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Computational isomorphisms in classical logic

(Extended Abstract)

Vincent Danos [a,1] Jean-Baptiste Joinet [a,2] Harold Schellinx [b,3]

[a] *Équipe de Logique Mathématique*
Université Paris VII

[b] *Mathematisch Instituut*
Universiteit Utrecht

Abstract

We prove that any pair of derivations, without structural rules, of $F \vdash G$ and $G \vdash F$, where F, G are first-order formulas 'without any qualities', in a constrained classical sequent calculus $\mathbf{LK}^{\eta}_{p}$, define a computational isomorphism up to an equivalence on derivations based upon reversibility properties of logical rules.

This result gives a rationale behind the success of Girard's denotational semantics for classical logic, in which all standard 'linear' boolean equations are satisfied.

1 Introduction

1.1 A patch of paradise to be broadened

In recent work [1] devoted to the proof theory of classical logic, we embarked on the project of overcoming the obstacles that prevent cut from being a *decent* binary operation on the set of classical sequent derivations. To clarify what we mean by *decency*, let us have a look at the world of simply typed λ-calculus, which, seen from a *normalization-as-computation* point of view, is something close to a patch of paradise.

Among the ingredients of 'computational decency' there, we not only encounter (1) a framework to represent proofs (intuitionistic natural deduction, **IND**) and (2) a *noetherian* and *confluent* cut-elimination scheme (β-reduction), but also (3) a quotient of the space of proofs (the η-quotient) where computational isomorphisms are realized.

[1] danos@logique.jussieu.fr; URA 753

[2] joinet@logique.jussieu.fr; Université Paris I (Panthéon-Sorbonne)

[3] schellin@math.ruu.nl; research part of the project 'The Geometry of Logic', financed by the Netherlands Organization for Scientific Research (NWO).

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As an example of a computational isomorphism, which gives a good impression of how members of this *intuitionistic triple* ($\mathbf{IND}$,β, $\overset{\eta}{\approx}$) cooperate, consider f's 'twister' $T[f] = \lambda y \lambda x((f)x)y$. Clearly:

$$f : A \to (B \to C) \vdash T[T[f]] : A \to (B \to C)$$

Note that the term $T[T[f]]$ has the effect of switching the order of the arguments of the function f, and then switching them back again. Does such a double switching have an effect in terms of computations? Of course one would like the answer to be a firm "no!". Otherwise said - we'll be more precise later - we want this kind of "commutativity" to be a *computational isomorphism*. By two β-reductions, $T[T[f]] = \lambda y \lambda x((\lambda y \lambda x((f)x)y)x)y$ becomes $\lambda y \lambda x((f)y)x$. So in order for this double switching to be an 'action without content or meaning', we need to identify the terms $\lambda y \lambda x((f)y)x$ and f. But that gap between terms is exactly the one that is closed by η-equivalence!

1.2 The classical triple

In [1] we constructed a *classical triple*: ($\mathbf{LK}^{\eta}_{p}$,tq,$\overset{s}{\approx}$). It is an extension of the intuitionistic triple, because the standard embedding of natural deduction into sequent calculus actually sends β-equivalent derivations to tq-equivalent ones and η-equivalent ones to strongly equivalent ones:

$$(\mathbf{IND}, \beta, \overset{\eta}{\approx}) \subset (\mathbf{LK}^{\eta}_{p}, tq, \overset{s}{\approx}).$$

To build this classical triple, we start from a very general calculus for classical logic, baptized $\mathbf{LK}^{tq}$ (which includes logical rules in 'all styles': *multiplicative, additive*), and equip it with a normalization scheme (tq) which asks of each cut formula a "colour", t or q, to decide which sub-proof is to be moved first. This quite general scheme is shown to be noetherian and confluent using embeddings of classical logic into linear logic.

Just as asking that the above "commutativity" be a computational isomorphism *forces* η-equivalence on $\mathbf{IND}$-derivations, asking the boolean equivalences we consider to be computational isomorphisms forces *strong equivalence* on $\mathbf{LK}^{tq}$-derivations. Strongly equivalent proofs differ only with respect to reversion of ... reversible logical rules. However, in $\mathbf{LK}^{tq}$ pure, tq-reduction *breaks* $\overset{s}{\approx}$-classes! The quotient induced by $\overset{s}{\approx}$ consequentially is degenerated: all derivations having the same conclusion are identified. For $\overset{s}{\approx}$ to become compatible with the tq scheme, we need to (1) narrow the space of $\mathbf{LK}^{tq}$ proofs (the resulting fragment we call $\mathbf{LK}^{\eta}$) and (2) restrict the normalization-space of $\mathbf{LK}^{\eta}$ by *polarizing* derivations, i.e. by subordinating "colours" (hence normalization steps) to reversibility properties of connectives.

Both $\mathbf{LK}^{\eta}$ and $\mathbf{LK}^{\eta}_{p}$ are complete with respect to classical provability and closed under tq-normalization; and, by design, $\mathbf{LK}^{\eta}_{p}$ realizes "linear" boolean equivalences as computational isomorphisms.

1.3 Relationships between $\boldsymbol{LK}_p^\eta$ and $\boldsymbol{LC}$

Up to the stoup/no stoup formulation of the syntax, **LC**, Girard's calculus for classical logic [2], is but a fragment of $\mathbf{LK}_p^\eta$ where one imposes a coordination between styles and colours, and second, our strong normalization is a syntactic materialization of the identifications achieved by Girard's denotational semantics for **LC**, which by the way works for the whole of $\mathbf{LK}_p^\eta$.

1.4 An abstract criterion for isomorphisms

Now let us become more precise about what exactly we mean by a *computational isomorphism*. And for that, let us concentrate on sequent calculus, where composition appears via an explicit rule, the cut-rule.

Given a sequent calculus **L** with a confluent and noetherian normalization scheme, for any proofs π and π' in **L** of $\Gamma \vdash \Delta, F$ and $F, \Gamma' \vdash \Delta'$ respectively, we can define $\pi \odot_F \pi'$ to be *the* normal form of the derivation obtained by cutting π and π' on F. Let id_X denote the axiom $X \vdash X$ which we suppose is a unit w.r.t. $\odot_X$. Let now $\approx$ be an equivalence relation on **L**-proofs, such that any two equivalent proofs π and π' in **L** have equivalent normal forms (in which case we say the equivalence is *compatible* with the normalization scheme).

Definition 1.1 *A pair of* **L***-derivations* ϕ *and* χ *of* $F \vdash G$ *and* $G \vdash F$ *define a* computational isomorphism *between* F *and* G *with respect to* $\approx$*, if* $\phi \odot_G \chi \approx id_F$ *and* $\chi \odot_F \phi \approx id_G$*.*

The aim of the present paper is to provide a sufficient condition for a pair of derivations of $F \vdash G$ and $G \vdash F$ to define a computational isomorphism in $\mathbf{LK}_p^\eta$ with respect to $\stackrel{s}{\approx}$. Our criterion, which replaces empirical checkings of the kind we saw in the example given before, is quite general. Let us say a formula F is 'without any qualities' when all relation symbols in F are distinct. Then: in $\mathbf{LK}_p^\eta$, any pair of derivations, without structural rules, of $F \vdash G$ and $G \vdash F$, where F, G are first-order formulas without any qualities, define a computational isomorphism with respect to strong equivalence.

The only difficulty in the proof is to show (theorem 4.1) that for such formulas F, an $\mathbf{LK}_p^\eta$-derivation, with no structural rules, of the sequent $F \vdash F$ always is strongly equivalent to id_F.

Linear derivations, which can be considered as **MALL** derivations (**LL** derivations without exponentials), seem to play a distinguished rôle in the search for the algebraic structure behind classical logic considered as a computational system.

2 Preliminaries

2.1 Strong equivalence

As will be clear from the introduction, the present paper heavily relies upon our earlier work and we will freely use notions and notations from [1]. More specifically, we refer to [1] for the definitions of $\mathbf{LK}^{tq}$, $\mathbf{LK}^{\eta}$, $\mathbf{LK}^{\eta}_p$, the definition of tq-reduction, proofs of strong normalization and confluence, etcetera. For reference we added the 'all-style' sequent calculus **LK** that underlies these systems as an appendix.

The notion of *strong equivalence* of **LK**-derivations comes from reversibility properties of logical operators. All unary multiplicative rules, binary additive rules, both negation rules, right universal rules and left existential rules, are *reversible*: one can always permute them down with any other rule, that is, except when the reversible formula is active in the rule below. So two **LK**-derivations π and π' are said strongly equivalent, if they are the same up to such permutations of reversible logical rules and canonical expansion of identity-axioms.

Visually, strong equivalence can be thought of as the equivalence relation induced by the 'continuous' process of 'opening' and 'closing' in a proof occurrences of formulas that have a reversible main connective: if you think of the main-active interspaces of such formulas as *zippers* in the proof, then 'opening' the formula (permuting the reversible rule downwards) *unzips* the proof, 'closing' it (permuting the rule upwards, which is not always possible) *zips* it.

Only within $\mathbf{LK}^{\eta}_p$, as was proved in [1], is strong equivalence 'computationally meaningful':

Proposition 2.1 *If two $\mathbf{LK}^{\eta}_p$ proofs are strongly equivalent, then so are their normal forms.*

2.2 Archetypes, linear derivations, and other characters

We consider a first-order language for classical logic built from a set of variables $x_1, x_2, \ldots$, a set of n-ary function symbols $f_1, f_2, \ldots$, a set of n-ary relation symbols $R_1, R_2, \ldots$ (where $n = 0, 1, \ldots$ and each function and relation symbol is supposed to come with a fixed arity; 0-ary relation symbols are sometimes referred to as propositional variables; for each n the set of n-ary function, relation symbols is supposed to be infinite), negation $\neg$, quantifiers $\forall, \exists$ and binary connectives $\wedge_a, \vee_a, \stackrel{a}{\rightarrow}, \wedge_m, \vee_m, \stackrel{m}{\rightarrow}$ (the *additive* and *multiplicative* versions of the connectives well-known from classical propositional logic).

Define as usual the set of *terms* inductively by: all variables are terms, and whenever $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term; and the set of formulas by: if R is an n-ary relation symbol and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a(n atomic) formula and, whenever F, G are formulas and x a variable, then $\neg F$, $\mathcal{Q}xF$, $F \circ G$ are formulas (where $\mathcal{Q}$ ranges over quantifiers and $\circ$ over binary connectives).

Definition 2.2 *A first-order formula F is an* archetype *iff all relation symbols occurring in F are distinct, and, whenever $\mathcal{Q}xG$ is a subformula of F, then x is a free variable of G (i.e., there is no vacuous binding).*

For example, $(\forall x R_1(x)) \vee_a R_2(y, z)$ is an archetype, but both $\forall x R(z)$ and $R(f(t, t')) \wedge_a (\exists x R(x, z))$ are not.

Definition 2.3 *A* linear derivation *of $F \vdash F$ for some first-order formula F (notation: τ_F) is a normal derivation of $F \vdash F$ in the 'all-style' sequent calculus **LK**, that does not make use of structural rules.*

Clearly, given some F, we can in general not expect τ_F to be unique. Indeed, if F is not atomic, then obvious distinct examples of τ_F are the proof consisting in nothing but the identity axiom $F \vdash F$ (the *trivial* τ_F, written as id_F), and iterations of the derivation called η_F in [1]:

Definition 2.4 *If F_i is (are) the immediate subformula(s) of F, an* iterated η-proof *of $F \Rightarrow F$ (notation: η_F), consists in axiom(s) $F_i \Rightarrow F_i$ and/or iterated η-proofs of $F_i \Rightarrow F_i$, followed by precisely one instance of each of the logical rules introducing F's main connective.*

3 Linear derivations of archetypical identities are units

In what follows we will characterize the derivations τ_F, and show that, in case F is an archetype, *any* τ_F necessarily ends in an application of a reversible rule. Also, every τ_F, by permutations of instances of reversible rules, can be transformed in an iterated η-proof of $F \vdash F$. As a consequence we get that: for archetypes F, any linear derivation of an identity $F \vdash F$ is strongly equivalent to id_F, a result which we then relativize to $\mathbf{LK}^{\eta}_{p}$.
We start by assuming the archetype to be *propositional*, and then will use the characterization of τ_F for propositional archetypes F in order to extend the characterization to first-order archetypes.

3.1 The propositional case

In the proof of the propositional case several times the following simple property is invoked, which states that whenever a sequent $\Gamma \vdash \Delta$ is provable in non-exponential propositional *linear* logic (without constants) (**MALL**), in a cut-free derivation, because of the absence of weakening, every formula X in $\Gamma \cup \Delta$ can be 'traced upwards' to *at least one* identity axiom (otherwise said: every formula X has *at least one* atomic subformula whose tree of ancestors has a leaf in an identity axiom).
Let σ_i's be variables over $\{l, r\}$. We will use the following convenient notation: $A_1^{\sigma_1}, \ldots, A_n^{\sigma_n}$ will denote the sequent $\Gamma \Rightarrow \Delta$ where Γ is the submultiset of $A_1^{\sigma_1}, \ldots, A_n^{\sigma_n}$ containing $A_i^{\sigma_i}$'s such that $\sigma_i = l$ and Δ the complementary submultiset. We use a 'bar' to indicate transposition within $\{l, r\}$ i.e. $\bar{l} := r$, $\bar{r} := l$.

Lemma 3.1 *Let π be a normal **MALL**-derivation (without constants) of $X^{\sigma}, \Delta^{\sigma_i}$. Then there is at least one atomic subformula p of X that occurs positively (negatively) in X and negatively (positively) in $X \cup \Delta$. Hence if a sequent Γ is provable, then any formula in the multi-set Γ contains at least one atom p that occurs more than once in Γ.*

Any propositional formula F is of the form $\neg^m F'$, where $\neg^m$ denotes $m \geq 0$

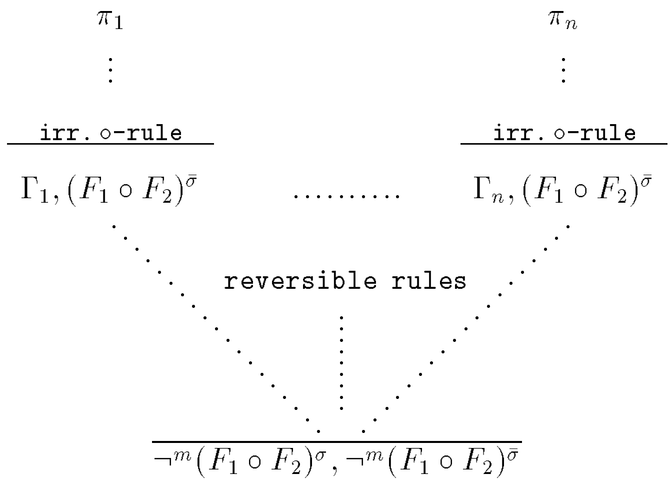

Fig. 1. A non-trivial τ_F

negation signs and F' is either atomic or of the form $F_1 \circ F_2$ for some binary connective $\circ$.

Lemma 3.2 *Let F be a propositional archetype. Then any non-trivial τ_F ends in an application of the reversible rule introducing F's main connective.*

By the above lemma we know that, for F a non-atomic propositional archetype, a non-trivial τ_F *necessarily* ends in an application of the reversible rule introducing F's main connective.
But we also know what are the lowest occurrences of irreversible rules:

Lemma 3.3 *Let $F \equiv \neg^m(F_1 \circ F_2)$ be an archetype. All lowest occurrences of irreversible rules in τ_F introduce the principal connective of $F_1 \circ F_2$. Moreover, all passive formulas occurring in a premise of such a rule are subformulas of the active formula.*

Consequently, a non-trivial τ_F deriving $\neg^m(F_1 \circ F_2)^\sigma, \neg^m(F_1 \circ F_2)^{\bar\sigma}$, where $(F_1 \circ F_2)^\sigma$, say, is on the reversible side, necessarily is of the form as in figure 1. There all formulas in Γ_i are proper subformulas of $F_1 \circ F_2$. We will speak of the *irreversible bar* in τ_F; the reversible rules below are called τ_F's *closing rules*. Clearly the number of closing rules in any non-trivial τ_F is at least 1.
Observe also that, for $F \equiv \neg G$, by a permutation of closing rules we can bring τ_F in the form

$$\cfrac{\cfrac{\begin{matrix}\tau_G\\ \vdots\\ G^{\bar\sigma}, G^\sigma\end{matrix}}{G^\sigma, (\neg G)^\sigma}}{(\neg G)^{\bar\sigma}, (\neg G)^\sigma}$$

which ends 'just like an η'.

Definition 3.4 *Let F_i be the immediate subformula(s) of F. We say that τ_F is* locally η *iff it is the identity axiom $F \vdash F$ or consists in derivations τ_{F_i} followed by precisely one instance of each of the logical rules introducing F's* *main connective.*

We already saw that any $\tau_{\neg G}$ can be transformed in a derivation that is locally η. The following lemma shows that this can always be done, for whatever archetype F.

Lemma 3.5 *Let $F \equiv F_1 \circ F_2$ be a propositional archetype. Then for any τ_F there exists a permutation of its closing rules with the irreversible bar such that the resulting derivation is locally η.*

The following theorem then is an immediate corrollary:

Theorem 3.6 *Any linear derivation of $F \vdash F$, with F an archetype, is strongly equivalent to id_F.*

Proof. By induction on the complexity of F, using (the zipping) lemma 3.5. As reversion of $F \vdash F$ can introduce structural rules, the converse does not hold: there are non-linear derivations strongly equivalent to the identity axiom. □

Let τ'_F denote any derivation obtained from a τ_F by removing zero or more of its closing rules. Also the following proposition is a corollary to the above.

Proposition 3.7 *Let F be an archetype. Then any sub-derivation of τ_F is of the form τ'_G for some subformula G of F; moreover, any sequent in τ_F is of the form $\Gamma, \neg^n H$, where H is either atomic or has an irreversible main connective, and all formulas in Γ are subformulas of $\neg^n H$.*

Let us mention another corollary, which is often used in the proofs of the lemmas in the next section.

Lemma 3.8 *Suppose τ'_F derives Γ, F. Then, for every atomic formula p occurring in Γ there is an axiom $A \vdash A$ in τ'_F, such that A contains p.*

3.2 Extension to first-order

In order to extend the above characterization to the first-order case it suffices to extend lemmas 3.2, 3.3 and 3.5 to first-order archetypes. We will make use of the fact that first-order formulas have an obvious underlying *propositional* structure.

Definition 3.9 *We inductively define a mapping $(\cdot)^\flat$ ('flat') from first-order formulas to propositional formulas by:*

$$
\begin{aligned}
R_i(t_1, \ldots, t_n)^\flat &:= r_i \\
(\neg F)^\flat &:= \neg F^\flat \\
(QxF)^\flat &:= F^\flat \\
(F_1 \circ F_2)^\flat &:= F_1^\flat \circ F_2^\flat
\end{aligned}
$$

where r_i is a (new) propositional variable, and call $F^\flat$ the propositional collapse *of F. (Cf.* [5], chapter 9.*)*

The propositional collapse of first-order formulas extends in an obvious way to first-order proofs; if π is a first-order proof of Γ, then, (modulo possible repetitions of sequents due to erasing quantifiers) $\pi^\flat$ is propositional proof of $\Gamma^\flat$.

We are going to use the following trivial property of the $(\cdot)^\flat$-mapping:

Lemma 3.10 *Let F be a first-order archetype and τ_F a linear proof of $F \vdash F$. Then $F^\flat$ is a propositional archetype and $(\tau_F)^\flat$ a linear proof $\tau_{F^\flat}$ of $F^\flat \vdash F^\flat$.*

Now, using the results in the propositional case, one shows that lemmas 3.2 and 3.3 continue to hold in the first-order case.

Lemma 3.11 *Let F be a first-order archetype. Then any non-trivial τ_F ends in an application of the reversible rule introducing F's main connective.*

Lemma 3.12 *Let $F \equiv \neg^m G$ (with G non-atomic, not starting with a negation) be a first-order archetype. All lowest occurrences of irreversible rules in τ_F introduce the principal connective or quantifier of G. Moreover, all passive formulas occurring in a premise of such a rule are subformulas of the active formula.*

Similarly, with due care as to the possibility of, when necessary, renaming variables and terms, one may verify that also lemma 3.5 continues to hold:

Lemma 3.13 *Let F be a first-order archetype. Then for any τ_F there exists a permutation of its closing rules with the irreversible bar such that the resulting derivation is locally η.*

We therefore find:

Theorem 3.14 *Theorem 3.6 and proposition 3.7 hold for all first-order archetypes.*

4 Classical isomorphisms

4.1 Back to $\boldsymbol{LK}_p^\eta$

Linear derivations of archetypical identities, hence, are strongly equivalent to identity axioms; shown while pretending to be 'colour-blind', this property of course continues to hold in $\mathbf{LK}^{tq}$ for coloured archetypes.

Note that if π is a derivation in $\mathbf{LK}^\eta$, or $\mathbf{LK}_p^\eta$, then *zipping* it can always be done *within* $\mathbf{LK}^\eta$, or $\mathbf{LK}_p^\eta$. Hence the last theorem can be relativized to $\mathbf{LK}^\eta$ or $\mathbf{LK}_p^\eta$, thus:

Theorem 4.1 *Any linear $\boldsymbol{LK}_p^\eta$ derivation of $F \vdash F$, with F an archetype, is strongly equivalent to id_F.*

In $\mathbf{LK}_p^\eta$ linear derivations of $F \vdash F$, for polarized first-order archetypes, are, of a strikingly simple form. E.g., the structure of the fully expanded τ_F (all occurring identity axioms are atomic) in $\mathbf{LK}_p^\eta$ is the following:

(i) do *all* possible reversible rules, starting from the reversible rule introducing F's main connective (be careful: only *one* of the negation-rules is reversible

in $\mathbf{LK}_p^\eta$ and while in $\mathbf{LK}^{tq}$ derivations the negation-rules are in some sense 'roaming free', in $\mathbf{LK}_p^\eta$ they are strictly localized), until you are left with only atomic formulas or formulas with an irreversible main connective; (ii) then decompose the 'irreversible' F, up to the 'duals' of the formulas left in (i).
After step (ii) all leaves are of the form $F_i \vdash F_i$, and the process starts over again.
The result of step (i) is unique up to possible permutations of 'independent' reversible rules, but this is the *only* degree of freedom.

4.2 The criterion: linearity - The harvesting: classical isomorphisms.

The following theorem gives a sufficient condition for the existence of a computational isomorphism between F and G:

Theorem 4.2 *Suppose ϕ and χ are linear $\boldsymbol{LK}_p^\eta$-derivations of $F \vdash G$ and $G \vdash F$ respectively, where F and G are first-order archetypes. Then ϕ and χ define a computational isomorphism between F and G w.r.t. $\stackrel{s}{\approx}$.*

Proof. As being linear is stable under tq-reduction, we find $\phi \odot_G \chi = \tau_F \stackrel{s}{\approx} id_F$ and $\chi \odot_F \phi = \tau_G \stackrel{s}{\approx} id_G$. Hence ϕ and χ define a computational isomorphism.□

As a by-product of the above analysis we recover most linear boolean equivalences : commutativity and associativity of conjunction and disjunction, involutivity of negation, de Morgan laws, etc.
However observe that we cannot *always* use the condition of theorem 4.2 above to 'catch' isomorphisms. An example is given by the distributivity $A \wedge_m (B \vee_a C) \Leftrightarrow (A \wedge_m B) \vee_a (A \wedge_m C)$, for which there is a computational isomorphism; but of course the formulas are not archetypical.

4.3 Linear isomorphisms

Definition 4.3 *An isomorphism (ϕ, χ) is linear whenever ϕ and χ are (strongly equivalent to) linear derivations.*

The following proposition expresses a necessary condition for the existence of linear isomorphisms between F^ϵ and $G^{\epsilon'}$ (where the superscripts indicate the colour of the formulas, cf. [1]), in case F^ϵ, $G^{\epsilon'}$ are first-order archetypes.

Proposition 4.4 *Let $F^\epsilon, G^{\epsilon'}$ be first-order archetypes. If $\epsilon \neq \epsilon'$, then there are no linear isomorphisms between F^ϵ and $G^{\epsilon'}$.*

Proof. Suppose ϕ and χ are $\mathbf{LK}_p^\eta$ derivations of the sequents $F \vdash G$ and $G \vdash F$ respectively, where F and G are first-order archetypes of opposite polarities, then they must be both attractive in one of the sequents, say $F \vdash G$, and both non-attractive in the other. Because of the absence of structural rules, one of F and G, say G, has to be logically main in ϕ's last rule. Then $\chi \odot_F \phi$ must end in this same rule since F is attractive in ϕ, so that the structural step will carry χ above G's last rule. But then $\chi \odot_F \phi$ can't be a τ_G since its last rule is an irreversible one, which would contradict lemma 3.11.□

Now here is an example of a 'good taste' corollary to our approach, namely the *unicity* of the archetypical, linear computational isomorphisms in $\mathbf{LK}_p^\eta$ caught by means of our criterion:

Theorem 4.5 *Let F, G be archetypes, ϕ and χ linear $\boldsymbol{LK}_p^\eta$ derivations of $F \vdash G$ and $G \vdash F$. Any linear $\boldsymbol{LK}_p^\eta$ derivation ϕ' of $F \vdash G$ is strongly equivalent to ϕ.*

Proof. Using proposition 4.4 and lemma 4 of [1] we have $\phi \odot_G (\chi \odot_F \phi') = (\phi \odot_G \chi) \odot_F \phi'$, that is these two cuts commute. By linearity of ϕ' and theorem 4.1, $\chi \odot_F \phi' \overset{s}{\approx} id_G$ and since $\phi \odot_G \chi \overset{s}{\approx} id_F$, $\phi \overset{s}{\approx} \phi'$. □

Conversely, the necessary condition in proposition 4.4, shows that a certain number of equivalences can't be recovered at the computational level: style-switchings, prenexifications and some distributivities, etc.
Granted that the maximization of isomorphisms reduce the 'noise' of the syntax, that is the amount of syntactic details which blur the actual computational phenomenon, then our classical triple should be a good calculus in which to examine the computational content of classical proofs.

5 The same result in $\lambda\mu$-calculus

We now re-contextualize our result in the frame of typed $\lambda\mu$-calculus (see [8–10] for definitions).

5.1 Embedding typed $\lambda\mu$-calculus into $\boldsymbol{LK}_p^\eta$

Terms in this calculus denote deductions in Parigot's Classical Natural Deduction (**CND** for short) restricted to the multiplicative implication and universal quantifiers of first and second order. This natural deduction is embeddable in **LK** in the usual way, that is introduction rules are read off as right rules and elimination rules as cuts against the left rule, e.g.:

$$\dfrac{\dfrac{\vdots}{\vdash A \to B} \qquad \dfrac{\dfrac{\vdots}{\vdash A} \qquad \dfrac{}{B' \vdash B}}{A \to B \vdash B}}{\vdash A}$$

Observe that if all formulas are polarized (that is, in this case, chosen of colour t), the proof above does satisfy the η-constraint, for it is the 'primed' B that should be main and it is. Hence this embedding maps **CND** into $\mathbf{LK}_p^\eta$ and this embedding even happens to be a homomorphism with respect to normalization (up to the equivalence induced by the 'delocalization' of structural rules, cf. [6]).

5.2 *Reversion of a $\lambda\mu$-term*

Let R, the *reversion*, be the mapping of unnamed terms of type $A \rightarrow B$ to unnamed terms of the form $\lambda x^A \mu\beta^B\, t$, defined by induction as follows:

(i) $R(x^{A\rightarrow B}) = \lambda z^A \mu\beta^B\, [\beta](x)z$;

(ii) $R(\lambda x^A u^B) = \lambda x^A \mu\beta^B\, [\beta]u^B$;

(iii) $R((u)^{C\rightarrow A\rightarrow B} v^C) = \lambda z'^A \mu\beta^B\, [\beta](\lambda z^C \mu\epsilon'^B\, u'')v^C$, if $R(u) = \lambda z^C \mu\epsilon^{A\rightarrow B}\, u'$ and $R(\mu\epsilon^{A\rightarrow B}\, u') = \lambda z'^A \mu\epsilon'^B\, u''$;

(iv) $R(\mu\alpha^{A\rightarrow B}\, t) = \lambda x^A \mu\beta^B\, t[[\beta]u'[\beta/\epsilon, x/z]/[\alpha]u]$, if $R(u) = \lambda z^A \mu\epsilon^B\, u'$.

This application can be extended to a mapping of unnamed terms of type $\forall X\, A$ to unnamed terms of the form $\forall X \mu\beta^A\, t$.

5.3 *Guess*

Now for a plausible guess: 1) the equivalence relation generated by R is compatible with $\lambda\mu$-normalization and 2) our main result still holds, that is, two linear $\lambda\mu$-terms proving an equivalence between archetypes compose in both directions to a unit in the quotient.

Reversion, which is just η-expansion in the intuitionistic case, was already (independently) considered by Parigot as a preliminary transformation in the problem of reading back $\lambda\mu$ integers; also Herbelin in [4,3] dealt with fully reversed terms (which is possible only in the absence of second order quantification) in his game-theoretic interpretation of $\lambda\mu$; finally Ong, in a recent paper [7], proposes a $\mu\zeta$-rule, which might define the same equivalence relation as ours, and proves its soundness by model-theoretic means. Concerning this last study it would be interesting to see whether our result is expressible in his categorical framework.

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Appendix: LK, classical logic

Identity axiom and cut rule:

$$(\mathrm{Ax})\quad A \vdash A \qquad\qquad (\mathrm{cut})\quad \frac{\Gamma_1 \vdash \Delta_1, A \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Axioms for the constants:

$$(\top_m)\quad \vdash \top_m \qquad\qquad (\top_a)\quad \Gamma \vdash \top_a, \Delta$$

$$(\bot_m)\quad \bot_m \vdash \qquad\qquad (\bot_a)\quad \Gamma, \bot_a \vdash \Delta$$

Negation rules:

$$(\mathrm{L}\neg)\ \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad\qquad (\mathrm{R}\neg)\ \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

Multiplicative logical rules:

$$(\mathrm{L}\overset{m}{\to})\ \frac{\Gamma_1 \vdash \Delta_1, A \qquad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \overset{m}{\to} B \vdash \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\overset{m}{\to})\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \overset{m}{\to} B, \Delta}$$

$$(\mathrm{L}\vee_m)\ \frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee_m B \vdash \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\vee_m)\ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee_m B, \Delta}$$

$$(\mathrm{R}\wedge_m)\ \frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge_m B, \Delta_1, \Delta_2} \qquad\qquad (\mathrm{L}\wedge_m)\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge_m B \vdash \Delta}$$

Additive logical rules:

$$(\mathrm{R}\overset{a}{\to})\ \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A \overset{a}{\to} B, \Delta} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \overset{a}{\to} B, \Delta} \qquad\qquad (\mathrm{L}\overset{a}{\to})\ \frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{\Gamma, A \overset{a}{\to} B \vdash \Delta}$$

$$(\mathrm{R}\vee_a)\ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee_a B, \Delta} \quad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee_a B, \Delta} \qquad\qquad (\mathrm{L}\vee_a)\ \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee_a B \vdash \Delta}$$

$$(\mathrm{L}\wedge_a)\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge_a B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge_a B \vdash \Delta} \qquad\qquad (\mathrm{R}\wedge_a)\ \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge_a B, \Delta}$$

Rules for quantifiers (y not free in Γ, Δ):

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x\, A \vdash \Delta} \qquad\qquad (\mathrm{R}\forall)\ \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x\, A, \Delta}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x\, A \vdash \Delta} \qquad\qquad (\mathrm{R}\exists)\ \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x\, A, \Delta}$$

Structural rules:

$$(\mathrm{LW})\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad\qquad (\mathrm{RW})\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

$$(\mathrm{LC})\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad\qquad (\mathrm{RC})\ \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$$