle is also 0, and hence that F is differentiable at $\bar{c}$ with $F'(\bar{c}) = m$. □

Remark 3.1. *The requirement that $F : C \to \mathbb{R}$ be globally sandwiched between L and U on all of C can be relaxed. The function F can be restricted to any domain $C' \subseteq C$ such that $\bar{c} \in \operatorname{int}(C')$. Therefore, F only needs to be locally sandwiched between L and U for the Differentiable Sandwich Lemma to apply.*

We informally discuss the role of the support functions in the one-dimensional case. If a function F has a lower support function L, then this rules out "upward kinks" in which the left derivative is greater than the right derivative. Similarly, if F has an upper support function U, then "downward kinks" are ruled out. If F is fully sandwiched between L and U, then it has neither upward nor downward kinks, and is differentiable. Differentiable upper and lower support functions are related to Fréchet sub- and superderivatives. We refer to the appendix for a discussion.

The sandwich approach avoids the use of directional derivatives as they do not exist in general (see the previous section). Previous envelope theorems imposed assumptions such as concavity,[13] Lipschitz continuity,[14] equidifferentiability,[15] or supermodularity[16] to ensure the existence of directional derivatives. But, these assumptions can be avoided with our method.

3.2 Maximum Lemma

For the Differentiable Sandwich Lemma to be useful, there must be a way to construct differentiable upper and lower support functions. The following lemma gives a simple construction – a horizontal line or plane – that is a differentiable upper support function above the maximum of any function (see Figure 4).

[13] Benveniste and Scheinkman (1979)
[14] Clarke (1975)
[15] Milgrom and Segal (2002)
[16] Amir *et al.* (1991)

Lemma 2 (Maximum Lemma). *Let $\phi : C \to \mathbb{R}$ be a function. If $\hat{c} \in \text{int}(C)$ maximises ϕ, then ϕ has a differentiable upper support function at $\hat{c}$.*

Proof. $U(c) = \phi(\hat{c})$ has derivative $U'(c) = 0$ for all c. □

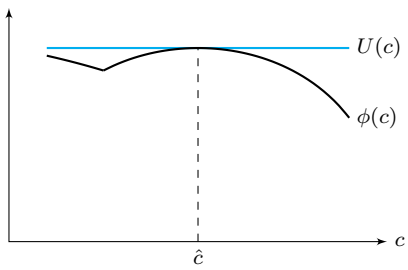


Figure 4: Maximum Lemma

3.3 Reverse Calculus

Since the construction of a differentiable upper support function is straightforward, it might seem that establishing first-order conditions merely involves finding a differentiable lower support function. However, there is an additional difficulty. For example, consider the objective $\phi(c) = F(c) + G(c)$, where F and G have differentiable lower support functions f and g. At an optimal choice $\hat{c}$, ϕ is differentiable because it is sandwiched between $U(c) = \phi(\hat{c})$ and $L(c) = f(c) + g(c)$. This establishes the first-order condition $\phi'(\hat{c}) = U'(\hat{c}) = 0$. However, in most optimisation problems, such a first-order condition is unhelpful. Economic insights are typically obtained by expanding the marginal objective ϕ' to arrive at a first-order condition such as $F'(\hat{c}) + G'(\hat{c}) = 0$. However, this expansion is *not* justified until we can establish F and G are differentiable at $\hat{c}$.

Calculus solves the opposite problem. Calculus involves rules such as "if F and G are differentiable at $\bar{c}$, then $H(c) = F(c) + G(c)$ is also differentiable at $\bar{c}$." We wish to show the converse, that because $\phi(c) = F(c) + G(c)$ is differentiable, both F and G are differentiable at $\bar{c}$.

In the above simple addition problem, in fact F must be differentiable at $\hat{c}$, because F is sandwiched between f and $U(c) = \phi(c) - g(c)$. A similar sandwich can be constructed for G. Therefore, we can indeed conclude that if $\hat{c}$ is an interior maximiser of ϕ, then $F'(\hat{c}) + G'(\hat{c}) = 0$.

The following lemma generalises this logic to several standard mathematical operations. It is important because we will need it to establish that all endogenous functions, included in our first-order conditions, are in fact differentiable.

Lemma 3 (Reverse Calculus). *Suppose $F : C \to \mathbb{R}$ and $G : C \to \mathbb{R}$ have differentiable lower support functions at $\bar{c}$.*

(i) If $H(c) = F(c) + G(c)$ is differentiable at $\bar{c}$, then F is differentiable at $\bar{c}$.

(ii) If $H(c) = F(c)G(c)$ is differentiable at $\bar{c}$ and $F(\bar{c}) > 0$ and $G(\bar{c}) > 0$, then F is differentiable at $\bar{c}$.

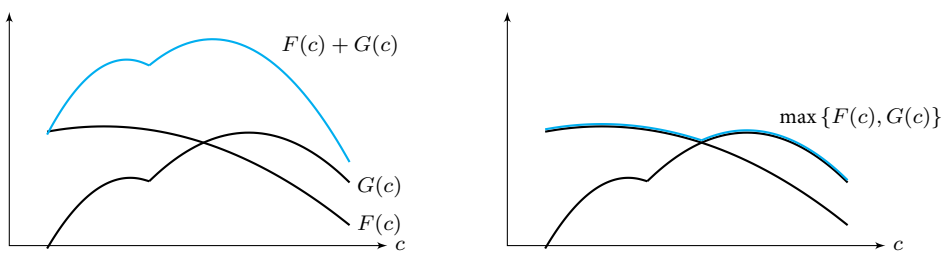
(iii) If $H(c) = \max\{F(c), G(c)\}$ is differentiable at $\bar{c}$ and $F(\bar{c}) = H(\bar{c})$, then F is differentiable at $\bar{c}$.

(iv) If $H(c) = J(F(c))$ and $J : \mathbb{R} \to \mathbb{R}$ are differentiable at $\bar{c}$ and $F(\bar{c})$ respectively with $J'(F(\bar{c})) \neq 0$, then F is differentiable at $\bar{c}$.[17]

Proof. Let f and g be differentiable lower support functions of F and G at $\bar{c}$. For (i)–(iii), we sandwich F between f and an appropriate differentiable upper support function U and apply the Differentiable Sandwich Lemma (Lemma 1). Appropriate upper support functions are (i) $U(c) = H(c) - g(c)$, (ii) $U(c) = H(c)/g(c)$, and (iii) $U(c) = H(c)$.

For (iv), $F(c) = J^{-1}(H(c))$ is differentiable at $\bar{c}$ by the inverse function theorem and the chain rule. □

These rules have simple geometric interpretations as illustrated in Figure 5. The first rule says that if a differentiable function is the sum of two functions that have no upward kinks, then they have no downward kinks either.



(a) Addition (b) Maximum

Figure 5: Illustration of the Reverse Calculus Lemma

[17] A generalisation is possible that accommodates J only having a differentiable lower support function.

3.4 Illustration: Indivisible Labour

Our tools above may be applied directly to economic models to establish first-order conditions. We now present a recipe for applying them to an indivisible labour choice problem. In the next section, we show that the same recipe applies to a general class of models. The two sections can be read in parallel – as they follow the same logic – albeit in very different settings.

Problem. Each period, a worker chooses consumption c, savings a' which bring a return of $\theta' a'$, and an indivisible labour supply h' which pays a wage w. The rate of return on savings $\theta \in \{\theta_g, \theta_b\}$ follows a Markov process with transition probability $p(\theta'|\theta)$. We assume that the worker's utility function $u(c, h)$ is differentiable with respect to consumption c for all labour choices h. The worker's value function is

$$\begin{aligned} V(a,\theta) &= \max_{c,a',h} u(c,h) + \sum_{\theta' \in \{\theta_g,\theta_b\}} p(\theta'|\theta)\beta V(a',\theta') \\ \text{s.t.} \quad & c + a' = \theta a + wh \quad \text{and} \quad h \in \{0,1\}\,. \end{aligned} \tag{3}$$

Note that V is neither globally differentiable nor concave due to the presence of the indivisible labour choice (see Section 2). Nevertheless, we will show that $V(\cdot, \theta)$ is differentiable at optimal choices. First, it will be convenient to reformulate the worker's objective as

$$\phi(a'; a, \theta) = \max\{u(\theta a + w - a', 1), u(\theta a - a', 0)\} + \sum_{\theta'} p(\theta'|\theta)\beta V(a', \theta'). \tag{4}$$

Differentiable Lower Support Functions. To apply our envelope theorem, the main task is to construct differentiable lower support functions for the endogenous functions $V(\cdot, \theta_g)$ and $V(\cdot, \theta_b)$. To this end, we use a "lazy" value function based on a construction employed by Benveniste and Scheinkman (1979). Consider a lazy worker who does not know his optimal policy functions $\hat{a}''(a', \theta')$ and $\hat{h}'(a', \theta')$ for all states (a', θ'). Rather, he only knows the optimal choices for a particular state, $(\bar{a}', \bar{\theta}')$, namely $\bar{a}'' = \hat{a}''(\bar{a}', \bar{\theta}')$ and $\bar{h}' = \hat{h}'(\bar{a}', \bar{\theta}')$. If he discovers that he is in a different state, (a', θ'), then he is too lazy to reconsider his choice and chooses $(\bar{a}'', \bar{h}')$. This lazy worker's value function is

$$L(a', \bar{\theta}'; \bar{a}') = u(\bar{\theta}' a' + w\bar{h}' - \bar{a}'', \bar{h}') + \sum_{\theta''} p(\theta''|\bar{\theta}')\beta V(\bar{a}'', \theta''). \tag{5}$$

Since a' only enters this function in the first term in a simple way, the lazy value function is differentiable at $\bar{a}'$ with

$$L_1(\bar{a}', \bar{\theta}'; \bar{a}') = u_1(\bar{\theta}'\bar{a}' + w\bar{h}' - \bar{a}'', \bar{h}'). \tag{6}$$

First-Order Conditions. We provide three first-order conditions, all of which hold at the optimal choices $\hat{c} = \hat{c}(a, \theta) > 0$ and $\hat{a}' = \hat{a}'(a, \theta) > 0$. First, we claim that $\phi(\cdot; a, \theta)$ is differentiable at $\hat{a}'$ with

$$\phi_1(\hat{a}'; a, \theta) = 0. \tag{7}$$

Second, we claim that then $V(\cdot, \theta_g)$ and $V(\cdot, \theta_b)$ are differentiable at $\hat{a}'$ with

$$u_1(\hat{c}, \hat{h}(a, \theta)) = \sum_{\theta'} p(\theta'|\theta)\beta V_1(\hat{a}', \theta'). \tag{8}$$

Finally, we claim that the Euler equation holds:

$$u_1(\hat{c}, \hat{h}(a, \theta)) = \sum_{\theta'} p(\theta'|\theta)\beta u_1(\hat{c}'(\hat{a}', \theta'), \hat{h}'(\hat{a}', \theta')). \tag{9}$$

The recursive first-order condition (8) involves an endogenous value function, whereas the Euler equation (9) does not. These two first-order conditions are complementary. The first is based on the dynamic programming notion of focusing on one trade-off at a time, and is often more intuitive and well-suited to numerical calculations.[18] On the other hand, in this particular example, the second first-order condition is simpler in the sense that it does not involve any endogenous functions. In other applications, such as the debt application of Section 4.1, both types of first-order conditions include the derivative of an endogenous function.

Proof. We show how to apply our tools to establish the results above. First, we establish the objective ϕ is differentiable at any interior optimal choice $\hat{a}' = \hat{a}'(\bar{a}, \bar{\theta})$. We apply Lemma 1 by sandwiching $\phi(\cdot; \bar{a}, \bar{\theta})$ between a horizontal upper support function (Lemma 2) and a differentiable lower support function based on the lazy worker construction,

$$\psi(a'; \bar{a}, \bar{\theta}) = u(\theta\bar{a} + w\hat{h}(\bar{a}, \bar{\theta}) - a', \hat{h}(\bar{a}, \bar{\theta})) + \sum_{\theta'} p(\theta'|\bar{\theta})\beta L(a', \theta'; \hat{a}'). \tag{10}$$

This establishes that the first-order condition

$$\phi_1(\hat{a}'(\bar{a}, \bar{\theta}); \bar{a}, \bar{\theta}) = 0$$

is well-defined and holds for all $(\bar{a}, \bar{\theta})$.

Second, we expand this first-order condition. We must establish that $V(\cdot, \theta_g)$ and $V(\cdot, \theta_b')$ are differentiable before including their derivatives in the first-order conditions. We repeatedly apply Lemma 3 to prove in turn that the following functions are differentiable functions of a' at $\hat{a}'$:

[18] Fella (2013) and Iskhakov *et al.* (2012) apply our theorem to generalize the endogenous grid method by Carroll (2006).

(i) $\max\{u(\theta a + w - a', 1), u(\theta a - a', 0)\}$ and $\sum_{\theta'} p(\theta'|\theta)\beta V(a', \theta')$,

(ii) $p(\theta' = \theta_g|\theta)\beta V(a', \theta_g)$ and $p(\theta' = \theta_b|\theta)\beta V(a', \theta_b)$,

(iii) $V(a', \theta_g)$ and $V(a', \theta_b)$.

Since all of these functions are differentiable at $\hat{a}'$, standard rules of calculus imply the first-order condition (8).

Finally, we establish the Euler equation (9). By Lemma 1, the derivatives of the endogenous functions $V(\cdot, \theta_g)$ and $V(\cdot, \theta_b)$ coincide with those of their lower support functions. Therefore, we may substitute the derivative of the lower support functions (6) into (8) to obtain the Euler equation.

Discussion. We are not aware of any other envelope theorem for establishing the recursive first-order condition (8). Even though we established that the value functions are differentiable at the optimal choices, they are neither globally concave nor globally differentiable. On the other hand, the Euler equation (9) in this illustration can be obtained without using our method. Since $\hat{a} = \hat{a}'(a, \theta)$ maximises the differentiable function $\psi(\cdot; a, \theta)$, the first-order condition $\psi_1(\hat{c}; a, \theta) = 0$ holds, and implies (9). Therefore, the role of our envelope theorem – like other envelope theorems – is to establish first-order conditions involving endogenous functions.

There are three important antecedents of our approach for one-dimensional continuous choice spaces (i.e. when $C = \mathbb{R}$). However, all of them make use of left and right derivatives – which do not exist in general (see Section 2) – rather than differentiable sandwiches to prove that the value function is differentiable at optimal choices. Dechert and Nishimura (1983, Corollary 2) and Amir *et al.* (1991, Lemma 3.4) supply specialized results in the context of non-convex growth models, the latter by assuming supermodularity.[19] Milgrom and Segal (2002, Corollary 2) applies more generally than these earlier results and accommodates discrete choices without any topological or monotonicity assumptions. However, as discussed in the introduction, it imposes superfluous requirements of equidifferentiability and bounded derivatives to ensure that left and right derivatives exist.[20] This impedes their theorem from being applied to dynamic programming problems with unbounded marginal utilities.

Another approach by Morand, Reffett and Tarafdar (2012) applies the envelope theorem of Clarke (1975) to non-convex and non-smooth dynamic programming programs. Similar to Clarke (1975), they impose local Lipschitz continuity. This paper is weakly related to our approach in that they use Dini derivatives (see the appendix). All of the above envelope theorems only apply to value functions and not to other endogenous functions.

[19] The supermodularity approach has recently been applied by Menzio, Shi and Sun (2013) to a discrete choice problem in a model of non-degenerate distribution of money holdings.

[20] Cotter and Park (2006) establish an envelope theorem for smooth dynamic programming problems with non-concave utility functions using the approach of Milgrom and Segal (2002).

3.5 Theorem

Our recipe presented above applies generally. We now show that if an objective is constructed out of endogenous functions using standard mathematical operations, then those functions' derivatives may be included in first-order conditions provided that they have differentiable lower support functions.

To establish this result, we must be more precise about what it means to construct a function out of other functions. In the illustration above, the objective $\phi(\cdot; a, \theta)$ is constructed from the three endogenous functions $a' \mapsto -a'$, $V(\cdot, \theta^G)$, and $V(\cdot, \theta^B)$ using four operations: function addition, function multiplication, function composition, and taking the upper envelope of a set of functions. We define an envelope algebra as the set of all functions that may be constructed from a set of (endogenous) functions. Our definition is recursive to accommodate the idea that once we construct a function, we can use that function to construct other functions.

Let $\mathcal{F}(C)$ be the set of functions with domain C and co-domain $\mathbb{R}$.

Definition 3. *We say $\mathcal{E} \subseteq \mathcal{F}(C)$ is an **envelope algebra** if:*

(i) $F + G \in \mathcal{E}$ *for all* $F, G \in \mathcal{E}$,

(ii) $FG \in \mathcal{E}$ *for all* $F, G \in \mathcal{E}$ *with* $F, G : C \to \mathbb{R}_{++}$,

(iii) $H(c) = \max_{G \in \mathcal{G}} G(c)$ *is in* $\mathcal{E}$ *for all* $\mathcal{G} \subseteq \mathcal{E}$ *provided it is well-defined, and*

(iv) $J \circ F \in \mathcal{E}$ *for all* $F \in \mathcal{E}$ *and all differentiable* $J : \mathbb{R} \to \mathbb{R}$ *with* $J_1 : \mathbb{R} \to \mathbb{R}_{++}$.

Definition 4. *The **generated envelope algebra** $\mathcal{E}(\mathcal{F})$ is the smallest envelope algebra generated by $\mathcal{F} \subseteq \mathcal{F}(C)$ that contains $\mathcal{F}$.*

For our purposes, the envelope algebra consists of all of the functions we can construct out of the endogenous functions $\mathcal{F}$. In the illustration above, $\mathcal{F} = \{a' \mapsto -a', V(\cdot, \theta_g), V(\cdot, \theta_b)\}$ and $\mathcal{E}(\mathcal{F})$ is an infinite set of functions. In particular, $\phi(\cdot; a, \theta) \in \mathcal{E}(\mathcal{F})$ for all (a, θ).

The following lemma establishes that if all of the endogenous functions $\mathcal{F}$ have differentiable lower support functions, then so do all of the functions constructed out of them. In particular, this means that the objective $\phi(\cdot; a, \theta)$ has a differentiable lower support function.

Lemma 4. *Let $\mathcal{F} \subseteq \mathcal{F}(C)$ be a set of functions that have a differentiable lower support function at $\bar{c} \in \text{int}(C)$. Then every $F \in \mathcal{E}(\mathcal{F})$ has a differentiable lower support at $\bar{c}$.*

Now, we turn our attention to applying the Reverse Calculus Lemma (Lemma 3). This process begins with the knowledge that the objective ϕ is differentiable, and proceeds to establish that its components are also differentiable. Therefore, the recursion must proceed in the opposite direction from before. Moreover, some components may not be

locally relevant for the objective if that component is not on the upper envelope. For example, the worker's effort choice is irrelevant if he decides to stay at home. We call the relevant components the active envelope set.

Definition 5. *Fix any $(\mathcal{E}, \phi, \bar{c})$ such that $\mathcal{E}$ is an envelope algebra, $\phi \in \mathcal{E}$, and $\bar{c} \in C$. We define the* ***active envelope set*** *$\mathcal{A}(\mathcal{E}, \phi, \bar{c})$ as the smallest set $\mathcal{A} \subseteq \mathcal{E}$ such that*

(i) $\phi \in \mathcal{A}$.

(ii) If $F, G \in \mathcal{E}$ and $F + G \in \mathcal{A}$, then $F, G \in \mathcal{A}$.

(iii) If $F, G \in \mathcal{E}$ and $F, G : C \to \mathbb{R}_{++}$ and $FG \in \mathcal{A}$, then $F, G \in \mathcal{A}$.

(iv) If $F \in \mathcal{G} \subseteq \mathcal{E}$ and $H(c) = \sup_{G \in \mathcal{G}} G(c)$ is in $\mathcal{A}$ and $F(\bar{c}) = H(\bar{c})$, then $F \in \mathcal{A}$.

(v) If $J \circ F \in \mathcal{A}$ where $J : \mathbb{R} \to \mathbb{R}$ is differentiable and $J_1 : \mathbb{R} \to \mathbb{R}_{++}$, then $F \in \mathcal{A}$.

Finally, we can state our main result. Informally speaking, the theorem says the following. Suppose an objective function ϕ is constructed out of functions, all of which have differentiable lower support functions. Then, at any interior optimal choice, (i) the objective and the relevant constituent functions are differentiable, and (ii) a first-order condition holds. In the illustration above, the theorem establishes that the endogenous functions $V(\cdot, \theta_g)$ and $V(\cdot, \theta_b)$ are differentiable at any optimal choice $\hat{a}'$.

Theorem 1 (Envelope Theorem). *Let $\mathcal{F} \subseteq \mathcal{F}(C)$ be a set of functions that have a differentiable lower support function at $\hat{c} \in \operatorname{int}(C)$. If $\phi \in \mathcal{E}(\mathcal{F})$ and $\hat{c} \in \arg\max_{c \in C} \phi(c)$, then (i) every function in the active function set $\mathcal{A}(\mathcal{E}(\mathcal{F}), \phi, \hat{c})$ is differentiable at $\hat{c}$, and (ii) $\phi_1(\hat{c}) = 0$.*

Proof. Since $\phi \in \mathcal{E}(\mathcal{F})$ and the envelope algebra $\mathcal{E}(\mathcal{F})$ is generated from functions with differentiable lower support functions at $\hat{c}$, Lemma 4 implies that ϕ has a differentiable lower support function at $\hat{c}$. Since $\hat{c}$ maximises ϕ, Lemma 2 establishes that ϕ has a differentiable upper support function at the maximum $\hat{c}$. Therefore, ϕ is sandwiched between two differentiable functions, so Lemma 1 implies that it is differentiable at $\hat{c}$. Moreover, $\phi'(\hat{c})$ coincides with the derivative of its upper support function, which is 0.

We prove by induction that every function in the active set $\mathcal{A} = \mathcal{A}(\mathcal{E}(\mathcal{F}), \phi, \hat{c})$ is differentiable at $\hat{c}$. We set $\mathcal{A}^1 = \{\phi\}$. To construct $\mathcal{A}^{n+1}$, we examine each $H \in \mathcal{A}^n$. For each part of Lemma 3, we select appropriate functions F and G from $\mathcal{E}(\mathcal{F})$, and conclude that F is differentiable at $\hat{c}$. We do this for every possible combination of F and G, and include each such F in $\mathcal{A}^{n+1}$. We repeat this a countable number of times, and observe that $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{A}^n$. □

Remark 3.2. *Active functions share their derivatives with their lower support functions. It is often easier to calculate the derivatives of endogenous functions by differentiating their lower support functions.*

Theorem 1 establishes the method from Section 3.4 applies to a wide class of optimisation problems. In the next section, we apply the method to three important problems in which first-order conditions previously seemed inapplicable. While the general setting of the theorem is quite abstract, the method itself is quite intuitive. Therefore, we find it clearer to repeat the logic of applying the lemmas each time rather than applying the abstract theorem.

4 Applications

4.1 Unsecured Credit

Our first application is about unsecured debt contracts where borrowers may decide to either repay in full or to default. We focus on markets without collateral such as sovereign debt markets. The punishment for default is exclusion from the credit market thereafter. Nevertheless, default occasionally occurs so interest paid by the borrower must compensate for the default risk.[21] For this reason, the interest charged is non-linear and determined by a recursive relationship with the borrower's value function. If the interest rates are low, then the borrower's value of honouring debt contracts is high because rolling over debt is cheap. Conversely, if the borrower's value of repaying is high tomorrow, then the default risk today is low. This recursive relationship determines interest rates as a function of loan size and the credit limit.

The borrower's decision problem is poorly behaved for two related reasons. First, the discrete repayment choice leads to jumps in the marginal value of owing debt. Second, following marginal changes in debt, these jumps lead to kinks in the default risk and hence kinks in the interest rate. In other words, neither the value function nor the budget constraint are globally differentiable. Nevertheless, we apply our envelope theorem to establish that both endogenous functions – the value function and the interest rate – are differentiable at optimal debt choices (except for choices at the endogenous risk-free credit limit). Hence, first-order conditions apply and we can establish an Euler equation involving a marginal interest rate and a marginal continuation value. We then apply our envelope theorem to characterise the borrower's credit limit and reach our conclusion that the borrower never exhausts his endogenous credit limit.

We build on the unsecured credit analysis by Arellano (2008) which is in the tradition of Eaton and Gersovitz (1981). Arellano carefully analyses it theoretically and numerically. She also sketches a Laffer curve for the debt choice, but – without first-order conditions – does not characterise borrower behaviour along it. The following three papers apply some Euler equations, with the first explicitly acknowledging that they lack justification for differentiating the interest rates with respect to loan size. We provide a

[21] Default need not be inefficient compared to risk-free debt, as it implements risk-sharing.

justification. Aguiar and Gopinath (2006) dropped a detailed discussion of their heuristic (but now verified) Euler equation from their NBER working paper version. Similarly, we verify the heuristic Euler equations that Arellano and Ramanarayanan (2012) use to compare maturity structures of loans. Finally, Hatchondo and Martinez (2009) discuss an Euler equation, implicitly assuming differentiability of interest rates. None of these papers use first-order conditions to investigate credit limits, nor deduce our result that borrowers never exhaust their credit limits.

Model. A risk-averse borrower has a differentiable utility function u and discount factor $\beta \in (0, 1)$. The borrower's marginal value of consumption at zero is infinite, i.e. $\lim_{c \to 0^+} u_1(c) = \infty$. Every period, the borrower receives an endowment x which is independently and identically distributed with density $f(\cdot)$ on the support $[x^{\min}, x^{\max}]$. We assume the borrower's endowment is bounded away from zero, i.e. $x^{\min} > 0$. To smooth out endowment shocks, the borrower may take out loans from a lender with deep pockets. We focus our attention on debt contracts of the following form. The borrower offers to pay a lender b' in the following period, but only promises to honour this debt obligation if the endowment tomorrow, x', lies in the set H'. Thus, a debt contract consists of (b', H'), both of which are chosen by the borrower. The lender is risk-neutral, discounts time at the same rate, and is therefore willing to pay $\beta \int_{H'} f(x')\, dx' b'$ in return for the promise. If the borrower defaults, he is excluded from credit markets thereafter. We also accommodate an additional exogenous sanction of $s \geq 0$ units of consumption every period for defaulting, which reflects the difficulty of settling non-financial transactions without credit.[22] The borrower's autarky value after defaulting is

$$W_{\text{aut}}(x) = u(x - s) + \beta \int_{[x^{\min}, x^{\max}]} W_{\text{aut}}(x') f(x') dx'. \tag{11}$$

The lender only agrees to the contract (b', H') if the borrower has an incentive to honour the promise for the proposed endowments H'. Specifically, the borrower's value of repaying b' at an honour endowment $x' \in H'$, denoted $W_{\text{hon}}(b', x')$, should not be less

[22] Exogenous sanctions are often included in unsecured credit models, so we include them to show the generality of our technique. Without them, Bulow and Rogoff (1989) show that exclusion from credit markets alone is an insufficient punishment for enforcing debt contracts if the borrower can make private investments.

than the autarky value $W_{\text{aut}}(x)$. The borrower's value of honouring debts is therefore[23]

$$\begin{aligned} W_{\text{hon}}(b,x) = \max_{c,b',H'} \; & u(c) + \beta \int_{H'} W_{\text{hon}}(b',x') f(x') dx' + \beta \int_{[x^{\min},x^{\max}]\setminus H'} W_{\text{aut}}(x') f(x') dx', \\ \text{s.t.} \quad & c + b = x + \left[\beta \int_{H'} f(x')\, dx' \right] b', \\ & W_{\text{hon}}(b',x') \geq W_{\text{aut}}(x') \text{ for all } x' \in H', \\ & b' \leq b^{\text{ponzi}}. \end{aligned} \tag{12}$$

The last constraint rules out Ponzi schemes and the b^{ponzi} parameter may be arbitrarily large.

Reformulation. We reformulate this problem by making two simplifications. First, Arellano (2008, Proposition 3) established that because x is IID, the honour set H' chosen by the borrower is determined by a cut-off rule $y(\cdot)$ so that the borrower honours his debt at state (b', x') if and only if $x' \geq y(b')$. In other words, the borrower only ever chooses debt contracts of the form $(b', H') = (b', [y(b'), x^{\max}])$, so debt contracts are characterised by b' alone. This means we may denote the price of debt $q(b')$ as a function of b'. Second, it is convenient to work with the net value function $W(b,x) = W_{\text{hon}}(b,x) - W_{\text{aut}}(x)$. The reformulated problem becomes

$$W(b,x) = \max_{b' \leq b^{\text{ponzi}}} u(x + q(b')b' - b) - u(x - s) + \beta V(b'), \tag{13}$$

where

$$V(b') = \int_{[y(b'),x^{\max}]} W(b',x') f(x') dx', \tag{14a}$$

$$q(b') = \beta[1 - F(y(b'))], \tag{14b}$$

$$y(b') = \begin{cases} x^{\min} & \text{if } W(b',x^{\min}) > 0, \\ x^{\max} & \text{if } W(b',x^{\max}) < 0, \\ \min\{x' \in [x^{\min}, x^{\max}] : W(b',x') = 0\} & \text{otherwise.} \end{cases} \tag{14c}$$

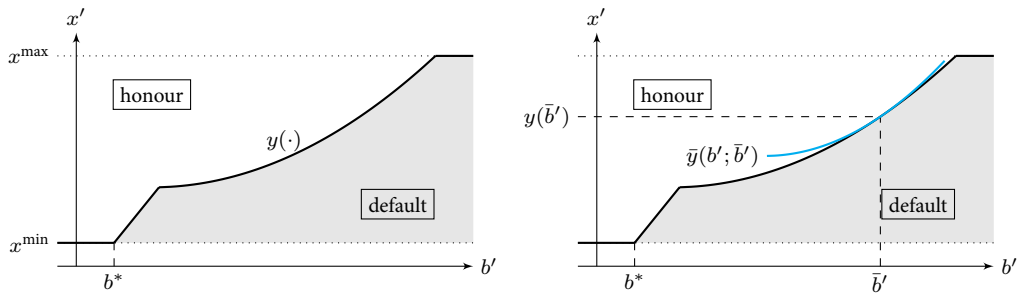
In this reformulation, the borrower's only choice is his future debt obligation b'. We denote optimal policy functions by $\hat{b}'(b,x)$.[24]

The objective (13) has two endogenous functions, q and V, which we will show are not globally differentiable. The value function has downward kinks at states of indifference between honouring and defaulting, as in the value function of the indivisible labour

[23] We mention some technicalities: (i) the borrower should be constrained to choosing a measurable honour set, and (ii) the Bellman operator is well-defined for continuous value functions.

[24] The borrower might be indifferent between several optimal policies.

choice illustration. Moreover, we have no a priori knowledge of the differentiability of the debt price. We will construct differentiable lower support functions for q and V and hence show that they both do not exhibit upward kinks at any choice, with one exception: The debt price exhibits an upward kink at the risk-free credit limit.



(a) Borrowers default when $x' < y(b')$

(b) A "lazy" borrower undervalues honouring debts, and defaults too much

Figure 6: The default cut-off rule

Differentiable Lower Support Functions. The problem of constructing a differentiable lower support function for the debt price $q(\cdot)$ is equivalent to that of constructing a differentiable upper support function for the cut-off rule $y(\cdot)$, illustrated in Figure 6a. For debts below some threshold b^*, the borrower always honours his obligations, so the cut-off $y(\cdot)$ is constant and hence differentiable on $[-\infty, b^*)$. At each debt level $\bar{b}' > b^*$, we now construct a differentiable upper support function for $y(\cdot)$. We consider a lazy borrower that – as a consequence of his laziness – undervalues honouring debts, and hence uses a higher cut-off than $y(\cdot)$. Specifically we consider a lazy borrower who incorrectly anticipates the state to be $(b', x') = (\bar{b}', y(\bar{b}'))$, i.e. he anticipates his state will be on the cut-off. In unanticipated states, he chooses his debt to be $\hat{b}''(\bar{b}', y(\bar{b}'))$ independently of the realized endowment x'. His consumption is adjusted by the differences from the anticipated endowment and debt. This lazy borrower's net value function is

$$L(b', x'; \bar{b}') = u(x' - b' + q(\bar{b}'')\bar{b}'') - u(x' - s) + \beta V(\bar{b}''). \tag{15}$$

Since the lazy borrower undervalues honouring debts, his honour cut-off $\bar{y}(\cdot; \bar{b}')$ implicitly defined by

$$L(b', \bar{y}(b'; \bar{b}'); \bar{b}') = 0 \tag{16}$$

provides an upper support function for the cut-off $y(\cdot)$ at $\bar{b}'$, depicted in Figure 6b. Since the lazy borrower's value function is differentiable, the implicit function theorem implies

that $\bar{y}(\cdot;\bar{b}')$ is differentiable with $y_1(\bar{b}';\bar{b}') > 1$ for all $\bar{b}' > b^*$.[25]

Thus far, we have established that the slope of the cut-off $y(\cdot)$ is zero approaching the risk-free limit b^* from the left, but greater than one approaching b^* from the right. Therefore, the cut-off has a downward kink at b^*, so it has no differentiable upper support function at this point. This means we have established:

Lemma 5. *At every $\bar{b}' \neq b^*$, there exists a differentiable upper support function $\bar{y}(\cdot;\bar{b}')$ for $y(\cdot)$, and hence a differentiable lower support function $\underline{q}(\cdot;\bar{b}')$ for $q(\cdot)$. Moreover, $y(\cdot)$ has an downward kink at b^* with $0 = y'(b^*-) < 1 < y'(b^*+)$.*

To construct a differentiable lower support function for V, we begin by constructing a differentiable lower support function for $W(b', x')$. However, this time, we use a different lazy borrower's value function from the one used to construct (15). This time, the lazy borrower correctly anticipates x', but incorrectly anticipates b' to be $\bar{b}'$. He takes on a debt of $\bar{b}''(x') = \hat{b}''(\bar{b}', x')$ independently of his previous obligation of b'. His value function is

$$M(b', x'; \bar{b}') = u(x' - b' + q(\bar{b}''(x'))\bar{b}''(x')) - u(x' - s) + \beta V(\bar{b}''(x')). \tag{17}$$

This means that,

$$\underline{V}(b'; \bar{b}') = \int_{\bar{y}(b';\bar{b}')}^{x^{\max}} M(b', x'; \bar{b}') f(x')\, dx' \tag{18}$$

is a lower support function for V at $\bar{b}'$. We would like to establish that $\underline{V}(\cdot;\bar{b}')$ is differentiable. First, $M(\cdot, x'; \bar{b}')$ is continuously differentiable for all $(x', \bar{b}')$. Second, we note that without loss of generality, we may assume some optimal policy $\hat{b}''(\cdot,\cdot)$ is measurable, and hence the resulting lazy policy $\bar{b}''(\cdot)$ is also measurable.[26] Third, the measurability of the lazy policy implies that $M_1(b', \cdot; \bar{b}')$ is measurable for all $(b', \bar{b}')$. Moreover, it is possible to show that $M_1(b', \cdot; \bar{b}')$ is uniformly bounded for all b' in some open neighbourhood of $\bar{b}'$. Hence the Leibniz rule for differentiating under the integral sign implies that $\underline{V}(\cdot;\bar{b}')$ is differentiable at $b' = \bar{b}'$ with[27]

$$\underline{V}_1(b'; \bar{b}') = \int_{\bar{y}(b';\bar{b}')}^{x^{\max}} M_1(\bar{b}', x'; \bar{b}') f(x')\, dx'. \tag{19}$$

This means we have established:

Lemma 6. *At every $\bar{b}'$, there exists a differentiable lower support function $\underline{V}(\cdot;\bar{b}')$ for V.*

[25] Apply the implicit function theorem on the lazy borrower's value function to get

$$\bar{y}_1(\bar{b}'; \bar{b}') = \frac{u_1(\bar{c}'(\bar{b}', y(\bar{b}')))}{u_1(\bar{c}'(\bar{b}', y(\bar{b}'))) - u_1(x' - s)} > 1.$$

[26] See for example the Measurable Maximum Theorem in Aliprantis and Border (2006, Theorem 18.19).

[27] See for example Weizsäcker (2008, Theorem 4.6).

First-Order Conditions. We can now return to the original problem (13). If $\hat{b}'$ is an optimal debt choice at the state (b, x), then it maximises

$$\phi(b'; b, x) = u(x - b + q(b')b') - u(x - s) + \beta V(b'). \tag{20}$$

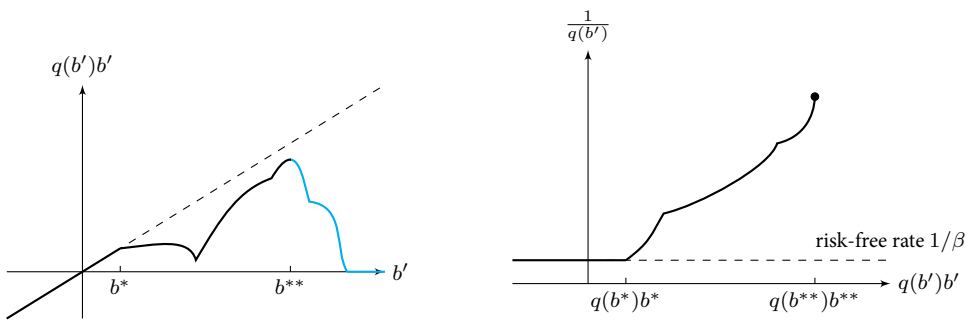
Using $\underline{q}(\cdot; \bar{b}')$ and $\underline{V}(\cdot; \bar{b}')$, we can construct a differentiable lower support for this objective at any $\bar{b}'$. By the Differentiable Sandwich Lemma (Lemma 1), the borrower's objective is differentiable at the optimal debt choice $\hat{b}'$. Moreover, by repeatedly applying the Reverse Calculus Lemma (Lemma 3), we deduce that q and V are differentiable at $\hat{b}'$. We conclude:

Corollary 1. *Suppose $\hat{b}'(\cdot, \cdot)$ is an optimal policy function, fix any state (b, x), and set $\hat{b}' = \hat{b}'(b, x)$. If $\hat{b}' \neq b^*$, then the following first-order condition holds and the endogenous functions q and V that appear in it are differentiable at $\hat{b}'$:*

$$u_1(\hat{c}(b, x))(q(\hat{b}') + q_1(\hat{b}')\hat{b}') = \beta V_1(\hat{b}') = \beta \int_{y(\hat{b}')}^{x^{max}} u_1(\hat{c}(\hat{b}', x'))f(x')\, dx', \tag{21}$$

where $\hat{c}(b, x) = x - b + q(\hat{b}'(b, x))\hat{b}'(b, x)$.

The borrower equates the marginal benefit of owing debt with the marginal cost. The marginal benefit consists of the marginal utility of consumption times the marginal revenue from promising an extra unit to the lender. The marginal cost consists of the expected marginal utility of the foregone consumption when repaying the following period (when the endowment shock is above the default cut-off).



(a) Laffer curve for debt

(b) Endogenous interest rate

Figure 7: Characterisation of endogenous borrowing

Credit Limits. We now turn our attention to the borrower's behaviour near the credit limit. The amount the lender is willing to pay, $q(b')b'$ in return for a promise of b' is not an increasing function. This is because there are two types of empty promises: $b' = 0$, and b' so large it is never honoured. The borrower's return on promises therefore follows a Laffer curve, depicted in Figure 7a. The borrower's credit limit is the maximum of this curve, $q(b^{**})b^{**}$, where

$$b^{**} = \arg\max_{b'} q(b')b'. \tag{22}$$

If $b^{**} > b^*$, then we have already constructed a differentiable lower support function for q, so the Differentiable Sandwich Lemma (Lemma 1) together with the Reverse Calculus Lemma (Lemma 3) imply that q is differentiable at b^{**} with

$$q(b^{**}) + q_1(b^{**})b^{**} = 0. \tag{23}$$

Substituting this into the Euler equation (21), we see that the marginal benefit of taking on debt at b^{**} is zero, while the marginal cost is positive. Therefore, we conclude

Corollary 2. *For any given model primitives, either*

(i) the overall and risk-free credit limits coincide, i.e. $b^{**} = b^*$, or

(ii) the overall credit limit is higher and exhausting it is suboptimal, i.e. $b^{**} > b^*$ and $\hat{b}'(b, x) < b^{**}$ for all states (b, x).

This conclusion is a logical generalisation of behaviour in Aiyagari's (1994) model. Both here and there, the borrower reaches the risk-free credit limit with positive probability. In the model we study, the overall credit limit is potentially higher, as the borrower has the additional possibility of taking out risky loans. However, behaviour near the two credit limits is strikingly different. Below the risk-free limit, the interest rate $1/q(b')$ remains constant as the loan size $q(b')b'$ increases. Above the risk-free limit, the interest rate increases as the borrower takes on more debt and increases the default risk, as depicted in Figure 7b. This difference accounts for why borrowers might exhaust their risk-free limit, but not their overall limit.

Arellano (2008, Figure 2) plots a similar Laffer curve as in Figure 7a. Possibly for computational reasons, her curve is smooth and does not depict the upward kink of the Laffer curve at the risk-free limit, b^*. She does not apply first-order conditions along the Laffer curve.

Final Remarks. Despite our results regarding first-order conditions, credit limits, and the Laffer curve, some questions remain. First, we do not know if the Laffer curve is single-peaked. Second, the IID shock assumption was important for Arellano (2008) to establish that the default policy is a cut-off rule. More generally, persistent shocks cause

interest rates to depend on the shock in addition to the size of the loan, which is crucial for understanding how credit markets operate when borrowers are distressed. Nevertheless, we believe our analysis can be generalised. Chatterjee *et al.* (2007, Theorem 3) established that persistent shocks lead to two-sided cut-off rules. We conjecture that it is possible to construct differentiable support functions for the two cut-offs, and use this to construct a differentiable upper support function for the repayment probability. Finally, we believe that first-order conditions will be central to exploring extensions of the model to study issues such as partial default and optimal term structure.

4.2 Adjustment Costs

Firms are slow to adjust prices, labour forces, and capital stocks in reaction to changes in market conditions. One explanation for this is that firms face adjustment costs such as fixed costs or other non-convex costs. There is a large literature investigating how shocks propagate in the presence of adjustment costs and whether or not adjustment costs amplify shocks; see the surveys by Khan and Thomas (2008a), Leahy (2008), and Caplin and Leahy (2010). However, most of this literature is purely empirical, because the theory of adjustment costs faces two important obstacles. One is the complexity of optimal policy functions. Both theoretical and empirical analysis has only been tractable thus far when optimal polices involve smooth cut-off rules for determining when adjustments take place.[28] The other is the difficulty in deriving recursive first-order conditions, as the value of adjustment is not differentiable in general. Caballero and Engel (1999) use shocks that enter linearly into the production function to smooth out the kinks in the value function. Under this specific structure, they are able to take first-conditions to characterise optimal adjustments. To make this operational, they conjecture that adjustments follow a smooth two-sided (S, s) policy, but only verify this numerically.[29] Gertler and Leahy (2008) study a quadratic approximation of the firm's objective function in which the non-differentiable terms in the continuation value of adjustment vanish and optimal policies are smooth two-sided (S, s). They establish low error bounds for this approximation for an appropriate range of adjustment cost and shock parameters. Elsby and Michaels (2012) use first-order conditions under the conjecture that the optimal adjustment policy is a smooth two-sided (S, s) policy, also without providing sufficient conditions on primitives for this conjecture to hold. For the purposes of illustration, Cooper and Haltiwanger (2006, Section 3.2) and Khan and Thomas (2008b, Appendix B) provide derivatives only in the absence of fixed costs; we show these derivatives hold generally. An alternative approach is to assume that information arrives gradually over continuous time; see Harrison, Sellke and Taylor (1983), Stokey (2008), and Golosov and Lucas

[28] Specifically, we say that a policy is a smooth two-sided (S, s) policy if (i) for every capital (or labour or price) level, the set of shocks for which the firm makes an adjustment is an interval and (ii) the upper and lower end points of this interval are differentiable functions of the capital level.

[29] Caballero and Engel (1999, Footnote 16)

(2007).

The fundamental problem is that if a firm invests more today, then it might defer subsequent investment longer. Thus a small change in today's choice may lead to a lumpy change in a later choice, giving a non-differentiable and non-concave value of investment. We show that at optimal adjustment choices, the value function is differentiable so that recursive first-order conditions are applicable. We require only very weak assumptions on the primitives. In particular, our result remains true even when optimal policies are not two-sided (S, s) (see for example Bar-Ilan, 1990).

Model. In a general formulation, a firm is endowed with a capital stock k and shock z. Shocks evolve according to a Markov process with conditional distribution $P(z'|z)$. In each period, the firm's flow profit is $\pi(k, z)$; for example $\pi(k, z) = pf(k, z) - rk$ where p is output price, f is the production function, and r is the rental rate of capital. The firm pays an adjustment cost $c(k', k, z)$; non-adjustment is costless. We assume the flow profit $\pi(\cdot, z)$ is differentiable for all z, and that the adjustment cost $c(\cdot, \cdot, z)$ function is differentiable at all points (k, k', z) such that $k' \neq k$. For example, this accommodates the pure fixed-cost function, $c(k', k, z) = I(k' \neq k)$. The firm's value before adjusting its capital stock at state (k, z) is $V(k, z)$. Its value after adjusting its capital stock to k' is $W(k', z)$. These two value functions are related by the following two Bellman equations:

$$V(k, z) = \max_{k'} \pi(k, z) - c(k', k, z) + \beta W(k', z), \tag{24a}$$

$$W(k', z) = \int V(k', z')\, dP(z'|z). \tag{24b}$$

Our goal is to establish the first-order condition for the capital choice k'

$$c_1(k', k, z) = \beta W_1(k', z) \tag{25}$$

and to derive a formula for the marginal value of investment $W_1(k', z)$ at the optimal choice $k' = \hat{k}'(k, z)$. If there is a fixed cost of an adjustment, then this formula will only be satisfied when the agent makes an adjustment, i.e. at shocks z lying in the optimal adjustment set

$$\hat{A}(k) = \left\{ z : \hat{k}'(k, z) \neq k \right\}. \tag{26}$$

Differentiable Lower Support Functions. We construct a differentiable lower support function for the value function V by considering a lazy manager who knows the optimal policy when he begins with a familiar capital stock of $k = \bar{k}$. The obvious lazy manager policy of sticking to the same capital choice when $k \neq \bar{k}$ is not useful here, because it leads to a discontinuous lazy value function.[30] Instead, we consider a lazy manager who

[30] This obvious lazy manager makes an extra adjustment even if the capital stock is only slightly different from the familiar level.

uses the familiar adjustment set and adjustment level for unfamiliar capital stocks, i.e. he waits until he draws a shock $z \in \hat{A}(\bar{k})$ and adjusts to $\hat{k}'(\bar{k}, z)$. Thereafter, his choices coincide with the rational manager. His value function is

$$L(k, z; \bar{k}) = \pi(k, z) + \begin{cases} \beta \int L(k, z'; \bar{k})\, dP(z'|z) & \text{if } z \notin \hat{A}(\bar{k}), \\ -c(\hat{k}'(\bar{k}, z), k, z) + \beta W(\hat{k}'(\bar{k}, z), z) & \text{if } z \in \hat{A}(\bar{k}). \end{cases} \tag{27}$$

It is straightforward to calculate the lazy manager's marginal value of capital, because the capital stock k does not affect any subsequent choices:[31]

$$L_1(k, z; \bar{k}) = \pi_1(k, z) + \begin{cases} \beta \int L_1(k, z'; \bar{k})\, dP(z'|z) & \text{if } z \notin \hat{A}(\bar{k}), \\ -c_2(\hat{k}'(\bar{k}, z), k, z) & \text{if } z \in \hat{A}(\bar{k}). \end{cases} \tag{28}$$

First-Order Conditions. If $\hat{k}'$ is an optimal choice at the state (k, z), then $\hat{k}'$ maximises

$$\phi(k'; k, z) = \pi(k, z) - c(k', k, z) + \beta W(k', z). \tag{29}$$

By substituting in (27) and (24b), we may construct a differentiable lower support function for $\phi(\cdot; k, z)$ at $\hat{k}'$. Lemma 2 provides a differentiable upper support function, so Lemma 1 establishes the following corollary.

Corollary 3. *If making an adjustment is optimal at state (k, z), i.e. $z \in \hat{A}(k)$, then the investment value W is differentiable in capital at $(\hat{k}'(k, z), z)$ and*

$$c_1(\hat{k}'(k, z), k, z) = \beta W_1(\hat{k}'(k, z), z) = \beta \int \tilde{L}_1(\hat{k}'(k, z'), z')\, dP(z'|z), \tag{30a}$$

$$\textit{where } \tilde{L}_1(k, z) = \pi_1(k, z) + \begin{cases} \beta \int \tilde{L}_1(k, z')\, dP(z'|z) & \textit{if } z \notin \hat{A}(k), \\ -c_2(\hat{k}'(k, z), k, z) & \textit{if } z \in \hat{A}(k). \end{cases} \tag{30b}$$

The first equation says that the marginal adjustment cost should equal the marginal value of investment, which is the same for both rational and lazy managers. The second equation says that the marginal value of increasing investment equals the expected marginal increase in profit until the next adjustment plus the marginal decrease in the subsequent adjustment cost. We have thus shown that first-order conditions are generally valid even if the optimal adjustment policies are not (S,s). In other words, we have established that the applicability of first-order conditions is not an obstacle to the theoretical analysis of the implications of adjustment costs to prices, labour forces, and capital stocks. The only remaining obstacle is understanding when optimal policies are (S,s).

[31] The lazy manager's marginal value follows from the chain rule applied to (i) the expected discounted profit as a function of all state-contingent capital choices, holding adjustment times fixed, and (ii) the lazy capital choices as a function of initial capital k only.

4.3 Social Insurance

Governments run public health, unemployment and disability insurance programs, and private companies offer insurance contracts. These are constrained by frictions such as hidden information, adverse selection, and moral hazard. Informal insurance arises within well-connected families and communities when they can partially overcome these frictions. There is a large literature studying informal insurance, and the interaction of informal insurance with other forms of insurance.[32] In the dynamic insurance models of Thomas and Worrall (1988, 1990) and Kocherlakota (1996), the main issue is how cross-subsidisation may be self-enforcing. Agents with good luck subsidise those with bad luck in return for promises of future payments and insurance. These papers study smooth convex environments in which the Benveniste and Scheinkman (1979) theorem provides a formula for the marginal cost of making promises.[33] However, some important insurance problems involve non-smooth settings. We focus on a setting similar to that of Morten (2012), which is an extension of Ligon *et al.*'s (2002) model of self-enforcing dynamic insurance. Villagers share risk among themselves by both sharing divisible output and sending some members of the community to find temporary work in cities. The temporary migration decisions are inherently discrete as they involve a fixed cost of moving to the city and back. Other examples of indivisible items in village economies include livestock, medical treatments, agricultural land (due to high legal costs), and houses. This environment is non-smooth and non-concave, so the marginal cost of promises does not exist globally. Nevertheless, our envelope theorem applies and allows us to characterise optimal insurance policies in terms of the marginal cost of promises. Optimal policies involve sharing risk through allocating indivisible temporary work obligations; divisible consumption is then allocated to smooth out the marginal utility of consumption across states.

Model. Consider the following dynamic risk-sharing game between two households $h \in \{1, 2\}$. Each period begins with a Markov shock $s \in S$ with transition function $p(s'|s)$. The shock determines each household's endowment of a divisible consumption good, $C_h(s)$. The aggregate endowment is $C(s) = C_1(s) + C_2(s)$. In addition, each household may produce M units of the consumption good from temporary migrant work in a city. We write $d_h = 1$ if the household migrates, and $d_h = 0$ otherwise. We assume that the utility from consumption $u(\cdot, d_h)$ is differentiable, and that the marginal utility approaches infinity as consumption approaches zero. The autarky value of each

[32] Apart from the papers we discuss, Townsend (1994), Attanasio and Ríos-Rull (2000), and Krueger and Perri (2006) are important papers.

[33] Kocherlakota (1996) mistakenly claims his value function is differentiable. Koeppl (2006) amends his Bellman equation along the lines of Thomas and Worrall (1988). See also Ljungqvist and Sargent (2004, Chapter 20), and Rincón-Zapatero and Santos (2009, Section 4.2) for further discussion.

household is

$$V_h^{\text{aut}}(s) = \max_{d_h} u(C_h(s) + Md_h, d_h) + \beta \sum_{s'} p(s'|s) V_h^{\text{aut}}(s'). \tag{31}$$

Before investigating the social insurance arrangements with autarky constraints, we present the social planner's problem with Negishi weights η_1 and η_2:

$$W(s) = \max_{c_1, d_1} \eta_1 u(c_1, d_1) + \eta_2 u(c_2, d_2) + \beta \sum_{s' \in S} p(s'|s) W(s') \tag{32a}$$

$$\text{where} \quad c_1(s) + c_2(s) = C(S) + (d_1 + d_2)M. \tag{32b}$$

The first-order condition with respect to c_1 gives the Borch (1962) equation

$$\frac{u_1(c_1, d_1)}{u_1(c_2, d_2)} = \frac{\eta_2}{\eta_1}. \tag{33}$$

This means that after the social planner allocates the migration decisions, she adjusts the consumption good until the planner's marginal rate of substitution between the households is equal to the ratio of Negishi weights at all states and dates.

Now, we add in autarky constraints to study the optimal incentive-compatible social insurance contract. The value function for household 1 can be formulated recursively in terms of a principal-agent problem in which household 1 acts as an insurer and is able to promise future utility to household 2. This promised utility is a state variable, and has a corresponding promise-keeping constraint. Both households can leave the contract at any time, so there is an autarky constraint for each of them.

$$\begin{aligned} V(s, v_2) = \max_{c_1, d_1, d_2, v_2'(s')} & u(c_1, d_1) + \beta \sum_{s' \in S} p(s'|s) V(s', v_2'(s')) \\ \text{s.t.} \quad (\text{PK}_2) \quad & u(c_2, d_2) + \beta \sum_{s' \in S} p(s'|s) v_2'(s') = v_2, \\ (\text{A}_1\text{-}s') \quad & V(s', v_2'(s')) \geq V_1^{\text{aut}}(s') \text{ for all } s' \in S, \\ (\text{A}_2\text{-}s') \quad & v_2'(s') \geq V_2^{\text{aut}}(s') \text{ for all } s' \in S, \\ \text{where} \quad & c_2 = C(s) + (d_1 + d_2)M - c_1. \end{aligned} \tag{34}$$

Differentiable Lower Support Functions. We construct a lower support function of V using the lazy insurer's value function as follows. At a familiar promised utility $\bar{v}_2$, the lazy insurer knows the optimal migration allocation and future promised utilities, which we denote $\bar{d}_1$ and $\bar{v}_2'(\cdot)$. The lazy insurer makes these familiar choices even at unfamiliar promised utilities $v_2 \neq \bar{v}_2$, and only adjusts consumption of the consumption good c_1

in order to satisfy the promise-keeping constraint (PK₂). Thus, the lazy insurer's value function is

$$L(s, v_2) = u(c_1(\bar{v}_2'), \bar{d}_1) + \beta \sum_{s'} p(s'|s) V(s', \bar{v}_2'(s')), \tag{35}$$

where $c_1(\bar{v}_2')$ is defined implicitly by (PK₂). The lazy insurer's marginal value of promising utility $\bar{v}_2$ to the other household is

$$L_{v_2}(s, \bar{v}_2) = u_1(c_1(\bar{v}_2'), \bar{d}_1) \frac{dc_1(\bar{v}_2')}{dv_2} = -\frac{u_1(\bar{c}_1, \bar{d}_1)}{u_1(\bar{c}_2, \bar{d}_2)}, \tag{36}$$

where dc_1/dv_2 was calculated with the implicit function theorem.

First-Order Conditions. It is tempting to apply our envelope theorem to the choices of all promised utilities $v_2'(\cdot)$. However, some of these choices may be boundary choices, i.e. at some states s', one of the autarky constraints (A₁-s') or (A₂-s') may bind. Technically speaking, if any choice is on the boundary, then the choice vector is a boundary choice. Our solution is to focus on interior choices by considering each choice separately. Suppose $(\hat{d}_1, \hat{v}_2'(\cdot))$ are optimal choices at state s. If $\hat{v}_2'(s')$ is an interior optimal choice at state s', then this choice maximises

$$\phi(v_2'; \hat{d}_1, s') = u(c_1(v_2'), \hat{d}_1) + \beta p(s'|s) V(s', v_2'), \tag{37}$$

where $c_1(\cdot)$ is the same function defined above in terms of the promise-keeping constraint (PK₂). Notice that the terms for the other future states were dropped, as they are unaffected by the choice of v_2'. We may now apply the logic from Section 3.4 (which is summarised in Theorem 1).

Corollary 4. *Fix some state s', and suppose that*

(i) $(\hat{c}_1, \hat{c}_2, \hat{d}_1, \hat{d}_2, \hat{v}_2'(\cdot))$ *are optimal choices at* (s, v_2)*,*

(ii) no autarky constraints bind for the choice of $\hat{v}_2'(s')$ *for state* s'*, and*

(iii) $(\hat{c}_1', \hat{c}_2', \hat{d}_1, \hat{d}_2))$ *are optimal choices at* $(s', \hat{v}_2'(s'))$*.*

Then the value function $V(s', \cdot)$ *is differentiable at* $\hat{v}_2'(s')$ *with*

$$-\frac{u_1(\hat{c}_1, \hat{d}_1)}{u_1(\hat{c}_2, \hat{d}_2)} = V_{v_2}(s', \hat{v}_2'(s')) = -\frac{u_1(\hat{c}_1', \hat{d}_1')}{u_1(\hat{c}_2', \hat{d}_2')}. \tag{38}$$

Discussion. This equation is the Borch (1962) equation which characterises perfect insurance – the social planner's marginal rate of substitution is equated across states and time periods. This means we have shown that with both divisible and indivisible choices, there is perfect insurance between households at all states and times for which the autarky constraints are lax. When an autarky constraint binds, the Negishi weights are adjusted and perfect insurance continues until an autarky constraint binds in the future. This generalises the conclusion drawn by Thomas and Worrall (1988) when indivisible choices are absent.

4.4 Benveniste-Scheinkman Envelope Theorem

Our approach leads to an elementary proof of the Benveniste and Scheinkman (1979) envelope theorem. This theorem establishes global differentiability of the value function in convex settings (without any discrete choices).

Problem 1. *Consider the following dynamic programming problem:*

$$V(c) = \sup_{c' \in \{\bar{c}' : (c, \bar{c}') \in \Gamma\}} u(c, c') + \beta V(c'), \tag{39}$$

where the domain of V is C. We assume that (i) Γ is a convex subset of $C \times C$, (ii) u is concave, and (iii) $u(\cdot, c')$ and $u(c, \cdot)$ are differentiable, respectively.

Corollary 5 (Benveniste-Scheinkman Theorem)**.** *If $\hat{c}'$ is an optimal choice at state $c \in$* $\text{int}(\{\bar{c} : (\bar{c}, \hat{c}') \in \Gamma\})$*, then V is differentiable at c with $V_1(c) = u_1(c, \hat{c}')$.*

Proof. V is concave because u is concave and Γ is convex. Hence, the supporting hyperplane theorem can be applied to the hypograph of V to construct a linear upper support function U that touches V at c. We construct the differentiable lower support function $L(c) = u(c, \hat{c}') + \beta V(\hat{c}')$. Lemma 1 delivers the conclusions. □

The Differentiable Sandwich Lemma in our proof plays a similar role as Rockafellar (1970, Theorem 25.1) in the original proof of Benveniste and Scheinkman (1979).[34] Concavity implies the existence of a differentiable upper support function and Benveniste and Scheinkman use a lazy agent's value function as a differentiable lower support function. Mirman and Zilcha (1975, Lemma 1) provide a one-dimensional antecedent based on directional derivatives rather than the sandwich approach.

[34] We show that Rockafellar's result contains a superfluous concavity assumption; the lower support function in the sandwich need not be concave.

5 Conclusion

All envelope theorems have a sandwich idea at their core. Previous proofs were structured around sandwiches of inequalities of directional derivatives. By restructuring around sandwiches of differentiable upper and lower support functions, we gain two things. First, we do not require any of the strong technical conditions from previous envelope theorems, and can accommodate primitives with Inada conditions. Second and more importantly, our approach potentially applies to any type of endogenous functions that might need to be differentiated in a first-order condition.

Our method gains us a straightforward way of mixing and matching different constructions of upper and lower halves of sandwiches. We used five constructions throughout, namely (i) horizontal lines above maxima, (ii) supporting hyperplanes above concave functions, (iii) reverse calculus, (iv) lazy value functions below rational value functions, and (v) lazy cut-off rules. Of these, only the reverse calculus construction is truly unprecedented. The power of our approach derives from the ability to combine these constructions. For example, the unsecured credit application uses all but the supporting hyperplane construction. There are also other possibilities that we did not explore. Decision makers can be "lazy" in ways that lead to upper support functions, such as being lazily optimistic about future opportunities. In bargaining games, a lower support function for one player's value function leads to an upper support function for the other player's value function.

To conclude, our new approach reveals that trade-offs which previously seemed poorly behaved in fact have smooth structures within them that lead to first-order characterisations of optimal decisions.

A Support Functions and Subdifferentials

The notion of a differentiable lower support function generalises the classic ideas from convex analysis of supporting hyperplanes and subdifferentials. In this appendix, we establish a tight equivalence between differentiable lower support functions and Fréchet subdifferentials. These were once seen as a promising way to generalise the classical techniques of convex optimisation described by Rockafellar (1970) beyond convex settings. However, according to Kruger (2003), these were abandoned because of "rather poor calculus" as Fréchet subdifferentials do not sum, i.e. $\partial_F(f+g)(x) \neq \partial_F f(x) + \partial_F g(x)$. In light of our developments, we believe that Fréchet subdifferentials may have other applications to optimisation theory.

Suppose $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$.

Definition 6. *A function $f : C \to \mathbb{R}$ is **Fréchet subdifferentiable** at $\bar{c}$ if there is some*

$m^* \in C^*$ *such that*

$$\liminf_{\Delta c \to 0} \frac{f(\bar{c} + \Delta c) - f(\bar{c}) - m^* \Delta c}{\|\Delta c\|} \geq 0. \tag{40}$$

Such an m^* is called a Fréchet subderivative of f at $\bar{c}$, and the set of all subderivatives is called the Fréchet subdifferential of f at $\bar{c}$, denoted $\partial_F f(\bar{c})$. Definitions for Fréchet superdifferentiable, superderivatives, and superdifferentials are analogous.

Theorem 2. *m^* is a Fréchet subderivative of $f : C \to \mathbb{R}$ at $\bar{c}$ if and only if f has a differentiable lower support function L at $\bar{c}$ such that $L_1(\bar{c}) = m^*$.*

Proof. If L is such a differentiable lower support function, then $L_1(\bar{c}) = m^*$, i.e.

$$\lim_{\Delta c \to 0} \frac{L(\bar{c} + \Delta c) - f(\bar{c}) - m^* \Delta c}{\|\Delta c\|} = 0. \tag{41}$$

Since $f(\bar{c} + \Delta c) \geq L(\bar{c} + \Delta c)$ for all Δc, it follows that

$$\liminf_{\Delta c \to 0} \frac{f(\bar{c} + \Delta c) - f(\bar{c}) - m^* \Delta c}{\|\Delta c\|} \geq 0 \tag{42}$$

and hence m^* is a Fréchet subderivative of f at $\bar{c}$.

Conversely, suppose that m^* is a subderivative of f at $\bar{c}$. We claim that

$$L(c) = \min \{f(c), f(\bar{c}) + m^*(c - \bar{c})\} \tag{43}$$

is a differentiable lower support function of f at $\bar{c}$. By construction, L is a lower support function. Moreover, the function $U(c) = f(\bar{c}) + m^*(c - \bar{c})$ is a differentiable upper support function of L at $\bar{c}$; by the first part of the theorem, $U_1(\bar{c}) = m^*$ is a superderivative of L at $\bar{c}$. On the other side, m^* is a subderivative of L at $\bar{c}$ because

$$\begin{aligned} &\liminf_{\Delta c \to 0} \frac{L(\bar{c} + \Delta c) - f(\bar{c}) - m^* \Delta c}{\|\Delta c\|} \\ &= \min \left\{ 0, \liminf_{\Delta c \to 0} \frac{f(\bar{c} + \Delta c) - f(\bar{c}) - m^* \Delta c}{\|\Delta c\|} \right\} \\ &\geq 0. \end{aligned} \tag{44}$$

Therefore, L is differentiable at $\bar{c}$ with $L_1(\bar{c}) = m^*$. □

Lemma 1 then becomes a classic result.

Lemma 7. *If m^* is a Fréchet subderivative of $f : C \to \mathbb{R}$ at $\bar{c}$ and M^* is a superderivative of f at $\bar{c}$, then f is differentiable at $\bar{c}$ with $f'(\bar{c}) = m^* = M^*$.*

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